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Some Properties of a class of Stochastic Heat Equations

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DEDICATION

Dedicated to God Almighty for seeing me through; for His provision, mercy and grace.

To my family, my parents Mr. Sampson and Mrs. Philomina Igwurube; my uncles- Nelson, Michael and Kingsley Ebe; my brothers and sisters- Chigozie, Amobi, Chinero, Ebuka, Anayo, Izu etc; and to my lovely wife Justina Onyinyechi Omaba.

Omaba, M. Ejighikeme.

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CONTENTS

ABSTRACT

We study stochastic heat equations of the forms $[\partial_t u - \mathcal{L}u]\mathrm{d}t\mathrm{d}x = \lambda\int_{\mathbf{R}} \sigma(u,h)\tilde{N}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}h)$, and $[\partial_t u - \mathcal{L}u]\mathrm{d}t\mathrm{d}x = \lambda\int_{\mathbf{R}^d} \sigma(u,h)N(\mathrm{d}t,\mathrm{d}x,\mathrm{d}h)$. Here, $u(0,x) = u_0(x)$ is a non-random initial function, N a Poisson random measure with its intensity $\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}h)$ and $\nu(\mathrm{d}h)$ a Lévy measure; $\tilde{N}$ is the compensated Poisson random measure and $\mathcal{L}$ a generator of a Lévy process. The function $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and $\lambda > 0$ the noise level. The above discontinuous noise driven equations are not always easy to handle. They are discontinuous analogues of the equation introduced in [44] and also more general than those considered in [10]. We do not only compare the growth moments of the two equations with each other but also compare them with growth moments of the class of equations studied in [44]. Some of our results are significant generalisations of those given in [10] while the rest are completely new. Second and first growth moments properties and estimates were obtained under some linear growth conditions on σ. We also consider $\mathcal{L} := -(-\Delta)^{\alpha/2}$, the generator of α-stable processes and use some explicit bounds on its corresponding fractional heat kernel to obtain more precise results. We also show that when the solutions satisfy some non-linear growth conditions on σ, the solutions cease to exist for both compensated and non-compensated noise terms for different conditions on the initial function $u_0(x)$. We consider also fractional heat equations of the form $\partial_t u(t,x) = -(-\Delta)^{\alpha/2}u(t,x) + \lambda\sigma(u(t,x)\dot{F}(t,x)$, for $x \in \mathbf{R}^d$, $t > 0$, $\alpha \in (1,2)$, where $\dot{F}$ denotes the Gaussian coloured noise. Under suitable assumptions, we show that the second moment $\mathrm{E}|u(t,x)|^2$ of the solution grows exponentially with time. In particular we give an affirmative answer to the open problem posed in [32]: given u_0 a positive function on a set of positive measure, does $\sup_{x\in\mathbf{R}^d} \mathrm{E}|u(t,x)|^2$ grow exponentially with time? Consequently we give the precise growth rate with respect to the parameter λ.

CHAPTER 1

INTRODUCTION

Systems of Partial differential equations (PDEs) best describe at a macroscopic level majority of physical phenomena (some modelling quantities) like densities, temperatures, concentrations, etc of many natural, human/biological, chemical, mechanical, economical/financial systems and processes.

In stochastic partial differential equations (SPDEs), white noise $\dot{W}(t, x)$ has been one of the most commonly used noise terms. For past few decades, there have been significantly much advancements in the study of random field solutions to SPDEs driven by the general Wiener/Brownian noises. Researchers have focused mainly on the analysis of heat and wave equations perturbed by Gaussian white noise in time with spatial correlations [99, 37, 38, 42, 7]. Whereas, the SPDEs driven by Gaussian noise have been well studied for a long time, the SPDEs driven by Lévy (Poisson) noise have only been investigated more extensively and intensively quite of recent [1, 2, 9, 13, 6, 77, 73, 39, 51]. In its recent development, SPDEs driven by fractional type noises have received great attention too [8, 11, 14, 15, 93, 73, 25, 26]. Though the white noise term possesses many attractive modelling properties, Lévy noise has better modelling characteristics [23, 24]. Stochastic PDEs driven by jump processes or jump type noises (known as stochastic forcing terms) such as Lévy-type or Poisson-type perturbations have become important and popular for modelling physical, biological and financial phenomena. The Lévy-type perturbations produce a better modelling result and performance of those natural occurrences and phenomena of some real world modelling, capturing some large moves and unpredictable events unlike Brownian motion perturbation that has many imperfections. Lévy noise $\tilde{N}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}h)$ or $N(\mathrm{d}t, \mathrm{d}x, \mathrm{d}h)$ has a very rich and vast applications in Finance, Economics, Physics, to mention but a few. The jump processes of the Lévy noise can be used to model market behaviours of price processes. The jumps are particularly relevant for the purpose of modelling the price process of financial assets: structure of futures and forward prices, interest rate models, and so on. They can describe more accurately the observed realities of financial markets. The Lévy noise has a representation that consists of a small jump term and a big jump term. The small jump term describes the daily jitter that causes minor fluctuations in stock prices while the big jump term describes the large stock price movements caused by major market upsets arising from weather conditions, natural disasters like flood, earthquake, tornado, hurricane and volcanic eruptions.

This work was inspired by [44] and the references therein, where a non-linear parabolic SPDEs of the form $\partial_t u = \mathcal{L}\, u + \sigma(u)\dot{W}$ with $\dot{W}$ as the space-time white noise was considered. The function $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and $\mathcal{L}$ the L^2-generator of a Lévy process. In what follows, we consider some discontinuous analogue of results in [44] which are of the general Lévy-type space-time white noises $\tilde{N}(\mathrm{d}t,\, \mathrm{d}x,\, \mathrm{d}h)$ and $N(\mathrm{d}t,\, \mathrm{d}x,\, \mathrm{d}h)$. To understand the full behaviour of the solutions and to have an explicit estimate for the generator of the process, we consider the fractional Laplacian as a special case of the generator of a Lévy process. Some precise conditions for existence and uniqueness of the solutions were given and we show that the solutions grow in time at most a precise exponential rate at some time interval; and if the solutions satisfy some non-linear conditions then it ceases to exist at some finite time t. Albeverio and Wu in [1] studied the parabolic SPDEs driven by Poisson white noise with $\mathcal{L} := \frac{1}{2}\Delta$; Δ the Laplacian, where they established the existence and uniqueness of the solution. Fournier in [43] also studied the case of $\mathcal{L} := \Delta$ of the parabolic SPDE driven by a white noise and a compensated Poisson measure, where he proved the existence and the uniqueness of the weak solution and also studied its Malliavin calculus. We proved the existence and uniqueness of solution to the parabolic SPDE driven by both compensated and non-compensated Poisson measures for $\mathcal{L}$ the L^2 generator of a Lévy process on the space $(L^p,\, \|.\|_{p,\,\beta})$ for a family of p-norms as defined in [44] for $p = 1,\, 2$.

There are three major approaches to solving an SPDEs that appear in literature: the Martingale measure approach [99, 34], the Variational approach [21, 82, 7, 91, 92] and the Semigroup theory approach [33, 84, 85, 20, 74]. A stochastic partial differential equation, like a partial differential equation, can be viewed in two major ways: firstly, one can consider its solution as a real-valued function of t and x, where t is the time parameter, and x (which varies depending on the nature of the domain say $\Omega \subset \mathbf{R}^d$) is a space parameter; one can also consider the solution as a function of t with values in a space of functions of x, say $L^2(\Omega)$ (Da Prato-Zabczyk Approach). Secondly a solution of a stochastic partial differential equation can be considered either as a real-valued random field indexed by t and x, or as a stochastic process indexed by t with values in an infinite dimensional space (Walsh Approach). The variational approach and the semigroup approach are mainly concerned with giving a rigorous meaning to solutions of stochastic differential equations in an infinite dimensional spaces. Walsh's Martingale approach involves representing the solution as an integral equations with respect to martingale measures, which makes use of tools in Measure theory, Potential theory, Harmonic analysis and general stochastic analysis and this is the approach we will be adopting in this work.

1.1 Formulation of the solution(s)

Here, we formulate the research problems and give underlying conditions on the equations' parameters.

1.1.1 The compensated equation

Consider the following stochastic heat equations driven by a compensated Poisson noise

$$\left[\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x)\right]\mathrm{d}x\mathrm{d}t = \lambda \int_{\mathbf{R}} \sigma(u(t,\,x),\,h)\tilde{N}(\mathrm{d}t,\,\mathrm{d}x,\,\mathrm{d}h), \tag{1.1.1}$$

with initial condition $u(0,x) = u_0(x)$. Here and throughout, $u_0 : \mathbf{R} \to \mathbf{R}_+$ is a non-random function, and $\mathcal{L}$ is the L^2- generator of a Lévy process.

Definition 1.1.1. We say that a process $\{u(t,\,x)\}_{x\in\mathbf{R},t>0}$ is a mild solution of (1.1.1) if a.s, the following is satisfied

$$\begin{aligned} u(t,x) &= \int_{\mathbf{R}} p(t,\,x,\,y)u_0(y)\,\mathrm{d}y \\ &+ \lambda \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} p(t-s,\,x,\,y)\sigma(u(s,\,y),h)\tilde{N}(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s), \end{aligned} \tag{1.1.2}$$

where $p(t,\,.,\,.)$ is the heat kernel. If in addition to the above, $\{u(t,\,x)\}_{x\in\mathbf{R},t>0}$ satisfies the following condition

$$\sup_{0\leq t\leq T} \sup_{x\in\mathbf{R}} \mathrm{E}|u(t,\,x)|^2 < \infty, \tag{1.1.3}$$

for all $T > 0$, then we say that $\{u(t,\,x)\}_{x\in\mathbf{R},t>0}$ is a random field solution to (1.1.1).

In order to state our theorem, we make the following notation. Define

$$\Upsilon(\beta) := \frac{1}{2\pi} \int_{\mathbf{R}} \frac{\mathrm{d}\xi}{\beta + 2\mathcal{R}\mathrm{e}\Psi(\xi)} \qquad \text{for all } \beta > 0, \tag{1.1.4}$$

where Ψ is the characteristic exponent for the Lévy process. A result of Dalang [34] shows that equation (1.1.1) has a unique solution with the requirement that $\Upsilon(\beta) < \infty$ for all $\beta > 0$ which forces $d = 1$ and coincides to a similar situation to that in [44] since $\tilde{N}$ is a martingale valued Poisson measure. Fix some $x_0 \in \mathbf{R}$ and define the *upper p th-moment Liapunov exponent* $\bar{\gamma}(p)$ of u [at x_0] as

$$\bar{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{E}\left[|u(t\,,x_0)|^p\right] \quad \text{for all } p \in (0\,,\infty), \tag{1.1.5}$$

and say that u is (see [44])

1. **weakly intermittent** if, regardless of the value of x_0,

$$\bar{\gamma}(2) > 0 \text{ and } \bar{\gamma}(p) < \infty \quad \text{for all } p > 2,$$

2. **fully intermittent** if, regardless of the value of x_0, the map

$$p \mapsto \frac{\bar{\gamma}(p)}{p} \text{ is strictly increasing for all } p \geq 2.$$

The study of exponential behaviours of solution as $t \to \infty$, can best be interpreted by the notion of intermittency. The concept means that, as $t \to \infty$, the solution exhibits a spatially extremely irregular structure consisting of islands of high peaks which are located far from each other. The solution u is influenced by the interaction or competition between the generator of a semigroup $\mathcal{L}$ which has a smoothing effect, and the noise potential $\tilde{N}(\mathrm{d}h,\, \mathrm{d}x,\, \mathrm{d}t)$, which makes the solution spatially irregular. This is a long-time behaviour of a system exhibiting an intermittency effect. The notion of intermittency arose originally in the study of turbulent flow, firstly as a phenomenon in Physics and Statistical particle Physics. It is a concept of instabilities in random media that arose as a result of high value quantity growth of some structures. Intermittency connotes random deviations from smooth and regular behaviour (see [90, 103, 104] and their references for details). Intermittency for the parabolic Anderson problem was studied in [22] while intermittency properties in a hyperbolic Anderson problem was studied in [36].

Remark 1.1.2. Unfortunately, we do not have any result for $\overline{\gamma}(p)$ for $p \geq 2$ and we therefore cannot talk about intermittency properties of the solution defined and studied for the white noise case [44]. The reason is that here, we do not have an appropriate Burkholder's inequality to use. But under some further assumptions, one can have $\overline{\gamma}(p) < \infty$. We however do not pursue this here.

1.1.2 The non-compensated equation

Consider the non-compensated equation.

$$\left[\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x)\right]\mathrm{d}x\mathrm{d}t = \lambda \int_{\mathbf{R}^d} \sigma(u(t,\,x),\,h)N(\mathrm{d}h,\,\mathrm{d}x,\,\mathrm{d}t), \tag{1.1.6}$$

with initial condition $u(0,x) = u_0(x)$. Here again $\mathcal{L}$ is the L^2- generator of a Lévy process.

Remark 1.1.3. Unlike the compensated noise term $\tilde{N}(\mathrm{d}t,\,\mathrm{d}x,\,\mathrm{d}h)$, the non-compensated noise $N(\mathrm{d}t,\,\mathrm{d}x,\,\mathrm{d}h)$ is not a martingale-valued Poisson random measure. The existence and uniqueness of the solution to (1.1.1) does not depend on the integrability condition (1.1.5) and the first moment of the solution exists; hence the existence and uniqueness for all $d \geq 1$.

Definition 1.1.4 (of random field solution)**.**

We seek a mild solution to equation (1.1.6) of the form.

$$\begin{aligned} u(t,\,x) &= \int_{\mathbf{R}^d} p(t,\,x,\,y)u_0(y)\,\mathrm{d}y \\ &+ \lambda \int_0^t \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} p(t-s,\,x,\,y)\sigma(u(s,\,y),h)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s), \end{aligned} \tag{1.1.7}$$

with $p(t,\,.,\,.)$ the heat kernel. We impose the following integrability condition on the solution.

$$\sup_{t>0}\sup_{x\in\mathbf{R}^d} \mathrm{E}|u(t,\,x)| < \infty.$$

Let us define a Poisson random measure $N = \sum_{i\geq 1}\delta_{(T_i,\,X_i,\,Z_i)}$ on $\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d$ defined on a probability space $(\Omega,\,\mathcal{F},\,P)$ with intensity measure $\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}h)$ where ν is a Lévy measure on $\mathbf{R}^d$; that is, it satisfies the following

$$\int_{\mathbf{R}^d}(1\wedge h^2)\nu(\mathrm{d}h) < \infty.$$

According to [6], let $(\varepsilon_j)_{j\geq 0}$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$ and $1 = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots$. Let

$$\Gamma_j = \{h \in \mathbf{R}^d; \varepsilon_j < |h| < \varepsilon_{j-1}\},\; j \geq 1 \text{ and } \Gamma_0 = \{h \in \mathbf{R}^d; |h| > 1\}.$$

Then for any set $B \in \mathcal{B}(\mathbf{R}_+ \times \mathbf{R}^d)$, define

$$\int_{B\times\Gamma_j} hN(\mathrm{d}t,\,\mathrm{d}x,\,\mathrm{d}h) = \sum_{(T_i,\,X_i)\in B} Z_i I_{\{Z_i\in\Gamma_j\}}, \quad j \geq 0,$$

with the following property: $\mathrm{E}[N(B\times\Gamma_j)] = |B|\nu(\Gamma_j)$ and

$$N(B\times\Gamma_j) = \#\{i\geq 1; (T_i,\,X_i,\,Z_i)\in B\times\Gamma_j\} < \infty \ \ a.s.$$

Therefore the above integral is finite since the sum contains finitely many terms. We are going to make sense of the above Poisson noise integrals in details later in the next chapter.

Consider the mild solution (1.1.7) where $u_0 = 1$, $\sigma(.,h) = 1$ with $N = \sum_{i=1}^{\mu}\delta_{(T_i,\,X_i,\,Z_i)}$. Then (see [43])

$$u(t,x) = 1 + \sum_{i=1}^{\mu} p(t-T_i,x,X_i)I_{\{t>T_i\}}.$$

We observe that for each $\omega\in\Omega$ with $\mu(\omega)\geq 1$, the map $t\mapsto u(t,\,X_i)(\omega)$ explodes when t tends to T_i from the right for each $i\geq 1$ and hence not cádlág. Therefore our mild solutions (1.1.1) and (1.1.6) can either be viewed as weak predictable processes (a version of the process which will be predictable) as defined in [43] or as modified cádlág processes(a cádlág version or modification of the process) in time t [1]. Following [10], we define the mild solution $u(t,x)$ to (1.1.1) as a progressively measurable process such that for any $x\in\mathbf{R}$,

$$\int_0^t \mathrm{E}|u(s,x)|^2\mathrm{d}s < \infty, \text{ that is, } u\in L^2([0,t]).$$

Similarly, the mild solution $u(t,x)$ to (1.1.6) is a progressively measurable process such that for any $x\in\mathbf{R}^d$,

$$\int_0^t \mathrm{E}|u(s,x)|\mathrm{d}s < \infty, \text{ that is, } u\in L^1([0,1]).$$

1.1.3 The white noise and coloured noise equations

Consider the stochastic heat equation

$$\frac{\partial}{\partial t}u(t,\,x) = \mathcal{L}u(t,\,x) + \lambda\sigma(u(t,\,x))\dot{w}(t,\,x), \ \ x\in\mathbf{R}, \text{ and } t>0 \tag{1.1.8}$$

with $u(0,\,x) = u_0(x)$, for all $x\in\mathbf{R}$. The function $u_0:\mathbf{R}\to\mathbf{R}_+$ is a non-random function, $\sigma:\mathbf{R}\to\mathbf{R}$ a Lipschitz continuous function and $\dot{w}(t,\,x)$ denotes white noise on $(0,\infty)\times\mathbf{R}$. We take $\mathcal{L} := -(-\Delta)^{\alpha/2}$, $\alpha>1$. It is given in [44] that as time goes to infinity, the second moment of the mild solution, $\mathrm{E}|u(t,x)|^2$ grows like $\exp(ct)$ for some positive constant c whenever the initial condition u_0 is bounded below. Whether it is possible for one to get rid of this assumption and prove the exponential growth when $u_0(x)$ is not bounded below has been a hard open problem posed in [32]. This question has been addressed for different class of equations. One of the main results here is to give an affirmative answer to the above open problem, by showing that the second moment of the solution grows exponentially with time even if the initial function is not bounded below. Other results include a non-linear noise growth-index of L^2- energy of the solution for time $t > 0$ to (1.1.8). It is known in [66], that intermittency can be associated to non-linear noise excitation which in its informal observation is equivalent to the existence of a non-linear noise excitation. The above equation (1.1.8) has been proved to be intermittent, see [44]. Excitability can be described as a dynamic phenomenon of systems far from equilibrium. All excitable systems exhibit the following states; the existence of a "rest" state, an "excited" (or "firing") state, and a "refractory" (or "recovery") state depending on the amount of external perturbations on the systems. It measures the rate of an exponential growth of the solution with respect to the noise level λ.

In what follows, we extend the results for the case of a coloured noise. Instead of looking at the above equation, we consider the following stochastic heat equation driven by a coloured noise on $\mathbf{R}^d$,

$$\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x) + \lambda\sigma(u(t,x))\dot{F}(t,\, x), \tag{1.1.9}$$

with the initial condition $u(0,\, x) = u_0(x)$, $x \in \mathbf{R}^d$. The parameter σ satisfies same underlying Lipschitz assumption and $\lambda > 0$ is the level of the noise. The term $\dot{F}$ is a spatially-coloured, temporally white, Gaussian noise; a generalised Gaussian random field whose covariance kernel is described as follows

$$\mathrm{E}[\dot{F}(t,x)\dot{F}(s,y)] = \delta_0(t-s)f_\beta(x,y)$$

where the correlation function f_β is the Riesz kernel given by

$$f_\beta(x,y) = \frac{1}{|x-y|^\beta},$$

with parameter $\beta \in (0,d)$, $d \geq 1$ the dimension. The initial function u_0 is assumed to be a bounded non-negative function such that

$$\int_A u_0(x)\mathrm{d}x > 0, \text{ for some } A \subset \mathbf{R}^d.$$

That is, we define u_0 as any measurable function $u_0 : \mathbf{R}^d \to \mathbf{R}_+$ which is positive on a set of positive measure. This assumption implies that the set $A = \{x : u_0(x) > \frac{1}{n}\} \subset \mathbf{R}^d$ has positive measure for all but finite many n. Thus by Chebyshev's inequality,

$$\int_{\mathbf{R}^d} u_0(x)\mathrm{d}x \geq \int_{\{x:\, u_0(x)>\frac{1}{n}\}} u_0(x)\mathrm{d}x \geq \frac{1}{n}\mu\left\{x : u_0(x) > \frac{1}{n}\right\} > 0,$$

where μ is a Lebesgue measure. Following Walsh [99], one defines the mild solution to (1.1.9) by the following integral equation

$$u(t,x) = (P_t u_0)(x) + \int_0^t \int_{\mathbf{R}^d} p(t-s,x,y)\sigma(u(s,y))F(\mathrm{d}y,\,\mathrm{d}s),$$

where

$$(P_t u_0)(x) := \int_{\mathbf{R}^d} p(t,x,y)u_0(y)\mathrm{d}y$$

is the semigroup and $p(t,x,y)$ denotes the fractional heat kernel. We will also be interested in random field solutions which require that the mild solution satisfies the following integrability condition

$$\sup_{x\in\mathbf{R}^d}\sup_{t>0} \mathrm{E}|u(t,x)|^2 < \infty.$$

This further impose that $\beta \leq \alpha$, see [41]. Existence and uniqueness are well known for the equations studied here as given in [44] and the references therein. The constant c with subscripts or superscripts appearing in our results or their proofs will denote some generic constants that we do not keep track of. The main results of the thesis are summarised below.

- We make sense of equations (1.1.1) and (1.1.6), show that their solutions are well defined by establishing their existence and uniqueness under some suitably defined conditions on the parameter $\sigma : \mathbf{R} \to \mathbf{R}$, the non-random function $u_0 : \mathbf{R} \to \mathbf{R}_+$ measurable and bounded.

- The growth moments of the solutions to (1.1.1) and (1.1.6) were established. Whereas second moment estimate was given to the compensated equation (1.1.1), the first moment growth of the non-compensated equation (1.1.6) was estimated and we show that both solutions grow at an exponential rate with time t under some linear growth conditions on σ.

- Given some non-linear growth conditions on the parameter σ, that's, if σ grows faster than linear growth, then there is no random field solutions to (1.1.1) and (1.1.6). While we proved that the compensated equation ceases to exist with the initial function u_0 bounded below, the solution to the non-compensated counterpart on the other hand fails to exist both when the initial data u_0 is a positive function and when it is bounded below .

- Let the initial function $u_0 : \mathbf{R} \to \mathbf{R}_+$ be positive on a set of positive measure. We show that the excitation index of the second moment of the solution u to (1.1.8) at time t is given by $\frac{2\alpha}{\alpha-1}$.
- As an extension, we prove that the excitation index of the second moment of the solution u to (1.1.9) at time t is given by $\frac{2\alpha}{\alpha-\beta}$ with the initial condition u_0 assumed to be a positive function on a set of positive measure.

In a nutshell, we give a plan of this thesis. In chapter one, the research problem(s) were introduced; chapter two surveys some basic definitions and concepts used in the work. Our results are in three main parts. Whereas Chapter three focuses on some properties of the heat equation with respect to compensated Poisson noise, chapter four discusses the results on the heat equation driven by non-compensated Poisson noise. Chapter five is devoted to the proofs of nonlinear noise excitation growth index of the solution for the space-time white noise driven equations and its extension to Riesz kernel spatial correlated noise.

CHAPTER 2

PRELIMINARIES

Now, we give some basic definitions, some concepts on PDEs and Stochastic processes.

Partial Differential Equations

"...Partial differential equations are the basis of all physical theorem."
Bernhard Riemann(1826-1866).

Definition 2.0.5. Let $L^2(\mathbf{R}^d, \mathrm{d}x)$ be a Hilbert space with an inner product

$$(f, g)_{L^2(\mathbf{R}^d, \mathrm{d}x)} = \int_{\mathbf{R}^d} f(x)\, g(x) \mathrm{d}x$$

with $\mathrm{d}x$ a Lebesgue measure on $\mathbf{R}^d$. For all $p \geq 1$, $L^p(\mathbf{R}^d, \mathrm{d}x)$ are Banach spaces with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbf{R}^d} |f(x)|^p \mathrm{d}x \right)^{1/p}.$$

Let the Fourier transform of f be denoted by $\hat{f} := \mathcal{F}f$, where we define the (normalised) Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} \mathrm{e}^{-i\xi x} f(x) \mathrm{d}x, \text{ for all}\, \xi \in \mathbf{R}^d \text{ and } f \in L^1(\mathbf{R}^d).$$

Theorem 2.0.6. ***(Plancherel)*** *The Fourier transform $\mathcal{F} : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ is a unitary map. For all $f, g \in L^2(\mathbf{R}^d)$,*

$$(\hat{f}, \hat{g}) = (f, g).$$

In particular,

$$\|f\|_{L^2(\mathbf{R}^d)} = \|\hat{f}\|_{L^2(\mathbf{R}^d)}.$$

2.1 Fundamental solution of Heat equation

Consider the following heat equation

$$\partial_t u(t,x) = (\mathcal{L}u)(t,x) \text{ such that } u(0,x) = \delta_0(x),$$

where $\mathcal{L}$ is the generator of a Lévy process X_t with characteristic exponent Ψ. Take Fourier transform in x of both sides,

$$\partial_t \hat{u}(t,\xi) = -\Psi(\xi)\hat{u}(t,\xi) \text{ such that } \hat{u}(0,\xi) = 1.$$

Therefore

$$\hat{u}(t,\xi) = \mathrm{e}^{-t\Psi(\xi)} = \hat{P}_t(\xi),$$

and the solution is measure-valued, that's,

$$u(t,x) := \mathrm{P}(X_t \in \mathrm{d}x) := P_t(\mathrm{d}x),$$

where $(P_t)_{t>0}$ is known as the semigroup of the Lévy process X_t.

For $\mathcal{L} = \kappa\Delta$, $\kappa > 0$ and Δ the Laplace operator, then $\hat{P}_t(\xi) = \mathrm{e}^{-t\kappa\xi^2}$ and

$$p(t,x) = \frac{1}{\sqrt{4\kappa\pi t}} \mathrm{e}^{-\frac{x^2}{4\kappa t}}.$$

Definition 2.1.1. (The generator of a Lévy process) Define the domain of $\mathcal{L}$ by

$$\mathcal{D}[\mathcal{L}] = \left\{ \phi \in L^2(\mathbf{R}^d) : \Psi\hat{\phi} \in L^2(\mathbf{R}^d) \right\}.$$

Let $\mathcal{L}$ be the generator of a Lévy process. If $(P_t)_{t>0}$ is the semigroup of a Lévy process X_t, then

$$\mathcal{L}\phi = \lim_{t\to 0} \frac{P_t\phi - \phi}{t} \quad \text{in } L^2(\mathbf{R}^d),\ \forall\, \phi \in L^2(\mathbf{R}^d).$$

For all $\phi \in \mathcal{S}(\mathbf{R}^d)$ (Schwartz space of test functions), then

$$\widehat{\mathcal{L}\phi} = \lim_{t\to 0} \frac{\widehat{P_t\phi} - \hat{\phi}}{t} = \lim_{t\to 0} \frac{\mathrm{e}^{-t\Psi} - 1}{t}\hat{\phi} = -\Psi\hat{\phi}.$$

Therefore $\hat{\mathcal{L}} = -\Psi$.

The $L^2(\mathbf{R}^d)$-generator of the semigroup $\{P_t\}_{t\geq 0}$ is the fractional Laplace operator $-(-\Delta)^{\alpha/2},\ \alpha \in (0, 2)$. The operator with the domain

$$\mathcal{D}[(-\Delta)^{\alpha/2}] = \Big\{\phi \in L^2(\mathbf{R}^d) : |\xi|^\alpha\hat{\phi} \in L^2(\mathbf{R}^d),\ \ 0 < \alpha < 2\Big\},$$

defined by

$$\mathcal{F}\Big((-\Delta)^{\alpha/2}\phi(\xi)\Big) = |\xi|^\alpha\hat{\phi}(\xi),$$

is the fractional Laplacian of order $\alpha/2$.

2.1.1 Mild solution

Consider the non-linear heat equation with a discontinuous noise process

$$\big[\partial_t u(t,x) - \kappa\Delta u(t,x)\big]\mathrm{d}x\mathrm{d}t = \lambda \int_{\mathbf{R}} \sigma(u(t,x), h)N(\mathrm{d}h,\, \mathrm{d}x,\, \mathrm{d}t), \tag{2.1.1}$$

$\kappa > 0$, $t > 0$, $x \in \mathbf{R}$ and $u(0,x) = u_0(x)$, $t > 0$. We define for all smooth function $\phi : \mathbf{R} \to \mathbf{R}$, the semigroup $(P_t)_{t>0}$ as follows,

$$(P_t\phi)(y) := \int_{\mathbf{R}} p(t,x,y)\phi(x)\mathrm{d}x, \tag{2.1.2}$$

for which the integral is defined (exists) and $p(t,x,y)$ given by

$$p(t,x,y) = \frac{1}{\sqrt{4\kappa\pi t}} \exp\big(-\frac{|x-y|^2}{4\kappa t}\big)$$

is the solution of the homogeneous equation (that's at $\sigma = 0$) except at $t = 0$, $(P_0\phi)(y) = \phi(y)$. It follows by multiplying through by ϕ that

$$\int_0^t \int_{\mathbf{R}} \partial_s p(s,x,y)\phi(x)\mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbf{R}} \partial_{xx} p(s,x,y)\phi(x)\mathrm{d}x\mathrm{d}s.$$

By Fubini and integrating by parts, it follows that

$$\int_{\mathbf{R}} p(t,x,y)\phi(x)\mathrm{d}x - \int_{\mathbf{R}} p(0,x,y)\phi(x)\mathrm{d}x = \int_0^t \int_{\mathbf{R}} p(s,x,y)\phi_{xx}(x)\mathrm{d}x\mathrm{d}s.$$

Then by (2.1.2), we have for all test functions ϕ

$$(P_t\phi)(y) = \phi(y) + \int_0^t (P_s\phi_{xx})(y)\mathrm{d}s. \tag{2.1.3}$$

We therefore pose the problem in weak form. Let $\phi \in C_0^\infty(\mathbf{R})$, then by multiplying (2.1.1) by ϕ and integrating in $[\mathrm{d}x\mathrm{d}t]$,

$$\begin{aligned}\int_{\mathbf{R}} u(t,x)\phi(x)\mathrm{d}x &= \int_{\mathbf{R}} u_0(x)\phi(x)\mathrm{d}x + \int_0^t \int_{\mathbf{R}} u(s,x)\phi_{xx}(x)\mathrm{d}x\mathrm{d}s \\ &+ \lambda \int_0^t \int_{\mathbf{R}}\int_{\mathbf{R}} \sigma(u(s,x),h)\phi(x)N(\mathrm{d}h,\,\mathrm{d}x,\,\mathrm{d}s)\end{aligned} \tag{2.1.4}$$

Next, we extend (2.1.4) to smooth functions $\psi(t,x)$ of two variables. Then similarly as in ϕ, let $\psi \in C_0^\infty([0,\infty)\times\mathbf{R})$. Multiply (2.1.1) by ψ and integrate in $[\mathrm{d}x\mathrm{d}t]$,

$$\begin{aligned}\int_{\mathbf{R}} u(t,x)\psi(t,x)\mathrm{d}x &= \int_{\mathbf{R}} u_0(x)\psi(0,x)\mathrm{d}x + \int_0^t \int_{\mathbf{R}} u(s,x)\big[\psi_{xx}(s,x)+\psi_s(s,x)\big]\mathrm{d}x\mathrm{d}s \\ &+ \lambda \int_0^{t^+} \int_{\mathbf{R}}\int_{\mathbf{R}} \sigma(u(s,x),h)\psi(s,x)N(\mathrm{d}h,\,\mathrm{d}x,\,\mathrm{d}s).\end{aligned} \tag{2.1.5}$$

By uniqueness of the solutions, (2.1.4) must satisfy (2.1.5) since they both solve (2.1.1), we fix t and let $\psi(s,y) = (P_{t-s}\phi)(y)$. Then $\psi(t,y) = \phi(y)$ and by (2.1.3), we have that

$$\begin{cases} \psi_{xx}(s,x)+\psi_s(s,x) = 0, & x\in\mathbf{R},\ s>0 \\ \psi(0,x) = (P_t\phi)(x), & x\in\mathbf{R}. \end{cases}$$

Hence, the solution (2.1.5) becomes

$$\begin{aligned}\int_{\mathbf{R}} u(t,x)\phi(x)\mathrm{d}x &= \int_{\mathbf{R}} u_0(y)(P_t\phi)(y)\mathrm{d}y \\ &+ \lambda \int_0^t \int_{\mathbf{R}}\int_{\mathbf{R}} \sigma(u(s,y),h)(P_{t-s}\phi)(y)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s).\end{aligned} \tag{2.1.6}$$

Let ϕ approach a delta function, for example, if one takes ϕ of the form of an approximate identity in $\mathcal{S}(\mathbf{R})$ (Schwartz space of test functions), for example the Gaussian approximate identity $\phi_\epsilon(x) = \frac{1}{(2\pi\epsilon)^{1/2}}\exp\big(-\frac{|x|^2}{2\epsilon}\big)$. As $\epsilon\to 0^+$, then ϕ_ϵ converges weakly to δ and the above equation (2.1.6) for Lebesgue-almost all (t, x) will tend to

$$\begin{aligned} u(t,x) &= \int_{\mathbf{R}} u_0(y)p(t,x,y)\mathrm{d}y \\ &+ \lambda \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} \sigma(u(s,y),h)p(t-s,x,y)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s). \end{aligned}$$

2.2 Stochastic processes

"... Paul Lévy was a painter in the probability world. Like the very great painting geniuses, his palette was his own and his paintings transmuted forever our vision of reality." M. Loéve, in 1971.

A stochastic process with state space S is defined as a collection of random variables $(X_t)_{t\in T}$ defined on the triple $(\Omega, \mathcal{F}, \mathrm{P})$ known as a probability space. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space with a right-continuous filtrations $\{\mathcal{F}_t\}_{t\geq 0}$, such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

Definition 2.2.1. A real-valued stochastic process $(X_t)_{t\geq 0}$ is said to have left (right) limits if for P-a.e. $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ has left (right) limits. Simply put, the paths of the process X have P-a.s. left (right) limits.

Definition 2.2.2. A stochastic process $(X_t)_{t\geq 0}$ is said to be left-continuous (right-continuous) if for P-a.e. $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ is left-continuous (right-continuous).

Definition 2.2.3. (Cádlág process) Let $(X_t)_{t\geq 0}$, be a real-valued stochastic process. Then the process X_t is cádlág (continue á droite, á gauche) or RCLL (right continuous with left limits) if it is a right continuous process with paths having left limits.

Definition 2.2.4. Two real-valued stochastic processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ are called modifications (or versions) of one another if,

$$\text{for all } \ t \in T, \qquad \mathrm{P}\big(X_t = Y_t\big) = 1.$$

That is, for all $t \geq 0$, there exists a null set $\mathcal{N}_t \subset \Omega$ such that

$$X_t(\omega) = Y_t(\omega), \ \forall\, \omega \notin \mathcal{N}_t.$$

Definition 2.2.5. Let $(X_t)_{t\geq 0}$ be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with values in $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$. The process $(X_t)_{t\geq 0}$ is said to be measurable if it is measurable as a function defined on $[0,\infty)\times\Omega$ (with the σ-algebra $\mathcal{B}([0,\infty))\otimes\mathcal{F}$) and values in $\mathbf{R}^d$. That's, let $(\Omega_1,\mathcal{F}_1)$ and $(\Omega_2,\mathcal{F}_2)$ be measurable spaces. Then $X_t:\Omega_1\to\Omega_2$ for $t\geq 0$ is said to be measurable with respect to the σ-algebras $\mathcal{F}_i$, $i=1,2$, if and only if $X_t^{-1}(A)\in\mathcal{F}_1$ for each $A\in\mathcal{F}_2$. The process $(X_t)_{t\geq 0}$ is said to be progressively measurable if for every $T\geq 0$ it is, when viewed as a function $X(t,\omega)$ on the product space $[0,T]\times\Omega$, measurable relative to the product σ-algebra $\mathcal{B}([0,T])\otimes\mathcal{F}_T$.

Definition 2.2.6. (Adapted process) A sequence $(X_t)_{t\geq 0}$ of random variables is said to be adapted to a filtration $(\mathcal{F}_s)_{0\leq s\leq t}$ if, for each s, the random variable X_t is $\mathcal{F}_s$ measurable.

Definition 2.2.7. (predictable σ-algebra) Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathrm{P})$ be a filtered probability space. The σ-algebra on $[0,\infty)\times\Omega$ generated by all sets of the form $\{0\}\times A$, $A\in\mathcal{F}_0$, and $(s,t]\times A$, $0\leq s<t$, $A\in\mathcal{F}_s$, is said to be the predictable σ-algebra for the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Definition 2.2.8. (Predictable process) A real-valued process $(X_t)_{t\geq 0}$ is called predictable with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, or $\mathcal{F}_t$-predictable, if as a mapping from $[0,\infty)\times\Omega\to\mathbf{R}$ it is measurable with respect to the predictable σ-algebra generated by this filtration.

Definition 2.2.9. (Convergence of random variables) Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables and X be another real-valued random variable, all defined on the same probability space $(\Omega,\mathcal{F},\mathrm{P})$. Then

(1) X_n is said to converge almost surely to X, i.e., $X_n\xrightarrow{a.s.}X$ if

$$\mathrm{P}\Big(\big\{\omega\in\Omega:\lim_{n\to\infty}X_n(\omega)=X(\omega)\big\}\Big)=1.$$

(2) X_n converges in L^p-sense to X for all $1\leq p<\infty$, i.e., $X_n\xrightarrow{L^p}X$ if

$$\lim_{n\to\infty}\mathrm{E}_{\mathrm{P}}[|X_n(\omega)-X(\omega)|^p]=0.$$

(3) X_n is said to be convergent in probability to X, i.e., $X_n\xrightarrow{\mathrm{P}}X$ if for every $\epsilon>0$,

$$\lim_{n\to\infty}\mathrm{P}\Big(\big\{\omega\in\Omega:|X_n(\omega)-X(\omega)|\geq\epsilon\big\}\Big)=0.$$

(4) X_n is said to converge in distribution (or weakly) to X, i.e., $X_n\xrightarrow{d}X$ if

$$\lim_{n\to\infty}\mathrm{P}\Big(\big\{\omega\in\Omega:X_n(\omega)\leq x\big\}\Big)=\mathrm{P}\Big(\big\{\omega\in\Omega:X(\omega)\leq x\big\}\Big).$$

2.2.1 Lévy process

Definition 2.2.10. (Lévy Process) Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. A Lévy process on this space is a map $X_t(\omega) : \mathbf{R}_+ \times \Omega \to \mathbf{R}$ with the following properties

(1) $X_0 = 0$

(2) X_t has stationary increments, that's, $X_t - X_s$ has the same distribution as X_{t-s} for all $0 \leq s < t < \infty$.

(3) X_t has independent increments, that's, $X_t - X_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.

(4) X_t has cádlág paths.

(5) X_t is stochastically continuous, that's, for all $t \geq 0$ and some number $\epsilon > 0$,

$$\lim_{s \to t} \mathrm{P}(|X_t - X_s| > \epsilon) = 0.$$

2.2.2 Poisson process

A Poisson process can be defined as a stochastic process X_t having discontinuous realisations (sample paths), stationary, independent increments and follows a Poisson distribution.

Definition 2.2.11. (Poisson process) A poisson process with intensity $\lambda > 0$ is an integer-valued, continuous time stochastic process $\{X_t, t \geq 0\}$ such that

(1) $X_0 = 0$

(2) X_t has an independent increments, that is: for all $0 = t_0 < t_1 < \ldots < t_n$, the increments $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent random variables,

(3) X_t has stationary increments: for all $t \geq s \geq 0$, $X_{t+s} - X_s$ equals X_t in distribution. Simply put; for all $t \geq s \geq 0$, and non-negative integers k, the increment follows poisson distribution:

$$\mathrm{P}[X_{t+s} - X_s = k] = \mathrm{P}[X_t = k] = \frac{(\lambda t)^k \mathrm{e}^{-\lambda t}}{k!}.$$

2.2.3 Poisson random measures

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space and $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ be a measurable space. Let $\mathbb{M}$ be the space of all $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$-valued measures on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ and consider the measurable space $(\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ with

$$\mathcal{B}_{\mathbb{M}} := \sigma\big(\nu(A) : A \in \mathcal{B}(\mathbf{R}^d)\big), \text{ for each } \nu \in \mathbb{M}.$$

Definition 2.2.12. (Poisson random measure) Let ν be a σ-finite measure on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$. A random variable $N : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ with intensity measure ν is called a Poisson random measure on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ if the following conditions hold.

(1) For all $A \in \mathcal{B}$, $N(A) : \Omega \to \bar{\mathbb{Z}}_+$ is Poisson distributed with parameter $\mathrm{E}[N(A)] = \nu(A)$, that is:

$$P[N(A) = n] = \frac{[\mathrm{E}[N(A)]]^n \exp(-\mathrm{E}[N(A)])}{n!}, \; n \in \mathrm{N} \cup \{0\}.$$

If $E[N(A)] = +\infty$ then $N(A) = +\infty$ P-a.s.

(2) If $A_1, \ldots, A_k$ are pairwise disjoint then $N(A_1), \ldots, N(A_k)$ are independent.

Next we state some existence theorems for the Poisson random measure. The first one states that given a σ-finite measure on a space X, we can find or construct a Poisson random measure N and it is given below.

Theorem 2.2.13. *Given a σ-finite measure ν on $(X, \mathcal{B}(X))$, there exists a Poisson random measure N such that $\mathrm{E}[N(A)] = \nu(A)$ for $A \in \mathcal{B}(X)$. If $N(A) = \infty$, then $\nu(A) = \infty$.*

Proof. The proof can be found in Ikeda and Watanabe [60, 61]. □

2.2.4 Point processes and Poisson point processes

We now define the concept of Point processes and Poisson point processes. These are needed for the definition of stochastic Poisson integral. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space and $(X, \mathcal{B}(X))$ a measurable space.

Definition 2.2.14. (Point function) A Point function p is defined via the mapping $p : D_p \to X$, where X is some measurable space and D_p is a countable subset of $[0, \infty]$. The point function p defines a counting measure on $(0, \infty) \times X$ by the following expression: $N_p((0, t] \times A) := \#\{s \leq t;\, s \in D_p;\, p(s) \in A\}$, for any $A \in \mathcal{B}(X)$. Basically, $N_p((0, t] \times A)$ counts the number of times before t that $p(s)$ is in A.

Let the set of all point functions taking values in X, be denoted by Π_X and $\mathcal{B}(\Pi_X)$ the smallest σ-algebra such that every mapping $p \to N_p((0, t] \times A)$ for all $A \in \mathcal{B}(X)$ is measurable, that's,

$$\mathcal{B}(\Pi_X) := \sigma\big(\Pi_X \ni p \mapsto N_p((0, t] \times A) : A \in \mathcal{B}(X), \, t > 0\big).$$

Definition 2.2.15. (Point process) A Point process p on X is a $(\Pi_X, \mathcal{B}(\Pi_X))$-valued random variable. In other words, p is defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$, measurable and it spits out a point function from Π_X, that's, a random variable $p : (\Omega, \mathcal{F}) \to (\Pi_X, \mathcal{B}(\Pi_X))$.

Definition 2.2.16. (Poisson point process) A Point process p is said to be a Poisson point process if the corresponding counting measure $N_p(\mathrm{d}t\,\mathrm{d}x)$ is a Poisson random measure on $(0, \infty) \times X$.

The second existence theorem characterises a stationary Point process.

Theorem 2.2.17. *Given a σ-finite measure $n(\mathrm{d}x)$ on $(X, \mathcal{B}(X))$, then there exists a stationary Poisson point process p if the random measure $N_p(\mathrm{d}t,\, \mathrm{d}x)$ is of the form* $\mathrm{E}[N_p(\mathrm{d}t,\, \mathrm{d}x)] = n_p(\mathrm{d}t,\, \mathrm{d}x) = \mathrm{d}t\, n(\mathrm{d}x)$.

Proof. Ikeda and Watanabe [60]. □

Now applying the above theorem with $X := \mathbf{R}^d \times \mathbf{R}^d$ and $\mathcal{B}(X) := \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}^d)$. We will take $n(\mathrm{d}x,\, \mathrm{d}h) := \mathrm{d}x\, \nu(\mathrm{d}h)$. One set of the vectors will play the role of position while the other will play the role of "jumps". By the above theorem, we have a Poisson point process $p(s) \in \mathbf{R}^d \times \mathbf{R}^d$. The Poisson random measure is thus given by the following

$$N_p((0, t], A \times B) := \#\{s \leq t;\, s \in D_p;\, p(s) \in A \times B\}.$$

Definition 2.2.18. (Jump of a Lêvy Process) The jump process ΔX_t at time $t \geq 0$ is defined by $\Delta X_t := X_t - X_{t^-}$ where X_{t^-} is the left limit of the process X_t at the point t.

Definition 2.2.19. (Jump measure) Let $(\Delta X_t \neq 0,\, t > 0)$ be the jump process and the set $A \in \mathcal{B}(\mathbf{R}^d)$ bounded below, then one defines the jump measure by

$$N(t, A) = \#\{0 \leq s \leq t : \Delta X_s \in A\} = \sum_{0<s\leq t} I_A(\triangle X_s).$$

The jump measure counts the number of jumps of the process between 0 and t such that their sizes fall into A.

Definition 2.2.20. (Compensated Poisson process) For a Poisson process, $N((0,\, t], A \times B)$ such that

$$\mathrm{E}[N((0,\, t], A \times B)] = t|A|\nu(B) \text{ for all } A,\, B \in \mathcal{B}(\mathbf{R}^d),$$

one defines the compensated Poisson process by

$$\tilde{N}((0,\, t], A \times B) := N((0,\, t], A \times B) - t|A|\nu(B),$$

for any $t > 0$ and any $A,\ B \in \mathcal{B}(\mathbf{R}^d)$ provided that $|A|\nu(B) < \infty$.

Definition 2.2.21. (Lévy measure) Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space. The measure ν defined by

$$\nu(A) = \mathrm{E}[N((0,1], A)] = \mathrm{E}\big[\sum_{0<s\leq 1} I_A(\Delta X_s)\big]$$

for all $A \in \mathcal{B}(\mathbf{R}^d)$ is said to be a Lévy measure of the process X with E an expectation with respect to the measure P. Suppose that ν is a Lévy measure on $\mathbf{R}^d$; then it satisfies the following

$$\int_{\mathbf{R}^d} \big(1 \wedge h^2\big)\nu(\mathrm{d}h) < \infty.$$

2.2.5 The Poisson Discontinuous Integrals

We now make sense of the discontinuous integrals.

2.2.5.1 Definition of the integral for Point Processes

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ be a complete filtered Probability space and $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ be a measurable space. Let $\mathcal{F}_t$ be defined by

$$\mathcal{F}_t := \sigma\big(N_p([0,t],\ A\times\ B, \cdot) : A \times B \in \mathcal{B}(\mathbf{R}^d) \times \mathcal{B}(\mathbf{R}^d)\big) \vee \mathcal{N},$$

where $t > 0$ and $\mathcal{N}$ denotes the null set of $\mathcal{F}$. We can write the Poisson random measure as

$$N_p((0,t], A \times B) := \sum_{s\in D_p, s\leq t} I_{A\times B}(p_x(s), p_h(s)),$$

where we define $p(s) := (p_x(s), p_h(s))$.

Recall that in our case, we have $\mathrm{E}[N_p((0,t], A \times B)] = t|A|\nu(B)$. We now describe the stochastic integral with respect to this Poisson random measure. We will need to define the class of integrand precisely.

Definition 2.2.22. (The non-compensated Integral)

$$H_p^1 := \Big\{ f(t,x,h) : f \text{ is } \{\mathcal{F}_t\}\text{-predictable and } \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \mathrm{E}|f(s,\ x,\ h)|\mathrm{d}s\mathrm{d}x\nu(\mathrm{d}h) < \infty \Big\}.$$

The following integral can now be defined for all $f \in H_p^1$

$$\int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(s,\, x,\, h, .) N_p(\mathrm{d}s,\, \mathrm{d}x,\, \mathrm{d}h) = \sum_{s \leq t, s \in D_p} f(s,\, p_x(s),\, p_h(s))$$

as the a.s sum of the following absolutely convergent sum.

Definition 2.2.23. (The compensated Integral) Define, similarly, for f satisfying the square-integrability condition

$$H_p^2 = \left\{ f(t,x,h) : f \text{ is } \{\mathcal{F}_t\}\text{-predictable and } \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \mathrm{E}|f(s,\, x,\, h)|^2 \mathrm{d}s \mathrm{d}x \nu(\mathrm{d}h) < \infty \right\}.$$

Then for all $f \in H_p^2$, one defines the integral as follows

$$\begin{aligned} \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(s,\, x,\, h) \tilde{N}_p(\mathrm{d}s,\, \mathrm{d}x,\, \mathrm{d}h) &= \sum_{s \leq t, s \in D_p} f(s,\, p_x(s),\, p_h(s)) \\ &- \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} f(s,\, x,\, h) \mathrm{d}s \mathrm{d}x \nu(\mathrm{d}h) \end{aligned}$$

as the a.s sum of the following absolutely convergent sum.

2.2.5.2 Definition of the integral for deterministic functions.

Definition 2.2.24. Let N be a Poisson random measure on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ with intensity measure ν. Given that N has the following representation

$$N(A)(\omega) = \sum_{k=1}^{\infty} \delta_{x_k(\omega)}(A), \quad \omega \in \Omega,\ A \in \mathcal{B}(\mathbf{R}^d)$$

for a properly chosen sequence (x_k) of random elements in $\mathbf{R}^d$. We define the integral with respect to N as follows: suppose that f is a real-valued function defined on $\mathbf{R}^d$ then

$$\int_{\mathbf{R}^d} f(x) N(\mathrm{d}x) = \sum_{k=1}^{\infty} f(x_k),$$

provided that the series is convergent P-a.s.

2.2.5.3 Definition of the integral for measurable functions

Definition 2.2.25. Let $f : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ be a Borel measurable function and A, B bounded below, then for each $t > 0$, we define the Poisson integral of f as a random finite sum by

$$\int_{A\times B} f(x,\, h)N(t, \mathrm{d}x,\, \mathrm{d}h) = \sum_{x,\, h\in A\times B} f(x,\, h)N(t, \{x\},\, \{h\}),$$

which is an $\mathbf{R}^d$-valued random variable. Then, since $N(t, \{x\},\, \{h\}) \neq 0$, that's, $N(t, \{x\}) \neq 0 \Leftrightarrow \Delta X_s^x = x$ and $N(t, \{h\}) \neq 0 \Leftrightarrow \Delta X_s^h = h$ for at least one $0 \leq s \leq t$, we have

$$\int_{A\times B} f(x,\, h)N(t, \mathrm{d}x,\, \mathrm{d}h) = \sum_{0\leq s\leq t} f(\Delta X_s^x,\, \Delta X_s^h) I_{A\times B}(\Delta X_s^x,\, \Delta X_s^h).$$

Generally, let $P = (P_t)_{t\geq 0}$ be a compound Poisson process and define for each $t \geq 0$, $P_t^x(A) = \int_A xN(t,\, \mathrm{d}x)$ and $P_t^h(B) = \int_B hN(t,\, \mathrm{d}h)$ for all $A,\, B \in \mathcal{B}(\mathbf{R}^d)$. Then for a predictable function f such that $\mathrm{E}\int_0^t \int_{A\times B} |f(s,\, x,\, h)|\nu(\mathrm{d}h)\mathrm{d}x\mathrm{d}s < \infty$,

$$\int_0^t \int_{A\times B} f(s,\, x,\, h)N(\mathrm{d}s,\, \mathrm{d}x,\, \mathrm{d}h) = \sum_{0\leq s\leq t} f(s,\, \Delta P_s^x,\, \Delta P_s^h) I_{A\times B}(\Delta P_s^x,\, \Delta P_s^h)$$

as a random finite sum, and if f is square integrable then

$$\begin{aligned}\int_0^t \int_{A\times B} f(s,\, x,\, h)\tilde{N}(\mathrm{d}s,\, \mathrm{d}x,\, \mathrm{d}h) &= \int_0^t \int_{A\times B} f(s,\, x,\, h)N(\mathrm{d}s,\, \mathrm{d}x,\, \mathrm{d}h) \\ &- \int_0^t \int_{A\times B} f(s,\, x,\, h)\nu(\mathrm{d}h)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Any measurable function may be approximated by simple functions.

Theorem 2.2.26. *Let $f : \Omega \to \mathbf{R}_+ \cup \{+\infty\}$ be a nonnegative $\mathcal{F}$-measurable functions. Then there exists a sequence of simple $\mathcal{F}$-measurable functions (f_n) such that $0 \leq f_1 \leq \ldots \leq f_n \leq f_{n+1} \leq \ldots$ and $\lim_{n\to\infty} f_n = f$ (that's, there is a monotone increasing sequence (f_n) of nonnegative simple functions that converges pointwise to f). If f is bounded, then f_n converges to f uniformly.*

Corollary 2.2.27. *If $f : (\Omega, \mathcal{F}) \to \mathbf{R} \cup \{+\infty\}$ is $\mathcal{F}$-measurable then it is the limit of a sequence of simple $\mathcal{F}$-measurable functions.*

Now we give definitions of the integral for simple functions (and processes). Let N be the Poisson random measure associated to a Lévy process X_t, $t \geq 0$.

Definition 2.2.28. Let $(\Omega_1,\, \mathcal{F}_1)$ and $(\Omega_2,\, \mathcal{F}_2)$ be measurable spaces. Then $f : \Omega_1 \times \Omega_2 \to \mathbf{R} \cup \{+\infty\}$ is said to be a simple function if and only if f is Borel measurable and takes on only finite many distinct values. That's f has the form

$$f = \sum_{k=1}^{n} a_k I_{A_k \times B_k}$$

for $(a_k)_{1\le k\le n} \in \mathbf{R}$ and the measurable sets $(A_k)_{1\le k\le n} \in \mathcal{F}_1$, $(B_k)_{1\le k\le n} \in \mathcal{F}_2$ such that $A_i \cap A_k = \emptyset$, $B_i \cap B_k = \emptyset$, $i \neq k$ with $\cup_{k=1}^{n} A_k = \Omega_1$ and $\cup_{k=1}^{n} B_k = \Omega_2$. Given that $N(A_k)$, $N(B_k) < \infty$, P-a.s., or equivalently $\nu(A_k)$, $\nu(B_k) < \infty$, we define

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} f(x,\, h) N(t,\, \mathrm{d}x,\, \mathrm{d}h) = \sum_{k=1}^{n} a_k N(t,\, A_k \times B_k).$$

Definition 2.2.29. For a nonnegative Borel measurable function f on a measurable space $(\Omega,\, \mathcal{F})$ with the intensity measure ν,

$$\int\int f(x,\, h) N(t,\, \mathrm{d}x,\, \mathrm{d}h) = \sup\left\{ \int\int g(x,\, h) N(t,\, \mathrm{d}x,\, \mathrm{d}h),\ 0 \le g \le f,\ \ g \ \text{ simple} \right\}.$$

Theorem 2.2.30. (Monotone Convergence) *Suppose that (f_n) is a monotone increasing sequence of nonnegative $\mathcal{F}$-measurable functions and f its pointwise limit,*

$$f(x,\, h) = \lim_{n\to\infty} f_n(x,\, h).$$

Then

$$\lim_{n\to\infty} \int_{\Omega_1 \times \Omega_2} f_n(x,\, h) N(t,\, \mathrm{d}x,\, \mathrm{d}h) = \int_{\Omega_1 \times \Omega_2} f(x,\, h) N(t,\, \mathrm{d}x,\, \mathrm{d}h)$$

in $L^1(\mathrm{P})$ sense.

Next we give definition of the integral for simple processes.

Definition 2.2.31. (Random step function) An adapted process $f(t,\, x,\, h)$ is said to be simple, if there exists a partition $0 = t_0 < t_1 < ... < t_n = T$ of $[0, T]$ such that we have the random step function

$$f(t,\, x,\, h) = \sum_{i=0}^{n-1} \sum_{k=1}^{m} \phi_{ik} I_{(t_i, t_{i+1}]}(t) I_{A_k \times B_k}(x,\, h)$$

where ϕ_{ik} are some bounded $\mathcal{F}_{t_i}$-measurable random variables and A_k, B_k pairwise disjoint subsets with $|A_k|$, $\nu(B_k) < \infty$. Therefore we define the integral process by

$$\int_0^t \int_{\mathbf{R}^d \times \mathbf{R}^d} f(s,\, x,\, h) N(\mathrm{d}s,\, \mathrm{d}x,\, \mathrm{d}h) = \sum_{i=0}^{n-1} \sum_{k=1}^{m} \phi_{ik} [N(t_{i+1} \wedge t,\, A_k \times B_k) - N(t_i \wedge t,\, A_k \times B_k)].$$

For a compensated random measure $\tilde{N}(t, A \times B) = N(t, A \times B) - t|A|\nu(B)$, we define

$$\int_0^t \int_{\mathbf{R}^d \times \mathbf{R}^d} f(s,\, x,\, h)\tilde{N}(\mathrm{d}s,\, \mathrm{d}x,\, \mathrm{d}h) = \sum_{i=0}^{n-1} \sum_{k=1}^{m} \phi_{ik}[\tilde{N}(t_{i+1} \wedge t,\, A_k \times B_k) - \tilde{N}(t_i \wedge t,\, A_k \times B_k)],$$

with the following properties:

- **Martingale preservation**

$$\mathrm{E}\Big[\int_0^t \int_{\mathbf{R}^d \times \mathbf{R}^d} f(s,\, x,\, h)\tilde{N}(\mathrm{d}s,\; \mathrm{d}x,\, \mathrm{d}h)\Big] = 0$$

- **Itô Isometry**

$$\mathrm{E}\Big[\Big|\int_0^t \int_{\mathbf{R}^d \times \mathbf{R}^d} f(s,\, x,\, h)\tilde{N}(\mathrm{d}s,\; \mathrm{d}x,\, \mathrm{d}h)\Big|^2\Big] = \int_0^t \int_{\mathbf{R}^d \times \mathbf{R}^d} \mathrm{E}|f(s,\, x,\, h)|^2 \nu(\mathrm{d}h)\mathrm{d}x\mathrm{d}s.$$

Thus we extend the compensated integral to all square integrable functions such that $\mathrm{E}\int_0^t \int_{\mathbf{R}^d \times \mathbf{R}^d} |f(s,\, x,\, h)|^2 \nu(\mathrm{d}h)\, \mathrm{d}x\, \mathrm{d}s < \infty$, from simple processes f_n by

$$\mathrm{E}\int_0^t \int_{\mathbf{R}^d \times \mathbf{R}^d} |f_n(s,\, x,\, h) - f(s,\, x,\, h)|^2 \nu(\mathrm{d}h)\, \mathrm{d}x\, \mathrm{d}s \to 0 \quad \text{as} \;\; n \to \infty.$$

2.3 Symmetric α-stable processes

Definition 2.3.1. (Stable process) A random variable X is said to be stable if there exist real valued sequences $(c_n, n \in \mathbb{N})$ and $(d_n, n \in \mathbb{N})$ with each $c_n > 0$ such that

$$X_1 + X_2 + \,\dots\, + X_n =^d c_n X + d_n \tag{2.3.1}$$

where $X_1 + X_2 + \,\dots\, + X_n$ are independent copies of X. The random variable X is said to be strictly stable if each $d_n = 0$.

It has been shown (Feller (1971)) that the only choice of c_n in (2.3.1) is of the form

$$c_n = \sigma n^{\frac{1}{\alpha}}, \qquad 0 < \alpha \leq 2.$$

The parameter α plays a key role in the investigation of stable random variables and it is called the "index of stability". The operator $-(-\Delta)^{\alpha/2}$ is the fractional Laplacian of the L^2 - generator of a symmetric stable process X_t of order α.

Definition 2.3.2. (Symmetric stable process) A symmetric α-stable process X on $\mathbf{R}^d$ is a Lévy process whose transition density $p(t,x)$ relative to Lebesgue measure is uniquely determined by its Fourier transform:

$$\mathrm{E}[\exp(i\xi X_t)] = \int_{\mathbf{R}^d} \mathrm{e}^{i<x,\,\xi>} p(t,\,x)\mathrm{d}x = \mathrm{e}^{-t|\xi|^\alpha} \quad \xi \in \mathbf{R}^d.$$

We present some required properties of $p(t,x)$ which come in handy in the proof of our results [96, 44].

$$\begin{aligned} p(t,\,x) &= t^{-d/\alpha} p(1,\,t^{-1/\alpha}x) \\ p(st,\,x) &= t^{-d/\alpha} p(s,\,t^{-1/\alpha}x). \end{aligned} \tag{2.3.2}$$

From the above relation, $p(t,\,0) = t^{-d/\alpha}\,p(1,\,0)$, is a decreasing function of t. The heat kernel $p(t,\,x)$ is also a decreasing function of $|x|$, that's

$$|x| \geq |y| \quad \text{implies that} \quad p(t,\,x) \leq p(t,\,y).$$

This and equation (2.3.2) imply that for all $t \geq s$,

$$\begin{aligned} p(t,\,x) = p(t,\,|x|) &= p\Big(s.\frac{t}{s},|x|\Big) = \Big(\frac{t}{s}\Big)^{-d/\alpha} p\Big(s,\Big(\frac{t}{s}\Big)^{-1/\alpha}|x|\Big) \\ &\geq \Big(\frac{s}{t}\Big)^{d/\alpha} p(s,|x|) \qquad \Big(\text{since}\,\Big(\frac{t}{s}\Big)^{-1/\alpha}|x| \leq |x|\Big) \\ &= \Big(\frac{s}{t}\Big)^{d/\alpha} p(s,\,x). \end{aligned}$$

Proposition 2.3.3. *Let $p(t,\,x)$ be the transition density of a strictly α-stable process. If $p(t,\,0) \leq 1$ and $a \geq 2$, then*

$$p\big(t,\,\frac{1}{a}(x-y)\big) \geq p(t,\,x)p(t,\,y)\ \forall\, x,\,y \in \mathbf{R}^d.$$

Proof. Given that

$$\frac{1}{a}|x-y| \leq \frac{2}{a}|x| \vee \frac{2}{a}|y| \leq |x| \vee |y|,$$

then it follows from the above that,

$$\begin{aligned} p\big(t, \frac{1}{a}(x-y)\big) &\geq p(t,\, |x| \vee |y|) \\ &\geq p(t,\, |x|) \wedge p(t,\, |y|) \\ &\geq p(t,\, |x|)p(t,\, |y|) \\ &= p(t,\, x)p(t,\, y). \end{aligned}$$

□

The transition density also satisfies the following Chapman-Kolmogorov equation,

$$\int_{\mathbf{R}^d} p(t,x)p(s,x)\mathrm{d}x = p(t+s,0).$$

Let $T(t,\, .,\, .)$ for all $t \geq 0$ be a semigroup given by

$$T(t,x,y) := \big(\frac{1}{4\pi t}\big)^{d/2} \exp\big(-\frac{|x-y|^2}{4t}\big)$$

the fundamental solution of a heat equation and define

$$f_{t,\alpha/2}(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathrm{e}^{zs - tz^{\alpha/2}}\mathrm{d}z \ (a>0,\, t>0,\, s\geq 0,\, 0<\alpha<2) \tag{2.3.3}$$

by the inversion formula of a Laplace transform. That is, let $(Y_t)_{t>0}$ be a $\alpha/2$-stable subordinator given by the Laplace transform

$$\exp(-tz^{\alpha/2}) = \mathrm{E}[\exp(-zY_t)] = \int_0^\infty \mathrm{e}^{-zs} f_{t,\alpha/2}(s)\mathrm{d}s,$$

where $f_{t,\alpha/2}(s),\ t>0, s\geq 0$ is its one dimensional density function. It has been shown (see [96, 101]) that

$$p(t,x,y) = \begin{cases} \int_0^\infty f_{t,\alpha/2}(s)T(s,x,y)\mathrm{d}s & \text{for } 0<\alpha<2 \\ T(t,\, x,\, y) & \text{for } \alpha = 2. \end{cases}$$

We quickly mention here, for clarity and consistency that the following notations $p(t,\, x,\, y) = p(t,\, x-y) = p_t(x,\, y)$ will be adopted and have their usual meaning and definition.

Lemma 2.3.4. *Suppose that $p(t,\, x)$ denotes the heat kernel for a strictly stable process of order α. Then the following estimate holds.*

$$p(t,\, x,\, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \quad \textit{for all} \quad t > 0 \quad \textit{and} \quad x,\, y \in \mathbf{R}^d.$$

Here and in the sequel, for two non-negative functions f, g, $f \asymp g$ means that there exists a positive constant $c > 1$ such that $c^{-1} g \le f \le c\, g$ on their common domain of definition.

Proof. We give a detailed proof of a well known estimate on a fractional heat kernel [11, 15, 18]. Let $0 < \alpha < 2$ and recall that $p(t, x, y)$ is given by

$$p(t,\, x,\, y) = \int_0^\infty f_{t,\alpha/2}(s) T(s, x, y) \mathrm{d}s \tag{2.3.4}$$

where $f_{t,\alpha/2}(s)$ is as defined in (2.3.3) and there exist some positive constants c_1, c_2, c_3, c_4 (see [18]) such that

$$c_1 s^{-d/2} \exp(-c_2 \frac{|x-y|^2}{s}) \le T(s, x, y) \le c_3 s^{-d/2} \exp(-c_4 \frac{|x-y|^2}{s}). \tag{2.3.5}$$

We have the following scaling property for $f_{t,\alpha/2}(s)$ given by

$$f_{t,\alpha/2}(s) = t^{-2/\alpha} f_{1,\alpha/2}(t^{-2/\alpha} s), \;\; t, s > 0. \tag{2.3.6}$$

Next, we state the behaviour of $f_{1,\alpha/2}(s)$ for large values of s (see [11]) given by

$$\lim_{s\to\infty} f_{1,\alpha/2}(s) s^{1+\alpha/2} = \frac{\alpha}{\Gamma(1-\alpha/2)}. \tag{2.3.7}$$

Then by the boundedness of $f_{1,\alpha/2}(.)$, the above behaviour and the scaling property, the following estimates follow

$$f_{t,\alpha/2}(s) \le c_5 t s^{-(1+\alpha/2)}, \;\; t,\, s > 0 \tag{2.3.8}$$

and

$$f_{t,\alpha/2}(s) \ge c_6 t s^{-(1+\alpha/2)}, \;\; t > 0,\, s > s_0 t^{2/\alpha}, \tag{2.3.9}$$

where s_0 depends only on α. First we prove the lower bound of the estimate, let us define for all $t > 0$, $x,\, y \in \mathbf{R}^d$, $d(t, x, y) := |x-y|^2 t^{-2/\alpha}$ and substitute $v = c_2 |x-y|^2 s^{-1}$, then by (2.3.5), (2.3.9) and (2.3.4), it follows that

$$\begin{aligned} p(t,x,y) &\geq ct\int_{s_0 t^{2/\alpha}}^{\infty} s^{-d/2}\mathrm{e}^{-c_2\frac{|x-y|^2}{s}} s^{-(1+\alpha/2)}\mathrm{d}s \\ &= ct\int_{s_0 t^{2/\alpha}}^{\infty} s^{-(d+\alpha)/2}\mathrm{e}^{-c_2\frac{|x-y|^2}{s}} s^{-1}\mathrm{d}s \\ &= ct|x-y|^{-(d+\alpha)}\int_0^{c_2 s_0^{-1}d(t,x,y)} v^{(d+\alpha)/2-1}\mathrm{e}^{-v}\mathrm{d}v \qquad (2.3.10) \\ &\geq ct|x-y|^{-(d+\alpha)}\mathrm{e}^{-c_7 d(t,x,y)}\int_0^{c_7 d(t,x,y)} v^{(d+\alpha)/2-1}\mathrm{d}v \\ &= ct^{-d/\alpha}\mathrm{e}^{-c_7 d(t,x,y)}. \end{aligned}$$

If $t \geq |x-y|^\alpha$, that's, $\frac{|x-y|^\alpha}{t} \leq 1$, then $d(t,x,y) \leq 1$ and $\mathrm{e}^{-c_7 d(t,x,y)} \geq \mathrm{e}^{-c_7}$. It follows that $p(t,x,y) \geq c_7 t^{-d/\alpha}$ and therefore

$$p(t,\, x,\, y) \geq c_7 \min\big(t^{-d/\alpha},\, \frac{t}{|x-y|^{d+\alpha}}\big).$$

On the other hand, if $t < |x-y|^\alpha$ then $d(t,x,y) > 1$ so that the integral in (2.3.10) is bounded away from 0 and hence

$$p(t,\, x,\, y) \geq c_8 t|x-y|^{-(d+\alpha)} \geq c_8 \min\big(t^{-d/\alpha},\, \frac{t}{|x-y|^{d+\alpha}}\big).$$

We follow similar steps in proving the upper bound estimate by using (2.3.8) and (2.3.5), thus

$$\begin{aligned} p(t,\, x,\, y) &\leq c_9\, t\int_0^{\infty} s^{-(d/2+\alpha/2+1)}\mathrm{e}^{-c_4\frac{|x-y|^2}{s}}\mathrm{d}s \\ &= c_9\, \frac{t}{|x-y|^{d+\alpha}}\Gamma(\frac{d+\alpha}{2}), \end{aligned}$$

which establishes the first term under the minimum. To estimate the other term, it suffices to verify the following estimate

$$f_{t,\alpha/2}(s) \leq c\, t\, s^{-(1+\alpha/2)}\mathrm{e}^{-t\, s^{-\alpha/2}}, \quad s,t > 0. \qquad (2.3.11)$$

For $t = 1$ in (2.3.3), thus

$$f_{1,\alpha/2}(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \exp(zs - z^{\alpha/2})\mathrm{d}z$$

for $a > 0$. The above integral is not easy to evaluate, so we approximate it. We state a technique known as Laplace approximation used for saddle point approximation (from Taylor series to Saddle points). Consider a positive function $f(x)$ and suppose that one wants to approximate its value at some point x_0 via Taylor series expansion of its first few terms. Let $h(x) \equiv \log f(x)$, then writing $f(x) = \exp[h(x)]$ and choosing x_0 as the point to expand around, one obtains

$$f(x) \approx \exp\left\{h(x_0) + (x - x_0)h'(x_0) + \frac{(x - x_0)^2}{2}h''(x_0)\right\}.$$

The above approximation simplifies if one chooses $x_0 = \hat{x}$, where $h'(\hat{x}) = 0$, and

$$\int f(x)\mathrm{d}x \approx \int \exp\left\{h(\hat{x}) + \frac{(x - \hat{x})^2}{2}h''(\hat{x})\right\}\mathrm{d}x.$$

Therefore by letting $z_0 = (\frac{\alpha}{2s})^{2/(2-\alpha)}$, $a = z_0$ and substituting for $\xi = z/z_0$,

$$f_{1,\alpha/2}(s) = \frac{z_0}{2\pi i}\int_{1-i\infty}^{1+i\infty} \exp[\phi(\xi)]\mathrm{d}\xi$$

with $\phi(\xi) = -z_0^{\alpha/2}\xi^{\alpha/2} + sz_0\xi$. We have the following for the value of z_0 above,

$$\begin{aligned}
\phi(1) &= -\big(1 - \frac{\alpha}{2}\big)\big(\frac{\alpha}{2}\big)^{\alpha/(2-\alpha)}s^{-\alpha/(2-\alpha)}, \\
\phi'(1) &= 0, \\
\phi''(1) &= \frac{\alpha(2-\alpha)}{4}\big(\frac{\alpha}{2}\big)^{\alpha/(2-\alpha)}s^{-\alpha/(2-\alpha)}.
\end{aligned}$$

Hence,

$$\phi(\xi) \doteqdot \phi(1) + \frac{(\xi - 1)^2}{2}\phi''(1)$$

and, if we let $\xi = 1 + iy$ with y real, then as $y \to 0^+$,

$$\phi(\xi) \doteqdot \phi(1) - \frac{y^2}{2}\phi''(1).$$

Thus, by the saddle point approximation, as $s \to 0^+$,

$$\begin{aligned}
f_{1,\alpha/2}(s) &\approx \frac{z_0}{2\pi i}\exp\big[\phi(1)\big]i\int_{\mathbf{R}} \exp\big(-\frac{1}{2}\phi''(1)y^2\big)\mathrm{d}y \\
&= \frac{z_0}{2\pi}\exp\big[\phi(1)\big]\sqrt{\frac{2}{\phi''(1)}}\int_{\mathbf{R}} \mathrm{e}^{-x^2}\mathrm{d}x \\
&= \frac{z_0}{2\pi}\exp\big[\phi(1)\big]\sqrt{\frac{2\pi}{\phi''(1)}}.
\end{aligned}$$

Therefore defining $c_{10} = (1 - \frac{\alpha}{2})(\frac{\alpha}{2})^{\alpha/(2-\alpha)}$ and $c = \frac{(\alpha/2)^{1/(2-\alpha)}}{[2\pi(1-\alpha/2)]^{1/2}}$, we have that

$$f_{1,\alpha/2}(s) \le c\, s^{(\alpha-4)/(4-2\alpha)} \mathrm{e}^{-c_{10} s^{-\alpha/(2-\alpha)}}.$$

Also, $\alpha/(2-\alpha) > \alpha/2$ and $\frac{-\alpha}{2(2-\alpha)} - 1 < -1 - \alpha/2$, that's, $\frac{\alpha-4}{4-2\alpha} < -(1+\alpha/2)$. Then it follows (see [15]) that

$$s^{(\alpha-4)/(4-2\alpha)} \mathrm{e}^{-c_{10} s^{-\alpha/(2-\alpha)}} = o(s^{-(1+\alpha/2)} \mathrm{e}^{-s^{-\alpha/2}}), \ \text{as}\, s \to 0.$$

Therefore, this and (2.3.8) imply that

$$f_{1,\alpha/2}(s) \le c\, s^{-(1+\alpha/2)} \mathrm{e}^{-s^{-\alpha/2}}, \ \ s > 0.$$

Then by applying the scaling property in (2.3.6), the estimate in (2.3.11) follows. Hence, making use of the established (2.3.11), and substituting for $v = t\, s^{-\alpha/2}$, we have that

$$\begin{aligned} p(t,\, x,\, y) &\le c\,t \int_0^\infty s^{-d/2} s^{-(1+\alpha/2)} \mathrm{e}^{-t\, s^{-\alpha/2}} \mathrm{d}s \\ &= c\,t^{-d/\alpha} \int_0^\infty v^{d/\alpha} \mathrm{e}^{-v} \mathrm{d}v \\ &= c\,t^{-d/\alpha} \Gamma(d/\alpha + 1) = c_1\, t^{-d/\alpha}, \end{aligned}$$

and that completes the proof. □

Now we show that the first term $(P_t u_0)(x)$ of the mild solution to (1.1.9) grows or decays but only polynomially fast with time. Recall that

$$(P_t u_0)(x) := \int_{\mathbf{R}^d} p(t,\, x,\, y) u_0(y) \mathrm{d}y.$$

With the assumption that the initial condition u_0 is positive on a set of positive measure, we then have the following.

Proposition 2.3.5. *There exists a $T > 0$ and a constant c_1 such that for all $t > T$ and all $x \in B(0,\, t^{1/\alpha})$,*

$$(P_t u_0)(x) \ge \frac{c_1}{t^{d/\alpha}}.$$

Proof. Applying Lemma 2.3.4,

$$\begin{aligned}(P_t u_0)(x) &\geq \int_{B(x,\,t^{1/\alpha})} p(t,x,y)u_0(y)\mathrm{d}y \\ &\geq \frac{c_1}{t^{d/\alpha}} \int_{B(x,\,t^{1/\alpha})\cap B(0,\,t^{1/\alpha})} u_0(y)\mathrm{d}y.\end{aligned}$$

Since $x \in B(0,\, t^{1/\alpha})$, then by the assumption on u_0, we can find $T > 0$ large enough so that for all $t > T$,

$$\int_{B(x,\,t^{1/\alpha})\cap B(0,\,t^{1/\alpha})\cap A} u_0(y)\mathrm{d}y \geq c_2,$$

where A is the set of positive measure where we are assuming u_0 is positive and the result follows. □

The next proposition is similar to but more general than Proposition 2.3.5.

Proposition 2.3.6. *Given the assumption on the initial function u_0. Then for $t_0 \geq 1$, $\eta > 0$, there exists $c(t_0) > 0$ such that*

$$\int_{\mathbf{R}^d} p(t+t_0,\, x,\, y)u_0(y)\mathrm{d}y \geq c(t_0)p(t+\eta,\, x).$$

Proof. Choose $t_0 > 0$ such that $p(t_0, 0) \leq 1$, then by Kolmogorov property, proposition 2.3.3 and the scaling property of the heat kernel (2.3.2), we have that

$$\begin{aligned}&\int_{\mathbf{R}^d} p(t+t_0,\, x-y)u_0(y)\mathrm{d}y \\ &\qquad = \int_{\mathbf{R}^d} \mathrm{d}y \Big\{ \int_{\mathbf{R}^d} p(t,\, x-z)p(t_0,\, z-y)\mathrm{d}z \Big\} u_0(y) \\ &\qquad = \int_{\mathbf{R}^d} \mathrm{d}z\, p(t, x-z). \int_{\mathbf{R}^d} p(t_0, \frac{1}{2}(2z-2y))u_0(y)\mathrm{d}y \\ &\qquad \geq \int_{\mathbf{R}^d} \mathrm{d}z\, p(t, x-z). \int_{\mathbf{R}^d} p(t_0,\, 2z)p(t_0, 2y)u_0(y)\mathrm{d}y \\ &\qquad = 2^{-d} \int_{\mathbf{R}^d} \mathrm{d}z\, p(t, x-z)\, p(t_0/2^\alpha,\, z) \int_{\mathbf{R}^d} p(t_0, 2y)u_0(y)\mathrm{d}y.\end{aligned}$$

Let $c(t_0) := 2^{-d} \int_{\mathbf{R}^d} p(t_0, 2y)u_0(y)\mathrm{d}y$ which is finite by Proposition 2.3.5. Denote $\eta := t_0/2^\alpha > 0$ and the result follows. □

In what follows, we will need the Gamma function denoted by $\Gamma(.)$. Here, $\mathbf{Z}_+$ denotes the set of all non-negative integers and N the set of all positive integers. We give a slightly different proof of the lemma proved in([66] page 38).

Lemma 2.3.7. *Let $0 < \rho \leq 1$, then there exists a positive constant c_1 such that for all $b \geq (\mathrm{e}/\rho)^\rho$,*

$$\sum_{j=0}^{\infty} \left(\frac{b}{j^\rho}\right)^j \geq \exp\left(c_1 b^{1/\rho}\right).$$

Proof. We begin by writing

$$\begin{aligned} \sum_{j=0}^{\infty} \left(\frac{b}{j^\rho}\right)^j &= 1 + \sum_{j\in \mathrm{N},\, j\rho<1} \left(\frac{b}{j^\rho}\right)^j + \sum_{j\in \mathrm{N},\, j\rho\geq 1} \left(\frac{b}{j^\rho}\right)^j \\ &\geq 1 + \sum_{j\in \mathrm{N},\, j\rho\geq 1} \left(\frac{b}{j^\rho}\right)^j. \end{aligned} \tag{2.3.12}$$

By Stirling's formula for $j\rho$ integers, we have that $\Gamma(j\rho+1) = (j\rho)! \approx \sqrt{2\pi j\rho}(j\rho/\mathrm{e})^{j\rho}$. It follows that for $j\rho \geq 1$, $\Gamma(j\rho+1) \geq (j\rho/\mathrm{e})^{j\rho}$ and

$$\sum_{j\in \mathrm{N},\, j\rho\geq 1} \left(\frac{b}{j^\rho}\right)^j \geq \sum_{j\in \mathbf{Z}_+,\, j\rho\geq 1} \frac{\left(b^{1/\rho}(\rho/\mathrm{e})\right)^{j\rho}}{\Gamma(j\rho+1)}. \tag{2.3.13}$$

Since $0 < \rho \leq 1$ and $j \in \mathrm{N}$, then for each positive integer $k \geq 2$, we can always find a distinct product $j\rho$ such that $\Gamma(j\rho+1) \leq \Gamma(\lceil j\rho \rceil + 1) = (\lceil j\rho \rceil)! = k!$ and $j\rho \geq 1$. Let $\lceil j\rho \rceil$ denote the smallest integer greater than $j\rho$ and denote $\lfloor j\rho \rfloor$ to be the greatest integer less than $j\rho$. Substitute these into (2.3.13), since $b \geq (\mathrm{e}/\rho)^\rho$ we have that $b^{1/\rho}(\rho/\mathrm{e}) \geq 1$. Thus

$$\begin{aligned} \sum_{j\in \mathrm{N},\, j\rho\geq 1} \frac{\left(b^{1/\rho}(\rho/\mathrm{e})\right)^{j\rho}}{\Gamma(j\rho+1)} &\geq \sum_{j\in \mathrm{N},\, j\rho\geq 1} \frac{\left(b^{1/\rho}(\rho/\mathrm{e})\right)^{\lfloor j\rho \rfloor}}{(\lceil j\rho \rceil)!} \\ &\geq \sum_{k=2}^{\infty} \frac{\left(b^{1/\rho}(\rho/\mathrm{e})\right)^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{\left(b^{1/\rho}(\rho/\mathrm{e})\right)^{k}}{(k+1)!} \\ &\geq \sum_{k=1}^{\infty} \frac{2^{-k}\left(b^{1/\rho}(\rho/\mathrm{e})\right)^{k}}{k!} \\ &= \exp\left(b^{1/\rho}(\rho/2\mathrm{e})\right) - 1. \end{aligned}$$

Substituting this into (2.3.12) gives the result. $\square$

The next result gives the reverse of the above inequality proved in the above Lemma. The proof follows same notations as in the proof of the above Lemma.

Lemma 2.3.8. *Let $0 < \rho \le 1$, then there exist positive constants c_1 and c_2 such that*

$$\sum_{j=0}^{\infty} \frac{b^j}{\Gamma(j\rho+1)} \le c_1 \exp\left(c_2 b^{1/\rho}\right) \text{ for all } \ b > 0.$$

Proof. Just as in the above Lemma, we start by writing

$$\sum_{j=0}^{\infty} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} = \sum_{j\in\mathbf{Z}_+,\, j\rho<1} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} + \sum_{j\in\mathbf{Z}_+,\, j\rho\ge1} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)}. \tag{2.3.14}$$

We first consider the case when $0 < b < 1$. With same notations from the above Lemma, we write

$$\begin{aligned}
\sum_{j\in\mathbf{Z}_+,\, j\rho\ge1} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} &\le \sum_{j\in\mathbf{Z}_+,\, j\rho\ge1} \frac{(b^{1/\rho})^{\lfloor j\rho\rfloor}}{(\lfloor j\rho\rfloor)!} \\
&\le c_1 \sum_{k=2}^{\infty} \frac{(b^{1/\rho})^{k-1}}{(k-1)!} = c_1 \sum_{k=1}^{\infty} \frac{(b^{1/\rho})^{k}}{k!} = c_1\big(\exp(b^{1/\rho}) - 1\big),
\end{aligned}$$

and

$$\sum_{j\in\mathbf{Z}_+,\, j\rho<1} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} \le c_2,$$

where $\Gamma(j\rho+1)$ is bounded below by a constant for all $j\rho < 1$. Substituting both sums into (2.3.14), therefore we come up with

$$\sum_{j=0}^{\infty} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} \le c_3\Bigg(\exp(b^{1/\rho}) + 1\Bigg) \le c_4 \exp(b^{1/\rho}).$$

Next, we consider the case of $b \ge 1$ where we have

$$\begin{aligned}
\sum_{j\in\mathbf{Z}_+,\, j\rho\ge1} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} &\le \sum_{j\in\mathbf{Z}_+,\, j\rho\ge1} \frac{(b^{1/\rho})^{\lceil j\rho\rceil}}{(\lfloor j\rho\rfloor)!} = \sum_{j\in\mathbf{Z}_+,\, j\rho\ge1} \frac{(b^{1/\rho})^{\lfloor j\rho\rfloor+1}}{(\lfloor j\rho\rfloor)!} \\
&\le c_5 \sum_{k=2}^{\infty} \frac{(b^{1/\rho})^{k}}{(k-1)!} = c_5 \sum_{k=1}^{\infty} \frac{(b^{1/\rho})^{k+1}}{k!} \\
&\le c_6 \sum_{k=2}^{\infty} \frac{(2b^{1/\rho})^{k}}{k!} = c_6\big(\exp(2b^{1/\rho}) - 2b^{1/\rho} - 1\big).
\end{aligned}$$

We also have on the other hand that

$$\sum_{j \in \mathbf{Z}_+,\, j\rho<1} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} \leq 2c_7 b^{1/\rho}.$$

Therefore substituting for both cases into (2.3.14), we obtain

$$\sum_{j=0}^{\infty} \frac{(b^{1/\rho})^{j\rho}}{\Gamma(j\rho+1)} \leq c_8 \exp(c_9 b^{1/\rho}).$$

□

CHAPTER 3

ON A STOCHASTIC HEAT EQUATION DRIVEN BY COMPENSATED POISSON NOISE

We look at the following stochastic heat equations driven by discontinuous processes.

$$\left[\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x)\right]\mathrm{d}x\mathrm{d}t = \lambda \int_{\mathbf{R}} \sigma(u(t,\,x),\,h)\tilde{N}(\mathrm{d}t,\,\mathrm{d}x,\,\mathrm{d}h), \tag{3.0.1}$$

with initial condition $u_0(x)$. Here and throughout, $u_0 : \mathbf{R} \to \mathbf{R}_+$ is a non-random function, and $\mathcal{L}$ is the L^2-generator of a Lévy process. For existence and uniqueness, we need the following condition on σ. Essentially this condition says that σ is globally Lipschitz in the first variable and bounded by another function in the second variable.

Condition 3.0.9. There exist a positive function J and a finite positive constant, Lip_σ such that for all $x,\,y,\,h \in \mathbf{R}$, we have

$$|\sigma(0,h)| \leq J(h) \quad \text{and} \quad |\sigma(x,\,h) - \sigma(y,\,h)| \leq J(h)\mathrm{Lip}_\sigma|x-y|. \tag{3.0.2}$$

The function J is assumed to satisfy the following integrability condition:

$$\int_{\mathbf{R}} J(h)^2\nu(\mathrm{d}h) \leq \mathrm{K}, \tag{3.0.3}$$

where K is some finite positive constant.

3.1 Existence and Uniqueness result

With the linear growth condition on σ, we give proof of the existence and uniqueness result for the compensated equation. Roughly speaking we have the following theorem with the assumption that $u_0 : \mathbf{R} \to \mathbf{R}_+$ is measurable and bounded.

Theorem 3.1.1. *Under condition 3.0.9, there exists a unique random field solution to* (3.0.1) *satisfying*

$$\bar{\gamma}(2) \leq \inf \left\{ \beta > 0 : \Upsilon(\beta) < \frac{1}{\lambda^2 \mathrm{KLip}_\sigma^2} \right\} < \infty,$$

where $\bar{\gamma}(p)$ is as defined in (1.1.5)*.*

This result is a direct analogue of Theorem 2.1 of [44] and it gives an upper bound on the rate of growth of the second moment $\mathrm{E}|u(t,x)|^2$ of the solution. It also generalises Theorem 1.3.1 of [10].

Example 3.1.2. For $\mathcal{L} = \Delta$, the characteristic exponent is given by $\Psi(\xi) = \xi^2$ and $\sigma(u,h) = u\,h$ satisfies condition 3.0.9 with $\mathrm{Lip}_\sigma = 1$ and $J(h) = |h|$ such that

$$\int_{\mathbf{R}} |h|^2 \nu(\mathrm{d}h) < \infty.$$

Also,

$$\Upsilon(\beta) = \frac{1}{2\pi} \int_{\mathbf{R}} \frac{\mathrm{d}\xi}{\beta + 2\mathcal{R}\mathrm{e}\Psi(\xi)} = \frac{1}{2\pi} \int_{\mathbf{R}} \frac{\mathrm{d}\xi}{\beta + 2\xi^2} = \frac{1}{2\pi} . \frac{\pi}{\sqrt{2\beta}} = \frac{1}{\sqrt{8\beta}} \quad \text{for all } \beta > 0.$$

Estimates for the proof(s) of results are presented as follow.

3.1.1 Some estimates for the compensated equation

Throughout this section we set

$$(\mathcal{A}u)(t,\,x) := \lambda \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} p(t-s,\,x,\,y) \sigma(u(s,\,y), h) \tilde{N}(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s). \tag{3.1.1}$$

We will also use the following norm:

$$\|u\|_{2,\beta} := \left\{ \sup_{t>0} \sup_{x \in \mathbf{R}} \mathrm{e}^{-\beta t} \mathrm{E}[|u(t,\,x)|^2] \right\}^{1/2}. \tag{3.1.2}$$

Lemma 3.1.3. *For all $\beta > 0$,*

$$\sup_{t>0} \mathrm{e}^{-\beta t} \int_0^t \int_{\mathbf{R}} |p(s,\,x,\,y)|^2 \mathrm{d}y\, \mathrm{d}s \leq \Upsilon(\beta).$$

Proof.

$$\begin{aligned}\sup_{t>0} \mathrm{e}^{-\beta t}\int_0^t \int_{\mathbf{R}} |p(s,\,x,\,y)|^2 \mathrm{d}y\,\mathrm{d}s &\le \int_0^\infty \int_{\mathbf{R}} \mathrm{e}^{-\beta s}|p(s,\,x,\,y)|^2 \mathrm{d}y\,\mathrm{d}s \\ &= \int_0^\infty \int_{\mathbf{R}} \mathrm{e}^{-\beta s}|\hat{p}(s,\xi)|^2 \mathrm{d}\xi\,\mathrm{d}s \\ &= \int_{\mathbf{R}} \frac{\mathrm{d}\xi}{\beta + 2\mathcal{R}\mathrm{e}\Psi(\xi)} = \Upsilon(\beta).\end{aligned}$$

□

The above result is true for $\mathbf{R}^d$ and will be severally used in the proof of the following Lemma(s) below. We then have the following estimates.

Proposition 3.1.4. *Suppose that u admits a weak predictable version and that $\|u\|_{2,\,\beta} < \infty$ for $\beta > 0$. Then there exists some positive constant $c_4 = \sqrt{c_1 \mathrm{K}}\lambda$ such that*

$$\|\mathcal{A}u\|_{2,\beta} \le c_4[\mathrm{Lip}_\sigma \|u\|_{2,\beta} + 1]\sqrt{\Upsilon(\beta)}.$$

Proof. We use the Itô isometry to write

$$\mathrm{E}|\mathcal{A}u(t,\,x)|^2 = \lambda^2 \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} |p(t-s,\,x,\,y)|^2 \mathrm{E}|\sigma(u(s,\,y),h)|^2 \nu(\mathrm{d}h)\,\mathrm{d}y\,\mathrm{d}s.$$

We now use the first part of condition 3.0.9 to see that there exists a constant c_1 such that $|\sigma(x,h)|^2 \le c_1 J^2(h)(\mathrm{Lip}_\sigma |x|^2 + 1)$. This and the above display yield the following bound

$$\begin{aligned}&\mathrm{E}|(\mathcal{A}u)(t,\,x,\,.)|^2 \\ &\le c_2 \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} |p(t-s,\,x,\,y)|^2 J^2(h)(\mathrm{Lip}_\sigma^2 \mathrm{E}|u(s,\,y)|^2 + 1)\nu(\mathrm{d}h)\,\mathrm{d}y\,\mathrm{d}s \\ &\le c_3 \int_{\mathbf{R}} J^2(h)\nu(\mathrm{d}h)\mathrm{e}^{\beta t}\Upsilon(\beta)[\mathrm{Lip}_\sigma^2 \|u\|_{2,\beta}^2 + 1].\end{aligned}$$

We now use the second part of condition 3.0.9 and rearrange the bound in the above display to end up with

$$\begin{aligned}\|\mathcal{A}u\|_{2,\beta} &\le c_4 \sqrt{\Upsilon(\beta)[\mathrm{Lip}_\sigma^2 \|u\|_{2,\beta}^2 + 1]} \\ &\le c_4[\mathrm{Lip}_\sigma \|u\|_{2,\beta} + 1]\sqrt{\Upsilon(\beta)}.\end{aligned}$$

for some constant c_4. □

Proposition 3.1.5. *Let $\beta > 0$ and let u and v be two weak predictable random field solutions satisfying $\|u\|_{2,\beta} + \|v\|_{2,\beta} < \infty$. Then*

$$\|\mathcal{A}u - \mathcal{A}v\|_{2,\beta} \leq \lambda \mathrm{Lip}_\sigma \sqrt{\mathrm{K}\Upsilon(\beta)}\|u - v\|_{2,\beta}.$$

Proof. The proof is very similar to the proof of the previous result. We start by applying Itô-isometry and the first part of condition 3.0.9, to obtain

$$\mathrm{E}|\mathcal{A}u(t,\, x) - \mathcal{A}v(t, x)|^2 \leq \mathrm{K}\lambda^2 \mathrm{Lip}_\sigma^2 \int_0^t \int_{\mathbf{R}} \mathrm{E}|u(s, y) - v(s, y)|^2 |p(t - s,\, x,\, y)|^2 \mathrm{d}y\mathrm{d}s.$$

Following same procedure as the preceding Proposition, we have that

$$\begin{aligned}
&\|\mathcal{A}u - \mathcal{A}v\|_{2,\beta}^2 \\
&\qquad \leq \sup_{x\in\mathbf{R}} \sup_{t>0} \mathrm{e}^{-\beta t} \lambda^2 \mathrm{K}\mathrm{Lip}_\sigma^2 \int_0^t \int_{\mathbf{R}} \mathrm{E}\left[|u(s, y) - v(s, y)|^2\right] |p(t - s,\, x,\, y)|^2 \mathrm{d}y\mathrm{d}s \\
&\qquad \leq \lambda^2 \mathrm{K}\mathrm{Lip}_\sigma^2 \|u - v\|_{2,\beta}^2 \int_0^\infty \int_{\mathbf{R}} \mathrm{e}^{-\beta s} |\widehat{p}(s, \xi)|^2 \mathrm{d}\xi\mathrm{d}s
\end{aligned}$$

by Plancherel's identity. Hence,

$$\begin{aligned}
\|\mathcal{A}u - \mathcal{A}v\|_{2,\beta}^2 &\leq \lambda^2 \mathrm{K}\mathrm{Lip}_\sigma^2 \|u - v\|_{2,\beta}^2 \int_0^\infty \int_{\mathbf{R}} \mathrm{e}^{-\beta s} \mathrm{e}^{-2s\mathcal{R}\mathrm{e}\Psi(\xi)} d\xi\mathrm{d}s \\
&= \lambda^2 \mathrm{K}\mathrm{Lip}_\sigma^2 \|u - v\|_{2,\beta}^2 \Upsilon(\beta).
\end{aligned}$$

Therefore,

$$\|\mathcal{A}u - \mathcal{A}v\|_{2,\beta} \leq \lambda \mathrm{Lip}_\sigma \sqrt{\mathrm{K}\Upsilon(\beta)}\|u - v\|_{2,\beta}.$$

□

We now use the above estimates to prove the result.

Proof of the existence and uniqueness part of Theorem 3.1.1. We prove the existence of the solution by an iterative schemes. Let's define $v_0(t, x) := u_0(x)$ for all $t \geq 0$ and $x \in \mathbf{R}$. Since u_0 is assumed to be bounded, so $\|v_0\|_{2,\beta} < \infty$ for all $\beta > 0$. Iteratively, we set

$$\begin{cases}
v_{n+1}(t, x) &= \mathcal{A}v_n(t, x) + (P_t u_0)(x) \\
v_n(t, x) &= \mathcal{A}v_{n-1}(t, x) + (P_t u_0)(x).
\end{cases}$$

From the above, we have that for sufficiently large β

$$\|\mathcal{A}v_{n+1}\|_{2,\beta} = \|\mathcal{A}(\mathcal{A}v_n)\|_{2,\beta}.$$

Then by Proposition 3.1.4, we have that

$$\|\mathcal{A}v_{n+1}\|_{2,\beta} = \|\mathcal{A}(\mathcal{A}v_n)\|_{2,\beta} \le c_4\sqrt{\Upsilon(\beta)}\big[\mathrm{Lip}_\sigma\|\mathcal{A}v_n\|_{2,\beta} + 1\big]. \tag{3.1.3}$$

Since $\lim_{\beta\to\infty}\Upsilon(\beta) = 0$, then we can always choose and fix $\beta > 0$ such that

$$c_4\sqrt{\Upsilon(\beta)}\mathrm{Lip}_\sigma < 1 \;\Leftrightarrow\; c_4^2\Upsilon(\beta)\mathrm{Lip}_\sigma^2 < 1$$

since $c_4\sqrt{\Upsilon(\beta)}\mathrm{Lip}_\sigma > 0$. It follows that

$$\Upsilon(\beta) < \frac{1}{\big[c_4\,\mathrm{Lip}_\sigma\big]^2} < \infty, \;\forall\,\beta > 0.$$

From (3.1.3) we have $\|\mathcal{A}v_{n+1}\|_{2,\beta} \le c_4\sqrt{\Upsilon(\beta)}\mathrm{Lip}_\sigma\|\mathcal{A}v_n\|_{2,\beta} + c_4\sqrt{\Upsilon(\beta)}$. Taking sup of both sides over n, therefore,

$$\sup_{n\ge 0}\|\mathcal{A}v_{n+1}\|_{2,\beta} - c_4\sqrt{\Upsilon(\beta)}\mathrm{Lip}_\sigma\sup_{n\ge 0}\|\mathcal{A}v_n\|_{2,\beta} \le c_4\sqrt{\Upsilon(\beta)}.$$

But $\sup_{n\ge 0}\|\mathcal{A}v_{n+1}\|_{2,\beta} = \sup_{n\ge 0}\|\mathcal{A}v_n\|_{2,\beta}$ since $\|\mathcal{A}v_0\|_{2,\beta}$ is significantly small for sufficiently large β, then

$$\sup_{n\ge 0}\|\mathcal{A}v_n\|_{2,\beta} \le \frac{c_4\sqrt{\Upsilon(\beta)}}{1 - c_4\mathrm{Lip}_\sigma\sqrt{\Upsilon(\beta)}} < \infty.$$

Also $v_k(t,x) = \mathcal{A}v_{k-1}(t,x) + (P_tu_0)(x)$ and the uniform bound on $P_tu_0(x)$,

$$\begin{aligned}\|v_k\|_{2,\beta} &= \|\mathcal{A}v_{k-1}\|_{2,\beta} + \|P_tu_0(x)\|_{2,\beta}\\ &\le \|\mathcal{A}v_{k-1}\|_{2,\beta} + \sup_{r\in\mathbf{R}}|u_0(r)|.\end{aligned}$$

Then we have that

$$\begin{aligned}\sup_{k\ge 1}\|v_k\|_{2,\beta} &\le \sup_{k\ge 1}\|\mathcal{A}v_{k-1}\|_{2,\beta} + \sup_{r\in\mathbf{R}}|u_0(r)|\\ &\le \frac{c_4\sqrt{\Upsilon(\beta)}}{1 - c_4\mathrm{Lip}_\sigma\sqrt{\Upsilon(\beta)}} + \sup_{r\in\mathbf{R}}|u_0(r)| < \infty.\end{aligned}$$

Furthermore, $\|v_{n+1} - v_n\|_{2,\beta} = \|\mathcal{A}v_n - \mathcal{A}v_{n-1}\|_{2,\beta}$ and by Proposition 3.1.5, we have that for all $n \ge 1$

$$\|v_{n+1} - v_n\|_{2,\beta} \le c_4\mathrm{Lip}_\sigma\sqrt{\Upsilon(\beta)}\|v_n - v_{n-1}\|_{2,\beta}.$$

Since we can choose $\beta > 0$ such that $c_4 \operatorname{Lip}_\sigma \sqrt{\Upsilon(\beta)} < 1$, then by contraction mapping principle we prove the existence of the solution u such that

$$\lim_{n\to\infty} \|v_n - u\|_{2,\beta} = \lim_{n\to\infty} \|\mathcal{A}v_n - \mathcal{A}u\|_{2,\beta} = 0, \text{ and } u(t,x) = \mathcal{A}u(t,x) + (P_t u_0)(x).$$

Therefore, $\|u\|_{2,\beta} \le \|\mathcal{A}u\|_{2,\beta} + \sup_{r\in\mathbf{R}} u_0(r) < \infty$ and $\|u_n - \mathcal{A}u_n - P_t u_0\|_{2,\beta} = 0$, hence

$$\mathrm{E}\big[|u(t,x) - \mathcal{A}u(t,x) - P_t u_0(x)|^2\big] = 0, \;\; t \ge 0 \text{ and } x \in \mathbf{R}.$$

That's, for $j = 1, \dots n$, we have that

$$\begin{aligned}
&\|v_{n+1} - v_n\|_{2,\beta} \\
&\quad \le c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)} \|v_n - v_{n-1}\|_{2,\beta} \le \dots \le \left(c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)}\right)^j \|v_{n-(j-1)} - v_{n-j}\|_{2,\beta} \\
&\quad \le \left(c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)}\right)^n \|v_1 - v_0\|_{2,\beta}.
\end{aligned}$$

Choose $\epsilon > 0$ and pick $N_\epsilon \in \mathrm{N}$ such that $\left(c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)}\right)^{N_\epsilon} \|v_1 - v_0\|_{2,\beta} < \epsilon$. Since $c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)} < 1$ then for any $n + 1 > n \ge N_\epsilon$,

$$\|v_{n+1} - v_n\|_{2,\beta} \le \left(c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)}\right)^n \|v_1 - v_0\|_{2,\beta} \le \left(c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)}\right)^{N_\epsilon} \|v_1 - v_0\|_{2,\beta} < \epsilon.$$

Therefore $\{v_n\}$ is a Cauchy sequence. Since $(L^2,\, \|.\|_{2,\beta})$ is complete, the $v_n's$ converge in $(L^2,\, \|.\|_{2,\beta})$. So $u = \lim_{n\to\infty} v_n$ belongs to $(L^2,\, \|.\|_{2,\beta})$.

Next, we prove the uniqueness of the solution up to modification. Let u_1 and u_2 be solutions and assume for contradiction that $u_1 \ne u_2$ such that

$$\begin{cases} u_1(t,x) &=& \mathcal{A}u_1(t,x) + (P_t u_0)(x) \\ u_2(t,x) &=& \mathcal{A}u_2(t,x) + (P_t u_0)(x). \end{cases}$$

Therefore, $\|u_1 - u_2\|_{2,\beta} = \|\mathcal{A}u_1 - \mathcal{A}u_2\|_{2,\beta}$, $\beta > 0$. Then by Proposition 3.1.5 we have that $\|u_1 - u_2\|_{2,\beta} \le c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)} \|u_1 - u_2\|_{2,\beta}$, and $\|u_1 - u_2\|_{2,\beta}[1 - c_4 \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)}] \le 0$. This implies that $\|u_1 - u_2\|_{2,\beta} \le 0$ (since $1 - c_4 \operatorname{Lip}_\sigma \sqrt{\Upsilon(\beta)} > 0$) which implies that $\|u_1 - u_2\|_{2,\beta} = 0$, and follows that $u_1 = u_2$. This contradicts the assumption that $u_1 \ne u_2$, hence $u_1 = u_2$ and thus a unique solution. Therefore u_1 and u_2 are modification of each other. The proof of the upper bound result of the theorem follows similar argument as in the proof of Theorem 3.2.2 below. □

3.2 Growth of second moment of the solution

Here, we estimate bound on growth moment and show that our solution grows exponentially for the compensated equation. For the lower bound result, we will need the following extra condition on σ.

Condition 3.2.1. There exist a positive function $\bar{J}$ and a constant, L_σ such that for all $x,\, h \in \mathbf{R}$, we have

$$|\sigma(x,\, h)| \geq L_\sigma \bar{J}(h)|x| \tag{3.2.1}$$

The function $\bar{J}$ is assumed to satisfy the following integrability condition.

$$\kappa \leq \int_{\mathbf{R}} \bar{J}(h)^2 \nu(\mathrm{d}h) \leq \mathrm{K}, \tag{3.2.2}$$

where K is the constant from (3.0.3) and κ is another positive, finite constant.

Let's recall that

$$\overline{\gamma}(2) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{E}|u(t,x)|^2,\ x \in \mathbf{R}.$$

Then under some further assumptions, we have the following

Theorem 3.2.2. *If we further assume that condition 3.2.1 holds and* $\inf_{x\in\mathbf{R}} u_0(x) > 0$, *then*

$$\overline{\gamma}(2) \geq \Upsilon^{-1}\big(\frac{1}{\mathcal{K}^2}\big) > 0,$$

where $\Upsilon^{-1}(t) := \sup\big\{\beta > 0 : \Upsilon(\beta) > t\big\}$ *and* $\mathcal{K} = \sqrt{\kappa} L_\sigma \lambda$.

Example 3.2.3. For $\mathcal{L} = \Delta,\ \sigma = u\, h$, then $L_\sigma = 1,\ J(h) = |h|$ and

$$\Upsilon^{-1}\left(\frac{1}{\kappa\lambda^2}\right) = \sup\left\{\beta > 0 : \frac{1}{\sqrt{8\,\beta}} > \frac{1}{\kappa\lambda^2}\right\}.$$

Proof of Theorem 3.2.2. As in the proof of Theorem 2.7 of [44], it is sufficient for us to show that

$$\int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}|u(t,x)|^2 \mathrm{d}t = \infty,\ \forall\, t > 0$$

whenever $\Upsilon(\beta) \geq \frac{1}{\mathcal{K}^2}$ with $\mathcal{K} := \sqrt{\kappa} L_\sigma \lambda$. We prove by contradiction by assuming that

$$\limsup_{t\to\infty} \mathrm{e}^{-\alpha t}\mathrm{E}|u(t,x)|^2 < \infty$$

with $\Upsilon(\alpha) < \frac{1}{\mathcal{K}^2}$ and $x \in \mathbf{R}$, which is equivalent to say that $\mathrm{E}|u(t,x)|^2 = O(\exp(\alpha t))$ as

$t \to \infty$, since $\mathrm{e}^{-\alpha t} \geq 0$, $\forall\, t \geq 0$ and $\alpha \in \mathbf{R}$. It follows from the above that

$$\int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}|u(t,x)|^2\mathrm{d}t \leq C\int_0^\infty \mathrm{e}^{-\beta t}\mathrm{e}^{\alpha t}\mathrm{d}t = C\int_0^\infty \mathrm{e}^{-(\beta-\alpha)t}\mathrm{d}t < \infty,$$

for all $\beta \in (\alpha, \Upsilon^{-1}(\frac{1}{\mathcal{K}^2}))$ which contradicts the assumption that

$$\int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}|u(t,x)|^2\mathrm{d}t = \infty,\ \forall\, t > 0$$

provided that $\Upsilon(\beta) \geq \frac{1}{\mathcal{K}^2}$ and hence the claim. By condition 3.2.1, we obtain that

$$\begin{aligned}
&\mathrm{E}|u(t,x)|^2\\
&\quad = |(P_t u_0)(x)|^2 + \lambda^2\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}} \mathrm{E}|\sigma(u(s,y),h)|^2|p(t-s,x-y)|^2\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s\\
&\quad \geq |(P_t u_0)(x)|^2 + \lambda^2 L_\sigma^2\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}} \mathrm{E}(|u(s,y)|^2)\bar{J}^2(h)|p(t-s,x-y)|^2\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s\\
&\quad \geq |(P_t u_0)(x)|^2 + \kappa\lambda^2 L_\sigma^2\int_0^t\int_{\mathbf{R}} \mathrm{E}|u(s,y)|^2|p(t-s,x-y)|^2\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

The last inequality follows by the assumption on ν. Apply Laplace transforms to both for all $\beta > 0$ and $x \in \mathbf{R}$,

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}|u(t,x)|^2\mathrm{d}t\\
&\geq \int_0^\infty \varepsilon^2\mathrm{e}^{-\beta t}\mathrm{d}t + \kappa\lambda^2 L_\sigma^2\int_0^\infty \mathrm{e}^{-\beta t}\int_0^t\int_{\mathbf{R}} \mathrm{E}(|u(s,y)|^2)|p(t-s,x-y)|^2\mathrm{d}y\mathrm{d}s\mathrm{d}t\\
&\geq \int_0^\infty e^{-\beta t}|(P_t u_0)(x)|^2)dt + \kappa\lambda^2 L_\sigma^2\int_{\mathbf{R}}\int_0^\infty \mathrm{e}^{-\beta s}|p(s,x-y)|^2\mathrm{d}s\int_0^\infty \mathrm{e}^{-\beta s}\mathrm{E}|u(s,y)|^2\mathrm{d}s\mathrm{d}y.
\end{aligned}$$

Then,

$$F_\beta(x) \geq G_\beta(x) + \mathcal{K}^2\int_{\mathbb{R}} H_\beta(x-y)\left\{\int_0^\infty \mathrm{e}^{-\beta s}\mathrm{E}(|u(s,y)|^2)\mathrm{d}s\right\}\mathrm{d}y,$$

where we define the following

$$F_\beta(x) := \int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}|u(t,x)|^2\mathrm{d}t,\ G_\beta(x) := \int_0^\infty \mathrm{e}^{-\beta t}|(P_t u_0)(x)|^2\mathrm{d}t,$$

$$H_\beta(x) := \int_0^\infty \mathrm{e}^{-\beta t}|p(t,x)|^2\mathrm{d}t \quad \text{and } \mathcal{K}^2 = \kappa\lambda^2 L_\sigma^2.$$

Therefore,

$$F_\beta(x) \geq G_\beta(x) + \mathcal{K}^2(F_\beta * H_\beta)(x).$$

Let's define a linear operator $\mathcal{H}$ (a convolution operator) by

$$(\mathcal{H}f)(x) := \mathcal{K}^2(H_\beta * f)(x), \tag{3.2.3}$$

we deduce that

$$\mathcal{H}^n F_\beta(x) - \mathcal{H}^{n+1}F_\beta(x) \geq \mathcal{H}^n G_\beta(x)$$

where $n \geq 0$ is the number of convolutions. Also,

$$\sum_{n=0}^{N} \big(\mathcal{H}^n F_\beta - \mathcal{H}^{n+1}F_\beta\big)(x) \geq \sum_{n=0}^{N} \big(\mathcal{H}^n G_\beta\big)(x),$$

which implies that

$$F_\beta - \mathcal{H}^{N+1}F_\beta(x) \geq \sum_{n=0}^{N} \big(\mathcal{H}^n G_\beta\big)(x)$$

and

$$F_\beta \geq \mathcal{H}^{N+1}F_\beta(x) + \sum_{n=0}^{N} \big(\mathcal{H}^n G_\beta\big)(x) \geq \sum_{n=0}^{N} \big(\mathcal{H}^n G_\beta\big)(x).$$

Therefore,

$$F_\beta \geq \sum_{n=0}^{N} \big(\mathcal{H}^n G_\beta\big)(x) \quad \text{and} \quad F_\beta \geq \sum_{n=0}^{\infty} \big(\mathcal{H}^n G_\beta\big)(x) \quad \text{as} \quad N \to \infty.$$

Denote $\varepsilon := \inf_{x\in\mathbf{R}} u_0(x)$ then it follows that $(P_t u_0)(x) \geq \varepsilon$ and hence $G_\beta(x) \geq \frac{\varepsilon^2}{\beta}$. From equation (3.2.3) we have that

$$\begin{aligned}(\mathcal{H}G_\beta)(x) &= \mathcal{K}^2(H_\beta * G_\beta)(x) = \mathcal{K}^2 \int_0^\infty \mathrm{e}^{-\beta t}|(P_t u_0)(x)|^2 \mathrm{d}t. \int_{\mathbb{R}} H_\beta(x-y)\mathrm{d}y \\ &\geq \frac{\mathcal{K}^2\varepsilon^2}{\beta}\int_{\mathbb{R}} H_\beta(x-y)\mathrm{d}y = \frac{\mathcal{K}^2\varepsilon^2}{\beta}\int_{\mathbb{R}} H_\beta(y)\mathrm{d}y.\end{aligned}$$

Therefore,

$$(\mathcal{H}G_\beta)(x) \geq \frac{\mathcal{K}^2\varepsilon^2}{\beta}\Upsilon(\beta).$$

Iterating the above argument for n times and taking an infinite sum, it follows that:

$$\sum_{n=0}^{\infty} \big(\mathcal{H}^n G_\beta\big)(x) \geq \frac{\varepsilon^2}{\beta}\sum_{n=0}^{\infty}(\mathcal{K}^2\Upsilon(\beta))^n \text{ and } F_\beta(x) \geq \sum_{n=0}^{\infty} \big(\mathcal{H}^n G_\beta\big)(x) \geq \frac{\varepsilon^2}{\beta}\sum_{n=0}^{\infty}(\mathcal{K}^2\Upsilon(\beta))^n.$$

That is;

$$F_\beta(x) \geq \frac{\varepsilon^2}{\beta}\sum_{n=0}^{\infty}(\mathcal{K}^2\Upsilon(\beta))^n \quad \text{and} \quad F_\beta(x) = \infty \quad \text{whenever } \mathcal{K}^2\Upsilon(\beta) \geq 1,$$

and the result follows. □

In the next result, we will restrict our attention to the special case when $\mathcal{L} := -(-\Delta)^{\alpha/2}$ which is the generator of α-stable process. This will enable us to get more precise information about the behaviour of the growth of the second moment of the solution to (3.0.1). Note that $\Upsilon(\beta) < \infty$ requires that $1 < \alpha \leq 2$ which will be in force.

Theorem 3.2.4. *Suppose that* $\mathcal{L} := -(-\Delta)^{\alpha/2}$, $1 < \alpha \leq 2$ *and that condition 3.2.1 together with* $\inf_{x\in\mathbf{R}} u_0(x) > 0$ *hold, then*

$$\mathrm{E}|u(t,x)|^2 \geq c_2 \exp(c_3\lambda^{\frac{2\alpha}{\alpha-1}}t), \text{ for all } t > 0,$$

where c_2 *and* c_3 *are some positive constants.*

Proof. Starting with Itô isometry

$$\mathrm{E}|u(t,x)|^2 = |(P_t u_0)(x)|^2 + \lambda^2\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}}|p(t-s,x-y)|^2\mathrm{E}|\sigma(u(s,y),h)|^2\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s.$$

Then using $\varepsilon := \inf_{x\in\mathbf{R}} u_0(x)$, and condition 3.2.1 to write

$$\begin{aligned}\mathrm{E}|u(t,x)|^2 &\geq \varepsilon^2 + \kappa\lambda^2 L_\sigma^2\int_0^t\int_{\mathbf{R}}|p(t-s,x-y)|^2\mathrm{E}|u(s,y)|^2\mathrm{d}y\mathrm{d}s\\ &\geq \varepsilon^2 + \kappa\lambda^2 L_\sigma^2\int_0^t\inf_{y\in\mathbf{R}}\mathrm{E}|u(s,y)|^2 p(2(t-s),0)\mathrm{d}s.\end{aligned}$$

Setting $F(t) := \inf_{x\in\mathbf{R}}\mathrm{E}|u(t,x)|^2$, the above together with an upper bound on the heat kernel in Lemma 2.3.4 yield

$$\begin{aligned}F(t) &\geq \varepsilon^2 + c_1\kappa\lambda^2 L_\sigma^2\int_0^t\frac{F(s)}{(t-s)^{1/\alpha}}\mathrm{d}s\\ &= \varepsilon^2 + c_1\kappa\lambda^2 L_\sigma^2\int_0^t(t-s)^{\frac{\alpha-1}{\alpha}-1}F(s)\mathrm{d}s.\end{aligned}$$

This proves the theorem by applying renewal inequality. □

Next we drop the assumption that the initial condition u_0 is bounded below. This we compensate by paying the price of not getting an exponential growth of the solution for all times.

Theorem 3.2.5. *Suppose that $\mathcal{L} := -(-\Delta)^{\alpha/2}$ and that condition 3.2.1 holds. Fix any interval $[t_0, T]$, with $0 < t_0 < T < \infty$. If $u_0 \neq 0$, then*

$$\inf_{x\in[-1,1]} \mathrm{E}|u(t,\, x)|^2 \geq c_4 \exp(c_5 \kappa\lambda^2 L_\sigma^2 t), \text{ for all } t \in [t_0, T],$$

where c_4 and c_5 are some positive constants.

Proof. As before, take second moment of the solution,

$$\mathrm{E}|u(t,x)|^2 = |(P_t u_0)(x)|^2 + \lambda^2 \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} |p(t-s, x-y)|^2 \mathrm{E}|\sigma(u(s,y),h)|^2 \nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s.$$

Fix $t_0 > 0$, then use condition 3.2.1 to write

$$\begin{aligned}
\mathrm{E}|u(t+t_0,x)|^2 &\geq |(P_{t+t_0}u_0)(x)|^2 + \kappa\lambda^2 L_\sigma^2 \int_0^{t+t_0} \int_{\mathbf{R}} |p(t+t_0-s, x-y)|^2 \mathrm{E}|u(s,y)|^2 \mathrm{d}y\mathrm{d}s \\
&\geq |(P_{t+t_0}u_0)(x)|^2 + \kappa\lambda^2 L_\sigma^2 \int_{t_0}^{t+t_0} \int_{\mathbf{R}} |p(t+t_0-s, x-y)|^2 \mathrm{E}|u(s,y)|^2 \mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Make the following change of variable $s - t_0$, then set $v(t,x) := u(t+t_0, x)$ for a fixed $t_0 > 0$ together with Proposition 2.3.6 to write

$$\begin{aligned}
\mathrm{E}|v(t,x)|^2 &\geq |(P_{t+t_0}u_0)(x)|^2 + \kappa\lambda^2 L_\sigma^2 \int_0^t \int_{\mathbf{R}} |p(t-s, x-y)|^2 \mathrm{E}|v(s,y)|^2 \mathrm{d}y\mathrm{d}s \\
&\geq |c(t_0)p(t+\eta, x)|^2 + \kappa\lambda^2 L_\sigma^2 \int_0^t \int_{\mathbf{R}} |p(t-s, x-y)|^2 \mathrm{E}|v(s,y)|^2 \mathrm{d}y\mathrm{d}s \\
&\geq c_1 p^2(t+\eta, x) + \kappa\lambda^2 L_\sigma^2 \int_0^t \inf_{y\in[-1,1]} \mathrm{E}|v(s,y)|^2 \int_{-1}^1 p^2(t-s, x-y)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Upon setting $g(t) := \inf_{x\in[-1,1]} \mathrm{E}|v(t,x)|^2$, we have that

$$g(t) \geq c_2 (t+\eta)^{-2/\alpha} + c_3 \kappa\lambda^2 L_\sigma^2 \int_0^t g(s) \int_{-1}^1 (t-s)^{-2/\alpha} \mathrm{d}y\mathrm{d}s.$$

Since $t \leq T$, we obtain that

$$g(t) \geq c_4 + c_5 \kappa\lambda^2 L_\sigma^2 \int_0^t g(s)\mathrm{d}s,$$

where the constants c_4 and c_5 depend on T and the result follows. □

3.3 Non-existence of global solution

Much of the result on SPDEs assumes that the multiplicative non-linearity term is globally Lipschitz. We show that if the non-linearity term grows faster than linear growth, then our solution fails to exist. The global non-existence of the solution occurs for some non-linear conditions on σ. If instead of (3.2.1), we have the following condition.

Condition 3.3.1. There exists a constant $\beta > 1$ such that

$$|\sigma(x,\, h)| \geq L_\sigma \bar{J}(h)|x|^\beta, \tag{3.3.1}$$

where the constant L_σ and the function $\bar{J}$ are the same as in condition 3.2.1.

We then have the following result.

Theorem 3.3.2. *Suppose that $\mathcal{L} := -(-\Delta)^{\alpha/2}$ and that condition 3.3.1 is in force. Then there does not exist any random field solution to* (3.0.1) *.*

We now give the following proposition which establishes the fact that under the local Lipschitz continuity as stated in condition 3.3.1, there exists a unique solution up to a fixed time T.

Proposition 3.3.3. *Suppose that condition 3.3.1 holds. Then there exists a $T > 0$ such that* (3.0.1) *has a unique random field solution up to time T.*

Proof. We begin by defining

$$\sigma_N(x,\, h) = \begin{cases} \sigma(x,\, h) & \text{if} \quad x \leq N \\ \sigma(N,\, h) & \text{if} \quad x > N. \end{cases}$$

$\sigma_N(x,\, h)$ therefore satisfies (3.0.2) but with a different constant. Therefore by the proof of Theorem 3.1.1, there exists a unique solution $\{u_N(t,\, x)\}_{0\leq t\leq T, x\in\mathbf{R}}$ satisfying

$$\sup_{0<t<T} \sup_{x\in\mathbf{R}} \mathrm{E}|u_N(t,\, x)|^2 < \infty.$$

By Proposition A.1.1, for a fixed $x \in \mathbf{R}$, $\mathrm{E}|u_N(t,\, x)|^2$ is continuous in t. Since $\mathrm{E}|u(0,x)|^2$ is finite, there exists $T > 0$ such that for all $t < T$, $E|u_N(t,\, x)|^2$ is finite as well. We have therefore established short-time existence of the solution. □

We state one last result on the global non-existence of solutions to a class of ODE which the conclusion of our main result will be based on.

Proposition 3.3.4. *Consider the following ordinary differential equation.*

$$y'(t)t^b = y(t)^{1+a}$$

for all positive constants a, b *with initial condition* $y(t_0)$*. If* $b < 1$*, then the solution blows up for any positive initial datum* $y(t_0)$*. On the other hand, if* $b > 1$*, then the solution blows up for sufficiently large initial datum* $y(t_0)$*.*

Proof. For the simplest case when $b = 1$, an elementary calculus shows that

$$y(t)^{-a} = y(t_0)^{-a} - \ln\big(\frac{t}{t_0}\big)$$

solves the above equation. The above solution shows that $y(t)$ blows up in finite time for any positive initial function $y(t_0)$. Next for $a > 0$ and $b \neq 1$ the solution to the equation is given by

$$y(t)^{-a} = y(t_0)^{-a} + \frac{a}{b-1}\big(t^{1-b} - t_0^{1-b}\big).$$

The theorem follows for $b > 1$ whenever $y(t_0)^a > \frac{t_0^{b-1}(b-1)}{a}$ and when $b < 1$ for any positive initial condition. □

Proof of Theorem 3.3.2. We use Ito's isometry to write

$$\mathrm{E}|u(t,x)|^2 = |(P_t u_0)(x)|^2 + \lambda^2 \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} |p(t-s,x-y)|^2 \mathrm{E}|\sigma(u(s,y),h)|^2 \nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s.$$

We use the assumption that $u_0(x) > c_1$ for some positive constant c_1 and condition 3.3.1 to come up with

$$\begin{aligned} \mathrm{E}|u(t,x)|^2 &\geq c_1^2 + \kappa\lambda^2 L_\sigma^2 \int_0^t \int_{\mathbf{R}} p^2(t-s,x-y)\mathrm{E}|u(s,y)|^{2\beta}\mathrm{d}y\mathrm{d}s \\ &\geq c_1^2 + \kappa\lambda^2 L_\sigma^2 \int_0^t (\inf_{y\in\mathbf{R}} \mathrm{E}|u(s,y)|^2)^\beta p(2(t-s),0)\mathrm{d}s. \end{aligned}$$

Upon setting

$$F(t) = \inf_{x\in\mathbf{R}} \mathrm{E}|u(t,x)|^2,$$

the above inequality reduces to

$$F(t) \geq c_1^2 + \kappa\lambda^2 L_\sigma^2 \int_0^t p(2(t-s),0)F^\beta(s)\mathrm{d}s.$$

We now use Lemma 2.3.4 to find lower bounds on the heat kernel appearing in the above display.

$$\begin{aligned} F(t) &\geq c_2 + \kappa\lambda^2 L_\sigma^2 c_3 \int_0^t (t-s)^{-1/\alpha} F^\beta(s)\mathrm{d}s \\ &\geq c_2 + \kappa\lambda^2 L_\sigma^2 c_3 \int_0^t t^{-1/\alpha} F^\beta(s)\mathrm{d}s. \end{aligned}$$

Hence, multiplying through by $t^{1/\alpha}$

$$\begin{aligned} F(t)t^{\frac{1}{\alpha}} &\geq c_2 t^{1/\alpha} + \kappa\lambda^2 L_\sigma^2 c_3 \int_0^t F^\beta(s)\mathrm{d}s \\ &= c_2 t^{1/\alpha} + \kappa\lambda^2 L_\sigma^2 c_3 \int_0^t \frac{(s^{1/\alpha}F(s))^\beta}{s^{\beta/\alpha}}\mathrm{d}s. \end{aligned}$$

Let $Y(t) = F(t)t^{1/\alpha}$, then for all $t \geq 0$

$$\begin{aligned} Y(t) &\geq c_2 t^{1/\alpha} + \kappa\lambda^2 L_\sigma^2 c_3 \int_0^t \frac{Y^\beta(s)}{s^{\beta/\alpha}}\mathrm{d}s \\ &\geq \kappa\lambda^2 L_\sigma^2 c_3 \int_0^t \frac{Y^\beta(s)}{s^{\beta/\alpha}}\mathrm{d}s, \end{aligned}$$

which essentially implies by Proposition 3.3.4 that $Y(t)$ ceases to exist in finite time. □

CHAPTER 4

ON A STOCHASTIC HEAT EQUATION DRIVEN BY NON-COMPENSATED POISSON NOISE

We now look at the non-compensated equation.

$$\left[\frac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x)\right]\mathrm{d}x\mathrm{d}t = \lambda \int_{\mathbf{R}^d} \sigma(u(t,\,x),\,h)N(\mathrm{d}t,\,\mathrm{d}x,\,\mathrm{d}h), \tag{4.0.1}$$

with initial condition $u(0,x) = u_0(x)$. Here again $\mathcal{L}$ is the L^2- generator of a Lévy process. For the existence and uniqueness result, we make the following assumption.

Condition 4.0.5. There exist a positive function J and a finite positive constant, Lip_σ such that for all $x,\,y,\,h \in \mathbf{R}^d$, we have

$$|\sigma(0,h)| \leq J(h) \quad \text{and} \quad |\sigma(x,\,h) - \sigma(y,\,h)| \leq J(h)\mathrm{Lip}_\sigma |x-y|. \tag{4.0.2}$$

The function J is assumed to satisfy the following integrability condition.

$$\int_{\mathbf{R}^d} J(h)\nu(\mathrm{d}h) \leq \mathrm{K}, \tag{4.0.3}$$

where K is some finite positive constant.

4.1 Existence and Uniqueness result

As with the existence and uniqueness result of the compensated noise, we also assume that $u_0 : \mathbf{R}^d \to \mathbf{R}_+$ is both measurable and bounded.

Theorem 4.1.1. *Under condition 4.0.5, there exists a unique random field solution to* (4.0.1) *with*

$$\bar{\gamma}(1) \leq \lambda \mathrm{K}\mathrm{Lip}_\sigma.$$

The above result gives an upper bound on the growth of the first moment. Before we give the proof of Theorem 4.1.1, first some estimates.

4.1.1 Estimates for the non-compensated equation.

We define the following norm

$$\|u\|_{1,\beta} = \sup_{t\geq 0} \sup_{x\in\mathbf{R}^d} \mathrm{e}^{-\beta t}\mathrm{E}|u(t,x)|,$$

the first moment norm of the solution. Let

$$\mathcal{B}u(t,x) := \lambda \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p(t-s,\, x,\, y)\sigma(u(s,y),h)N(\mathrm{d}h,\, \mathrm{d}y,\, \mathrm{d}s),$$

and hence the following Lemma(s):

Lemma 4.1.2. *Suppose that u is weak predictable and $\|u\|_{1,\beta} < \infty$ for all $\beta > 0$ and $\sigma(u,h)$ satisfies condition 4.0.5, then $\|\mathcal{B}u\|_{1,\beta} \leq \frac{\lambda \mathrm{K}}{\beta}\big[1 + \mathrm{Lip}_\sigma \|u\|_{1,\beta}\big]$.*

Proof. Following similar steps as in the previous section,

$$\begin{aligned}
\mathrm{E}|\mathcal{B}u(t,\, x)| &= \lambda \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p(t-s,\, x,\, y)\mathrm{E}|\sigma(u(s,y),h)|\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s \\
&\leq \lambda \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p(t-s,\, x,\, y)J(h)\big[1 + \mathrm{Lip}_\sigma \mathrm{E}|u(s,y)|\big]\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s \\
&\leq \lambda \mathrm{K} \int_0^t \int_{\mathbf{R}^d} p(t-s,\, x,\, y)\big[1 + \mathrm{Lip}_\sigma \mathrm{E}|u(s,y)|\big]\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Multiply throught by the exponential factor

$$\begin{aligned}
&\mathrm{e}^{-\beta t}\mathrm{E}|\mathcal{B}u(t,\,x)| \\
&\leq \lambda \mathrm{K}\int_0^t\int_{\mathbf{R}^d}\mathrm{e}^{-\beta(t-s)}p(t-s,\,x,\,y)\big[\mathrm{e}^{-\beta s}+\mathrm{Lip}_\sigma\mathrm{e}^{-\beta s}\mathrm{E}|u(s,y)|\big]\mathrm{d}y\mathrm{d}s \\
&\leq \lambda \mathrm{K}\sup_{s\geq 0}\sup_{y\in\mathbf{R}^d}\big[\mathrm{e}^{-\beta s}+\mathrm{Lip}_\sigma\mathrm{e}^{-\beta s}\mathrm{E}|u(s,y)|\big]\int_0^t\int_{\mathbf{R}^d}\mathrm{e}^{-\beta(t-s)}p(t-s,\,x,\,y)\mathrm{d}y\mathrm{d}s \\
&= \lambda \mathrm{K}\big[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}\big]\int_0^t\int_{\mathbf{R}^d}\mathrm{e}^{-\beta(t-s)}p(t-s,\,x,\,y)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{B}u\|_{1,\beta} &\leq \lambda \mathrm{K}\big[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}\big]\int_0^\infty\int_{\mathbf{R}^d}\mathrm{e}^{-\beta s}p(s,\,x,\,y)\mathrm{d}y\mathrm{d}s \\
&= \lambda \mathrm{K}\big[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}\big]\int_0^\infty\mathrm{e}^{-\beta s}\mathrm{d}s,
\end{aligned}$$

and the result follows. □

Lemma 4.1.3. *Let u and v be weak predictable random field solutions satisfying $\|u\|_{1,\beta}+\|v\|_{1,\beta}<\infty$ for all $\beta>0$ and assume $\sigma(u,h)$ satisfies condition 4.0.5, then*

$$\|\mathcal{B}u-\mathcal{B}v\|_{1,\beta}\leq\frac{\lambda\mathrm{K}\mathrm{Lip}_\sigma}{\beta}\|u-v\|_{1,\beta}.$$

Proof. Follows as above Lemma 4.1.2. □

Proof of Theorem 4.1.1. We prove the existence of the solution by method of Picard-Iteration schemes by following same proof in Theorem 3.1.1. Iteratively, define

$$\begin{cases} v_{n+1}(t,x) &= \mathcal{B}v_n(t,x)+(P_tu_0)(x) \\ v_n(t,x) &= \mathcal{B}v_{n-1}(t,x)+(P_tu_0)(x) \end{cases}$$

where $v_0 =: \mathcal{B}v_{-1}$. Then for sufficiently large $\beta>0$

$$\|\mathcal{B}v_{n+1}\|_{1,\beta}=\|\mathcal{B}(\mathcal{B}v_n)\|_{1,\beta}\leq\frac{\lambda\mathrm{K}}{\beta}\big[1+\mathrm{Lip}_\sigma\|\mathcal{B}v_n\|_{1,\beta}\big].$$

Since $\lim_{\beta\to\infty}\frac{1}{\beta}=0$, then we choose and fix $\beta>0$ such that $\frac{\lambda\mathrm{K}}{\beta}\mathrm{Lip}_\sigma<1$. Take sup of both sides over n and given that $\sup_{n\geq 0}\|\mathcal{B}v_{n+1}\|_{1,\beta}=\sup_{n\geq 0}\|\mathcal{B}v_n\|_{1,\beta}$, since $\|\mathcal{B}v_0\|_{1,\beta}$ is significantly small for β sufficiently large. Then

$$\sup_{n\geq 0} \|\mathcal{B}v_n\|_{1,\beta} \leq \frac{\frac{\lambda \mathrm{K}}{\beta}}{1-\frac{\lambda \mathrm{K}\mathrm{Lip}_\sigma}{\beta}} < \infty.$$

Also, $v_k(t,x) = \mathcal{B}v_{k-1}(t,x) + (P_t u_0)(x)$ and since $(P_t u_0)(x)$ is bounded uniformly by $\sup_{r\in\mathbf{R}^d}|u_0(r)|$,

$$\|v_k\|_{1,\beta} = \|\mathcal{B}v_{k-1}\|_{1,\beta} + \|(P_t u_0)(x)\|_{1,\beta} \leq \|\mathcal{B}v_{k-1}\|_{1,\beta} + \sup_{r\in\mathbf{R}^d}|u_0(r)|.$$

Then it follows that

$$\begin{aligned}\sup_{k\geq 1}\|v_k\|_{1,\beta} &\leq \sup_{k\geq 1}\|\mathcal{B}v_{k-1}\|_{1,\beta} + \sup_{r\in\mathbf{R}^d}|u_0(r)| \\ &\leq \frac{\frac{\lambda \mathrm{K}}{\beta}}{1-\frac{\lambda \mathrm{K}\mathrm{Lip}_\sigma}{\beta}} + \sup_{r\in\mathbf{R}^d}|u_0(r)| < \infty.\end{aligned}$$

On the other hand, for all $n\geq 1$ and $j = 1,\dots n$,

$$\begin{aligned}\|v_{n+1}-v_n\|_{1,\beta} &= \|\mathcal{B}v_n - \mathcal{B}v_{n-1}\|_{1,\beta} \\ &\leq \frac{\lambda \mathrm{K}\mathrm{Lip}_\sigma}{\beta}\|v_n - v_{n-1}\|_{1,\beta} \leq \dots \leq \left(\frac{\lambda \mathrm{K}\mathrm{Lip}_\sigma}{\beta}\right)^j \|v_{n-(j-1)} - v_{n-j}\|_{1,\beta} \\ &\leq \left(\frac{\lambda \mathrm{K}\mathrm{Lip}_\sigma}{\beta}\right)^n \|v_1 - v_0\|_{1,\beta}.\end{aligned}$$

The existence of the solution u follows by contraction mapping principle since $\frac{\lambda \mathrm{K}}{\beta}\mathrm{Lip}_\sigma < 1$ such that

$$\lim_{n\to\infty}\|v_n - u\|_{1,\beta} = \lim_{n\to\infty}\|\mathcal{B}v_n - \mathcal{B}u\|_{1,\beta} = 0, \text{ where } u(t,x) = (P_t u_0)(x) + \mathcal{B}u(t,x).$$

Therefore, $\|u\|_{1,\beta} \leq \|\mathcal{B}u\|_{1,\beta} + \sup_{r\in\mathbf{R}^d} u_0(r) < \infty$ and $\|u_n - \mathcal{B}u_n - P_t u_0\|_{1,\beta} = 0$, hence

$$\mathrm{E}\big[|u(t,x) - \mathcal{B}u(t,x) - P_t u_0(x)|\big] = 0, \;\; t\geq 0 \,\text{and}\, x\in\mathbf{R}^d.$$

Next, we prove the uniqueness. Suppose for contradiction that there exist two solutions u, v such that $u \neq v$, then by Proposition 3.1.5,

$$\|u-v\|_{1,\beta} = \|\mathcal{B}u - \mathcal{B}v\|_{1,\beta} \leq \frac{\lambda \mathrm{K}\mathrm{Lip}_\sigma}{\beta}\|u-v\|_{1,\beta}.$$

Then $\|u-v\|_{1,\beta}[1-\frac{\lambda \mathrm{KLip}_\sigma}{\beta}] \leq 0$ and $\|u-v\|_{1,\beta} \leq 0$ since $\beta > \lambda\mathrm{KLip}_\sigma$. That's $\|u-v\|_{1,\beta} = 0$ which implies that $u - v = 0$ contradicting the assumption that $u \neq v$. Hence u and v are unique modifications of one another which proves the theorem. The second part of the theorem follows similar steps as in Theorem 4.2.2 below. □

4.2 Growth of first moment of the solution

We can also give the lower bound estimate on the growth of the first moment.

Condition 4.2.1. There exist a positive function $\bar{J}$ and a constant, L_σ such that for all $x,\, h \in \mathbf{R}^d$, we have

$$|\sigma(x,\, h)| \geq L_\sigma \bar{J}(h)|x| \tag{4.2.1}$$

The function $\bar{J}$ is assumed to satisfy the following integrability condition.

$$\kappa \leq \int_{\mathbf{R}^d} \bar{J}(h)\nu(\mathrm{d}h) \leq K, \tag{4.2.2}$$

where K is the constant from (4.0.3) and κ is another positive finite constant.

Next we define the upper 1st moment Lyapunov exponent as follows:

$$\overline{\gamma}(1) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{E}|u(t,x)|,$$

and then the lower bound result.

Theorem 4.2.2. *If we further assume that condition 4.2.1 holds but with* $\inf_{x\in R} u_0(x) > 0$, *then*

$$\inf_{x\in\mathbf{R}^d} \mathrm{E}|u(t,x)| \geq c_1 \mathrm{e}^{\kappa L_\sigma \lambda t},$$

where c_1 *is a positive constant. In particular, we have*

$$\bar{\gamma}(1) \geq \kappa L_\sigma \lambda.$$

Proof of Theorem 4.2.2. We start off with

$$\begin{aligned}
\mathrm{E}|u(t,x)| &\geq |(P_t u_0)(x)| + \kappa\lambda L_\sigma \int_0^t \int_{\mathbf{R}^d} |p(t-s,x-y)|\mathrm{E}|u(s,y)|\mathrm{d}y\mathrm{d}s \\
&\geq \varepsilon + \kappa\lambda L_\sigma \int_0^t \inf_{y\in\mathbf{R}^d} \mathrm{E}|u(s,y)| \int_{\mathbf{R}^d} p(t-s,x-y)\mathrm{d}y\mathrm{d}s \\
&= \varepsilon + \kappa\lambda L_\sigma \int_0^t \inf_{y\in\mathbf{R}^d} |\mathrm{E}|u(s,y)|\mathrm{d}s.
\end{aligned}$$

Upon setting $f(t) := \inf_{x\in\mathbf{R}^d} \mathrm{E}|u(t,\, x)|$, the above yields

$$f(t) \geq \varepsilon + \kappa\lambda L_\sigma \int_0^t f(s)\mathrm{d}s,$$

which immediately implies the result . The second part of the theorem follows readily from the first part. □

The next result gets rid of the assumption that the initial condition is bounded below. The price we pay is that we need some precise information about the heat kernel and we fail to get an exponential property for all times. The theorem follows.

Theorem 4.2.3. *Let $\mathcal{L} := -(-\Delta)^{\alpha/2}$. Suppose that condition 4.2.1 together with $\|u_0\|_{L^1(B(0,1))} > 0$ hold. Then there exits $t_0 > 1$ such that for all $t_0 < t < T < \infty$, we have*

$$\inf_{x\in B(0,1)} \mathrm{E}|u(t,x)| \geq c_4 \exp(c_5\kappa\lambda t), \text{ for all } t \in [t_0, T],$$

where c_4 and c_5 are some positive constants.

Proof. We follow same lines of proof of Theorem 3.2.5. Set $v(t,x) := u(t+t_0, x)$ for fixed $t_0 > 1$, then by condition 4.2.1 together with Lemma 2.3.6, we obtain

$$\begin{aligned}
\mathrm{E}|v(t,x)| &\geq (P_{t+t_0}u_0)(x) + \kappa\lambda L_\sigma \int_0^t \int_{\mathbf{R}^d} p(t-s, x-y)\mathrm{E}|v(s,y)|\mathrm{d}y\mathrm{d}s \\
&\geq c_1 p(t+\eta, x) + \kappa\lambda L_\sigma \int_0^t \inf_{y\in B(0,1)} \mathrm{E}|v(s,y)| \int_{B(0,1)} p(t-s, x-y)\mathrm{d}y\mathrm{d}s \\
&\geq c_2(t+\eta)^{-1/\alpha} + c_3\kappa\lambda L_\sigma \int_0^t \inf_{y\in B(0,1)} \mathrm{E}|v(s,y)| \int_{B(0,1)} (t-s)^{-1/\alpha}\mathrm{d}y\mathrm{d}s
\end{aligned}$$

Upon setting $g(t) := \inf_{x\in B(0,1)} \mathrm{E}|v(t,\, x)|$, the above gives

$$g(t) \geq c_4 + c_5\kappa\lambda L_\sigma \int_0^t g(s)\mathrm{d}s, \quad \text{for all } t_0 \leq t \leq T,$$

where the constants c_4 and c_5 are dependent on T. This proves the result. □

4.3 Non-existence of global solution

We show that if the function σ grows faster than linear growth, then the first moment of the solution to (4.0.1) ceases to exist for all time t. Instead of (4.2.1), we consider the following condition:

Condition 4.3.1. There exists a constant $\gamma > 1$ such that

$$|\sigma(x,\, h)| \geq L_\sigma \bar{J}(h)|x|^\gamma, \tag{4.3.1}$$

where the constant L_σ and the function J are the same as in condition 4.2.1.

We then have the following result with the initial condition $u_0 : \mathbf{R}^d \to \mathbf{R}_+$, a positive function on a set of positive measure.

Theorem 4.3.2. *Suppose that both conditions 4.0.5 and 4.3.1 are in force. Then there are no random field solutions to* (4.0.1) *whenever the non-negative initial condition* u_0 *is bounded below. Let* $\mathcal{L} := -(-\Delta)^{\alpha/2}$*, then under the same conditions, there are no random field solutions to* (4.0.1) *even if we only have* $u_0 \neq 0$.

We will also need the following proposition which establishes the fact that under the local Lipschitz continuity as stated in condition 4.3.1, there exists a unique solution up to a fixed time T.

Proposition 4.3.3. *Suppose that condition 4.3.1 holds. Then there exists a* $T > 0$ *such that* (4.0.1) *has a unique random field solution up to time* T.

Proof. We begin by defining

$$\sigma_N(x,\, h) = \begin{cases} \sigma(x,\, h) & \text{if} \quad x \leq N \\ \sigma(N,\, h) & \text{if} \quad x > N. \end{cases}$$

$\sigma_N(x,\, h)$ therefore satisfies (4.0.2) but with a different constant. Therefore by the proof of Theorem 4.1.1, there exists a unique solution $\{u_N(t,\, x)\}_{0\leq t\leq T,\, x\in\mathbf{R}^d}$ satisfying

$$\sup_{0<t<T} \sup_{x\in\mathbf{R}^d} \mathrm{E}|u_N(t,\, x)| < \infty.$$

By Proposition A.1.8, for a fixed $x \in \mathbf{R}$, $\mathrm{E}|u_N(t,\, x, .)|$ is continuous in t. Since $\mathrm{E}|u(0, x)|$ is finite, there exists $T > 0$ such that for all $t < T$, $E|u_N(t,\, x, .)|$ is finite as well. We have therefore established short-time existence of the solution. □

Here follows Jensen's inequality which shall be used in the proof of the blow-up result.

Lemma 4.3.4. *Given that* $p(\mathrm{d}x)$ *is a probability measure on* $\mathbf{R}^d$ *and suppose that the function* u *is non-negative. Then for all convex function* f *the following holds,*

$$\int_{\mathbf{R}^d} f(u(x))p(\mathrm{d}x) \geq f\bigg(\int_{\mathbf{R}^d} u(x)p(\mathrm{d}x)\bigg).$$

We now prove the finite time blow up for the non-compensated noise equation.

Proof of Theorem 4.3.2. We begin with the first part of the theorem. Now write

$$\mathrm{E}|u(t,x)| = |(P_t u_0)(x)| + \lambda \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |p(t-s,x-y)|\mathrm{E}|\sigma(u(s,y),h)|\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s.$$

We use the assumption that $u_0(x) > c_1$ for some positive constant c_1 and condition 4.3.1 to come up with

$$\begin{aligned} \mathrm{E}|u(t,x)| &\geq c_1 + \kappa\lambda L_\sigma \int_0^t \int_{\mathbf{R}^d} p(t-s,x-y)\mathrm{E}|u(s,y)|^\gamma \mathrm{d}y\mathrm{d}s \\ &\geq c_1 + \kappa\lambda L_\sigma \int_0^t \big(\inf_{y\in\mathbf{R}^d} \mathrm{E}|u(s,y)|\big)^\gamma \mathrm{d}s. \end{aligned}$$

Upon setting

$$F(t) = \inf_{x\in\mathbf{R}^d} \mathrm{E}|u(t,x)|,$$

the above inequality reduces to

$$F(t) \geq c_1 + \kappa\lambda L_\sigma \int_0^t F^\gamma(s)\mathrm{d}s,$$

which fails to converge in a finite time for all $\gamma > 1$.

Next, we consider when the initial function u_0 is positive but not necessarily bounded below and $\mathcal{L} = -(-\Delta)^{\alpha/2}$. Let $t_0 > 0$ be fixed, then

$$\begin{aligned} u(t+t_0,x) &= \int_{\mathbf{R}^d} p(t+t_0,x-y)u(0,y)\mathrm{d}y \\ &+ \lambda \int_0^{t+t_0} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p(t+t_0-s,x-y)\sigma(u(s,y),h)N(\mathrm{d}h,\, \mathrm{d}y,\, \mathrm{d}s). \end{aligned}$$

Taking first moment of both sides, then by Proposition 2.3.6 we have

$$\begin{aligned} \mathrm{E}|u(t+t_0,x)| &\geq \int_{\mathbf{R}^d} p(t+t_0,x-y)u(0,y)\mathrm{d}y \\ &+ \lambda \int_0^{t+t_0} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p(t+t_0-s,x-y)\mathrm{E}|\sigma(u(s,y),h)|\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s \\ &\geq c(t_0)\, p(t+\eta,x) \\ &+ \lambda L_\sigma \kappa \int_{t_0}^{t+t_0} \int_{\mathbf{R}^d} p(t+t_0-s,x-y)\mathrm{E}|u(s,y)|^\gamma \mathrm{d}y\mathrm{d}s. \end{aligned}$$

We perform a change of variable, by letting $\tau = s - t_0$, $\mathrm{d}\tau = \mathrm{d}s$ and therefore

$$\begin{aligned}\mathrm{E}|u(t+t_0,x)| &\geq c(t_0)p(t+\eta,x) \\ &+ \lambda L_\sigma \kappa \int_0^t \int_{\mathbf{R}^d} p(t-\tau,x-y)\mathrm{E}|u(\tau+t_0,y)|^\gamma \mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

Let $v(t,x) := u(t+t_0,x)$; then to show that $\mathrm{E}|u(t+t_0,x)|$ fails to exist in some finite time suffices to show the same for $\mathrm{E}|v(t,x)|$ and therefore

$$\mathrm{E}|v(t,x)| \geq c(t_0)\, p(t+\eta,x) + \lambda L_\sigma \kappa \int_0^t \int_{\mathbf{R}^d} p(t-\tau,x-y)(\mathrm{E}|v(\tau,y)|)^\gamma \mathrm{d}y\mathrm{d}\tau.$$

Multiply through by $p(t,x)$, and integrate in $[\mathrm{d}x]$, we obtain

$$\begin{aligned}&\int_{\mathbf{R}^d} \mathrm{E}|v(t,x)|p(t,x)\mathrm{d}x \\ &\geq c(t_0)\int_{\mathbf{R}^d} p(t+\eta,x)p(t,x)\mathrm{d}x + \lambda L_\sigma\kappa \int_0^t \mathrm{d}\tau \int_{\mathbf{R}^d} \mathrm{d}y(\mathrm{E}|v(\tau,y)|)^\gamma \int_{\mathbf{R}^d} p(t-\tau,x-y)p(t,x)\mathrm{d}x \\ &= c(t_0)\, p(2t+\eta,0) + \lambda L_\sigma\kappa \int_0^t \mathrm{d}\tau \int_{\mathbf{R}^d} \mathrm{d}y(\mathrm{E}|v(\tau,y)|)^\gamma p(2t-\tau,y).\end{aligned}$$

The last line follows by Kolmogorov property. Moreover, by properties of $p(t,x)$, Jensen's inequality and Lemma 4.3.4, we get

$$\begin{aligned}&\int_{\mathbf{R}^d} \mathrm{E}|v(t,x)|p(t,x)\mathrm{d}x \\ &\qquad\geq c(t_0)\, p(1,0)(2t+\eta)^{-\frac{d}{\alpha}} + \kappa L_\sigma\kappa \int_0^t \mathrm{d}\tau \int_{\mathbf{R}^d} \mathrm{d}y \left(\frac{\tau}{2t-\tau}\right)^{\frac{d}{\alpha}} p(\tau,y)(\mathrm{E}|v(\tau,y)|)^\gamma \\ &\qquad\geq c_0(2t+\eta)^{-\frac{d}{\alpha}} + \lambda L_\sigma\kappa \int_0^t \mathrm{d}\tau \left(\frac{\tau}{2t-\tau}\right)^{\frac{d}{\alpha}} \left(\int_{\mathbf{R}^d} (\mathrm{E}|v(\tau,y)|p(\tau,y)\mathrm{d}y\right)^\gamma.\end{aligned}$$

Set $F(t) = \int_{\mathbf{R}^d} \mathrm{E}|v(t,x)|p(t,x)\mathrm{d}x$ then

$$\begin{aligned}F(t) &\geq c_0(2t+\eta)^{-\frac{d}{\alpha}} + \lambda L_\sigma\kappa \int_0^t \left(\frac{\tau}{2t-\tau}\right)^{\frac{d}{\alpha}} F(\tau)^\gamma \mathrm{d}\tau \\ &\geq c_0(2t+\eta)^{-\frac{d}{\alpha}} + \lambda L_\sigma\kappa \int_0^t \left(\frac{\tau}{2t}\right)^{\frac{d}{\alpha}} F(\tau)^\gamma \mathrm{d}\tau.\end{aligned}$$

Multiply through by $t^{\frac{d}{\alpha}}$, therefore

$$F(t)t^{\frac{d}{\alpha}} \geq c_0\left(\frac{t}{2t+\eta}\right)^{d/\alpha} + \frac{\lambda L_\sigma \kappa}{2^{d/\alpha}} \int_0^t \frac{\left(\tau^{\frac{d}{\alpha}} F(\tau)\right)^\gamma}{\tau^{\frac{d(\gamma-1)}{\alpha}}} \mathrm{d}\tau.$$

Let $Y(t) = F(t)t^{\frac{d}{\alpha}}$ and assume further that $t \geq \delta$ for all $\delta \geq 0$, then

$$Y(t) \geq c_0\big(\frac{\delta}{2\delta+\eta}\big)^{d/\alpha} + \frac{\lambda L_\sigma \kappa}{2^{d/\alpha}} \int_\delta^t \frac{Y(\tau)^\gamma}{\tau^{\frac{d(\gamma-1)}{\alpha}}} \mathrm{d}\tau \geq \frac{\lambda L_\sigma \kappa}{2^{d/\alpha}} \int_\delta^t \frac{Y(\tau)^\gamma}{\tau^{\frac{d(\gamma-1)}{\alpha}}} \mathrm{d}\tau.$$

and the result follows by Proposition 3.3.4. □

CHAPTER 5

ON SOME PROPERTIES OF A CLASS OF FRACTIONAL STOCHASTIC HEAT EQUATIONS

We consider the following stochastic heat equation with Gaussian white noise

$$\frac{\partial}{\partial t}u(t,\, x) = \mathcal{L}u(t,\, x) + \lambda\sigma(u(t,\, x))\dot{w}(t,\, x), \;\; x \in \mathbf{R}, \text{ and } t > 0 \tag{5.0.1}$$

with $u(0, x) = u_0(x)$, for all $x \in \mathbf{R}$. The function $u_0 : \mathbf{R} \to \mathbf{R}_+$ is a non-random function that is positive on a set of positive measure, the function $\sigma : \mathbf{R} \to \mathbf{R}$ is a Lipschitz continuous function and $\dot{w}(t,\, x)$ denotes white noise on $(0, \infty) \times \mathbf{R}$. We need some explicit heat kernel estimates, so we restrict our attention to the case when $\mathcal{L}$ is the generator of a symmetric stable process, that is, $\mathcal{L} := -(-\Delta)^{\alpha/2}$ with $\alpha \in (1, 2)$. For a stochastic heat equation driven by a coloured noise on $\mathbf{R}^d$,

$$\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x) + \lambda\sigma(u(t,x))\dot{F}(t,x), \tag{5.0.2}$$

with the initial condition $u(0,x) = u_0(x)$, $x \in \mathbf{R}^d$. The parameter σ satisfies Lipschitz assumption and $\lambda > 0$ is the level of the noise. The term $\dot{F}$ is a spatially-coloured, temporally white, Gaussian noise; a generalised Gaussian random field whose covariance kernel is described as follows

$$\mathrm{E}[\dot{F}(t,x)\dot{F}(s,y)] = \delta_0(t-s)f_\beta(x,y)$$

where the correlation function f_β is the Riesz kernel given by

$$f_\beta(x,y) = \frac{1}{|x-y|^\beta},$$

with $\beta \in (0,d)$, $d \geq 1$ the dimension. We start with the following important estimate:

Lemma 5.0.5. *For any $t > 0$ and all $x, y \in \mathbf{R}^d$, there exists some positive constant c_1 such that*

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} p(t,x,y)p(t,y,z)f_\beta(y,z)\mathrm{d}y\mathrm{d}z \leq c_1 t^{-\beta/\alpha}.$$

Proof. By semigroup identity and Lemma 2.3.4, we write

$$\begin{aligned}
&\int_{\mathbf{R}^d \times \mathbf{R}^d} p(t,x,w)p(t,y,z)f_\beta(w,z)\mathrm{d}w\mathrm{d}z \\
&= \int_{\mathbf{R}^d} p(2t,w,x-y)f_\beta(w,0)\mathrm{d}w \\
&\leq C\int_{\mathbf{R}^d}\left((2t)^{-d/\alpha} \wedge \frac{2t}{|w-(x-y)|^{d+\alpha}}\right)|w|^{-\beta}\mathrm{d}w \\
&= C\int_{\mathbf{R}^d}\big((2t)^{-d/\alpha} \wedge \frac{2t}{|v|^{d+\alpha}}\big)|v+(x-y)|^{-\beta}\mathrm{d}v \\
&= C(2t)^{-d/\alpha}\int_{|v|\leq (2t)^{d/\alpha}} |v+(x-y)|^{-\beta}\mathrm{d}v + 2tC\int_{|v|>(2t)^{d/\alpha}} \frac{|v+(x-y)|^{-\beta}}{|v|^{d+\alpha}}\mathrm{d}v.
\end{aligned}$$

The lemma follows from evaluating the two integrals. □

We now present some results on the following renewal inequalities from ([55], chapter 7) and its converse. Here, we desire bounds on the functions involved rather than finding their asymptotic properties.

Proposition 5.0.6. *Let $\rho > 0$ and suppose $f(t)$ is a non-negative and locally integrable function satisfying*

$$f(t) \leq c_1 + \kappa\int_0^t (t-s)^{\rho-1}f(s)\mathrm{d}s \ \textit{for all}\ t > 0,$$

where c_1 is some positive number. Then we have

$$f(t) \leq c_2 \exp\left(c_3(\Gamma(\rho))^{1/\rho}\kappa^{1/\rho}t\right) \ \textit{for all}\ t > 0,$$

for some positive constants c_2 and c_3.

Proof. Let $(\mathcal{A}\psi)(t) = \kappa \int_0^t (t-s)^{\rho-1}\psi(s)\mathrm{d}s$ for locally integrable function ψ. Then $f(t) \leq c_1 + (\mathcal{A}f)(t)$. Let $k \geq 1$ be a fixed integer, then $(\mathcal{A}^k f)(t) := \kappa \int_0^t (t-s)^{\rho-1}(\mathcal{A}^{k-1}f)(s)\mathrm{d}s$ and set $1(s) := 1$ for all $0 \leq s \leq T$. Then it follows by iteration that

$$f(t) \leq c_1 \sum_{k=0}^{n-1} (\mathcal{A}^k 1)(t) + (\mathcal{A}^n f)(t)$$

and by induction,

$$(\mathcal{A}^n f)(t) = \frac{(\kappa\Gamma(\rho))^n}{\Gamma(n\rho)} \int_0^t (t-s)^{n\rho-1} f(s)\mathrm{d}s,$$

and consequently, we have

$$(\mathcal{A}^n 1)(t) = \frac{(\kappa\Gamma(\rho))^n t^{n\rho}}{\Gamma(n\rho+1)}.$$

As $n \to \infty$, we have that $(\mathcal{A}^n f)(t) \to 0$ and therefore,

$$f(t) \leq c_1 \sum_{k=0}^{\infty} (\mathcal{A}^k 1)(t) = c_1 \sum_{k=0}^{\infty} \frac{(\kappa\Gamma(\rho)t^\rho)^k}{\Gamma(k\rho+1)}.$$

The result follows by applying Lemma 2.3.8 with $b = \kappa\Gamma(\rho)t^\rho$. □

The converse of the above Proposition is given as follows.

Proposition 5.0.7. *Let $\rho > 0$ and suppose $f(t)$ is non-negative, locally integrable function satisfying*

$$f(t) \geq c_1 + \kappa \int_0^t (t-s)^{\rho-1} f(s)\mathrm{d}s \;\; \text{for all } t > 0,$$

where c_1 is some positive number. Then we have

$$f(t) \geq c_2 \exp\left(c_3 (\Gamma(\rho))^{1/\rho} \kappa^{1/\rho} t\right) \;\; \text{for all } t > \frac{\mathrm{e}}{\rho}(\Gamma(\rho)\kappa)^{-1/\rho},$$

for some positive constants c_2 and c_3.

Proof. Following same lines of proof as above Proposition 5.0.6, we have that

$$f(t) \geq c_1 \sum_{k=0}^{\infty} \frac{(\kappa\Gamma(\rho)t^\rho)^k}{\Gamma(k\rho+1)}.$$

Applying Lemma 2.3.7 with $b = \kappa\Gamma(\rho)t^\rho$ proves the result. □

5.1 White Noise Results

We start with some results on the white noise. Here follows our first main result.

Theorem 5.1.1. *Let $u(t,x)$ be the unique solution to equation* (5.0.1)*. Then there exists $T>0$, such that for all $t>T$,*

$$\inf_{x\in B(0,\,t^{1/\alpha})} \mathrm{E}|u(t,x)|^2 \geq c\exp\left(c'\lambda^{2\alpha/(\alpha-1)}t\right),$$

where $\tilde{c}$ and $\tilde{c}'$ are some positive constants. An immediate consequence of this gives

$$\liminf_{t\to\infty}\frac{1}{t}\log \mathrm{E}|u(t,x)|^2 \geq c'\lambda^{2\alpha/(\alpha-1)},$$

for any fixed $x\in\mathbf{R}$.

5.1.1 Proof of Theorem 5.1.1

To give a proof of the above theorem, we begin with the following proposition. Fix $t>0$ and set

$$g_t := \inf_{x\in B(0,\,t^{1/\alpha})} (P_tu_0)(x),$$

we then have the following.

Proposition 5.1.2. *Suppose that $\mathcal{L} := -(-\Delta)^{\alpha/2}$ and that condition 3.2.1 is in force. Then for any fixed $t>0$ and $x\in B(0,t^{1/\alpha})$,*

$$\mathrm{E}|u(t,x)|^2 \geq g_t^2\sum_{k=1}^{\infty}\left(\frac{c_3\lambda^2L_\sigma^2\alpha}{\alpha-1}\right)^k\left(\frac{t}{k}\right)^{k(\alpha-1)/\alpha}.$$

Proof. By Itô isometry, we obtain

$$\mathrm{E}|u(t,x)|^2 \geq |(P_tu_0)(x)|^2 + (\lambda L_\sigma)^2\int_0^t\int_{\mathbf{R}} p^2(t-s_1,x,y_1)\mathrm{E}|u(s_1,y_1)|^2\mathrm{d}y_1\mathrm{d}s_1.$$

Recursively,

$$\mathrm{E}|u(s_1,y_1)|^2 \geq |(P_{s_1}u_0)(y_1)|^2 + (\lambda L_\sigma)^2\int_0^{s_1}\int_{\mathbf{R}} p^2(s_1-s_2,y_1,y_2)\mathrm{E}|u(s_2,y_2)|^2\mathrm{d}y_2\mathrm{d}s_2.$$

Therefore iterating the integrals,

$$
\begin{aligned}
\mathrm{E}|u(t,x)|^2 &\geq |(P_t u_0)(x)|^2 + (\lambda L_\sigma)^2 \int_0^t \int_{\mathbf{R}} p^2(t-s_1,x,y_1)|(P_{s_1}u_0)(y_1)|^2 \mathrm{d}y_1 \mathrm{d}s_1 \\
&+ (\lambda L_\sigma)^4 \int_0^t \int_{\mathbf{R}} \int_0^{s_1} \int_{\mathbf{R}} p^2(t-s_1,x,y_1) p^2(s_1-s_2,y_1,y_2) \\
&\times \mathrm{E}|u(s_2,y_2)|^2 \mathrm{d}y_1 \mathrm{d}s_1 \mathrm{d}y_2 \mathrm{d}s_2.
\end{aligned}
$$

Setting $y_0 = x$ and $s_0 = t$ and continuing the recursion as above we obtain that

$$
\begin{aligned}
\mathrm{E}|u(t,x)|^2 &\geq |(P_t u_0)(x)|^2 \\
&+ \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_0^t \int_{\mathbf{R}} \int_0^{s_1} \int_{\mathbf{R}} \cdots \int_0^{s_{k-1}} \int_{\mathbf{R}} |(P_{s_k}u_0)(y_k)|^2 \\
&\times \prod_{i=1}^{k} p^2(s_{i-1}-s_i, y_{i-1}, y_i) \mathrm{d}y_{k+1-i} \mathrm{d}s_{k+1-i}.
\end{aligned}
$$

Set $B = B(0,\, t^{1/\alpha})$ then by reducing the spatial domain of integration we have

$$
|(P_{s_k}u_0)(y_k)|^2 \geq \inf_{y\in B} |(P_t u_0)(y)|^2 = g_t^2.
$$

Therefore,

$$
\begin{aligned}
\mathrm{E}|u(t,x)|^2 &\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_0^t \int_{\mathbf{R}} \int_0^{s_1} \int_{\mathbf{R}} \cdots \int_0^{s_{k-1}} \int_B \\
&\times \prod_{i=1}^{k} p^2(s_{i-1}-s_i, y_{i-1}, y_i) \mathrm{d}y_{k+1-i} \mathrm{d}s_{k+1-i} \\
&\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_{t-t/k}^t \int_{\mathbf{R}} \int_{s_1-t/k}^{s_1} \int_{\mathbf{R}} \cdots \int_{s_{k-1}-t/k}^{s_{k-1}} \int_B \\
&\times \prod_{i=1}^{k} p^2(s_{i-1}-s_i, y_{i-1}, y_i) \mathrm{d}y_{k+1-i} \mathrm{d}s_{k+1-i}.
\end{aligned}
$$

Also, making a change of variable for $s_{i-1} - s_i$, for all $i = 1, 2, \ldots, k$, then

$$
\begin{aligned}
\mathrm{E}|u(t,x)|^2 &\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_0^{t/k} \int_{\mathbf{R}} \int_0^{t/k} \int_{\mathbf{R}} \cdots \int_0^{t/k} \int_B \\
&\times \prod_{i=1}^{k} p^2(s_i, y_{i-1}, y_i) \mathrm{d}y_{k+1-i} \mathrm{d}s_{k+1-i}.
\end{aligned}
$$

We further restrict the domain of integration by choosing y_i for each $i = 1, 2, \ldots, k$, such that

$$y_i \in B(0,\, t^{1/\alpha}) \cap B(y_{i-1}, s_i^{1/\alpha}),$$

and denote $\mathcal{A}_i := \{B(0,\, t^{1/\alpha}) \cap B(y_{i-1},\, s_i^{1/\alpha})\}$ in order to get the required bound on the kernels. Since $|y_{i-1} - y_i| \leq s_i^{1/\alpha}$, it follows that $p(s_i,\, y_{i-1},\, y_i) \geq c_1 s_i^{-1/\alpha}$. Thus

$$\mathrm{E}|u(t,x)|^2 \geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_0^{t/k} \int_{\mathcal{A}_1} \int_0^{t/k} \int_{\mathcal{A}_2} \cdots \int_0^{t/k} \int_{\mathcal{A}_k} \prod_{i=1}^{k} s_i^{-2/\alpha} \mathrm{d}s_{k+1-i} \mathrm{d}y_{k+1-i}.$$

By the lower bounds on the area of $\mathcal{A}_i$, that's, $|\mathcal{A}_i| \geq c_2 s_i^{1/\alpha}$ we have

$$\mathrm{E}|u(t,x)|^2 \geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} c_3^k \int_0^{t/k} \int_0^{t/k} \cdots \int_0^{t/k} \prod_{i=1}^{k} s_i^{-1/\alpha} \mathrm{d}s_k \mathrm{d}s_{k-1} \ldots \mathrm{d}s_1.$$

Integrating term-wisely gives the following

$$\begin{aligned} \mathrm{E}|u(t,x)|^2 &\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma \sqrt{c_3})^{2k} \left(\frac{\alpha}{\alpha - 1}\right)^k \left(\frac{t}{k}\right)^{k(\alpha-1)/\alpha} \\ &= g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma \sqrt{c_3}\, c_4)^{2k} \left(\frac{t}{k}\right)^{k(\alpha-1)/\alpha}, \end{aligned}$$

with c_4 given by $c_4 = \sqrt{\frac{\alpha}{(\alpha-1)}}$. Setting

$$F_t(\lambda) = \inf_{x \in B} \mathrm{E}|u(t,x)|^2,$$

therefore

$$F_t(\lambda) \geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma \sqrt{c_3}\, c_4)^{2k} \left(\frac{t}{k}\right)^{k(\alpha-1)/\alpha},$$

and the result follows immediately. □

Proof of Theorem 5.1.1. Applying Lemma 2.3.7 with Proposition 5.1.2 gives the first statement of the theorem. For the second part of the theorem, we fix $x \in \mathbf{R}$. Clearly we have that $x \in B(0,\, 2|x|)$ and by the first part of the theorem, we have that for $t^{1/\alpha} \geq 2|x| \vee T$,

$$\mathrm{E}|u(t,x)|^2 \geq c \exp\left(c' \lambda^{2\alpha/(\alpha-1)} t\right).$$

By taking appropriate limit, we obtain the second part of the theorem. □

We now prove an upper bound result for the white noise equation. Let $t > 0$ be fixed, define

$$g_\lambda(t) = \sup_{x\in\mathbf{R}} \mathrm{E}|u(t,x)|^2.$$

Proposition 5.1.3. *For $t \geq 0$ fixed, there exist some positive constants c_2 and c_3 such that*

$$g_\lambda(t) \leq c_3 + (\lambda \mathrm{Lip}_\sigma)^2 c_2 \int_0^t (t-s)^{-1/\alpha} g_\lambda(s)\mathrm{d}s.$$

Proof. By the assumption that $u_0(x) \leq c_1$, then it follows by taking second moment and by semigroup identity, that

$$\mathrm{E}|u(t,x)|^2 \leq \left| c_1 \int_{\mathbf{R}} p(t,x,y)\mathrm{d}y \right|^2 + \lambda^2 \mathrm{Lip}_\sigma^2 \int_0^t \int_{\mathbf{R}} p^2(t-s,x,y)\mathrm{E}|u(s,y)|^2 \mathrm{d}y\mathrm{d}s.$$

Thus,

$$\begin{aligned} g_\lambda(t) &\leq c_1^2 + \lambda^2 \mathrm{Lip}_\sigma^2 \int_0^t p(2(t-s),0) g_\lambda(s)\mathrm{d}s \\ &= c_1^2 + \lambda^2 \mathrm{Lip}_\sigma^2 c_2 \int_0^t (t-s)^{-1/\alpha} g_\lambda(s)\mathrm{d}s \\ &= c_1^2 + \lambda^2 \mathrm{Lip}_\sigma c_2 \int_0^t (t-s)^{\frac{\alpha-1}{\alpha}-1} g_\lambda(s)\mathrm{d}s. \end{aligned}$$

□

5.2 Coloured Noise Results

We now give a corresponding relationship between the "dissipative" effect of the fractional Laplacian and the "growth" induced by the coloured noise term.

Proposition 5.2.1. *For all $t > 0$ fixed and $x \in B(0, t^{1/\alpha})$, then*

$$\mathrm{E}|u(t,x)|^2 \geq g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma c_1\, c_2)^{2k} \left(\frac{t}{k}\right)^{k(\alpha-\beta)/\alpha},$$

with $c_2 := \left(\sqrt{\frac{\alpha}{2^{\alpha(k-1)}k(\alpha-\beta)}}\right)^{1/k}$, c_1 some positive constants.

Proof. By taking second moment of the mild solution,

$$\begin{aligned}\mathrm{E}|u(t,x)|^2 &\geq |(P_t u_0)(x)|^2 \\ &+ \lambda^2 L_\sigma^2 \int_0^t \int_{\mathbf{R}^d\times\mathbf{R}^d} p(t-s_1,x,y_1)p(t-s_1,x,z_1)f_\beta(y_1,z_1) \\ &\times \mathrm{E}|u(s_1,y_1)u(s_1,z_1)|\mathrm{d}y_1\mathrm{d}z_1\mathrm{d}s_1.\end{aligned}$$

Recursively,

$$\begin{aligned}\mathrm{E}|u(s_1,y_1)u(s_1,z_1)| &\geq (P_{s_1}u_0)(y_1)(P_{s_1}u_0)(z_1) \\ &+ (\lambda L_\sigma)^2 \int_0^{s_1} \int_{\mathbf{R}^d\times\mathbf{R}^d} p(s_1-s_2,y_1,y_2)p(s_1-s_2,z_1,z_2)f_\beta(y_2,z_2) \\ &\times \mathrm{E}|u(s_2,y_2)u(s_2,z_2)|\mathrm{d}y_2\mathrm{d}z_2\mathrm{d}s_2.\end{aligned}$$

We follow the same steps from Proposition 5.1.2 to write

$$\begin{aligned}\mathrm{E}|u(t,x)|^2 &\geq |(P_t u_0)(x)|^2 \\ &+ \sum_{k=1}^{\infty}(\lambda L_\sigma)^{2k} \int_0^t \int_{\mathbf{R}^d\times\mathbf{R}^d}\int_0^{s_1} \int_{\mathbf{R}^d\times\mathbf{R}^d}\cdots\int_0^{s_{k-1}} \int_{\mathbf{R}^d\times\mathbf{R}^d} (P_{s_k}u_0)(y_k)(P_{s_k}u_0)(z_k) \\ &\times \prod_{i=1}^{k} p(s_{i-1}-s_i,y_{i-1},y_i)p(s_{i-1}-s_i,z_{i-1},z_i)f_\beta(y_i,z_i)\mathrm{d}y_{k+1-i}\mathrm{d}z_{k+1-i}\mathrm{d}s_{k+1-i}.\end{aligned}$$

Set $B = B(0,\, t^{1/\alpha})$ then by reducing the temporal domain of the integration,

$$(P_{s_k}u_0)(y_k)(P_{s_k}u_0)(z_k) \geq \inf_{y,z\in B}(P_t u_0)(y)(P_t u_0)(z) = g_t^2.$$

Therefore, after few lines of computations, (see Proposition 5.1.2) we have

$$\begin{aligned}\mathrm{E}|u(t,x)|^2 &\geq g_t^2 + g_t^2\sum_{k=1}^{\infty}(\lambda L_\sigma)^{2k} \int_0^{t/k} \int_{\mathbf{R}^d\times\mathbf{R}^d}\int_0^{t/k} \int_{\mathbf{R}^d\times\mathbf{R}^d}\cdots\int_0^{t/k} \int_{B\times B} \\ &\times \prod_{i=1}^{k} p(s_i,y_{i-1},y_i)p(s_i,z_{i-1},z_i)f_\beta(y_i,z_i)\mathrm{d}y_{k+1-i}\mathrm{d}z_{k+1-i}\mathrm{d}s_{k+1-i}.\end{aligned}$$

We further restrict the temporal domain, so we can get the required bounds on $p(s_i, y_{i-1}, y_i)$ and $p(s_i, z_{i-1}, z_i)$ by making the following definitions. Given that $x \in B(0, t^{1/\alpha})$ then choose y_i, z_i such that

$$\mathcal{A}_i := \{y_i \in B(x, s_1^{1/\alpha}/2) \cap B(y_{i-1}, s_i^{1/\alpha})\},$$

$$\mathcal{B}_i := \{z_i \in B(x, s_1^{1/\alpha}/2) \cap B(z_{i-1}, s_i^{1/\alpha})\},$$

where y_i, z_i both lie inside $B(0, t^{1/\alpha})$ for each $i = 1, 2, \ldots, k$. We make the following observation about the area of the restricted domain, thus,

$$\begin{aligned} |B(x, s_1^{1/\alpha}/2) \cap B(y_{i-1}, s_i^{1/\alpha})| &\geq c_1 \, s_i^{d/\alpha} \\ |B(x, s_1^{1/\alpha}/2) \cap B(z_{i-1}, s_i^{1/\alpha})| &\geq c_1 \, s_i^{d/\alpha} \end{aligned}$$

for all $c_1 > 0$ independent of i and

$$|y_i - z_i| \leq |y_i - x| + |x - z_i| \leq \frac{s_1^{1/\alpha}}{2} + \frac{s_1^{1/\alpha}}{2} = s_1^{1/\alpha},$$

which implies that $f_\beta(y_i,\, z_i) \geq s_1^{-\beta/\alpha}$. We also have $|y_i - y_{i-1}| \leq s_i^{1/\alpha}$ and $|z_i - z_{i-1}| \leq s_i^{1/\alpha}$ which imply that $p(s_i,\, y_i, y_{i-1}) \geq c_1 s_i^{-d/\alpha}$ and $p(s_i,\, z_i, z_{i-1}) \geq c_1 s_i^{-d/\alpha}$. Thus,

$$\begin{aligned} \mathrm{E}|u(t,x)|^2 &\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_0^{t/k} \int_{\mathcal{A}_1} \int_{\mathcal{B}_1} \int_0^{t/k} \int_{\mathcal{A}_2} \int_{\mathcal{B}_2} \cdots \int_0^{t/k} \int_{\mathcal{A}_k} \int_{\mathcal{B}_k} \\ &\times \prod_{i=1}^{k} p(s_i, y_{i-1}, y_i) p(s_i, z_{i-1}, z_i) f_\beta(y_i, z_i) \mathrm{d}y_{k+1-i} \mathrm{d}z_{k+1-i} \mathrm{d}s_{k+1-i} \\ &\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_0^{t/k} \int_{\mathcal{A}_1} \int_{\mathcal{B}_1} \int_0^{t/k} \int_{\mathcal{A}_2} \int_{\mathcal{B}_2} \cdots \int_0^{t/k} \int_{\mathcal{A}_k} \int_{\mathcal{B}_k} \\ &\times \prod_{i=1}^{k} s_i^{-2d/\alpha} s_1^{-\beta/\alpha} \mathrm{d}y_{k+1-i} \mathrm{d}z_{k+1-i} \mathrm{d}s_{k+1-i}. \end{aligned}$$

Applying the bounds on the areas of the bounded domain and the fact that $t/k \geq s_1$ for all k, we have that

$$\begin{aligned} \mathrm{E}|u(t,x)|^2 &\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma)^{2k} \int_0^{t/k} \int_0^{t/k} \cdots \int_0^{t/k} \prod_{i=1}^{k} c_1^2 s_1^{-\beta/\alpha} \mathrm{d}s_{k+1-i} \\ &= g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma c_1)^{2k} \int_0^{t/k} \int_0^{t/k} \cdots \int_0^{t/k} s_1^{-k\beta/\alpha} \mathrm{d}s_k \mathrm{d}s_{k-1} \ldots \mathrm{d}s_1 \\ &\geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma \, c_1)^{2k} \int_0^{t/k} s_1^{k-1} s_1^{-k\beta/\alpha} \mathrm{d}s_1 \\ &= g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma \, c_1)^{2k} \frac{\alpha}{k(\alpha - \beta)} \left(\frac{t}{k}\right)^{k(\alpha-\beta)/\alpha}. \end{aligned}$$

Setting

$$F_t(\lambda) = \inf_{x\in B(0,t^{1/\alpha})} \mathrm{E}|u(t,\,x)|^2$$

then

$$F_t(\lambda) \geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma\, c_1 c_2)^{2k} \left(\frac{t}{k}\right)^{k(\alpha-\beta)/\alpha}$$

with $c_2 := (\sqrt{\frac{\alpha}{k(\alpha-\beta)}})^{1/k}$. □

Next follows the upper bound estimate. Define

$$G_t(\lambda) = \sup_{x\in\mathbf{R}^d} \mathrm{E}|u(t,x)|^2.$$

Proposition 5.2.2. *For $t \geq 0$ fixed, there exist c_1 and c_2 such that*

$$G_t(\lambda) \leq c_1 + c_2(\lambda \mathrm{Lip}_\sigma)^2 \int_0^t \frac{G_s(\lambda)}{(t-s)^{\beta/\alpha}} \mathrm{d}s.$$

Proof. Taking second moment, semigroup identity and Hölder's inequality,

$$\begin{aligned}
\mathrm{E}|u(t,x)|^2 &\leq \left| c \int_{\mathbf{R}^d} p(t,x,y)\mathrm{d}y \right|^2 + \lambda^2 \int_0^t \int_{\mathbf{R}^d\times\mathbf{R}^d} \\
&\times\ p(t-s,x,y)p(t-s,x,z)\mathrm{E}|\sigma(u(s,y))\sigma(u(s,z))| f_\beta(x,y)\mathrm{d}y\mathrm{d}z\mathrm{d}s \\
&\leq\ c_1 + \lambda^2 \mathrm{Lip}_\sigma^2 \int_0^t \int_{\mathbf{R}^d\times\mathbf{R}^d} \\
&\times\ p(t-s,x,y)p(t-s,x,z)[\mathrm{E}|u(s,y)|^2]^{1/2}[\mathrm{E}|u(s,z)|^2]^{1/2} f_\beta(x,y)\mathrm{d}y\mathrm{d}z\mathrm{d}s.
\end{aligned}$$

Hence,

$$\begin{aligned}
G_t(\lambda) &\leq c_1 + (\lambda \mathrm{Lip}_\sigma)^2 \int_0^t G_s(\lambda) \int_{\mathbf{R}^d\times\mathbf{R}^d} p(t-s,x,y)p(t-s,x,z) f_\beta(x,y)\mathrm{d}y\mathrm{d}z\mathrm{d}s \\
&= c_1 + c_2(\lambda \mathrm{Lip}_\sigma)^2 \int_0^t (t-s)^{-\beta/\alpha} G_s(\lambda)\mathrm{d}s \\
&= c_1 + c_2(\lambda \mathrm{Lip}_\sigma)^2 \int_0^t (t-s)^{\frac{\alpha-\beta}{\alpha}-1} G_s(\lambda)\mathrm{d}s.
\end{aligned}$$

□

5.2.1 Proofs of Main Results

We now give proofs of main results of this chapter.

Theorem 5.2.3. *There exist constants c and c' such that*

$$\sup_{x\in\mathbf{R}^d} \mathrm{E}|u(t,x)|^2 \leq c\exp\left(c'\lambda^{2\alpha/(\alpha-\beta)}t\right) \text{ for all } t>0.$$

Also there exists $T>0$, such that for all $t>T$,

$$\inf_{x\in B(0,\,t^{1/\alpha})} \mathrm{E}|u(t,x)|^2 \geq \tilde{c}\exp\left(\tilde{c}'\lambda^{2\alpha/(\alpha-\beta)}t\right),$$

where $\tilde{c}$ and $\tilde{c}'$ are some positive constants. This immediately implies that for any fixed $x\in\mathbf{R}^d$,

$$\tilde{c}'\lambda^{2\alpha/(\alpha-\beta)} \leq \liminf_{t\to\infty}\frac{\log \mathrm{E}|u(t,x)|^2}{t} \leq \limsup_{t\to\infty}\frac{\log \mathrm{E}|u(t,x)|^2}{t} \leq \tilde{c}'\lambda^{2\alpha/(\alpha-\beta)}.$$

Proof. We first start with the upper bound. This readily follows as an immediate consequence of Propositions 5.2.2 and 5.0.6 with $\rho=(\alpha-\beta)/\alpha$ and $\kappa=c_2(\lambda\mathrm{Lip}_\sigma)^2$. For the lower bound, we make use of Proposition 2.3.5 which states that $g_t\geq c_1t^{-d/\alpha}$ for all $t>T$, where $T>0$. This together with Proposition 5.2.1 gives the following

$$\begin{aligned}\mathrm{E}|u(t,x)|^2 &\geq g_t^2+g_t^2\sum_{k=1}^{\infty}(\lambda L_\sigma c_1\,c_2)^{2k}\left(\frac{t}{k}\right)^{k(\alpha-\beta)/\alpha}\\ &\geq c_2t^{-2d/\alpha}\sum_{k=1}^{\infty}(\lambda L_\sigma c_1\,c_2)^{2k}\left(\frac{t}{k}\right)^{k(\alpha-\beta)/\alpha}.\end{aligned}$$

Therefore by taking T larger and applying Lemma 2.3.7 with $b=\left(\lambda L_\sigma c_1c_2\right)^2t^\rho$ and $\rho=(\alpha-\beta)/\alpha$, we have

$$\inf_{x\in B(0,\,t^{1/\alpha})} \mathrm{E}|u(t,x)|^2 \geq c_3\exp\left(c_4\lambda^{2\alpha/(\alpha-\beta)}t\right), \text{ for all } t>T$$

where $c_4:=\left(L_\sigma c_1c_2\right)^{2\alpha/(\alpha-\beta)}$ and c_3 are positive constants. Taking limit proves the immediate consequence. □

Next theorem gives the rate of growth of the mild solution with respect to the level of noise λ.

Theorem 5.2.4. *For any fixed $t > 0$ and $x \in \mathbf{R}^d$, we have*

$$\lim_{\lambda\to\infty} \frac{\log\log \mathrm{E}|u(t,x)|^2}{\log \lambda} = \frac{2\alpha}{\alpha-\beta}.$$

Proof. By the upper bound of Theorem 5.2.3, it immediately follows that

$$\limsup_{\lambda\to\infty} \frac{\log\log \mathrm{E}|u(t,x)|^2}{\log \lambda} \leq \frac{2\alpha}{\alpha-\beta}.$$

Next, we seek to prove the converse of the above inequality. Let $x \in \mathbf{R}^d$ be fixed. If $t > T$ large enough so that $x \in B(0,\, t^{1/\alpha})$ then by Proposition 5.2.1, we have

$$\mathrm{E}|u(t,x)|^2 \geq g_t^2 + g_t^2 \sum_{k=1}^{\infty} (\lambda L_\sigma c_1\, c_2)^{2k} \left(\frac{t}{k}\right)^{k(\alpha-\beta)/\alpha}.$$

Applying Lemma 2.3.7 with $b = \big(\lambda L_\sigma c_1 c_2\big)^2 t^\rho$ and $\rho = (\alpha-\beta)/\alpha$, we obtain

$$\mathrm{E}|u(t,x)|^2 \geq g_t^2 \exp\big(c_3 \lambda^{2\alpha/(\alpha-\beta)} t\big),$$

where $c_3 := \big(L_\sigma c_1 c_2\big)^{2\alpha/(\alpha-\beta)} > 0$. Take logarithm of both sides to obtain

$$\liminf_{\lambda\to\infty} \frac{\log\log \mathrm{E}|u(t,x)|^2}{\log \lambda} \geq \frac{2\alpha}{\alpha-\beta}.$$

Now if we take $x \notin B(0,\, t^{1/\alpha})$ and choose a constant $\kappa > 0$ such that $x \in B(0,\, \kappa t^{1/\alpha})$, then use the idea of Proposition 5.2.1 to end up with

$$\mathrm{E}|u(t,x)|^2 \geq g_{\kappa t}^2 \exp\big(c_3 \lambda^{2\alpha/(\alpha-\beta)} t\big),$$

and the result follows by same lemma. □

Define the energy of the solution $u(t,x)$ as follows,

$$\mathcal{E}_t(\lambda) := \sqrt{\int_{\mathbf{R}^d} \mathrm{E}|u(t,x)|^2 \mathrm{d}x},$$

when it does exist. Under suitable assumption on the initial condition, it can be shown to exist. For instance, when the initial condition is a bounded non-negative function which is compactly supported, it does exist. The excitation index of the solution is defined as follows

$$e(t) := \lim_{\lambda\to\infty} \frac{\log\log \mathcal{E}_t(\lambda)}{\log\lambda}.$$

Thus the following result.

Theorem 5.2.5. *The excitation index* $e(t)$ *of the solution to* (5.0.2) *is given by* $\frac{2\alpha}{\alpha-\beta}$.

Proof. We start by estimating the upper bound on $e(t)$. Taking second moment of the mild solution, one obtains

$$\begin{aligned}
\mathcal{E}_t(\lambda)^2 &\leq \int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} p(t,x,y)u_0(y)\mathrm{d}y \right|^2 \mathrm{d}x \\
&+ \lambda^2 \mathrm{Lip}_\sigma^2 \int_{\mathbf{R}^d}\int_0^t \int_{\mathbf{R}^d\times\mathbf{R}^d} p(t-s,x,y)p(t-s,x,z)f_\beta(y,z)\mathrm{E}[|u(s,y)u(s,z)|]\mathrm{d}y\mathrm{d}z\mathrm{d}s\mathrm{d}x \\
&= I_1 + I_2.
\end{aligned}$$

We seek for lower bounds on both integrals. By the assumption on u_0, we have that

$$\begin{aligned}
I_1 = \int_{\mathbf{R}^d} \left| \int_K p(t,x,y)u_0(y)\mathrm{d}y \right|^2 \mathrm{d}x& \\
&\leq \int_{\mathbf{R}^d} \left[\int_K p^2(t,x,y)\mathrm{d}y. \int_K u_0^2(y)\mathrm{d}y \right] \mathrm{d}x \\
&\leq \int_K \int_{\mathbf{R}^d} p^2(t,x,y)\mathrm{d}x\mathrm{d}y. \int_K u_0^2(y)\mathrm{d}y = \tilde{K}\int_K p(2t,y,y)\mathrm{d}y \leq c_1,
\end{aligned}$$

where $\tilde{K} = \int_K u_0^2(y)\mathrm{d}y$. Next, we find bound on I_2 as follows,

$$\begin{aligned}
I_2 &= \lambda^2\mathrm{Lip}_\sigma^2 \int_0^t \int_{\mathbf{R}^d\times\mathbf{R}^d} p(2(t-s),y,z)f_\beta(y,z)\mathrm{E}[|u(s,y)u(s,z)|]\mathrm{d}y\mathrm{d}z\mathrm{d}s \\
&\leq \lambda^2\mathrm{Lip}_\sigma^2 \int_0^t \sup_{x\in\mathbf{R}^d} \mathrm{E}|u(s,x)|^2 \int_{\mathbf{R}^d\times\mathbf{R}^d} p(2(t-s),y,z)f_\beta(y,z)\mathrm{d}y\mathrm{d}z\mathrm{d}s \\
&\leq c_2\lambda^2\mathrm{Lip}_\sigma^2 \int_0^t (t-s)^{-\beta/\alpha} \sup_{x\in\mathbf{R}^d} \mathrm{E}|u(s,x)|^2\mathrm{d}s.
\end{aligned}$$

Then by the upper bound of theorem 5.2.3,

$$I_2 \leq c_3\lambda^2\mathrm{Lip}_\sigma^2 \int_0^t (t-s)^{-\beta/\alpha} \exp\left(c'\lambda^{2\alpha/(\alpha-\beta)}s\right)\mathrm{d}s \leq c_4 \exp\left(c'\lambda^{2\alpha/(\alpha-\beta)}t\right).$$

Applying logarithm, we have therefore,

$$\limsup_{\lambda\to\infty}\frac{\log\log\mathcal{E}_t(\lambda)}{\log\lambda}\leq\frac{2\alpha}{\alpha-\beta}.$$

Next we seek for a lower bound on $e(t)$. Following same proof of Proposition 5.2.1, we obtain

$$\begin{aligned}\mathrm{E}|u(t,x)|^2 &\geq |(P_tu_0)(x)|^2\\ &+ \sum_{k=1}^{\infty}(\lambda L_\sigma)^{2k}\int_0^t\int_{\mathbf{R}^d\times\mathbf{R}^d}\int_0^{s_1}\int_{\mathbf{R}^d\times\mathbf{R}^d}\cdots\int_0^{s_{k-1}}\int_{\mathbf{R}^d\times\mathbf{R}^d}(P_{s_k}u_0)(y_k)(P_{s_k}u_0)(z_k)\\ &\times\prod_{i=1}^{k}p(s_{i-1}-s_i,y_{i-1},y_i)p(s_{i-1}-s_i,z_{i-1},z_i)f_\beta(y_i,z_i)\mathrm{d}y_{k+1-i}\mathrm{d}z_{k+1-i}\mathrm{d}s_{k+1-i}.\end{aligned}$$

Next, we integrate both sides in $[\mathrm{d}x]$ to obtain

$$\begin{aligned}\mathcal{E}_t(\lambda)^2 \geq &\int_{\mathbf{R}^d}\left|(P_tu_0)(x)\right|^2\mathrm{d}x\\ &+\int_{\mathbf{R}^d}\sum_{k=1}^{\infty}(\lambda L_\sigma)^{2k}\int_0^t\int_{\mathbf{R}^d\times\mathbf{R}^d}\int_0^{s_1}\int_{\mathbf{R}^d\times\mathbf{R}^d}\cdots\int_0^{s_{k-1}}\int_{\mathbf{R}^d\times\mathbf{R}^d}(P_{s_k}u_0)(y_k)(P_{s_k}u_0)(z_k)\\ &\times\prod_{i=1}^{k}p(s_{i-1}-s_i,y_{i-1},y_i)p(s_{i-1}-s_i,z_{i-1},z_i)f_\beta(y_i,z_i)\mathrm{d}y_{k+1-i}\mathrm{d}z_{k+1-i}\mathrm{d}s_{k+1-i}\mathrm{d}x.\end{aligned}$$

Reducing the spatial domain of the integration we obtain

$$\begin{aligned}\mathcal{E}_t(\lambda)^2 \geq &\int_{B(0,\,t^{1/\alpha})}\left|(P_tu_0)(x)\right|^2\mathrm{d}x\\ &+\int_{B(0,\,t^{1/\alpha})}|(P_tu_0)(x)|^2\sum_{k=1}^{\infty}(\lambda L_\sigma)^{2k}\int_0^t\int_{\mathbf{R}^d\times\mathbf{R}^d}\int_0^{s_1}\int_{\mathbf{R}^d\times\mathbf{R}^d}\cdots\int_0^{s_{k-1}}\int_{B(0,\,t^{1/\alpha})\times B(0,\,t^{1/\alpha})}\\ &\times\prod_{i=1}^{k}p(s_{i-1}-s_i,y_{i-1},y_i)p(s_{i-1}-s_i,z_{i-1},z_i)f_\beta(y_i,z_i)\mathrm{d}y_{k+1-i}\mathrm{d}z_{k+1-i}\mathrm{d}s_{k+1-i}\mathrm{d}x.\end{aligned}$$

By the same idea of proof of Proposition 5.2.1, we have

$$\mathcal{E}_t(\lambda)^2\geq c_3+c_3\sum_{k=1}^{\infty}(\lambda L_\sigma\,c_1c_2)^{2k}\left(\frac{t}{k}\right)^{k(\alpha-\beta)/\alpha}$$

with $c_2:=\Big(\sqrt{\frac{\alpha}{2^{\alpha(k-1)}k(\alpha-\beta)}}\Big)^{1/k}$, c_1 and c_3 some positive constants. This together with

Lemma 2.3.7 with $b = \big(\lambda L_\sigma c_1 c_2\big)^2 t^\rho$ and $\rho = (\alpha-\beta)/\alpha$, yield

$$\liminf_{\lambda\to\infty} \frac{\log\log \mathcal{E}_t(\lambda)}{\log\lambda} \geq \frac{2\alpha}{\alpha-\beta}.$$

The two inequalities prove the theorem. □

The next theorem gives a relationship between the excitation index of (5.0.2) and the continuity property of the solution.

Theorem 5.2.6. *Let $\eta < (\alpha-\beta)/2\alpha$. Then for every $x \in \mathbf{R}^d$, $\{u(t,x), t>0\}$ the solution to* (5.0.2) *has Hölder continuous trajectories with exponent η.*

The above shows that $\eta < \frac{1}{\mathrm{e}(t)}$, which showcases a link between noise excitability and continuity of the solution. Recall that

$$\hat{p}(t,\xi) := \mathrm{E}\mathrm{e}^{i\xi\cdot X_t} = \mathrm{e}^{-t|\xi|^\alpha}.$$

Proposition 5.2.7. *Let $q \in (0,\ \frac{\alpha-\beta}{2\alpha})$ and $r \in (0,\ 1)$, we then have*

$$\int_0^t \int_{\mathbf{R}^d} \big|\, \hat{p}(t-s+r,\xi) - \hat{p}(t-s,\xi) \,\big|^2 \frac{1}{|\xi|^{d-\beta}} \mathrm{d}\xi \mathrm{d}s \leq c_1 r^{2q},$$

for some positive constant c_1.

Proof. From the Fourier transform of the heat kernel,

$$\big|\, \hat{p}(t-s+r,\xi) - \hat{p}(t-s,\xi) \,\big|^2 = \mathrm{e}^{-2(t-s)|\xi|^\alpha}[\mathrm{e}^{-r|\xi|^\alpha} - 1]^2.$$

We now apply the following inequality that $1-\mathrm{e}^{-x} \leq x$ for all $x \geq 0$ and it implies that $|\mathrm{e}^{-x}-1| \leq x$. Therefore $|\mathrm{e}^{-r|\xi|^\alpha} - 1| \leq r|\xi|^\alpha$. Then from ([19], page 55), we have that for any $q \in (0,\ \frac{\alpha-\beta}{2\alpha}) \subset (0,1)$,

$$|\mathrm{e}^{-r|\xi|^\alpha} - 1| \leq 2^{1-q}|\mathrm{e}^{-r|\xi|^\alpha} - 1|^q \leq 2^{1-q} r^q |\xi|^{\alpha q}$$

Thus we observe that $\big|\,\mathrm{e}^{-r|\xi|^\alpha} - 1\,\big| \leq c\, r^q |\xi|^{\alpha q}$ to obtain the following bound

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} \big|\, \hat{p}(t-s+r,\xi) - \hat{p}(t-s,\xi) \,\big|^2 \frac{1}{|\xi|^{d-\beta}} \mathrm{d}\xi \mathrm{d}s \\
&\qquad = \int_0^t \int_{\mathbf{R}^d} \frac{\mathrm{e}^{-2(t-s)|\xi|^\alpha}[\mathrm{e}^{-r|\xi|^\alpha} - 1]^2}{|\xi|^{d-\beta}} \mathrm{d}\xi \mathrm{d}s \leq c\, r^{2q} \int_0^t \int_{\mathbf{R}^d} \frac{\mathrm{e}^{-2(t-s)|\xi|^\alpha}|\xi|^{2\alpha q}}{|\xi|^{d-\beta}} \mathrm{d}\xi \mathrm{d}s \\
&\qquad = c\, r^{2q} \bigg[\int_0^t \int_{|\xi|<1} \frac{\mathrm{e}^{-2(t-s)|\xi|^\alpha}|\xi|^{2\alpha q}}{|\xi|^{d-\beta}} \mathrm{d}\xi \mathrm{d}s + \int_0^t \int_{|\xi|\geq 1} \frac{\mathrm{e}^{-2(t-s)|\xi|^\alpha}|\xi|^{2\alpha q}}{|\xi|^{d-\beta}} \mathrm{d}\xi \mathrm{d}s \bigg].
\end{aligned}$$

The first integral appearing on the right hand side of the above is clearly bounded. So we do a bit more work for the second integral. Make the following change of variable $z := 2(t-s)|\xi|^\alpha$, we obtain the bound

$$\begin{aligned}\int_0^t \int_{|\xi|\geq 1} \frac{\mathrm{e}^{-2(t-s)|\xi|^\alpha}|\xi|^{2\alpha q}}{|\xi|^{d-\beta}}\mathrm{d}\xi\mathrm{d}s &= \frac{1}{2}\int_{|\xi|\geq 1} \frac{1}{|\xi|^{d+\alpha-\beta-2\alpha q}} \int_0^{2|\xi|^\alpha t} \mathrm{e}^{-z}\mathrm{d}z\mathrm{d}\xi \\ &\leq \int_{|\xi|\geq 1} \frac{1}{|\xi|^{d+\alpha-\beta-2\alpha q}}\mathrm{d}\xi.\end{aligned}$$

When $\alpha-\beta-2\alpha q > 0$ (since we assumed that $q < \frac{\alpha-\beta}{2\alpha}$), the above integral becomes finite. We now combine all the above estimates to obtain the desired result. □

Proof of Theorem 5.2.6. The proof makes use of Kolmogorov's continuity theorem. We will therefore look at the increment $\mathrm{E}|u(t+r,x)-u(t,x)|^p$ for $r \in (0,1)$ and $p \geq 2$. We have

$$\begin{aligned}u(t+r,x)-u(t,x) &= \int_{\mathbf{R}^d} [p(t+r,x,y)-p(t,x,y)]u_0(y)\mathrm{d}y \\ &+ \lambda \int_0^t \int_{\mathbf{R}^d} [p(t+r-s,x,y)-p(t-s,x,y)]\sigma(u(s,y))F(\mathrm{d}y,\,\mathrm{d}s) \\ &+ \lambda \int_t^{t+r} \int_{\mathbf{R}^d} p(t+r-s,x,y)\sigma(u(s,y))F(\mathrm{d}y,\,\mathrm{d}s).\end{aligned}$$

Since $(P_t u_0)(x)$ is in fact smooth for $t>0$, we will look for higher moments of the remaining terms. Recall that $\sup_{x\in\mathbf{R}^d} \mathrm{E}|u(t,x)|^p$ is finite for all $t>0$. We therefore use Burkholder's inequality together with Proposition 5.2.7 to write

$$\begin{aligned}&\mathrm{E}\left| \int_0^t \int_{\mathbf{R}^d} [p(t+r-s,x,y)-p(t-s,x,y)]\sigma(u(s,y))F(\mathrm{d}y,\,\mathrm{d}s)\right|^p \\ &\qquad \leq c_1 \mathrm{Lip}_\sigma^p \left| \int_0^t \int_{\mathbf{R}^d} \left|\hat{p}(t-s+r,x,\xi)-\hat{p}(t-s,x,\xi)\right|^2 \frac{1}{|\xi|^{d-\beta}}\mathrm{d}\xi\mathrm{d}s\right|^{p/2} \\ &\qquad \leq c_1 \mathrm{Lip}_\sigma^p r^{pq}.\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\mathrm{E}\left|\int_t^{t+r}\int_{\mathbf{R}^d} p(t+r-s,x,y)\sigma(u(s,y))F(\mathrm{d}y,\,\mathrm{d}s)\right|^p \\
&\qquad\le c_2\mathrm{Lip}_\sigma^p\left|\int_t^{t+r}\int_{\mathbf{R}^d}\left|\hat{p}(t-s+r,x,\xi)\right|^2\frac{1}{|\xi|^{d-\beta}}\mathrm{d}\xi\mathrm{d}s\right|^{p/2} \\
&\qquad= c_2\mathrm{Lip}_\sigma^p\left|\int_t^{t+r}\int_{\mathbf{R}^d}\mathrm{e}^{-2(t+r-s)|\xi|^\alpha}\frac{1}{|\xi|^{d-\beta}}\mathrm{d}\xi\mathrm{d}s\right|^{p/2} \\
&\qquad= c_2\mathrm{Lip}_\sigma^p\left|\frac{1}{2}\int_{\mathbf{R}^d}\frac{\mathrm{d}\xi}{|\xi|^{d-\beta+\alpha}}\int_0^{2r|\xi|^\alpha}\mathrm{e}^{-z}\mathrm{d}z\right|^{p/2} \\
&\qquad\le c_2\mathrm{Lip}_\sigma^p\left|\frac{1}{2}\int_{\mathbf{R}^d}\frac{1}{|\xi|^{d-\beta+\alpha}}|\mathrm{e}^{-2r|\xi|^\alpha}-1|\mathrm{d}\xi\right|^{p/2} \\
&\qquad\le c_3\mathrm{Lip}_\sigma^p\left|r^{2q}\int_{\mathbf{R}^d}\frac{1}{|\xi|^{d-\beta+\alpha-2\alpha q}}\mathrm{d}\xi\right|^{p/2}\le c_4\mathrm{Lip}_\sigma^p r^{pq},
\end{aligned}$$

with the assumption on q. Alternatively, we could use Lemma 5.0.5 to write

$$\begin{aligned}
&\mathrm{E}\left|\int_t^{t+r}\int_{\mathbf{R}^d} p(t+r-s,x,y)\sigma(u(s,y))F(\mathrm{d}y,\,\mathrm{d}s)\right|^p \\
&\le c_2\mathrm{Lip}_\sigma^p\left|\int_t^{t+r}\int_{\mathbf{R}^d\times\mathbf{R}^d} p(t-s+r,x,y)p(t-s+r,x,z)f(y,z)\mathrm{d}y\mathrm{d}z\mathrm{d}s\right|^{p/2} \\
&\le c_2\mathrm{Lip}_\sigma^p\left|\int_t^{t+r}(t-s)^{-\beta/\alpha}\mathrm{d}s\right|^{p/2}\le c_3 r^{(\alpha-\beta)p/2\alpha}.
\end{aligned}$$

Recall that $q<\frac{(\alpha-\beta)}{2\alpha}$, then combining the estimates above we see that

$$\mathrm{E}|u(t+r,x)-u(t,x)|^p\le c_5 r^{pq}=c_5 r^{1+(pq-1)}.$$

Now an application of Kolmogorov's continuity theorem with $0<\eta<\frac{pq-1}{p}\le q$ completes the proof. □

Conclusions. Existence and uniqueness of the solutions were given under some linear growth conditions on σ and also the bounds for the growth moments of the discontinuous integral solutions for both the compensated and non-compensated noise term.

For some non-linear growth conditions on σ, both the compensated and non-compensated equations fail to have any random field solutions for different conditions on the initial data $u_0(x)$.

A non-linear effect of noise on a class of stochastic heat equations was studied for a noisy case, that's, when λ is large enough. Therefore we established non-linear noise excitation growth indexes for both the Gaussian space-time white noise and Riesz spatial correlation noise for all time $t > 0$ and $\mathbf{R}^d$, and showed that as the level of the noise λ increases, the solution is bounded exponentially. This shows a non-linear long time effect of noise on the class of SPDEs. In what follows, the result establishes that the expected L^2-energy of the solution grows at an exponential index as the level of the noise increases. So far the above results were obtained with the initial condition being some functions which are positive on a set of positive measure. We can also extend the results with initial conditions which are more general. The only issue to achieve this extension is the existence and uniqueness of random field solution. The result of [29] can be used where this issue was settled whenever u_0 is any finite initial measure with white driven noise. Possible future work is to develop an appropriate Burholder's inequality in order to study the intermittent property of the compensated poisson noise equation.

APPENDIX A

APPENDIX

A.1 Continuity of the solution

We have the following a priori result about the continuity of the second moment of the solution to (3.0.1).

Proposition A.1.1. *Suppose that condition 3.0.9 holds, then for each* $x \in \mathbf{R}$*, the unique solution to* (3.0.1) *is mean square continuous in time. That is for each* $x \in \mathbf{R}$*, the function* $t \to \mathrm{E}[|u(t,\, x)|^2]$ *is continuous.*

The mild solution is given by

$$u(t,x) = \int_{\mathbf{R}} p(t,x,y)u(0,y)\mathrm{d}y + \lambda \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} p(t-s,x,y)\sigma(u(s,y),h)\tilde{N}(\mathrm{d}h,\, \mathrm{d}y,\, \mathrm{d}s).$$

We assume $0 < t_1 < t_2$, then for fixed $x \in \mathbf{R}$

$$\begin{aligned} u(t_2,x) - u(t_1,x) &= \int_{\mathbf{R}} [p(t_2,x,y) - p(t_1,x,y)]u(0,y)\mathrm{d}y \\ &+ \lambda \int_{\mathbf{R}} \int_{\mathbf{R}} \int_0^{t_1} [p(t_2-s,x,y) - p(t_1-s,x,y)]\sigma(u(s,y),h)\tilde{N}(\mathrm{d}h,\, \mathrm{d}y,\, \mathrm{d}s) \\ &+ \lambda \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{t_1}^{t_2} p(t_2-s,x,y)\sigma(u(s,y),h)\tilde{N}(\mathrm{d}h,\, \mathrm{d}y,\, \mathrm{d}s). \end{aligned}$$

We make the following definitions,

$$\begin{aligned}
D_0 &= \int_{\mathbf{R}} [p(t_2, x, y) - p(t_1, x, y)]u(0, y)\mathrm{d}y \\
D_1 &= \lambda \int_{\mathbf{R}} \int_{\mathbf{R}} \int_0^{t_1} [p(t_2 - s, x, y) - p(t_1 - s, x, y)]\sigma(u(s, y), h)\tilde{N}(\mathrm{d}h,\, \mathrm{d}y,\, \mathrm{d}s) \\
D_2 &= \lambda \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{t_1}^{t_2} p(t_2 - s, x, y)\sigma(u(s, y), h)\tilde{N}(\mathrm{d}h,\, \mathrm{d}y,\, \mathrm{d}s)
\end{aligned}$$

The proof of the above theorem will be a consequence of the following Lemma(s).

Lemma A.1.2. *For all* $\beta > 0$*,* $0 < t_1 < t_2$ *and* $x \in \mathbf{R}$*,*

$$|D_0|^2 \leq \frac{c_0}{2\pi} \int_{\mathbf{R}} \mathrm{e}^{-2t_1 \mathcal{R}\mathrm{e}\Psi(\xi)} |1 - \mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2 \mathrm{d}\xi.$$

Proof. We start by writing

$$\begin{aligned}
\mathrm{E}|D_0|^2 = |D_0|^2 &= |\int_{\mathbf{R}} [p(t_2, x, y) - p(t_1, x, y)]u(0, y)\mathrm{d}y|^2 \\
&\leq \int_{\mathbf{R}} |u(0, y)|^2 \mathrm{d}y \int_{\mathbf{R}} |p(t_2, x, y) - p(t_1, x, y)|^2 \mathrm{d}y \\
&\leq c_0 \int_{\mathbf{R}} |p(t_2, x, y) - p(t_1, x, y)|^2 \mathrm{d}y = c_0 \|p(t_2, .) - p(t_1, .)\|^2_{L^2(\mathbf{R})}.
\end{aligned}$$

By Plancherel's theorem

$$\|p(t_2, .) - p(t_1, .)\|^2_{L^2(\mathbf{R})} = \|\hat{p}(t_2, .) - \hat{p}(t_1, .)\|^2_{L^2(\mathbf{R})} = \frac{1}{2\pi} \int_{\mathbf{R}} \mathrm{e}^{-2t_1 \mathcal{R}\mathrm{e}\Psi(\xi)} |1 - \mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2 \mathrm{d}\xi.$$

Therefore,

$$\mathrm{E}|D_0|^2 \leq \frac{c_0}{2\pi} \int_{\mathbf{R}} \mathrm{e}^{-2t_1 \mathcal{R}\mathrm{e}\Psi(\xi)} |1 - \mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2 \mathrm{d}\xi.$$

□

Lemma A.1.3. *For all* $\beta > 0$*,* $0 < t_1 < t_2$ *and* $x \in \mathbf{R}$*,*

$$\mathrm{E}|D_1|^2 \leq \frac{\lambda^2 \mathrm{KLip}_\sigma^2}{2\pi} \|u\|^2_{2,\beta} \mathrm{e}^{\beta t_1} \int_{\mathbf{R}} \frac{|1 - \mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2}{\beta + 2\mathcal{R}\mathrm{e}\Psi(\xi)} \mathrm{d}\xi.$$

Proof. By Itô's isometry, we obtain

$$\begin{aligned}
\mathrm{E}|D_1|^2 &= \lambda^2 \int_{\mathbf{R}}\int_{\mathbf{R}}\int_0^{t_1} |p(t_2-s,x,y)-p(t_1-s,x,y)|^2 \mathrm{E}|\sigma(u(s,y),h)|^2 \nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s \\
&\le \lambda^2 \mathrm{KLip}_\sigma^2 \int_{\mathbf{R}}\int_0^{t_1} |p(t_2-s,x,y)-p(t_1-s,x,y)|^2 \mathrm{E}|u(s,y)|^2 \mathrm{d}y\mathrm{d}s \\
&\le \lambda^2 \mathrm{KLip}_\sigma^2 \|u\|_{2,\beta}^2 \int_0^{t_1} \mathrm{e}^{\beta s} \|\hat{p}(t_2-s,.)-\hat{p}(t_1-s,.)\|_{L^2(\mathbf{R})}^2 \mathrm{d}s.
\end{aligned}$$

But

$$\|\hat{p}(t_2-s,.)-\hat{p}(t_1-s,.)\|_{L^2(\mathbf{R})}^2 = \frac{1}{2\pi}\int_{\mathbf{R}} \mathrm{e}^{-2(t_1-s)\mathcal{R}\mathrm{e}\Psi(\xi)} |1-\mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2 \mathrm{d}\xi.$$

Therefore,

$$\begin{aligned}
\mathrm{E}|D_1|^2 &\le \frac{\lambda^2 \mathrm{KLip}_\sigma^2}{2\pi} \|u\|_{2,\beta}^2 \int_{\mathbf{R}} \mathrm{d}\xi \frac{|1-\mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2}{\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)} \left[1-\mathrm{e}^{-t_1\left(\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)\right)}\right] \\
&\le \frac{\lambda^2 \mathrm{KLip}_\sigma^2}{2\pi} \|u\|_{2,\beta}^2 \mathrm{e}^{\beta t_1} \int_{\mathbf{R}} \frac{|1-\mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2}{\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)} \mathrm{d}\xi.
\end{aligned}$$

□

Lemma A.1.4. *For all $\beta > 0$, $0 < t_1 < t_2$ and $x \in \mathbf{R}$,*

$$\mathrm{E}|D_2|^2 \le \frac{\lambda^2 \mathrm{KLip}_\sigma^2}{2\pi} \|u\|_{2,\beta}^2 \int_{\mathbf{R}} \frac{\mathrm{d}\xi}{\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)} .\mathrm{e}^{\beta t_2}\left[1-\mathrm{e}^{-(t_2-t_1)\left(\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)\right)}\right].$$

Proof. Take second moment of the solution

$$\begin{aligned}
\mathrm{E}|D_2|^2 &= \lambda^2 \int_{\mathbf{R}}\int_{\mathbf{R}}\int_{t_1}^{t_2} |p(t_2-s,x,y)|^2 \mathrm{E}|\sigma(u(s,y),h)|^2 \nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s \\
&\le \lambda^2 \mathrm{KLip}_\sigma^2 \|u\|_{2,\beta}^2 \int_{t_1}^{t_2} \mathrm{e}^{\beta s} \|\hat{p}(t_2-s,.)\|_{L^2(\mathbf{R})}^2 \mathrm{d}s \\
&\le \frac{\lambda^2 \mathrm{KLip}_\sigma^2}{2\pi} \|u\|_{2,\beta}^2 \int_{t_1}^{t_2} \mathrm{d}s \mathrm{e}^{\beta s} \int_{\mathbf{R}} \mathrm{e}^{-2(t_2-s)\mathcal{R}\mathrm{e}\Psi(\xi)} \mathrm{d}\xi \\
&= \frac{\lambda^2 \mathrm{KLip}_\sigma^2}{2\pi} \|u\|_{2,\beta}^2 \int_{\mathbf{R}} \frac{\mathrm{d}\xi}{\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)} .\mathrm{e}^{\beta t_2}\left[1-\mathrm{e}^{-(t_2-t_1)\left(\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)\right)}\right].
\end{aligned}$$

□

Proof of Theorem A.1.1. Combining Lemma A.1.2, A.1.3 and A.1.4, therefore

$$\begin{aligned}
\mathrm{E}|u(t_2,x)-u(t_1,x)|^2 &\leq \frac{C}{2\pi}\int_{\mathbf{R}} \mathrm{e}^{-2t_1\mathcal{R}\mathrm{e}\Psi(\xi)}|1-\mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2\mathrm{d}\xi \\
&+ \frac{\lambda^2\mathrm{KLip}_\sigma^2}{2\pi}\|u\|_{2,\beta}^2\mathrm{e}^{\beta t_1}\int_{\mathbf{R}}\frac{|1-\mathrm{e}^{-(t_2-t_1)\Psi(\xi)}|^2}{\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)}\mathrm{d}\xi \\
&+ \frac{\lambda^2\mathrm{KLip}_\sigma^2}{2\pi}\|u\|_{2,\beta}^2\int_{\mathbf{R}}\frac{\mathrm{d}\xi}{\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)}.\mathrm{e}^{\beta t_2}\left[1-\mathrm{e}^{-(t_2-t_1)\left(\beta+2\mathcal{R}\mathrm{e}\Psi(\xi)\right)}\right].
\end{aligned}$$

Then

$$\lim_{\delta\downarrow 0}\sup_{|t_1-t_2|<\delta}\mathrm{E}|u(t_2,x)-u(t_1,x)|^2\leq 0$$

and therefore

$$\lim_{t_1\uparrow t_2}\mathrm{E}|u(t_2,x)-u(t_1,x)|^2=0\,\text{for a fixed}\,x\in\mathbf{R}.$$

□

For the α-stable process, define

$$\mathcal{A}^\alpha u(t,x):=\lambda\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}p^\alpha(t-s,\,x,\,y)\sigma(u(s,y),h)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s).$$

Lemma A.1.5. *Suppose that u is predictable and $\|u\|_\beta<\infty$ for all $\beta>0$ and $\sigma(u,h)$ satisfies assumption* (3.0.9)*, then*

$$\|\mathcal{A}^\alpha u\|_{1,\beta}\leq \mathrm{C}_{d,\alpha,\beta}\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]$$

where $\mathrm{C}_{d,\alpha,\beta}:=\frac{2C(d,\alpha)}{d+\alpha-1}\frac{\Gamma(\gamma_1+2)}{\beta^{\gamma_1+2}}+2C(d,\alpha)\frac{\Gamma(\gamma_2+2)}{\beta^{\gamma_2+2}}$.

Proof. Taking first moment of the solution,

$$\begin{aligned}
\mathrm{E}|\mathcal{A}^\alpha u(t,x)| &= \lambda\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|p^\alpha(t-s,x,y)|\mathrm{E}|\sigma(u(s,y),h)|\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s \\
&\leq \lambda\mathrm{K}\int_0^t\int_{\mathbf{R}^d}|p^\alpha(t-s,x,y)|[1+\mathrm{Lip}_\sigma\mathrm{E}|u(s,y)|\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Multiply through by $\exp(-\beta t)$,

$$\begin{aligned} \mathrm{e}^{-\beta t}\mathrm{E}|\mathcal{A}^\alpha u(t,x)| &\leq \lambda\mathrm{K}\int_0^t\int_{\mathbf{R}^d}\mathrm{e}^{-\beta(t-s)}|p^\alpha(t-s,x,y)|\big\{\mathrm{e}^{-\beta s}[1+\mathrm{Lip}_\sigma\mathrm{E}|u(s,y)|]\big\}\mathrm{d}y\mathrm{d}s \\ &\leq \lambda\mathrm{K}\mathrm{Lip}_\sigma \sup_{s\geq 0}\sup_{y\in\mathbf{R}^d}\big\{\mathrm{e}^{-\beta s}[1+\mathrm{Lip}_\sigma\mathrm{E}|u(s,y)|]\big\} \\ &\times \int_0^t\int_{\mathbf{R}^d}\mathrm{e}^{-\beta(t-s)}|p^\alpha(t-s,\,x,\,y)|\mathrm{d}y\mathrm{d}s. \end{aligned}$$

Then,

$$\begin{aligned} \|\mathcal{A}^\alpha u\|_{1,\beta} &\leq \lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\sup_{t\geq 0}\int_0^t\int_{\mathbf{R}^d}\mathrm{e}^{-\beta(t-s)}|p^\alpha(t-s,\,x,\,y)|\mathrm{d}y\mathrm{d}s \\ &\leq \lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty\int_{\mathbf{R}^d}\mathrm{e}^{-\beta s}|p^\alpha(s,y)|\mathrm{d}y\mathrm{d}s \\ &\leq \lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty\int_{\mathbf{R}^d}\mathrm{e}^{-\beta s}\Big\{C\Big(\frac{s}{|y|^{d+\alpha}}\wedge s^{-\frac{d}{\alpha}}\Big)\Big\}\mathrm{d}y\mathrm{d}s. \end{aligned}$$

The last inequality follows by Lemma 2.3.4. Let's assume that $\frac{s}{|y|^{d+\alpha}}\leq s^{-\frac{d}{\alpha}}$ which holds only when $|y|^\alpha\geq s$. Therefore

$$\begin{aligned} \|\mathcal{A}^\alpha u\|_\beta &\leq C(d,\alpha)\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty\mathrm{d}s\mathrm{e}^{-\beta s}\Big\{s\int_{|y|\geq s^{-\alpha}}\frac{\mathrm{d}y}{|y|^{d+\alpha}} \\ &+ s^{-d/\alpha}\int_{|y|<s^{-\alpha}}\mathrm{d}y\Big\} \\ &= C(d,\alpha)\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty\mathrm{d}s\mathrm{e}^{-\beta s}\Big\{s\Big(-\int_{-\infty}^{s^{-\alpha}}y^{-(d+\alpha)}\mathrm{d}y \\ &+ \int_{s^{-\alpha}}^\infty y^{-(d+\alpha)}\mathrm{d}y\Big)+2s^{-(1-d)/\alpha}\Big\} \\ &= C(d,\alpha)\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty\mathrm{d}s\mathrm{e}^{-\beta s}\Big\{s\Big(-\frac{y^{-(d+\alpha-1)}}{1-d-\alpha}\,\Big|_{-\infty}^{s^{-\alpha}} \\ &+ \frac{y^{-(d+\alpha-1)}}{1-d-\alpha}\,\Big|_{s^{-\alpha}}^\infty\Big)+2s^{-(1-d)/\alpha}\Big\} \\ &= C(d,\alpha)\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty\mathrm{d}s\mathrm{e}^{-\beta s}\Big\{s\Big(-\frac{2}{1-d-\alpha}s^{\alpha(d+\alpha-1)}\Big) \\ &+ 2s^{-(1-d)/\alpha}\Big\} \\ &= C(d,\alpha)\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty\mathrm{d}s\mathrm{e}^{-\beta s}\Big\{\frac{2}{d+\alpha-1}s^{1+\alpha(d+\alpha-1)} \\ &+ 2s^{-(1-d)/\alpha}\Big\}. \end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{A}^\alpha u\|_{1,\beta} &\leq 2C(d,\alpha)\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\left\{\frac{1}{d+\alpha-1}\int_0^\infty s^{\gamma_1+1}\mathrm{e}^{-\beta s}\mathrm{d}s\right.\\ &+\left.\int_0^\infty s^{\gamma_2+1}\mathrm{e}^{-\beta s}\mathrm{d}s\right\}\end{aligned}$$

where $\gamma_1 := \alpha(d+\alpha-1)$ and $\gamma_2 := (\alpha-1-d)/\alpha$. Then we compute the integral $I_{\beta,\gamma} := \int_0^\infty s^{\gamma+1}\mathrm{e}^{-\beta s}\mathrm{d}s$. Let $\tau = \beta s,\ \mathrm{d}s = \frac{d\tau}{\beta}$, now therefore,

$$\begin{aligned}I_{\beta,\gamma} &= \frac{1}{\beta^{\gamma+2}}\int_0^\infty \tau^{\gamma+1}\mathrm{e}^{-\tau}\mathrm{d}\tau\\ &= \frac{1}{\beta^{\gamma+2}}\int_0^\infty \tau^{(\gamma+2)-1}\mathrm{e}^{-\tau}\mathrm{d}\tau = \frac{\Gamma(\gamma+2)}{\beta^{\gamma+2}}.\end{aligned}$$

Therefore

$$\|\mathcal{A}^\alpha u\|_{1,\beta} \leq 2C(d,\alpha)\lambda\mathrm{K}[1+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\left\{\frac{1}{d+\alpha-1}\frac{\Gamma(\gamma_1+2)}{\beta^{\gamma_1+2}}+\frac{\Gamma(\gamma_2+2)}{\beta^{\gamma_2+2}}\right\}.$$

□

Lemma A.1.6. *Suppose u and v are two predictable random field solutions satisfying $\|u\|_{1,\beta}+\|v\|_{1,\beta}<\infty$ for all $\beta>0$ and $\sigma(u,h)$ satisfies condition 3.0.9, then*

$$\|\mathcal{A}^\alpha u-\mathcal{A}^\alpha v\|_\beta \leq \mathrm{C}_{d,\alpha,\beta}\lambda\mathrm{K}\mathrm{Lip}_\sigma\|u-v\|_{1,\beta}.$$

Proof. Similar steps as Lemma A.1.3. □

Theorem A.1.7. *Suppose that $\mathrm{C}_{d,\alpha,\beta} < \frac{1}{\lambda\mathrm{K}\mathrm{Lip}_\sigma}$ for positive constants* K, Lip_σ*, then there exists a solution u that is unique up to modification.*

Here, we present the time continuity of the solution.

Proposition A.1.8. *Suppose that condition 4.0.5 holds, then for each $x\in\mathbf{R}^d$, the unique solution to* (4.0.1) *is mean continuous in time. That is for each $x\in\mathbf{R}^d$, the function $t\to\mathrm{E}[|u(t,x)|]$ is continuous.*

The solution is given by

$$u(t,x) = \int_{\mathbf{R}^d} p_t(y-x)u(0,y)\mathrm{d}y + \lambda\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} p(t-s,x,y)\sigma(u(s,y),h)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s).$$

We assume $0 < t_1 < t_2$, then for fixed $x \in \mathbf{R}^d$

$$\begin{aligned} u(t_2,x) - u(t_1,x) &= \int_{\mathbf{R}^d} [p(t_2,x,y) - p(t_1,x,y)]u(0,y)\mathrm{d}y \\ &+ \lambda \int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_0^{t_1} [p(t_2-s,x,y) - p(t_1-s,x,y)] \\ &\times \sigma(u(s,y,.),h)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s) \\ &+ \lambda \int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_{t_1}^{t_2} p(t_2-s,x,y)\sigma(u(s,y),h)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s). \end{aligned}$$

We make the following definitions,

$$\begin{aligned} D_3 &= \int_{\mathbf{R}^d} [p(t_2,x,y) - p(t_1,x,y)]u(0,y)\mathrm{d}y \\ D_4 &= \lambda \int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_0^{t_1} [p(t_2-s,x,y) - p(t_1-s,x,y)]\sigma(u(s,y),h)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s) \\ D_5 &= \lambda \int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_{t_1}^{t_2} p(t_2-s,x,y)\sigma(u(s,y),h)N(\mathrm{d}h,\,\mathrm{d}y,\,\mathrm{d}s). \end{aligned}$$

The proof of the above theorem is a follow up from the following lemmas.

Lemma A.1.9. *For all $\beta > 0$, $0 < t_1 < t_2$, $x \in \mathbf{R}^d$ then*

$$\begin{aligned} |D_3| &\leq 2c_0\left\{\frac{1}{d+\alpha-1}\left(t_2^{1+\alpha(d+\alpha-1)} - t_1^{1+\alpha(d+\alpha-1)}\right)\right. \\ &+ \left.\left(t_2^{(1-d)/\alpha} - t_1^{(1-d)/\alpha}\right)\right\}. \end{aligned}$$

Proof. Write,

$$\begin{aligned} \mathrm{E}|D_3| = |D_3| &= |\int_{\mathbf{R}^d} [p(t_2,x,y) - p(t_1,x,y)]u(0,y)\mathrm{d}y| \\ &\leq \sup_{y\in\mathbf{R}^d} |u(0,y)| \int_{\mathbf{R}^d} |p(t_2,x,y) - p(t_1,x,y)|\mathrm{d}y \\ &= c_0 \int_{\mathbf{R}^d} |p(t_2,x,y) - p(t_1,x,y)|\mathrm{d}y. \end{aligned}$$

For α-stable processes,

$$p^\alpha(t_2,x-y) - p^\alpha(t_1,x-y) \equiv \left(t_2^{-d/\alpha} \wedge \frac{t_2}{|x-y|^{d+\alpha}}\right) - \left(t_1^{-d/\alpha} \wedge \frac{t_1}{|x-y|^{d+\alpha}}\right)$$

Therefore,

$$\mathrm{E}|D_3| \leq c_0 \Big\{ \int_{\mathbf{R}^d} \big(t_2^{-d/\alpha} \wedge \frac{t_2}{|x-y|^{d+\alpha}}\big)\mathrm{d}y - \int_{\mathbf{R}^d} \big(t_1^{-d/\alpha} \wedge \frac{t_1}{|x-y|^{d+\alpha}}\big)\mathrm{d}y \Big\}$$

But

$$\begin{aligned} \int_{\mathbf{R}^d} \big(t_2^{-d/\alpha} \wedge \frac{t_2}{|x-y|^{d+\alpha}}\big)\mathrm{d}y &= t_2 \int_{|x-y| \geq t_2^{-\alpha}} \frac{\mathrm{d}y}{|x-y|^{d+\alpha}} \\ &+ t_2^{-d/\alpha} \int_{|x-y|<t_2^{-\alpha}} \mathrm{d}y \\ &= \frac{2}{d+\alpha-1} t_2^{1+\alpha(d+\alpha-1)} + 2t_2^{(1-d)/\alpha}. \end{aligned}$$

Doing same for the other integral on t_1, therefore

$$\begin{aligned} |D_3| &\leq 2c_0 \Big\{ \frac{1}{d+\alpha-1} \Big(t_2^{1+\alpha(d+\alpha-1)} - t_1^{1+\alpha(d+\alpha-1)} \Big) \\ &+ \Big(t_2^{(1-d)/\alpha} - t_1^{(1-d)/\alpha} \Big) \Big\}. \end{aligned}$$

□

Lemma A.1.10. *For all $\beta > 0$, $0 < t_1 < t_2$ and $x \in \mathbf{R}^d$,*

$$\begin{aligned} \mathrm{E}|D_4| &\leq 2\lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \Big\{ \frac{1}{d+\alpha-1} \Big(\mathrm{e}^{\beta t_2} \int_{t_2-t_1}^{t_2} z^{1+\alpha(d+\alpha-1)} \mathrm{e}^{-\beta z} \mathrm{d}z \\ &- \mathrm{e}^{\beta t_1} \int_0^{t_1} z^{1+\alpha(d+\alpha-1)} \mathrm{e}^{-\beta z} \mathrm{d}z \Big) \\ &+ \Big(\mathrm{e}^{\beta t_2} \int_{t_2-t_1}^{t_2} z^{(1-d)/\alpha} \mathrm{e}^{-\beta z} \mathrm{d}z - \mathrm{e}^{\beta t_1} \int_0^{t_1} z^{(1-d)/\alpha} \mathrm{e}^{-\beta z} \mathrm{d}z \Big) \Big\}. \end{aligned}$$

Proof. We begin by writing

$$\begin{aligned}
\mathrm{E}|D_4| &= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_0^{t_1} |p(t_2-s,x,y)-p(t_1-s,x,y)|\mathrm{E}|\sigma(u(s,y),h)|\nu(\mathrm{d}h)\mathrm{d}y\mathrm{d}s \\
&\leq \lambda \mathrm{KLip}_\sigma \int_{\mathbf{R}^d}\int_0^{t_1} |p(t_2-s,x,y)-p(t_1-s,x,y)|\mathrm{E}|u(s,y)|\mathrm{d}y\mathrm{d}s \\
&\leq \lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \int_{\mathbf{R}^d}\int_0^{t_1} \mathrm{e}^{\beta s}|p(t_2-s,x,y)-p(t_1-s,x,y)|\mathrm{d}s\mathrm{d}y \\
&\leq \lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \int_0^{t_1} \mathrm{d}s\mathrm{e}^{\beta s}\Big\{\int_{\mathbf{R}^d} \big((t_2-s)^{-d/\alpha}\wedge \frac{t_2-s}{|x-y|^{d+\alpha}}\big)\mathrm{d}y \\
&- \int_{\mathbf{R}^d} \big((t_1-s)^{-d/\alpha}\wedge \frac{t_1-s}{|x-y|^{d+\alpha}}\big)\mathrm{d}y\Big\}.
\end{aligned}$$

Similarly as above, therefore,

$$\begin{aligned}
\mathrm{E}|D_4| &\leq 2\lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \int_0^{t_1} \mathrm{d}s\mathrm{e}^{\beta s}\Big\{\frac{1}{d+\alpha-1}\Big((t_2-s)^{1+\alpha(d+\alpha-1)} \\
&- (t_1-s)^{1+\alpha(d+\alpha-1)}\Big) + \Big((t_2-s)^{(1-d)/\alpha} - (t_1-s)^{(1-d)/\alpha}\Big)\Big\}
\end{aligned}$$

and the result follows. □

Lemma A.1.11. *For all $\beta > 0$, $0 < t_1 < t_2$ and $x \in \mathbf{R}^d$,*

$$\begin{aligned} \mathrm{E}|D_5| &\leq 2\lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \mathrm{e}^{\beta t_2} \bigg\{ \frac{1}{d+\alpha-1} \int_0^{t_2-t_1} z^{1+\alpha(d+\alpha-1)} \mathrm{e}^{-\beta z} \mathrm{d}z \\ &+ \int_0^{t_2-t_1} z^{(1-d)/\alpha} \mathrm{e}^{-\beta z} \mathrm{d}z \bigg\}. \end{aligned}$$

Proof. Taking an expectation of the solution,

$$\begin{aligned} \mathrm{E}|D_5| &= \lambda \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \int_{t_1}^{t_2} |p^\alpha(t_2-s,x,y)| \mathrm{E}|\sigma(u(s,y),h)| \nu(\mathrm{d}h) \mathrm{d}y \mathrm{d}s \\ &\leq \lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \int_{\mathbf{R}^d} \int_{t_1}^{t_2} \mathrm{e}^{\beta s} |p^\alpha(t_2-s,x,y)| \mathrm{d}y \mathrm{d}s \\ &= \lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \int_{t_1}^{t_2} \mathrm{d}s \mathrm{e}^{\beta s} \bigg\{ (t_2-s) \int_{|x-y| \geq (t_2-s)^{-\alpha}} \frac{\mathrm{d}y}{|x-y|^{d+\alpha}} \\ &+ (t_2-s)^{-d/\alpha} \int_{|x-y| < (t_2-s)^{-\alpha}} \mathrm{d}y \bigg\} \\ &= \lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \bigg\{ \frac{2}{d+\alpha-1} \int_{t_1}^{t_2} \mathrm{e}^{\beta s} (t_2-s)^{1+\alpha(d+\alpha-1)} \mathrm{d}s \\ &+ 2 \int_{t_1}^{t_2} \mathrm{e}^{\beta s} (t_2-s)^{(1-d)/\alpha} \mathrm{d}s \bigg\} \\ &= 2\lambda \mathrm{KLip}_\sigma \|u\|_{1,\beta} \mathrm{e}^{\beta t_2} \bigg\{ \frac{1}{d+\alpha-1} \int_0^{t_2-t_1} z^{1+\alpha(d+\alpha-1)} \mathrm{e}^{-\beta z} \mathrm{d}z \\ &+ \int_0^{t_2-t_1} z^{(1-d)/\alpha} \mathrm{e}^{-\beta z} \mathrm{d}z \bigg\}. \end{aligned}$$

□

Proof of Theorem A.1.8. Combining Lemma A.1.9, A.1.10 and A.1.11, therefore

$$\begin{aligned}
\mathrm{E}|u(t_2,x)-u(t_1,x)| \;\le\;& 2c_0\Bigg\{\frac{1}{d+\alpha-1}\Big(t_2^{1+\alpha(d+\alpha-1)}-t_1^{1+\alpha(d+\alpha-1)}\Big)\\
&+\Big(t_2^{(1-d)/\alpha}-t_1^{(1-d)/\alpha}\Big)\Bigg\}\\
&+2\lambda \mathrm{KLip}_\sigma\|u\|_{1,\beta}\Bigg\{\frac{1}{d+\alpha-1}\Big(\mathrm{e}^{\beta t_2}\int_{t_2-t_1}^{t_2} z^{1+\alpha(d+\alpha-1)}\mathrm{e}^{-\beta z}\mathrm{d}z\\
&-\mathrm{e}^{\beta t_1}\int_0^{t_1} z^{1+\alpha(d+\alpha-1)}\mathrm{e}^{-\beta z}\mathrm{d}z\Big)\\
&+\Big(\mathrm{e}^{\beta t_2}\int_{t_2-t_1}^{t_2} z^{(1-d)/\alpha}\mathrm{e}^{-\beta z}\mathrm{d}z-\mathrm{e}^{\beta t_1}\int_0^{t_1} z^{(1-d)/\alpha}\mathrm{e}^{-\beta z}\mathrm{d}z\Big)\Bigg\}\\
&+2\lambda \mathrm{KLip}_\sigma\|u\|_{1,\beta}\mathrm{e}^{\beta t_2}\Bigg\{\frac{1}{d+\alpha-1}\int_0^{t_2-t_1} z^{1+\alpha(d+\alpha-1)}\mathrm{e}^{-\beta z}\mathrm{d}z\\
&+\int_0^{t_2-t_1} z^{(1-d)/\alpha}\mathrm{e}^{-\beta z}\mathrm{d}z\Bigg\}.
\end{aligned}$$

Then

$$\lim_{\delta\downarrow 0}\sup_{|t_1-t_2|<\delta}\mathrm{E}|u(t_2,x,.)-u(t_1,x,.)|\le 0$$

and therefore

$$\lim_{\delta\downarrow 0}\sup_{|t_1-t_2|<\delta}\mathrm{E}|u(t_2,x)-u(t_1,x)|=0\,\text{for a fixed}\,x\in\mathbf{R}^d.$$

□

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LIST OF CONFERENCES, TALKS AND PUBLICATIONS

- On some properties of a class of SPDEs driven by space-time Lévy noise-Seventh annual conference of Mathematics Research Students, Loughborough University, May 2013 (talk).
- May 2013 Science Matters, Loughborough University, UK. (Attended).
- Poster Presentation- March 2012 Science Matters, Loughborough University, UK.
- Weak Intermittency property of a class of SPDEs-Young Researchers in Mathematics, April 2012, University of Bristol (talk).
- Sixth annual conference of Mathematics Research Students, Loughborough University, May 2012 (attended).
- LMS-EPSRC Short Course- Topics in Probability, April 2011, University of Oxford, UK (attended).
- Young Researchers in Mathematics, April 2011, University of Warwick (attended).
- LMS-EPSRC Short Course- Duality, BSDEs and Malliavin Calculus, July 2011, University of Oxford, UK (attended).
- Stochastic Analysis: A UK-China Workshop, July 2011, Loughborough University, UK (attended).
- Equadiff August 2011, Loughborough University, UK (attended).

- Mohammud Foondun, Wei Liu and McSylvester E. Omaba. Moment bounds for a class of fractional stochastic heat equations. Preprint, 2014.