

Harmonic forms on manifolds with edges

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Abstract

Let (X, g) be a compact Riemannian stratified space with simple edge singularity. Thus a neighbourhood of the singular stratum is a bundle of truncated cones over a lower dimensional compact smooth manifold. We calculate the various polynomially weighted de Rham cohomology spaces of X , as well as the associated spaces of harmonic forms. In the unweighted case, this is closely related to recent work of Cheeger and Dai [5]. Because the metric g is incomplete, this requires a consideration of the various choices of ideal boundary conditions at the singular set. We also calculate the space of L^2 harmonic forms for any complete edge metric on the regular part of X .

1 Introduction

One of the early successes in the extension of Hodge theory to manifolds with singularities was the work of Cheeger in the early 1980's on manifolds with isolated conic singularities [3]. This provided the inspiration for, and one of the first corroborations of, conjectures made by him and Goresky and Macpherson relating Hodge theory on stratified spaces to intersection cohomology. Soon afterwards, Cheeger gave a general strategy to extend these results to singular spaces with an iterated stratified structure [4].

In the intervening years substantial progress has been made in this subject. The goal of much of this work is the following: consider a particular class of noncompact or singular spaces, and a natural class of metrics on them, and find a relationship between the space of L^2 harmonic forms and some purely topological invariants of the underlying space. Somewhat nonobviously, the dimension of this Hodge cohomology space is a quasi-isometry invariant of the metric, which makes this a feasible program. The problem is rather different for complete and for incomplete metrics since in the incomplete case one must also understand the contributions coming from the choice of boundary conditions at the singular locus.

Methods which have been successful for this include sheaf-theoretic (and related algebraic) techniques as well as both soft and hard analytic methods. As an example of the state of the art of the former we refer to Saper's recent work on locally symmetric spaces [23], [24]; not surprisingly, combining these various tools can be very effective, cf.

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our recent work with Hausel [13]. We do not attempt, however, to list the many other significant recent contributions to this area.

The specific problem we study here is to determine the topological meaning of the space of weighted L^2 harmonic forms on a compact manifold with ‘incomplete edge’ singularities. More precisely, let X be a compact stratified space with only two strata: an open dense top-dimensional stratum $X^{(n)} := M$ and a lower dimensional stratum B . We assume that a tubular neighbourhood of B in X is diffeomorphic to a bundle of cones over B with fibre a (truncated) cone $C_1(F)$ over a smooth compact manifold F . The Riemannian metric g on M restricts to a standard conic metric on each fibre.

We recast this. Let \overline{M} be a smooth compact manifold with boundary $Y = \partial\overline{M}$. Suppose that Y is the total space of a fibration $\phi : Y \rightarrow B$ with fiber F , and that x is a boundary defining function on \overline{M} , so $Y = \{x = 0\}$ and $dx \neq 0$ there. Write

$$n = \dim \overline{M}, \quad b = \dim B, \quad f = \dim F.$$

A metric g on the interior $M = \overline{M} \setminus Y$ is called an incomplete edge metric if in some collar neighborhood of the boundary $\mathcal{U} \cong (0, x_0) \times Y$ of ∂M , it is quasi-isometric to one of the form

$$g = dx^2 + \tilde{h} + x^2\kappa,$$

where \tilde{h} is the pullback to \mathcal{U} of a metric h on B (via the composition of projections $\mathcal{U} \rightarrow Y \rightarrow B$), and κ is a symmetric two-tensor on \mathcal{U} which restricts to a metric on each fiber F in $\partial\overline{M}$, i.e. at $x = 0$. The metric completion of M with respect to such a g is diffeomorphic in an appropriate sense to the stratified space X obtained by collapsing each fibre F at Y to a point. Natural examples of incomplete edge metrics include hyperbolic cone metrics with unbranched singular set, cf. [14].

Hodge theory on incomplete edge spaces, subject to the condition that f is odd, is also the subject of the paper [5] by Cheeger and Dai. Their aim, however, is primarily directed at the study of the signature on cone bundles and its relationship with Dai’s τ -invariant for the bundle $Y \rightarrow B$. The present paper came into being because we realized that the methods developed in our previous work [13] adapt directly to this setting, and can be used to determine the individual Hodge cohomology spaces also when f is even; it requires little extra effort to study this problem on an entire scale of polynomially weighted spaces. This extended setting is quite natural and in [15], the first author ties it to another interpretation of Dai’s invariant $\tau(Y)$. Of course, we fully acknowledge the overlap of the material here with that in [5], and are also grateful to X. Dai for several very useful conversations.

The main results in [13] identify the Hodge cohomology on a manifold M with the same differential topological structure as above, but endowed with a ‘fibred boundary’ or ‘fibred cusp’ metric, with certain intersection cohomology groups of the space X . These types of metrics are complete and occur frequently in interesting geometric contexts, e.g. as gravitational instanton metrics (the fibred boundary case) or locally symmetric metrics with \mathbb{Q} -rank 1 cusp ends (the fibred cusp case). The proofs there proceed by first showing that the appropriate intersection cohomology can be calculated in terms of weighted (conormal) L^2 cohomology, and then showing that these are identified with L^2 harmonic forms. The first step uses sheaf-theory and the second relies on analysis via a parametrix construction.

While the proofs here are similar, there are two important differences: first, the parametrix method in [13] draws on the fibred boundary pseudodifferential calculus from [19], while here we use the edge pseudodifferential calculus from [18]. From the reader's point of view, this substitution is only formal, since the results we need appear quite similar (although the intricacies of the parametrix constructions and analytic phenomena in the two calculi are quite different). However, since the metrics in one of the classes we consider in this paper are incomplete, we must pay more careful attention to the whole question of choices of closed extension for d , δ , $D = d + \delta$ and Δ .

We now describe our results in more detail. We first consider the weighted de Rham complex $(x^\alpha L^2 \Omega^*(M, g), d)$, where g is an incomplete edge metric. In general, if \mathcal{F} is any function space on the Riemannian manifold (M, g) , then we denote by $\mathcal{F} \Omega^*(M)$ the space of sections of the exterior bundle $\Lambda^* T^*(M)$ with this regularity. When there is dependence on the metric, it is indicated explicitly by writing $\mathcal{F} \Omega^*(M, g)$. A form α is in $x^\alpha L^2 \Omega^*(M, g)$ if $\alpha = x^\alpha \alpha'$ where $\alpha' \in L^2 \Omega^*(M, g)$. Some results are presented in §3.3 concerning when d has a unique closed extension to these weighted spaces. In particular we prove the

Proposition: (§3.3, Corollary 3) *Suppose that $((f-1)/2 - a, (f+1)/2 - a) \cap \mathbb{N} = \emptyset$ or else, if there exists an integer $q_a \in ((f-1)/2 - a, (f+1)/2 - a)$ then $H^{q_a}(F) = \{0\}$. (This is true in particular when $a = 0$ and either b is even, or else b is odd and $H^{f/2}(F) = \{0\}$.) Under either of these conditions, the operator d on $x^\alpha L^2 \Omega^*(M, g)$ has a unique closed extension in all degrees.*

This result is an analogue of, and generalizes, a result due to Cheeger in the conic case [4]. The proof involves the justification of a delicate integration by parts.

Although we state and prove this proposition separately, it is also a consequence of another result we prove later in this paper concerning closed extensions of the elliptic operator $D_a = d + \delta_a$, where δ_a is the codifferential on $x^\alpha L^2 \Omega^*(M, g)$. The more technical proof in this case uses the ellipticity of D_a and the existence of a parametrix for it in the edge calculus, as described in §4. We prove the

Proposition: (§4.3, Theorem 7): *Let (M, g) satisfy the hypotheses in the preceding proposition relative to the weight a . If in addition Δ_F (the Laplacian on the fibres F with respect to any one of the family of metrics κ) has no ‘small eigenvalues’, as defined in §4.3, then D_a is essentially self-adjoint on $x^\alpha L^2 \Omega^*(M, g)$. If the first hypothesis is satisfied, then it is always possible to achieve this extra small eigenvalue hypothesis with a metric g' which is quasi-isometric to g .*

These results concern special situations where there is a unique closed extension, but in general, there are two different canonical procedures to extend $(x^\alpha L^2 \Omega^*(M, g), d)$ to a Hilbert complex, known as the maximal or minimal extension of d , and these may be lead to quite different complexes. The associated de Rham cohomologies are denoted $H_{\max/\min}^k(M, g, a)$, respectively. In analogy with familiar terminology on manifolds with boundary (which is a special case of our setting when the fibre F is a point), we call the Hodge Laplacians associated to each of these complexes the absolute and relative Hodge Laplacians, and denote their nullspaces by $\mathcal{H}_{\text{abs/rel}}^*(M, g, a)$. It is almost a tautology that the maximal and minimal weighted de Rham cohomology spaces are identified with the absolute and relative weighted Hodge cohomology spaces. As for the topological

interpretation, we prove the

Theorem: (§3.5, Theorem 4) *Let (M, g) be a manifold with incomplete edge metric. The maximal and minimal weighted Hodge cohomology spaces are canonically identified with intersection cohomology for the stratified space (X, B) by:*

$$H_{\max}^k(M, g, a) = \begin{cases} IH_{\overline{\mathfrak{m}}+\ll a-1\gg}^k(X, B) & f \text{ odd} \\ IH_{\overline{\mathfrak{m}}+\ll a-1/2\gg}^k(X, B) & f \text{ even} \end{cases}$$

and

$$H_{\min}^k(M, g, a) = \begin{cases} IH_{\underline{\mathfrak{m}}+\langle a \rangle}^k(X, B) & f \text{ odd} \\ IH_{\underline{\mathfrak{m}}+\langle a-1/2 \rangle}^k(X, B) & f \text{ even} \end{cases} ;$$

here $\ll t \gg$ denotes the least integer strictly greater than t and $\langle t \rangle$ denotes the least integer greater than or equal to t , and $\underline{\mathfrak{m}}$, respectively $\overline{\mathfrak{m}}$, are the lower and upper middle perversities.

We single out two important special cases:

Corollary: *The maximal and minimal de Rham cohomologies when $a = 0$ correspond to upper and lower middle perversity intersection cohomology:*

$$\begin{aligned} H_{\max}^k(M, g, 0) &= IH_{\overline{\mathfrak{m}}}^k(X) \\ H_{\min}^k(M, g, 0) &= IH_{\underline{\mathfrak{m}}}^k(X) \end{aligned} \quad (1)$$

Moreover, when f is even, the maximal and minimal de Rham cohomologies at weights $\pm 1/2$ coincide, and again correspond to upper and lower middle perversity intersection cohomology:

$$\begin{aligned} H_{\max/\min}^k(M, g, -1/2) &= IH_{\overline{\mathfrak{m}}}^k(X) \\ H_{\max/\min}^k(M, g, 1/2) &= IH_{\underline{\mathfrak{m}}}^k(X). \end{aligned} \quad (2)$$

The notation $IH_{\mathfrak{p}}^*(X, B)$ is somewhat nonstandard, and indicates a slight generalization of these spaces (so as to include, for example, the case where F is a point), which we discuss in §3.2.

These results are, to some extent, ‘soft’ in that they do not require any serious use of elliptic theory, and for that reason we have separated them into the first few sections of the paper. The main ingredients in their proofs are some abstract functional analytic results involving Hilbert complexes from [2], reviewed in §2, the sheaf-theoretic characterization of intersection cohomology from [6], discussed in §3.2, and the appropriate Poincaré Lemmas (also known as ‘the local calculations’), which are developed in §3.4 and 3.5. Very helpful in our approach is the fact that we may restrict attention to conormal forms, but this is not strictly speaking necessary. This part of the paper is a recapitulation and extension of Cheeger’s original work on the Hodge theory on cones, with an attempt to present the argument as cleanly as possible in this slightly more general context, but the results could all also be proved using the techniques in [4].

On the other hand, we require more analytic information in order to study the minimal Hodge cohomology $\mathcal{H}_{\min}^*(M, g, a)$, which is defined as the common nullspace of $d_{\min, a}$ and $\delta_{\min, a}$. This terminology is slightly unfortunate, since \mathcal{H}_{\min}^* is not related to the minimal de Rham cohomology H_{\min}^* discussed above, but rather corresponds to the nullspace of the minimal extension of D_a . In any case, we prove the

Theorem: (§4.5, Theorem 8) *Let M be a manifold with an incomplete edge metric, g . The minimal weighted Hodge cohomology is given by*

$$\mathcal{H}_{\min}^k(M, g, a) = \begin{cases} \text{Im} (IH_{\underline{m}+\langle a \rangle}^k(X, B) \rightarrow IH_{\underline{m}+\ll a-1 \gg}^k(X, B)) & f \text{ odd} \\ \text{Im} (IH_{\underline{m}+\langle a-1/2 \rangle}^k(X, B) \rightarrow IH_{\underline{m}+\ll a-1/2 \gg}^k(X, B)) & f \text{ even.} \end{cases}$$

In particular, when $a = 0$,

$$\mathcal{H}_{\min}^k(M, g, 0) = \text{Im} (IH_{\underline{m}}^k(X, B) \rightarrow IH_{\underline{m}}^k(X, B)).$$

The proof requires two main analytic results: the conormal regularity of solutions in the minimal domain of the equation $D_a\omega = 0$, and the solvability (and regularity theory for the solution) of $D_a\zeta = \eta$ for suitable η . For these we invoke the theory of pseudodifferential edge operators, as developed in [18]. The results from this theory which we require are reviewed in §4.1.

Referring to that section for the following terminology, we note that the specific computations we must make in order to apply this more general theory are the calculation of the indicial roots of D_a and the injectivity of the normal operator $N(xD_a)$ on suitable weighted L^2 spaces. As we show in §4.2.1, the calculation of indicial roots for D_a on manifolds with edges is essentially identical to that on cones, and this partially explains the similarity of the results in the two cases. However, the extra role played by the model operator $N(xD_a)$, see §4.2.2 as well as Proposition 8 in §4.1, is not required in the conic case, but is the key fact needed in the parametrix construction in the edge calculus.

We now turn to some applications and extensions of our results. The first is a Bochner-type vanishing result. Recall that the Weitzenböck formula for the Hodge Laplacian on k -forms on M states that $\Delta_k = \nabla^*\nabla + \mathcal{R}_k$, where \mathcal{R}_k is a curvature operator, acting by endomorphisms on $\Lambda^k M$.

Theorem: *Suppose that M admits an incomplete edge metric g such that $\mathcal{R}_k \geq 0$ everywhere, and is strictly positive at some point of M . Then the minimal Hodge cohomology $\mathcal{H}_{\min}^k(M, g)$ (at weight $a = 0$), and hence*

$$\text{Im} (IH_{\underline{m}}^k(X, B) \rightarrow IH_{\underline{m}}^k(X, B)),$$

both vanish. If in addition f is odd or else if f is even but $H^{f/2}(F) = \{0\}$, then we may also deduce that $IH_{\underline{m}}^k(X, B) = IH_{\overline{m}}^k(X, B) = \{0\}$.

The proof is the usual one, and simply involves noting that when ω is in the nullspace of D_{\min} , then the integration by parts

$$\langle \Delta\omega, \omega \rangle = \|\nabla\omega\|^2 + \langle \mathcal{R}_k\omega, \omega \rangle$$

is justified.

We can extend this type of analysis significantly further. For example, the generalization to second order natural geometric operators L is essentially straightforward. By definition, such an operator is one of the form $L = \nabla^*\nabla + \mathcal{R}$ acting on sections of some subbundle E of the full tensor bundle over M , using the induced Levi-Civita connection (though we may also twist by any other bundle with connection); the symmetric

endomorphism \mathcal{R} on E is a generalized curvature operator. L is formally symmetric on $L^2(M; E)$, and we may ask the same questions about its domain, mapping properties and nullspace as we have for the Hodge Laplacian. The point we wish to make is that answers to these questions are readily deduced, and direct consequences of the edge theory, once one has calculated the indicial roots of L and determined the mapping properties of the normal operator $N(x^2L)$. Furthermore, the calculation of indicial roots for such an operator reduces directly to the analogous computation on the cone $C(F)$ endowed with the metric $dx^2 + x^2\kappa$. The presence of an extra twisting bundle may change the arithmetic of the indicial root computation and the spaces on which the normal operator is injective, but makes no difference otherwise. We shall not develop these remarks further here, but shall return to them in greater detail elsewhere. (At that time we shall also give a more careful account of parametrices in the edge calculus for the Laplacian with relative and absolute boundary conditions.) There are many interesting geometric consequences of such results. In particular, the infinitesimal rigidity of hyperbolic cone metrics with smooth singular set, as proved in [14] in 3 dimensions and in the recent paper [22] (for deformations amongst Einstein metrics rather than just hyperbolic metrics) in dimensions greater than 3 is a direct consequence. See [21] for further discussion.

Now consider a more topological application: When $f = \dim F$ is odd, the maximal and minimal cohomologies agree, as do the two middle perversities $\overline{\mathfrak{m}}$ and $\underline{\mathfrak{m}}$ for X . Our results in this case, when $a = 0$, agree with those in [5]. In this special case, and assuming that $n = 4\ell$, there is a well-defined pairing on middle degree forms; Cheeger and Dai prove the corresponding signature theorem. We extend their result to cover also the case f even. Define the L^2 -signature as the signature of the degenerate pairing on $\mathcal{H}_{\text{rel}}^{2\ell}(M, g)$ induced from the map

$$\mathcal{H}_{\text{rel}}^{2\ell}(M, g) \longrightarrow \mathcal{H}_{\text{abs}}^{2\ell}(M, g),$$

and the nondegenerate pairing between these spaces. Similarly, one can also define a topological signature $\sigma(M)$ as the signature of the degenerate pairing on $H_0^*(M)$ defined through its map to $H^*(M)$ and the nondegenerate pairing between these spaces. Finally, recall the invariant $\tau(Y)$ defined by Dai [7] for the total space of the fibration $Y = \partial M \rightarrow B$.

Theorem: *The L^2 -signature of the stratified Riemannian space X endowed with an incomplete edge metric g , is given by*

$$L^2 - \text{sgn}(M, g) = \sigma(M) + \tau.$$

Our final result concerns the Hodge cohomology of the manifold M endowed with a *complete* edge metric. By definition, and following the notation above, g is a complete edge metric if near ∂M it has the form

$$g = \frac{dx^2 + \tilde{h}}{x^2} + \kappa.$$

The prototype would be the product of a hyperbolic space (or any conformally compact metric) and a compact manifold F . It is of interest, as a generalization of the main result of [17], to calculate the Hodge cohomology for such manifolds. We obtain the

Theorem: (§5, Theorem 9) *Let (M, g) be a manifold with a complete edge metric, with associated compact stratified space X . Suppose that k is **not** of the form $j + (b + 1)/2$ where $\mathcal{H}^j(F) \neq \{0\}$. Then*

$$L^2\mathcal{H}^k(M, g) \cong IH_{f+\frac{b}{2}-k}^k(X, B).$$

In all other cases, where k does have this form, $L^2\mathcal{H}^k(M, g)$ is infinite dimensional. If $n = 4\ell$ and $L^2\mathcal{H}^{2\ell}(M, g)$ is finite dimensional, then its signature is equal to the L^2 signature of the conformally related incomplete edge metric, which in turn is equal to the sum of the topological signature of M and the τ invariant, as in the previous theorem.

In conclusion, let us remark that sorting out the detailed analysis of the Hodge Laplacian (and other natural geometric elliptic operators) for more general classes of stratified spaces, e.g. algebraic varieties, is a problem of great importance with many applications. Hodge theory on quite general real analytic manifolds is the subject of an ongoing project by D. Grieser and R. Melrose, using an approach closely related to (although more general than) the one used here. We appreciate their interest in and forbearance concerning the present paper.

2 Hodge-de Rham theory for Hilbert complexes

In this section we review some generalities about L^2 cohomology, based on the formalism of Hilbert complexes from [2], to which we refer in the interests of brevity for most of the proofs.

Consider a complex of the form

$$0 \rightarrow L_0 \xrightarrow{D_0} L_1 \xrightarrow{D_1} L_2 \dots \xrightarrow{D_{n-1}} L_n \rightarrow 0, \quad (3)$$

where each L_i is a separable Hilbert space, $D_i : L_i \rightarrow L_{i+1}$ is a closed operator with dense domain $\mathcal{D}(D_i)$ such that $\text{ran}(D_i) \subseteq \mathcal{D}(D_{i+1})$ and $D_{i+1} \circ D_i = 0$ for all i . Under these conditions, (3) is called a Hilbert complex, and is denoted by (L_*, D_*) .

Many familiar constructions in Hodge-de Rham theory carry over immediately to this setting, and we list in particular:

- i) There is a dual Hilbert complex

$$0 \rightarrow L_0 \xleftarrow{D_0^*} L_1 \xleftarrow{D_1^*} L_2 \dots \xleftarrow{D_{n-1}^*} L_n \rightarrow 0$$

defined using the Hilbert space adjoints of the differentials, $D_i^* : L_{i+1} \rightarrow L_i$;

- ii) The Laplacian $\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*$ is a self-adjoint operator on L_i with domain

$$\mathcal{D}(\Delta_i) = \{u \in \mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*) : D_i u \in \mathcal{D}(D_i^*), D_{i-1}^* u \in \mathcal{D}(D_{i-1})\}$$

and nullspace

$$\ker \Delta_i := \mathcal{H}^i(L_*, D_*) = \ker D_i \cap \ker D_{i-1}^*;$$

iii) There is a weak Kodaira decomposition

$$L_i = \mathcal{H}^i \oplus \overline{\text{ran}D_{i-1}} \oplus \overline{\text{ran}D_i^*};$$

iv) The cohomology of (L_*, D_*) is defined by

$$H^i(L_*, D_*) = \ker D_i / \text{ran}D_{i-1};$$

if this space is finite dimensional, then $\text{ran}D_{i-1}$ is necessarily closed and

$$H^i(L_*, D_*) = \mathcal{H}^i(L_*, D_*).$$

The main case of interest here is when (M, g) is a (not necessarily complete) Riemannian manifold, $L_i = L^2\Omega^i(M, g)$ and D_i is the exterior derivative operator. Later we shall also consider the somewhat more general case

$$L_i = e^{2w_i}L^2\Omega^i(M, g),$$

where $w_i \in C^\infty(M)$ is some weight function, but for the remainder of this section, to be concrete, we shall assume that $w_i \equiv 0$; it will be clear that everything extends to the general case in a straightforward manner.

To turn the ‘core’ de Rham complex $(C_0^\infty\Omega^*(M), d)$ into a Hilbert complex, we must specify a closed extension of d , and there may be more than one way to do this.

Definition 1. *The two canonical closed extensions of d are:*

- *The maximal extension d_{\max} ; this is the operator d acting on the domain*

$$\begin{aligned} \mathcal{D}(d_{\max}) &= \{\omega \in L^2\Omega^*(M) : d\omega \in L^2\Omega^*(M, g)\} \\ &= \{\omega \in L^2\Omega^*(M) : \exists \eta \in L^2\Omega^*(M, g) \\ &\quad \text{s.t. } \langle \omega, \delta\zeta \rangle = \langle \eta, \zeta \rangle \ \forall \zeta \in C_0^\infty\Omega^*(M)\}. \end{aligned}$$

In other words, $\mathcal{D}(d_{\max})$ is the largest set of forms ω in L^2 such that $d\omega$, computed distributionally, is also in L^2 .

- *The minimal extension d_{\min} ; this is given by the graph closure of d on $C_0^\infty\Omega^*(M)$, i.e.*

$$\begin{aligned} \mathcal{D}(d_{\min}) &= \{\omega \in L^2\Omega^*(M) : \exists \omega_j \in C_0^\infty\Omega^*(M), \ \omega_j \rightarrow \omega \text{ in } L^2\} \\ &\quad \text{and } d\omega_j \text{ also converges to some } \eta \in L^2\}, \end{aligned}$$

in which case $d_{\min}\omega = \eta$.

Maximal and minimal extensions of δ are defined in the same manner.

Clearly $\mathcal{D}(d_{\min}) \subseteq \mathcal{D}(d_{\max})$. An old result due to Gaffney [8] shows that these domains are the same when (M, g) is complete, but in many other cases of interest (for example, on a manifold with boundary) they may differ.

In order to show that $(L^2\Omega^*(M, g), d_{\max/\min})$ are both Hilbert complexes, we require the

Lemma 1.

$$d_{\max} : \mathcal{D}(d_{\max}) \rightarrow \mathcal{D}(d_{\max})$$

and

$$d_{\min} : \mathcal{D}(d_{\min}) \rightarrow \mathcal{D}(d_{\min}).$$

Proof. The fact that $(d_{\min})^2 = 0$ follows from the identity $d^2 = 0$ on $\mathcal{C}_0^\infty \Omega^*$, while the fact that $\delta^2 = 0$ on test forms and the definition of distributional derivatives shows that $(d_{\max})^2 = 0$. \square

The cohomologies of these complexes are denoted $H_{\max/\min}^*(M, g)$, respectively. Here and in the sequel we shall use notation like max/min in a hopefully self-explanatory manner to indicate statements which hold for each of the indicated extensions.

It is straightforward that the Hilbert complex adjoint of $(L^2 \Omega^*(M, g), d_{\max/\min})$ is $(L^2 \Omega^*(M, g), \delta_{\min/\max})$, i.e.

$$(d_{\max})^* = \delta_{\min} \quad \text{and} \quad (d_{\min})^* = \delta_{\max}.$$

There are three well-behaved weak Kodaira decompositions:

$$L^2 \Omega^j(M, g) = \mathcal{H}_{\text{abs/rel/max}}^j(M, g) \oplus \overline{\text{ran } d_{\max/\min/\min, j-1}} \oplus \overline{\text{ran } \delta_{\min/\max/\min, j}}, \quad (4)$$

with summands mutually orthogonal in each case. The first summand on the right, called the absolute, relative or maximal Hodge cohomology, respectively, is defined as the orthogonal complement of the other two summands. Since $(\text{ran } d_{\max})^\perp = \ker \delta_{\min}$, etc., we see that

$$\mathcal{H}_{\text{abs/rel/max}}^j(M, g) = \ker d_{\max/\min/\max, j} \cap \ker \delta_{\min/\max/\max, j-1}, \quad (5)$$

respectively. The third decomposition, incorporating both d_{\min} and δ_{\min} , is the original one defined by Kodaira. The corresponding Hodge cohomology \mathcal{H}_{\max}^j is often infinite dimensional, though. We do *not* consider a fourth weak Kodaira decomposition involving the ranges of d_{\max} and δ_{\max} , since these subspaces might not even be disjoint, let alone orthogonal. (This is due to the fact that $d_{\min} d_{\max}$ may not even be defined, let alone vanish.) Nonetheless we still define the minimal Hodge cohomology

$$\begin{aligned} \mathcal{H}_{\min}^j(M, g) &= L^2 \Omega^j(M, g) \ominus (\text{ran } d_{\max, j-1} + \text{ran } \delta_{\max, j}) \\ &= \ker d_{\min, j} \cap \ker \delta_{\min, j-1} \\ &= \ker d_{\min, j} / \left(\overline{\text{ran } d_{\max, j-1}} \cap \ker d_{\min, j} \right). \end{aligned} \quad (6)$$

Note also that

$$\mathcal{H}_{\min}^j(M, g) = \mathcal{H}_{\text{rel}}^j(M, g) \cap \mathcal{H}_{\text{abs}}^j(M, g)$$

The operators $d_{\min/\max}$ are both clearly quasi-isometry invariants, and the various Kodaira decompositions above then show that the minimal and maximal cohomologies $H_{\max/\min}^*(M, g)$, and their ‘reduced’ versions, the absolute and relative Hodge cohomologies $\mathcal{H}_{\text{abs/rel}}^*(M, g)$, are all quasi-isometry invariants. This invariance is also true for the maximal and minimal Hodge cohomologies $\mathcal{H}_{\max/\min}^*(M, g)$.

There are quite a few ‘Laplacians’ one might consider, most prominent amongst which are the absolute and relative Laplacians

$$\Delta_{\text{abs}} = \delta_{\min} d_{\max} + d_{\max} \delta_{\min}, \quad \Delta_{\text{rel}} = \delta_{\max} d_{\min} + d_{\min} \delta_{\max}.$$

These are self-adjoint and satisfy

$$\mathcal{H}_{\text{abs}}^j(M, g) = \ker \Delta_{\text{abs}} \quad \text{and} \quad \mathcal{H}_{\text{rel}}^j(M, g) = \ker \Delta_{\text{rel}}. \quad (7)$$

Furthermore, if $H_{\text{max/min}}^j(M, g)$ is finite dimensional, then the range of $d_{\text{max/min}, j-1}$ is closed, and $H_{\text{max/min}}^j(M, g) = \mathcal{H}_{\text{abs/rel}}^j(M, g)$. Consequently, these Hodge cohomology spaces may be computed using only tools from differential topology and general cohomology, e.g. sheaf theory, Mayer-Vietoris, etc.

One can also define

$$\Delta_{\text{mM/Mm}} = \delta_{\text{min/max}} d_{\text{max/min}} + d_{\text{min/max}} \delta_{\text{max/min}}.$$

These are symmetric, but not necessarily self-adjoint, invariant under the Hodge star, and satisfy

$$\mathcal{H}_{\text{max/min}}^j(M, g) = \ker \Delta_{\text{mM/Mm}}.$$

Note that

$$\mathcal{H}_{\text{min}}^j(M, g) = \ker d_{\min} \cap \ker \delta_{\min},$$

so this is consistent with our prior definition of $\mathcal{H}_{\text{min}}^j$.

We conclude this section by stating two more results, both true in the general Hilbert complex setting, but for simplicity we restrict to the setting of differential forms. The first concerns a Künneth-type theorem.

Proposition 1 ([2] Corollary 2.15). *Let (L', D') and $(L'' D'')$ be two Hilbert complexes. Form the completed tensor product Hilbert complex (L, D) :*

$$L_j = \bigoplus_{i+\ell=j} L'_i \hat{\otimes} L''_\ell,$$

$$D_j = \bigoplus_{i+\ell=j} (D'_i \otimes id_{L''_\ell} + (-1)^i id_{L'_i} \otimes D''_\ell).$$

Suppose that either D' or D'' has closed range in all degrees. Then

$$H^j(L, D) = \bigoplus_{i+\ell=j} H^i(L', D') \otimes H^\ell(L'' D'').$$

The other result concerns the possibility of computing one of these cohomology groups using a ‘core subcomplex’ of smooth (but not necessarily compactly supported) forms

$$\mathcal{D}_{\text{max/min}}^\infty \Omega^*(M, g) \subset L^2 \Omega^*(M, g)$$

consisting of all elements ω which are in the domain of $\Delta_{\text{abs/rel}}^\ell$ for every $\ell \geq 0$.

Proposition 2 ([2] Theorem 2.12). *The cohomology $H_{\max/\min}^*(M, g)$ is equal to the cohomology of the complex $(\mathcal{D}^\infty \Omega_{\max/\min}^*(M, g), d_{\max/\min})$.*

For example, when (M, g) is compact without boundary, this is simply the well-known result that $H^*(M)$ can be computed using the complex of smooth forms. When (M, g) is compact with boundary, then as discussed carefully in [2], $H_{\max/\min}^*(M, g)$ is equal to the cohomology of the complex of smooth forms continuous to the boundary which satisfy absolute/relative boundary conditions.

3 De Rham theory and edges

The context in which we shall adapt and develop the material from the last section is the category of manifolds with edge singularities. After defining these we briefly review the intersection cohomology theory for such spaces, and then turn to an analysis of the maximal and minimal cohomologies of the Hilbert complexes of weighted L^2 forms, and in particular the identification of weighted de Rham cohomologies with intersection cohomologies with different perversities.

3.1 Manifolds with edge singularities

We now begin to develop some of the ideas in the last section in the concrete setting of manifolds with conic or edge singularities.

Definition 2. *A pseudomanifold X of dimension n has simple edge singularities if it has a dense open stratum M , which is a smooth manifold of top dimension, and the singular strata $X_{\text{sing}} = X \setminus M$ are a disjoint union of closed smooth manifolds B_j (of possibly varying dimension) such that each B_j has a neighbourhood \mathcal{U}_j which is diffeomorphic to a bundle with base B_j and fibre a truncated cone $C_1(F_j)$ over a smooth link F_j .*

The boundary $\partial\mathcal{U}_j$ of each cone bundle neighbourhood \mathcal{U}_j is the total space of a bundle over B_j with fibre F_j . There are more complicated singular spaces with iterated edge singularities, which is why we call this class ‘simple’. However, for brevity, in this paper we shall refer to a space of this type as a manifold with edge singularities. Note that this class includes the case of manifolds with conic singularities, i.e. where some B_j are 0-dimensional.

Definition 3. *A metric g on a space X with simple edge singularities is said to be of incomplete edge type if it is an ordinary smooth metric away from the singular strata B_j , while in each cone bundle neighbourhood \mathcal{U}_j it is quasi-isometric to one of the form*

$$g = dx^2 + \pi_j^* h + x^2 \kappa;$$

here x is the polar distance on each cone $C_1(F_j)$, κ is a symmetric 2-tensor on $\partial\mathcal{U}_j$ which restricts to a metric on each fibre F_j , $\pi_j : \mathcal{U}_j \rightarrow B_j$ is the projection, and h is a metric on B_j .

A metric g on the principal stratum M of such a space X is of complete edge type if in each \mathcal{U}_j it has the form

$$g = \frac{dx^2 + \pi_j^* h}{x^2} + \kappa,$$

where x, κ, π_j and h are as above. We often drop the π_j^* in this notation and shall also frequently write $\tilde{g}(x) = h + x^2\kappa$.

We next recall some analytic and geometric properties of Riemannian submersions from [13] and discuss their relevance to de Rham theory for edge metrics.

Let $\phi : Y \rightarrow B$ be a fibration with fibre F , and suppose that it is endowed with a metric \tilde{g} of the form $\phi^*(h) + \kappa$, where h is a metric on B . We assume furthermore that $\phi : (Y, \tilde{g}) \rightarrow (B, h)$ is a Riemannian submersion. The tangent bundle TY splits into a vertical and horizontal subbundle, $T^V Y \oplus T^H Y$, where $T^V Y = \ker(d\phi)$ and $T^H Y$ is its orthogonal complement. Note that $T^H Y$ is canonically identified with ϕ^*TB , and in addition, because \tilde{g} is a Riemannian submersion, $T^H Y$ is also the subbundle annihilated by k . In any case, this induces a splitting of the form bundles on Y , and thus every differential form has a (horizontal, vertical) bidegree, i.e.

$$\Omega^{p,q}(Y) = \Omega^p(B) \hat{\otimes} \Omega^q(Y, T^V Y).$$

The space of harmonic forms on F is finite dimensional, and we let

$$\Pi_0^q : L^2\Omega^q(F) \longrightarrow L^2\mathcal{H}^q(F), \quad \Pi_\perp = I - \Pi_0$$

denote the natural orthogonal projectors; these extend naturally to each $L^2\Omega^{p,q}(Y)$.

Proposition 3. *The differential and codifferential on Y decompose as*

$$d_Y = d_F + \tilde{d}_B - \mathbb{I} + \mathbb{R}, \quad \delta_Y = \delta_F + (\tilde{d}_B)^* - \mathbb{I}^* + \mathbb{R}^*,$$

where d_F is the pullback of d to the fibre, \tilde{d}_B is the lift of d_B as a horizontal operator, and \mathbb{I} and \mathbb{R} are tensorial operators built from the second fundamental form of the fibres and the curvature of the bundle, respectively. These act as

$$\begin{aligned} d_F : \Omega^{p,q}(Y) &\rightarrow \Omega^{p,q+1}(Y), & \tilde{d}_B : \Omega^{p,q}(Y) &\rightarrow \Omega^{p+1,q}(Y) \\ \mathbb{I} : \Omega^{p,q}(Y) &\rightarrow \Omega^{p+1,q}(Y), & \mathbb{R} : \Omega^{p,q}(Y) &\rightarrow \Omega^{p+2,q-1}(Y). \end{aligned}$$

Now consider the degenerating family of metrics $\tilde{g}(x) = h + x^2\kappa$ ($0 < x \leq 1$). If α is a (p, q) -form, then

$$|\alpha|_{\tilde{g}(x)} = x^{-q} |\alpha|_{\tilde{g}(1)}.$$

Furthermore, as explained in [13], $\tilde{d}_B^x = \tilde{d}_B$, $\mathbb{I}^x = \mathbb{I}$ and $R^x = xR$; the x in the superscript signifies that the operator is to be calculated relative to the metric $\tilde{g}(x)$, and the operator without a superscript is calculated relative to $\tilde{g}(1)$. Hence

$$d_Y^x = d_F + \tilde{d}_B - \mathbb{I} + x\mathbb{R}, \quad \delta_Y^x = \delta_F + (\tilde{d}_B)^* - \mathbb{I}^* + x\mathbb{R}^*.$$

We define the operator

$$\mathfrak{d} = \Pi_0(\tilde{d}_B - \mathbb{I})\Pi_0;$$

this acts on the space of fibre-harmonic forms, or equivalently, we can think of this as acting on the space of forms on B with coefficients in the flat vector bundle of harmonic forms on F that comes from the fibre bundle Y . In [13] we proved the following useful lemmas:

Lemma 2. *The operator \mathfrak{d} and its adjoint \mathfrak{d}^* are differentials, i.e. $\mathfrak{d}^2 = (\mathfrak{d}^*)^2 = 0$.*

Corollary 1. *Let $\mathbb{D} = \mathfrak{d} + \mathfrak{d}^*$, and suppose that $\mathbb{D}\alpha = 0$ for some fibre-harmonic form α . Then $\mathfrak{d}\alpha = \mathfrak{d}^*\alpha = 0$, and so the terms $\alpha_{p,q}$ of pure bidegree also satisfy $\mathbb{D}\alpha_{p,q} = 0$.*

3.2 Intersection cohomology

Let X be a pseudomanifold which is a smoothly stratified space of real dimension n , with no codimension one stratum. For this subsection only, we allow X to be more singular than was considered earlier in this paper; namely, around any point $q \in X$ contained in the stratum X_ℓ of codimension ℓ is a neighbourhood \mathcal{U}_q diffeomorphic to $\mathcal{V} \times C(\Sigma)$, where \mathcal{V} is a Euclidean ball and $C(\Sigma)$ is the cone over a link Σ , which itself is a stratified space (of dimension $\ell - 1$).

A perversity \mathbf{p} is an n -tuple of natural numbers, $(p(1), p(2), \dots, p(n))$ satisfying $p(1) = p(2) = 0$ and $p(\ell - 1) \leq p(\ell) \leq p(\ell - 1) + 1$ for all $\ell \leq n$. Associated to such a space X and perversity \mathbf{p} is the intersection complex $IC_*^{\mathbf{p}}(X)$; roughly speaking, the integer $p(\ell)$ regulates the dimension of the intersection of chains (in general position) with the stratum of codimension ℓ . The homology of this complex is the intersection homology $IH_*^{\mathbf{p}}(X)$; the cohomology of the dual cochain complex is the intersection cohomology $IH_{\mathbf{p}}^*(X)$.

The following result, which asserts that the cohomology of any fine sheaf over X is equal to the intersection cohomology of X (with respect to some perversity \mathbf{p} , so long as the local sheaf cohomology in any sufficiently small neighbourhood equals the intersection cohomology of that neighbourhood.

Proposition 4 ([12]). *Let X be a stratified space and let (\mathcal{L}, d) be a complex of fine sheaves on X with cohomology $H^*(X, \mathcal{L})$. Suppose that if \mathcal{U} is a neighbourhood in the principal (smooth) stratum of X , then $H^*(\mathcal{U}, \mathcal{L}) = H^*(\mathcal{U}, \mathbb{C})$, while if q lies in a stratum of codimension ℓ , and $\mathcal{U} = \mathcal{V} \times C(\Sigma)$ as above, then*

$$H^k(\mathcal{U}, \mathcal{L}) \cong IH_{\mathbf{p}}^k(\mathcal{U}) = \begin{cases} IH_{\mathbf{p}}^k(\Sigma) & k \leq \ell - 2 - p(\ell) \\ 0 & k > \ell - 1 - p(\ell). \end{cases} \quad (8)$$

Then there is a natural isomorphism between the hypercohomology $\mathbb{H}^(X, \mathcal{L})$ associated to this complex of sheaves and $IH_{\mathbf{p}}^*(X)$, the intersection cohomology of perversity \mathbf{p} .*

The details and proof of this theorem can be found in [6] and [1]. We refer to equation (8) by saying that \mathcal{L} satisfies the correct *local calculation* for intersection cohomology with perversity \mathbf{p} . This result will be one of our primary tools below.

On any pseudomanifold X , there are two distinguished perversity functions: $\underline{\mathbf{m}}$ the lower middle, and $\overline{\mathbf{m}}$ the upper middle perversity. When all strata of X are even dimensional, the upper and lower middle perversity intersection cohomologies are equal and satisfy Poincaré duality. When not all the strata are even dimensional, then $IH_{\underline{\mathbf{m}}}(X) \neq IH_{\overline{\mathbf{m}}}(X)$ in general, but these spaces are Poincaré dual to one another.

In the case of an n -dimensional manifold with edge singularities, the only relevant values of the perversity function \mathbf{p} are those on the codimensions of the singular strata, i.e. the components of $X^{sing} \cong B$. Assume this is connected, for simplicity. If the fibre F has dimension f then this codimension is $f + 1$. If f is odd, then the upper and the lower middle perversities satisfy $\underline{\mathbf{m}}(f + 1) = \overline{\mathbf{m}}(f + 1) = (f - 1)/2$. If f is even, then $\underline{\mathbf{m}}(f + 1) = f/2$ and $\overline{\mathbf{m}}(f + 1) = f/2 - 1$.

These definitions and constructions may be extended in a straightforward way to include perversities such that $\mathbf{p}(f + 1) \leq 0$ or $\mathbf{p}(f + 1) \geq f$. From the local calculation

above, these give cohomologies

$$\begin{aligned} IH_{\mathfrak{p}}^*(X, B) &\cong H^*(X, X^{sing}) & \mathfrak{p}(f+1) &\geq f \\ IH_{\mathfrak{p}}^*(X, B) &\cong H^*(X - X^{sing}) & \mathfrak{p}(f+1) &\leq 0, \end{aligned}$$

respectively. With this extended definition, Poincaré duality is preserved in the sense that

$$IH_{\underline{\mathfrak{m}}+s}^{n/2-k} \cong IH_{\overline{\mathfrak{m}}-s}^{n/2+k},$$

but on the other hand we lose the property that the intersection cohomology groups are independent of stratification, which was the original reason for restricting the values of the perversity function. Since there is a natural stratification in the case of manifolds with edge singularities, this is of no concern.

3.3 Maximal and minimal domains for weighted complexes

From now on, we assume that X is a space with simple edge singularities and g is a metric on it which is of incomplete edge type. Let us fix a smooth function x on the principal stratum M which equals the polar distance function in a neighbourhood \mathcal{U}_j of each singular stratum B_j .

For any $a \in \mathbb{R}$, consider the complex of weighted L^2 forms

$$0 \rightarrow x^a L^2 \Omega^0(M, g) \xrightarrow{d} x^a L^2 \Omega^1(M, g) \xrightarrow{d} \dots \xrightarrow{d} x^a L^2 \Omega^n(M, g) \rightarrow 0.$$

As already explained in §2, there are (at least) two ways to turn this into a Hilbert complex, using either the maximal or minimal extensions of d . The corresponding complexes, $(x^a L^2 \Omega^*(M, g), d_{\max/\min, a})$ are called the maximal and minimal weighted cohomologies (with weight a), respectively, and have cohomologies and spaces of harmonic forms $H_{\max/\min}^*(M, g, a)$ and $\mathcal{H}_{\text{abs/rel}}^*(M, g, a)$. Note that the formal adjoint of d with respect to the $x^a L^2$ inner product has a term of order 0 depending on a .

It will be possible to simplify many of the calculations in the computations of these spaces below by applying Proposition 2. As we show later, it will ultimately suffice to consider forms which are ‘tangentially regular’ or conormal, as we now define. Let M be a compact manifold with boundary. The space $\mathcal{V}_b(M)$ of b -vector fields on M is, by definition, the space of all smooth vector fields which are constrained at ∂M to be tangent to the boundary. Thus, in any local coordinate system (x, y) , where x is a boundary defining function and y is a local coordinate system on the boundary, \mathcal{V}_b is generated over $\mathcal{C}^\infty(M)$ by the basis sections $x\partial_x, \partial_{y_j}$.

Definition 4. *Let $\gamma \in \mathbb{R}$. The space $\mathcal{A}^\gamma(M)$ of conormal functions on M of order γ (with respect to L^∞) is defined as*

$$\{u : V_1 \cdots V_\ell u \in x^\gamma L^\infty(M) \quad \forall \ell \text{ and } V_j \in \mathcal{V}_b\}.$$

We write $\mathcal{A}^{\gamma+}(M) = \cup_{\epsilon>0} \mathcal{A}^{\gamma+\epsilon}(M)$ and $\mathcal{A}^(M) = \cup_\gamma \mathcal{A}^\gamma(M)$. If E is any smooth vector bundle over M , then $\mathcal{A}^\gamma(M; E)$ is defined using any system of smooth trivializations of E ; in particular, the space of differential forms conormal of weight γ is denoted $\mathcal{A}^\gamma \Omega^*(M)$.*

Note in particular that if $u \in \mathcal{A}^\gamma(M; E)$, then $|u| = \mathcal{O}(x^\gamma)$ along with all its b derivatives, and is C^∞ in the interior of M . In the following, we frequently make use of the fact that if $u \in \mathcal{A}^*(M; E)$ and $\int |u|^2 x^b dx dy < \infty$ (with respect to any smooth nonsingular Hermitian metric on E), then $|u| = o(|x|^{-(b+1)/2})$ uniformly in $y \in \partial M$ as $x \rightarrow 0$. The proof is elementary and left to the reader.

We remark that $\mathcal{D}(d_{\min,a}) \cap \mathcal{A}^* \Omega^*$ is dense in $\mathcal{D}(d_{\min,a})$, and similarly $\mathcal{D}(d_{\max,a}) \cap \mathcal{A}^* \Omega^*$ is dense in $\mathcal{D}(d_{\max,a})$. The former of these is immediate from the definition, while the latter follows (in the cone or edge setting) using standard mollification arguments.

Let us now consider the problem of when $\mathcal{D}(d_{\max,a})$ equals $\mathcal{D}(d_{\min,a})$ on the truncated cone $C_1(F)$, with exact warped product conic metric $g = dx^2 + x^2 \kappa$. This is already contained in Cheeger [4] (when $a = 0$), but we present this argument to make it accessible for later generalization. By the remark above, it suffices to consider only conormal forms, and this simplifies the discussion somewhat.

Lemma 3. *The form $\omega = \alpha + dx \wedge \beta \in \mathcal{D}(d_{\max,a}) \cap \mathcal{A}^* \Omega^k(C_1(F))$ lies in $\mathcal{D}(d_{\min,a}) \cap \mathcal{A}^* \Omega^k(C_1(F))$ if and only if $|\Pi_0^k \alpha(x)|_g = o(x^{-k})$ (or equivalently, $|\Pi_0^k \alpha(x)|_\kappa = o(1)$) when $k \in ((f-1)/2 - a, (f+1)/2 - a)$. In particular, if k is in this range, and if $\Phi \in \mathcal{H}^k(F)$ and $\sigma(x) \in \mathcal{A}^*(\mathbb{R}^+)$, then $\sigma(x)\Phi \in \mathcal{D}(d_{\min,a})$ if and only if $\sigma(x) = o(1)$.*

Before commencing with the proof, we state the immediate and important consequence.

Corollary 2. *Suppose either that either $((f-1)/2 - a, (f+1)/2 - a) \cap \mathbb{Z} = \emptyset$ or else $((f-1)/2 - a, (f+1)/2 - a) \cap \mathbb{Z} = \{k_a\}$ and $\mathcal{H}^{k_a}(F) = \{0\}$. Then the extensions $d_{\max,a}$ and $d_{\min,a}$ coincide.*

Proof. Following the definitions, to show that the minimal and maximal domains are equal it suffices to show that for any conormal forms $\omega \in \mathcal{D}(d_{\max,a})$ and $\psi \in \mathcal{D}(d_{\min,a})$, with ψ vanishing near $x = 1$, we have $\langle d\omega, \psi \rangle_a = \langle \omega, \delta_a \psi \rangle_a$; so we must prove that this holds precisely under the given hypothesis.

Define $*_a$ by $\psi \wedge *_a \psi = |\psi|^2 x^{-2a} dV_g$. Then $*_{-a} *_a = \pm 1$ (depending on the parity of the degree), and $\delta_a = \pm *_a d *_a$. In addition, if $\psi \in x^a L^2 \Omega^*(C_1(F), g)$, then $*_a \psi \in x^{-a} L^2 \Omega^*(C_1(F), g)$.

Now

$$\langle d\omega, \psi \rangle_a = \int d\omega \wedge *_a \psi = \int d(\omega \wedge *_a \psi) - \int \omega \wedge *_a \delta_a \psi,$$

and so, writing $*_a \psi = \tau + dx \wedge \rho$, Stokes' theorem produces the boundary term

$$\int_{C_1(F)} d(\omega \wedge *_a \psi) = \lim_{x \rightarrow 0} \int_{\{x\} \times F} \alpha \wedge \tau. \quad (9)$$

Hence $\omega \in \mathcal{D}(d_{\min,a})$ if and only if this boundary term vanishes for all ψ .

Next, the weighted L^2 conditions

$$\int |\alpha|_h^2 x^{f-2k-2a} + |\tau|_h^2 x^{f-2(f-k)+2a} dx dV_h < \infty$$

and conormality yield

$$|\alpha(x)|_\kappa = o(x^{k-f/2+a-1/2}), \quad |\tau(x)|_\kappa = o(x^{(f-k)-f/2-a-1/2}),$$

but this gives only that (9) is $o(x^{-1})$, which is not enough.

We must now show that this boundary contribution vanishes if and only if $|\Pi_0^k \alpha(x)|_\kappa = o(1)$ when $k \in ((f-1)/2 - a, (f+1)/2 - a)$. To this end, introduce the Hodge decomposition on F

$$\begin{aligned}\alpha(x) &= d_F A_1(x) + \delta_F A_2(x) + \alpha_0(x) \\ \tau(x) &= d_F B_1(x) + \delta_F B_2(x) + \tau_0(x).\end{aligned}\tag{10}$$

Inserting these expressions into (9), six of the nine terms vanish, leaving only

$$\lim_{x \rightarrow 0} \left(\int_F A_1(x) \wedge \delta_F d_F B_1(x) + \int_F d_F \delta_F A_2(x) \wedge B_2(x) + \int_F \alpha_0(x) \wedge \tau_0(x) \right).\tag{11}$$

Since $A_1, B_1 \perp \ker d_F$, we can estimate $\|A_1(x)\|_{L^2(F)} \leq c\|\alpha\|_{L^2(F)}$, $\|B_1(x)\|_{L^2(F)} \leq \|\tau\|_{L^2(F)}$; in addition, $d_F \alpha(x) = d_F \delta_F A_2(x) \in x^a L^2 \Omega^{k+1}$, $d_F \tau(x) = d_F \delta_F B_2(x) \in x^{-a} L^2 \Omega^{f-k+1}$. Hence the first two terms here are $o(1)$, and it remains only to analyze the third.

Now decompose $\alpha_0(x) = \sum a_j(x) \phi_j$ and $*_F \tau_0(x) = \sum t_j(x) \phi_j$, where $\{\phi_j\}$ is an orthonormal basis for $\mathcal{H}^k(F)$, and where each $a_j(x)$ and $t_j(x)$ is conormal on $[0, 1]$. The boundary integral now reduces to the finite sum $\sum_j a_j(x) t_j(x)$, and we must show that each $a_j(x) t_j(x) = o(1)$. However,

$$\int_0^1 (a_j^2(x) + (a_j'(x))^2) x^{f-2k-2a} dx < \infty, \quad \int_0^1 (t_j^2(x) + (t_j'(x))^2) x^{f-2k+2a} dx < \infty,$$

and these imply that $a_j(x), a_j'(x) = o(x^p)$, $p = k - f/2 + a - 1/2$, and $t_j(x), t_j'(x) = o(x^q)$, $q = (f - k) - f/2 - a - 1/2$. The improvement comes by writing a_j, t_j as integrals of a_j', t_j' , respectively, where the other limit of integration is taken at 0 or 1 depending on p or q ; this gives

$$a_j(x) = \begin{cases} o(x^{p+1}) & p < -1 \\ o(\log x) & p = -1 \\ a_j(0) + o(x^{p+1}) & -1 < p < 0 \\ o(x^{p+1}) & p \geq 0 \end{cases} \quad t_j(x) = \begin{cases} o(x^{q+1}) & q < -1 \\ o(\log x) & q = -1 \\ t_j(0) + o(x^{q+1}) & -1 < q < 0 \\ o(x^{q+1}) & q \geq 0 \end{cases}$$

(In particular, $a_j(0), t_j(0)$ exist when $p, q \in (-1, 0)$.) Hence, if either $p \notin [-1, 0)$ or $q \notin [-1, 0)$, we get $a_j(x) t_j(x) = o(1)$. If $p = -1$ then $q = 0$ and we reach the same conclusion. If both $-1 < p, q < 0$, which occurs precisely when $k \in ((f-1)/2 - a, (f+1)/2 - a)$, then we obtain a well-defined, but not necessarily vanishing, boundary term $\int_F \alpha_0 \wedge \tau_0$. This completes the proof. \square

We wish to extend this result to incomplete edge metrics. Using the quasi-isometry invariance and localizability (proved in §3.4 below) of the maximal and minimal domains, it suffices to consider forms on $\mathcal{V} \times C_1(F)$, $\mathcal{V} \subset \mathbb{R}^b$, with warped product metric $dx^2 + h + x^2 \kappa = dx^2 + \tilde{g}(x)$, and we may as well assume that h is the Euclidean metric too. As before, Π_0 denotes the orthogonal projection onto fibre-harmonic forms, and we let $Y = \mathcal{V} \times F$.

Lemma 4. Let $\omega = \alpha(x) + dx \wedge \beta(x)$, where

$$\alpha(x) = \sum_k \alpha_{*,k}(x), \quad \beta(x) = \sum_k \beta_{*,k}(x)$$

are the decompositions into fibre degree. Then $\omega \in \mathcal{D}(d_{\min,a}) \cap \mathcal{A}^*$ if $\omega \in \mathcal{D}(d_{\max,a}) \cap \mathcal{A}^*$ and $|\Pi_0(\alpha_{*,k}(x))|_{\tilde{g}(x)} = o(x^{-k})$ whenever $k \in ((f-1)/2 - a, (f+1)/2 - a)$. In particular, if $\eta \in \Omega^{(*,k)}(Y)$ is fibre harmonic and $s(x)$ is conormal, then $s(x)\eta \in \mathcal{D}(d_{\min,a})$ if and only if $s(x) = o(1)$.

Proof. Following the same proof as above, the form $\omega = \alpha + dx \wedge \beta \in \mathcal{D}(d_{\max,a}) \cap \mathcal{A}^*$ is also in $\mathcal{D}(d_{\min,a})$ if and only if, for every $\gamma \in \mathcal{D}(\delta_{\max,a}) \cap \mathcal{A}^*$ with $*\gamma = \xi + dx \wedge \nu$, the boundary term

$$\int_M d\omega \wedge *\gamma - \int_M \omega \wedge \delta\gamma = \lim_{x \rightarrow 0} \int_Y \alpha(x) \wedge \xi(x)$$

vanishes. Decomposing into fibre degree, the boundary term becomes:

$$\lim_{x \rightarrow 0} \sum_{k=0}^f \int_Y \alpha_k(x) \wedge \xi_{f-k}(x).$$

We have

$$\alpha_k \in \mathcal{A}^{k-(f+1)/2+a+}(M, \Lambda^{*,k}T^*Y), \quad \xi_{f-k} \in \mathcal{A}^{(f-k)-(f+1)/2-a+}(M, \Lambda^{*,f-k}T^*Y),$$

so $|\alpha_k \wedge \xi_{f-k}|_{\tilde{g}(x)} = o(x^{-1})$ only at this stage. Continuing as before, we can reduce to the case where each α_k and ξ_{f-k} are fibre-harmonic, and write

$$(\alpha_k)_0 = \sum a_j(x, y)\phi_j, \quad (\xi_{f-k})_0 = \sum t_j(x, y)\phi_j$$

where $a_j, t_j \in \mathcal{A}^*$. In order to improve the order of vanishing of a_j and t_j as before, we observe that, for example, assuming each α_k is fibre-harmonic, then

$$\omega, d\omega \in x^a L^2 \Rightarrow \int (|\alpha_k|^2 + |\beta_k|^2 + |\partial_x \alpha_k - d_y \beta_k|^2) x^{f-2k-2a} dx dV_{\tilde{g}(1)} < \infty.$$

However, conormality already gives that

$$\int |d_y \beta_k|^2 x^{f-2k-2a} dx dV_{\tilde{g}(1)} < \infty,$$

and so we conclude that

$$\int |\partial_x \alpha_k|^2 x^{f-2k-2a} dx dV_{\tilde{g}(1)} < \infty.$$

A similar argument applies to ξ . The rest of the proof is now the same as in the conic case. \square

Corollary 3. Suppose that either $((f+1)/2 - a, (f-1)/2 - a) \cap \mathbb{N} = \emptyset$ or else if $q_a \in ((f+1)/2 - a, (f-1)/2 - a) \cap \mathbb{N}$ then $H^{q_a}(F) = \{0\}$. (This is true in particular when $a = 0$ if either b is even, or else b is odd and $H^{f/2}(F) = \{0\}$.) Then $\mathcal{D}(d_{\max,a}) = \mathcal{D}(d_{\min,a})$, and hence $H_{\max}^*(M, g, a) = H_{\min}^*(M, g, a) = \mathcal{H}^*(M, g, a)$ in every degree.

3.4 Preparations for the Poincaré lemma

We now prove several preliminary results which will be used in the computation of weighted de Rham cohomology on manifolds with edge singularities. Our arguments closely follow those in [4], but with simplifications since by Propositions 1 and 2 it suffices to work only with conormal forms.

To begin, define two complexes of sheaves, $\mathcal{L}_{\max/\min,a}$; these are induced by the presheaves obtained by restricting the forms in $\mathcal{D}(d_{\max/\min,a})$ to the cofinal sequence of coverings of X of the form $\{\mathcal{U}_\alpha\}$, where each \mathcal{U}_α is either disjoint from all of the singular strata B_j or else, if $\mathcal{U}_\alpha \cap B_j \neq \emptyset$, then \mathcal{U}_α is a product neighbourhood $\mathcal{V}_\alpha \times C_\epsilon(F_j)$; here $\mathcal{V}_\alpha \subset B_j$ and $C_\epsilon(F_j)$ is the truncation to $x \leq \epsilon$ of the cone $C(F_j)$.

Lemma 5. *The sheaves in each of the two complexes $\mathcal{L}_{\max/\min,a}$ over X are fine.*

Proof. We define a partition of unity $\{\chi_\alpha\}$ subordinate to $\{\mathcal{U}_\alpha\}$ in such a way that each $d\chi_\alpha$ is bounded. Indeed, away from the singular strata this is obvious, whereas if $\mathcal{U}_\alpha = \mathcal{V} \times C_\epsilon(F)$ then we can let χ_α be a product of cutoff functions in each factor. It is now standard that if $\omega \in \mathcal{D}(d_{\max/\min,a})$, then $\chi_\alpha \omega \in \mathcal{D}(d_{\max/\min,a})$ as well. This gives the result. \square

Now form the associated spectral sequence for either of the double deRham/Čech complexes $\mathcal{L}_{\max/\min,a}$. Its hypercohomology may be computed taking either the Čech or the de Rham differential first. In the former case, at the first stage this becomes the standard definition of $H_{\max/\min}^*(M, g, a)$; for the latter case, however, we must calculate the weighted de Rham cohomology of each neighbourhood \mathcal{U}_α , which is the main goal of this section. We shall see that these are the same as the local intersection cohomology with respect to a certain perversity function \mathfrak{p} depending on a , and this will prove the equality of weighted de Rham and intersection cohomologies.

By a slight abuse of notation, we denote the local cohomology of these sheaves by $H_{\max/\min}^*(\mathcal{U}_\alpha, g, a)$, respectively. If \mathcal{U}_α is disjoint from all singular strata, then

$$H^j(\mathcal{U}_\alpha, \mathcal{L}_{\max/\min,a}) = \begin{cases} \mathbb{R} & j = 0 \\ 0 & j > 0. \end{cases}$$

Neither the weight function x^a nor the metric g play a role. On the other hand, first note that

$$(\mathcal{L}_{\max/\min,a}(\mathcal{V} \times C_1(F)), d_{\max/\min,a}) = (L^2\Omega^*(\mathcal{V}), d_{\max}) \widehat{\otimes} (L^2\Omega^*(C_1(F), g, a), d_{\max/\min,a}).$$

Furthermore, $\dim H^j(\mathcal{V}) < \infty$ so the range of d_{\max} on $L^2\Omega^*(\mathcal{V})$ is closed. Hence by Proposition 1 and quasi-isometry invariance,

$$\begin{aligned} H_{\max/\min}^j(\mathcal{V} \times C_1(F), g, a) &= \bigoplus_{\ell=0}^j H^\ell(\mathcal{V}) \otimes H_{\max/\min}^{j-\ell}(C_1(F), dx^2 + x^2\kappa, a) \\ &= H_{\max/\min}^j(C_1(F), dx^2 + x^2\kappa, a). \end{aligned}$$

We have now reduced the computation of weighted de Rham cohomology on X to that of the truncated cone. For the next step we use the following regularity result

Lemma 6.

$$\bigcap_{\ell=1}^{\infty} \mathcal{D}(\Delta_{\text{abs/rel}}^{\ell}, a) \subset \mathcal{A}^* \Omega^*(C_1(F)).$$

Proof. The proof relies on the mapping properties of a parametrix for $\Delta_{\text{abs/rel}, a}$; this parametrix is an element $G \in x^2 \Psi_b^{-2, \mathcal{E}}(M, \Omega^*)$, the calculus of b -pseudodifferential operators on M , such that $G\Delta = I - R$ for some residual element R which satisfies $R : L^2 \Omega^* \rightarrow \mathcal{A}^* \Omega^*$. (Both G and R depend on a and the choice of extension.) Suppose that $u \in \mathcal{D}(\Delta_{\text{max}, a})$, so in particular $u, \Delta u \in x^a L^2 \Omega^*$. Then $u = Gf + Ru$, hence $u \in x^{a+2} H_b^2 \Omega^* + \mathcal{A}^*$. Induction on ℓ gives the result. We refer to [18] and [16] for more details. \square

Now we prove three lemmas taken from [4]. In the following, $g = dx^2 + x^2 \kappa$ on $C_1(F)$.

Proposition 5. *Let $r : C_1(F) \setminus \{0\} \rightarrow F$ be the canonical retraction map. Then*

$$r^* : L^2 \Omega^k(F) \cap \ker d \longrightarrow \mathcal{D}(d_{\text{max}, a}) \cap x^a L^2 \Omega^k(C_1(F))$$

is well-defined and bounded if and only if $k < \frac{f+1}{2} - a$.

Proof. Let $\alpha \in L^2 \Omega^k(F)$; then

$$\int_{C_1(F)} |\alpha|_g^2 x^{-2a} dV_g = \|\alpha\|_{L^2(F)}^2 \int_0^1 x^{f-2k-2a} dx,$$

and this is finite if and only if $k < \frac{f+1}{2} - a$. Since $dr^* \alpha = r^* d\alpha = 0$, the image of r^* lies in $\mathcal{D}(d_{\text{max}, a})$. \square

Proposition 6. *There exists a $K > 0$ such that for any $\omega = \alpha + dx \wedge \beta \in x^a L^2 \Omega^k(C(F))$ there exists some $c \in (1/2, 1)$ for which*

$$\|\alpha(c)\|_{L^2(F)}^2 \leq K \|\omega\|_{x^a L^2(C_1(F))}^2.$$

Proof. If not, then for any $N > 0$, there exists some (nontrivial) $\omega \in x^a L^2 \Omega^k(C_1(F))$ such that $N \|\omega\|_{x^a L^2(C_1(F))}^2 < \|\alpha(x)\|_{L^2(F)}^2$ for all $x \in (1/2, 1)$. However, if this is the case, then for this ω ,

$$\begin{aligned} \|\omega\|_{x^a L^2(C_1(F))}^2 &\geq \|\alpha\|_{x^a L^2(C_1(F))}^2 \geq \int_{1/2}^1 \|\alpha(x)\|_{L^2(F)}^2 x^{f-2k-2a} dx \\ &> N \|\omega\|_{x^a L^2(C_1(F))}^2 \int_{1/2}^1 x^{f-2k-2a} dx. \end{aligned}$$

This is a contradiction when N is large. \square

Proposition 7. *If $k < \frac{f+3}{2} - a$, then for any $c \in (1/2, 1)$, the map*

$$x^a L^2 \Omega^k(C_1(F)) \ni \omega = \alpha + dx \wedge \beta \longrightarrow K_c(\omega) = \int_c^x \beta(s) ds \in x^a L^2 \Omega^{k-1}(C_1(F))$$

is bounded.

Proof. First,

$$\|K_c \omega\|_{x^a L^2(C_1(F))}^2 = \int_0^1 \int_F \left| \int_c^x \beta(s) ds \right|_{\kappa}^2 x^{f-2k+2-2a} dx dV_{\kappa}. \quad (12)$$

Next, for any real number b ,

$$\begin{aligned} \left| \int_c^x \beta(s) ds \right|_{\kappa}^2 &\leq \left(\int_c^x |\beta(s)|_{\kappa} ds \right)^2 = \left(\int_c^x s^{-b} |s^b \beta(s)|_{\kappa} ds \right)^2 \\ &\leq \int_c^x s^{-2b} ds \int_c^x |\beta(s)|_{\kappa}^2 s^{2b} ds = \begin{cases} \frac{x^{1-2b} - c^{1-2b}}{1-2b} \int_c^x |\beta(s)|_{\kappa}^2 s^{2b} ds & b \neq 1/2 \\ (\log(x) - \log(c)) \int_c^x |\beta(s)|_{\kappa}^2 s^{2b} ds & b = 1/2. \end{cases} \end{aligned}$$

Setting $2b = f - 2k + 2 - 2a$ and using this in (12) gives

$$\begin{aligned} &\|K_c \omega\|_{x^a L^2(C_1(F))}^2 \\ &\leq \begin{cases} \int_0^1 \frac{x-c(x/c)^{f-2k+2-2a}}{1-(f-2k+2-2a)} \int_c^x |\beta(s)|_{\kappa}^2 s^{f-2k+2-2a} ds dV_{\kappa} dx & k \neq \frac{f+1}{2} - a \\ \int_0^1 x(\ln(x) - \ln(c)) \int_c^x |\beta(s)|_{\kappa}^2 s^{f-2k+2-2a} ds dV_{\kappa} dx & k = \frac{f+1}{2} - a \end{cases} \\ &\leq \begin{cases} \left(\int_0^1 \frac{x-c(x/c)^{f-2k+2-2a}}{1-(f-2k+2-2a)} dx \right) \|\beta\|_{x^a L^2(C_1(F))}^2 & k \neq \frac{f+1}{2} - a \\ \left(\int_0^1 x(\ln(x) - \ln(c)) dx \right) \|\beta\|_{x^a L^2(C_1(F))}^2 & k = \frac{f+1}{2} - a. \end{cases} \end{aligned}$$

Since c is bounded away from 0, both of these coefficients on the right are uniformly bounded when $k \leq \frac{f+3}{2} - a$. \square

3.5 Poincaré lemma

We now compute the weighted cohomologies of the truncated cone.

Lemma 7.

$$H_{\max}^k(C_1(F), g, a) = \begin{cases} H^k(F) & k < (f+1)/2 - a \\ 0 & k \geq (f+1)/2 - a, \end{cases}$$

and

$$H_{\min}^k(C_1(F), g, a) = \begin{cases} H^k(F) & k \leq (f-1)/2 - a \\ 0 & k > (f-1)/2 - a. \end{cases}$$

Proof. By Lemma 6, we may work exclusively with conormal forms. First let $k < \frac{f+1}{2} - a$. We wish to define a map

$$R : H_{\max}^k(C_1(F), g, a) \longrightarrow H^k(F),$$

and show that it is an isomorphism. Let $[\omega] \in H_{\max}^k(C_1(F), g, a)$, and choose a conormal representative $\omega = \alpha + dx \wedge \beta \in \mathcal{D}(d_{\max}, g, a)$. For any $c \in (1/2, 1)$, set $R([\omega]) = [\alpha(c)]$.

To check that R is defined independently of all choices, first note that $d\omega = d_F\alpha + dx \wedge (\alpha' - d_F\beta) = 0$, so $d_F\alpha(c) = 0$ for any c . Next, if $\tilde{\omega} = \tilde{\alpha} + dx \wedge \tilde{\beta}$ is another conormal representative of $[\omega]$, then there exists $\eta = \mu + dx \wedge \nu \in x^a L^2 \Omega^{k-1}(C_1(F)) \cap \mathcal{A}^*$ with $\tilde{\omega} = \omega + d\eta$. This implies in particular that $\tilde{\alpha}(c) = \alpha(c) + d_F\mu(c)$, so $[\alpha(c)] = [\tilde{\alpha}(c)]$. Similarly, to prove that this class is independent of c , note that $\int_c^{c'} \beta(s) ds \in L^2 \Omega^{k-1}(F)$ and hence $\alpha' = d_F\beta$, which implies that

$$d \int_c^{c'} \beta(s) ds = \int_c^{c'} d_F \beta(s) ds = \int_c^{c'} \alpha'(s) ds = \alpha(c') - \alpha(c);$$

thus $[\alpha(c)] = [\alpha(c')]$.

R is certainly linear; it is bounded by Proposition 6 and surjective by Proposition 5, so we must only show that it is injective. Suppose $\omega = \alpha + dx \wedge \beta \in \mathcal{A}^*$ is closed and $R([\alpha]) = [0]$. Then $\alpha(c) = d_F\eta$ for some $\eta \in \mathcal{C}^\infty \Omega^{k-1}(F)$. By Proposition 5, $r^*\eta \in \mathcal{D}(d_{\max}, g, a)$, while Proposition 7 gives that $\int_c^x \beta(s) ds \in x^a L^2 \Omega^{k-1}(C_1(F)) \cap \mathcal{A}^*$. Thus

$$\begin{aligned} d \left(\eta + \int_c^x \beta(s) ds \right) &= d_F\eta + dx \wedge \beta(x) + \int_c^x d_F\beta(s) ds \\ &= \alpha(c) + dx \wedge \beta(x) + \int_c^x \alpha'(s) ds = \omega. \end{aligned}$$

Since $\omega \in x^a L^2 \Omega^k(C_1(F))$, this implies that $\eta + \int_c^x \beta(s) ds \in \mathcal{D}(d_{\max}, g, a)$, hence ω is exact in the maximal complex, i.e. $[\omega] = [0]$, as desired.

We next show that when $k \geq \frac{f+1}{2} - a$, any $[\omega] \in H_{\max}^k(C_1(F), g, a)$ is trivial. Thus, for any representative $\omega = \alpha + dx \wedge \beta \in \mathcal{D}(d_{\max}, g, a) \cap \mathcal{A}^*$, we must find a $(k-1)$ -form $\eta \in \mathcal{D}(d_{\max}, g, a) \cap \mathcal{A}^*$ with $\omega = d_{\max, a}\eta$. Assume $|\omega|_\kappa = O(x^p)$ for some p . The condition $\int_0^1 |\omega|_\kappa^2 x^{f-2k-2a} dx dV_\kappa < \infty$ gives that $p > a + k - \frac{f+3}{2}$. Furthermore, by assumption, $a + k - \frac{f+3}{2} > -1$, so $K_0(\omega) = \int_0^x \beta(s) ds$ is defined. Using $|\beta(s)|_\kappa = O(s^{p+1})$, we deduce that the integral

$$\|K_0\omega\|_{x^a L^2(C_1(F))}^2 = \int_0^1 \int_F \left| \int_0^x \beta(s) ds \right|_\kappa^2 x^{f-2k+2-2a} dx dV_\kappa$$

is finite. Now, $\alpha \in \mathcal{A}^* \cap x^a L^2 \Omega^k(C_1(F))$, so $|\alpha|_\kappa = O(x^q)$ where $2q + f - 2k - 2a > -1$, i.e. $q > 0$, so

$$d(K_0(\omega)) = dx \wedge \beta(x) + \int_0^x \alpha'(s) ds = \omega(x),$$

as desired. We have now shown that $K_0(\omega) \in \mathcal{D}(d_{\max}, g, a)$ and hence $[\omega] = 0$ in $H_{\max}^k(C_1(F), g, a)$. This completes the computation of $H_{\max}^*(C_1(F), g, a)$.

The computation of $H_{\min}^*(C_1(F), g, a)$ proceeds identically when $k \leq (f-1)/2 - a$ or $k \geq (f+3)/2 - a$, but the remaining cases are treated slightly differently.

Claim 1. *If $k \in ((f-1)/2 - a, (f+1)/2 - a)$ and $\omega = \alpha + dx \wedge \beta \in x^a L^2 \Omega^k(C_1(F)) \cap \mathcal{D}(d_{\min,a}) \cap \mathcal{A}^*$ is closed, then $\omega = d\eta$ for some $\eta \in \mathcal{D}(d_{\min,a}) \cap \mathcal{A}^*$. As a consequence, $H_{\min}^k(C_1(F), g, a) = 0$.*

Proof. As above, $\alpha' = d_F \beta$, so if $c, c' \in (0, 1)$,

$$\Pi_0(\alpha(c') - \alpha(c)) = \Pi_0 \int_c^{c'} \alpha'(s) ds = \Pi_0 d_F \int_c^{c'} \beta(x) dx = 0.$$

Since $\omega \in \mathcal{D}(d_{\min,a})$, $\Pi_0(\alpha(x)) = o(1)$, so $\Pi_0(\alpha(x)) = 0$ for all x . This gives that

$$d \int_c^x \beta(s) ds = dx \wedge \beta + \alpha(x) - \alpha(c) = \omega - \alpha(c) = \omega - \Pi_{\perp} \alpha(c) = \alpha - d(r^* \eta)$$

for some $\eta \in L^2 \Omega^{k-1}(F) \cap \mathcal{D}(d)$. Thus $\alpha = d(\eta + \int_c^x \beta(s) ds)$.

As for its domain, it suffices by Lemma 3 to show that $\eta + \int_c^x \beta(s) ds \in \mathcal{D}(d_{\max,a})$ since $k-1 < (f-1)/2 - a$. But on the one hand, $d(\eta + \int_c^x \beta(s) ds) = \omega \in x^a L^2 \Omega^k(C_1(F))$; furthermore, $(\eta + \int_c^x \omega(s) ds) \in L^2 \Omega^{k-1}(C(F), g, a)$ since $r^* \eta \in x^a L^2 \Omega^{k-1}(C_1(F))$, again because $k-1 < (f+1)/2 - a$, and $\beta = o(x^{(2k-2+2a-f-1)/2}) = o(x^p)$ for some $p > -1$, so $\int_c^x \beta(s) ds \in x^a L^2 \Omega^{k-1}(C_1(F))$. This proves the claim. \square

Claim 2. *The map $H_{\min}^k(C_1(F), g, a) \rightarrow H^k(F)$ is injective when $k \in ((f+1)/2 - a, (f+3)/2 - a)$; hence, for k in this range, $H_{\min}^k(C_1(F), g, a) \cong H_{\max}^i(C(F), g, a) \cong 0$.*

Proof. If $\omega = \alpha + dx \wedge \beta \in \mathcal{A}^*$ represents a class in $H_{\min}^k(C_1(F), g, a)$, then we already know that $\omega = d\eta$ for some $\eta \in \mathcal{D}(d_{\max,a}) \cap \mathcal{A}^*$. So we must show that we can arrange for η to lie in $\mathcal{D}(d_{\min,a})$ as well. Since $k-1 \in ((f-1)/2 + a, (f+1)/2 - a)$, $\beta(x) = o(x^p)$ for some $\nu > -1$, and so

$$\Pi_0 \int_c^0 \beta(s) ds$$

is defined. We can choose ζ which solves $\alpha(c) = d_F \zeta$ by specifying that $\Pi_0(\zeta) = -\Pi_0 \int_c^0 \beta(s) ds$. Now let $\eta = \zeta + \int_0^x \beta(s) ds$. As before, $\eta \in \mathcal{A}^*$ and $d\eta = \omega$. Furthermore, $\Pi_0(\eta(0)) = 0$ and $\partial_x(\Pi_0(\eta)) = \Pi_0(\beta(x)) = o(x^{(2(k-1)+2a-f-1)/2}) = o(x^p)$ for $p > -1$, so $\Pi_0(\eta) = o(1)$. This means that $\eta \in \mathcal{D}(d_{\min,a})$. \square

\square

This completes the calculation of the local cohomology for the sheaves $\mathcal{L}_{\max/\min,a}$ on X . By Proposition 4, we now obtain one of our main results:

Theorem 4. *If (M, g) is a manifold with an incomplete edge metric and X is the associated stratified space, then*

$$H_{\max}^*(M, g, a) = \begin{cases} IH_{\mathfrak{m}+\ll a-1 \gg}^*(X, B) & f \text{ odd} \\ IH_{\mathfrak{m}+\ll a-1/2 \gg}^*(X, B) & f \text{ even} \end{cases}$$

and

$$H_{\min}^*(M, g, a) = \begin{cases} IH_{\mathfrak{m}+\langle a \rangle}^*(X, B) & f \text{ odd} \\ IH_{\mathfrak{m}+\langle a-1/2 \rangle}^*(X, B) & f \text{ even} \end{cases} ;$$

here $\ll t \gg$ denotes the least integer strictly greater than t and $\langle t \rangle$ denotes the least integer greater than or equal to t .

There are two important special cases which we single out:

Corollary 5. *The maximal and minimal de Rham cohomologies at weight zero correspond to upper and lower middle perversity intersection cohomology.*

$$\begin{aligned} H_{\max}^*(M, g, 0) &= IH_{\overline{m}}^*(X) \\ H_{\min}^*(M, g, 0) &= IH_{\underline{m}}^*(X) \end{aligned} \tag{13}$$

Moreover, when f is even, the maximal and minimal de Rham cohomologies at weights $\pm 1/2$ coincide, and again correspond to upper and lower middle perversity intersection cohomology.

$$\begin{aligned} H_{\max/\min}^*(M, g, -1/2) &= IH_{\overline{m}}^*(X) \\ H_{\max/\min}^*(M, g, 1/2) &= IH_{\underline{m}}^*(X). \end{aligned} \tag{14}$$

From §2, these weighted de Rham cohomology spaces are identified with the nullspaces of the associated (absolute and relative) Laplacians, and we conclude that the nullspaces of $\Delta_{\text{abs/rel},a}$ are (finite dimensional and) identified with particular intersection cohomology spaces. Note that we are *not* asserting anything about the nullspaces of the ‘ordinary’ Laplacians $\Delta_{\text{abs/rel},0}$ on the weighted spaces $x^a L^2 \Omega^*$; indeed, these nullspaces are either infinite dimensional when $a \ll 0$ or vanish identically when $a \gg 0$, cf. [18].

4 Elliptic edge operators and minimal Hodge cohomology

To proceed further in the study of these weighted de Rham complexes, we must use elliptic methods. More specifically, we still wish to study the question of when there is a unique closed extension for d on $x^a L^2 \Omega^*(M)$ for incomplete edge metrics, and we also wish to compute the minimal Hodge cohomology. We shall study both of these questions using the formally symmetric operator $D_a = d + \delta_a$ on $x^a L^2 \Omega^*(M)$. The proper context for this analysis is the calculus of pseudodifferential edge operators, and in the next subsection we review the generalities of this theory. After that we show how it applies to the specific problems at hand.

4.1 Edge operators

We now review the general theory of elliptic edge operators. This is the correct context to study D_a for an incomplete edge metric (and also the corresponding operator for a complete edge metric). This theory is developed fully in [18], and we refer there for more details

Fix a local coordinate system (y_1, \dots, y_b) on B and (z_1, \dots, z_f) on F , so that $w = (x, y, z)$ is a local coordinate system in some neighbourhood of a singular stratum in X . By including the hypersurface $\{x = 0\}$, we are blowing up B in X ; the resulting manifold with boundary is denoted \overline{M} and its interior is denoted M .

A differential operator L on M is called an edge operator of order m if it can be expressed in the form

$$L = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha}(x, y, z) (x \partial_x)^j (x \partial_y)^\alpha \partial_z^\beta, \tag{15}$$

where the (scalar or matrix-valued) coefficients are bounded. We shall assume that these coefficients are smooth in these variables, down to $x = 0$. For example, if g is a *complete* edge metric, then the scalar or Hodge Laplacian is an operator of this type; similarly, if g is an *incomplete* edge metric, then its Laplacian is of the form $x^{-2}L$, where L is an edge operator of order 2.

4.1.1 Ellipticity and model operators

There is a well-defined symbol in this setting:

$$\sigma(L)(x, y, z; \xi, \eta, \zeta) := \sum_{j+|\alpha|+|\beta|=m} a_{j,\alpha}(x, y, z) \xi^j \eta^\alpha \zeta^\beta,$$

and we say that L is elliptic in the edge calculus provided $\sigma(L)(x, y, z; \xi, \eta, \zeta)$ is invertible when $(\xi, \eta, \zeta) \neq 0$.

Ellipticity alone does not guarantee that L is Fredholm between appropriate function spaces; one must also require that certain model operators for L also be invertible. There are two such operators:

- The *normal operator* of L is defined by

$$N(L) := \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(0, y, z) (s\partial_s)^j (s\partial_u)^\alpha \partial_z^\beta \quad (s, u) \in \mathbb{R}^+ \times \mathbb{R}^b;$$

here $y \in B$ enters only parametrically and the operator acts on functions on $\mathbb{R}^+ \times \mathbb{R}^b \times F$. This operator can be regarded as L with its coefficients frozen (in an appropriate sense) at $x = 0$, acting on functions (or sections of an appropriate bundle) on the space $\mathbb{R}_s^+ \times \mathbb{R}_u^b \times F_z$.

- The *indicial operator* of L is defined by

$$I(L) := \sum_{j+|\beta|\leq m} a_{j,0,\beta}(0, y, z) (s\partial_s)^j \partial_z^\beta.$$

For example, the normal and indicial operators associated to the scalar Laplacian for the complete edge metric $x^{-2}(dx^2 + h) + \kappa$ are

$$N(\Delta_g) = s^2 \partial_s^2 + (1-b) s \partial_s + s^2 \Delta_u + \Delta_\kappa, \quad I(\Delta_g) = s^2 \partial_s^2 + (1-b) s \partial_s + \Delta_\kappa.$$

The indicial operator captures some fundamental invariants associated to L :

Definition 5. *The number $\gamma \in \mathbb{C}$ is said to be an indicial root of L at $y_0 \in B$ if there exists a function $v(z)$ on F such that (in local coordinates where y_0 corresponds to $y = 0$)*

$$I(L)_{y_0}(s^\gamma v(z)) = \left(\sum_{j+|\beta|\leq m} a_{j,0,\beta}(0, 0, z) (s\partial_s)^j \right) s^\gamma v(z) = \mathcal{O}(s^{\gamma+1}).$$

Indicial roots may often be calculated in terms of eigenvalues for an induced elliptic operator on the fibre F_{y_0} , and might depend on $y_0 \in B$.

The operator L acts naturally on weighted Sobolev spaces. Let M be a manifold with complete edge metric G . For $\ell \in \mathbb{N}$ and $\delta \in \mathbb{R}$, define

$$x^\delta H_e^\ell(M) = \{u = x^\delta v : (x\partial_x)^j (x\partial_y)^\alpha \partial_z^\beta v \in L^2(M, dV_G) \forall j + |\alpha| + |\beta| \leq \ell\}.$$

(By interpolation and duality, these spaces can be defined for any $\ell \in \mathbb{R}$.) Clearly, if L is any edge operator of order m , then

$$L : x^\delta H_e^{\ell+m}(M) \longrightarrow x^\delta H_e^\ell(M) \quad (16)$$

for any δ, ℓ . Further hypotheses, beyond the ellipticity of L are required to ensure that this mapping is well-behaved.

The first instance of this is that the indicial roots of L yield weights δ for which (16) does not have closed range; these are precisely the weights δ for which an indicial root γ ‘just fails’ to lie in $x^\delta L^2$ near $x = 0$, i.e. where $x^\gamma \in x^{\delta-\epsilon} L^2$ for any $\epsilon > 0$ but $x^\gamma \notin x^\delta L^2$. We denote this critical weight δ associated to a given indicial root γ as $\delta(\gamma)$. With respect to the measure $dx dy dz$, $\delta(\gamma) := \operatorname{Re} \gamma + 1/2$. However, the measure appearing in our application below is $x^{f-2a} dx dy dz$, and $x^\gamma \in x^\delta L^2(x^{f-2a} dx dy dz)$ near $x = 0$ if and only if $\delta < \gamma + (f+1)/2 - a$, so that we shall define $\delta(\gamma) = \operatorname{Re} \gamma + (f+1)/2 - a$.

Even when δ is not equal to one of these critical values, the behaviour of the normal operator at weight δ plays another very important role.

Proposition 8. *Let L be an elliptic differential edge operator of order m . Fix δ such that $\delta \neq \delta(\gamma(b))$ for any indicial root $\gamma(b)$, $b \in B$. Suppose also that $N(L) : s^\delta H_e^m \rightarrow s^\delta L^2$ is surjective (for all $b \in B$). Then (16) is essentially surjective, in the sense that its range is closed and of finite codimension. On the other hand, if $N(L)$ is injective on $s^\delta L^2$, then any element of the nullspace of L is necessarily conormal.*

There are many more refined statements one can make about the mapping properties of L . For later applications, we state only one very special result. We shall restrict to a special setting, which is what arises in our applications below. The hypothesis that the the normal operator $N(L)$ is surjective is equivalent to the injectivity of the normal operator for the adjoint L^* . This adjoint depends on the choice of measure, and we shall assume (as in our applications) that the adjoint of L on $x^\delta L^2$ corresponds to the same operator L on a ‘dually weighted’ space $x^{\delta^*} L^2$, for some $\delta^* > \delta$. The fact that L has closed range implies the existence of a generalized inverse $G : x^\delta L^2 \rightarrow x^\delta H_e^m$ which satisfies $LG = I - P$, where P is the orthogonal projector onto the cokernel. By duality, elements of this cokernel are identified with elements of the nullspace of L on $x^{\delta^*} L^2$, and by the result above, these are conormal.

Proposition 9. *Let L satisfy the special assumptions of the preceding paragraph. Suppose furthermore that the interval (δ, δ^*) contains a finite set of indicial roots γ_j , $j = 1, \dots, N$, all of which are constant in $b \in B$. Let $f \in x^{\delta^*} L^2 \cap \mathcal{A}^*$. Then $u = Gf$ satisfies $Lu = f - \phi$ where $\phi \in x^{\delta^*} L^2 \cap \mathcal{A}^*$, $L\phi = 0$, and $u = \sum_{j=1}^N u_j(y, z) x^{\gamma_j} + v$; where each $u_j(y, z) \in \mathcal{C}^\infty$ solves the indicial equation $I(L)(s^{\gamma_j} u_j(y, z)) = 0$, and the error term $v \in \mathcal{A}^* \cap x^{\delta^*} L^2$.*

The proofs rely on the construction of a pseudodifferential parametrix G for L , depending on δ . This is an element in the calculus of pseudodifferential edge operators $\Psi_e^*(M)$. We do not define this calculus here, but remark only that these operators are described by specifying the precise asymptotic behaviour of their Schwartz kernels, near the diagonal and also near the boundaries and corner of $M \times M$. We refer as before to [18]. (We should note also that the results stated here are slightly more general than what is written explicitly in that source because we are allowing the possibility of variable indicial roots outside the critical interval; however, these can be derived easily from the same techniques.)

4.2 Edge analysis of D_a

We now proceed to apply the methods of the last subsection to the analysis of D_a . The first tasks are the calculation of the indicial roots of D_a and the analysis of the normal operator $N(xD_a)$.

4.2.1 Indicial roots of D_a

For simplicity, first consider the calculation of the indicial roots for D_a for the metric $g = dx^2 + x^2\kappa$ on the cone $C_1(F)$. To do this, we decompose this operator as much as possible. Thus, first regard D_a as a 2×2 matrix acting on pairs $(\alpha, \beta) \leftrightarrow \alpha + dx \wedge \beta$, where $\alpha(x), \beta(x) \in \Omega^*(F)$ for each x ; normalize by writing the k -form part of α as $x^k\alpha_k$, and similarly for β . A short calculation then shows that, acting on pairs (α_k, β_k) ,

$$I(xD_a)|_{\Omega^k \oplus \Omega^k} = \begin{pmatrix} \frac{1}{x}D_F & -\partial_x - \frac{f-k-2a}{x} \\ \partial_x + \frac{k}{x} & -\frac{1}{x}D_F \end{pmatrix};$$

the full indicial operator is the direct sum over k of these matrices. Similarly, the indicial family is the direct sum of matrices

$$I(xD_a)_k(\gamma) = \begin{pmatrix} D_F & -\gamma - (f - k + 2a) \\ \gamma + k & -D_F \end{pmatrix} \quad (17)$$

This can be reduced further using the eigendecomposition for Δ_F . In particular, we see that the operator in (17) is noninvertible if and only if for some eigenvalue λ^2 for Δ_F ,

$$\begin{pmatrix} \lambda & -\gamma - (f - k - 2a) \\ \gamma + k & -\lambda \end{pmatrix}$$

is singular, or equivalently

$$\gamma^2 + (f - 2a)\gamma + k(f - k - 2a) - \lambda^2 = 0.$$

Hence the indicial roots come in pairs:

$$\gamma_{\lambda,k}^{\pm} = a - \frac{f}{2} \pm \frac{1}{2} [(f - 2a - 2k)^2 + 4\lambda^2]^{1/2}. \quad (18)$$

The extension of these calculations to incomplete edge metrics requires only some mild alterations. Write $g = dx^2 + \tilde{g}(x)$, where $\tilde{g}(x) = h + x^2\kappa$ is a degenerating family of metrics on $Y = \partial M$. When α is a (p, q) -form on Y , its pointwise norm satisfies

$$|\alpha|_{\tilde{g}(x)} = x^{-q} |\alpha|_{\tilde{g}(1)}.$$

Denote by D_Y^x the operator D on Y for the metric G_x . By Proposition 3, with respect to the metric $\tilde{g}(1)$ on Y ,

$$d_Y = d_F + \tilde{d}_B - \mathbb{I} + \mathbb{R}, \quad \delta_Y = \delta_F + (\tilde{d}_B)^* - \mathbb{I}^* + \mathbb{R}^*;$$

hence, $D_Y = D_F + \overline{D}_B + \overline{R}$ where

$$\overline{D}_B = \tilde{d}_B + (\tilde{d}_B)^* - \mathbb{I} - \mathbb{I}^*, \quad \text{and} \quad \overline{R} = \mathbb{R} + \mathbb{R}^*.$$

A quick review of the definitions shows that

$$D_Y^x = \frac{1}{x} D_F + \overline{D}_B + x \overline{R},$$

where all the components on the right are the corresponding operators at $x = 1$.

Now let $Z \rightarrow B$ be the bundle with fibre $C_1(F)$ obtained from $Y \rightarrow B$, with metric $g = dx^2 + \tilde{g}(x)$. Decompose any form $\omega = \alpha + dx \wedge \beta$ on Z as $\alpha = \sum x^k \alpha_k$, where α_k is of type $(*, k)$ on Y , and similarly for β . Thus the pointwise norms satisfy $|\omega|_g^2 = \sum \left(|\alpha_k|_{\tilde{g}(1)}^2 + |\beta_k|_{\tilde{g}(1)}^2 \right)$. In terms of these decompositions and normalizations, the restriction of the operator D_a to pairs of $(*, k)$ -forms on Z is given by

$$D_a = \begin{pmatrix} \frac{1}{x} D_F + \overline{D}_B + x \overline{R} & -\partial_x - \frac{f-k-2a}{x} \\ \partial_x + \frac{k}{x} & -\left(\frac{1}{x} D_F + \overline{D}_B + x \overline{R} \right) \end{pmatrix}. \quad (19)$$

From this expression, we see that neither \overline{D}_B nor \overline{R} appear in the indicial operator $I(xD_a)$. Hence the computation of the indicial roots is exactly the same as in the conic case; in other words, all indicial roots are of the form (18). Note, however, that the eigenvalues λ^2 may depend on $b \in B$, hence the same may be true of these indicial roots.

Notice that if $\omega \in \Omega^k$ and $|\omega|_g \sim x^\gamma$, then $\omega \in x^a L^2 \Omega^k$ (near $x = 0$) if and only if $\gamma > a - f/2$. The indicial roots which lie near to (and above) this ' $x^a L^2$ cutoff' are the ones which cause the difference between minimal and maximal domains. We explain this later, but for now record the

Corollary 6. *The indicial roots of the operator D_a contained in the interval $(a - (f + 1)/2, a - (f - 1)/2)$ correspond to the eigenvalues λ^2 of Δ_F on k -forms such that $(f - 2a + 2k)^2 + 4\lambda^2 < 1$. In order for this condition to be nonvacuous, it is necessary that $k \in ((f - 1)/2 - a, (f + 1)/2 - a)$. Note in particular that $\gamma_{0,k}^\pm = a - f/2 \pm (f/2 - a - k) = -k, k + 2a - f \in (a - (f + 1)/2, a - (f - 1)/2)$ precisely when $k \in ((f - 1)/2 - a, (f + 1)/2 - a)$ and $H^k(F) \neq 0$.*

4.2.2 The normal operator of D_a

According to the discussion in the final paragraphs of §4.1, we must also study the mapping properties of the normal operator $N(xD_a)$. Before doing so, we address some 'duality' issues. The main point is that D_a is formally symmetric on $x^a L^2 \Omega^*$, hence the adjoint of the (closed range) operator

$$D_a : x^a L^2 \Omega^* \longrightarrow x^{a-1} L^2 \Omega^* \quad (20)$$

is identified with

$$D_a : x^{a+1}L^2\Omega^* \longrightarrow x^aL^2\Omega^*. \quad (21)$$

Recall also that, according to the computations of the preceding subsection, there are at most two indicial roots γ_0^\pm in the interval $(a - (f + 1)/2, a - (f - 1)/2)$, and these are symmetric around the midpoint $a - f/2$. We shall assume that the metric g is such that Δ_F has no small nonzero eigenvalues, so that no other indicial roots intersect the closed interval $[a - (f + 1)/2, a - (f - 1)/2]$.

By (19),

$$N(xD_a) = sD_{C(F),a} + sD_{\mathbb{R}^b}, \quad (22)$$

where the first operator on the right is the analogous weighted operator on the complete cone $C(F)$ and the second is on Euclidean space. In fact, the identification of $s^{-1}N(xD_a)$ with $D_{C(F),a} + D_{\mathbb{R}^b}$ may also be seen by naturality, since the operator on the left must equal, at $b \in B$, the Hodge-de Rham operator on $\mathbb{R}_s^+ \times \mathbb{R}_u^b \times F$ with respect to $g_b = ds^2 + s^2|du|^2 + \kappa_b$.

Proposition 10.

$$N(xD_a) : s^{a+1}L^2\Omega^* \longrightarrow s^{a+1}L^2\Omega^*$$

is injective, and hence (21) has a finite dimensional nullspace consisting of conormal forms

Proof. Suppose that $\omega \in s^\alpha L^2\Omega^*$ is in the nullspace of this operator. Take the Fourier transform in the u direction; denoting the dual variable by η , then whenever $\eta \neq 0$ we can rescale, setting $t = s|\eta|$, $\hat{\eta} = \eta/|\eta|$. Then

$$(D_{C(F),a} + i \operatorname{cl}(\hat{\eta})) \hat{\omega} = 0,$$

where $\operatorname{cl}(\hat{\eta})$ is Clifford multiplication $\hat{\eta} \wedge \cdot + \iota(\hat{\eta}) \cdot$. Apply $D_{C(F),a} + i \operatorname{cl}(\hat{\eta})$ to this equation to deduce that

$$(\Delta_{C(F),a} + 1) \hat{\omega} = 0.$$

It is not hard to show, cf. [18], that any solution of this equation either grows or decays exponentially as $t \rightarrow \infty$, and the L^2 hypothesis prohibits the former. Furthermore, solutions are polyhomogeneous as $t \rightarrow 0$, and hence decay at some indicial weight t^γ with $\gamma > a - (f - 1)/2$. Hence both $N(d)\hat{\omega}$ and $N(\delta_a)\hat{\omega}$ decay like $t^{\gamma-1}$, and in particular are still in $t^\alpha L^2$. This means we can integrate by parts to obtain

$$0 = \langle (\Delta_{C(F),a} + 1)\hat{\omega}, \hat{\omega} \rangle_a = \|N(d)\hat{\omega}\|_a^2 + \|N(\delta_a)\hat{\omega}\|_a^2 + \|\hat{\omega}\|_a^2;$$

all boundary terms vanish. This gives $\hat{\omega} = 0$, as desired.

When $\eta = 0$, the problem reduces to showing that $\Delta_{C(F),a}$ has no nullspace in $t^{a+1}L^2$ on the entire cone $C(F)$, which is even more easily verified to be true (e.g. by separation of variables). \square

Following the discussion from the end of §4.1, if $(a - (f + 1)/2, a - (f - 1)/2)$ contains no indicial roots for D_a , $N(xD_a)$ is injective on $s^a L^2\Omega^*$.

There are no forms in the nullspace of the normal operator which lie in $t^a L^2\Omega^* \cap \mathcal{D}(d_{\max,a}) \cap \mathcal{D}(\delta_{\min,a})$ or $t^a L^2\Omega^* \cap \mathcal{D}(d_{\min,a}) \cap \mathcal{D}(\delta_{\max,a})$. From this it is possible to show

that $\omega \in \mathcal{H}_{\text{abs/rel}}^*(M, g, a) \subset \mathcal{A}^*\Omega^*$, i.e. such harmonic forms are conormal. Unfortunately, these results rely on a slightly more elaborate parametrix construction than is available in the literature, so at present we are only asserting this informally. At the end of the next subsection, however, we show that forms in $\mathcal{H}_{\text{min}}^*(M, g, a)$ are conormal.

4.3 Closed extensions of D_a

It is a general fact that closed extensions of d on $x^a L^2\Omega^*$ are in bijective correspondence with the self-adjoint extensions of D_a on this space. To see this, first note that if \bar{d} is any closed extension of d , and if \bar{d}_a^* is its Hilbert space adjoint, then $\overline{D_a} = \bar{d} + \bar{d}_a^*$ is a self-adjoint extension of D_a . Conversely, any self-adjoint extension of D_a determines an associated closed extension for d , cf. [2, Lemma 2.3]. We summarize this in the

Proposition 11. *If d has more than one closed extension on $x^a L^2\Omega^*(M, g)$, then D_a has more than one self-adjoint extension on this space; equivalently, if D_a is essentially self-adjoint, then $d_{\text{max},a} = d_{\text{min},a}$.*

Notice that since d always has closed extensions, D_a always has self-adjoint extensions. However, if D_a is not essentially self-adjoint, then it will have closed extensions which are not necessarily self-adjoint; the relationship between these and the closed extensions of d is somewhat more complicated, and we shall not attempt to describe it.

Theorem 7. *The symmetric operator D_a is essentially self-adjoint on $x^a L^2\Omega^*(M, g)$ if and only if there exists no indicial root for D_a in the interval $(a - (f+1)/2, a - (f-1)/2)$. As explained earlier, this is equivalent to the nonexistence of small eigenvalues λ^2 for Δ_F on k -forms such that $k \in (f/2 - a - \frac{1}{2}\sqrt{1-4\lambda^2}, f/2 - a + \frac{1}{2}\sqrt{1-4\lambda^2})$. In this case d also has a unique closed extension.*

Proof. Assume that there are no small eigenvalues, as described in the statement of the theorem. Fix a parametrix G for D_a relative to the space $x^a L^2\Omega^*$. This is an element of order -1 in the edge calculus. If $\omega \in \mathcal{D}(D_{\text{max},a})$, then $f = D_a\omega \in x^a L^2\Omega^*(M, g)$, and applying G gives that in fact $\omega \in x^{a+1}H_e^1\Omega^*$. We recall that in general, Gf would be the sum of two terms, the first corresponding to these small indicial roots and the second an error term in $x^{a+1}H_e^1\Omega^*$, but by our hypothesis, the former of these is absent. It is now straightforward to check that ω may be smoothly approximated in the D_a -graph norm, i.e. that there exists a sequence of smooth compactly supported forms ϕ_j such that $\phi_j \rightarrow \omega$, $D_a\phi_j \rightarrow D_a\omega$ in $x^a L^2\Omega^*$. This shows that $\omega \in \mathcal{D}(D_{\text{min},a})$.

Conversely, if there do exist indicial roots in the critical range, then these may be used to construct nontrivial elements in $\mathcal{D}(D_{\text{max},a}) \setminus \mathcal{D}(D_{\text{min},a})$, and by the general abstract theory, there will be more than one self-adjoint extension of D_a . Because this is not central to our discussion, we leave details to the reader (and refer to [10] for a thorough discussion of the conic case). \square

In the conic case (when $a = 0$) this result is due to Cheeger [3]; cf. also [16]. The analysis needed in that case is simpler than the edge analysis used here, though this is not apparent ‘on the surface’.

We turn now to a description of $\mathcal{D}(D_{\text{max},a})$ in the more general case where this vanishing condition is no longer satisfied. By definition, this domain is the set of all

$\omega \in x^a L^2 \Omega^*$ such that $D_a \omega \in x^a L^2 \Omega^*$. We have already remarked that the mapping (20) has closed range, which we denote \mathcal{R}_a . Its cokernel is finite dimensional since, by Proposition (10), the adjoint mapping (21) has a finite dimensional nullspace. Hence we can choose a generalized inverse

$$G_a : x^{a-1} L^2 \Omega^*(M, g) \longrightarrow x^a L^2 \Omega^*(M, g);$$

this is a pseudodifferential edge operator of order -1 which satisfies $D_a G_a = I - P$ on $x^{a-1} L^2 \Omega^*$, where P is the orthogonal projector onto the cokernel. The condition $f \in \mathcal{R}_a$ is equivalent to $\langle f, \gamma \rangle = 0$ for all γ in the nullspace of (21). Notice also that $\mathcal{R}_a \cap x^a L^2 \Omega^*$ is dense in \mathcal{R}_a . Therefore,

$$\mathcal{D}(D_{\max, a}) = G_a (\mathcal{R}_a \cap x^a L^2 \Omega^*) + \mathcal{N}_a,$$

where \mathcal{N}_a is the nullspace of (20).

Finally, if $\gamma \in \mathcal{H}_{\min}^*(M, g, a)$, then $\langle D_{\max, a} \eta, \gamma \rangle = 0$ for all $\eta \in \mathcal{D}(D_{\max, a})$, i.e. $\langle f, \gamma \rangle = 0$ for all $f \in \mathcal{R}_a \cap x^a L^2 \Omega^*$. Using the density statement above, this shows that γ lies in the nullspace of (21), and is thus conormal.

We note in conclusion that any $\omega \in \mathcal{D}(D_{\max, a})$ has a ‘weak’ asymptotic expansion of the form

$$\omega \sim \omega_0^-(y, z) x^{\gamma_0^-} + \omega_0^+(y, z) x^{\gamma_0^+} + \omega'$$

where $\omega_0^\pm \in \mathcal{D}'(Y)$ and $\omega' = \mathcal{O}(x^{a+1})$, but only when paired with any test form on Y . We refer to [18] for more details.

4.4 The minimal Hodge cohomology

We are now in a position to prove the

Theorem 8. *Let M be a manifold with an incomplete edge metric g . Then*

$$\mathcal{H}_{\min}^k(M, g, a) = \begin{cases} \text{Im} (IH_{\underline{m}+<a>}^k(X, B) \rightarrow IH_{\underline{m}+\ll a-1 \gg}^k(X, B)) & f \text{ odd} \\ \text{Im} (IH_{\underline{m}+<a-1/2>}^k(X, B) \rightarrow IH_{\underline{m}+\ll a-1/2 \gg}^k(X, B)) & f \text{ even.} \end{cases}$$

In particular, when $a = 0$,

$$\mathcal{H}_{\min}^k(M, g, 0) = \text{Im} (IH_{\underline{m}}^k(X, B) \longrightarrow IH_{\underline{m}}^k(X, B)).$$

Proof. Recalling that \mathcal{H}_{\min}^* is quasi-isometry invariant, we may as well assume that Δ_F has no small nonzero eigenvalues, in the sense of the preceding subsections. If there are no small eigenvalues at all, i.e. either when $(a - (f + 1)/2, a - (f - 1)/2) \cap \mathbb{N} = \emptyset$ (which holds, for example, when $a = 0$ and f is odd), or else if there exists $q_a \in (a - (f + 1)/2, a - (f - 1)/2)$ but $H^{q_a}(F) = \{0\}$, then the result follows directly from what we have already done, since then $H_{\max/\min}^k(M, g, a)$ and $\mathcal{H}_{\max/\min/\text{abs}/\text{rel}}^k(M, g, a)$ are all equal, cf. Corollary 3. Thus we suppose that there exists $q_a \in (a - (f + 1)/2, a - (f - 1)/2)$ such that $H^{q_a}(F) \neq \{0\}$.

According to Theorem 4, the space appearing on the right in the statement of this theorem, for f even or odd, is identified with $\text{Im} (H_{\min}^k(M, g, a) \rightarrow H_{\max}^k(M, g, a))$. For simplicity, we denote it as $\mathcal{J}^k(M, g, a)$.

We claim first that there is a natural injective map

$$\mathcal{H}_{\min}^k(M, g, a) \longrightarrow \mathcal{J}^k(M, g, a).$$

To see this, recall that any form $\omega \in \mathcal{H}_{\min}^k(M, g, a) = \mathcal{H}_{\text{abs}}^k(M, g, a) \cap \mathcal{H}_{\text{rel}}^k(M, g, a)$ naturally represents a class in $H_{\min}^k(M, g, a)$. If $[\omega] = 0$ in $H_{\max}^k(M, g, a)$, then $\omega = d\zeta$ for some $\zeta \in \mathcal{D}(d_{\max, a})$. But this would imply that $\|\omega\|_a^2 = \langle \omega, d\zeta \rangle_a = 0$ since $\omega \in \ker(\delta_{\min, a})$. This proves the claim.

The issue, then, is to prove that any class $[\eta] \in \mathcal{J}^k(M, g, a)$ is represented by an element of $\mathcal{H}_{\min}^k(M, g, a)$. Choose a representative $\eta \in \mathcal{A}^\gamma \Omega^k$, $\gamma > a - (f - 1)/2$, for this class. Now use a generalized inverse G for $D_a : x^a L^2 \Omega^*(M, g) \rightarrow x^{a-1} L^2 \Omega^*(M, g)$ (acting on its maximal domain). According to Propositions 8 and 9, this gives $\zeta \in x^a L^2 \Omega^* \cap \mathcal{A}^{\gamma_0^-}$ and an element of the cokernel, which by duality corresponds to an element $\omega \in \ker D_a \cap x^{a+1} L^2 \Omega^* = \mathcal{H}_{\min}^k(M, g, a)$, such that $\eta = D_a \zeta + \omega$. (Recall that γ_0^- is the lower of the two indicial roots in the critical interval.) The theorem will follow once we show that $\delta_a \zeta = 0$. Now

$$\|\delta_a \zeta\|_a^2 = \langle \delta_a \zeta, \eta - d\zeta - \omega \rangle_a.$$

Integrating by parts formally this should vanish, so it remains to show that each integration by parts is valid. First, $\langle \delta_a \zeta, \eta \rangle_a = 0$ since both terms are conormal and $d\eta = 0$. Similarly $\langle \delta_a \zeta, \omega \rangle_a = 0$ since $\omega \in \ker d_{\min, a}$. To show that the remaining term vanishes, observe that $\zeta = x^{\gamma_0^-} \zeta_0^- + x^{\gamma_0^+} \zeta_0^+ + \zeta'$ where $\zeta' \in \mathcal{A}^\gamma \Omega^*$, $\gamma > a - (f - 1)/2$. Writing $\zeta_0^\pm = \mu_0^\pm + dx \wedge \nu_0^\pm$, then $\mu_0^\pm, \nu_0^\pm \in \mathcal{C}^\infty$ and in the nullspace of Δ_F . A closer inspection of the equation $D_a \zeta = \eta - \omega$ shows that

$$\mathfrak{d}\mu_0^\pm = \mathfrak{d}\nu_0^\pm = \mathfrak{d}^* \mu_0^\pm = \mathfrak{d}^* \nu_0^\pm = 0.$$

(The operators \mathfrak{d} , \mathfrak{d}^* are the differential and codifferential for the (fibre-harmonic) projected complex for the Riemannian submersion metric $(Y, \tilde{g}(x))$.) In any case, the identity $\langle d\zeta, \delta_a \zeta \rangle_a = 0$ is now immediate. This proves the remaining assertion, and hence the theorem. \square

5 Hodge theory for complete edge metrics

We are also able to determine the dimensions of the spaces of the spaces $L^2 \mathcal{H}^k(M, g)$ when (M, g) is a manifold with a complete edge metric. Unlike the incomplete case, this Hodge cohomology may well be infinite dimensional in certain degrees, so that there will be an infinite dimensional space of L^2 harmonic forms in certain cases. The simplest example of this is when (M, g) is the n -dimensional hyperbolic space, or indeed any conformally compact manifold, and $k = n/2$, cf. [20].

Theorem 9. *Let (M^n, g) be a manifold with a complete edge metric. Let X be the compact stratified space defined in §3. Suppose that k is **not** of the form $j + (b + 1)/2$ where $\mathcal{H}^j(F) \neq \{0\}$. Then*

$$L^2 \mathcal{H}^k(M, g) \cong IH_{f+\frac{b}{2}-k}^k(X, B).$$

In all other cases, where k does not have this form, $L^2\mathcal{H}^k(M, g)$ is infinite dimensional.

In this complete setting, if $n = 4\ell$ and $L^2\mathcal{H}^{2\ell}(M, g)$ is not infinite dimensional, the L^2 signature for the complete metric is equal to the L^2 signature theorem for M endowed with the conformally equivalent incomplete edge metric x^2g , and hence is given by the same topological data as in that case.

Proof. There are several viable ways to proceed: one could use a parametrix construction based on the edge calculus to do a global Hodge theoretic argument as in [20]; one could also, as in the incomplete case, use sheaf theory, calculations of local cohomologies, etc., ab initio; we take a shorter intermediate route, reducing to the incomplete edge case using the conformal invariance of the space of middle degree L^2 harmonic forms.

It suffices to consider forms of degree $k \geq n/2$. Define $k = (n + r)/2$ with $0 \leq r \leq n$. For later use, we also set $\sigma = (f + r)/2$, $\sigma' = (f - r)/2$. Now, since $2k = n + r$, k is the middle degree on the manifold $\widetilde{M} = M \times S^r$. Endow \widetilde{M} with the product metric $G_r = g + \kappa_r$ (where κ_r is the standard metric on S^r); in a neighbourhood of $\partial\widetilde{M}$

$$G_r = \frac{dx^2 + h}{x^2} + (\kappa + \kappa_r).$$

In other words, G_r is still a complete edge metric with the same base (B, h) as g , but with fibre $(F \times S^r, \kappa + \kappa_r)$. There is a corresponding *incomplete* edge metric $\hat{g} = x^2\tilde{g}$ on \widetilde{M} . We denote its compactification, obtained by pinching the fibres $F \times S^r$ at the boundary, by \widehat{X} .

We first claim that

$$L^2\mathcal{H}^k(\widetilde{M}, \tilde{g}) = \mathcal{H}_{\max}^k(\widetilde{M}, \hat{g}, 0). \quad (23)$$

The verification is straightforward; by conformal invariance of the L^2 condition and the operator $\delta = \pm * d*$ in the middle degree, harmonic forms on the complete manifold are in the maximal domains of d and δ (and indeed their nullspaces) on the incomplete manifold, and conversely.

By the L^2 Künneth theorem,

$$L^2\mathcal{H}^k(\widetilde{M}, \tilde{g}) = L^2\mathcal{H}^k(M, g) \oplus L^2\mathcal{H}^{k-r}(M, g).$$

On the other hand, while we have not shown how to compute $\mathcal{H}_{\max}^k(\widetilde{M}, \hat{g}, 0)$ in general, and indeed have noted that it is sometimes infinite dimensional, we now show that under certain hypotheses, it equals $\mathcal{H}_{\min}^k(\widetilde{M}, \hat{g}, 0)$; thus we can then apply Theorem 3 to calculate the right side of (23) as $IH_{\mathfrak{m}}^k(\widehat{X})$. (Under these hypotheses, the intersection cohomologies with upper or lower middle perversity are the same, so we just write \mathfrak{m} .)

So, let us suppose that either b is even, or else if b is odd then $H^\sigma(F)$ (and hence $H^{\sigma'}(F)$) is trivial; note this last condition is automatic when $r > f$. We claim that under these conditions, the minimal and maximal de Rham complexes on \widetilde{M} coincide, and thus $\mathcal{H}_{\max}^k(\widetilde{M}, \hat{g}, 0) = \mathcal{H}_{\min}^k(\widetilde{M}, \hat{g}, 0)$. Indeed, this follows directly from Corollary 3: we have $\dim M = \tilde{n} = 2k$ even; if b is even, then $\tilde{f} = f + r = \tilde{n} - b - 1$ is odd, while if b is odd, then

$$H^{\tilde{f}/2}(F \times S^r) = H^\sigma(F) \oplus H^{\sigma'}(F) = \{0\}.$$

Taking these facts together, and assuming this vanishing of the fibre cohomology when b is odd, we have proved that

$$L^2\mathcal{H}^k(M, g) \oplus L^2\mathcal{H}^{k-r}(M, g) = IH_{\mathfrak{m}}^k(\tilde{X}). \quad (24)$$

It remains to compute the final term on the right of (24). We decompose

$$\tilde{X} = M \times S^r \sqcup_{Y \times S^r} \tilde{Z},$$

where \tilde{Z} is the cone bundle over B with fibre $F \times S^r$ and boundary $Y \times S^r$. Of course,

$$H^k(M \times S^r) = H^k(M) \oplus H^{k-r}(M).$$

On the other hand,

$$IH_{\mathfrak{m}}^\ell(\tilde{Z}) = IH_{\mathfrak{m}}^\ell(Z) \oplus IH_{\mathfrak{m}}^{\ell-r}(Z).$$

To prove this, we return to the sheaf-theoretic description. For product neighbourhoods $\mathcal{U} = \mathcal{V} \times C_1(F \times S^r)$, we have

$$IH_{\mathfrak{m}}^j(\mathcal{U}; \mathcal{L}) = IH_{\mathfrak{m}}^j(C_1(F \times S^r)) = \begin{cases} H^j(F) \oplus H^{j-r}(F), & j \leq (f+r-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have combined the conditional inequality in this last step, which should depend on the parity of $f+r$, into one condition. This condition is correct as stated if $f+r$ is odd; if $f+r$ is even, then one would expect the condition $j \leq (f+r)/2 - 1 = \sigma - 1$ or $j \leq (f+r)/2 = \sigma$, depending on whether one was using upper or lower middle perversity. However, the hypothesis $H^\sigma(F) = 0$ guarantees that we get the same result in either case. Thus we see that the spectral sequence whose hypercohomology computes the intersection cohomology of \tilde{Z} decouples into two noninteracting pieces.

We have now proved that for a fixed k , assuming the hypotheses above,

$$L^2\mathcal{H}^k(M) \oplus L^2\mathcal{H}^{k-r}(M) = IH_{\mathfrak{m}}^k(X) \oplus IH_{\mathfrak{m}}^{k-r}(X).$$

We would, of course, like the summands to be equal separately; this can be seen simply by noting that the correspondence takes place on the level of forms, and we can separate out the terms with like degree.

It remains to show that in the remaining cases, i.e. when b is odd and $H^\sigma(F) \neq \{0\}$, $\sigma = (2k - b - 1)/2$, $L^2\mathcal{H}^k(M)$ is infinite dimensional. This follows from two assertions: that 0 is in the essential spectrum of Δ_k , and that there is a spectral gap at 0 for Δ_k . We content ourselves with sketching the proofs briefly. The first step relies on the observation that up to quasiisometry, some neighbourhood of infinity looks like the product of half of a hyperbolic space and a compact manifold, specifically $\mathbb{H}_+^{b+1} \times F$. Since $b+1$ is even, there is an infinite dimensional family of L^2 harmonic forms on the first factor. Since $k = (b+1)/2 + \sigma$, we can take suitable truncations of these, coupled with harmonic forms of degree σ on F , to produce a Weyl sequence on M . For the second step, we construct a parametrix in the edge calculus for Δ_k . Its normal operator is given by

$$N(\Delta_k) = \sum_{j=0}^k N(\Delta_{\mathbb{H}^{b+1}, j}) + \Delta_{F, k-j}.$$

Both of these operators have spectral gaps at zero (even when $j = (b + 1)/2$), so it is possible to construct a parametrix for $\Delta_k - \lambda$ with compact remainder when λ is small but nonzero. We refer to [17] and [20] for more complete descriptions of such proofs in a slightly simpler context. \square

References

- [1] A. Borel et al., *Intersection Cohomology*, Birkhauser (1994).
- [2] J. Brüning and M. Lesch, *Hilbert Complexes*, J. Func. Anal. **108**, 88-132, (1992).
- [3] J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. **76** (1979), 2103-2106.
- [4] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds in Geometry of the Laplace operator* Proc. Symp. Pure Math Vol. XXXVI, Amer. Math. Soc., Providence, (1980),91-146.
- [5] J. Cheeger and X. Dai, *L^2 cohomology of a non-isolated conical singularity and nonmultiplicativity of the signature*. Preprint.
- [6] J. Cheeger, M. Goresky and R. McPherson, *L^2 -cohomology and intersection homology of singular algebraic varieties* in *Seminar on Differential Geometry*, Annals of Mathematical Studies 102, Princeton Univ. Press, Princeton, N.J., (1982), 303–340.
- [7] X. Dai, *Adiabatic limits, nonmultiplicativity of signature, and the Leray spectral sequence*, J. Amer. Math. Soc. **4** (1991), no. 2, 265–321.
- [8] G. deRham, *Differentiable manifolds*. Grundlehren der Mathematischen Wissenschaften, 266, Springer-Verlag, Berlin, (1984).
- [9] M. P. Gaffney, *A special Stokes's theorem for complete Riemannian manifolds*, Ann. Math. **60** no. 2, (1954), 140–145.
- [10] J. Gil and G. Mendoza, *Adjoint of elliptic cone operators*, Amer. J. Math **125** (2003), 357–408.
- [11] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology, **19** (1980), no. 2, 135–162.
- [12] M. Goresky and R. MacPherson, *Intersection homology II*, Invent. Math. **72** no. 1 (1983), 77–129.
- [13] T. Hausel, E. Hunsicker, and R. Mazzeo, *Hodge cohomology of gravitational instantons*. Duke Math. J. **122** no. 3 (2004), 485–548.
- [14] C. Hodgson, S. Kerckhoff *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Diff. Geom. **48** (1998), 1–59.
- [15] E. Hunsicker, *Hodge and signature theorems for a class of manifolds with fibration boundary*, in preparation.

- [16] M. Lesch, *Differential Operators of Fuchs Type, Conical Singularities, and Asymptotic Methods*. Teubner Texte für Mathematik **136**, Teubner-Verlag, Leipzig (1997).
- [17] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Diff. Geom. **28** (1988), 309–339.
- [18] R. Mazzeo *Elliptic theory of differential edge operators I*, Comm. PDE. **16** (1991), no. 10, 1616–1664.
- [19] R. Mazzeo and R. Melrose, *Pseudodifferential operators on manifolds with fibred boundaries* in “Mikio Sato: a great Japanese mathematician of the twentieth century.”, Asian J. Math. **2** (1998), no. 4, 833–866.
- [20] R. Mazzeo and R. S. Phillips, *Hodge theory on hyperbolic manifolds*, Duke Math. J. **60** (1990), no. 2, 509–559.
- [21] R. Mazzeo and H. Weiss, *The Teichmüller theory of conic surfaces*, in preparation.
- [22] G. Montcouquiol, *Rigidité infinitésimale de cônes-variétés Einstein à courbure négative*. arXiv:math.DG/0503195
- [23] L. Saper *L^2 cohomology of locally symmetric spaces I*, arXiv:math.RT/0412353
- [24] L. Saper *On the cohomology of locally symmetric spaces and their compactifications*, in *Current developments in mathematics 2002*, Eds. D. Jerison et al., International Press (2003), 219–289.