Limit cycles in Liénard equations

Giovanni Tosi and Simon S. Wang*
Department of Aeronautical and Automotive Engineering,
Loughborough University, Loughborough, LE11 3TU

Key words Limit cycle, Liénard equation, Planar polynomial vector field, Hilbert

1. Introduction

An analytical estimation of the existence and characteristics of limit cycles in a given planar polynomial vector field represents a significant progress towards the complete answer to the second part of Hilbert's 16th problem. In a very recent work [1], the second author of this present paper has developed a theory to fulfil this purpose. One major conclusion of the theory is that the number of limit cycles nested around a critical point in a general planar polynomial vector field is bounded by the Hilbert number $H(n) = n^2 - 1$ where n is the order of the vector field. It is well known that linear vector fields have no limit cycles and this, of course agrees with the conclusion. Shi [2] shows that there are maximum three limit cycles nested around a critical point in quadratic vector fields. Again, it is in an agreement with the conclusion. For cubic vector fields results from previous studies [3,4,5] are also in an agreement with the conclusion whilst the result from the work [6] is in a disagreement although there exists some doubt about the result. In this present work, a detailed study is given to the limit cycles in a fifteenth order Liénard equation by using both the theory [1] and numerical simulations to check the validity of the theory. The method of analysis is briefly given in Section 2. An application example and conclusions are presented in Section 3 and 4, respectively.

2. Method of analysis

2.1 Liénard equation and the theory of Wang [1]

The Liénard equation takes the following form:

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = \omega^2 x_1 - \varepsilon \sum_1^m \mu_i x_2^i \end{cases} \tag{1}$$

where m is an integer called the order of the equation, ω , ε , μ are numerical parameters and the μ terms are usually called damping terms. The origin (*0,0*) is its only critical point. For this particular planar polynomial vector field, the theory [1] shows that its number of limit cycles is bounded by the number of the positive real roots of the following polynomial equation in terms of a^2 in the whole space of parameter μ ,

$$\int_0^{2\pi/\omega} \sum_0^n (a\omega)^{2i} \mu_{2i+1} (\sin \omega\tau)^{2i+2} d\tau = 0 \tag{2}$$

That is, the maximum number of limit cycles in Liénard equations of order m is n when $m=2n+1$ or $2n+2$. It is seen that the existence of limit cycles in Liénard equations depends on the odd damping terms only. The even damping terms, however will have influences on the physical behaviour of the limit cycles [1]. When the parameter ε is small the theory predicts the number, amplitudes and

*Corresponding author: e-mail: s.wang@lboro.ac.uk, phone: 0044 1509227252, Fax: 0044 15097275

frequencies of limit cycles. When ε is large the theory gives the upper bound for the number of limit cycles.

2.2 Construction of Liénard equations and numerical simulation

To verify the theory, an inverse process is adopted in this work. That is, a group of *n* linear algebraic equations in terms of *n+1* odd μ parameters is obtained first by substituting a group of *n* given values of *a* into the polynomial equation (2). The parameters μ is then solved from this group of equations. A Liénard system of order *m* is, therefore constructed with *n* limit cycles whose amplitudes and frequencies should have the values that have been given when the parameter ε is small. When the parameter ε is large, the number of limit cycles of the system should be bounded by *n*. It is noted that without loosing any generality the parameter ω is taken to be a unit value throughout the following analysis. In the next section, limit cycles in a fifteenth order Liénard equation are studied.

3. Application example

A fifteenth order Liénard equation is considered here. Therefore, equation (2) becomes

$$\frac{6435}{16384}\mu_{15}a^{14}+\frac{429}{1024}\mu_{13}a^{12}+\frac{231}{512}\mu_{11}a^{10}+\frac{63}{128}\mu_9a^8+\frac{35}{64}\mu_7a^6+\frac{5}{8}\mu_5a^4+\frac{3}{4}\mu_3a^2+\mu_1=0 \quad (3)$$

This equation can only have seven positive real roots at most – hence resulting in seven limit cycles. In the reverse construction process, the amplitudes of these seven limit cycles are set to be $a=0.5^{0.5},1.0^{0.5},1.5^{0.5},2.0^{0.5},2.5^{0.5},3^{0.5},3.5^{0.5}$ and $\mu_1=1$ to achieve a set of optimised parameters μ which are given as

$$\begin{cases}\mu_1=1\\ \mu_3=-6.9143\\ \mu_5=16.6755\\ \mu_7=-19.647\\ \mu_9=12.642\\ \mu_{11}=-4.531\\ \mu_{13}=0.849\\ \mu_{15}=-0.0647\end{cases} \quad (4)$$

Therefore, a fifteenth order Liénard equation is constructed as

$$\begin{cases}\dot{x}_1=-x_2\\ \dot{x}_2=x_1-\varepsilon\left(-0.0647x_2^{15}+0.849x_2^{13}-4.531x_2^{11}+12.642x_2^9-19.647x_2^7+16.6755x_2^5-6.9143x_2^3+x_2\right)\end{cases} \quad (5)$$

Comparison of theoretical (denoted by a subscript t) amplitudes and frequencies of all the limit cycles for six ε values is given in Table 1. It is observed that $a(x_2)$ for all the limit cycles can always be accurately predicted by the theory regardless of the ε values whilst the $a(x_1)$ and ω_a are accurately predicted when ε is not large. It is noted that when ε is smaller than 3.9 there are seven limit cycles as predicted by the theory. The portraits of the seven cycles are shown in Figure 1. When ε approaches to 3.8 the two inner most cycles start to impact to each other as shown in Figure 2 and finally collapse at $\varepsilon=3.9$ as shown in Figure 3. Thus, there only five limit cycles left. It is noted that the first 'bubble burst' occurs around the threshold $\varepsilon=4.0.$ When $\varepsilon=5.1$ the second 'bubble burst' happens as shown in Figure 4. Therefore, there are only three limit cycles left. When ε approaches to

9 the two inner cycles of the remaining three start to impact to each other as shown in Figure 5 resulting the third 'bubble burst' at $\varepsilon=10$. Finally, there is only one limit cycle left. Its phase portrait is shown in Figure 6. It is noted from Table 1 that all the collisions happens due to the more rapid growth of the inner cycle in the dimension of x_1.

Table 1. Parameter comparison between theoretical and simulated values for 15th order Liénard Equation

	$\varepsilon=0.5$						$\varepsilon=3.8$					
Cycle	$a(x_1)$	$a(x_1)_t$	$a(x_2)$	$a(x_2)_t$	ω_a	ω_{at}	$a(x_1)$	$a(x_1)_t$	$a(x_2)$	$a(x_2)_t$	ω_a	ω_{at}
1	0.706	0.707	0.704	0.707	0.997	1	0.954	0.707	0.868	0.707	0.861	1
2	0.986	1	0.986	1	0.997	1	0.975	1	0.901	1	0.873	1
3	1.194	1.225	1.194	1.225	1.005	1	1.282	1.225	1.261	1.225	0.959	1
4	1.373	1.414	1.373	1.414	0.997	1	1.406	1.414	1.391	1.414	0.974	1
5	1.536	1.581	1.536	1.581	0.997	1	1.597	1.581	1.587	1.581	0.982	1
6	1.724	1.732	1.724	1.732	0.997	1	1.747	1.732	1.739	1.732	0.989	1
7	1.872	1.871	1.872	1.871	1.005	1	1.980	1.871	1.875	1.871	0.982	1

	$\varepsilon=3.9$						$\varepsilon=5.1$					
Cycle	$a(x_1)$	$a(x_1)_t$	$a(x_2)$	$a(x_2)_t$	ω_a	ω_{at}	$a(x_1)$	$a(x_1)_t$	$a(x_2)$	$a(x_2)_t$	ω_a	ω_{at}
1	1.285	1.225	1.263	1.225	0.959	1	1.614	1.581	1.594	1.581	0.967	1
2	1.405	1.414	1.389	1.414	0.969	1	1.755	1.732	1.738	1.732	0.967	1
3	1.598	1.581	1.587	1.581	0.980	1	2.431	1.871	1.872	1.871	0.967	1
4	1.748	1.732	1.738	1.732	0.983	1						
5	1.984	1.871	1.875	1.871	0.980	1						

	$\varepsilon=9$						$\varepsilon=10$					
Cycle	$a(x_1)$	$a(x_1)_t$	$a(x_2)$	$a(x_2)_t$	ω_a	ω_{at}	$a(x_1)$	$a(x_1)_t$	$a(x_2)$	$a(x_2)_t$	ω_a	ω_{at}
1	1.754	1.581	1.661	1.581	0.843	1	2.355	1.871	1.867	1.871	0.885	1
2	1.789	1.732	1.709	1.732	0.867	1						
3	2.279	1.871	1.869	1.871	0.904	1						

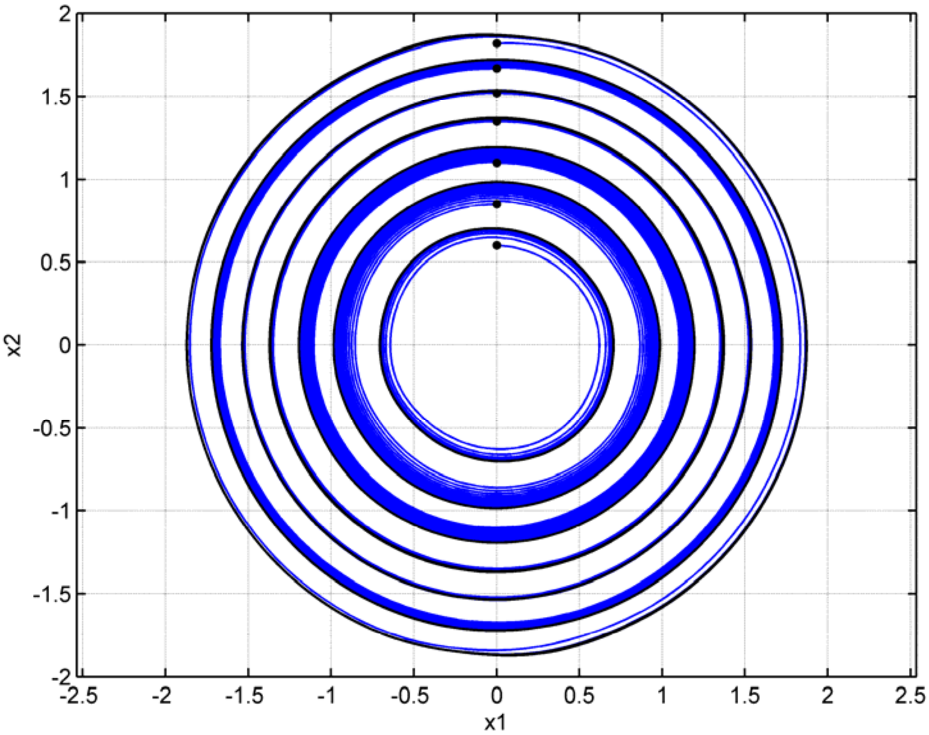

Figure 1– Phase Portrait of 15th Order Liénard Equation, $\varepsilon = 0.5$

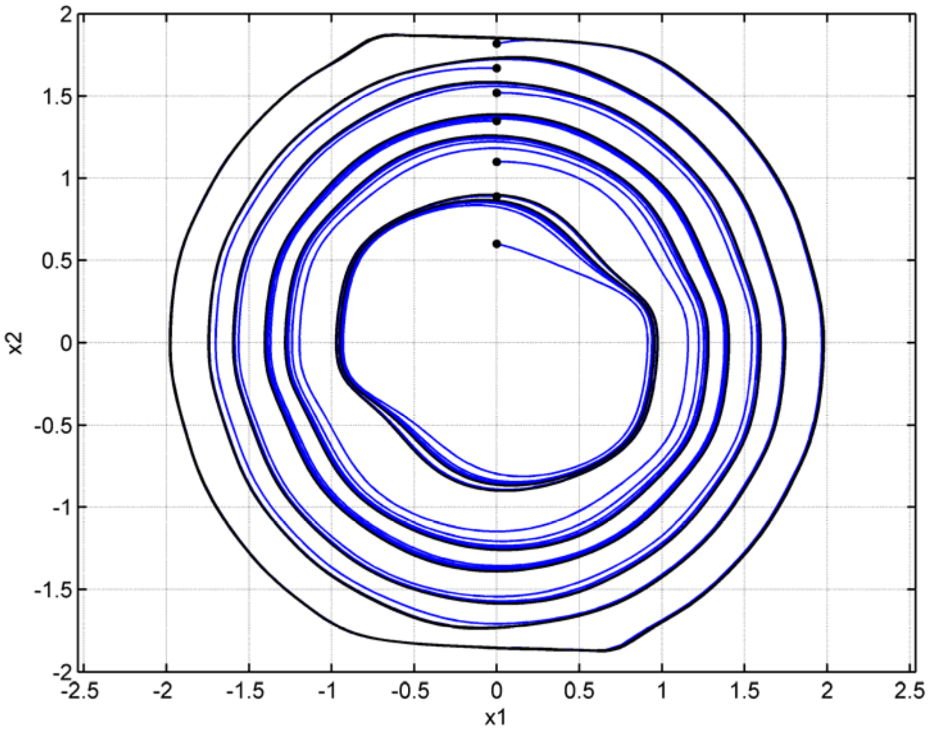

Figure 2 Phase Portrait of 15th Order Liénard Equation, $\varepsilon = 3.8$

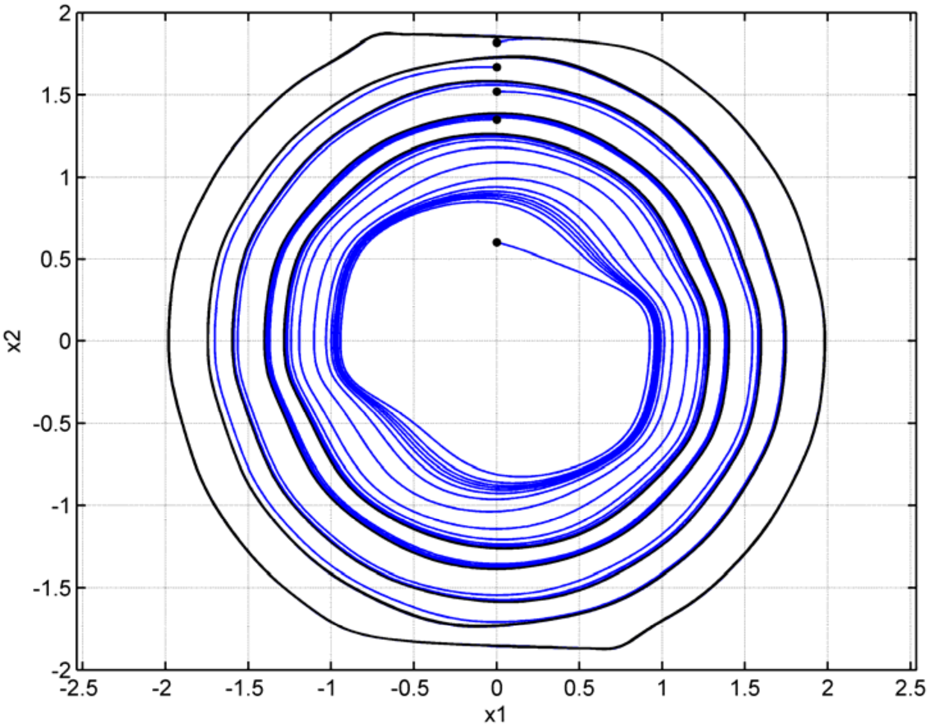

Figure 3 – Phase Portrait of 15th Order Liénard Equation, ε=3.9

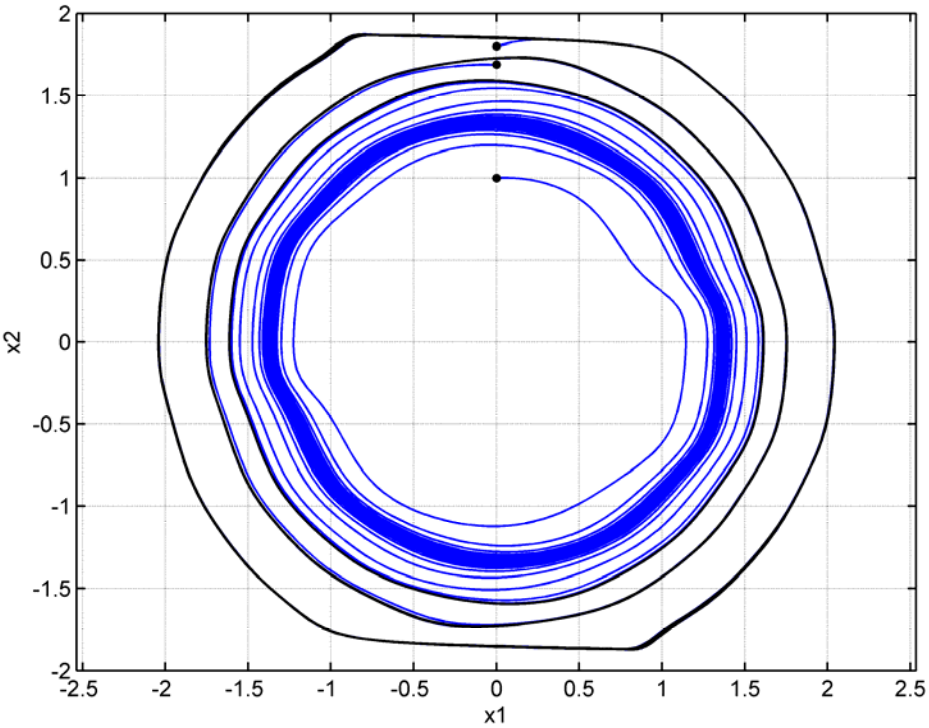

Figure 4 – Phase Portrait of 15th Order Liénard Equation, ε=5.1

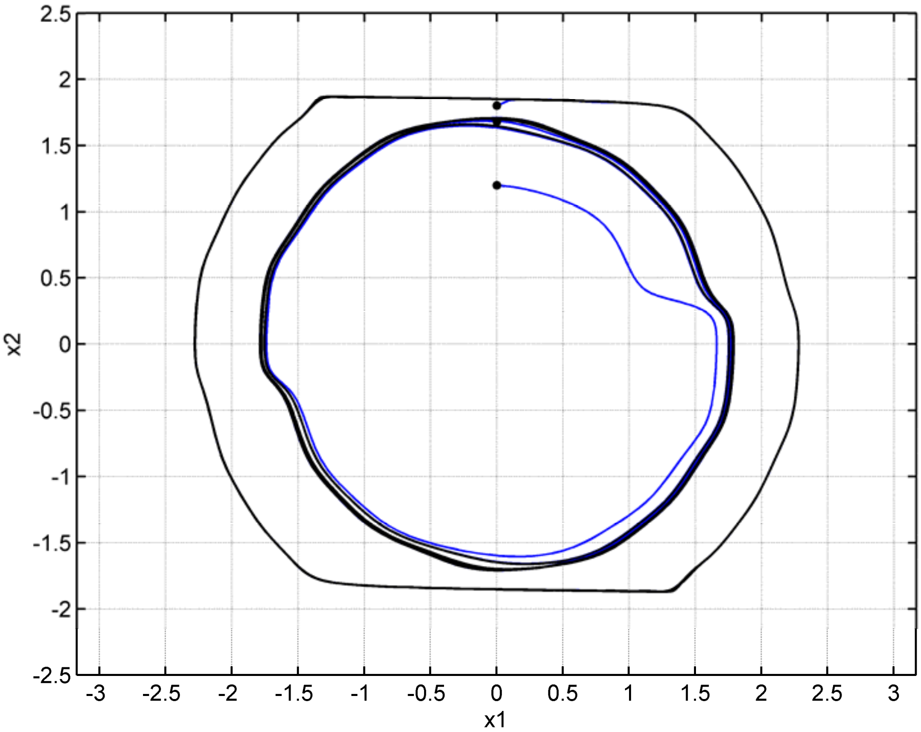

Figure 5 – Phase Portrait of 15th Order Liénard Equation, ε=9

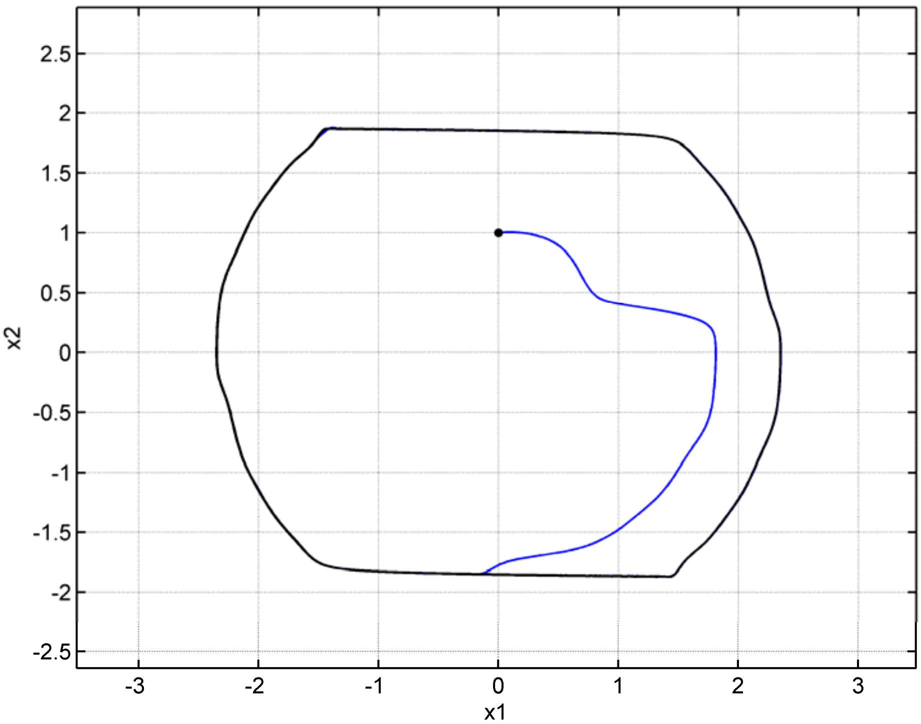

Figure 6 – Phase Portrait of 15th Order Liénard Equation, ε=10

4. Conclusions

The theory of Wang [1] for the solution of the second part of Hilbert's 16th problem is applied to study the limit cycles in Liénard equation, a particular planar polynomial vector field. A fifteenth order Liénard system is considered. Computer simulated results show that the theory gives the exact number of limit cycles and accurate predictions of their amplitudes and frequencies when the system parameter ε is not large. When ε is large the theory can still give accurate predictions for the amplitudes $a(x_2)$ for all the limit cycles whilst there exist discrepancies between the theoretical predictions and simulated results for amplitudes $a(x_1)$ and frequencies ω_a. The discrepancy increases with the order of limit cycles and the increase of the parameter ε. It is also found that the frequencies of limit cycles decrease with the increase of the order of limit cycles due to the larger orbits. Moreover, with the increase of the parameter ε all the cycles become larger and their frequencies, therefore smaller. Therefore, the theoretical values of the frequencies provide the upper bound for the frequencies of all the limit cycles. It is also found that it is always that the two inner most cycles impact to each other due to the more rapid growth of the inner one in the dimension of x_1 and collapse resulting the 'bubble burst' phenomenon when system parameter ε increases. The simulated results prove that the theory [1] provides the upper bound for the number of limit cycles in Liénard equations.

References

[1] S. Wang, Towards the answer to the second part of Hilbert's 16th problem. *World Journal of Engineering.* **1**, 32 (2004) (also, An invited lecture in the Fourth World Congress of Nonlinear Analysis, June 30 – July 7, 2004, Orlando, Florida, USA).

[2] S. Shi, A concrete example of the existence of four limit cycles for plane quadratic systems. *Sci. China Ser* **A11**, 1051 (1979).

[3] N. G. Lloyd, J. M. Pearson, E. Saez, I. Szanto, limit cycles of a cubic Kolomogorov system. *Appl. Math. Lett* , **9**, 15 (1996).

[4] P. Yu, M. Han, Twelve limit cycles in a cubic order planar system with z_2 - symmetry. Communications on Pure and Applied Analysis. **3**, 515 (2004).

[5] M. Han, Y. Lin, P. Yu, A study of the existence of limit cycles of a planar system with third degree polynomials, *International Journal of Bifurcation and Chaos*, **14**, 41 (2004).

[6] H. Zoladek, Eleven small limit cycles in a cubic vector field. *Nonlinearity.* **8**, 843 (1995).