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L^2 Harmonic Forms for a Class of Complete Kähler Metrics

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1. Introduction

The Hodge theorem for compact manifolds states that every real cohomology class of a compact manifold M is represented by a unique harmonic form. That is, the space of solutions to the differential equation $(d+d^*)\phi = 0$ on L^2 forms over M, a space that depends on the metric on M, is canonically isomorphic to the purely topological real cohomology space of M. This isomorphism is enormously useful because it provides a way to transform theorems from geometry into theorems in topology and vice versa. No such result holds in general for complete noncompact manifolds, but in many specific cases there are Hodge-type theorems. One of the oldest is the description, due to Atiyah, Patodi, and Singer [1], of the space of L^2 harmonic forms on a manifold with complete cylindrical ends. By calculating the solutions to the equation for harmonic forms on the cylindrical ends, they showed that the space of L^2 harmonic forms is isomorphic to the image of the relative cohomology of the manifold in the absolute cohomology. Another Hodge-type result was found by Zucker [14] for a natural class of metrics called Poincaré metrics. These metrics, first constructed by Cornalba and Griffiths [4], are complete Kähler metrics with hyperbolic cusp-type singularities at isolated points on a Riemann surface. Zucker showed that the space of L^2 forms on a Riemann surface that are harmonic with respect one of these metrics is isomorphic to the standard cohomology of the surface. This result was extended by Cattani, Kaplan, and Schmid [3] to analogous metrics on bundles over projective varieties with singularities along a divisor. These metrics can be thought of as complete Kähler metrics on the noncompact manifold given by removing the divisor.

There is a larger natural class of complete, noncompact Kähler metrics on the complement of a divisor in a projective variety. They are of interest both because of their relation to the Poincaré metrics and because other examples of them have arisen in papers by Tian and Yau [12; 13] as starting points for the construction of metrics solving the Kähler–Einstein problem, and the final metrics in these papers are quasi-isometric to the starting metrics. In this paper, we study the space of L^2 harmonic forms on manifolds with such metrics and its relation to cohomology of the original projective variety, especially with a view to how the spaces relate to Hodge diamond structures on subspaces of the cohomology of the original variety.

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Let $\bar{M}$ be a complete smooth algebraic variety of complex dimension n, and let $\mathcal{D}$ be an ample divisor with normal crossings in $\bar{M}$. Let $\|\cdot\|$ be a norm of $[\mathcal{D}]$ with positive definite curvature form such that $\|S\| < 1$ on $\bar{M}$, where S is the holomorphic section of $[\mathcal{D}]$ defining $\mathcal{D}$. Choose a function f such that $f'(x) > c_1 > -1$ for sufficiently large x and $f''(x) + f'(x)e^{-2x} > c_2/x^2 > 0$. Then define the Kähler form

$$\omega_f = \frac{i}{2\pi}\{\partial\bar{\partial}f(-\log(\|S\|^2))\} + K\omega_0, \tag{1}$$

where ω_0 is the Kähler form on $\bar{M}$ and K is a constant chosen sufficiently large to ensure that the form is positive definite away from the divisor. Then the metric g_f associated to ω_f will be a complete Kähler metric on M. Examples include the case $f(x) = -\ln(x)$, when these are Poincaré-type metrics as in [3]. If $f(x) = x^a$ $(a > 1)$, the resulting metrics are quasi-isometric to the metrics constructed in [12], and if $f(x) = e^{bx}$ $(b > 0)$, then they are quasi-isometric to the metrics constructed in [13]. In this paper we consider these three cases, as well as the case where $f(x) = -x^a$ $(0 < a < 1)$ because it interpolates between the Poincaré metrics and the metrics in [12]. That is to say, the growth of the function $f(x) = -x^a$ $(0 < a < 1)$ lies in between the growth of $f(x) = -\ln(x)$ and $f(x) = x^a$ $(a > 1)$. If we consider the case where the divisor $\mathcal{D}$ is smooth and look at the asymptotic form of the metrics near $\mathcal{D}$, we can see how the convexity of f relates to the growth of the corresponding metrics near the divisor, given here by $x = 0$. The Poincaré metrics look like

$$x^2\left(\frac{dx^2}{x^4} + T^2 + \frac{g_{\mathcal{D}}}{x^2}\right),$$

where $g_{\mathcal{D}}$ is any metric on $\mathcal{D}$ and T is any connection 1-form on the spherical normal bundle of $\mathcal{D}$ in $\bar{M}$. The metrics in [12] look like

$$x^{2(2-a)}\left(\frac{dx^2}{x^6} + T^2 + \frac{g_{\mathcal{D}}}{x^2}\right),$$

where $a > 1$ is the a in $f(x) = x^a$, and the interpolating metrics have the form

$$x^2\left(\frac{dx^2}{x^{2(2+\varepsilon)}} + T^2 + \frac{g_{\mathcal{D}}}{x^2}\right),$$

where $\varepsilon \to 0$ as $a \to 0$ in $f(x) = -x^a$ and where $\varepsilon \to 1$ as $a \to 1$. If $f(x) = x$ then $f''(x) = 0$, so this case degenerates and gives not a complete metric but rather the restriction to M of a Kähler metric on $\bar{M}$. We focus on these cases because of their applications, but the results in this paper should extend to metrics with more general defining functions f.

We obtain the following results.

THEOREM 1. *If M and g_f are as previously described, then $\mathcal{H}^i_{(2)}(M, g_f) \cong \{0\}$ for $i \neq n$ if $f(x) = x^a$ $(a > 1)$ or if $f(x) = e^{bx}$ $(b > 0)$.*

No similar vanishing theorem will hold for metrics with $f(x) = x^a$ $(0 < a < 1)$ or for the Poincaré metrics because they have finite volume; hence constant functions are L^2 harmonic 0-forms. Further, if $\bar{M}$ is a $2n$-dimensional torus then there are L^2 harmonic forms in all degrees for the finite-volume metrics, regardless of the divisor $\mathcal{D}$ chosen.

For Kähler forms arising from the function $f(x) = e^{bx}$ when the divisor is smooth, it is possible to determine the middle-degree L^2 harmonic forms. These have the following description in terms of a subset of cohomology on M already known to carry a Hodge structure.

THEOREM 2. *If M and g_f are as above, $f(x) = e^{bx}$ $(b > 0)$, and $\mathcal{D}$ is smooth, then $\mathcal{H}^n_{(2)}(M, g_f)$ is isomorphic to the middle-degree primitive cohomology of $\bar{M}$.*

If the divisor has crossings or if $f(x) \neq e^{bx}$, then we don't have a general theorem describing the L^2 harmonic forms. However, we can understand an important subspace of them, the L^2 holomorphic and antiholomorphic forms, from the following theorem.

THEOREM 3. *If $M = \bar{M} - \mathcal{D}$ is a complete Kähler manifold with a metric quasi-isometric to one of the metrics g_f described in equation* (1)*, then the L^2 holomorphic and antiholomorphic forms on M extend across the divisor $\mathcal{D}$ to give all holomorphic or antiholomorphic forms on $\bar{M}$, except in the cases and degrees where* (*by Theorem 1*) *there are no L^2 harmonic forms.*

These theorems, while incomplete in their description of L^2 harmonic forms for the general class of metrics, give a sense of what those spaces of forms might look like, which we will discuss in the conclusion to this paper.

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2. Notation, Definitions, and Background

First, let's briefly review some definitions and background. If M is a manifold of real dimension m with a metric g, let

$$L^i_{(2)}(M, g) = \left\{ \sigma \in \Gamma(\Lambda^i T^*M) \mid \int_M |d\sigma|^2_g \, d\mathrm{vol}_g < \infty \right\}.$$

Then $(L^*_{(2)}(M, g), d)$ is a complex, so we can define the cohomology

$$H^i_{(2)}(M, g) = \frac{\ker(d\colon L^i_{(2)}(M, g) \to L^{i+1}_{(2)}(M, g))}{d(L^i_{(2)}(M, g))}.$$

The reduced L^2 cohmolology of M is defined similarly, but with the closure of the image of d replacing the image of d in the preceding fraction. If (M, g) is complete, then the space of L^2 harmonic forms is defined by

$$\mathcal{H}^i_{(2)}(M, g) = \{\sigma \in L^i_{(2)}(M, g) \mid d\sigma = d^*\sigma = 0\},$$

where $d^* = (-1)^{m(k-1)-1} * d *$. The weak Kodaira decomposition states that

$$L^2\Omega^k(M, g) = \overline{dC_0^\infty\Omega^{k-1}(M)} \oplus \overline{d^*C_0^\infty\Omega^{k+1}(M, g)} \oplus \mathcal{H}^k_{(2)}(M, g).$$

This implies that the space of L^2 harmonic forms on M is isomorphic to the reduced L^2 cohomology and is thus a quasi-isometry invariant of the manifold, up to isomorphism. This means we need really only study the metrics near the deleted divisor $\mathcal{D}$, and only up to quasi-isometry.

Therefore, we conclude this section by writing out the local form of the metrics given by the Kähler forms ω_f and by constructing quasi-isometric "model metrics" near the divisor:

$$\begin{aligned}
\omega_f &= \frac{i}{2\pi}\{\partial\bar\partial f(-\log(\|S\|^2))\} + K\omega_0 \\
&= \frac{i}{2\pi}\{\partial[f'(-\log\|S\|^2)\bar\partial(-\log\|S\|^2)]\} + K\omega_0 \\
&= \frac{i}{2\pi}\{f''(-\log\|S\|^2)\,\partial(-\log\|S\|^2)\bar\partial(-\log\|S\|^2) \\
&\qquad + f'(-\log\|S\|^2)\partial\bar\partial(-\log\|S\|^2)\} + K\omega_0.
\end{aligned}$$

Now consider ω_f near $\mathcal{D} = \{z_1 = \cdots = z_r = 0\}$. By the choice of norm on $[\mathcal{D}]$ we have $\|S\|^2 = e^{-\beta}|z_1|^2 \cdots |z_r|^2$, where $\frac{i}{2\pi}\partial\bar\partial\beta$ is positive definite on $\bar M$, so

$$-\log\|S\|^2 = \beta - \log|z_1|^2 - \cdots - \log|z_r|^2,$$

so

$$\bar\partial(-\log\|S\|^2) = \bar\partial\beta - \sum_{i=1}^r \frac{d\overline{z_i}}{\overline{z_i}} \quad \text{and} \quad \partial\bar\partial(-\log\|S\|^2) = \partial\bar\partial\beta.$$

Thus

$$\begin{aligned}
\omega_f = \frac{i}{2\pi}\Bigg\{ f''(-\log\|S\|^2)\Bigg(\partial\beta - \sum_{i=1}^r \frac{dz_i}{z_i}\Bigg) \wedge \Bigg(\bar\partial\beta - \sum_{i=1}^r \frac{d\overline{z_i}}{\overline{z_i}}\Bigg) \\
+ f'(-\log\|S\|^2)\partial\bar\partial\beta\Bigg\} + K\omega_0.
\end{aligned}$$

Up to quasi-isometry, this is equivalent to

$$\begin{aligned}
\omega'_f = \frac{i}{2\pi}\Bigg\{ f''(-\log\|S\|^2) \sum_{i=1}^r \frac{dz_i \wedge d\overline{z_i}}{\|z_i\|^2} \\
+ (1 + f'(-\log\|S\|^2)) \sum_{i=r+1}^n dz_i \wedge \overline{dz_i}\Bigg\}. \quad (2)
\end{aligned}$$

On the interior of M, the form can be written locally around any point as

$$\omega_f = (1 + f'(-\log\|S\|^2)) \sum_{i=1}^{n} dz_i \wedge \overline{dz_i}.$$

Away from the divisor, $0 < \|S\| < 1$, so $-\log\|S\|^2$ is bounded away from zero and infinity. Therefore, if $f'(x) > 0$ for all $x > 0$ (which is true for the two infinite-volume metrics we study), then dropping the 1 from $(1 + f'(-\log\|S\|^2))$ yields a quasi-isometric metric. In particular, the form remains positive definite. Dropping the 1 corresponds to eliminating the $K\omega_0$ term from the original definition of ω_f, so in these cases we can drop this term entirely. This is not possible in the finite-volume cases.

3. Proofs of Theorems

3.1. Proof of Theorem 1

Gromov [5] showed that if a Kähler manifold has a Kähler form $\omega = d\eta$, where η is a form whose norm was bounded with respect to the metric, then the strong Lefschetz theorem could be used to prove that the manifold had no L^2 forms outside the middle dimension. This result has been generalized in [2; 7; 10]. McNeal [10] considers manifolds M endowed with Kähler metrics given by forms $\omega = i\partial\bar{\partial}\lambda$. If

$$\|\partial\lambda\|_\omega^2 \leq A + B\lambda$$

for some constants $A > 0$ and $B \geq 0$, where $\|\cdot\|_\omega$ is the norm induced by the Kähler form ω, he says λ *dominates its gradient*. McNeal shows that if λ is an exhaustion function that dominates its gradient, then there are no L^2 harmonic forms for (M, ω) outside the middle degree. So, in order to prove Theorem 1, we begin by noting that if $f(x) = x^a$ $(a > 1)$ and $f(x) = e^{bx}$ $(b > 0)$, then eliminating the $K\omega_0$ term in the Kähler form construction in equation (1) yields a quasi-isometric Kähler metric given by the Kähler form $\omega = i\partial\bar{\partial}(1/2\pi) f(-\log\|S(x)\|^2)$. Since $\lambda = (1/2\pi) f(-\log\|S(x)\|^2)$ is an exhaustion function, Theorem 1 then follows from the following claim.

CLAIM 1. *The function* $\lambda = (1/2\pi) f(-\log\|S(x)\|^2)$ *dominates its gradient for* $f(x) = x^a$ $(a > 1)$ *and* $f(x) = e^{bx}$ $(b > 0)$.

Proof. We can do these both at once, taking the norms with respect to the quasi-isometric metrics g_f associated to the forms given in equation (2). With respect to these metrics, which are diagonal in local coordinates near the divisor $\mathcal{D} = \{z_1 = \cdots = z_r = 0\}$, we have $\|dz_i\|_{g_f}^2 = f''(-\log\|S\|^2)^{-1}$ for $i \leq r$ and $\|dz_i\|_{g_f}^2 = f'(-\log\|S\|^2)^{-1}$ for $i > r$. We will supress the g_f subscripts in the following calculation. So in either case, near the divisor we have

$$
\begin{aligned}
\|\partial\lambda\|^2_{g_f} &= \left\| \frac{1}{2\pi}\partial f(-\log\|S(x)\|^2) \right\|^2 \\
&= \left\| \frac{1}{2\pi} f'(-\log\|S\|^2)\,\partial(-\log\|S\|^2) \right\|^2 \\
&= \left(\frac{1}{2\pi} f'(-\log\|S\|^2) \right)^2 \left\| -\partial\beta - \sum_{i=1}^{r} \frac{dz_i}{z_i} \right\|^2 \\
&\le c(f'(-\log\|S\|^2))^2 \left(\sum_{i=1}^{r} \frac{\|dz_i\|^2}{\|z_i\|^2} + \sum_{i=r+1}^{n} \|dz_i\|^2 \right) \\
&\le c(f'(-\log\|S\|^2))^2 \left(\sum_{i=1}^{r} (f''(-\log\|S\|^2))^{-1} + \sum_{i=r+1}^{n} (f'(-\log\|S\|^2))^{-1} \right) \\
&\le c_2 \left[\frac{f'(-\log\|S\|^2))^{-1}}{f''(-\log\|S\|^2)} + f'(-\log\|S\|^2) \right] \\
&\le \frac{c_3}{2\pi} f(-\log\|S\|^2) = c_3\lambda
\end{aligned}
$$

for either choice of $f(x)$, since

$$\frac{(f'(x))^2}{f(x)f''(x)} + \frac{f'(x)}{f(x)}$$

is bounded by a constant for $x > 1$ (i.e., near enough the divisor) for either choice of $f(x)$. Hence, on the entire manifold we can find some constants $A > 0$ and $B \ge 0$ such that $\|\partial\lambda\|^2_\omega \le A + B\lambda$. □

3.2. *Proof of Theorem 2*

Because of the quasi-isometry invariance of $\mathcal{H}^i_{(2)}(M, g)$, it suffices to consider the metrics in Theorem 2 near the deleted divisor. We can reparametrize as in [13] to show the metric is quasi-isometric in a neighborhood of the deleted divisor to the simpler metric:

$$ds^2 = e^{2bR}(dR^2 + ds_P^2), \tag{3}$$

where P is the unit normal bundle of $\mathcal{D}$ in $\bar{M}$, ds_P^2 is any metric on P, and the divisor is given by $R = 0$.

For forms of middle degree on a complete manifold, both L^2 and $d + d^*$ are conformally invariant. This means that the space of L^2 harmonic forms for the metric g_f on M is isomorphic to the space of L^2 harmonic forms for a metric equal to the product metric $dR^2 + ds_P^2$ near the end. By the results in [1], we know that this second space is isomorphic to $\mathrm{Im}(H^i_o(M) \to H^i(M))$, the image of the compact cohomology of M in the full (not L^2) cohomology of M.

In order to get the description in terms of primitive cohomology of M, we use the exact sequence for a pair twice. Note that $H^n_o(M)$ can also be thought of as $H^n(\bar{M}, \mathcal{D})$, so we obtain the exact sequence

$$\cdots \to H^n(\bar{M}, \mathcal{D}) \to H^n(\bar{M}) \to H^n(\mathcal{D}) \to \cdots;$$

we also have the exact sequence

$$\cdots \to H^n(\bar{M}, M) \to H^n(\bar{M}) \to H^n(M) \to \cdots.$$

By Poincaré duality, we know that $H^n(\mathcal{D}) \cong H_{n-2}(\mathcal{D})$ and that $H^{n+2}(\bar{M}) \cong H_{n-2}(\bar{M})$. Then, by the Lefschetz hyperplane theorem, $H_{n-2}(\mathcal{D}) \cong H_{n-2}(\bar{M})$. Similarly, by the Thom isomorphism, $H^n(\bar{M}, M) \cong H^{n-2}(\mathcal{D})$ and, by the Lefschetz hyperplane theorem, $H^{n-2}(\bar{M}) \cong H^{n-2}(\mathcal{D})$. So the two exact sequences just displayed become

$$\begin{array}{ccccc} & & H^n(\bar{M}, \mathcal{D}) & & \\ & & \downarrow{\scriptstyle j^*} \quad \searrow & & \\ H^{n-2}(\bar{M}) & \xrightarrow{\bigcup[\mathcal{D}]} & H^n(\bar{M}) & \xrightarrow{i^*} & H^n(M) \\ & & \downarrow{\scriptstyle \bigcup[\mathcal{D}]} & & \\ & & H^{n+2}(\bar{M}). & & \end{array}$$

Thus, the image of j^* in $H^n(\bar{M})$ is the kernel of $\bigcup[\mathcal{D}]$, whereas the kernel of i^* is the image of $\bigcup[\mathcal{D}]$. By the Lefschetz hyperplane theorem, the map

$$\bigcup[\mathcal{D}]^2 \colon H^{n-2}(\bar{M}) \to H^{n+2}(\bar{M})$$

is an isomorphism, so this means that

$$\mathrm{Im}(H^n(\bar{M}, \mathcal{D}) \to H^n(M)) \cong \mathrm{Im}(H^n(\bar{M}, \mathcal{D}) \to H^n(\bar{M})),$$

which consists of the primitive classes in $H^n(\bar{M})$ since $\mathcal{D}$ is ample. This concludes the proof of Theorem 2. □

3.3. *Proof of Theorem 3*

It suffices to prove this result for holomorphic forms, since the result for anti-holomorphic forms follows by conjugation. We'll do the proof, which is based on Hartog's theorem, by cases. In each case, we will establish estimates locally around a point on the divisor to show that functions that are holomorphic on a punctured polydisk around the point are in L^2 for the relevant metrics if and only if they have no singularities. The punctured polydisk, which we will assume to have radius $1/2$, is denoted $(\Delta^*)^r \times (\Delta)^{n-r}$, and the divisor is given locally in holomorphic coordinates by $\mathcal{D} = \{z_1 = \cdots = z_r = 0\}$. The metric has no effect on whether forms are holomorphic, and the L^2 condition is preserved by quasi-isometry. Therefore, to simplify calculations, instead of using g_f itself we will calculate using a quasi-isometric metric, g_f', on the punctured polydisk. In the case of the infinite-volume metrics, where we are concerned only with middle-degree

forms, we will take the metric associated to the form ω_f' identified in equation (2). In the case of the finite-volume metrics, we will further simplify the metric to

$$g_f' = (-\ln(|z_1|^2 \cdots |z_r|^2))^{a-2} \sum_{j=1}^{r} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2} + \sum_{j=r+1}^{n} dz_j \wedge d\bar{z}_j,$$

where $a = 0$ for the metrics with $f(x) = -\ln(x)$.

Case 1: Infinite-volume metrics. If $f(x) = x^a$ $(a > 1)$ or $f(x) = e^{bx}$ $(b > 0)$ then, by the vanishing in Theorem 1, we need only consider holomorphic forms of degree n. The volume element for g_f' is given in terms of the local holomorphic coordinates by

$$d\text{vol} = \frac{d\mathbf{z} \wedge d\bar{\mathbf{z}}}{\|dz_1\|^2 \cdots \|dz_n\|^2},$$

where $d\mathbf{z} \wedge d\bar{\mathbf{z}}$ is the Euclidean volume element in the polydisk, $dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. Any holomorphic n-form on the punctured polydisk is given by $\sigma = \sigma(\mathbf{z})d\mathbf{z}$, where $\sigma(\mathbf{z})$ is a holomorphic function on the punctured polydisk. This form is L^2 with respect to the metric g_f' if and only if $\sigma(\mathbf{z})$ has no poles along the divisor, since

$$\begin{aligned}\int_{(\Delta^*)^r \times (\Delta)^{n-r}} \|\sigma\|^2 \, d\text{vol} &= \int_{(\Delta^*)^r \times (\Delta)^{n-r}} |\sigma(\mathbf{z})|^2 \|d\mathbf{z}\|^2 \frac{d\mathbf{z} \wedge d\bar{\mathbf{z}}}{\|dz_1\|^2 \cdots \|dz_n\|^2} \\ &= \int_{(\Delta^*)^r \times (\Delta)^{n-r}} |\sigma(\mathbf{z})|^2 \, d\mathbf{z} \wedge d\bar{\mathbf{z}},\end{aligned}$$

which is finite if and only if $\sigma(\mathbf{z})$ has no poles. Thus, every holomorphic form on the punctured polydisk that is L^2 with respect to g_f extends to the whole polydisk, and every holomorphic form on the whole polydisk restricts to a form on the punctured polydisk that is L^2 with respect to g_f. This completes the proof in Case 1.

Case 2: Finite-volume metrics. If $f(x) = -\ln(x)$ or $f(x) = -x^a$ $(0 < a < 1)$, then we must prove the theorem for holomorphic forms of any degree $0 \leq k \leq n$. On the punctured polydisk $(\Delta^*)^r \times (\Delta)^{n-r}$ we derive that, for the metric g_f',

$$\|dz_j\|^2 = (-\ln(|z_1|^2 \cdots |z_r|^2))^{2-a} |z_j|^2$$

for $j \leq r$ and $\|dz_j\|^2 = 1$ for $j > r$, and the volume form is

$$d\text{vol} = \frac{d\mathbf{z} \wedge d\bar{\mathbf{z}}}{\|dz_1\|^2 \cdots \|dz_r\|^2}.$$

A holomorphic k-form σ on the punctured polydisk can be written in the local holomorphic coordinates as $\sigma = \sum_{|I|=k} \sigma_I(\mathbf{z}) d\mathbf{z}_I$, where $I = \{I_1, \ldots, I_k\}$ is a set of distinct indicies between 1 and n, $d\mathbf{z}_I = dz_{I_1} \wedge \cdots \wedge dz_{I_k}$, and the functions $\sigma_I(\mathbf{z})$ are all holomorphic. The form σ extends to a holomorphic form on the entire disk if and only if the $\sigma_I(\mathbf{z})$ have no poles along the divisor.

Consider the L^2 norm with respect to g'_f of σ:

$$\|\sigma\|^2_{g'_f} = \int_{(\Delta^*)^r\times(\Delta)^{n-r}} \Big\| \sum_{|I|=k} \sigma_I(\mathbf{z})\, d\mathbf{z}_I \Big\|^2 d\text{vol}$$
$$= \int_{(\Delta^*)^r\times(\Delta)^{n-r}} \sum_{|I|=k,\,|J|=k} \sigma_I(\mathbf{z})\overline{\sigma_J(\mathbf{z})}\langle d\mathbf{z}_I, d\mathbf{z}_J\rangle\, d\text{vol}.$$

Since the 1-forms dz_j are orthogonal, so are the k-forms $d\mathbf{z}_I$. Hence the preceding equals

$$\sum_{I=k} \int_{(\Delta^*)^r\times(\Delta)^{n-r}} |\sigma_I(\mathbf{z})|^2 \|d\mathbf{z}_I\|^2\, d\text{vol}.$$

We can therefore consider one index I at a time:

$$\|\sigma_I(\mathbf{z})d\mathbf{z}_I\|^2_{g'_f} = \int_{(\Delta^*)^r\times(\Delta)^{n-r}} |\sigma_I(\mathbf{z})|^2 \|d\mathbf{z}_I\|^2\, d\text{vol}$$
$$= \int_{(\Delta^*)^r\times(\Delta)^{n-r}} |\sigma_I(\mathbf{z})|^2 \frac{\|dz_{I_1}\|^2\cdots\|dz_{I_k}\|^2}{\|dz_1\|^2\cdots\|dz_r\|^2}\, d\mathbf{z}\wedge d\bar{\mathbf{z}}.$$

Assume that q of the indices for I are indices corresponding to punctured disks and that the others are indices corresponding to entire disks. Without loss of generality, by reordering the holomorphic coordinates we can even assume that the first q indices are $\{1, 2, \ldots, q\}$. Then this integral becomes

$$\int_{(\Delta^*)^r\times(\Delta)^{n-r}} |\sigma_I(\mathbf{z})|^2 \frac{d\mathbf{z}\wedge d\bar{\mathbf{z}}}{\|dz_{q+1}\|^2\cdots\|dz_r\|^2}$$
$$= \int_{(\Delta^*)^r\times(\Delta)^{n-r}} |\sigma_I(\mathbf{z})|^2 \frac{d\mathbf{z}\wedge d\bar{\mathbf{z}}}{|z_{q+1}|^2\cdots|z_r|^2(-\log(|z_1|^2\cdots|z_r|^2)^{(r-q)(2-a)}}.$$

In polar coordinates, where $z_j = \rho_j e^{i\theta_j}$, this becomes

$$\int_0^{1/2}\cdots\int_0^{1/2}\int_0^{2\pi}\cdots\int_0^{2\pi} |\sigma_I(\rho,\theta)|^2 \frac{\rho_1\cdots\rho_n\, d\rho_1\wedge\cdots\wedge d\rho_n}{\rho_{q+1}^2\cdots\rho_r^2(-\log(\rho_1^2\cdots\rho_r^2))^{(r-q)(2-a)}}\, d\theta.$$

The integrals in θ are all finite, so we are left with

$$\int_0^{1/2}\cdots\int_0^{1/2} |\tilde{\sigma}_I(\rho)|^2 \frac{\rho_1\cdots\rho_n\, d\rho_1\wedge\cdots\wedge d\rho_n}{\rho_{q+1}^2\cdots\rho_r^2(-\log(\rho_1^2\cdots\rho_r^2))^{(r-q)(2-a)}}.$$

If σ_I has any poles, then the integral in the direction of the pole is infinite. If σ_I has no poles, then the integral is

$$\le c\int_0^{1/2}\cdots\int_0^{1/2} \frac{d\rho_{q+1}\wedge\cdots\wedge d\rho_r}{\rho_{q+1}\cdots\rho_r(-\log(\rho_1^2\cdots\rho_r^2))^{(r-q)(2-a)}}.$$

As we construct each of these integrals, the exponent on the log term is reduced by 1. Each iterated integral is finite as long as the exponent is greater than 1. Hence, since there are $r-q$ integrals to do and since $2-a>1$, the overall integral is

finite. Therefore, holomorphic forms restricted from the entire polydisk are the same as holomorphic forms in L^2 with respect to g_f for the punctured polydisk. This finishes Case 2. □

4. Conclusion

These theorems suggest what the L^2 harmonic forms might look like in general for the metrics discussed in this paper. In the case where the divisor is smooth, for example, the description of the space of L^2 harmonic forms for the $f(x) = e^{bx}$ $(b > 0)$ metrics in terms of primitive cohomology still makes sense in the case of more general divisors, and it still explains the Hodge substructure. This suggests that it may still be correct in the more general case. We also have a general sense that metrics whose asymptotic behavior lies between the two extremes of the $f(x) = -\ln(x)$ and $f(x) = e^{bx}$ $(b > 0)$ should have spaces of L^2 harmonic forms that also lie in between the spaces of L^2 harmonic forms for these metrics. This, together with the strong condition that the spaces carry a pure Hodge structure, suggests more concretely that, for the metrics with $f(x) = -x^a$ $(0 < a < 1)$, the space of L^2 harmonic forms is isomorphic in all degrees to the standard cohomology of the compact manifold $\bar{M}$, something we already know (by Theorem 3) to be true for the holomorphic and antiholomorphic forms and their Poincaré duals. Finally, the missing metric (where $f(x) = x$) corresponds in the smooth divisor case to a cylindrical metric, so the metrics where $f(x) = x^a$ $(a > 1)$ lie in between the cylindrical case and the generalized Poincaré metric case, which have the same middle-degree spaces of L^2 harmonic forms, as described in Theorem 2. It seems likely, therefore, that this is also the correct description of the middle-degree L^2 harmonic forms in the case $f(x) = x^a$ $(a > 1)$.

The complete results about Poincaré metrics in [3; 14] are obtained using the fact that, for these metrics, the space of L^2 harmonic forms is isomorphic to the standard L^2 cohomology, which has a nice description in terms of sheaves. Therefore, in the Poincaré case, the space of L^2 harmonic forms can be understood by finding a quasi-isomorphism from the L^2 sheaf to one on M. The other metrics considered in this paper do not in general have such an isomorphism. In these cases, the spectrum of the Laplacian comes down to zero, as can be seen in the $n = 1$ case as a consequence of results in [15] about warped product metrics; it is thus impossible to apply the sheaf-theoretic arguments of [3] and [15].

The proofs of Theorems 2 and 3 in this paper ultimately rely on the fact that we understand the asymptotic struture of harmonic forms near the deleted divisor. It seems possible, then, that techniques of geometric microlocal analysis could shed light on our conjectures. This approach was developed by Melrose [11] as a new philosophy for understanding the results in [1]. It gives a rough road map for turning information about the asymptotic structure of a singular metric near the boundary of a manifold into information about the asymptotics of solutions to elliptic differential equations on the manifold. The general idea has been applied in many subsequent papers (see e.g. [6, 9; 11]) to metrics similar in asymptotic structure to the metrics considered here.

References

[1] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Cambridge Philos. Soc. 77 (1975), 43–69.

[2] J. Cao and F. Xavier, *Kähler parabolicity and the Euler number of compact manifolds of non-positive sectional curvature*, Math. Ann. 319 (2001), 483–491.

[3] E. Cattani, A. Kaplan, and W. Schmid, *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. Math. 87 (1987), 217–252.

[4] M. Cornalba and P. Griffiths, *Analytic cycles and vector bundles on non-compact algebraic varieties*, Invent. Math. 28 (1975), 1–106.

[5] M. Gromov, *Kähler hyperbolicity and L^2-Hodge theory*, J. Differential Geom. 33 (1991), 263–292.

[6] A. Hassell and A. Vasy, *The spectral projections and the resolvent for scattering metrics*, J. Anal. Math. 79 (1999), 241–298.

[7] J. Jost and Y. L. Xin, *Vanishing theorems for L^2-cohomology groups*, J. Reine Angew. Math. 525 (2000), 95–112.

[8] R. Mazzeo, *Elliptic theory of differential edge operators, I*, Comm. Partial Differential Equations 16 (1991), 1615–1664.

[9] R. Mazzeo and R. Melrose, *Pseudodifferential operators on manifolds with fibred boundaries. Mikio Sato: A great Japanese mathematician of the twentieth century*, Asian J. Math. 2 (1998), 833–866.

[10] J. McNeal, *L^2 Harmonic forms on some complete Kähler manifolds*, Math. Ann. (to appear).

[11] R. Melrose, *The Atiyah–Patodi–Singer index theorem*, Res. Notes Math., 4, A K Peters, Wellesley, MA, 1993.

[12] G. Tian and S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature, I*, J. Amer. Math. Soc. 3 (1990), 579–609.

[13] ———, *Complete Kähler manifolds with zero Ricci curvature, II*, Invent. Math. 106 (1991), 27–60.

[14] S. Zucker, *Hodge theory with degenerating coefficients. L_2 cohomology in the Poincaré metric*, Ann. of Math. (2) 109 (1979), 415–476.

[15] ———, *L_2-cohomology of warped products and arithmetic groups*, Invent. Math. 70 (1982), 169–218.

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