Generalised hydrodynamic reductions of the kinetic equation for soliton gas

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Abstract

We derive generalised multi-flow hydrodynamic reductions of the nonlocal kinetic equation for a soliton gas and investigate their structure. These reductions not only provide further insight into the properties of the new kinetic equation but also could prove to be representatives of a novel class of integrable systems of hydrodynamic type, beyond the conventional semi-Hamiltonian framework.

1 Introduction

The generalised soliton-gas kinetic equation represents an integro-differential system [2]

$$f_t + (sf)_x = 0, \tag{1}$$

$$s(\eta) = S(\eta) + \frac{1}{\eta}\int_0^\infty G(\eta,\mu) f(\mu)[s(\mu) - s(\eta)]d\mu. \tag{2}$$

Here $f(\eta) \equiv f(\eta, x, t)$ is the distribution function and $s(\eta) \equiv s(\eta, x, t)$ is the associated transport velocity. The (given) functions $S(\eta)$ and $G(\eta, \mu)$ do not depend on x and t. The function $G(\eta, \mu)$ is assumed to be symmetric, i.e. $G(\eta, \mu) = G(\mu, \eta)$.

System (1), (2) with

$$S(\eta) = 4\eta^2 , \qquad G(\eta, \mu) = \log \left| \frac{\eta - \mu}{\eta + \mu} \right| \tag{3}$$

was derived in [1] as an infinite-genus thermodynamic limit of the Whitham modulation equations associated with the KdV equation, $\varphi_t - 6\varphi\varphi_x + \varphi_{xxx} = 0$, and was shown to describe macroscopic dynamics of a soliton gas, a disordered infinite-soliton ensemble of finite density [4]. In the KdV context, $\eta \geq 0$ is a real-valued spectral parameter and the function $f(\eta, x, t)$ is the distribution function of solitons over the spectrum so that $\kappa = \int_0^\infty f(\eta) d\eta = \mathcal{O}(1)$ is the spatial density of solitons. If $\kappa \ll 1$, the first order approximation of (2), (3) yields Zakharov's kinetic equation for a dilute gas of KdV solitons [11]. The quantity $S(\eta)$ in (2) has a natural meaning of the velocity of an isolated (free) soliton with the spectral parameter η and the function $G(\eta, \mu)/\eta$ is the expression for a phase shift of this soliton occurring after its collision with another soliton having the spectral parameter $\mu < \eta$. Then $s(\eta, x, t)$ acquires the meaning of the self-consistently defined mean local velocity of solitons with the spectral parameter close to η. A straightforward physical derivation of the kinetic equation (1), (2) for integrable systems, based on the original Zakharov [11] phase-shift reasoning was proposed in [3].

In recent paper [2], the multi-flow hydrodynamic reductions of the kinetic equation (1), (2) were studied using the so-called 'cold-gas' ansatz

$$f(\eta, x, t) = \sum_{m=1}^{N} f^m(x, t) \delta(\eta - \eta^m), \tag{4}$$

where the 'spectral' components $\eta^N > \eta^{N-1} > \cdots > \eta^1 > 0$ are arbitrary numbers. These 'isospectral' cold-gas reductions were shown to have the form of systems of hydrodynamic conservation laws

$$u^i_t = (u^i v^i)_x , \qquad i = 1, \ldots, N , \tag{5}$$

where the conserved 'densities' $u^i = \eta^i f(\eta^i, x, t)$ and the associated velocities $v^i = -s(\eta^i, x, t)$ are related algebraically:

$$v^i = \xi_i + \sum_{m \neq i} \epsilon^{im} u^m (v^m - v^i) , \quad \epsilon^{ik} = \epsilon^{ki} . \tag{6}$$

Here

$$\xi_i = -S(\eta^i) , \qquad \epsilon^{ik} = \frac{G(\eta^i, \eta^k)}{\eta^i \eta^k} , \qquad i \neq k . \tag{7}$$

The isospectral cold-gas reductions (5), (6) were proven in [2] to represent integrable (semi-Hamiltonian [9]) linearly degenerate hydrodynamic type systems (see [5], [7]) for arbitrary N, which is a strong indication that the full kinetic equation (1), (2) could constitute an integrable system in the sense yet to be explored.

The present paper is devoted to a more general multi-flow hydrodynamic approximation of the kinetic equation (1), (2), which we derive by considering an ansatz (see, for instance, [10])

$$f(\eta, x, t) = \sum_{m=1}^{N} f^m(x,t)\delta(\eta - \eta^m(x,t)) \tag{8}$$

with the 'spectral components' $\eta^k = \eta^k(x,t)$ being (unknown) functions of x and t rather than arbitrary constants as in (4). We show that the corresponding N-flow non-isospectral hydrodynamic reductions have the form of $2N$-component hydrodynamic type systems

$$u^i_t = (u^i v^i)_x, \quad \eta^i_t = v^i \eta^i_x, \quad i = 1, 2, \dots, N, \tag{9}$$

where the functions $u^i(x,t)$, $v^i(x,t)$ and $\eta^i(x,t)$ are related algebraically by the same equations (6), (7) provided certain restrictions on the behaviour of the kernel function $G(\eta, \mu)$ for $\eta \to \mu$ are satisfied.

System (9), (5), (6) is not integrable by the standard Tsarev generalized hodograph method, because it possesses just N Riemann invariants and has double characteristic velocities. However, having in mind that this system is obtained as an exact reduction of an integrable system (at least for $S(\eta)$, $G(\eta, \mu)$ defined by (3) — the KdV case), one can expect that the multi-flow reductions (9) will be integrable by some modification of the generalised hodograph method [9]. This could lead to an extension of the conventional notion of an integrable system of hydrodynamic type. We are going to investigate this problem in detail in future publications.

2 Generalised hydrodynamic reductions

2.1 Evolution equations

Substituting (8) into (1) we obtain (hereafter we shall be using a shorthand notation η^i for $\eta^i(x,t)$)

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^{N} f^i(x,t)\delta(\eta - \eta^i)\right) + \frac{\partial}{\partial x}\left(s(\eta, x, t)\sum_{i=1}^{N} f^i(x,t)\delta(\eta - \eta^i)\right) = 0,$$

Differentiating and collecting the terms for $\delta(\eta - \eta^i)$ and $\delta'(\eta - \eta^i)$ we obtain

$$\sum_{n=1}^{N}[f^n_t + (s(\eta, x, t)f^n)_x]\delta(\eta - \eta^n) - \sum_{n=1}^{N}[f^n\eta^n_t + s(\eta, x, t)f^n\eta^n_x]\delta'(\eta - \eta^n) = 0. \tag{10}$$

Here $f^i \equiv f^i(x,t)$. Evaluating asymptotic behavior of this expression near each point η^i we arrive $2N$ component hydrodynamic type system (cf. (5))

$$f^i_t + (s(\eta^i, x, t)f^i)_x = 0, \quad \eta^i_t + s(\eta^i, x, t)\eta^i_x = 0, \quad n = 1, \dots, N\,. \tag{11}$$

It is instructive to derive the hydrodynamic reduction (11) by a direct calculation. This is done by integrating (10) with respect to η over a small vicinity of each point $\eta = \eta^i$ with the weights 1 and $(\eta - \eta^i)$, respectively.

Let us fix $x = x_0$. Then, assuming the ordering $\eta^N > \eta^{N-1} > \cdots > \eta^1 > 0$ to hold for all t in a small vicinity of x_0 we introduce N closed intervals $\sigma_i = [\eta^i - \varepsilon_i, \eta^i + \varepsilon_i]$ choosing $\varepsilon_i > 0$ in such a way that in the vicinity of x_0 one has $\eta^j(x,t) \in \sigma_i$ iff $j = i$.

We now integrate (10) over σ_i:

$$\int\limits_{\sigma_i} \left[\sum_{n=1}^{N} [f_t^n + (s(\eta,x,t)f^n)_x]\delta(\eta - \eta^n) - \sum_{n=1}^{N} [f^n \eta_t^n + s(\eta,x,t) f^n \eta_x^n] \delta'(\eta - \eta^n) \right] d\eta = 0,$$

which reduces, after integrating the term with δ'^{i}) by parts, to

$$\int\limits_{\sigma_i} \left[\sum_{n=1}^{N} [f_t^n + s(\eta,x,t) f_x^n + \frac{\partial s(\eta,x,t)}{\partial x} f^n + \frac{\partial s(\eta,x,t)}{\partial \eta} f^n \eta_x^n] \delta(\eta - \eta^n) \right] d\eta = 0\,. \tag{12}$$

Now, integration over σ_i immediately leads to the hydrodynamic conservation law:

$$f_t^i + (s(\eta^i,x,t) f^i)_x = 0\,, \tag{13}$$

which is valid in the small vicinity of x_0. If we assume that the above restrictions on the behaviour of functions $\eta^i(x,t)$ hold for any $x = x_0 \in \mathbb{R}$, $t > 0$, equation (13) will be valid on the entire real line. Setting $i = 1,\ldots,N$ in (12) we immediately obtain the first N equations in (11).

To derive the second set of equations (11) we multiply (10) by $(\eta - \eta^i)$ and integrate over σ_j to get

$$\begin{gathered} \int\limits_{\sigma_j} \left[\sum_{n=1}^{N} [f_t^n + (s(\eta,x,t)f^n)_x]\delta(\eta - \eta^n)(\eta - \eta^i) \right] d\eta \\ - \int\limits_{\sigma_j} \left[\sum_{n=1}^{N} [f^n \eta_t^n + s(\eta,x,t) f^n \eta_x^n](\eta - \eta^i)\delta'(\eta - \eta^n) \right] d\eta = 0. \end{gathered} \tag{14}$$

If $j = i$, the first integral in (14) vanishes, while the second one, after integrating by parts and utilising the fact that each interval σ_i contains only its "own" value η^i, yields

$$\begin{gathered} \int\limits_{\sigma_i} \frac{\partial}{\partial \eta} \left((\eta - \eta^i)[f^i \eta_t^i + s(\eta,x,t) f^i \eta_x^i] \right) \delta(\eta - \eta^i) d\eta \\ = \int\limits_{\sigma_i} \left([f^i \eta_t^i + s(\eta,x,t) f^i \eta_x^i] + (\eta - \eta^i) \frac{\partial s(\eta,x,t)}{\partial \eta} f^i \eta_x^i \right) \delta(\eta - \eta^i) d\eta = 0. \end{gathered} \tag{15}$$

Evaluating the integral in (15) we get

$$\eta_t^i + s(\eta^i,x,t)\eta_x^i = 0, \quad i = 1,\ldots,N\,. \tag{16}$$

It is not difficult to see that, if $j \neq i$, we recover equations (13). Thus, the compatibility of the non-isospectral ansatz (8) with the kinetic equation (1), (2) imposes restrictions (16) on the functions $\eta^i(x,t)$.

Generally, $2N$-component hydrodynamic type system (11) possesses N conservation laws

$$\partial_t(\varphi_i(\eta^i)f^i) + (s(\eta^i, x, t)\varphi_i(\eta^i)f^i)_x = 0,$$

where $\varphi_i(\eta^i)$ are arbitrary functions of a single variable. It is convenient to choose $\varphi_i(\eta^i) = \eta^i$ so that (11) reduces to (cf. (5))

$$u^i_t = (u^i v^i)_x, \quad \eta^i_t = v^i \eta^i_x, \qquad i = 1, \dots, N\,, \tag{17}$$

where $u^i = \eta^i f^i$, $v^i = -s(\eta^i, x, t)$.

2.2 Closure relations

The closure relations connecting the field variables u^i, v^i, and η^i in (17) are obtained by substituting the same ansatz (8) into the integral equation (2). Since we are going to use the variables u^i instead of f^i, we slightly modify ansatz (8) as follows

$$\eta f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\delta(\eta - \eta^i)\,. \tag{18}$$

Substitution of (18) into (2) yields

$$s(\eta, x, t) = S(\eta) + \sum_{m=1}^{N} u^m \frac{G(\eta, \eta^m)}{\eta\eta^m}[s(\eta^m, x, t) - s(\eta, x, t)]\,, \tag{19}$$

As in [2], we introduce (see (7))

$$\epsilon^{ik} = \frac{G(\eta^i, \eta^k)}{\eta^i\eta^k}, \qquad i \neq k\,. \tag{20}$$

There is an important point to be made. In the linearly degenerate reductions (5), (6) associated with the isospectral ansatz (4) involving arbitrary *constants* η^i, the dependencies of $\xi^i = -S(\eta^i)$ and ϵ^{ik} on the relevant components of the vector $\boldsymbol{\eta} = \{\eta^1, \eta^2, \dots, \eta^N\}$ are not important from the viewpoint of integrability — these only provide the connection with the original nonlocal equation (2) — see [2]. However, under the generalised ansatz (8), η^is become dependent *variables*, $\eta^i = \eta^i(x, t)$, so the aforementioned dependencies become essential for the structure of the corresponding hydrodynamic reductions.

Now we pass to the limit as $\eta \to \eta^i$ in (19) to obtain, assuming $\lim_{\eta\to\eta^i} s(\eta, x, t) = -v^i$ (continuity),

$$v^i = \sum_{m\neq i} \epsilon^{im} u^m (v^i - v^m) - S(\eta^i) + \frac{u^i}{(\eta^i)^2} \lim_{\eta\to\eta^i} G(\eta, \eta^i)(s(\eta, x, t) - s(\eta^i, x, t)). \tag{21}$$

If the limit

$$\lim_{\eta\to\eta^i} G(\eta, \eta^i)(s(\eta, x, t) - s(\eta^i, x, t)) \tag{22}$$

exists then (21) becomes

$$v^i = \sum_{m \neq i} \epsilon^{im} u^m (v^i - v^m) - S(\eta^i) + g_i(\mathbf{u}, \mathbf{v}, \boldsymbol{\eta}), \tag{23}$$

where

$$g_i(\mathbf{u}, \mathbf{v}, \boldsymbol{\eta}) = \frac{u^i}{(\eta^i)^2} \lim_{\eta \to \eta^i} G(\eta, \eta^i)[s(\eta, x, t) - s(\eta^i, x, t)].$$

The existence of the limit (22) implies that the function $G(\eta, \mu)$ has *at most* a simple pole singularity on the diagonal $\mu = \eta$.

If the limit (22) vanishes for all $i = 1, \dots, N$ (which happens if $G(\eta, \mu)$ either vanishes itself or has a singularity weaker than a simple pole as $\mu \to \eta$) then $g_i \equiv 0$ and equation (23) reduces to the closure conditions (6), (7) obtained for the isospectral cold-gas reduction. Below we restrict our consideration just to this, most important, case, which arises, in particular, in the case of the kinetic equation for the KdV solitons [1], when the kernel function $G(\eta, \mu)$ has only logarithmic singularity on the diagonal – see (3).

In conclusion of this section we note that nonexistence of the limit (22) for some given $G(\mu, \eta)$ signifies incompatibility of the delta function ansatz (18) with the integral equation (2) for that particular kernel $G(\mu, \eta)$.

3 The structure of generalised multi-flow hydrodynamic reductions

Motivated by the results of [2] for the isospectral cold-gas hydrodynamic reductions (5), (6) we introduce a symmetric matrix $\hat{\epsilon} = [\epsilon^{mn}]_{N \times N}$ with the off-diagonal elements $\epsilon^{ik}(\boldsymbol{\eta})$ defined by (20) and the diagonal elements ϵ^{kk} being some new field variables $r^k(\mathbf{u}, \boldsymbol{\eta})$.

Theorem 1. ([2]): *Algebraic system* (6) *admits the parametric solution*

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \quad v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi_m \beta_{mi}, \tag{24}$$

where symmetric functions $\beta_{ik}(\mathbf{r}, \boldsymbol{\eta})$ are the elements of the matrix $\hat{\beta} = [\beta_{mn}]_{N \times N}$ such that $\hat{\beta}\hat{\epsilon} = -\mathbf{1}$.

Proof: We replace (6) by an equivalent system

$$v^i = \xi_i + \sum_{m=1}^{N} \epsilon^{im} u^m (v^m - v^i). \tag{25}$$

(note that summation in (25) goes over all m including $m = i$ (cf. (6)). Then (25) can be re-written in the form

$$v^i \left(1 + \sum_{m=1}^{N} \epsilon^{im} u^m\right) = \xi_i + \sum_{m=1}^{N} \epsilon^{im} u^m v^m.$$

Substituting (24) into the above formula we obtain

$$v^i \left(1 + \sum_{m=1}^{N}\sum_{k=1}^{N}\beta_{mk}\epsilon^{ki}\right) = \xi_i + \sum_{m=1}^{N}\sum_{k=1}^{N}\xi_m\beta_{mk}\epsilon^{ki}. \tag{26}$$

Taking into account $\hat{\beta}\hat{\epsilon} = -\mathbf{1}$, one can see that expressions at both sides of (26) vanish independently. Thus (26) is an identity, hence the parametric representation (24) is consistent with system (6). The Theorem is proved.

Corollary: The field variables $r^k(\mathbf{u}, \boldsymbol{\eta})$ are rational functions of the conserved densities u^m, namely,

$$r^k = -\frac{1}{u^k}\left(1 + \sum_{m\neq k} u^m \epsilon^{mk}(\boldsymbol{\eta})\right), \quad k = 1, 2, ..., N. \tag{27}$$

Indeed, multiplying both sides of the first relationship in (24) by ϵ^{ik} and performing summation over i we obtain:

$$\sum_{m=1}^{N} u^m \epsilon^{mk} = \sum_{m=1}^{N}\sum_{n=1}^{N}\beta_{mn}\epsilon^{nk} = -1.$$

Thus,

$$\sum_{m\neq k} u^m \epsilon^{mk} + r^k u^k = -1,$$

which immediately yields (27).

Theorem 2: *Under parametrization (24) the $2N$-component hydrodynamic type system (17), (6), (7) reduces to a quasi-diaginal form:*

$$\eta^i_t = v^i \eta^i_x, \quad i = 1, \dots, N; \tag{28}$$

$$r^k_t = v^k r^k_x + \frac{1}{u^k}\left(\sum_{n\neq k} u^n (v^n - v^k)\frac{\partial \epsilon^{nk}}{\partial \eta^k} - \xi'_k\right)\eta^k_x, \quad k = 1, \dots, N\,. \tag{29}$$

Proof: The evolution equations (28) for $\eta^i(x,t)$ are the same as in (17) so we need only to derive equations (29) for $r^k(x,t)$, $k = 1, \dots, N$. Substituting parametric representation (24) into the conservation laws (17) we obtain

$$\partial_t\left(\sum_{m=1}^{N}\beta_{mi}\right) = \partial_x\left(\sum_{m=1}^{N}\xi_m\beta_{mi}\right).$$

Multiplying both sides by ϵ^{ik}, performing summation over the repeated index i and using the relationship $\hat{\beta}\hat{\epsilon} = -\mathbf{1}$, one arrives at the equation

$$\sum_{i=1}^{N}\sum_{m=1}^{N}\beta_{mi}\partial_t\epsilon^{ik} = \sum_{i=1}^{N}\sum_{m=1}^{N}\xi_m\beta_{mi}\partial_x\epsilon^{ik} - \partial_x\xi_k. \tag{30}$$

A simple but not entirely trivial calculation using the following obvious property of the matrix $\hat{\epsilon}(\mathbf{r},\boldsymbol{\eta})$:

$$\frac{\partial \epsilon^{nk}}{\partial r^s} = \delta_{nk}\delta_{ks}\,,$$

and the evolution equations (28) for η^k, shows that (30) reduces to

$$r^k_t = v^k r^k_x + \frac{1}{u^k}\sum_{s=1}^{N}\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\beta_{mn}(\xi_m - v^s)\frac{\partial \epsilon^{nk}}{\partial \eta^s} - \frac{\partial \xi_k}{\partial \eta^s}\right)\eta^s_x, \quad k = 1,\ldots,N. \tag{31}$$

Then taking into account (see (24))

$$\sum_{m=1}^{N}\beta_{mi} = u^i, \qquad \sum_{m=1}^{N}\xi_m\beta_{mi} = u^i v^i,$$

equations (31) reduce to the form (29). The Theorem is proved.

Eliminating u^i from (24), we obtain expressions relating $\mathbf{v}$ to $\boldsymbol{\eta}$ and $\mathbf{r}$:

$$v^i(\mathbf{r},\boldsymbol{\eta}) = \frac{\sum\limits_{m=1}^{N}\xi_m\beta_{mi}}{\sum\limits_{m=1}^{N}\beta_{mi}}\,, \tag{32}$$

while $u^i(\mathbf{r},\boldsymbol{\eta})$ are given by the first equation in (24). Now, system (28), (29) is closed.

For the isospectral case, when η^i, $i = 1, 2, \ldots, N$ are constants so $\partial_t\eta^i = \partial_x\eta^i = 0$ and we recover from (29) the Riemann invariant representation

$$r^i_t = v^i(\mathbf{r})r^i_x \tag{33}$$

of system (5), (6) obtained in [2] with the use of the linear degeneracy of the isospectral cold-gas hydrodynamic reductions. As a matter of fact, one can see now that the Riemann invariant equations (33) could be readily obtained directly from system (5), (6) by the substitution into it of the parametric solution (24).

Thus, the $2N$-component hydrodynamic reduction (17) admits parametrization (24) resolving algebraic system (6) and reducing the evolution equations to the form (28), (29). System (28), (29) has double characteristic velocities $v^k(\mathbf{r},\boldsymbol{\eta})$ (32). However, in the general case, **just** N **functions** $\eta^k(x,t)$ are Riemann invariants (i.e. only a half of the complete hydrodynamic system (17) can be written in diagonal form), while the field variables $r^k(x,t)$ become Riemann invariants only if the **corresponding** η^k =const.

In conclusion of this Section we note that linear degeneracy of system (33) proved in [2] implies that $\partial v^i(\mathbf{r})/\partial r^i = 0$ for all $i = 1,\ldots N$. The latter property clearly remains valid for the characteristic velocities $v^i(\mathbf{r},\boldsymbol{\eta})$ of the generalised hydrodynamic reductions (28), (29), however, now this is no longer associated with the notion of linear degeneracy of a hydrodynamic type system in the classical sense [6], [8] since r^k are no longer Riemann invariants and also $\partial v^i(\mathbf{r},\boldsymbol{\eta})/\partial \eta^i \neq 0$.

4 Conclusion

In this paper, we have derived the generalised hydrodynamic reductions of the nonlocal kinetic equation for a soliton gas (1), (2) by considering the non-isospectral multi-flow ansatz (8) for the distribution function. These new reductions have turned out to have rather unusual structure which we have revealed by using the parametric solution (24) to the algebraic closure conditions (6), (7). More precisely, the non-isospectral N-flow hydrodynamic reductions of the kinetic equation are shown to represent $2N$-component half-diagonal systems of hydrodynamic type (28), (29) with N Riemann invariants and N double characteristic velocities. The feature that makes the derived reductions deserving special attention is that, while they are clearly not integrable by Tsarev's generalised hodograph transform method [9], they could still prove to be integrable in some new sense yet to be understood. Indeed, having in mind that system (28), (29) can be derived as a generalised hydrodynamic reduction of the kinetic equation associated with an integrable equation (e.g. with the KdV equation — for $S(\eta)$, $G(\eta, \mu)$ defined by (3)), one can expect that this reduction will be integrable by some nontrivial extension of the generalised hodograph method.

As a by-product of our calculations we recover the Riemann invariant structure of the isospectral cold-gas reductions (5), (6), (7) studied in [2]. We note that our present compact derivation, unlike that in [2], does not make any use of the linear degeneracy property of the reductions under study.

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