ON THE CLASSIFICATION OF INTEGRABLE DIFFERENTIAL/DIFFERENCE EQUATIONS IN THREE DIMENSIONS

by

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Abstract

Integrable systems arise in nonlinear processes and, both in their classical and quantum version, have many applications in various fields of mathematics and physics, which makes them a very active research area.

In this thesis, the problem of integrability of multidimensional equations, especially in three dimensions (3D), is explored. We investigate systems of differential, differential-difference and discrete equations, which are studied via a novel approach that was developed over the last few years. This approach, is essentially a perturbation technique based on the so called 'method of dispersive deformations of hydrodynamic reductions'. This method is used to classify a variety of differential equations, including soliton equations and scalar higher-order quasilinear PDEs.

As part of this research, the method is extended to differential-difference equations and consequently to purely discrete equations. The passage to discrete equations is important, since, in the case of multidimensional systems, there exist very few integrability criteria. Complete lists of various classes of integrable equations in three dimensions are provided, as well as partial results related to the theory of dispersive shock waves. A new definition of integrability, based on hydrodynamic reductions, is used throughout, which is a natural analogue of the generalized hodograph transform in higher dimensions. The definition is also justified by the fact that Lax pairs –the most well-known integrability criteria– are given for all classification results obtained.

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Contents

List of Abbreviations

1+1D	one space and one time dimensions
2+1D	two space and one time dimensions
2D	two dimensions
3D	three dimensions
D△E	Differential-difference equation
dHD	deformed Harry–Dym
dKP	dispersionless Kadomtsev–Petviashvili
HD	Harry–Dym
KdV	Korteweg–de Vries
KP	Kadomtsev–Petviashvili
mKP	modified Kadomtsev–Petviashvili
mVN	modified Veselov–Novikov
NLS	Nonlinear Schrödinger
ODE	Ordinary differential equation
PDE	Partial differential equation
P△E	Partial discrete equation
VN	Veselov–Novikov

Chapter 1

Introduction

1.1 Outline of area of research

Integrable systems link together various areas of mathematics and mathematical physics, including differential geometry, complex analysis, spectral theory, and more.

Although the definition of integrability is a highly nontrivial subject, integrable (nonlinear) equations possess certain properties, that can provide some working criteria. Among these properties, is the existence of a special class of exact solutions of the equations, representing $n-$soliton interactions [20], and the solvability of the equation by the inverse scattering transform (IST), which is basically a nonlinear analogue of the Fourier transform [37]. Moreover, one can require the existence of an infinite number of symmetries, or a sufficient number of first integrals that are in involution (Liouville integrability) [7]. However, one of the most important integrability properties, that will be often used throughout this thesis, is the existence of a linear representation, known as *a Lax pair* [57], which yields the original nonlinear equation via the compatibility condition of two linear partial differential equations (PDEs), for the same auxiliary function.

The concept of integrability arose in the 18th century, in the context of finite-dimensional classical mechanics. Subsequently, various integrable systems where discovered, such as Euler's problem of two fixed centres, Jacobi's geodesic flow on ellipsoids, Neumann's problem, the Manakov top, etc. Their study was based on the Hamiltonian formalism, and their integrability appeared to be an ‘easy’ concept, based on the existence of sufficiently many, ‘well-behaved’ first integrals (constants of motion). The key result is the *Liouville-Arnold theorem*, which ensures that there exists a canonical transformation to, the so-called, *action-angle coordinates*, such that the transformed Hamiltonians depend only on the action variables. This way, one can explicitly solve (‘integrate’) Hamilton’s equations, provided that the action-angle map is explicitly known.

In contrast to finite-dimensional systems, a universally accepted definition of integrability does not exist in the case of infinite dimensions. Here one has to focus on the properties of the system and its solutions, and produce new techniques. The theory of these integrable equations, and their properties was initially developed in the case of two-dimensional (2D) models, such as the well-known Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad u = u(x,t),$$

a nonlinear equation describing waves on the surface of shallow water, the nonlinear Schrödinger (NLS) equation

$$i\psi_t = \psi_{xx} \pm 2|\psi|^2\psi, \quad \psi = \psi(x,t),$$

and many more. Then, the interest was transferred to multidimensional systems, and especially systems in three dimensions (3D), which constitute the subject of this thesis. We will be focusing on the study of nonlinear PDEs, such as Kadomtsev–Petviashvili (KP) equation

$$\left(u_t - \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}\right)_x = \frac{3}{4}u_{yy}, \quad u = u(x,y,t),$$

but also differential-difference equations (D△Es), such as the 2+1D Toda lattice

$$u_t = u\triangle_{\bar{y}}w, \quad w_x = \triangle_y u, \qquad u = u(x,y,t),$$

and purely discrete equations (P△Es), where the most well-known example is Hirota equation

$$\alpha T_1\tau T_{\bar{1}}\tau + \beta T_2\tau T_{\bar{2}}\tau + \gamma T_3\tau T_{\bar{3}}\tau = 0, \qquad \tau = \tau(x^1, x^2, x^3),$$

which gives various types of soliton equations and discrete analogues of them, via appropriate choice of the parameters α, β, γ and suitable limits [45]. Here T_i ($T_{\bar{i}}$) is the forward (backward) ϵ-shift in the i-direction and $\triangle_i = (T_i - 1)/\epsilon$ ($\triangle_{\bar{i}} = (1 - T_{\bar{i}})/\epsilon$), denotes the forward (backward) discrete derivative, respectively.

Discrete systems are of particular interest in mathematics, physics, numerical analysis, etc. They initially appeared as discretisations of continuous equations [1, 2, 16, 43, 44, 45], but recently their theoretical study started to develop independently. As in the case of continuous equations, an effective integrability criterion for discrete equations, is the existence of a Lax representation. Other techniques include the *singularity confinement* [40], which can be considered as the discrete analogue of the Painlevé property, and the *algebraic entropy* of an equation [9], which is a number defined by the growth of the degrees of the iterates of a map. However, the past few years, another important approach was introduced independently by the authors of [70, 10], called *multidimensional consistency*. Multidimensional consistency, which is probably the most significant method, is an extension of the 3D-consistency approach, which was used for the classification of discrete integrable equations in 2D.

Through this work, we aim to introduce a novel approach to the integrability of PDEs, D$\triangle$Es, and consequently P$\triangle$Es in three dimensions.

This novel approach originates from the theory of hydrodynamic type systems, so let us first go back to the continuous case, and give a brief description of this concept. Consider a homogeneous system of PDEs in 2D, of the form

$$u^i_t = v^i_j(u)u^j_x, \quad u = u(t,x), \tag{1.1}$$

where the standard summation rule over repeated indices is assumed, $u = (u^1, ..., u^n)$ is an n-component vector and v^i_j, which could also be considered as matrix elements of an $n \times n$ matrix V, are assumed to be smooth functions of $u^1, ..., u^n$ only. Systems of this type are called *hydrodynamic type systems*, or *1+1 dimensional quasilinear systems*, and arise in many different contexts, such as fluid mechanics and gas dynamics, general relativity, differential geometry, etc. S. P. Novikov formulated the conjecture that a quasilinear system in $1+1$ dimensions is integrable if it is diagonalisable, that is, transformable into the form $R^i_t = \lambda^i(R)R^i_x$, by a change of variables, and Hamiltonian. This conjecture was later proved by S. P. Tsarev, who also found that the Hamiltonian condition could be relaxed, by introducing the semi–Hamiltonian property. If a system satisfies the semi-Hamiltonian property, then it can be solved using the generalized hodograph method [82]. Classification of equations of this type has been a very active research topic and various results have been obtained over the years, see for example [21, 22, 36, 74, 82, 83, 84].

As already mentioned, our interest lies in equations in three dimensions. The most famous model is KP equation,

$$(u_t - 3/2uu_x - 1/4u_{xxx})_x = 3/4u_{yy}, \quad u = u(x, y, t), \tag{1.2}$$

which is a natural generalisation of KdV equation, and also arises in the context of modelling waves in ferromagnetic media, as well as matter–wave pulses in Bose–Einstein condensates [91]. Introducing a potential $w(x, y, t)$, KP can be rewritten in the form

$$u_t - 3/2uu_x - 1/4u_{xxx} = 3/4w_y, \quad w_x = u_y.$$

The linear representation (Lax pair) of this equation is given by the following overdetermined system,

$$\begin{aligned} \phi_y &= \phi_{xx} + u\phi, \\ \phi_t &= \phi_{xxx} + 3/2u\phi_x + 3/4u_x\phi + 3/4w\phi, \end{aligned} \tag{1.3}$$

where $\phi(x, y, t)$ is an auxiliary function. The consistency condition, $\phi_{yt} = \phi_{ty}$, modulo (1.3), yields KP equation. Following Zakharov [91], given a (2+1)-dimensional equation, one can rewrite it as a *dispersive* equation through the limiting procedure

$$\partial_x \to \epsilon\partial_x, \quad \partial_y \to \epsilon\partial_y, \quad \partial_t \to \epsilon\partial_t,$$

and the obtain its *dispersionless limit* by setting $\epsilon \to 0$. In the case of KP equation, this limiting procedure results in

$$u_t - 3/2uu_x - 1/4\epsilon^2 u_{xxx} = 3/4w_y, \quad w_x = u_y,$$

and as $\epsilon \to 0$, one obtains

$$u_t - 3/2uu_x = 3/4w_y, \quad w_x = u_y,$$

known as dispersionless KP (dKP) equation. The dKP arises in nonlinear acoustics, and the theory of Einstein-Weyl structures. In this case, the Lax representation is no longer linear. Instead, it transforms into a pair of nonlinear PDEs for the auxiliary function $S(x, y, t)$,

$$\begin{aligned} S_y &= S_x^2 + u, \\ S_t &= S_x^3 + 3/2uS_x + 3/4w. \end{aligned}$$

A straightforward check of the compatibility condition, $S_{yt} = S_{ty}$, shows that it is indeed satisfied, modulo dKP. To obtain this nonlinear representation, one should make the change $\partial_x \to \epsilon\partial_x, \partial_y \to \epsilon\partial_y, \partial_t \to \epsilon\partial_t$, use the substitution $\phi = \exp(S/\epsilon)$ in the original Lax pair for KP equation, and then take the limit as $\epsilon \to 0$.

All dispersionless integrable systems possess the so-called *dispersionless Lax pair* [91]. This is a pair of equations

$$S_t = G(S_x, u), \quad S_y = F(S_x, u), \tag{1.4}$$

where $u = (u^1, \ldots, u^n)$, and the function $S(x, y, t)$ is called *scalar pseudo-potential*. Dependence of the functions F and G on S_x may be nonlinear. The consistency condition $S_{ty} = S_{yt}$ is satisfied, modulo the original equation (or in other words, the Lax pair implies the original equation via the consistency condition). Moreover, dispersionless Lax pairs can be used to classify dispersionless integrable systems, as we will illustrate in the thesis.

In a series of recent works [27, 28, 31, 32, 46], it was realised that various three-dimensional dispersionless problems can be studied by a new method, called *the method of hydrodynamic reductions*. This method works for equations that, under a proper substitution, fit into the following general first-order (2+1)-dimensional quasilinear hydrodynamic type form,

$$A(u)u_x + B(u)u_y + C(u)u_t = 0, \tag{1.5}$$

where $u = (u^1, ..., u^m)^t$ is an m-component column vector of the dependent variables, and A, B, C are $l \times m$ matrices where l, the number of equations, is allowed to exceed the number of the unknowns, m. The key idea of the method of hydrodynamic reductions is to look for special class of exact solutions of (1.5), the so called *N-phase solutions*, $u = u(R^1, ..., R^N)$, where the 'phases' $R^1, ..., R^N$ solve a pair of commuting diagonal systems of hydrodynamic type

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x. \tag{1.6}$$

Since systems (1.6) are diagonal and commuting, they are automatically semi-Hamiltonian and, therefore, completely integrable by the generalized hodograph method. In other words, the idea is to decouple the (2+1)-dimensional problem (1.5) into a pair of diagonal (1+1)-dimensional hydrodynamic type systems. Then

Definition 1 *[27, 30] The original system is called integrable if, for any N, it possesses infinitely many N-component reductions of the type* (1.6)*, parametrised by N arbitrary functions of a single argument.*

This definition of integrability will be used throughout this thesis.
Hydrodynamic reductions of particular dispersionless equations were studied before [38], though only recently it was understood that the requirement of the existence of an infinity of such reductions is a strong and restrictive condition, providing a good definition of integrability for dispersionless systems. This requirement leads to a simple algorithmic way of verifying whether a given equation is integrable, and also provides an effective classification scheme.

After the classification of integrable hydrodynamic type systems, it seemed natural to look for *integrable dispersive deformations* of these systems. The basic idea is to require that all hydrodynamic reductions of the dispersionless system are 'inherited' by its dispersive counterpart [31, 32, 46], while at the same time the commutativity of the phase flows is preserved. Particularly, one seeks a *k-th* order dispersive deformation of equation (1.5) of the form

$$A(u)u_x + B(u)u_y + C(u)u_t + \epsilon(\dots) + \epsilon^2(\dots) + \cdots + \epsilon^k(\dots) + \cdots = 0, \tag{1.7}$$

where terms in the brackets are $m \times m$ matrices, whose entries are homogeneous differential polynomials in the x- and y-derivatives of u, of order $k+1$. Coefficients of these polynomials are allowed to be arbitrary functions of u. Then, we require that N-phase solutions can be deformed accordingly,

$$u = u(R^1, ..., R^N) + \epsilon u_1 + \cdots + \epsilon^k u_2 + O(\epsilon^{k+1}) \tag{1.8}$$

where u_i are assumed to be homogeneous polynomials of degree i in the x-derivatives of R^i's. Similarly, hydrodynamic reductions can be deformed as

$$\begin{aligned} R^i_y &= \mu^i(R)R^i_x + \epsilon a_1 + \cdots + \epsilon^k a_m + O(\epsilon^{k+1}), \\ R^i_t &= \lambda^i(R)R^i_x + \epsilon b_1 + \cdots + \epsilon^k b_m + O(\epsilon^{k+1}). \end{aligned} \tag{1.9}$$

where a_i, b_i are assumed to be homogeneous polynomials of degree $i+1$ in the x-derivatives of R^i's. Substituting (1.8) into (1.7), and using (1.9) along with the consistency conditions $R^i_{ty} = R^i_{yt}$, one arrives at a complicated set of relations, allowing one to reconstruct dispersive terms in (1.7).

Important Remark. The requirement of the inheritance of hydrodynamic reductions of an integrable dispersionless system by the corresponding dispersive equation, provides an efficient classification criterion. The reconstruction of dispersive terms is an algebraic procedure that is performed step–by–step, at the orders of the deformation parameter ϵ. It is an open problem to prove that this works at all orders of ϵ. Thus, throughout this thesis, when we say that an equation is integrable, using the definition stated earlier, we mean *integrable to a finite order of* ϵ, although, in the end, integrability in the usual sense is implied, by providing Lax pairs for the resulting equations.

Since the dispersive equation (1.7) can be considered as a formal series in ϵ, this means that we can apply the method of deformations of hydrodynamic reductions to (semi-) discrete equations, that are expressed in terms of ϵ-shift operators, $T_x f(x,y) = f(x+\epsilon, y)$, $T_{\bar{x}} f(x,y) = f(x-\epsilon, y)$, since $T_i = e^{\epsilon\partial_i}$. For example, the 2+1D Toda lattice

$$u_t = u\triangle_{\bar{y}} w, \quad w_x = \triangle_y u, \qquad u = u(x,y,t),$$

after expanding using Taylor's formula, can be written as

$$\begin{aligned} \frac{u_t}{u} &= w_y - \frac{\epsilon}{2}w_{yy} + \frac{\epsilon^2}{6}w_{yyy} + \ldots, \\ w_x &= u_y + \frac{\epsilon}{2}u_{yy} + \frac{\epsilon^2}{6}u_{yyy} + \ldots. \end{aligned}$$

Hence, the integrability of differential-difference and discrete equations will be explored using this new approach.

1.2 Main results of the thesis

This thesis is motivated by the work of the authors in [32], who considered an important class of equations, which includes the very well-known examples of KP, Gardner and Veselov-Novikov equations. Using the method of hydrodynamic reductions, as the main approach, they classified integrable equations of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(...) + \epsilon^2(...), \quad w_x = u_y, \tag{1.10}$$

where φ, ψ, η depend on the scalar field $u(x, y, t)$ and the nonlocal variable $w(x, y, t)$. Terms at ϵ and ϵ^2 are homogeneous differential polynomials of order two and three respectively in the x- and y- derivatives of u and w, with coefficients being arbitrary functions of u and w. Their main result is summarised in the following theorem.

Theorem 1.1 *[32] The following equations provide a complete list of integrable equations of the form* (1.10)*, with* $\eta \neq 0$*, whose dispersionless limit is linearly nondegenerate:*

KP equation	$u_t = uu_x + w_y + \epsilon^2 u_{xxx},$
mKP equation	$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx},$
Gardner equation	$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx},$
VN equation	$u_t = (uw)_y + \epsilon^2 u_{yyy},$
mVN equation	$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y,$
HD equation	$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$
deformed HD equation	$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$
E_5 equation	$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta uu_y + w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)],$
E_6 equation	$u_t = \frac{4}{3}\beta^2 u^3 u_x + (w - 3\beta u^2)u_y + uw_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)],$

where $B = \beta u D_x - D_y$, β*=const,* δ*=const.*

The dispersionless limits of these equations possess dispersionless Lax pairs of the form

$$F(S_x, S_y, u) = 0, \qquad S_t = G(S_x, S_y, u, w),$$

with F quadratic and G cubic in S_x, S_y. This information can be used to classify integrable dispersionless equations using dispersionless Lax pairs, and in fact, we prove how one can re-derive the classification list above. Moreover, using the same technique, we study equations with one extra nonlocal variable v, in the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_y, \quad w_x = u_y, \quad v_x = f(u, w)_y, \tag{1.11}$$

where $\tau f_w \neq 0$, and we prove that

Theorem 1.2 *Integrable equations of the form* (1.11)*, are higher flows of dispersionless KP, mKP, Gardner, HD and deformed HD equations.*

If we consider integrable equations of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x, \quad w_x = u_y, \quad v_y = f(u, w)_x,$$

where $\eta, \tau f_u \neq 0$, we can show that all such equations are commuting flows of the dispersionless VN equation. We extend the problem by considering equations with several nonlocal variables, where the result of theorem 1.2 is repeated, when up to four nonlocalities are added. Also, we prove the following

Theorem 1.3 *Commuting flows of dispersionless equations from theorem 1.1 and system (1.11), result in higher flows of dispersionless KP, mKP, Gardner, HD and deformed HD equations.*

Then, part of the thesis, addresses the problem of classifying integrable differential-difference equations in 2+1 dimensions with one/two discrete variables. These equations are of the general form

$$u_t = F(u, w), \quad u = u(x, y, t), \quad w = w(x, y, t), \tag{1.12}$$

where u is a scalar field, w is the nonlocal variable, and F is a differential/difference expression of u, w, and their derivatives (the form of F and the nonlocality are specified in the corresponding chapter 4). All the nonlocalities considered, reduce to $w_x = u_y$ in the dispersionless limit $\epsilon \to 0$. We use the standard notation explained in the previous section, and the method of deformations of hydrodynamic reductions. We focus on various classes of equations generalising the intermediate long wave and Toda type equations, and we consider nonlocalities of intermediate long wave, Toda and fully discrete type. The functions $\varphi, \psi, \eta, \tau, f, g, p, q, h, k$ that appear here depend on u, w. Our first result is

Theorem 1.4 *The following examples constitute a complete list of integrable equations of the form $u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(...) + \epsilon^2(...)$, where dots denote terms which are homogeneous polynomials of degree two and three in the x- and y-derivatives of u and w, whose coefficients are allowed to be functions of u and w, with the nonlocality of intermediate long wave type $\triangle_x w = \frac{T_x+1}{2} u_y$:*

$$\begin{aligned} u_t &= uu_y + w_y, \\ u_t &= (w + \alpha e^u)u_y + w_y, \\ u_t &= u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} u_{yyy}, \\ u_t &= u^2 u_y + (uw)_y + \frac{\epsilon^2}{12}\left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y. \end{aligned}$$

where $\alpha = const.$

The first equation is known in the literature as a differential-difference analogue of the KP equation [16]. It can also be viewed as a $2+1$ dimensional integrable version of the intermediate long wave equation [92]. The third equation is known as a differential-difference version of the Veselov-Novikov equation [75], while the last can be viewed as a differential-difference version of the modified Veselov-Novikov equation. Using the same form of nonlocality, but changing the structure of the equation, we obtain

Theorem 1.5 *The following examples constitute a complete list of integrable equations of the form $u_t = \psi u_y + \eta w_y + f\triangle_x g + p\triangle_{\bar{x}} q$, with the nonlocality of intermediate long wave* *type* $\triangle_x w = \frac{T_x+1}{2} u_y$*:*

$$\begin{aligned} u_t &= uu_y + w_y, \\ u_t &= (w + \alpha e^u)u_y + w_y, \\ u_t &= wu_y + w_y + \frac{\triangle_x + \triangle_{\bar{x}}}{2} e^{2u}, \\ u_t &= wu_y + w_y + e^u(\triangle_x + \triangle_{\bar{x}})e^u. \end{aligned}$$

where $\alpha = const.$

The third example first appeared in [60]. In the case of the Toda type nonlocality, we obtain

Theorem 1.6 *The following examples constitute a complete list of integrable equations of the form* $u_t = \varphi u_x + f\triangle_y g + p\triangle_{\bar{y}} q$*, with the nonlocality of Toda type* $w_x = \triangle_y u$*:*

$$\begin{aligned} u_t &= u\triangle_{\bar{y}} w, \\ u_t &= (\alpha u + \beta)\triangle_{\bar{y}} e^w, \\ u_t &= e^w\sqrt{u}\triangle_y\sqrt{u} + \sqrt{u}\triangle_{\bar{y}}(e^w\sqrt{u}), \end{aligned}$$

here $\alpha, \beta = const.$

The first example is the 2+1 dimensional Toda equation, which can also be written in the form $(\ln u)_{xt} = \triangle_y \triangle_{\bar{y}} u$, while the second is equivalent to the Volterra chain when $\alpha \neq 0$, or to the Toda chain when $\alpha = 0$. A more general class of equations is when both x and y are discrete. Then

Theorem 1.7 *The following examples constitute a complete list of integrable equations of the form* $u_t = f\triangle_x g + h\triangle_{\bar{x}} k + p\triangle_y q + r\triangle_{\bar{y}} s$*, with the fully discrete nonlocality* $\triangle_x w = \triangle_y u$*:*

$$\begin{aligned} u_t &= u\triangle_{\bar{y}}(u - w), \\ u_t &= u(\triangle_x + \triangle_{\bar{y}})w, \\ u_t &= (\alpha e^{-u} + \beta)\triangle_{\bar{y}} e^{u-w}, \\ u_t &= (\alpha e^u + \beta)(\triangle_x + \triangle_{\bar{y}})e^w, \\ u_t &= \sqrt{\alpha - \beta e^{2u}}\left(e^{w-u}\triangle_y\sqrt{\alpha - \beta e^{2u}} + \triangle_{\bar{y}}(e^{w-u}\sqrt{\alpha - \beta e^{2u}})\right), \end{aligned}$$

here $\alpha, \beta = const.$

In equivalent form, the last example is known as the $2+1$ dimensional analogue of the modified Volterra lattice [88]. Also, for a class of equations of type (1.12), we attempt to classify nonlocalities of the form $\epsilon w_x = Bu$, where B is a constant-coefficient pseudo-differential operator of the form $B = \epsilon\partial_y + \epsilon^2(...) + \epsilon^3(...) + \dots$, and the coefficient at ϵ^k is a polynomial in ∂_x, ∂_y of degree k. The result is the following

Theorem 1.8 *The examples below constitute a complete list of integrable equations of the form $u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y$, with the nonlocality $\epsilon w_x = A(\partial_x, \partial_y)u_y$*

$$\begin{aligned}
u_t &= uu_y + w_y, \qquad \triangle_x w = \frac{T_x+1}{2}u_y, \\
u_t &= (w+\alpha e^u)u_y + w_y, \qquad \triangle_x w = \frac{T_x+1}{2}u_y, \\
u_t &= uw_y, \qquad w_x = (\partial_y^{-1}\triangle_y\triangle_{\bar{y}})u, \\
u_t &= e^w w_y, \qquad w_x = (\partial_y^{-1}\triangle_y\triangle_{\bar{y}})u, \\
u_t &= e^{u-w}(w_y - u_y), \quad w_x = u_y + \epsilon^2\partial_y(\partial_x - \partial_y)^2 u + O(\epsilon^4).
\end{aligned}$$

where $\alpha = const.$

The nonlocality of the first two equations is that of the intermediate long wave type, while from the third one can set $w \to \partial_y^{-1}\triangle_{\bar{y}}w$, to recover the familiar form of the Toda equation. The nonlocality for the last equation requires further investigation.

Finally, we consider fully discrete equations in 3D and address the problem of classification of such integrable equations. The method of deformations of hydrodynamic reductions is again the main approach: we require that hydrodynamic reductions of the corresponding dispersionless limits are 'inherited' by the discrete equations. We study two particularly interesting subclasses, namely integrable discrete conservation laws, and discrete integrable quasilinear equations, as well as differential-difference degenerations of them (we refer to chapter 5 for references). The case of discrete conservation laws leads to the following

Theorem 1.9 *Integrable discrete conservation laws,* $\triangle_1 f + \triangle_2 g + \triangle_3 h = 0$*, where* f, g, h *are functions of* $\triangle_1 u, \triangle_2 u, \triangle_3 u$*, are naturally grouped into seven three-parameter families,*

$$aI + \beta J + \gamma K = 0,$$

where a, β, γ *are arbitrary constants, while* I, J, K *denote left hand sides of three linearly independent discrete conservation laws of the seven octahedron-type equations listed below. In each case we give explicit forms of* I, J, K*, as well as the underlying octahedron equation.*

Case 1.

Conservation Laws	*Octahedron equation*
$I = \triangle_1 e^{\triangle_2 u} + \triangle_3 \left(e^{\triangle_2 u - \triangle_1 u} - e^{\triangle_2 u}\right) = 0$	$\frac{T_2\tau - T_{12}\tau}{T_{23}\tau} = T_1\tau\left(\frac{1}{T_{13}\tau} - \frac{1}{T_3\tau}\right)$
$J = \triangle_1 e^{-\triangle_3 u} + \triangle_2 \left(e^{\triangle_1 u - \triangle_3 u} - e^{-\triangle_3 u}\right) = 0$	*(setting* $\tau = e^{u/\epsilon}$*)*
$K = \triangle_2 \left(\triangle_3 u - \ln(1 - e^{\triangle_1 u})\right) +$	
$+\triangle_3 \left(\ln(1 - e^{\triangle_1 u}) - \triangle_1 u\right) = 0$	

Case 2.

Conservation Laws	*Octahedron equation*
$I = \triangle_2 \ln \triangle_1 u + \triangle_3 \ln\left(1 - \frac{\triangle_2 u}{\triangle_1 u}\right) = 0$	$T_{12}uT_{13}u + T_2uT_{23}u + T_1uT_3u$
$J = \triangle_1 \ln \triangle_2 u + \triangle_3 \ln\left(\frac{\triangle_1 u}{\triangle_2 u} - 1\right) = 0$	$= T_{12}uT_{23}u + T_1uT_{13}u + T_2uT_3u$
$K = \triangle_1 \left(\frac{(\triangle_2 u)^2}{2} - \triangle_2 u \triangle_3 u\right) +$	
$+\triangle_2 \left(\triangle_1 u \triangle_3 u - \frac{(\triangle_1 u)^2}{2}\right) = 0$	

Case 3. Generalised lattice Toda *(depending on a parameter* α*)*

Conservation Laws	*Octahedron equation*
subcase $\alpha \neq 0$	
$I = \triangle_1(e^{\triangle_2 u - \triangle_3 u} + \alpha e^{-\triangle_3 u}) - \triangle_2(e^{\triangle_1 u - \triangle_3 u} + \alpha e^{-\triangle_3 u}) = 0$	$\frac{T_{23}\tau}{T_3\tau} + \frac{T_{12}\tau}{T_2\tau} + \alpha\frac{T_{12}\tau T_{23}\tau}{T_2\tau T_3\tau} =$
$J = \triangle_2 \ln\left(e^{\triangle_1 u} + \alpha\right) + \triangle_3\left(\ln \frac{e^{\triangle_1 u} - e^{\triangle_2 u}}{e^{\triangle_1 u} + \alpha} - \triangle_2 u\right) = 0$	$\frac{T_{12}\tau}{T_1\tau} + \frac{T_{13}\tau}{T_3\tau} + \alpha\frac{T_{12}\tau T_{13}\tau}{T_1\tau T_3\tau}$
$K = \triangle_1 \ln\left(e^{\triangle_2 u} + \alpha\right) + \triangle_3\left(\ln \frac{e^{\triangle_1 u} - e^{\triangle_2 u}}{e^{\triangle_2 u} + \alpha} - \triangle_1 u\right) = 0$	*(setting* $\tau = e^{-u/\epsilon}$*)*
subcase $\alpha = 0$	*lattice Toda equation*
$I = \triangle_1 e^{\triangle_2 u - \triangle_3 u} - \triangle_2 e^{\triangle_1 u - \triangle_3 u} = 0$	$(T_1 - T_3)\frac{T_2\tau}{\tau} = (T_2 - T_3)\frac{T_1\tau}{\tau}$
$J = \triangle_2 \triangle_1 u + \triangle_3(\ln(1 - e^{\triangle_2 u - \triangle_1 u}) - \triangle_2 u) = 0$	*(setting* $\tau = e^{-u/\epsilon}$*)*
$K = \triangle_1 e^{-\triangle_2 u} - \triangle_2 e^{-\triangle_1 u} + \triangle_3(e^{-\triangle_1 u} - e^{-\triangle_2 u}) = 0$	

Case 4. Lattice KP

Conservation Laws	*Octahedron equation*
$I = \triangle_1((\triangle_3 u)^2 - (\triangle_2 u)^2) + \triangle_2((\triangle_1 u)^2$	$(T_1u - T_2u)T_{12}u + (T_3u - T_1u)T_{13}u$
$\quad - (\triangle_3 u)^2) + \triangle_3((\triangle_2 u)^2 - (\triangle_1 u)^2) = 0$	$+(T_2u - T_3u)T_{23}u = 0$
$J = \triangle_1 \ln(\triangle_3 u - \triangle_2 u) - \triangle_3 \ln(\triangle_2 u - \triangle_1 u) = 0$	
$K = \triangle_2 \ln(\triangle_1 u - \triangle_3 u) - \triangle_3 \ln(\triangle_2 u - \triangle_1 u) = 0$	

Case 5. Lattice mKP

Conservation Laws	*Octahedron equation*
$I = \triangle_1(e^{\triangle_2 u} - e^{\triangle_3 u}) + \triangle_2(e^{\triangle_3 u} - e^{\triangle_1 u}) +$	$\frac{T_{13}\tau - T_{12}\tau}{T_1\tau} + \frac{T_{12}\tau - T_{23}\tau}{T_2\tau}$
$\quad +\triangle_3(e^{\triangle_1 u} - e^{\triangle_2 u}) = 0$	$+\frac{T_{23}\tau - T_{13}\tau}{T_3\tau} = 0$
$J = \triangle_1 \ln\left(e^{\triangle_3 u} - e^{\triangle_2 u}\right) - \triangle_2 \ln\left(e^{\triangle_3 u} - e^{\triangle_1 u}\right) = 0$	*(setting* $\tau = e^{u/\epsilon}$*)*
$K = \triangle_2 \ln\left(e^{\triangle_3 u} - e^{\triangle_1 u}\right) - \triangle_3 \ln\left(e^{\triangle_2 u} - e^{\triangle_1 u}\right) = 0$	

Case 6. Schwarzian KP

Conservation Laws	*Octahedron equation*
$I = \triangle_2 \ln\left(1 - \frac{\triangle_3 u}{\triangle_1 u}\right) - \triangle_3 \ln\left(\frac{\triangle_2 u}{\triangle_1 u} - 1\right) = 0$	$(T_2\triangle_1 u)(T_3\triangle_2 u)(T_1\triangle_3 u)$
$J = \triangle_3 \ln\left(1 - \frac{\triangle_1 u}{\triangle_2 u}\right) - \triangle_1 \ln\left(\frac{\triangle_3 u}{\triangle_2 u} - 1\right) = 0$	$= (T_2\triangle_3 u)(T_3\triangle_1 u)(T_1\triangle_2 u)$
$K = \triangle_1 \ln\left(1 - \frac{\triangle_2 u}{\triangle_3 u}\right) - \triangle_2 \ln\left(\frac{\triangle_1 u}{\triangle_3 u} - 1\right) = 0$	

Case 7. Lattice spin

Conservation Laws	*Octahedron equation*
Hyperbolic version	*lattice-spin equation*
$I = \triangle_1 \ln \frac{\sinh \triangle_3 u}{\sinh \triangle_2 u} + \triangle_2 \ln \frac{\sinh \triangle_1 u}{\sinh \triangle_3 u} + \triangle_3 \ln \frac{\sinh \triangle_2 u}{\sinh \triangle_1 u} = 0$	$\left(\frac{T_{12}\tau}{T_2\tau} - 1\right)\left(\frac{T_{13}\tau}{T_1\tau} - 1\right)\left(\frac{T_{23}\tau}{T_3\tau} - 1\right)$
$J = \triangle_1 \ln \frac{\sinh(\triangle_2 u - \triangle_3 u)}{\sinh \triangle_2 u} - \triangle_3 \ln \frac{\sinh(\triangle_1 u - \triangle_2 u)}{\sinh \triangle_2 u} = 0$	$= \left(\frac{T_{12}\tau}{T_1\tau} - 1\right)\left(\frac{T_{13}\tau}{T_3\tau} - 1\right)\left(\frac{T_{23}\tau}{T_2\tau} - 1\right)$
$K = \triangle_2 \ln \frac{\sinh(\triangle_3 u - \triangle_1 u)}{\sinh \triangle_1 u} - \triangle_3 \ln \frac{\sinh(\triangle_1 u - \triangle_2 u)}{\sinh \triangle_1 u} = 0$	*(setting* $\tau = e^{2u/\epsilon}$*)*
Trigonometric version	*Sine-Gordon equation*
$I = \triangle_1 \ln \frac{\sin \triangle_3 u}{\sin \triangle_2 u} + \triangle_2 \ln \frac{\sin \triangle_1 u}{\sin \triangle_3 u} + \triangle_3 \ln \frac{\sin \triangle_2 u}{\sin \triangle_1 u} = 0$	$(T_2 \sin \triangle_1 u)(T_3 \sin \triangle_2 u)(T_1 \sin \triangle_3 u)$
$J = \triangle_1 \ln \frac{\sin(\triangle_2 u - \triangle_3 u)}{\sin \triangle_2 u} - \triangle_3 \ln \frac{\sin(\triangle_1 u - \triangle_2 u)}{\sin \triangle_2 u} = 0$	$= (T_2 \sin \triangle_3 u)(T_3 \sin \triangle_1 u)(T_1 \sin \triangle_2 u)$
$K = \triangle_2 \ln \frac{\sin(\triangle_3 u - \triangle_1 u)}{\sin \triangle_1 u} - \triangle_3 \ln \frac{\sin(\triangle_1 u - \triangle_2 u)}{\sin \triangle_1 u} = 0$	

A similar result is obtain for integrable equations of the form $\triangle_1 f + \triangle_2 g + \partial_3 h = 0$, where f, g, h are functions of $\triangle_1 u, \triangle_2 u, u_3$. For discrete integrable quasilinear equations, we prove that

Theorem 1.10 *There exists a unique nondegenerate discrete second order quasilinear equation in 3D of the form* $\sum_{i,j=1}^{3} f_{ij}(\triangle u)\triangle_{ij} u = 0$, *where* f_{ij} *are functions of* $\triangle_1 u, \triangle_2 u, \triangle_3 u$, *known as lattice KP equation,*

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u + (\triangle_3 u - \triangle_1 u)\triangle_{13} u + (\triangle_2 u - \triangle_3 u)\triangle_{23} u = 0.$$

In the case of semi-discrete quasilinear equations, we show that

Theorem 1.11 *There exists a unique nondegenerate second order equations of the type* $f_{11}\triangle_{11} u + f_{12}\triangle_{12} u + f_{22}\triangle_{22} u + f_{13}\triangle_1 u_3 + f_{23}\triangle_2 u_3 + f_{33} u_{33} = 0$, *where* f_{ij} *are functions of* $\triangle_1 u, \triangle_2 u, u_3$, *known as semi-discrete Toda lattice,*

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u - \triangle_1 u_3 + \triangle_2 u_3 = 0.$$

1.3 Organisation of the thesis

The main results of this thesis are distributed to chapters 3, 4 and 5. Chapter 2 has an introductory character, chapter 3 includes results on quasilinear PDEs, chapter 4 focuses on differential-difference equations, while chapter 5 on discrete equations. Results of chapters 4 and 5 appear in the articles [34] and [35], respectively.
Particularly, the thesis is organised as follows.

In **chapter 2** we discuss the main ideas of the theory of hydrodynamic type systems in $1+1$ dimensions, which are the basis of the theory in $2+1$ dimensions. As already mentioned, S. P. Tsarev [82] proved S. P. Novikov's conjecture, that a quasilinear system in $1+1$ dimensions is integrable if it is diagonalisable and Hamiltonian. In fact, he relaxed the Hamiltonian condition, by introducing the semi–Hamiltonian property, and showed that it is necessary and sufficient condition for integrability. Hence, in this chapter, we recall the basic tools of this theory. We explain that the Riemann invariants are variables in which a hydrodynamic type system is diagonal, and give a coordinate-invariant criterion of diagonalisability, using the so called Haantjes tensor. Then, focusing on the Hamiltonian approach, we discuss the semi-Hamiltonian property, but also the concept of conservation laws and the existence of infinite number of commuting flows of semi-Hamiltonian hydrodynamic systems. Finally, we explain the generalised hodograph method, which can be used to solve this type of systems.

Chapter 3 is devoted to quasilinear PDEs in $2+1$ dimensions. These equations contain nonlocal variables, and we postulate a specific form for them. Classification of dispersionless equations within this class is performed using the method of hydrodynamic reductions, or the approach using dispersionless Lax pairs. Particularly, in this chapter, we explicitly describe the method of hydrodynamic reductions, and we also show a way to reconstruct dispersive deformations for a given integrable dispersionless system, by deforming hydrodynamic reductions. The method of hydrodynamic reductions can be applied to equations whose dispersionless limit is nondegenerate, and these nondegeneracy conditions are explained in detail. Then, we introduce the concept of dispersionless Lax pairs, and we show how they can be used to classify dispersionless integrable systems. Classification results in the case of equations with one, two, and more than two (nested) nonlocalities, are distributed across three sections. In the end, we discuss the existence of commuting flows of the systems under consideration.

In **chapter 4**, we address the problem of classifying integrable differential-difference equations in $2+1$ dimensions with one/two discrete variables. We briefly remind the nondegeneracy conditions that need to be met in order to obtain classification results for this type of equations. We apply the method of hydrodynamic reductions and dispersive deformations of dispersionless limits, as it was explained in the previous chapter, by using the example of Toda equation, while in the rest of the chapter, we present the classification results for various classes of equations generalising the intermediate long wave and Toda type equations. Among the classes that were studied, we first present some classification results, in the case where the nonlocal variables are expressed in terms of pseudo-differential operators. We also classify equations, which are named after the type of nonlocality that is considered, namely the intermediate long wave, Toda and fully discrete type nonlocality. For all the resulting equations, the corresponding Lax pair is given.

In **chapter 5**, we consider discrete equations in 3D and address the problem of classification of such integrable equations, within various particularly interesting subclasses. We list various well-known examples of discrete integrable 3D equations, which we call Hirota-type, and we give their $\triangle$-representation. The reason for this representation is that their dispersionless limits become more clearly seen. A brief summary of the method of deformations of hydrodynamic reductions is described, using an example of a discrete wave-type equation. Then, we provide the classification result of integrable discrete conservation laws and discrete integrable quasilinear equations, and we also study differential-difference degenerations of them. In the last section, we perform some numerical simulations using Mathematica. Choosing a certain discrete equation, we compare its solution with the solution of the corresponding dispersionless equation and we show how the phenomenon of a dispersive shock wave appears. In fact, this phenomenon can be observed in very simple equations, and such an example is given in the end.

Finally, in **chapter 6** we provide a general summary of the thesis, and some remarks on future work.

Chapter 2

Hydrodynamic type systems in 1+1 dimensions

In this first chapter we discuss some important ideas from the theory of $1+1$ dimensional hydrodynamic type systems. These ideas are necessary in order to be able to extend our study to higher dimensional systems.

In section 2.1, we recall the one-dimensional hydrodynamic type systems by listing some simple well-known examples while in the next two sections, 2.2 and 2.3, we present some criteria in order to establish if a given system is diagonalisable and semi-Hamiltonian. Specifically, in section 2.2 we explain the Riemann invariants, which are the variables in which the general hydrodynamic system is diagonal, and in section 2.3 we consider Hamiltonian approach for studying hydrodynamic systems, and introduce the semi-Hamiltonian property. The notion of conservation laws and commuting flows is introduced in section 2.4, where we define hydrodynamic type first integrals. Finally, in the last section 2.5, we briefly discuss the generalized hodograph method for solving diagonalisable semi-Hamiltonian systems.

What lies beneath these ideas is the following

Conjecture. *A quasilinear system in* $1+1$ *dimensions is integrable if it is diagonalisable and Hamiltonian,*

which was formulated by S. P. Novikov and was later proved by S. P. Tsarev [82].

2.1 Equations of hydrodynamic type in 1+1 dimensions

Consider a homogeneous system of PDEs of the form

$$u^i_t = v^i_j(u)u^j_x, \tag{2.1}$$

here the standard summation rule with respect to j is assumed, for the functions $u^1(t,x), ...,$ $u^n(t,x)$, where $u = (u^1, ..., u^n)$ is an n-component vector and v^i_j, which could also be considered as matrix elements of an $n \times n$ matrix V, are assumed to be smooth, generally nonlinear functions of $u^1, ..., u^n$ only. Systems of this type are called *hydrodynamic type systems*, or *1+1 dimensional quasilinear systems*, and arise in many different contexts, such as fluid mechanics and gas dynamics, general relativity, differential geometry, etc. Here are some simple examples of equations of this type [82].

Example 2.1. The equations of motion for an ideal barotropic gas

$$\begin{aligned} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x + p_x/\rho = 0, \end{aligned} \tag{2.2}$$

where u is the speed of the gas, ρ is the density, and $p = p(\rho)$ is the equation of state. This system can be written in the form (2.1) as follows

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ p_\rho/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0,$$

which means that

$$u = \begin{pmatrix} \rho \\ u \end{pmatrix}, \qquad V = -\begin{pmatrix} u & \rho \\ p_\rho/\rho & u \end{pmatrix}.$$

Example 2.2. Benney's equations [93]

$$\begin{aligned} \eta^i_t + (u^i\eta^i)_x = 0, \\ u^i_t + u^i u^i_x + f\left(\sum_{i=1}^n \eta^i\right)_x = 0, \end{aligned}$$

which describe a multi-layered system of fluids, with η^i being the height and u^i the velocity of each layer.

We can now give the following

Definition 2 *The system* (2.1) *is called strictly hyperbolic if all eigenvalues of the matrix* v^i_j *are real and distinct.*

Note that all systems under consideration will be assumed strictly hyperbolic.

2.2 Riemann invariants

Consider again system (2.1),

$$u^i_t = v^i_j(u)u^j_x.$$

This system is invariant under the (local) change of variables $u = u(w)$, where $u = (u^1, ..., u^n)$ and $w = (w^1, ..., w^n)$. Indeed, if we apply the chain rule

$$\frac{\partial u^i}{\partial w^j} w^j_t = v^i_k(u(w)) \frac{\partial u^k}{\partial w^l} w^l_x,$$

we obtain

$$w^j_t = \frac{\partial w^j}{\partial u^i} \frac{\partial u^k}{\partial w^l} v^i_k(u(w)) w^l_x = v^j_l w^l_x,$$

which shows that the matrix v^i_j transforms as a $(1,1)$-tensor. If there exists a change of variables $u^i = u^i(R)$, with $R = (R^1, ..., R^n)$, such that the matrix v^i_j becomes diagonal in the coordinates R^i, we say that the system (2.1) is *diagonalisable* and we can bring it in the form

$$R^i_t = v^i(R) R^i_x,$$

where now there is **no summation** over repeated indices. The coordinates R^i are called *Riemann invariants* and if they exist, there is an algorithmic way to construct them. Let $v^1, ..., v^n$ be n real and distinct roots of the characteristic equation $\det(v^i_j - v^i \delta^i_j) = 0$, and let ξ^p_j be the corresponding left eigenvectors of the matrix v^i_j

$$\xi^p_i v^i_j = v^p \xi^p_j, \quad p = 1, ..., n.$$

Suppose that for each eigenvector ξ^p_j there exists an integrating factor c^p such that

$$c^p \xi^p_j = \partial R^p / \partial u^j.$$

Then $c^p \xi^p_j$ appear to be the components of a gradient, and the functions R^i are the desired Riemann invariants since

$$R^p_t = \frac{\partial R^p}{\partial u^j} \frac{\partial u^j}{\partial t} = c^p \xi^p_j \frac{\partial u^j}{\partial t} = c^p \xi^p_j v^j_k \frac{\partial u^k}{\partial x} = v^p c^p \xi^p_k \frac{\partial u^k}{\partial x} = v^p \frac{\partial R^p}{\partial u^k} \frac{\partial u^k}{\partial x} = v^p R^p_x. \tag{2.3}$$

We illustrate how this algorithmic procedure works with the following examples.

Example 2.3. Consider the system of equations of gas dynamics (2.2) in the case of polytropic equation of state $p(\rho) = \rho^\gamma$,

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + u u_x + \gamma \rho^{\gamma - 2} \rho_x &= 0. \end{aligned} \tag{2.4}$$

Solving the characteristic equation $\gamma \rho^{\gamma-1} - (u - \lambda)^2 = 0$ we find

$$\lambda_{1,2} = u \pm (\gamma \rho^{\gamma - 1})^{1/2}.$$

In order to find the Riemann invariants $R^i, i = 1, 2$, we require that

$$\begin{pmatrix} R^i_\rho & R^i_u \end{pmatrix} \begin{pmatrix} u - \lambda_i & \rho \\ \gamma \rho^{\gamma-2} & u - \lambda_i \end{pmatrix} = 0.$$

We can then find R^1 and R^2 in terms of u and ρ

$$R^1 = u + \frac{2\gamma^{1/2} \rho^{(\gamma-1)/2}}{\gamma - 1}, \qquad R^2 = u - \frac{2\gamma^{1/2} \rho^{(\gamma-1)/2}}{\gamma - 1},$$

and the eigenvalues λ_1, λ_2 in terms of R^1 and R^2

$$\lambda_1 = \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^1 - R^2)}{4}, \quad \lambda_2 = \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^2 - R^1)}{4}.$$

This way the initial system can be written in the diagonal form

$$\begin{aligned} R^1_t + \left(\frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^1 - R^2)}{4} \right) R^1_x &= 0, \\ R^2_t + \left(\frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^2 - R^1)}{4} \right) R^2_x &= 0, \end{aligned} \tag{2.5}$$

and it can be verified that system (2.4) can be brought in the form (2.5) by the change of variables

$$\frac{R^1+R^2}{2}=u, \qquad R^1-R^2=\frac{4\gamma^{1/2}\rho^{(\gamma-1)/2}}{\gamma-1}.$$

Example 2.4. Consider the equations of ideal chromatography, describing the flow of an n-component mixture through an absorbing medium (see [36, 82])

$$cu^i_x+(a^i(u)+u^i)_t=0, \quad i=1,...,n,$$

where $c=const$ and u^i and a^i are the concentrations of nonabsorbed and absorbed ith component respectively. In variables x and $\tau=ct-x$ these equations simplify to

$$u^i_x+a^i(u)_\tau=0, \tag{2.6}$$

which may be rewritten in the hydrodynamic type form $u^i_x-v^i_j(u)u^j_\tau=0$. To define this system completely, one needs to specify an isotherm, an explicit form of dependence $a^i=a^i(u)$. For example, in the case of a classical Langmuir isotherm

$$a^i=k_iu^i/V, \quad V:=1+\sum_{s=1}^n k_su^s, \tag{2.7}$$

where k_i are constants, the characteristic equation $\det(v^i_j-\lambda\delta^i_j)=0$ takes the form

$$V=\sum_{p=1}^n\frac{k_p^2u^p}{k_p-\lambda V}. \tag{2.8}$$

For each root λ^i of this equation, we define a function $R^i=\lambda^iV$, our candidate for a Riemann invariant corresponding to the eigenvalue λ_i. Straightforward calculations of the derivatives R^i_x and R^i_τ brings (2.6) into the form

$$R^i_x+\frac{R^i}{V}R^i_\tau=0. \tag{2.9}$$

In this system the coefficients R^i/V are expressed in terms of the functions u^i via $V(u)$. To eliminate V we proceed as follows. After multiplying both sides of (2.8) by $\prod_{p=1}^n(k_p-R)$, one gets an algebraic equation of order n with respect to R. This equation has $(-1)^nV$ and $\prod_p k_p$ as the coefficients at the highest and zero power of R, respectively. Therefore by

Viéte's formulas, which are formulas that relate the coefficients of a polynomial to sums and products of its roots, $\prod_p R^p = \frac{\prod_p k_p}{V}$. This gives the way to eliminate V from the last equation. Finally, the diagonal representation of the original problem takes the form

$$R^i_x + R^i \frac{\prod_p R^p}{\prod_p k_p} R^i_\tau = 0, \tag{2.10}$$

which justifies the choice of the quantities $R^i = \lambda^i V$ to be the Riemann invariants.

Notice that for the procedure described above we required that the initial hydrodynamic type systems are strictly hyperbolic, i.e. all eigenvalues of the matrix v^i_j are real and distinct. Then we are able to calculate the roots λ_p of the characteristic equation, find corresponding left eigenvectors ξ^p_j and compute all integrating factors c^p. However it is not always possible to have the explicit diagonal representation of a given system. There exists a useful criterion for the diagonalisability of systems of the form (2.1), where given the matrix v^i_j of the system one constructs the Nijenhuis tensor [66, 36]

$$N^i_{jk} := v^s_j \partial_s v^i_k - v^s_k \partial_s v^i_j - v^i_s(\partial_j v^s_k - \partial_k v^s_j),$$

and the Haantjes tensor [41, 36]

$$H^i_{jk} := (N^i_{qp} v^q_k - N^q_{kp} v^i_q) v^p_j - v^i_p (N^p_{qj} v^q_k - N^q_{kp} v^i_q).$$

Then

Theorem 2.1 *[41] A matrix $v^i_j(u)$ with real mutually distinct eigenvalues is diagonalisable by point transformations, if and only if the corresponding Haantjes tensor H^i_{jk} is identically zero.*

The theorem was stated in [41], in purely geometric terms as a condition of diagonalisability of a $(1,1)$-tensor field, but was also proved in [74]. It was first applied in the field of integrable systems to classify isotherms of absorption for which the equations of chromatography possess Riemann invariants [36].

2.3 The semi-Hamiltonian property

This section is dedicated to the study of the Hamiltonian theory for systems of equations of hydrodynamic type. We will outline the basic ideas of this theory in order to introduce semi-Hamiltonian systems [22, 82].

System (2.1) is called *Hamiltonian*, if there exists *a Poisson bracket* $\{\cdot,\cdot\}$ defined on a space of functions $u^i(x)$, as well as *a Hamiltonian*, which is a functional H, such that the system possesses the following representation,

$$u^i_t = \{u^i(x), H(x)\}. \tag{2.11}$$

These equations generate a Hamiltonian flow on the phase space of functions $u^i(x)$. A *first integral* of the above system is a functional $F(x)$, that satisfies the condition $\{F, H\} = 0$.

Definition 3 *A Poisson bracket on the space of functions $u^i(x)$ is called a bracket of hydrodynamic type, if it has the form*

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + b^{ij}_k(u(x))u^k_x\delta(x-y), \tag{2.12}$$

for some smooth functions $g^{ij}(u)$ and $b^{ij}_k(u)$.

In this case, for any two functionals I, J we have

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} A^{ij} \frac{\delta J}{\delta u^j(x)} dx, \qquad A^{ij} = g^{ij}(u(x))\frac{d}{dx} + b^{ij}_k(u(x))u^k_x.$$

Definition 4 *A Hamiltonian of hydrodynamic type is a functional $H(x) = \int h(u(x))dx$ with density $h(u)$ depending on u, but not the derivatives u^i_x, u^i_{xx}, etc. The system of equations*

$$u^i_t = \{u^i(x), H\} = (g^{ij}\partial_k\partial_j h + b^{ij}_k \partial_j h)u^k_x = v^i_k(u)u^k_x, \quad \partial_i = \partial/\partial u^i, \tag{2.13}$$

generated by these functionals, and the corresponding Poisson bracket (2.12) *will be called a Hamiltonian system of hydrodynamic type.*

The following theorem due to Dubrovin and Novikov holds.

Theorem 2.2 *[22]*

1. *Under local changes of coordinates* $u = u(w)$ *the coefficients* g^{ij} *transform as components of a* $(2,0)-$*tensor. Moreover if* $\det g^{ij} \neq 0$ *the quantities* Γ^j_{sk} *defined from the equation* $b^{ij}_k = -g^{is}\Gamma^j_{sk}$ *are transformed as components of Christoffel symbols corresponding to the metric* g_{ij}.

2. *The bracket* (2.12) *is antisymmetric if and only if* g_{ij} *is symmetric (meaning that it can be considered as a pseudo-Riemannian metric on the space of the variables* u*, if* $\det g^{ij} \neq 0$*) and the connection* Γ^j_{sk} *is compatible with this metric:* $\nabla_k g^{ij} = 0$.

3. *When* $\det g^{ij} \neq 0$*, the bracket* (2.12) *satisfies the Jacobi identity if and only if the connection* Γ^j_{sk} *is torsion free, i.e* $\Gamma^j_{sk} = \Gamma^j_{ks}$*, and the curvature tensor is zero.*

When these conditions are met, the matrices

$$v^i_k = g^{ij}\partial_k\partial_j h + b^{ij}_k\partial_j h = g^{is}\nabla_s\nabla_k h = \nabla^i\nabla_k h,$$

where $\nabla_s(h_k) = h_{sk} - \Gamma^j_{sk}h_j$, are called *Hamiltonian matrices.*

Lemma 2.3 *The matrix* $v^i_j(u)$ *is a matrix of a Hamiltonian system of hydrodynamic type, if and only if there exist a nondegenerate flat metric* g_{ij} *such that*

a) $g_{ik}v^k_j = g_{jk}v^k_i$, *and*

b) $\nabla_i v^k_j = \nabla_j v^k_i$, *where* ∇ *is the Levi-Civita connection corresponding to the metric* g_{ij}.

Proof of lemma 2.3:

Let $v^i_j(u) = \nabla^i\nabla_j h$ for a flat connection. Since we have a flat metric we have $g_{ik}v^k_j = \nabla_i\nabla_j h = \nabla_j\nabla_i h = g_{jk}v^k_i$. The same can be done for $\nabla_i v^k_j = \nabla_j v^k_i = \nabla_i\nabla^k\nabla_j h = \nabla_j\nabla^k\nabla_i h$. This finishes the proof of the Lemma. ■

We do not impose the Hamiltonian property for the integrability of the systems under consideration. Instead, we will require a weaker condition to be satisfied for our purposes.

Consider a diagonal system of hydrodynamic type,

$$u^i_t = v_i(u)u^i_x, \tag{2.14}$$

where from now on we will be using the notation above, i.e no summation will be implied over i. For the diagonal matrix $v^i_j(u) = v_j(u)\delta^i_j$, since the system is hyperbolic, $v_j(u)$ are distinct. Applying Lemma 2.3 to the diagonal matrices v^i_j we find that

$$0 = \nabla_i v^k_j - \nabla_j v^k_i = \partial_i v_j \delta^k_j - \partial_j v_i \delta^k_i + \Gamma^k_{ij}(v_j - v_i),$$

which means that $\Gamma^k_{ij} = 0$ for $i \neq j \neq k$ and

$$\frac{\partial_i v_k}{v_i - v_k} = \Gamma^k_{ki}, \quad i \neq k, \tag{2.15}$$

(no summation over repeated indices). Moreover the metric is diagonal due to

$$0 = g_{ik}v^k_j - g_{jk}v^k_i = g_{ik}v_j\delta^k_j - g_{jk}v_i\delta^k_i = g_{ij}(v_j - v_i),$$

with $v_i \neq v_j$ for $i \neq j$ (since the system is hyperbolic), and leads to

$$\Gamma^k_{ki} = \frac{1}{2}\partial_i \log g_{kk}, \tag{2.16}$$

because $\Gamma^k_{ij} = \frac{1}{2}g^{ks}(\partial_j g_{si} + \partial_i g_{sj} - \partial_s g_{ij})$ and $g^{ij}g_{jk} = \delta^i_k$. Therefore, the conditions $\partial_j \Gamma^k_{ki} = \partial_i \Gamma^k_{kj}$ are equivalent to

$$\partial_i \left(\frac{\partial_j v_k}{v_j - v_k} \right) = \partial_j \left(\frac{\partial_i v_k}{v_i - v_k} \right), \qquad i \neq j \neq k. \tag{2.17}$$

The conditions (2.15)-(2.17) can be viewed as the linear system of equations for $v_i(u)$ and if we compute the compatibility conditions we find that

$$\partial_i\partial_j v_k - \partial_j\partial_i v_k = v_i R^k_{kji} + v_j R^k_{kij} - v_k \left(\partial_i \Gamma^k_{kj} - \partial_j \Gamma^k_{ki} \right), \tag{2.18}$$

where

$$R^k_{kij} = \partial_i \Gamma^k_{kj} - \Gamma^k_{kj}\Gamma^j_{ji} - \Gamma^k_{ki}\Gamma^i_{ij} + \Gamma^k_{ki}\Gamma^k_{kj},$$

are components of the curvature tensor R^i_{jkl}. For the Hamiltonian matrix $v^i_j(u)$ these compatibility conditions are satisfied as the metric g_{ij} is flat, and the last bracket in (2.18) is zero due to (2.16).

In general, the solution of a consistent linear system of the type

$$\frac{\partial w_i}{\partial u^k} = \sum_{s=1}^{n} f_{ik}^s(u) w_s, \quad i \neq k, \ f_{ik}^s \in C^r(\mathcal{D}), \tag{2.19}$$

depends on n functions of a single argument, i.e one can formulate the Goursat-type problem, where w_k is defined on u^k-axis only.

The following summarises the ideas above.

Theorem 2.4 *[85] The metric associated with a diagonal Hamiltonian matrix $v_j^i(u)$ of a hyperbolic system is diagonal, and the variables u form a curvilinear orthogonal system of coordinates in flat space. Also, for each flat, curvilinear orthogonal system of coordinates there exists a family of Hamiltonian matrices, which are diagonal in these coordinates and this family is parametrised locally by n functions of a single argument. For the matrix v_j^i relations* (2.17) *hold*

$$\partial_i \left(\frac{\partial_j v_k}{v_j - v_k} \right) = \partial_j \left(\frac{\partial_i v_k}{v_i - v_k} \right), \qquad\qquad i \neq j \neq k.$$

The consistency of conditions (2.15) is a consequence of the property (2.17), i.e if v_i are the coefficients of a diagonal system (2.14) (not necessarily Hamiltonian) such that they are distinct and satisfy (2.17), then the system

$$\partial_i w_k = \Gamma_{ki}^k (w_i - w_k), \quad \Gamma_{ki}^k := \frac{\partial_i v_k}{v_i - v_k}, \quad i \neq k, \tag{2.20}$$

for the functions $w_i(u)$, $i = 1, ..., n$, is consistent. So for a given system of equations (2.14) with n diagonal elements v_i, we can find another n functions $w_1, ..., w_n$ from the overdetermined system (2.20). The consistency of the last system follows from the fact that

$$R_{kji}^k = \partial_i \Gamma_{kj}^k - \Gamma_{kj}^k \Gamma_{ji}^j - \Gamma_{ki}^k \Gamma_{ij}^i + \Gamma_{ki}^k \Gamma_{kj}^k = \tag{2.21}$$

$$= -\frac{v_i - v_k}{v_i - v_j} \left[\partial_i \left(\frac{\partial_j v_k}{v_j - v_k} \right) - \partial_j \left(\frac{\partial_i v_k}{v_i - v_k} \right) \right] = 0. \tag{2.22}$$

Definition 5 *A diagonal quasilinear system* (2.14) *is called semi-Hamiltonian if it is hyperbolic and its coefficients v_i satisfy the relation* (2.17)*. For $n \leq 2$ any hyperbolic system* (2.14) *is semi-Hamiltonian.*

Diagonal Hamiltonian systems are automatically semi-Hamiltonian, but the converse is not true. The semi-Hamiltonian property is necessary and suffcient for the integrability of a system of hydrodynamic type. In order to check the integrability one needs to verify that the system is diagonalisable, then bring it in a diagonal form (as discussed in the previous section) and finally check the semi-Hamiltonian property (2.17).

2.4 Conservation laws and commuting flows

A finite dimensional Hamiltonian system is said to be integrable when there exists a certain number of first integrals (as many as the dimension of the system), that are in involution. In this section we explain how this idea is understood for $1+1$ dimensional hydrodynamic type systems, and for this purpose we first give the definition of a first integral.

A first integral of the system (2.1), is a functional of the form

$$I = \int P(u)dx, \qquad u = (u^1, ..., u^n), \tag{2.23}$$

with density $P(u)$ independent of the spatial derivatives of the variables u, i.e $u^i_x, u^i_{xx}, ...$, which commutes with the Hamiltonian. This functional I is called *hydrodynamic type first integral* of (2.1), and together with the action of the Poisson bracket generates the flow

$$u^i_\tau = \{u^i(x), I\} = w^i_j(u)u^j_x, \tag{2.24}$$

which commutes with the flow (2.13). Since the integral I and the Hamiltonian commute, $\{H, I\} = 0$, it follows from the Jacobi identity that $u_{t\tau} = u_{\tau t}$. Commuting flows are sometimes refered to as *symmetries*. The following lemma establishes a connection between commuting Hamiltonian flows and conservation laws.

Lemma 2.5 *The functional* (2.23) *is an integral of the Hamiltonian system* (2.13) *if and only if the matrix* $w^i_j(u) = \nabla^i\nabla_j P$ *of the Hamiltonian flow* (2.24) *generated by* I *(and the same Poisson bracket) commutes with the matrix* v^i_j*, i.e.* $v^i_k w^k_j = w^i_k v^k_j$.

Proof of lemma 2.5:

Consider the identity

$$I_t = \int \partial_t P dx = \int \partial_i P v^i_j u^j_x dx = 0.$$

Its variational derivative

$$\frac{\delta}{\delta u^i} \int \partial_i P v^i_j u^j_x dx \equiv 0,$$

must be trivially zero, but from Lemma 2.3 and the relation $\partial_i \partial_j P = \nabla_i \nabla_j P - \Gamma^k_{ij} \partial_k P$ we obtain

$$\left[\partial_k(\partial_i P v^i_j) - \partial_j(\partial_i P v^i_k)\right] u^j_x = \left[(\nabla_k \nabla_i P) v^i_j - (\nabla_j \nabla_i P) v^i_k\right] u^j_x = 0,$$

which due to $g^{lk} v^i_l = g^{li} v^k_l$ becomes

$$g^{kl}\left[(\nabla_l \nabla_i P) v^i_j - (\nabla_j \nabla_i P) v^i_l\right] = (\nabla^k \nabla_i P) v^i_j - v^k_l (\nabla^l \nabla_j P) = w^k_i v^i_j - v^k_l w^l_j = 0.$$

Conversely, if v^i_j and w^i_j commute, the previous argument shows that $\partial_i(\partial_k P v^k_j) = \partial_j(\partial_k P v^k_i)$, which implies the existence of a function $Q(u)$ such that $\partial_j Q = \partial_i P v^i_j$, i.e.

$$I_t = \int \partial_k P v^k_j u^j_x dx = \int \frac{d}{dx} Q(u) dx = 0, \tag{2.25}$$

and the lemma is proved. ■

If a Hamiltonian system of the type $u_t = \{u, H\}$ possesses a hydrodynamic integral (2.23), then due to (2.25) there exists a function $Q(u)$ such that the system also possesses a *conservation law* of the form

$$P(u)_t = Q(u)_x. \tag{2.26}$$

We will now show that any semi-Hamiltonian diagonal system (2.14) has infinitely many independent hydrodynamic integrals (2.23), locally parametrized by n functions of one variable. From the relation $\partial_i(\partial_k P v^k_j) = \partial_j(\partial_k P v^k_i)$, one can show that in order for (2.23) to be a first integral of the semi-Hamiltonian system (2.14) it is necessary and sufficient that

$$\partial_i \partial_j P - \Gamma^i_{ij} \partial_i P - \Gamma^j_{ji} \partial_j P = 0, \quad i \neq j, \tag{2.27}$$

where $\Gamma^k_{ki} = \frac{\partial_i v_k}{v_i - v_k}$ as defined in the previous section. Introducing new variables $z_i = \partial_i P$, we can rewrite the system in the form

$$\partial_i z_j = \Gamma^i_{ij} z_i + \Gamma^j_{ji} z_j. \tag{2.28}$$

The consistency conditions for this system,

$$\partial_k(\partial_i z_j) - \partial_i(\partial_k z_j) = z_i R^k_{kji} + z_k R^k_{kij} - z_j \left(\partial_k \Gamma^j_{ji} - \partial_i \Gamma^j_{jk}\right) = 0, \tag{2.29}$$

are satisfied due to (2.21). The solutions of a system (2.28) are parametrized by n functions of one variable as this system fits into the class (2.19). Each hydrodynamic integral of a semi-Hamiltonian system obviously generates a conservation law (2.26).

Theorem 2.6 *[82] A semi-Hamiltonian diagonal system* (2.14) *has infinitely many commuting flows, parametrized locally by n functions of one variable. These flows commute with each other, their matrices are diagonal and all the hydrodynamic integrals of the original semi-Hamiltonian system are their integrals as well.*

Proof of theorem 2.6:

Commuting the flows (2.14) and (2.24) and denoting $v^i_j = v_j \delta^i_j$ we have

$$u^i_{\tau t} - u^i_{t\tau} = (\partial_k w^i_j v^k_p - \partial_k v^i_j w^k_p) u^p_x u^j_x + (w^i_j \partial_k v^j_q - v^i_j \partial_k w^j_q) u^q_x u^p_x + (w^i_j v^j_p - v^i_j w^j_p) u^p_{xx}.$$

We want this expression to be trivially zero, so if we consider it as a polynomial in u^i_x, u^i_{xx}, etc, each of the coefficients of this polynomial should be zero. This means that the coefficient of u^p_{xx} is zero, $(w^i_j v^j_p - v^i_j w^j_p) = 0$, and since $v^i_j = v_j \delta^i_j$ then w^i_j is also diagonal and we can denote it as $w^i_j = w_j \delta^i_j$. Then, from the remaining terms

$$u^i_{\tau t} - u^i_{t\tau} = \sum_{k \neq i} \left(\partial_k v_i (w_k - w_i) - \partial_k w_i (v_k - v_i)\right) u^i_x u^k_x = 0, \quad i \neq k, \tag{2.30}$$

we get the system of equations (2.20). Any two, diagonal flows satisfying (2.30) and (2.20), automatically commute. Finally, any hydrodynamic integral with the density P of the original semi-Hamiltonian system is also an integral of the symmetries, as one can see from

$$(w_j - w_i)\left(\partial_i \partial_j P - \Gamma^i_{ij} \partial_i P - \Gamma^j_{ji} \partial_j P\right) = (w_j - w_i)\partial_i \partial_j P - \partial_i P \partial_j w_i - \partial_j P \partial_i w_j = 0. \tag{2.31}$$

This finishes the proof of the theorem. ■

2.5 Generalized hodograph method

In this section we discuss the generalized hodograph method, which can be used to find solutions of semi-Hamiltonian systems [82].

Consider the hyperbolic system (2.14) and suppose that u is a two-dimensional vector $u = (u^1, u^2)$. Then the original system is automatically semi-Hamiltonian and its solutions can be constructed using the *classical hodograph method*. Applications of this method, in the context of fluid dynamics, were considered by Riemann [76] who introduced functions $r = r(x,t)$ and $s = s(x,t)$ (the Riemann invariants) and then expressed x and t in terms of r and s.

In order to show how the hodograph method can be applied, consider a system of two equations of the form

$$u^i_t = v^i_j u^i_x,$$

for the unknown functions $u^1 = u(t,x)$ and $u^2 = w(t,x)$. If both functions are locally invertible we can express variables t and x in terms of u and w, i.e $t = t(u,w)$ and $x = x(u,w)$. Then, using the chain rule, we expand the relations $x_x = 1, x_t = 0$ and $t_x = 0, t_t = 1$ and reduce them modulo the initial system. What we obtain is a linear system of equations for the functions x_u, x_w, t_u, t_w.

For example, for the shallow-water equations

$$u_t + uu_x + h_x = 0,$$

$$h_t + (hu)_x = 0,$$

following the procedure described above, the corresponding linear system for the functions $x(u,h)$, $t(u,h)$ is

$$x_h = ut_h - t_u,$$

$$-x_u = ht_h - ut_u,$$

or equivalently

$$(x - ut)_h = -t_u,$$

$$(x - ut)_u = -ht_h - t.$$

This way the quasilinear system of two equations is reduced to a linear one.

For a diagonal, n-component, $n > 2$, semi-Hamiltonian system of the form (2.14), solutions are given by the *generalized hodograph method* [82]. As we have discussed, any diagonal semi-Hamiltonian system (2.14) with nondegenerate metric has infinitely many commuting flows $u^i_\tau = w_i u^i_x$ where the coefficients $w_i(u)$ satisfy the linear system

$$\frac{\partial_i w_k}{w_i - w_k} = \frac{\partial_i v_k}{v_i - v_k}, \quad i \neq k. \tag{2.32}$$

We can construct the system of n equations for the n unknowns u^i

$$w_i(u) = v_i(u)t + x, \tag{2.33}$$

where x and t are parameters, v_i are the coefficients of the initial semi-Hamiltonian system, and w_i are coefficients of the corresponding commuting flow (satisfying (2.32)). Then

Theorem 2.7 *[82] Any smooth solution $u^i(t,x)$ of* (2.33) *is a solution of the diagonal semi-Hamiltonian system* (2.14). *Moreover, any solution of a given system* (2.14) *may be locally represented as a solution of* (2.33) *in a neighbourhood of a point (t_0, x_0) such that $u^i_x(t_0, x_0) \neq 0$ for every i.*

Proof of theorem 2.7:

To prove this theorem we first differentiate (2.33) with respect to t and x

$$\sum_{k=1}^{n} M_{ik} u^k_t = v_i(u), \quad \sum_{k=1}^{n} M_{ik} u^k_x = 1, \tag{2.34}$$

where $M_{ik} = \partial_k w_i(u) - \partial_k v_i(u)t$. If we take into account (2.33) and the semi-Hamiltonian property (2.32) we have $M_{ik} = 0$ for $i \neq k$, $M_{ii} u^i_t = v_i(u)$ and $M_{ii} u^i_x = 1$. It then follows that $u^i_t = v_i(u) u^i_x$. Note that we have the condition $u^i_x \neq 0$.

Conversely, let $u^i(x,t)$ be a solution of (2.14), and assume $u^i_x \neq 0$ in a neighbourhood of (x_0, t_0). The initial conditions $u^i_0(x) = u^i(x, t_0)$ induce the initial Cauchy data

$$w_i(u_0(x)) = v_i(u_0(x))t + x, \tag{2.35}$$

on the curve $u_0(x)$ for the problem (2.32). As $(u^i_0)_x(x_0) \neq 0$ by assumption, in a neighbourhood of $u_0(x)$ there exists an unique solution of (2.32) with initial conditions (2.35).

For these $w_i(u)$ the system (2.33) has a unique solution $\tilde{u}^i(x,t)$ in the neighbourhood of a point (x_0, t_0, u_0^i) since the Jacobian of (2.33) is diagonal at this point and $M_{ii} = (u_0^i)_x^{-1} \neq 0$. Since $\tilde{u}^i(x,t)$ is a solution of (2.14) and $\tilde{u}^i(x,t_0) = u_0^i(x)$, by the uniqueness of the solution of the Cauchy problem we have $\tilde{u}^i = u^i$ in a neighbourhood of (x_0, t_0) and the theorem is proved. ■

Chapter 3

Quasilinear Partial Differential Equations in 2+1D

Consider integrable $(1+1)$-dimensional scalar evolutionary equations of the form

$$u_t = F(u),$$

where $u(x,t)$ is a scalar potential, and F denotes a differential expression depending on x-derivatives of u up to a finite order. The classification of equations of this type has been a very active research topic, and various results have been obtained under additional assumptions for the expression F [62].

In this chapter, we are interested in studying a similar problem in $2+1$ dimensions, where now F contains nonlocal variables. In this direction, classification results are very few. Equations of the form

$$u_t = F(u, w),$$

where $u(x,y,t)$ is a scalar field and $w(x,y,t)$ is the nonlocal variable, which is assumed to have a simple form $w = D_x^{-1}D_y u$ or $w = D_y^{-1}D_x u$, were considered in [32]. We assume that the right hand side of the equation, F, is polynomial in the x- and y-derivatives of u and w, while the dependence on u and w itself is allowed to be arbitrary.

Initially, we review the case of equations with the simple nonlocality of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots), \qquad w_x = u_y, \tag{3.1}$$

where dots denote terms which are homogeneous polynomials of degree two and three in the x- and y-derivatives of u and w, whose coefficients are allowed to be functions of u and w. Equations of this type where thoroughly studied in [32], using the method of hydrodynamic reductions (which will be explicitly explained in this chapter). Here, we review the main results of this study: we discuss the integrability of the corresponding dispersionless equations (as $\epsilon \to 0$)

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y, \tag{3.2}$$

and state the classification theorem of integrable dispersive equations of type (3.1). Our study is then extended to equations where a second nonlocality is added, in the following way

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_y, \qquad w_x = u_y, \qquad v_x = f(u,w)_y. \tag{3.3}$$

Classification of integrable equations within this class, via the method of hydrodynamic reductions, turns out to be a computationally hard task. For this reason, we introduce an alternative approach which is based on Lax pairs. This approach allows us to easily classify integrable equations of type (3.3), and also reproduce the classifications result for equations (3.2). Moreover, we can make some remarks in the case of systems with more than two nonlocalities, which we call *nested nonlocalities*.

Particularly, the method of hydrodynamic reductions is described in section 3.1. This method is a way to decouple a quasilinear $(2+1)$-dimensional system into a pair of $1+1$ hydrodynamic type equations. The dispersionless Kadomtsev–Petviashvili (dKP) equation is used as an example. In section 3.2, we show a way to reconstruct dispersive deformations for a given integrable dispersionless system, by deforming hydrodynamic reductions, and as an example we use Kadomtsev–Petviashvili (KP) equation. The method of hydrodynamic reductions can be applied to equations whose dispersionless limit is *nondegenerate*, and these nondegeneracy conditions are explained in section 3.3. In section 3.4, we briefly introduce the concept of the so called dispersionless Lax pairs, and we show how they can be used to classify dispersionless integrable systems. The next three sections, 3.5, 3.6, 3.7, contain the main classification results in the case of equations with one, two, and more than two (nested) nonlocalities, respectively. Finally, in section 3.8, we find commuting flows of systems (3.2) and (3.3). As an example, we present the commuting flows of dKP, and a class of equations with nested nonlocalities.

3.1 The method of hydrodynamic reductions

The theory of integrability of one-dimensional hydrodynamic type systems provides the framework for studying the integrability of higher dimensional hydrodynamic type systems. Based on the ideas and results of the previous chapter, the scheme [27], which is known as the method of hydrodynamic reductions, provides a way to study the integrability of higher dimensional systems.

We consider a $(2+1)$-dimensional system of hydrodynamic type in the form

$$A(u)u_x + B(u)u_y + C(u)u_t = 0, \tag{3.4}$$

where $u = (u^1, ..., u^m)^t$ is an m-component column vector of dependent variables, and A, B, C are square $m \times m$ matrices.

Remark Generally, A, B, C could be $l \times m$ matrices where l, the number of equations, is allowed to exceed the number of the unknowns, m. For example, equation

$$u_{xt} - u_x u_{xx} = u_{yy},$$

if we make the change $a = u_x, b = u_y, c = u_t$, takes the form

$$a_y = b_x, \quad a_t = c_x, \quad b_t = c_y, \quad a_t - a a_x = b_y,$$

which means that we have four equations for the three unknowns a, b, c.

The key construction in the method of hydrodynamic reductions is the following ansatz. We seek multi-phase exact solutions of the form

$$u(x, y, t) = u(R^1, ..., R^n)$$

where the 'phases' $R^i(x, y, t)$ are the Riemann invariants satisfying a pair of commuting diagonal $(1+1)$-dimensional systems of hydrodynamic type,

$$R^i_y = \mu^i(R) R^i_x, \qquad R^i_t = \lambda^i(R) R^i_x. \tag{3.5}$$

In other words, given a $(2+1)$-dimensional system (3.4) we can decouple it into a pair of commuting $(1+1)$-dimensional systems of hydrodynamic type (3.5), which explains the name hydrodynamic reductions. Solutions of this type are also known as "non-linear interactions of n planar simple waves", or "multi - phase solutions", or "multiple waves" and have been thoroughly investigated in gas dynamics and later in the context of dispersionless KP and Toda hierarchies. Now, for system (3.5)

$$R^i_{yt} = \frac{\partial \mu^i}{\partial R^j} R^j_t R^i_x + \mu^i (\frac{\partial \lambda^i}{\partial R^j} R^j_x R^i_x + \lambda^i R^i_{xx}) = \frac{\partial \mu^i}{\partial R^j} \lambda^j R^j_x R^i_x + \mu^i (\frac{\partial \lambda^i}{\partial R^j} R^j_x R^i_x + \lambda^i R^i_{xx}),$$
$$R^i_{ty} = \frac{\partial \lambda^i}{\partial R^j} R^j_y R^i_x + \lambda^i (\frac{\partial \mu^i}{\partial R^j} R^j_x R^i_x + \mu^i R^i_{xx}) = \frac{\partial \lambda^i}{\partial R^j} \mu^j R^j_x R^i_x + \lambda^i (\frac{\partial \mu^i}{\partial R^j} R^j_x R^i_x + \mu^i R^i_{xx}),$$

which means that the consistency condition, $R^i_{yt} = R^i_{ty}$, is equivalent to the linear system for the diagonal elements λ^i and μ^i:

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j, \quad \partial_i = \partial / \partial_{R^i}. \tag{3.6}$$

By direct substitution of the ansatz (3.5)-(3.6) into (3.4), we arrive at the equations

$$(A + \mu^i B + \lambda^i C)\partial_i u = 0, \quad i = 1, ..., n. \tag{3.7}$$

which (in our case of square matrices A, B and C) implies that λ^i and μ^i satisfy the dispersion relation

$$det(A + \mu B + \lambda C) = 0. \tag{3.8}$$

When (3.6) holds, the solution of (3.4) is given by the generalised hodograph formula

$$v^i(R) = x + \lambda^i(R)t + \mu^i(R)y, \tag{3.9}$$

where v^i is the general solution of the linear system

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j. \tag{3.10}$$

Combining the equations (3.6) and (3.7), we end up with the system of equations for u, $\lambda^i(R)$ and $\mu^i(R)$ (so called *Gibbons-Tsarev system*).

Definition 6 *[27, 30] A system (3.4) is said to be integrable if, for any number of phases n, it possesses infinitely many n-component reductions parametrised by n arbitrary functions of a single argument.*

The example of dKP equation discussed below explains the freedom of n arbitrary functions parametrising n-component reductions, or $2n$ arbitrary functions parametrising n-phase solutions.

Here are some important remarks based on this theory.

- The procedure described above works when the dispersion relation (3.8) describes an irreducible algebraic curve, in the sense that (3.8) can not be factorised.

- In the case $n = 1$, we have $u = u(R)$, with R being a solution of

$$R_t = \lambda(R)R_x, \qquad R_y = \mu(R)R_x,$$

  and we automatically have $R_{ty} = R_{yt}$. The hodograph formula in the scalar case gives $f(R) = x + \lambda(R)t + \mu(R)y$ with $f(R)$ being an arbitrary function. This means that the surfaces $R = const$ are planes so that the solution $u = u(R)$ is constant along a one-parameter family of planes. This kind of solutions exist for all multi-dimensional quasilinear systems and cannot be used to check integrability.

- When we consider two-component reductions, i.e. $u = u(R^1, R^2)$ with R^1 and R^2 satisfying (3.5) then the general solution is given by the formula

$$v^1(R) = x + \lambda^1(R)t + \mu^1(R)y, \qquad v^2(R) = x + \lambda^2(R)t + \mu^2(R)y.$$

  When $R = const$ we have a two -parameter family of lines in the space (x, y, t) which means u is constant along the lines of a two parameter family.

- The existence of three component reductions is a very strong condition. This is evident if we consider the Gibbons - Tsarev system (3.6), (3.7). The compatibility conditions of this system involve triplets of indices $i \neq j \neq k$ which is very restrictive. Thus, the existence of infinitely many three-phase reductions guarantees the existence of higher phase reductions, and hence integrability.

In the case $m > 2$ we have a simple necessary condition for integrability, which can be obtained in the following way: Introduce the $m \times m$ matrix

$$V = (\alpha A + \beta B + \gamma C)^{-1}(\tilde{\alpha} A + \tilde{\beta} B + \tilde{\gamma} C)$$

where α, β, γ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are arbitrary constants. Recall the diagonalisation criterion from chapter 2: From the $(1,1)$-tensor $V = [v^i_j]$, introduce the Nijenhuis tensor

$$\mathcal{N}^i_{jk} = v^p_j \partial_{u^p} v^i_k - v^p_k \partial_{u^p} v^i_j - v^i_p (\partial_{u^j} v^p_k - \partial_{u^k} v^p_j),$$

and the Haantjes tensor

$$\mathcal{H}^i_{jk} = \mathcal{N}^i_{pr} v^p_j v^r_k - \mathcal{N}^p_{jr} v^i_p v^r_k - \mathcal{N}^p_{rk} v^i_p v^r_j + \mathcal{N}^p_{jk} v^i_r v^r_p.$$

Both those tensors can be computed using computer algebra. Then

Theorem 3.1 *[29] The vanishing of the Haantjes tensor (for any value of α, β, γ, $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$) is a necessary condition for the integrability of the quasilinear system (3.4).*

This necessary condition, in most cases, turns out to be sufficient.

3.1.1 The example of dKP equation

Let us apply the method of hydrodynamic reductions to dKP equation

$$u_t = uu_x + w_y, \quad w_x = u_y$$

written in a hydrodynamic form. Let us look for solutions in the form $u = u(R^1, \dots, R^n)$, $w = w(R^1, \dots, R^n)$, where the Riemann invariants R^i satisfy (3.5)

$$R^i_y = \mu^i(R) R^i_x, \qquad R^i_t = \lambda^i(R) R^i_x.$$

Substituting this ansatz into dKP we obtain

$$\partial_i w = \mu^i \partial_i u, \qquad \lambda^i - u - (\mu^i)^2 = 0,$$

where from the compatibility conditions $\partial_i \partial_j w = \partial_j \partial_i w$, we have

$$\partial_i \partial_j u = \frac{\partial_j \mu^i}{\mu^j - \mu^i} \partial_i u - \frac{\partial_i \mu^j}{\mu^j - \mu^i} \partial_j u. \tag{3.11}$$

Also the commutativity condition $R^i_{yt} = R^i_{ty}$ leads to

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \quad i \neq j. \tag{3.12}$$

Substituting the last equation into (3.11) gives the system for $u(R)$ and $\mu^i(R)$ (the Gibbons-Tsarev system),

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \quad \partial_i \partial_j u = 2\frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}, \quad i \neq j. \tag{3.13}$$

The consistency of this system is equivalent to the existence of infinitely many hydrodynamic reductions (3.5) of dKP. For the general solution of this system, one should prescribe $2n$ functions of a single variable as the Goursat data along the R^i-axes, precisely $\mu^i(R^i)$ and $u(R^i)$. As the last system is invariant under reparametrisation $f^i(R^i) \to R^i$ where f^i are arbitrary functions of their arguments, the parametric freedom reduces to n functions of a single variable. A general solution of the system (3.5) given by the generalized hodograph method, brings extra n arbitrary functions to the parametric freedom of a n-phase solution $u(R^1, ..., R^n)$ of the dKP equation.

3.2 Dispersive deformations of integrable dispersionless systems

Given a dispersionless system, we want to reconstruct dispersive terms. This can be done by requiring that all hydrodynamic reductions of the dispersionless system are inherited by its dispersive counterpart [31, 32].

In general, we proceed as follows: consider the $(2+1)$-dimensional hydrodynamic type system (3.4)

$$A(u)u_x + B(u)u_y + C(u)u_t = 0,$$

where $u = (u^1, ..., u^m)^t$ is an m-component column vector of dependent variables, and A, B, C are square $m \times m$ matrices. Following the method of hydrodynamic reductions described in the previous section, we seek multi-phase solutions of the form

$$u(x, y, t) = u(R^1, ..., R^n)$$

where $R^i(x, y, t)$ satisfy a pair of commuting $(1+1)$-dimensional hydrodynamic type systems

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x.$$

We now seek a *k-th* order dispersive deformation of equation (3.4) of the form

$$A(u)u_x + B(u)u_y + C(u)u_t + \epsilon(\dots) + \epsilon^2(\dots) + \cdots + \epsilon^k(\dots) + \cdots = 0, \tag{3.14}$$

where terms in the brackets are $m \times m$ matrices, whose entries are homogeneous differential polynomials in the x- and y-derivatives of u, of order $k+1$. Coefficients of these polynomials are allowed to be arbitrary functions of u. Then, we require that multi-phase solutions can be deformed accordingly,

$$u = u(R^1, ..., R^n) + \epsilon u_1 + \cdots + \epsilon^k u_k + O(\epsilon^{k+1}) \tag{3.15}$$

where u_i are assumed to be homogeneous polynomials of degree i in the x-derivatives of R^i's. Similarly, hydrodynamic reductions can be deformed as

$$\begin{aligned} R^i_y &= \mu^i(R)R^i_x + \epsilon a_1 + \cdots + \epsilon^k a_k + O(\epsilon^{k+1}),\\ R^i_t &= \lambda^i(R)R^i_x + \epsilon b_1 + \cdots + \epsilon^k b_k + O(\epsilon^{k+1}). \end{aligned} \tag{3.16}$$

where a_k, b_k are assumed to be homogeneous polynomials of degree $k+1$ in the x-derivatives of R^i's. Substituting (3.15) into (3.14), and using (3.16) along with the consistency conditions $R^i_{ty} = R^i_{yt}$, one arrives at a complicated set of relations, allowing one to uniquely reconstruct dispersive terms in (3.14).

Remark. In most cases, we can get the necessary classification results working with one-component reductions only so, for our purposes, we will be using only those throughout this thesis. This is because although one-component reductions are not enough to provide results for dispersionless equations (all dispersionless equations possess one-component reductions), they turn out to be sufficient when working with deformations.

In this chapter, we want to review a particular class of equations, namely equations of type

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y,$$

which where thoroughly studied in [32]. Using the method of hydrodynamic reductions we find that one-component reductions are of the form $u = R$, $w = w(R)$, where $R(x, y, t)$

satisfies a pair of Hopf-type equations

$$R_y = \mu R_x, \qquad R_t = (\varphi + \psi\mu + \eta\mu^2)R_x,$$

with μ being an arbitrary function of R, $w' = \mu$ and $\lambda = \varphi + \psi\mu + \eta\mu^2$ being the dispersion relation. For these equations, we seek a third order (in $x-, y-$ derivatives) dispersive deformation of the form

$$\begin{aligned} u_t &= \varphi u_x + \psi u_y + \eta w_y + \epsilon(...) + \epsilon^2(...), \\ w_x &= u_y, \end{aligned} \tag{3.17}$$

where the terms at ϵ and ϵ^2 are homogeneous differential polynomials in the x- and y-derivatives of u and w of the order two and three, respectively, whose coefficients are allowed to be arbitrary functions of u and w. Following the methodology described above, we require that one-phase solutions can be deformed accordingly,

$$\begin{aligned} u &= u(R) + \epsilon u_1 + \cdots + \epsilon^m u_2 + O(\epsilon^{m+1}), \\ w &= w(R) + \epsilon w_1 + \cdots + \epsilon^m w_2 + O(\epsilon^{m+1}), \end{aligned} \tag{3.18}$$

where u_i, w_i are assumed to be homogeneous polynomials of degree i in the x-derivatives of R's (thus, both R_{xx} and R_x^2 have degree two, etc). Expansions (3.18) are invariant under Miura-type transformations of the form $R \to R + \epsilon r_1 + \epsilon^2 r_2 + \dots$, where r_i denote terms which are polynomial of degree i in the x-derivatives of R's. These transformations can be used to simplify calculations. For instance in our case of one-phase solutions we can assume that u remains undeformed, i.e $u = R$ [23]. Also, the hydrodynamic reductions can be deformed as

$$\begin{aligned} R_y &= \mu R_x + \epsilon a_1 + \cdots + \epsilon^m a_m + O(\epsilon^{m+1}), \\ R_t &= (\varphi + \psi\mu + \eta\mu^2)R_x + \epsilon b_1 + \cdots + \epsilon^m b_m + O(\epsilon^{m+1}). \end{aligned} \tag{3.19}$$

with a_i, b_i being homogeneous polynomials of degree $i + 1$ in the x-derivatives of R's. Substituting (3.18) into (3.17), and using (3.19) along with the consistency conditions $R_{ty} = R_{yt}$, we can reconstruct dispersive terms in (3.17). This procedure is required to work for arbitrary μ: whenever one obtains a differential polynomial in μ which has to vanish due to the consistency conditions, all its coefficients have to be set equal to zero independently.

Remark. The reconstruction procedure does not necessarily lead to a unique dispersive extension. One and the same dispersionless system may possess essentially non-equivalent dispersive extensions. A simple example illustrating this (see section 3.5), is the dispersionless equation

$$u_t = (uw)_y, \qquad w_x = u_y,$$

which leads to two non-equivalent dispersive extensions, namely Veselov-Novikov (VN) and modified Veselov-Novikov (mVN) equation respectively:

$$u_t = (uw)_y + \epsilon^2 u_{yyy}, \qquad w_x = u_y,$$
$$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y, \quad w_x = u_y.$$

3.2.1 The example of KP equation

Consider KP equation written in the form

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \quad w_x = u_y.$$

The dispersionless KP equation,

$$u_t = uu_x + w_y, \quad w_x = u_y,$$

possesses one-phase solutions of the form $u = R$, $w = w(R)$, with $R(x, y, t)$ satisfying the equations

$$R_y = \mu R_x, \quad R_t = (\mu^2 + R)R_x, \tag{3.20}$$

where $\mu(R)$ is an arbitrary function, $w' = \mu$ and $\lambda = \mu^2 + R$. Then, applying the procedure described above, one obtains the following deformed one-phase solutions

$$u = R, \quad w = w(R) + \epsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3)R_x^2\right) + O(\epsilon^4), \tag{3.21}$$

and the deformed equations (3.20) take the form [31]

$$\begin{aligned} R_y =& \mu R_x \\ &+ \epsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3) R_x^2 \right)_x + O(\epsilon^4), \\ R_t =& (\mu^2 + R) R_x \\ &+ \epsilon^2 \left((2\mu\mu' + 1) R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2) R_x^2 \right)_x + O(\epsilon^4), \end{aligned} \tag{3.22}$$

where all coefficients are uniquely determined by μ. This means that KP equation can be decoupled into a pair of $(1+1)$-dimensional equations (3.22) in infinitely many ways, since μ is arbitrary. Note that only one component reductions were used, although KP equation is known to possess infinitely many n-component reductions for arbitrary n [38, 52].

3.3 Linearly degenerate systems

As mentioned earlier, our method applies to dispersionless systems whose dispersion relation, i.e $det(A+\mu B+\lambda C)=0$, defines an irreducible curve (is not factorisable). Moreover, we exclude from our studies systems which are *totally linearly degenerate*. Our theory of hydrodynamic reductions does not apply to those equations, and they need to be treated in a different way.

Consider a quasilinear system

$$u_t + A(u)u_x = 0,$$

where $u = (u^1, ..., u^n)$ is the vector of dependent variables, A is an $n \times n$ matrix, and x, t are independent variables.

Definition 7 *[33] A matrix A is said to be linearly degenerate if its eigenvalues, assumed real and distinct, are constant in the direction of the corresponding eigenvectors. Explicitly, $L_{\xi^i}\lambda^i = 0$, no summation, where L_{ξ^i} is the Lie derivative of the eigenvalue λ^i in the direction of the corresponding eigenvector ξ^i.*

Remark. There exists a simple invariant criterion of linear degeneracy which does not appeal to eigenvalues/eigenvectors [25]. Introducing the characteristic polynomial of A,

$$det(\lambda I - A(u)) = \lambda^n + f_1(u)\lambda^{n-1} + f_2(u)\lambda^{n-2} + \cdots + f_n(u),$$

we write down the following covector

$$\nabla f_1 A^{n-1} + \nabla f_2 A^{n-2} + \cdots + \nabla f_n, \tag{3.23}$$

where ∇ is the gradient, $\nabla f = \left(\frac{\partial f}{\partial u^1}, \ldots, \frac{\partial f}{\partial u^n}\right)$, and A^k denotes k-th power of the matrix A.

Proposition [25] The system $u_t + A(u)u_x = 0$ is weakly nonlinear if and only if the covector (3.23) is identically zero.

In order to show how linear degeneracy conditions are obtained, we discuss an example in the 2×2 case, and then extend it from $(1+1)$ to $(2+1)$ dimensions. We refer to [33] for more details. Consider a system of the form

$$u_t = f(w)u_x, \quad w_t = g(u)w_x. \tag{3.24}$$

Equations (3.24) can be rewritten in a matrix form $U_t = LU_x$, where $U = (u, w)$ is a column vector and the matrix L is given by

$$L = \begin{pmatrix} f(w) & 0 \\ 0 & g(u) \end{pmatrix}.$$

The characteristic polynomial is given by

$$det(L - \lambda I) = \lambda^2 - trL\lambda + detL,$$

with $trL = f(w) + g(u)$ and $detL = f(w)g(u)$. Then, for this two-component, $1+1$ dimensional system, condition (3.23) simplifies to

$$(\nabla trL)L = \nabla detL. \tag{3.25}$$

It is very easy to check that condition (3.25) is satisfied for system (3.24). Indeed

$$(f_u + g_u, f_w + g_w)\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = (f_u g + f g_u, f_w g + f g_w).$$

In the $(2+1)$-dimensional case things are quite similar: a PDE is said to be linearly degenerate, if all its traveling wave reductions to two dimensions are linearly degenerate.

Consider, for example, the equations

$$u_t = \varphi u_x + \psi u_y + \tau w_y, \qquad w_x = u_y, \tag{3.26}$$

which can be rewritten in the matrix form

$$AU_t + BU_x + CU_y = 0, \tag{3.27}$$

where $U = (u, w)$ and matrices A, B and C are given by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \varphi & 0 \\ 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} \psi & \tau \\ 1 & 0 \end{pmatrix}.$$

We seek travelling wave solutions, namely we assume that $U = (u, w) = (u(\xi, \eta), w(\xi, \eta))$, with $\xi = x + mt$, $\eta = y + kt$, and $m, k = const$. Then equation (3.27) takes the form

$$A(mU_\xi + kU_\eta) + BU_\xi + CU_\eta = 0,$$

or

$$U_\xi = (B + mA)^{-1}(C + kA)U_\eta.$$

Now condition (3.25) must be satisfied for the matrix

$$L = (B + mA)^{-1}(C + kA) = \begin{pmatrix} \frac{\psi - k}{\varphi - m} & \frac{\tau}{\varphi - m} \\ -1 & 0 \end{pmatrix}.$$

This leads to the following constraints

$$\psi_u + \varphi_w = 0, \qquad \varphi_u = 0, \qquad \tau_w = 0, \qquad \tau_u + \psi_w = 0. \tag{3.28}$$

When all equations above are satisfied simultaneously then the original system is considered to be linearly degenerate. In other words, expressions (3.28), that appear as denominators in the deformation procedure, are required not to be all simultaneously zero.

3.4 Dispersionless Lax pairs

All dispersionless integrable systems possess the so-called dispersionless Lax pairs [91]. The dispersionless Lax pair is a pair of equations

$$S_t = G(S_x, u), \quad S_y = F(S_x, u), \tag{3.29}$$

here $u = (u^1, \dots, u^m)$, which imply the original equation via the consistency condition $S_{ty} = S_{yt}$.

Example 3.1. The pair of equations

$$S_y = \frac{1}{2}S_x^2 + u, \qquad S_t = \frac{1}{3}S_x^3 + uS_x + w,$$

when we check the consistency condition $S_{ty} = S_{yt}$, yields dKP equation

$$u_t = uu_x + w_y, \qquad w_x = u_y.$$

Example 3.2. The pair

$$e^{S_y}S_x = u, \qquad S_t = -e^{S_y} + w,$$

yields the dispersionless Toda equation

$$u_t = uw_y, \qquad w_x = u_y.$$

The function $S(x, y, t)$ is called *scalar pseudo-potential.* Dependence of the functions F and G on S_x may be nonlinear. The problem of finding an appropriate quantisation of the corresponding dispersionless Lax pairs, was also addressed in [91].

Normally, during the classification procedure, we do not assume the existence of Lax pairs from the beginning, but it is something which can be obtained by direct computation once the results are obtained.

Moreover, dispersionless Lax pairs can be used to classify dispersionless limits. This will become more clear in the following sections where, for a certain class of equations, we will derive the same result using both the method of hydrodynamic reductions and the method introduced by using Lax pairs.

3.4.1 Classification of integrable dispersionless equations via Lax pairs

Here, we explain how Lax pairs can be used to classify integrable dispersionless limits. The first step is to impose from the beginning the structure of the Lax pair with arbitrary coefficients. Then, as we will see, one can straightforwardly derive the list of integrable dispersionless equations, by requiring that the compatibility condition of the Lax pair is satisfied. The only information used throughout this process is the original equation itself and its dispersion relation. Moreover, this method is not only fast, but it immediately excludes the non-deformable cases that may occur.

The equations of interest in this chapter are of third order (in $x-, y-$ derivatives of u, v, w). This means that they possess Lax pairs of the form

$$F(S_x, S_y, u) = 0, \quad S_t = G(S_x, S_y, u, v, w), \tag{3.30}$$

with F quadratic and G cubic in S_x, S_y. Similarly, if we were interested in fifth order equations, then F would be cubic and G of fifth order in S_x, S_y (see [73]). Now, back in our case, the the fact that F is quadratic in S_x implies that there are only three types of pairs that need to be studied:

Type I: $S_y = A(u)S_x^2 + B(u)S_x + C(u), \quad with \quad A(u) \neq 0.$

Type II: $S_y = A(u)S_x + B(u) + \frac{C(u)}{S_x+D(u)}, \quad with \quad C(u) \neq 0.$

Type III: $S_y^2 = A(u)S_xS_y + B(u)S_x^2 + C(u)S_x + D(u)S_y + E(u),$
with nondegeneracy condition $4BE + BD^2 + A^2E - C^2 - ACD \neq 0$. This nondegeneracy condition is obtained by requiring that the determinant of the coefficient matrix is nonzero.

The second equation, for all types, is of the form

$$S_t = a_1S_x^3 + a_2S_x^2S_y + a_3S_xS_y^2 + a_4S_y^3 + a_5S_x^2 + a_6S_xS_y + a_7S_y^2 + a_8S_x + a_9S_y + a_{10},$$

where $a_i, \quad i = 1, \ldots, 10$ are arbitrary functions of u, v and w.

In the following section, this method will be used on particular classes of equations. We follow the same procedure every time: we first state the hydrodynamic type system under consideration, we take the Lax pair, and compute the compatibility condition $S_{yt} = S_{ty}$ in each case. From this condition, we obtain the precise form of the Lax pair, and hence the resulting integrable, dispersionless equations.

3.5 Classification of integrable equations with one nonlocality

In this section we consider a class of third order integrable dispersive equations with one simple nonlocality of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(...) + \epsilon^2(...), \quad w_x = u_y, \tag{3.31}$$

where φ, ψ, η depend on the scalar field $u(x, y, t)$ and the nonlocal variable $w(x, y, t)$. The terms at ϵ and ϵ^2 are homogeneous differential polynomials of order two and three respectively in the $x-$ and $y-$ derivatives of u and w, with coefficients being arbitrary functions of u and w.

This is a very important problem since the well-known examples of KP, Gardner and Veselov-Novikov equations belong in this class. A detailed analysis of this problem, including proofs and calculations can be found in [28, 32].

First, we review some basic facts of this problem. We state the integrability conditions of the dispersionless system

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y \tag{3.32}$$

and then, we discuss the resulting classification theorem for the dispersive equations (3.31). This classification was obtained using the method of deformations of hydrodynamic reductions, that we already introduced. Also, using the method of Lax pairs, we will show how to classify integrable dispersionless equations within this class and, namely, obtain exactly the dispersionless equations that are listed in the classification theorem.

3.5.1 Integrability conditions of the dispersionless system

Given an equation of the form (3.31), the corresponding dispersionless limit is (3.32)

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y,$$

and can be rewritten in matrix form (3.4) as follows:

$$\begin{pmatrix} -1/\varphi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}_x + \begin{pmatrix} \psi/\varphi & \eta/\varphi \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}_y = 0.$$

The integrability conditions reduce to a system of second order partial differential equations for the coefficients φ, ψ and η, which can be derived from the general integrability conditions for 2×2 systems of hydrodynamic type in $2+1$ dimensions as obtained in [28]:

$$\begin{aligned}
\varphi_{uu} &= -\frac{\varphi_w^2 + \psi_u\varphi_w - 2\psi_w\varphi_u}{\eta}, \\
\varphi_{uw} &= \frac{\eta_w\varphi_u}{\eta}, \\
\varphi_{ww} &= \frac{\eta_w\varphi_w}{\eta}, \\
\psi_{uu} &= \frac{-\varphi_w\psi_w + \psi_u\psi_w - 2\varphi_w\eta_u + 2\eta_w\varphi_u}{\eta}, \\
\psi_{uw} &= \frac{\eta_w\psi_u}{\eta}, \\
\psi_{ww} &= \frac{\eta_w\psi_w}{\eta}, \\
\eta_{uu} &= -\frac{\eta_w\left(\varphi_w - \psi_u\right)}{\eta}, \\
\eta_{uw} &= \frac{\eta_w\eta_u}{\eta}, \\
\eta_{ww} &= \frac{\eta_w^2}{\eta};
\end{aligned} \tag{3.33}$$

we assume $\eta \neq 0$: this is equivalent to the requirement that the dispersion relation of the system (3.32) defines an irreducible curve since the condition $\det(\lambda A + B + \mu C) = 0$ is equivalent to $\lambda = \varphi + \psi\mu + \eta\mu^2$. The system (3.33) is in involution and straightforward to solve. First of all, the equations for η imply that, up to translations and rescalings, $\eta = 1$, $\eta = u$ or $\eta = e^w h(u)$. All three possibilities are considered but it is proved [32] that the third case of $\eta = e^w h(u)$ cannot arise as a dispersionless limit of an integrable equation. Notice that φ and ψ are defined up to additive constants which can always be set equal to zero via linear transformations of the initial equation (3.32). Moreover, the system (3.33) is form-invariant under transformations of the form

$$\tilde{x} = x - sy, \quad \tilde{y} = y, \quad \tilde{u} = u, \quad \tilde{w} = w + su.$$

All classification results are formulated modulo this equivalence.

3.5.2 Classification result of dispersive equations

Once system (3.33) is solved for the functions φ, ψ and η one can apply the deformation scheme described in section 3.2 to obtain the following

Theorem 3.2 *[32] The following equations provide a complete list of integrable equations of the form (3.31) with $\eta \neq 0$, whose dispersionless limit is linearly nondegenerate:*

KP equation	$u_t = uu_x + w_y + \epsilon^2 u_{xxx},$
mKP equation	$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx},$
Gardner equation	$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx},$
VN equation	$u_t = (uw)_y + \epsilon^2 u_{yyy},$
mVN equation	$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y,$
HD equation	$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$
deformed HD equation	$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$
Equation E_5	$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)],$
Equation E_6	$u_t = \frac{4}{3}\beta^2 u^3 u_x + (w - 3\beta u^2)u_y + uw_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)],$

where $B = \beta u D_x - D_y$, β=const, δ=const (and $w_x = u_y$).

Dispersionless limits of these equations possess a Lax pair of the form

$$F(S_x, S_y, u) = 0,$$
$$S_t = G(S_x, S_y, u, w), \tag{3.34}$$

where F is quadratic and G is cubic in S_x, S_y. These Lax pairs are represented in the following table

Equation	*Dispersionless limit*	*Dispersionless Lax pair*
KP	$u_t = uu_x + w_y$ $w_x = u_y$	$S_y = \frac{1}{2}S_x^2 + u$ $S_t = \frac{1}{3}S_x^3 + uS_x + w$
mKP	$u_t = (w - u^2/2)u_x + w_y$ $w_x = u_y$	$S_y = \frac{1}{2}S_x^2 + uS_x$ $S_t = \frac{1}{3}S_x^3 + uS_x^2 + (w + u^2/2)S_x$
$Gardner$	$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y$ $w_x = u_y$	$S_y = S_x^2 + (\beta u - \frac{\delta}{\beta})S_x$ $S_t = \frac{4}{3}S_x^3 + 2\left(\beta u - \frac{\delta}{\beta}\right)S_x^2$ $+(w\beta + \frac{u^2\beta^2}{2} - u\delta + \frac{\delta^2}{\beta^2})S_x$
VN	$u_t = (uw)_y$ $w_x = u_y$	$S_y = u/S_x$ $S_t = \frac{uw}{S_x} - \frac{1}{3S_x^3}u^3$
mVN	$u_t = (uw)_y$ $w_x = u_y$	$S_y = u/S_x$ $S_t = \frac{uw}{S_x} - \frac{1}{3S_x^3}u^3$
HD	$u_t = -2wu_y + uw_y$ $w_x = u_y$	$S_y = S_x^2/u^2$ $S_t = -2w\frac{S_x^2}{u^2} + \frac{4}{3}\frac{S_x^3}{u^3}$
dHD	$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y$ $w_x = u_y$	$S_y = \frac{S_x^2+\lambda}{u^2}$ $S_t = -2w\frac{S_x^2+\lambda}{u^2} + \frac{4}{3}\frac{S_x^3+\lambda S_x}{u^3}$
E_5	$u_t = (\beta w + \beta^2u^2)u_x - 3\beta uu_y + w_y$ $w_x = u_y$	$S_xS_y = \beta uS_x^2 + \frac{1}{3}$ $S_t = \beta^3u^3S_x^3 - S_y^3 + \beta wS_x$
E_6	$u_t = \frac{4}{3}\beta^2u^3u_x + (w - 3\beta u^2)u_y + uw_y$ $w_x = u_y$	$S_xS_y = \beta uS_x^2 + \frac{u}{3}$ $S_t = \beta^3u^3S_x^3 - S_y^3 + \frac{\beta^2}{3}u^3S_x + wS_y$

Again, β, δ are arbitrary constants and $\lambda = 3\delta/4$. Further dependence on extra parameters has been eliminated through transformations

$$S \to kS + lx + my + nt + su, \qquad k, l, m, n, s = const.$$

Note that VN and mVN equations have the same dispersionless limit, also HD equation can be obtained from the deformed HD when $\delta = 0$, and mKP can be obtained from Gardner equation with the choice $\beta = 1, \delta = 0$.

3.5.3 Classification via Lax pairs

Here, we will re-derive the classification results discussed above using the alternative approach of Lax pairs. Recall the hydrodynamic type equations with one nonlocality (3.32) have the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y,$$

where $\eta \neq 0$, and the procedure described in Section 3.4.1: the equations possess Lax pairs of the form

$$F(S_x, S_y, u) = 0, \quad S_t = G(S_x, S_y, u, w),$$

with F being quadratic in S_x, S_y. This implies three different types for the first equation, while the second equation, for all types, is of the form

$$S_t = a_1 S_x^3 + a_2 S_x^2 S_y + a_3 S_x S_y^2 + a_4 S_y^3 + a_5 S_x^2 + a_6 S_x S_y + a_7 S_y^2 + a_8 S_x + a_9 S_y + a_{10},$$

where $a_i, \quad i = 1, \ldots, 10$ are arbitrary functions of u and w. Using the equivalence group of this system, results of this method lead to the the dispersionless equations of Theorem 3.2. Indeed:

$$\textbf{Type I:}\ S_y = A(u) S_x^2 + B(u) S_x + C(u), \quad \textit{with} \quad A(u) \neq 0.$$

The consistency condition, $S_{yt} = S_{ty}$, leads to

Case 1: $A'(u) \neq 0$. Then $S_y = \frac{1}{u^2} S_x^2 + \frac{\delta}{u^2}$ which leads to the deformed HD equation and when $\delta = 0$ to the HD equation.

Case 2a: $A'(u) = 0$ and $B'(u) \neq 0$. Then $S_y = S_x^2 + (\beta u + \delta) S_x$ which leads to the Gardner equation and when $\delta = 0, \beta = 1$ to the modified KP equation.

Case 2b: $A'(u) = 0$ and $B'(u) = 0$. Then $S_y = S_x^2 + u$ which corresponds to the dKP equation.

$$\textbf{Type II:}\ S_y = A(u) S_x + B(u) + \frac{C(u)}{S_x + D(u)}, \quad \textit{with} \quad C(u) \neq 0.$$

From the consistency condition we obtain $D(u) = \textit{constant}$ and

Case 1: $A'(u) \neq 0$. Then either $S_y = u S_x + \frac{u}{S_x}$ which leads to the second new equation E_6 or $S_y = u S_x + \frac{c}{S_x}$ which leads to the first new equation E_5.

Case 2a: $A'(u) = 0$ and $B'(u) \neq 0$. In this case there are no examples because the condition $C\eta B' = 0$ which appears can not be satisfied.

Case 2b: $A'(u) = 0$ and $B'(u) = 0$. Then $S_y = \frac{cu}{S_x}$ which corresponds to the VN and mVN equations.

$$\textbf{Type III: } S_y^2 = A(u)S_xS_y + B(u)S_x^2 + C(u)S_x + D(u)S_y + E(u),$$

with $4BE + BD^2 + A^2E - C^2 - ACD \neq 0$ $(*)$. Then

Case 1: $A(u)^2 + 4B(u) \neq 0$. In this case $E(u) = \frac{1}{4}(r^2(A(u)^2 + 4B(u)) - D(u)^2)$, $C(u) = \frac{1}{2}(r(A(u)^2 + 4B(u)) - A(u)D(u))$ and $D(u) = rA(u) + \delta$, where r, δ are constants. But then condition $(*)$ is not satisfied.

Case 2a: $A(u)^2 + 4B(u) = 0$ and $2C(u) + A(u)D(u) = 0$. In this case there are no equations because the condition $(*)$ is not satisfied.

Case 2b: $A(u)^2+4B(u) = 0$ and $2C(u)+A(u)D(u) \neq 0$. In this case $E(u) = \frac{1}{4}(s(2C(u)+A(u)D(u)) - D(u)^2)$ and $D(u) = \frac{s}{2}A(u) + \delta$, where s, δ are constants. It turns out that $2C(u) + A(u)D(u)$ must be a non-zero constant, which leads to $A(u)D(u) = constant$ and after further calculations we obtain that $A(u) = constant$. But then all coefficients $A(u), B(u), C(u), D(u), E(u)$ are also constants, so there is no equation in this case either.

3.6 Classification of integrable equations with two nonlocalities

In this section, the aim is to study a class of hydrodynamic type equations with two nonlocalities of the form

$$\begin{aligned} &u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_y + \epsilon(...) + \epsilon^2(...),\\ &w_x = u_y,\\ &v_x = f(u, w)_y, \end{aligned} \tag{3.35}$$

in the same way as it was done for the equations with one nonlocality in the previous section. Here $\varphi, \psi, \eta, \tau$ are functions of u, v and w, and the terms at ϵ and ϵ^2 are homogeneous differential polynomials of order two and three respectively in the $x-$ and $y-$ derivatives of u, v and w.

For this class, we focus on the corresponding dispersionless system

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_y, \quad w_x = u_y, \quad v_x = f(u, w)_y.$$

We start by listing the integrability conditions of this system, but, in order to classify integrable dispersionless equations, we don't use the method of hydrodynamic reductions. Instead, we use the Lax pairs approach, where the resulting equations are higher flows of the dispersionless KP, Gardner and HD hierarchies. Also, we consider a similar class of equations with a slightly different second nonlocality, of type

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x, \quad w_x = u_y, \quad v_y = f(u, w)_x,$$

where classification via Lax pairs leads to commuting flows of the dispersionless VN hierarchy.

3.6.1 Integrability conditions of the dispersionless system

The corresponding dispersionless limit of equations (3.35) is

$$\begin{aligned} u_t &= \varphi u_x + \psi u_y + \eta w_y + \tau v_y, \\ w_x &= u_y, \\ v_x &= f(u, w)_y. \end{aligned} \tag{3.36}$$

Rewriting (3.36) in the matrix form

$$A(U)U_t + B(U)U_x + C(U)U_y = 0, \tag{3.37}$$

where $U = (u, v, w)$, then from the dispersion relation $D(\mu, \lambda) = det(\lambda A + B + \mu C) = 0$ we find that

$$\lambda = \phi + \mu\psi + \mu^2(\eta + \tau f_u) + \mu^3 \tau f_w. \tag{3.38}$$

The requirement that the dispersion relation defines an irreducible cubic implies $\tau f_w \neq 0$. The integrability conditions can be easily obtained from the Haantjes criterion, using computer algebra, as illustrated in figure 3.1 below.

Step1. Define the variables *u*, *v*, *w*.

```
In[1]:= u[1] = u;
        u[2] = v;
        u[3] = w;
```

Step2. Define the matrices $A(u)$, $B(u)$, $C(u)$ (C is denoted by C1):

```
In[4]:= A := {{-1, 0, 0}, {0, 0, 0}, {0, 0, 0}}
        A // MatrixForm
        B := {{φ[u[1], u[2], u[3]], 0, 0}, {0, 0, -1}, {0, -1, 0}}
        B // MatrixForm
        C1 := {{ψ[u[1], u[2], u[3]], τ[u[1], u[2], u[3]], η[u[1], u[2], u[3]]},
          {1, 0, 0}, {D[f[u[1], u[3]], u[1]], 0, D[f[u[1], u[3]], u[3]]}}
        C1 // MatrixForm
```

Step3. Define the matrix $V = v^i_j = (\lambda A + B)^{-1} (\mu A + C)$, where λ, μ are arbitrary constants:

```
In[10]:= V := (Inverse[λ A + B]) . (μ A + C1)
         V // MatrixForm
```

Step4. Define Nijehnius Tensor of the matrix V. Then print the results for i, j, $k = 1, 2, 3$.

```
In[12]:= NT[i_, j_, k_] :=
          FullSimplify[Sum[(V[[p, j]] D[V[[i, k]], u[p]] - V[[p, k]] D[V[[i, j]], u[p]] -
              V[[i, p]] (D[V[[p, k]], u[j]] - D[V[[p, j]], u[k]])), {p, 1, 3}]]

In[13]:= Do[Print[StringForm["i= ``, j= ``, k= ``, NT= ``", i, j, k, NT[i, j, k]]],
          {i, 1, 3}, {j, 1, 3}, {k, 1, 3}]
```

Step5. Define Haantjes Tensor. Then print the results for i, j, $k = 1, 2, 3$.

```
In[14]:= HT[i_, j_, k_] := FullSimplify[
           Sum[NT[i, p, r] V[[p, j]] V[[r, k]] - NT[p, j, r] V[[i, p]] V[[r, k]] - NT[p, r, k]
              V[[i, p]] V[[r, j]] + NT[p, j, k] V[[i, r]] V[[r, p]], {p, 1, 3}, {r, 1, 3}]]

In[15]:= Do[Print[StringForm["i= ``, j= ``, k= ``, HT= ``", i, j, k, HT[i, j, k]]],
          {i, 1, 3}, {j, 1, 3}, {k, 1, 3}]
```

Step6. From the equations derive the integrability conditions by requiring that the components of the Haantjes tensor vanish identically.

Figure 3.1: Computation of the components of Haantjes tensor.

This leads to a system of ten equations for the unknown functions ϕ, ψ, η, τ and f

$$\tau_v = 0,$$

$$\tau_w = \eta_v,$$

$$
\begin{aligned}
&\tau f_{ww} = f_w \tau_w, \\
&\tau f_{uw} = f_w \tau_u, \\
&\eta \tau_w - \tau(\eta_w + f_u \tau_w - 2 f_w \psi_v) = 0, \\
&\tau^2 f_{uu} + 2\tau f_w \phi_v - \tau \eta_u + \eta \tau_u = 0, \\
&\tau^2 f_{uu} - \eta \psi_v + 2\eta \tau_u + \tau \psi_w + \tau f_w \phi_v - 2\tau \eta_u = 0, \\
&(\eta\tau + \tau^2 f_u)\phi_w - (\eta \tau f_u + \eta^2)\phi_v - \tau^2 f_w \phi_u = 0, \\
&(\eta^2 + \eta\tau f_u) f_{uu} + f_w(\eta\phi_w + 2\eta f_u \phi_v - \eta \psi_u + \tau f_w \phi_u - \tau f_u \phi_w) = 0, \\
&\eta\tau f_{uu} - \eta f_w \phi_v + \eta f_u(\psi_v - 2\tau_u) - \tau f_u(\psi_w - 2\eta_u) + \tau f_w(2\phi_w \psi_u) = 0,
\end{aligned}
\tag{3.39}
$$

where we have taken into account the condition $\tau f_w \neq 0$.

In order to classify integrable hydrodynamic type equations of the form discussed above we have to solve the integrability conditions. This solution leads to a number of equations which may arise as dispersionless limits of integrable soliton equations. Then one has to reconstruct dispersive terms using the method of deformations [32]. Although one could proceed with the solution of system (3.39), this leads to an exhaustive number of cases, where most of them in the end will not be deformable. Therefore, we will use the approach based on Lax pairs, which will eliminate non-deformable cases from the very beginning.

3.6.2 Classification via Lax pairs

Recall the hydrodynamic type equations with two nonlocalities (3.36),

$$
\begin{aligned}
u_t &= \varphi u_x + \psi u_y + \eta w_y + \tau v_y, \\
w_x &= u_y, \\
v_x &= f(u,w)_y,
\end{aligned}
$$

where $\tau f_w \neq 0$, and the procedure described in Section 3.4.1. The equations possess Lax pairs of the form

$$
F(S_x, S_y, u) = 0, \quad S_t = G(S_x, S_y, u, v, w),
$$

with F being quadratic in S_x, S_y. This implies three different types for the first equation. The second equation, for all types, is of the form

$$S_t = a_1 S_x^3 + a_2 S_x^2 S_y + a_3 S_x S_y^2 + a_4 S_y^3 + a_5 S_x^2 + a_6 S_x S_y + a_7 S_y^2 + a_8 S_x + a_9 S_y + a_{10},$$

where $a_i, \quad i = 1, \ldots, 10$ are arbitrary functions of u, v and w. Using the equivalence group of this system we can prove that only type I Lax pairs give results, and these results are higher flows of dispersionless KP, mKP, Gardner, HD and deformed HD equations. It is important to note that the equations in this case are exactly the commuting flows that will be found in section 3.8 (where the equations for the higher flows will be given explicitly). Indeed

Type I: $S_y = A(u)S_x^2 + B(u)S_x + C(u), \quad with \quad A(u) \neq 0.$

Then the consistency condition, $S_{yt} = S_{ty}$, leads to

Case 1: $A'(u) \neq 0$. Then $S_y = \frac{1}{u^2}S_x^2 + \frac{\delta}{u^2}$ which leads to the higher deformed HD equation and when $\delta = 0$ to the HD equation.

Case 2a: $A'(u) = 0$ and $B'(u) \neq 0$. Then $S_y = S_x^2 + (\beta u + \delta)S_x$ which leads to the higher Gardner equation and when $\delta = 0, \beta = 1$ to the higher modified KP equation.

Case 2b: $A'(u) = 0$ and $B'(u) = 0$. Then $S_y = S_x^2 + u$ which corresponds to the higher dKP equation.

Type II: $S_y = A(u)S_x + B(u) + \frac{C(u)}{S_x + D(u)}, \quad with \quad C(u) \neq 0.$

From the consistency condition we obtain $D(u) = constant$ and

Case 1: $A'(u) \neq 0$. Then $b'(u) = dA'(u)$ and $C(u)A'(u) + A(u)C'(u) = 0$ but also $C(u)^2 \tau f_w C'(u) = 0$. This leads to $C'(u) = 0$ which means that $A'(u) = 0$, so there is no equation in this case.

Case 2a: $A'(u) = 0$ and $B'(u) \neq 0$. In this case there are no equations because the condition $C(u)^2 \tau f_w B'(u) = 0$ that appears can not be satisfied.

Case 2b: $A'(u) = 0$ and $B'(u) = 0$. There are no equations here either because the condition $C(u)^2 \tau f_w C'(u) = 0$ that appears can only be satisfied if $A(u), B(u), C(u)$ are all constants.

Type III: $S_y^2 = A(u)S_xS_y + B(u)S_x^2 + C(u)S_x + D(u)S_y + E(u)$,

with $4BE + BD^2 + A^2E - C^2 - ACD \neq 0 \ \ (*)$. Then

Case 1: $A(u)^2 + 4B(u) \neq 0$. In this case $E(u) = r^2(A(u)^2 + 4B(u)) - \frac{1}{4}D(u)^2$, $C(u) = r(A(u)^2 + 4B(u)) - \frac{1}{2}A(u)D(u)$ and $D(u) = 2rA(u) + \delta$, where r, δ are constants. But then condition $(*)$ is not satisfied.

Case 2a: $A(u)^2 + 4B(u) = 0$ and $2C(u) + A(u)D(u) = 0$. In this case there are no equations because the condition $(*)$ is not satisfied.

Case 2b: $A(u)^2 + 4B(u) = 0$ and $2C(u) + A(u)D(u) \neq 0$. In this case $E(u) = s(2C(u) + A(u)D(u)) - \frac{1}{4}D(u)^2$ and $D(u) = 2sA(u) + \delta$, where s, δ are constants. It turns out that $2C(u) + A(u)D(u)$ must be a non-zero constant, which leads to $A(u)D(u) = \mathit{constant}$ and consequently to $A(u) = \mathit{constant}$. But then all coefficients $A(u), B(u), C(u), D(u), E(u)$ are constants, so there is no equation in this case either.

3.6.3 Classification via Lax pairs: A different second nonlocality

We consider again a class of hydrodynamic type equations with two nonlocalities like (3.36) but now in the second nonlocality we interchange the variables x and y

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x,$$
$$w_x = u_y,$$
$$v_y = f(u,w)_x,$$

where $\eta, \tau f_u \neq 0$. We can prove that, for those equations, only type II Lax pairs give results and these results are commuting flows of the dispersionless VN equation. Explicitly, the equation for the commuting flows is

$$u_t = -2\tau_1\lambda u_x + \eta_1(uw)_y + \tau_1(uv)_x,$$
$$w_x = u_y,$$
$$v_y = (\kappa u + \lambda w)_x,$$

where $\tau_1, \eta_1, \kappa, \lambda$ are constants, and arises as a compatibility condition of the Lax pair

$$S_y = \frac{u}{S_x},$$
$$S_t = -\frac{1}{3}\kappa\tau_1 S_x^3 + (\tau_1 v - \lambda u)S_x + \eta_1 \frac{uw}{S_x} - \frac{\eta_1}{3}\frac{u^3}{S_x^3}.$$

Indeed

$$\textbf{Type I: } S_y = A(u)S_x^2 + B(u)S_x + C(u), \quad \textit{with} \quad A(u) \neq 0.$$

The consistency condition, $S_{yt} = S_{ty}$, leads to

Case 1: $A'(u) \neq 0$. Then the condition $\tau f_u A'(u) = 0$ has to be satisfied, which is not possible.

Case 2a: $A'(u) = 0$ and $B'(u) \neq 0$. But then $\tau f_u B'(u) = 0$ which is not possible.

Case 2b: $A'(u) = 0$ and $B'(u) = 0$. Then $\tau C'(u) = 0$ which yields that $A(u), B(u), C(u)$ are all constants.

$$\textbf{Type II: } S_y = A(u)S_x + B(u) + \tfrac{C(u)}{S_x + D(u)}, \quad \textit{with} \quad C(u) \neq 0.$$

From the consistency condition we obtain the following

Case 1: $A(u) \neq 0$. Then either $A'(u) \neq 0$ which leads to $C^3 \tau f_u A'(u) = 0$ and cannot be satisfied or $A'(u) = 0$ which in the end implies that $A(u), B(u), C(u), D(u)$ are all constants. So there is no equation in this case.

Case 2a: $A(u) = 0$ and $B'(u) \neq 0$. Which leads to a contradiction, namely $B'(u) = 0$.

Case 2b: $A(u) = 0$ and $B'(u) = 0$. Then either $C'(u) = 0$ which means that $A(u), B(u)$, $C(u), D(u)$ are all constants or $C'(u) \neq 0$ which eventually leads to $S_y = \frac{cu}{S_x}$, with $c = const$, and to the commuting flow of Veselov-Novikov equation.

$$\textbf{Type III: } S_y^2 = A(u)S_xS_y + B(u)S_x^2 + C(u)S_x + D(u)S_y + E(u),$$

with $4BE + BD^2 + A^2E - C^2 - ACD \neq 0 \quad (*)$. Then

Case 1: $A(u)^2 + 4B(u) \neq 0$. In this case $E(u) = s(A(u)^2 + 4B(u)) - \frac{1}{4}D(u)^2$, $C(u) = r(A(u)^2 + 4B(u)) - \frac{1}{2}A(u)D(u)$ and $D(u) = 2rA(u) + \delta$, where r, δ are constants. But then $r^2 = s$ which violates the condition $(*)$.

Case 2a: $A(u)^2 + 4B(u) = 0$ and $2C(u) + A(u)D(u) = 0$. In this case there are no equations because the condition $(*)$ is not satisfied.

Case 2b: $A(u)^2 + 4B(u) = 0$ and $2C(u) + A(u)D(u) \neq 0$. In this case $E(u) = s(2C(u) + A(u)D(u)) - \frac{1}{4}D(u)^2$ and $D(u) = 4rA(u) + \frac{2C(u)}{A(u)}\delta$, where r, s, δ are constants. All the subcases that arise lead to the result that $A(u), B(u), C(u), D(u), E(u)$ must be constants, so there is no equation in this case either.

3.7 Classification of integrable equations with nested nonlocalities

In this section, we extend the problem by adding nonlocalities to the equation. We perform classification using Lax pairs, and briefly mention the results when three, four and five nonlocalities are added to the problem. In the case of these *nested* nonlocalities, the result is higher flows of dispersionless KP, Gardner and HD equations, when we have three and four nonlocalities, while there exists no integrable equation, in the case of five nonlocalities.

Three nonlocalities

Consider the class of hydrodynamic type equations of the form

$$\begin{aligned} u_t &= \varphi u_x + \psi u_y + \eta w_y + \tau v_y + \zeta p_y, \\ w_x &= u_y, \\ v_x &= f(u,w)_y, \\ p_x &= g(u,v,w)_y. \end{aligned} \tag{3.40}$$

where $\varphi, \psi, \eta, \tau, \zeta$ are functions of u, v, w and p. The condition that the dispersion relation is irreducible, implies that $\zeta f_w g_v \neq 0$. Using the Lax pairs to classify equations of the form (3.40), we can prove that only type I Lax pairs give results, so

$$\begin{aligned} S_y &= A(u)S_x^2 + B(u)S_x + C(u), \\ S_t &= a_6 S_x^6 + a_5 S_x^5 + a_4 S_x^4 + a_3 S_x^3 + a_2 S_x^2 + a_1 S_x + a_0. \end{aligned}$$

Then these results are again higher flows of dispersionless KP, Gardner and HD equations, as we had in the classification of equations with two nonlocalities.

Four nonlocalities

Following the above scheme we add one extra nonlocality and we consider dispersionless hydrodynamic type equations with four nonlocalities,

$$\begin{aligned}
&u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_y + \zeta p_y + \xi q_y,\\
&w_x = u_y,\\
&v_x = f(u,w)_y,\\
&p_x = g(u,v,w)_y,\\
&q_x = h(u,v,w,p)_y,
\end{aligned} \tag{3.41}$$

where $\varphi, \psi, \eta, \tau, \zeta, \xi$ are functions of u, v, w, p and q and the dispersion relation implies that $\xi f_w g_v h_p \neq 0$.

Still the result we obtain is higher flows of dispersionless KP, Gardner and HD equations.

Five nonlocalities

If we add a fifth nonlocality to the equations (3.41) above,

$$\begin{aligned}
&u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_y + \zeta p_y + \xi q_y + \nu r_y,\\
&w_x = u_y,\\
&v_x = f(u,w)_y,\\
&p_x = g(u,v,w)_y,\\
&q_x = h(u,v,w,p)_y,\\
&r_x = k(u,v,w,p,q)_y,
\end{aligned} \tag{3.42}$$

where $\varphi, \psi, \eta, \tau, \zeta, \xi, \nu$ functions of u, v, w, p, q, r and $\nu f_w g_v h_p k_q \neq 0$ from the dispersion relation, then it is proved that there exists no equation in the classification list.

3.8 Commuting flows

In this last section, we are interested in finding commuting flows of the integrable dispersionless equations (3.32) and system (3.36). Due to the fact that those two classes of equations possess Lax pairs with the same structure this can be done as follows: Introduce the Lax pair whose first equation is given by

$$F(S_x, S_y, u) = 0, \tag{3.43}$$

where F is quadratic in S_x, S_y, (in fact F will be the S_y expression of the Lax pair, of the nine integrable dispersionless equations listed in theorem [32]), and define the second equation to be of the form

$$S_T = a_1 S_x^3 + a_2 S_x^2 S_y + a_3 S_x S_y^2 + a_4 S_y^3 + a_5 S_x^2 + a_6 S_x S_y + a_7 S_y^2 + a_8 S_x + a_9 S_y + a_{10} \tag{3.44}$$

where $a_i, \quad i = 1, \ldots, 10$ are arbitrary functions of u, v and w. The consistency condition $S_{yT} = S_{Ty}$, must be satisfied modulo (3.43) and the system (3.36). This condition leads to a number of constraints on the functions $a_i, \varphi, \psi, \eta, \tau$ and f.

Theorem 3.3 *Commuting flows of the integrable dispersionless equations (3.32) and system (3.36), turn out to be higher flows of dispersionless KP, mKP, Gardner, HD and deformed HD equations.*

Particularly,

dKP equation. The Lax pair takes the form

$$S_y = \frac{1}{2} S_x^2 + u,$$
$$S_T = c S_x^4 + k_1 S_x^3 + 4cu S_x^2 + (4cw + 3uk_1) S_x + 4cu^2 + v\tau + 3wk_1,$$

while the resulting equations are

$$u_T = (\tau w + 3k_1 u) u_x + 2u\tau u_y + 3k_1 w_y + \tau v_y,$$
$$w_x = u_y,$$
$$v_x = w_y,$$

where $k_1 = const, \tau = 4c = const$, and $\tau f_w = 4c \neq 0$.

Dispersionless Gardner equation. The Lax pair is

$$S_y = S_x^2 + \left(\beta u - \frac{\delta}{\beta}\right) S_x,$$

$$S_T = cS_x^4 + (2\beta cu + k_2)S_x^3 + \left(cw\beta + cu^2\beta^2 + cu\delta + \frac{3}{2}u\beta k_2\right) S_x^2 + (\frac{3}{4}cu^2\beta\delta + \frac{cu\delta^2}{2\beta} + v\beta\tau + \frac{3}{8}u^2\beta^2 k_2 + \frac{3}{4}u\delta k_2 + \frac{1}{4}w\left(2cu\beta^2 + 4c\delta + 3\beta k_2 + 2cu\beta^2\right))S_x,$$

and the resulting equations are

$$u_T = \varphi u_x + \psi u_y + \eta w_y + \tau v_y,$$

$$w_x = u_y,$$

$$v_x = \left(\frac{cw}{2\tau} - \frac{cu^2\beta}{4\tau}\right)_y,$$

where $k_2 = const, \tau = c/2 = const$, β, δ are constants, $\tau f_w = c/2 \neq 0$, and the unknown functions φ, ψ, η are given by

$$\eta = \frac{1}{4}\left(2cu\beta + \frac{6c\delta}{\beta} + 3k_2\right),$$

$$\psi = c\left(w\beta - \frac{1}{2}u^2\beta^2 + u\delta + \frac{3\delta^2}{2\beta^2}\right) + \frac{3\delta k_2}{2\beta},$$

$$\varphi = 2cw\delta - \frac{3}{4}cu^2\beta\delta + \frac{3cu\delta^2}{2\beta} + \frac{c\delta^3}{2\beta^3} + v\beta\tau + \frac{3}{4}w\beta k_2 - \frac{3}{8}u^2\beta^2 k_2 + \frac{3}{4}u\delta k_2 + \frac{3\delta^2 k_2}{4\beta^2}.$$

The choice $\delta = 0, \beta = 1$ leads to the commuting flow of the mKP equation.

Dispersionless deformed Harry-Dym equation. The Lax pair is given by

$$S_y = \frac{S_x + \lambda}{u^2},$$

$$S_T = \frac{c}{u^4}S_x^4 + \frac{k_3 - 2cw}{u^3}S_x^3 + \frac{4cu^2w^2 + 4c\lambda - 3u^2wk_3 - 4u^2v\tau_1}{2u^4}S_x^2 + \frac{-2cw\lambda + \lambda k_3}{u^3}S_x + \frac{1}{2u^4}(4cu^2w^2\lambda + 2c\lambda^2 - 3u^2w\lambda k_3 - 4u^2v\lambda\tau_1),$$

and the resulting flows are

$$u_T = \frac{\lambda(k_3 - 2cw)}{u^3}u_x + \left(4cw^2 - 3wk_3 - 4v\tau + \frac{4c\lambda}{2u^2}\right) u_y - \frac{1}{4}u(8cw - 3k_3)w_y + \tau uv_y,$$

$$w_x = u_y,$$

$$v_x = \left(\frac{cuw}{2\tau}\right)_y,$$

where τ, c, k_3 are constants, and $\tau u f_w = (c/2)u^2 \neq 0$. The choice $\lambda = 0$ leads to the commuting flow of HD equation.

This scheme is invariant under translations and scalings of u, v, w and S and linear transformations which preserve the structure of equations (3.36)

$$\tilde{x} = x - sy, \quad \tilde{y} = y, \quad \tilde{u} = u, \quad \tilde{w} = w + su, \quad \tilde{v} = v + sf(\tilde{u}, \tilde{w}),$$

here $s = const$. Modulo this equivalence, one can eliminate τ, c, k_1, k_2, k_3, the arbitrary constants, that appear in the equations.

3.8.1 Commuting flows of the dKP equation

As an example, we will find commuting flows of dKP equation and the integrable dispersionless equations with two, three and four nonlocalities, which were introduced earlier. Consider again KP equation written in the form

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \qquad w_x = u_y,$$

The corresponding dispersionless limit

$$u_t = uu_x + w_y, \qquad w_x = u_y.$$

possesses the Lax pair

$$S_y = \frac{1}{2}S_x^2 + u, \quad S_t = \frac{1}{3}S_x^3 + uS_x + w. \tag{3.45}$$

Now since dKP equation and the hydrodynamic type equations with two, three and four nonlocalities, possess Lax pairs with the same structure, namely one quadratic and one cubic in S_x, S_y, we can apply the procedure described in section 3.4.1 with the Lax pair of the form

$$S_y = \frac{1}{2}S_x^2 + u,$$
$$S_\tau = a_6 S_x^6 + a_5 S_x^5 + a_4 S_x^4 + a_3 S_x^3 + a_2 S_x^2 + a_1 S_x + a_0$$

where $a_i, \quad i = 0, \ldots, 6$ are arbitrary functions of u, v, w, p and q, depending on which equation we consider.

The results can be summarized as follows: dKP equation and its commuting flows can be obtained from the Lax pair

$$\begin{aligned} S_y &= \frac{1}{2}S_x^2 + u, \\ S_\tau &= 3m_3KP_3 + 4m_4KP_4 + 5m_5KP_5 + 6m_6KP_6. \end{aligned} \tag{3.46}$$

where m_3, m_4, m_5, m_6 are arbitrary constants and

$$\begin{aligned} KP_3 &= \frac{1}{3}S_x^3 + uS_x + w, \\ KP_4 &= \frac{1}{4}S_x^4 + uS_x^2 + wS_x + v + u^2, \\ KP_5 &= \frac{1}{5}S_x^5 + uS_x^3 + wS_x^2 + \left(v + \frac{3u^2}{2}\right)S_x + p + 2uw, \\ KP_6 &= \frac{1}{6}S_x^6 + uS_x^4 + wS_x^3 + (v + 2u^2)S_x^2 + (p + 3uw)S_x + q + 2uv + w^2 + \frac{4}{3}u^3. \end{aligned}$$

When we set $m_6 = m_5 = m_4 = 0$ and $m_3 \neq 0$ in (3.46) then we obtain the Lax pair for dKP equation (3.45).

For the cases described below, the constants m_i that appear in the Lax pair have been removed from the equations, using the corresponding equivalence group, i.e. Galilean transformations, scalings and translations of the dependent variables and also transformations of the form $S \to \lambda S + \mu$, here λ, μ are constants.

Equations with two nonlocalities

When we set $m_6 = m_5 = 0$ in (3.46) then

$$\begin{aligned} S_y &= \frac{1}{2}S_x^2 + u, \\ S_\tau &= m_4S_x^4 + m_3S_x^3 + 4um_4S_x^2 + (3um_3 + 4wm_4)S_x + 4vm_4 + 3m_3w + 4u^2m_4 \end{aligned}$$

where $m_4 \neq 0$, and corresponds to the hydrodynamic type equation with two nonlocalities

$$\begin{aligned} u_\tau &= wu_x + 2uu_y + v_y, \\ w_x &= u_y, \\ v_x &= w_y. \end{aligned}$$

Equations with three nonlocalities

In the case that $m_6 = 0$ then from (3.46) we get

$$S_y = \frac{1}{2}S_x^2 + u,$$
$$S_\tau = m_5 S_x^5 + m_4 S_x^4 + (m_3 + 5um_5)S_x^3 + (4um_4 + 5wm_5)S_x^2 + (3um_3 + 4wm_4 + \frac{15u^2 m_5}{2} + 5vm_5)S_x + 4m_4 v + 5pm_5 + 3wm_3 + 4u^2 m_4 + 10uwm_5,$$

where $m_5 \neq 0$, which corresponds to the hydrodynamic type equation with three nonlocalities

$$u_\tau = \left(v + \frac{3}{2}u^2\right)u_x + 2wu_y + 2uw_y + p_y,$$
$$w_x = u_y,$$
$$v_x = w_y,$$
$$p_x = (v + \frac{1}{2}u^2)_y.$$

Equations with four nonlocalities

Finally, when all the constants in (3.46) are nonzero we obtain

$$S_y = \frac{1}{2}S_x^2 + u,$$
$$S_\tau = m_6 S_x^6 + m_5 S_x^5 + (m_4 + 6um_6)S_x^4 + (m_3 + 5um_5 + 6wm_6)S_x^3 + (4um_4 + 5wm_5 + 12u^2 m_6 + 6vm_6)S_x^2 + \left(3um_3 + 4wm_4 + \frac{15u^2 m_5}{2} + 5vm_5 + 6pm_6 + 18uwm_6\right)S_x + 3wm_3 + 4u^2 m_4 + 4vm_4 + 5pm_5 + 10uwm_5 + 6qm_6 + 8u^3 m_6 + 12uvm_6 + 6w^2 m_6,$$

where $m_6 \neq 0$, which corresponds to the hydrodynamic type equation with four nonlocalities

$$u_\tau = (p + 3uw)u_x + 2(v + 2u^2)u_y + 2ww_y + 2uv_y + q_y,$$
$$w_x = u_y,$$
$$v_x = w_y,$$
$$p_x = (v + \frac{1}{2}u^2)_y,$$
$$q_x = (p + uw)_y.$$

Chapter 4

Differential-Difference equations in 2+1D

In this chapter we address the problem of classifying integrable differential-difference equations in 2+1 dimensions with one/two discrete variables. We consider equations of the general form

$$u_t = F(u, w), \tag{4.1}$$

where $u(x, y, t)$ is a scalar field, $w(x, y, t)$ is the nonlocal variable, and F is a differential/difference operator in the independent variables x and y. The explicit form of w and F will be specified in what follows, but it is important to note that all the nonlocalities considered in this chapter reduce to $w_x = u_y$ in the dispersionless limit $\epsilon \to 0$.

We use the following standard notation for the ϵ-shift operators

$$T_x f(x, y) = f(x + \epsilon, y), \quad T_{\bar{x}} f(x, y) = f(x - \epsilon, y),$$

and the forward/backward discrete derivatives

$$\triangle_x = \frac{T_x - 1}{\epsilon}, \quad \triangle_{\bar{x}} = \frac{1 - T_{\bar{x}}}{\epsilon},$$

same for $T_y,\ T_{\bar{y}},\ \triangle_y,\ \triangle_{\bar{y}}$.

In section 4.1 we list some examples of integrable equations of the type (4.1) which were previously known. In section 4.2 we briefly remind the nondegeneracy conditions that need to be met in order to obtain the classification results. Then, in section 4.3, we apply the method of hydrodynamic reductions and dispersive deformations of dispersionless limits as it was explained in the previous chapter by using the example of Toda equation, while in the final section 4.4, we present the classification results for various classes of equations generalising the intermediate long wave and Toda type equations. Among the classes that were studied, we first present some classification results, in the case that the nonlocalities are expressed in terms of pseudo-differential operators. Namely, in section 4.4.1, we present the classification of **nonlocalities of the form**

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y, \quad w_x = A(\partial_x) u_y,$$

and, in section 4.4.2, the classification of **nonlocalities of the form**

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y, \quad \epsilon w_x = A(\partial_x, \partial_y) u_y.$$

In the remaining sections (4.4.3-4.4.6), we classify the following type of equations, which are named after the type of nonlocality that is considered:
the **Intermediate Long Wave (type 1)**

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots), \qquad \triangle_x w = \frac{T_x + 1}{2} u_y,$$

the **Intermediate Long Wave (type 2)**

$$u_t = \psi u_y + \eta w_y + f \triangle_x g + p \triangle_{\bar{x}} q, \qquad \triangle_x w = \frac{T_x + 1}{2} u_y,$$

the **Toda type**

$$u_t = \varphi u_x + f \triangle_y g + p \triangle_{\bar{y}} q, \qquad w_x = \triangle_y u,$$

and the **Fully discrete type**

$$u_t = f \triangle_x g + h \triangle_{\bar{x}} k + p \triangle_y q + r \triangle_{\bar{y}} s, \qquad \triangle_x w = \triangle_y u,$$

where functions f, g, h, k, p, q and $\varphi, \psi, \eta, \tau$ depend on u, w. For all the equations here we present their corresponding Lax pair. It is important to note that we don't assume the existence of the Lax pair from the beginning, but it is something that follows from direct computations after we obtain the classification results. In Appendix B, we illustrate this computation, using a particular example. All results in this chapter were obtained in joint work with Prof E. V. Ferapontov and Dr V. Novikov [34].

4.1 Examples

There exist various known differential-difference equations in 2+1 dimensions, which have appeared in equivalent form in the literature. Let us recall some of them. The equation

$$u_t = uu_y + w_y, \qquad w = \frac{\epsilon}{2}\frac{T_x + 1}{T_x - 1}u_y,$$

appeared in [16] as a differential-difference analogue of the KP equation, see also [80]. It can be viewed as a $2+1$ dimensional integrable version of the intermediate long wave equation [92]. Another example is

$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12}u_{yyy}, \qquad w = \frac{\epsilon}{2}\frac{T_x + 1}{T_x - 1}u_y,$$

which can be viewed as a differential-difference version of the Veselov-Novikov equation, and appeared previously in [75]. Note that in both examples above, the nonlocal equation $w = \frac{\epsilon}{2}\frac{T_x+1}{T_x-1}u_y$, can be equivalently written as $\triangle_x w = \frac{T_x+1}{2}u_y$. One of the most important examples is the well-known Toda equation [63]

$$u_t = u\triangle_{\bar{y}} w, \qquad w_x = \triangle_y u,$$

and also the equation

$$u_t = (\alpha u + \beta)\triangle_{\bar{y}} e^w, \qquad w_x = \triangle_y u,$$

which is equivalent to the Volterra chain or the Toda chain when $\alpha \neq 0$ or $\alpha = 0$ respectively [78]. Also, in an equivalent form, the equation

$$u_t = \sqrt{\alpha - \beta e^{2u}}\left(e^{w-u}\triangle_y\sqrt{\alpha - \beta e^{2u}} + \triangle_{\bar{y}}(e^{w-u}\sqrt{\alpha - \beta e^{2u}})\right), \quad \triangle_x w = \triangle_y u,$$

appeared as the $2+1$ dimensional analogue of the modified Volterra lattice [88]. In what follows we show how all these examples can be obtained and we also list a number of equations, that to the best of our knowledge, appear to be new.

4.2 Nondegeneracy conditions

All equations discussed here possess dispersionless limits of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y, \tag{4.2}$$

where the functions φ, ψ and η depend on u and w. Integrable dispersionless systems of the form (4.2), were discussed already in chapter 3. These limits will be assumed to be *nondegenerate* in the following sense:

(i) The coefficient η is nonzero: this is equivalent to the requirement that the corresponding dispersion relation, $\lambda(R) = \varphi + \psi\mu(R) + \eta\mu(R)^2$, defines an irreducible conic.

(ii) The dispersionless limit (4.2) is not *totally linearly degenerate*. Recall that totally linearly degenerate systems are characterised by the relations [32]

$$\eta_w = 0, \quad \psi_w + \eta_u = 0, \quad \varphi_w + \psi_u = 0, \quad \varphi_u = 0.$$

A simple example of a totally linearly degenerate system is the linear system,

$$u_t = w_y, \quad w_x = u_y.$$

Dispersive deformations of degenerate systems do not inherit hydrodynamic reductions, at least not in the sense explained here, and require a different approach which is beyond the scope of this thesis. We point out that most of the integrable examples of interest are nondegenerate, or can be brought into a nondegenerate form.

4.3 The method of hydrodynamic reductions

Recall that we consider equations of the form (4.1),

$$u_t = F(u, w),$$

where F is a differential/difference operator in the variables x and y. Now since the right hand side of this equation can be expressed as an infinite series in ϵ, (as we will see later when F will be specified), this means that we can use the hydrodynamic reductions and the dispersive deformations of dispersionless limits essentially by following the pattern outlined in chapter 3. In order to illustrate our approach we will use the 2+1 version of Toda equation and present the classification scheme using a more general class of Toda-type equations.

4.3.1 The example of Toda equation

Consider the 2+1 dimensional Toda equation written in the form

$$u_t = u\triangle_{\bar{y}}w, \quad w_x = \triangle_y u.$$

Expanding the right hand sides using Taylor's formula one obtains

$$\begin{aligned} \frac{u_t}{u} &= w_y - \frac{\epsilon}{2}w_{yy} + \frac{\epsilon^2}{6}w_{yyy} + \ldots, \\ w_x &= u_y + \frac{\epsilon}{2}u_{yy} + \frac{\epsilon^2}{6}u_{yyy} + \ldots. \end{aligned} \tag{4.3}$$

Recall that $\triangle_i = (T_i - 1)/\epsilon$, with $T_i = e^{\epsilon\partial_i}$. The corresponding dispersionless limit results upon setting $\epsilon = 0$:

$$u_t = uw_y, \qquad w_x = u_y. \tag{4.4}$$

This dispersionless system admits exact solutions of the form

$$u = R, \qquad w = w(R), \tag{4.5}$$

where $R(x, y, t)$ satisfies the pair of Hopf-type equations,

$$R_y = \mu R_x, \qquad R_t = \mu^2 R R_x. \tag{4.6}$$

Here $\mu(R)$ is an arbitrary function, and $w' = \mu$. Solutions of this type are known as one-phase solutions (or planar simple waves, or one-component hydrodynamic reductions).

One can show that both solutions (4.5) and reductions (4.6) of the dispersionless system can be deformed into solutions and reductions for the full Toda equation in the form

$$u = R, \qquad w = w(R) + \epsilon w_1 R_x + \epsilon^2(w_2 R_{xx} + w_3 R_x^2) + O(\epsilon^3), \tag{4.7}$$

and

$$\begin{aligned} R_y =& \mu R_x + \epsilon^2(\alpha_1 R_{xxx} + \alpha_2 R_x R_{xx} + \alpha_3 R_x^3) + O(\epsilon^4), \\ R_t =& \mu^2 R R_x + \epsilon^2(\beta_1 R_{xxx} + \beta_2 R_x R_{xx} + \beta_3 R_x^3) + O(\epsilon^4), \end{aligned} \tag{4.8}$$

where w_i, α_i, β_i are certain functions of R. We point out that, modulo the Miura group $R \to R + \epsilon r_1 + \epsilon^2 r^2 + \ldots$, where r_i are $i-$th order polynomials in x derivatives of R [23], the relation $u = R$ can be assumed undeformed. Furthermore, one can show that odd order ϵ-corrections in the equations (4.8) (but not (4.7)) must vanish identically. Substituting (4.7) into (4.3), using (4.8) and the compatibility condition $R_{yt} = R_{ty}$, one can explicitly calculate all coefficients in (4.7) and (4.8) in terms of μ and its derivatives by collecting terms at different powers of ϵ [31]. This gives

$$
\begin{aligned}
w_1 =& \frac{1}{2}\mu^2, \\
w_2 =& \frac{1}{12}\mu^2 \left(2\mu + R\mu'\right), \\
w_3 =& \frac{1}{24}\left(R\left(\mu'\right)^2\left(2\mu - R\mu'\right) + \mu^2\left(11\mu' + R\mu''\right)\right), \\
\alpha_1 =& \frac{1}{12}R\mu^2\mu', \\
\alpha_2 =& \frac{1}{12}R\left(\left(\mu'\right)^2\left(4\mu - R\mu'\right) + 2\mu^2\mu''\right), \\
\alpha_3 =& \frac{1}{24}R\left(3\mu'\mu''\left(2\mu - R\mu'\right) + \mu^2\mu^{(3)}\right), \\
\beta_1 =& \frac{1}{12}R\mu^3\left(\mu + 2R\mu'\right), \\
\beta_2 =& \frac{1}{12}R\mu\left(R\left(\mu'\right)^2\left(11\mu - 2R\mu'\right) + 4\mu^2\left(3\mu' + R\mu''\right)\right), \\
\beta_3 =& \frac{1}{12}R\left(R\left(\mu'\right)^3\left(2\mu - R\mu'\right) + 8R\mu^2\mu'\mu'' + \mu\left(\mu'\right)^2\left(11\mu - 3R^2\mu''\right) + \mu^3\left(4\mu'' + R\mu^{(3)}\right)\right),
\end{aligned}
$$

etc. We point out that this calculation is an entirely algebraic procedure. Thus, all one-phase solutions of the dispersionless system are ‘inherited’ by the original dispersive equation, at least to the order $O(\epsilon^4)$: it is still an open problem to prove their inheritance to all orders in the deformation parameter ϵ. It is important for the integrability, that this works for *arbitrary* $\mu(R)$. The requirement of the inheritance of all hydrodynamic reductions of the dispersionless limit by the full dispersive equation is very restrictive (even to the order $O(\epsilon^2)$), and can be used as an efficient classification criterion in the search for integrable equations. In all examples considered so far, the existence of such deformations to the order ϵ^4 was already sufficient for integrability (in many cases, even the order ϵ^2 was enough), and implied the existence of conventional Lax pairs.

4.3.2 The example of Toda-type equations

We can now illustrate the classification scheme using a class of Toda-type equations of the form

$$u_t = f \triangle_{\bar{y}} g, \quad w_x = \triangle_y u,$$

where $f(u,w)$ and $g(u,w)$ are two arbitrary functions. The corresponding dispersionless limit,

$$u_t = f g_y, \qquad w_x = u_y, \tag{4.9}$$

admits exact solutions of the form

$$u = R, \qquad w = w(R), \tag{4.10}$$

where $R(x,y,t)$ satisfies the pair of Hopf-type equations,

$$R_y = \mu R_x, \qquad R_t = f(g_u \mu + g_w \mu^2) R_x. \tag{4.11}$$

Again, $\mu(R)$ is an arbitrary function, and $w' = \mu$. Imposing the requirement that all one-phase solutions of the corresponding dispersionless limit are inherited by the full dispersive equation, we obtain very strong constraints for the functions f and g. Indeed, at the order ϵ we get the system

$$g_u = 0, \quad f_u f_w = 0, \quad f_w \left(f g_{ww} + g_w f_w\right) = 0.$$

The case $f_u = 0, g_u = 0$ leads to an equation whose dispersionless limit

$$u_t = f(w) g'(w) w_y, \quad w_x = u_y,$$

is totally linearly degenerate, hence this case must be excluded. So at order ϵ we already have that $g_u = 0,\ f_w = 0$. At the order ϵ^2 one obtains two additional constraints:

$$f''(u) = 0, \quad g''(w)^2 - g'(w) g'''(w) = 0.$$

These are two simple second order ordinary differential equations that modulo elementary changes of variables lead to the cases $f(u) = u,\ g(w) = w$ and $f(u) = \alpha u + \beta,\ g(w) = e^w$, which correspond to the Toda and Volterra chains, respectively (see section 3.5). Note that all these constraints appear at the order ϵ^2, and are already sufficient for the integrability, implying the existence of Lax pairs.

4.4 Classification Results

Here we present various results which were obtained using the classification scheme described above. First, in the first two sections (4.4.1, 4.4.2), we look at the classification of nonlocalities in the case of shifts in one/two directions where the equation is local. More precisely, the classification of **nonlocalities of the form** $w_x = A(\partial_x)u_y$

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y,$$
$$w_x = A(\partial_x)u_y = (1 + \epsilon\alpha\partial_x + \epsilon^2\beta\partial_x^2 + \epsilon^3\gamma\partial_x^3 + \epsilon^4\delta\partial_x^4 + \dots)u_y,$$

here $\alpha, \beta, \gamma, \delta, \dots$ are constants, and the classification of **nonlocalities of the form** $w_x = A(\partial_x, \partial_y)u_y$

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y,$$
$$\epsilon w_x = A(\partial_x, \partial_y)u_y = (\epsilon\partial_y + \epsilon^2(...) + \epsilon^3(...) + \dots)u,$$

where the coefficients at ϵ^k are polynomials in ∂_x, ∂_y of degree k.

In the remaining sections (4.4.3-4.4.6), we classify certain classes of equations which are named after the type of nonlocality that is considered. Namely,
the **Intermediate Long Wave (type 1)**

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots), \qquad \triangle_x w = \frac{T_x + 1}{2}u_y,$$

the **Intermediate Long Wave (type 2)**

$$u_t = \psi u_y + \eta w_y + f\triangle_x g + p\triangle_{\bar{x}} q, \qquad \triangle_x w = \frac{T_x + 1}{2}u_y,$$

the **Toda type**

$$u_t = \varphi u_x + f\triangle_y g + p\triangle_{\bar{y}} q, \qquad w_x = \triangle_y u,$$

and finally the **Fully discrete type**

$$u_t = f\triangle_x g + h\triangle_{\bar{x}} k + p\triangle_y q + r\triangle_{\bar{y}} s, \qquad \triangle_x w = \triangle_y u,$$

where functions f, g, h, k, p, q and $\varphi, \psi, \eta, \tau$ depend on u, w. For all examples we calculate the corresponding dispersionless Lax pair based on Zakharov's idea that all integrable dispersionless systems posses a dispersionless Lax pair. Once these are computed we find the corresponding dispersive Lax pair by a suitable quantisation of the dispersionless Lax pair. Dispersionless and dispersive Lax pairs are related through the transformation $\psi = e^{S/\epsilon}$. Classification results are presented modulo Galilean transformations, and transformations of the form $u \to au + b,\ w \to aw + c$.

4.4.1 Classification of nonlocalities of the form $w_x = A(\partial_x)u_y$

Here we consider equations of the form

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y, \tag{4.12}$$

with the nonlocality $w_x = A(\partial_x)u_y$, where A is a constant-coefficient pseudo-differential operator of the form

$$A = 1 + \epsilon\alpha\partial_x + \epsilon^2\beta\partial_x^2 + \epsilon^3\gamma\partial_x^3 + \epsilon^4\delta\partial_x^4 + \dots, \tag{4.13}$$

here $\alpha, \beta, \gamma, \delta, \dots$ are constants. Again in the limit $\epsilon \to 0$ the nonlocality becomes $w_x = u_y$.

It can be proven that all odd terms in (4.13) vanish, while all the constants appearing in the even terms depend on β as follows

$$w_x = (1 + \epsilon^2\beta\partial_x^2 - \epsilon^4\frac{\beta^2}{5}\partial_x^4 + \epsilon^6\frac{2\beta^3}{35}\partial_x^6 + \dots)u_y.$$

Thus we obtain the following result

Theorem 4.1 *There exists only one possible nonlocality of the form (4.13) for equations (4.12) and this nonlocality is the intermediate long wave type nonlocality*

$$w = \frac{\epsilon}{2}\frac{T_x + 1}{T_x - 1}u_y.$$

Indeed

$$w_x = \partial_x\frac{\epsilon}{2}\frac{T_x + 1}{T_x - 1}u_y = \partial_x\frac{\epsilon}{2}\ \frac{2 + \epsilon\partial_x + \frac{\epsilon^2}{2}\partial_x^2 + \frac{\epsilon^3}{6}\partial_x^3 + \dots}{\epsilon\partial_x + \frac{\epsilon^2}{2}\partial_x^2 + \frac{\epsilon^3}{6}\partial_x^3 + \dots}u_y = \frac{1 + \frac{\epsilon}{2}\partial_x + \frac{\epsilon^2}{4}\partial_x^2 + \frac{\epsilon^3}{12}\partial_x^3 + \dots}{1 + \frac{\epsilon}{2}\partial_x + \frac{\epsilon^2}{6}\partial_x^2 + \frac{\epsilon^3}{24}\partial_x^3 + \dots}u_y,$$

and expanding the last fraction using

$$\frac{1}{1+s} = 1 - s + s^2 - s^3 + O(s^4),$$

we obtain

$$w_x = (1 + \frac{\epsilon^2}{12}\partial_x^2 - \frac{\epsilon^4}{720}\partial_x^4 + \frac{\epsilon^6}{30240}\partial_x^6 + \dots)u_y,$$

$(\beta = \frac{1}{12})$.

4.4.2 Classification of nonlocalities of the form $w_x = A(\partial_x, \partial_y)u_y$

Here we classify integrable equations of the same form as before,

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y, \tag{4.14}$$

where φ, ψ, τ and η depend on u, w, but now the nonlocality w is defined as $w_x = A(\partial_x, \partial_y)u_y$ or $\epsilon w_x = Bu$ where B is a constant-coefficient pseudo-differential operator of the form

$$B = \epsilon\partial_y + \epsilon^2(...) + \epsilon^3(...) + ...,$$

so that in the dispersionless limit one gets $w_x = u_y$. Here the coefficient at ϵ^k is a polynomial in ∂_x, ∂_y of degree k. For instance, the Toda equation from section 4.3 can be written in the equivalent form

$$u_t = uw_y, \quad w_x = (\partial_y^{-1}\triangle_y\triangle_{\bar{y}})u.$$

Indeed one can set $w \to \partial_y^{-1}\triangle_{\bar{y}}w$ to recover the familiar form of the Toda equation.

The result is the following

Theorem 4.2 *The examples below constitute a complete list of integrable equations of the form (4.14) with the nonlocality $\epsilon w_x = Bu$*

$$u_t = uu_y + w_y, \qquad \triangle_x w = \frac{T_x + 1}{2}u_y, \tag{4.15}$$

$$u_t = (w + \alpha e^u)u_y + w_y, \qquad \triangle_x w = \frac{T_x + 1}{2}u_y, \tag{4.16}$$

$$u_t = uw_y, \qquad w_x = (\partial_y^{-1}\triangle_y\triangle_{\bar{y}})u, \tag{4.17}$$

$$u_t = e^w w_y, \qquad w_x = (\partial_y^{-1}\triangle_y\triangle_{\bar{y}})u, \tag{4.18}$$

$$u_t = e^{u-w}(w_y - u_y), \quad w_x = u_y + \epsilon^2\partial_y(\partial_x - \partial_y)^2 u + O(\epsilon^4), \tag{4.19}$$

For the first two equations (4.15) and (4.16) the nonlocality is that of the intermediate long wave type, equation (4.17) can be brought to the Toda form, while last equation's nonlocality, (4.19)

$$w_x = u_y + \epsilon^2\partial_y(\partial_x - \partial_y)^2 u - \frac{\epsilon^4}{5}\partial_y(\partial_x - \partial_y)^2(\partial_x^2 + 2\partial_x\partial_y - 2\partial_y^2) + O(\epsilon^6),$$

needs further investigation, as we were not able to write it in a closed form, and it is the only case that the pseudo differential operator involves derivatives in both x and y directions.

4.4.3 Intermediate Long Wave nonlocality (type 1)

First we classify integrable equations of the form

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(...) + \epsilon^2(...), \tag{4.20}$$

where w is the nonlocality of the intermediate long wave type, $\triangle_x w = \frac{T_x+1}{2} u_y$, or, equivalently, $w = \frac{\epsilon}{2}\frac{T_x+1}{T_x-1} u_y$. Here dots denote terms which are homogeneous polynomials of degree two and three in the x- and y-derivatives of u and w, whose coefficients are allowed to be functions of u and w. One can show that all ϵ-terms, as well as all terms containing derivatives with respect to x, in particular φ and τ, must vanish identically.

Theorem 4.3 *The following examples constitute a complete list of integrable equations of the form (4.20) with the nonlocality of intermediate long wave type:*

$$u_t = uu_y + w_y, \tag{4.21}$$

$$u_t = (w + \alpha e^u) u_y + w_y, \tag{4.22}$$

$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} u_{yyy}, \tag{4.23}$$

$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12}\left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y . \tag{4.24}$$

Sketch of the proof of Theorem 4.3:

The proof follows the method outlined in section 4.3. In order to be able to reproduce the classification list, one should work with the Mathematica program given in Appendix A. Equation (4.20) is

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon F(u,w) + \epsilon^2 G(u,w),$$

$$w_x + \tfrac{\epsilon}{2} w_{xx} + \tfrac{\epsilon^2}{6} w_{xxx} + \cdots = u_y + \tfrac{\epsilon}{2} u_{xy} + \tfrac{\epsilon^2}{4} u_{xxy} + \ldots,$$

where $F(u,w)$ is a homogeneous polynomial of degree two in the x- and y-derivatives of u and w, while $G(u,w)$ is a homogeneous polynomial of degree three in these derivatives. The coefficients of these polynomials, f_i, g_i, are functions of u, w. In fact the polynomials can be simplified by expressing derivatives of w through the nonlocality $w = \frac{\epsilon}{2}\frac{T_x+1}{T_x-1}u_y$. The corresponding dispersionless limit is

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y, \quad w_x = u_y.$$

This limit possesses exact solutions of the form $u = R(x,y,t)$, $w = w(R(x,y,t))$ where R solves a pair of commuting equations,

$$R_y = \mu(R)R_x, \quad R_t = \Big(\varphi + \mu(R)(\psi+\tau) + \mu^2(R)\eta\Big)\, R_x, \tag{4.25}$$

here $\mu(R) = w'(R)$ is an arbitrary function. The dispersionless equation is non degenerate (in the sense described in section 4.2), when $\eta \neq 0$ and the relations

$$\eta_w = 0, \quad \psi_w + \tau_w + \eta_u = 0, \quad \varphi_w + \psi_u + \tau_u = 0, \quad \varphi_u = 0. \tag{4.26}$$

are not satisfied simultaneously. We require that all reductions (4.25) can be deformed into formal solutions of the original equation (4.20)

$$u = R, \qquad w = w(R) + \epsilon w_1 R_x + \epsilon^2(w_2 R_{xx} + w_3 R_x^2) + O(\epsilon^3),$$

and

$$\begin{aligned} R_y =& \mu R_x + \epsilon^2(\alpha_1 R_{xxx} + \alpha_2 R_x R_{xx} + \alpha_3 R_x^3) + O(\epsilon^4),\\ R_t =& \mu^2 R R_x + \epsilon^2(\beta_1 R_{xxx} + \beta_2 R_x R_{xx} + \beta_3 R_x^3) + O(\epsilon^4), \end{aligned}$$

where w_i, α_i, β_i are functions of R. Substituting the deformed solutions and reductions in the original equation (4.20), using the compatibility condition $R_{yt} = R_{ty}$ we obtain the following results:

At order ϵ we get a system of 15 equations which, when solved, leads to either that all coefficients $f_i(u,w) = 0$ or that the equation satisfies the linearly degenerate relations (4.26). Thus, at this order the result is that $F(u,w) = 0$. At the next order ϵ^2 we have a system of 36 equations, that leads to two essential branches. The first branch is when all coefficients of the polynomial $G(u,w)$ are zero. This basically means that the equation is in hydrodynamic form. The solution of the system of the 36 equations yields

$$\eta_u = \eta_w = 0, \quad \tau = 0, \quad \varphi_u = \varphi_w = 0, \quad \psi_{ww} = 0, \quad \psi_{uw} = 0, \quad \tau\psi_{uu} - \psi_u\psi_w = 0$$

hence $\eta(u,w) = \eta(const)$, $\varphi(u,w) = \varphi(const)$ and $\psi(u,w) = \psi_1(u) + w\psi_2$, with $\psi_2 = const$, and $\psi_1(u)$ satisfying the second order ODE $-\psi_2\psi_1'(u) + \eta\psi_1''(u) = 0$. The solution of this ODE leads to two cases:

$$\psi_2 = 0 \quad \psi_1(u) = c_1 u + c_2, \qquad or \qquad \psi_2 \neq 0, \quad \psi_1(u) = \frac{c_1\eta}{\psi_2} e^{\frac{\psi_2}{\eta}u} + c_2.$$

Up to scalings and translations of the dependent variables u, w we obtain the equations (4.21) and (4.22) from the theorem. The second branch at order ϵ^2 is when at least one of the coefficients of the polynomial $G(u,w)$ is non-zero. Similarly, solving the system of 36 equations we can prove that all terms containing derivatives with respect to x must vanish identically, and after scalings and translations of the dependent variables obtain the last two equations of the Theorem, (4.23) and (4.24).

We note that it is sufficient to perform calculations up to the order ϵ^2, and also that at this stage all the terms in the deformed solutions and reductions are given explicitly, in terms of the arbitrary function $\mu(R)$ and its derivatives. ■

Although the first two equations are of the first order, they should be viewed as dispersive: the dispersion is contained in the equation for nonlocality. The equation (4.21), which can be written in the form

$$u_t = uu_y + \frac{\epsilon}{2}\frac{T_x + 1}{T_x - 1}u_{yy},$$

first appeared in [16] as a differential-difference analogue of the KP equation, see also [80] (we point out that its dispersionless limit does not coincide with that of KP). It can also be viewed as a $2+1$ dimensional integrable version of the intermediate long wave equation [92]. The equation (4.23) is a differential-difference version of the Veselov-Novikov equation discussed in [75]. The last example can be viewed as a differential-difference version of the modified Veselov-Novikov equation. To the best of our knowledge, equations (4.22) and (4.24) are new.

Lax pairs, dispersionless limits and dispersionless Lax pairs for the equations from Theorem 4.3 are provided in the table below (note that equations (4.23) and (4.24) have coinciding dispersionless limits/Lax pairs). Here and in what follows, Lax pairs were obtained by the quantisation of dispersionless Lax pairs as discussed in [91]. First of all, using the method of hydrodynamic reductions, we produce every time the list of integrable equations. Then, we consider the dispersionless limits of these equations and compute the dispersionless Lax pairs. Finally, bearing in mind that $\psi = e^{S/\epsilon}$, we quantise this Lax pair appropriately, to obtain the Lax pair of the initial differential-difference equation.

Eqn	*Lax pair*	*Disp/less limit*	*Disp/less Lax pair*
(4.21)	$T_x\psi = \epsilon\psi_y - u\psi$ $\epsilon\psi_t = \frac{\epsilon^2}{2}\psi_{yy} + (w - \frac{\epsilon}{2}u_y)\psi$	$u_t = uu_y + w_y$ $w_x = u_y$	$e^{S_x} = S_y - u$ $S_t = \frac{1}{2}S_y^2 + w$
(4.22)	$T_x\psi = \epsilon e^{-u}\psi_y - \alpha\psi$ $\psi_t = \frac{\epsilon}{2}\psi_{yy} + (w - \frac{\epsilon}{2}u_y)\psi_y$	$u_t = (w + \alpha e^u)u_y + w_y$ $w_x = u_y$	$e^{S_x} = e^{-u}S_y - \alpha$ $S_t = \frac{1}{2}S_y^2 + wS_y$
(4.23)	$\epsilon(T_x - 1)\psi_y = -2u(T_x + 1)\psi$ $\psi_t = \frac{\epsilon^2}{12}\psi_{yyy} + (w - \frac{\epsilon}{2}u_y)\psi_y$	$u_t = u^2u_y + (uw)_y$ $w_x = u_y$	$\frac{e^{S_x}-1}{e^{S_x}+1}S_y = -2u$ $S_t = \frac{1}{12}S_y^3 + wS_y$
(4.24)	$\epsilon(T_x - 1)\psi_y = \frac{\epsilon}{2}\frac{u_y}{u}(T_x - 1)\psi-$ $2u(T_x + 1)\psi$ $\psi_t = \frac{\epsilon^2}{12}\psi_{yyy} + (w - \frac{\epsilon}{2}u_y)\psi_y+$ $\frac{1}{2}(w_y - \frac{\epsilon}{2}u_{yy})\psi$	$u_t = u^2u_y + (uw)_y$ $w_x = u_y$	$\frac{e^{S_x}-1}{e^{S_x}+1}S_y = -2u$ $S_t = \frac{1}{12}S_y^3 + wS_y$

Remark. Equations (4.21) and (4.22) are related by a (rather non-trivial) gauge transformation. Let us begin with the dispersionless limit of (4.22),

$$u_t = (w + \alpha e^u)u_y + w_y, \quad w_x = u_y,$$

with the corresponding Lax pair

$$S_t = \frac{1}{2}S_y^2 + wS_y, \quad e^{S_x} = e^{-u}S_y - \alpha.$$

Let h be a potential such that $u = h_x,\ w = h_y$. One can verify that the new variables $\tilde{u} = w + \alpha e^u,\ \tilde{w} = h_t - \frac{w^2}{2},\ \tilde{S} = S + h$ satisfy the dispersionless equation (4.21),

$$\tilde{u}_t = \tilde{u}\tilde{u}_y + \tilde{w}_y, \quad \tilde{w}_x = \tilde{u}_y,$$

along with the corresponding Lax pair

$$\tilde{S}_t = \frac{1}{2}\tilde{S}_y^2 + \tilde{w}, \quad e^{\tilde{S}_x} = \tilde{S}_y - \tilde{u},$$

thus establishing the required link at the dispersionless level (it is sufficient to perform this calculation at the level of Lax pairs: the equations for $\tilde{u}, \tilde{w}$ will be automatic). The dispersive version of this construction is as follows. We take the equation (4.22),

$$u_t = (w + \alpha e^u)u_y + w_y, \quad w = \frac{\epsilon}{2}\frac{T_x + 1}{T_x - 1}u_y,$$

with the corresponding Lax pair

$$\psi_t = \frac{\epsilon}{2}\psi_{yy} + (w - \frac{\epsilon}{2}u_y)\psi_y, \quad T_x\psi = \epsilon e^{-u}\psi_y - \alpha\psi.$$

Let H be a potential such that $u = \frac{T_x-1}{\epsilon}H,\ w = \frac{T_x+1}{2}H_y$. One can verify that the new variables $\tilde{u} = H_y + \alpha e^u,\ \tilde{w} = H_t - \frac{H_y^2}{2} + \frac{\alpha\epsilon}{2}e^u\triangle_x^+ H_y,\ \tilde{\psi} = e^{H/\epsilon}\psi$ satisfy the equation (4.21),

$$\tilde{u}_t = \tilde{u}\tilde{u}_y + \tilde{w}_y, \quad \tilde{w} = \frac{\epsilon}{2}\frac{T_x + 1}{T_x - 1}\tilde{u}_y,$$

with the corresponding Lax pair

$$\epsilon\tilde{\psi}_t = \frac{\epsilon^2}{2}\tilde{\psi}_{yy} + (\tilde{w} - \frac{\epsilon}{2}\tilde{u}_y)\tilde{\psi}, \quad T_x\tilde{\psi} = \epsilon\tilde{\psi}_y - \tilde{u}\tilde{\psi}.$$

Again, it is sufficient to perform this calculation at the level of Lax pairs. Due to the complexity of this transformation we prefer to keep both equations in the list of Theorem 4.3 as separate cases.

4.4.4 Intermediate Long Wave nonlocality (type 2)

Another interesting class of equations with the nonlocality of intermediate long wave type is

$$u_t = \psi u_y + \eta w_y + f\triangle_x g + p\triangle_{\bar{x}} q, \tag{4.27}$$

where $\triangle_x w = \frac{T_x+1}{2}u_y$, and ψ, η, f, g, p, q are functions of u and w.

Theorem 4.4 *The following examples constitute a complete list of integrable equations of the form (4.27) with the nonlocality of intermediate long wave type:*

$$\begin{aligned} u_t &= uu_y + w_y, \\ u_t &= (w + \alpha e^u)u_y + w_y, \\ u_t &= wu_y + w_y + \frac{\triangle_x + \triangle_{\bar{x}}}{2} e^{2u}, && (4.28) \\ u_t &= wu_y + w_y + e^u(\triangle_x + \triangle_{\bar{x}})e^u. && (4.29) \end{aligned}$$

The proof of this theorem follows the procedure described in Theorem 4.3 and is omitted.

Here the first two equations are the same as in Theorem 4.3h, the third example first appeared in [60], while the fourth is apparently new. Lax pairs, dispersionless limits and dispersionless Lax pairs for equations from Theorem 4.4 are provided in the table below:

Eqn	*Lax pair*	*Dispersionless limit*	*Dispersionless Lax pair*
(4.28)	$\epsilon\psi_y = (T_x e^u)T_x\psi + e^u T_{\bar{x}}\psi$ $\epsilon\psi_t = \frac{1}{2}e^{T_x(1+T_x)u}T_x^2\psi - \frac{1}{2}e^{(1+T_{\bar{x}})u}T_{\bar{x}}^2\psi +$ $T_x(we^u)T_x\psi + we^u T_{\bar{x}}\psi$	$u_t = 2e^{2u}u_x +$ $wu_y + w_y$ $w_x = u_y$	$S_y = 2e^u \cosh S_x$ $S_t = e^{2u}\sinh 2S_x +$ $2we^u \cosh S_x$
(4.29)	$\epsilon\psi_y = e^u(T_x\psi + T_{\bar{x}}\psi)$ $\epsilon\psi_t = \frac{1}{2}e^{(1+T_x)u}T_x^2\psi - \frac{1}{2}e^{(1+T_{\bar{x}})u}T_{\bar{x}}^2\psi +$ $we^u(T_x\psi + T_{\bar{x}}\psi) + \frac{\epsilon}{2}e^u[(\triangle_x + \triangle_{\bar{x}})e^u]\psi$	$u_t = 2e^{2u}u_x +$ $wu_y + w_y$ $w_x = u_y$	$S_y = 2e^u \cosh S_x$ $S_t = e^{2u}\sinh 2S_x +$ $2we^u \cosh S_x$

Note that equations (4.28) and (4.29) have coinciding dispersionless limits.

4.4.5 Toda type nonlocality

In this section we classify integrable equations of the form

$$u_t = \varphi u_x + f\triangle_y g + p\triangle_{\bar{y}} q, \tag{4.30}$$

where the nonlocality w is defined as $w_x = \triangle_y u$, and φ, f, g, p, q are functions of u and w.

Theorem 4.5 *The following examples constitute a complete list of integrable equations of the form (4.30) with the nonlocality of Toda type:*

$$u_t = u\triangle_{\bar{y}} w, \tag{4.31}$$

$$u_t = (\alpha u + \beta)\triangle_{\bar{y}} e^w, \tag{4.32}$$

$$u_t = e^w\sqrt{u}\triangle_y\sqrt{u} + \sqrt{u}\triangle_{\bar{y}}(e^w\sqrt{u}), \tag{4.33}$$

here $\alpha, \beta = const.$

We skip the details of the proof, as it follows the procedure described in Theorem 4.3. Equation (4.31) is the 2+1 dimensional Toda equation, which can also be written in the form $(\ln u)_{xt} = \triangle_y\triangle_{\bar{y}} u$, while equation (4.32) is equivalent to the Volterra chain when $\alpha \neq 0$, or to the Toda chain when $\alpha = 0$. Lax pairs, dispersionless limits and dispersionless Lax pairs for the equations from Theorem 4.5 are provided in the table at the end of this subsection.

Remark. One can show that there exist no nondegenerate integrable equations of the form

$$u_t = \eta w_y + f\triangle_x g + p\triangle_{\bar{x}} q,$$

where the nonlocality w is defined as $\triangle_x w = u_y$, and η, f, g, p, q are functions of u and w. Indeed, the integrability requirement implies the condition $\eta = 0$, which corresponds to degenerate systems.

Eqn	*Lax pair*	*Disp/less limit*	*Disp/less Lax pair*
(4.31)	$\epsilon T_y\psi_x = u\psi$ $\epsilon\psi_t = -T_y\psi + (T_{\bar{y}}w)\psi$	$u_t = uw_y$ $w_x = u_y$	$e^{S_y}S_x = u$ $S_t = -e^{S_y} + w$
(4.32)	$\epsilon T_y\psi_x = (\alpha T_y u + \beta)\psi - (T_y u)T_y\psi$ $\epsilon\psi_t = -e^w T_y\psi + \alpha e^w\psi$	$u_t = (\alpha u + \beta)e^w w_y$ $w_x = u_y$	$e^{S_y}S_x = \alpha u + \beta - ue^{S_y}$ $S_t = -e^w e^{S_y} + \alpha e^w$
(4.33)	$\epsilon T_y\psi_x = \epsilon\sqrt{\frac{T_y u}{u}}\psi_x - (T_y u)T_y\psi$ $- \sqrt{uT_y u}\ \psi$ $\epsilon\psi_t = \frac{1}{2}e^w T_y\psi - \frac{1}{2}(T_{\bar{y}}e^w)T_{\bar{y}}\psi$	$u_t = e^w u_y + ue^w w_y$ $w_x = u_y$	$e^{S_y}S_x = S_x - ue^{S_y} - u$ $S_t = e^w \sinh S_y$

4.4.6 Fully discrete type nonlocality

In this last section we classify integrable equations of the form

$$u_t = f\triangle_x g + h\triangle_{\bar{x}}k + p\triangle_y q + r\triangle_{\bar{y}}s, \tag{4.34}$$

where the nonlocality w is defined as $\triangle_x w = \triangle_y u$, and the functions f, g, h, k, p, q, r, s depend on u and w.

Theorem 4.6 *The following examples constitute a complete list of integrable equations of the form (4.34) with the fully discrete nonlocality:*

$$u_t = u\triangle_{\bar{y}}(u - w), \tag{4.35}$$

$$u_t = u(\triangle_x + \triangle_{\bar{y}})w, \tag{4.36}$$

$$u_t = (\alpha e^{-u} + \beta)\triangle_{\bar{y}}e^{u-w}, \tag{4.37}$$

$$u_t = (\alpha e^u + \beta)(\triangle_x + \triangle_{\bar{y}})e^w, \tag{4.38}$$

$$u_t = \sqrt{\alpha - \beta e^{2u}}\left(e^{w-u}\triangle_y\sqrt{\alpha - \beta e^{2u}} + \triangle_{\bar{y}}(e^{w-u}\sqrt{\alpha - \beta e^{2u}})\right), \tag{4.39}$$

here $\alpha, \beta = const.$

The proof follows the procedure described in Theorem 4.3 and is omitted.

In equivalent form, equation (4.39) is known as the $2+1$ dimensional analogue of the modified Volterra lattice [88]. Lax pairs, dispersionless limits and dispersionless Lax pairs for the equations from Theorem 4.6 are provided in the table below:

Eqn	*Lax pair*	*Dispersionless limit*	*Disp/less Lax pair*
(4.35)	$T_xT_y\psi = -T_y\psi + (T_yu)T_x\psi$ $\epsilon\psi_t = T_y\psi - w\psi$	$u_t = u(u_y - w_y)$ $w_x = u_y$	$e^{S_x+S_y} = ue^{S_x} - e^{S_y}$ $S_t = e^{S_y} - w$
(4.36)	$T_xT_y\psi = T_y\psi - u\psi$ $\epsilon\psi_t = T_y\psi + (T_{\bar{y}}w)\psi$	$u_t = u(u_y + w_y)$ $w_x = u_y$	$e^{S_x+S_y} = e^{S_y} - u$ $S_t = e^{S_y} + w$
(4.37)	$T_{\bar{y}}\psi = \frac{e^u}{\alpha+\beta e^u}T_{\bar{x}}\psi + \frac{1}{\alpha+\beta e^u}\psi$ $\epsilon T_{\bar{x}}\psi_t = -\epsilon e^{-u}\psi_t -$ $\alpha e^{-w}T_{\bar{x}}\psi + \beta e^{-w}\psi$	$u_t = (\alpha + \beta e^u)e^{-w}(u_y - w_y)$ $w_x = u_y$	$e^{-S_y} = \frac{e^u e^{-S_x}+1}{\alpha+\beta e^u}$ $e^{-S_x}S_t = -e^{-u}S_t -$ $\alpha e^{-w}e^{-S_x} + \beta e^{-w}$
(4.38)	$T_{\bar{y}}\psi = -\frac{e^u}{\alpha e^u+\beta}T_x\psi + \frac{1}{\alpha e^u+\beta}\psi$ $\epsilon T_x\psi_t = \epsilon e^{-u}\psi_t -$ $\beta(T_xe^w)T_x\psi - \alpha(T_xe^w)\psi$	$u_t = (\alpha e^u + \beta)e^w(u_y + w_y)$ $w_x = u_y$	$e^{-S_y} = \frac{-e^u e^{S_x}+1}{\alpha e^u+\beta}$ $e^{S_x}S_t = e^{-u}S_t -$ $\beta e^w e^{S_x} - \alpha e^w$
(4.39)	$T_xT_y\psi = \frac{\alpha}{\beta}(T_ye^{-u})T_y\psi +$ $\frac{T_y(e^{-u}\sqrt{\alpha-\beta e^{2u}})}{\sqrt{\alpha-\beta e^{2u}}}(T_x\psi - e^u\psi)$ $\epsilon\psi_t = \beta(T_{\bar{y}}e^w)T_{\bar{y}}\psi - \alpha e^wT_y\psi$	$u_t = \alpha(e^{w-u})_y - \beta(e^{w+u})_y$ $w_x = u_y$	$e^{S_x+S_y} = \frac{\alpha}{\beta}e^{-u}e^{S_y} +$ $e^{-u}e^{S_x} - 1$ $S_t = \beta e^w e^{-S_y} -$ $\alpha e^w e^{S_y}$

Remark 1. The continuum limit of the modified Volterra lattice (4.39) in $x-$direction, namely $x \to hx,\ u \to hu$ and $h \to 0$, gives the Toda-type lattice (4.33). Similarly, in the same limit equations (4.35) and (4.36) give the Toda equation (4.31), while the remaining

two, (4.37) and (4.38), lead to the equation (4.32) with $\alpha = 0$.

Remark 2. We point out that there exist other types of integrable equations with the nonlocality $\triangle_x w = \triangle_y u$, which are not covered by Theorem 4.6. One of such examples is the first flow of the discrete modified Veselov-Novikov hierarchy constructed in [90],

$$u_t = \sqrt{-(T_{\bar{y}}\triangle_x e^{2w})(\triangle_x e^{-2w})}, \quad \triangle_x w = \triangle_y u.$$

This equation is not of the form (4.34), furthermore, its dispersionless limit is degenerate:

$$u_t = 2w_x, \ w_x = u_y.$$

Chapter 5

Discrete equations in 3D

In this chapter, we are considering discrete 3D equations and address the problem of classification of such integrable equations, within various particularly interesting subclasses. The method of deformations of hydrodynamic reductions, as introduced in the previous chapters, can be applied in the same way: we require that hydrodynamic reductions of the corresponding dispersionless limits are 'inherited' by the discrete equations, and the only constraint is that of the nondegeneracy of the dispersionless limit. This method proposes a novel approach to the classification of integrable discrete equations in 3D, a problem which, until now, was treated via the *multidimensional consistency* [5, 86]. Multidimensional consistency is an extension of the 3D-consistency approach, which was proposed independently by the authors of [10, 70], for the classification of discrete equations in 2D. They considered equations on quad-graphs,

$$Q(u, T_1u, T_2u, T_{12}u; a, b) = 0,$$

here $a, b \in \mathbb{C}$, and the fields $u, T_1u, T_2u, T_{12}u$ can be attached to the vertices of a square. If the equation above can be generalised in a consistent way on the faces of a cube, then it is said to be 3D-consistent. Similarly, for discrete equations in 3D [5], classification is performed based on the consistency around the 4D cube. It is important to note that our approach to the integrability in 3D is essentially intrinsic: it applies directly to a given equation, and does not require its embedding into a compatible hierarchy living in a higher dimensional space.

The main observation is that various 3D difference equations can be obtained as 'naive' discretisations of second order quasilinear PDEs, by simply replacing partial derivatives ∂ by discrete derivatives $\triangle$. Although this recipe should by no means preserve the integrability in general, it does apply to a whole range of interesting examples. Thus, the dispersionless PDE

$$(u_1 - u_2)u_{12} + (u_3 - u_1)u_{13} + (u_2 - u_3)u_{23} = 0,$$

[59], gives rise to the lattice KP equation [16, 68, 67],

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u + (\triangle_3 u - \triangle_1 u)\triangle_{13} u + (\triangle_2 u - \triangle_3 u)\triangle_{23} u = 0.$$

Similarly, the dispersionless PDE

$$\partial_1 \left(\ln \frac{u_3}{u_2}\right) + \partial_2 \left(\ln \frac{u_1}{u_3}\right) + \partial_3 \left(\ln \frac{u_2}{u_1}\right) = 0$$

results in the Schwarzian KP equation [11, 12, 19, 53, 68],

$$\triangle_1 \left(\ln \frac{\triangle_3 u}{\triangle_2 u}\right) + \triangle_2 \left(\ln \frac{\triangle_1 u}{\triangle_3 u}\right) + \triangle_3 \left(\ln \frac{\triangle_2 u}{\triangle_1 u}\right) = 0.$$

Notation is similar to the one introduced in Chapter 4, so in what follows $u(x^1, x^2, x^3)$ is a function of three (continuous) variables. We use subscripts for partial derivatives of u with respect to the independent variables x^i: $u_i = u_{x^i},\ u_{ij} = u_{x^i x^j},\ \partial_i = \partial_{x^i}$, etc. Forward/backward ϵ-shifts and discrete derivatives in x^i-direction are denoted $T_i,\ T_{\bar{i}}$ and $\triangle_i, \triangle_{\bar{i}}$, respectively: $\triangle_i = \frac{T_i - 1}{\epsilon},\ \triangle_{\bar{i}} = \frac{1 - T_{\bar{i}}}{\epsilon}$. We also use multi-index notation for multiple shifts/derivatives: $T_{ij} = T_i T_j,\ \triangle_{i\bar{j}} = \triangle_i \triangle_{\bar{j}}$, etc.

In section 5.1 we list various well-known examples of discrete integrable 3D equations, which we call Hirota-type, and we give their $\triangle$-representation. The reason for this representation is that their dispersionless limits become more clearly seen. A brief summary of the method of deformations of hydrodynamic reductions is described in section 5.2. The two subsequent sections are devoted to the study of two interesting subclasses of equations that were considered. More precisely, in section 5.4, we provide a classification of **Integrable discrete conservation laws** of the form

$$\triangle_1 f + \triangle_2 g + \triangle_3 h = 0,$$

where f, g, h are functions of $\triangle_1 u, \triangle_2 u, \triangle_3 u$ and in section 5.4 we classify **Discrete integrable quasilinear equations** of the form

$$\sum_{i,j=1}^{3} f_{ij} \triangle_{ij} u = 0,$$

where f_{ij} are again functions of $\triangle_1 u, \triangle_2 u, \triangle_3 u$. We also study differential-difference degenerations of the above. In the last section, 5.6, we perform some numerical simulations using Mathematica. Choosing a certain discrete equation we compare its solution with the solution of the corresponding dispersionless equation and we show how the phenomenon of a dispersive shock wave appears. In fact this phenomenon can be observed in very simple equations and such an example is given in the end of the section. All results of this chapter were obtained in a joint work with Prof E. V. Ferapontov and Dr V. Novikov [35].

5.1 △-forms of discrete integrable equations

Below we list △-forms of various 3D discrete integrable equations which have been discussed in the literature. The advantage of △-representation is that the corresponding dispersionless limits become more clearly seen. Although these equations have appeared under different names, most of them are related via various gauge/Miura/Bäcklund type transformations. It is verified that all equations listed below inherit hydrodynamic reductions of their dispersionless limits, at least to the order ϵ^2.

Hirota equation [45]:

$$\alpha T_1\tau T_{\bar{1}}\tau + \beta T_2\tau T_{\bar{2}}\tau + \gamma T_3\tau T_{\bar{3}}\tau = 0.$$

Dividing by τ^2 and setting $\tau = e^{u/\epsilon^2}$ we can rewrite it in the form

$$\alpha e^{\triangle_{1\bar{1}} u} + \beta e^{\triangle_{2\bar{2}} u} + \gamma e^{\triangle_{3\bar{3}} u} = 0.$$

Its dispersionless limit is

$$\alpha e^{u_{11}} + \beta e^{u_{22}} + \gamma e^{u_{33}} = 0.$$

Hirota-Miwa equation [65]:

$$\alpha T_1\tau T_{23}\tau + \beta T_2\tau T_{13}\tau + \gamma T_3\tau T_{12}\tau = 0.$$

Dividing by $T_1\tau T_2\tau T_3\tau/\tau$ and setting $\tau = e^{u/\epsilon^2}$ we can rewrite it in the form

$$\alpha e^{\triangle_{23}u} + \beta e^{\triangle_{13}u} + \gamma e^{\triangle_{12}u} = 0.$$

Its dispersionless limit is

$$\alpha e^{u_{23}} + \beta e^{u_{13}} + \gamma e^{u_{12}} = 0.$$

Gauge-invariant Hirota equation, or Y-system [56, 89]:

$$\frac{T_2 v T_{\bar{2}} v}{T_1 v T_{\bar{1}} v} = \frac{(1+T_3 v)(1+T_{\bar{3}} v)}{(1+T_1 v)(1+T_{\bar{1}} v)}.$$

Taking log of both sides we obtain

$$(\triangle_{2\bar{2}} - \triangle_{1\bar{1}}) \ln v = (\triangle_{3\bar{3}} - \triangle_{1\bar{1}}) \ln(1+v).$$

Setting $v = e^u$ we get

$$\triangle_{2\bar{2}}\ u = \triangle_{1\bar{1}}\ [u - \ln(e^u+1)] + \triangle_{3\bar{3}}\ [\ln(e^u+1)],$$

its dispersionless limit is

$$u_{22} = [u - \ln(e^u+1)]_{11} + [\ln(e^u+1)]_{33}.$$

Lattice KP equation [16, 68, 67]:

$$(T_1 u - T_2 u)T_{12}u + (T_3 u - T_1 u)T_{13}u + (T_2 u - T_3 u)T_{23}u = 0.$$

In equivalent form,

$$(\triangle_1 u - \triangle_2 u)\triangle_{12}u + (\triangle_3 u - \triangle_1 u)\triangle_{13}u + (\triangle_2 u - \triangle_3 u)\triangle_{23}u = 0.$$

Its dispersionless limit is

$$(u_1 - u_2)u_{12} + (u_3 - u_1)u_{13} + (u_2 - u_3)u_{23} = 0.$$

Schwarzian KP equation [11, 12, 19, 53, 68]:

$$(T_2\triangle_1 u)(T_3\triangle_2 u)(T_1\triangle_3 u) = (T_2\triangle_3 u)(T_3\triangle_1 u)(T_1\triangle_2 u).$$

Taking log of both sides we obtain

$$\triangle_1\left(\ln\frac{\triangle_3 u}{\triangle_2 u}\right)+\triangle_2\left(\ln\frac{\triangle_1 u}{\triangle_3 u}\right)+\triangle_3\left(\ln\frac{\triangle_2 u}{\triangle_1 u}\right)=0.$$

Its dispersionless limit is

$$u_3(u_2-u_1)u_{12}+u_2(u_1-u_3)u_{13}+u_1(u_3-u_2)u_{23}=0.$$

Lattice spin equation [69]:

$$\left(\frac{T_{12}\tau}{T_2\tau}-1\right)\left(\frac{T_{13}\tau}{T_1\tau}-1\right)\left(\frac{T_{23}\tau}{T_3\tau}-1\right)=\left(\frac{T_{12}\tau}{T_1\tau}-1\right)\left(\frac{T_{13}\tau}{T_3\tau}-1\right)\left(\frac{T_{23}\tau}{T_2\tau}-1\right).$$

On multiplication by $T_1\tau T_2\tau T_3\tau$ it reduces to the Schwarzian KP equation. An alternative representation can be obtained by taking log of both sides and setting $\tau=e^{u/\epsilon}$. This gives

$$\triangle_1\ln\frac{e^{\triangle_3 u}-1}{e^{\triangle_2 u}-1}+\triangle_2\ln\frac{e^{\triangle_1 u}-1}{e^{\triangle_3 u}-1}+\triangle_3\ln\frac{e^{\triangle_2 u}-1}{e^{\triangle_1 u}-1}=0.$$

Its dispersionless limit is

$$\frac{e^{u_2}-e^{u_1}}{(e^{u_1}-1)(e^{u_2}-1)}u_{12}+\frac{e^{u_1}-e^{u_3}}{(e^{u_1}-1)(e^{u_3}-1)}u_{13}+\frac{e^{u_3}-e^{u_2}}{(e^{u_2}-1)(e^{u_3}-1)}u_{23}=0.$$

Sine-Gordon equation [53]:

$$(T_2\sin\triangle_1 u)(T_3\sin\triangle_2 u)(T_1\sin\triangle_3 u)=(T_2\sin\triangle_3 u)(T_3\sin\triangle_1 u)(T_1\sin\triangle_2 u).$$

Taking log of both sides we obtain

$$\triangle_1\left(\ln\frac{\sin\triangle_3 u}{\sin\triangle_2 u}\right)+\triangle_2\left(\ln\frac{\sin\triangle_1 u}{\sin\triangle_3 u}\right)+\triangle_3\left(\ln\frac{\sin\triangle_2 u}{\sin\triangle_1 u}\right)=0.$$

Its dispersionless limit is

$$(\cot u_2-\cot u_1)u_{12}+(\cot u_1-\cot u_3)u_{13}+(\cot u_3-\cot u_2)u_{23}=0.$$

This example is nothing but trigonometric version of the lattice spin equation.

Lattice mKP equation [69]:

$$\frac{T_{13}\tau-T_{12}\tau}{T_1\tau}+\frac{T_{12}\tau-T_{23}\tau}{T_2\tau}+\frac{T_{23}\tau-T_{13}\tau}{T_3\tau}=0.$$

Setting $\tau = e^{u/\epsilon}$ we obtain

$$\triangle_1(e^{\triangle_3 u} - e^{\triangle_2 u}) + \triangle_2(e^{\triangle_1 u} - e^{\triangle_3 u}) + \triangle_3(e^{\triangle_2 u} - e^{\triangle_1 u}) = 0,$$

its dispersionless limit is

$$(e^{u_1} - e^{u_2})u_{12} + (e^{u_3} - e^{u_1})u_{13} + (e^{u_2} - e^{u_3})u_{23} = 0.$$

Toda equation [55, 89]:

$$\alpha T_1\tau T_2\tau + \beta\tau T_{12}\tau + \gamma T_{1\bar{3}}\tau T_{23}\tau = 0.$$

Dividing by $T_1\tau T_2\tau$ and setting $\tau = e^{u/\epsilon^2}$ we get

$$\alpha + \beta e^{\triangle_{12} u} + \gamma e^{\triangle_{23} u - \triangle_{1\bar{3}} u + \triangle_{3\bar{3}} u} = 0,$$

its dispersionless limit is

$$\alpha + \beta e^{u_{12}} + \gamma e^{u_{23} - u_{13} + u_{33}} = 0.$$

Lattice Toda equation [69]:

$$(T_1 - T_3)\frac{T_2\tau}{\tau} = (T_2 - T_3)\frac{T_1\tau}{\tau}.$$

Setting $\tau = e^{u/\epsilon}$ we get

$$\triangle_1(e^{\triangle_2 u}) - \triangle_2(e^{\triangle_1 u}) + \triangle_3(e^{\triangle_1 u} - e^{\triangle_2 u}) = 0,$$

its dispersionless limit is

$$(e^{u_2} - e^{u_1})u_{12} + e^{u_1}u_{13} - e^{u_2}u_{23} = 0.$$

Lattice mToda equation [69]:

$$\left(\frac{T_{13}\tau}{T_1\tau} - 1\right)\left(\frac{T_{23}\tau}{T_3\tau} - 1\right) = \left(\frac{T_{12}\tau}{T_1\tau} - 1\right)\left(\frac{T_{23}\tau}{T_2\tau} - 1\right).$$

Taking log of both sides and setting $\tau = e^{u/\epsilon}$ we get

$$\triangle_1 \ln\frac{e^{\triangle_3 u} - 1}{e^{\triangle_2 u} - 1} - \triangle_2 \ln(e^{\triangle_3 u} - 1) + \triangle_3 \ln(e^{\triangle_2 u} - 1) = 0.$$

Its dispersionless limit is

$$-\frac{e^{u_2}}{e^{u_2}-1}u_{12}+\frac{e^{u_3}}{e^{u_3}-1}u_{13}+\frac{e^{u_3}-e^{u_2}}{(e^{u_2}-1)(e^{u_3}-1)}u_{23}=0.$$

Toda equation for rotation coefficients [17]:

$$(T_2-1)\frac{T_1\tau}{\tau}=T_1\frac{T_2\tau}{T_{\bar{3}}\tau}-\frac{T_{23}\tau}{\tau}.$$

This equation appeared in the theory of Laplace transformations of discrete quadrilateral nets. Setting $\tau=e^{u/\epsilon}$ we obtain

$$\triangle_2(e^{\triangle_1 u})=(\triangle_1-\triangle_3)e^{\triangle_2 u+\triangle_{\bar{3}}u}.$$

Its dispersionless limit is

$$e^{u_1}u_{12}=e^{u_2+u_3}(u_{12}+u_{13}-u_{23}-u_{33}).$$

One more version of the Toda equation [11]:

$$T_{\bar{1}3}\tau+\alpha T_2\tau=T_{\bar{1}}\tau T_3\tau\left(\frac{1}{\tau}+\alpha\frac{1}{T_{\bar{1}\bar{2}3}\tau}\right).$$

Setting $\tau=e^{-u/\epsilon}$ we obtain

$$\triangle_3 e^{\triangle_{\bar{1}}u}=\alpha(\epsilon\triangle_{\bar{1}\bar{2}}-\triangle_{\bar{1}}-\triangle_{\bar{2}})e^{\triangle_3 u-\triangle_2 u}.$$

Its dispersionless limit is

$$e^{u_1}u_{13}+\alpha e^{u_3-u_2}(u_{13}+u_{23}-u_{12}-u_{22})=0.$$

Schwarzian Toda equation [11, 12]:

$$(T_1\triangle_3 u)(T_2(\triangle_1+\triangle_{\bar{2}})u)(T_3\triangle_{\bar{2}}u)=(\triangle_3 u)(T_3(\triangle_1+\triangle_{\bar{2}})u)(T_1\triangle_2 u).$$

Taking log of both sides we obtain

$$\triangle_1\ln\triangle_3 u+(\triangle_2-\triangle_3)\ln(\triangle_1+\triangle_{\bar{2}})u+\triangle_3\ln\triangle_{\bar{2}}u-\triangle_1\ln\triangle_2 u+\frac{1}{\epsilon}\ln\left(1-\epsilon\frac{\triangle_{2\bar{2}}u}{\triangle_2 u}\right)=0.$$

Its dispersionless limit is

$$\frac{u_2}{u_1u_3}(u_1+u_2-u_3)u_{13}-u_{12}-u_{22}+u_{23}=0.$$

BKP equation in Miwa form [65, 71]:

$$\alpha T_1\tau T_{23}\tau+\beta T_2\tau T_{13}\tau+\gamma T_3\tau T_{12}\tau+\delta\tau T_{123}\tau=0.$$

This equation can be interpreted as the permutability theorem of Moutard transformations [71]. Dividing by $T_1\tau T_2\tau T_3\tau/\tau$ and setting $\tau=e^{u/\epsilon^2}$ we get

$$\alpha e^{\triangle_{23}u}+\beta e^{\triangle_{13}u}+\gamma e^{\triangle_{12}u}+\delta\ e^{\epsilon\triangle_{123}u+\triangle_{23}u+\triangle_{13}u+\triangle_{12}u}=0.$$

Its dispersionless limit is

$$\alpha e^{u_{23}}+\beta e^{u_{13}}+\gamma e^{u_{12}}+\delta e^{u_{23}+u_{13}+u_{12}}=0.$$

BKP equation in Hirota form [65]:

$$\alpha T_1\tau T_{\bar{1}}\tau+\beta T_2\tau T_{\bar{2}}\tau+\gamma T_3\tau T_{\bar{3}}\tau+\delta T_{123}\tau T_{\bar{1}\bar{2}\bar{3}}\tau=0.$$

Dividing by τ^2 and setting $\tau=e^{u/\epsilon^2}$ we get

$$\alpha e^{\triangle_{1\bar{1}}u}+\beta e^{\triangle_{2\bar{2}}u}+\gamma e^{\triangle_{3\bar{3}}u}+\delta e^{\epsilon(\triangle_{123}u-\triangle_{\bar{1}\bar{2}\bar{3}}u)+S}=0,$$

where

$$S=(\triangle_{1\bar{1}}u+\triangle_{2\bar{2}}u+\triangle_{3\bar{3}}u)+(\triangle_{12}u+\triangle_{\bar{1}\bar{2}}u)+(\triangle_{13}u+\triangle_{\bar{1}\bar{3}}u)+(\triangle_{23}u+\triangle_{\bar{2}\bar{3}}u).$$

Its dispersionless limit is

$$\alpha e^{u_{11}}+\beta e^{u_{22}}+\gamma e^{u_{33}}+\delta e^{u_{11}+u_{22}+u_{33}+2u_{12}+2u_{13}+2u_{23}}=0.$$

Schwarzian BKP equation [54, 72, 86]:

$$\frac{(T_1u-T_2u)(T_{123}u-T_3u)}{(T_2u-T_3u)(T_{123}u-T_1u)}=\frac{(T_{13}u-T_{23}u)(T_{12}u-u)}{(T_{12}u-T_{13}u)(T_{23}u-u)}.$$

Taking log of both sides we get

$$\triangle_3 \ln \frac{\epsilon\triangle_{12}u + \triangle_1 u + \triangle_2 u}{\triangle_1 u - \triangle_2 u} = \triangle_1 \ln \frac{\epsilon\triangle_{23}u + \triangle_2 u + \triangle_3 u}{\triangle_3 u - \triangle_2 u}.$$

Its dispersionless limit is [13]:

$$u_3(u_2^2 - u_1^2)u_{12} + u_2(u_1^2 - u_3^2)u_{13} + u_1(u_3^2 - u_2^2)u_{23} = 0.$$

It was shown in [86] that the Schwarzian BKP equation is the only nonlinearisable affine linear discrete equation consistent around a 4D cube.

BKP version of the sine-Gordon equation [54, 72]:

$$\frac{\sin(T_1u - T_2u)\sin(T_{123}u - T_3u)}{\sin(T_2u - T_2u)\sin(T_{123}u - T_1u)} = \frac{\sin(T_{13}u - T_{23}u)\sin(T_{12}u - u)}{\sin(T_{12}u - T_{13}u)\sin(T_{23}u - u)}.$$

Taking log of both sides we get

$$\triangle_3 \ln \frac{\sin(\epsilon\triangle_{12}u + \triangle_1 u + \triangle_2 u)}{\sin(\triangle_1 u - \triangle_2 u)} = \triangle_1 \ln \frac{\sin(\epsilon\triangle_{23}u + \triangle_2 u + \triangle_3 u)}{\sin(\triangle_3 u - \triangle_2 u)}.$$

Its dispersionless limit is

$$\sin 2u_3(\sin^2 u_2 - \sin^2 u_1)u_{12} + \sin 2u_2(\sin^2 u_1 - \sin^2 u_3)u_{13} +$$
$$\sin 2u_1(\sin^2 u_3 - \sin^2 u_2)u_{23} = 0.$$

CKP equation [48, 77]:

$$(\tau T_{123}\tau - T_1\tau T_{23}\tau - T_2\tau T_{13}\tau - T_3\tau T_{12}\tau)^2 =$$
$$4(T_1\tau T_2\tau T_{13}\tau T_{23}\tau + T_2\tau T_3\tau T_{12}\tau T_{13}\tau + T_1\tau T_3\tau T_{12}\tau T_{23}\tau - T_1\tau T_2\tau T_3\tau T_{123}\tau - \tau T_{12}\tau T_{13}\tau T_{23}\tau).$$

Multiplying by $[\tau/(T_1\tau T_2\tau T_3\tau)]^2$ and setting $\tau = e^{u/\epsilon^2}$ we obtain

$$(e^{\epsilon\triangle_{123}u + \triangle_{23}u + \triangle_{13}u + \triangle_{12}u} - e^{\triangle_{23}u} - e^{\triangle_{13}u} - e^{\triangle_{12}u})^2 =$$
$$4(e^{\triangle_{13}u + \triangle_{23}u} + e^{\triangle_{12}u + \triangle_{13}u} + e^{\triangle_{12}u + \triangle_{23}u} - e^{\epsilon\triangle_{123}u + \triangle_{23}u + \triangle_{13}u + \triangle_{12}u} - e^{\triangle_{23}u + \triangle_{13}u + \triangle_{12}u}).$$

Its dispersionless limit is

$$(e^{u_{23}+u_{13}+u_{12}} - e^{u_{23}} - e^{u_{13}} - e^{u_{12}})^2 = 4(e^{u_{13}+u_{23}} + e^{u_{12}+u_{13}} + e^{u_{12}+u_{23}} - 2e^{u_{23}+u_{13}+u_{12}}).$$

It is remarkable that this dispersionless equation decouples into the product of four dispersionless BKP-type equations: setting $u = 2v$ we obtain

$$(e^{v_{23}+v_{13}+v_{12}} + e^{v_{23}} + e^{v_{13}} + e^{v_{12}})(e^{v_{23}+v_{13}+v_{12}} - e^{v_{23}} - e^{v_{13}} + e^{v_{12}})\times$$

$$(e^{v_{23}+v_{13}+v_{12}} - e^{v_{23}} + e^{v_{13}} - e^{v_{12}})(e^{v_{23}+v_{13}+v_{12}} + e^{v_{23}} - e^{v_{13}} - e^{v_{12}}) = 0.$$

One can show that hydrodynamic reductions of each BKP-branch of the dispersionless equation are inherited by the full CKP equation. Multidimensional consistency of the CKP equation, interpreted as the Cayley hyperdeterminant, was established in [87, 18]. An alternative form of the CKP equation was proposed earlier in [48].

5.2 Method of dispersive deformations

As already mentioned, this method applies to dispersive equations possessing a nondegenerate dispersionless limit, and is based on the requirement that all hydrodynamic reductions of the dispersionless limit are 'inherited' by the full difference equation, at least to some finite order in the deformation parameter ϵ [31, 32, 34, 46]. Our experience suggests that in most cases it is sufficient to perform calculations up to the order ϵ^2, the necessary conditions for integrability obtained at this stage usually prove to be sufficient, and imply the existence of conventional Lax pairs, etc. Let us illustrate this approach by classifying integrable discrete wave-type equations of the form

$$\triangle_{t\bar{t}}\ u - \triangle_{x\bar{x}}\ f(u) - \triangle_{y\bar{y}}\ g(u) = 0, \tag{5.1}$$

where f and g are functions to be determined. Using expansions of the form

$$\triangle_{t\bar{t}} = \frac{(e^{\epsilon\partial_t} - 1)(1 - e^{-\epsilon\partial_t})}{\epsilon^2} = \partial_t^2 + \frac{\epsilon^2}{12}\partial_t^4 + \dots,$$

we can represent (5.1) as an infinite series in ϵ,

$$u_{tt} - f(u)_{xx} - g(u)_{yy} + \frac{\epsilon^2}{12}[u_{tttt} - f(u)_{xxxx} - g(u)_{yyyy}] + \dots = 0.$$

The corresponding dispersionless limit $\epsilon \to 0$ results in the quasilinear wave-type equation

$$u_{tt} - f(u)_{xx} - g(u)_{yy} = 0. \tag{5.2}$$

This equation possesses exact solutions of the form $u = R(x, y, t)$ where R solves a pair of Hopf-type equations,

$$R_t = \lambda(R)R_x, \quad R_y = \mu(R)R_x,$$

with the characteristic speeds λ, μ satisfying the dispersion relation $\lambda^2 = f' + g'\mu^2$. Solutions of this type are known as one-phase hydrodynamic reductions, or planar simple waves. Let us require that all such reductions can be deformed into formal solutions of the original equation (5.1) as follows:

$$\begin{aligned} R_y &= \mu(R)R_x + \epsilon(\dots) + \epsilon^2(\dots) + \dots, \\ R_t &= \lambda(R)R_x + \epsilon(\dots) + \epsilon^2(\dots) + \dots, \end{aligned} \tag{5.3}$$

here dots at ϵ^k denote terms which are polynomial in the x-derivatives of R of the order $k+1$. The relation $u = R(x, y, t)$ remains undeformed, this can always be assumed modulo Miura-type transformations of the form $R \to R + \epsilon r_1 + \epsilon^2 r_2 + \dots$. We emphasise that such deformations are required to exist for *any* function $\mu(R)$. Direct calculation demonstrates that all terms of the order ϵ vanish identically, while at the order ϵ^2 we get the following constraints for f and g:

$$f'' + g'' = 0, \quad g''(f' - 1) - g'f'' = 0, \quad f''^2(1 + 2f') - f'(f' + 1)f''' = 0.$$

Without any loss of generality one can set $f(u) = u - \ln(e^u + 1)$, $g(u) = \ln(e^u + 1)$, resulting in the difference equation

$$\triangle_{t\bar{t}}\, u - \triangle_{x\bar{x}}\, [u - \ln(e^u + 1)] - \triangle_{y\bar{y}}\, [\ln(e^u + 1)] = 0, \tag{5.4}$$

which is yet another equivalent form of the Hirota equation, known as the 'gauge-invariant form' [89], or the 'Y-system', see section 5.1 (we refer to [56] for a review of its applications). Its dispersionless limit,

$$u_{tt} - [u - \ln(e^u + 1)]_{xx} - [\ln(e^u + 1)]_{yy} = 0, \tag{5.5}$$

appeared recently in the classification of integrable equations possessing the 'central quadric ansatz' [26]. In this case the expansions (5.3) take the explicit form

$$R_y = \mu(R)\ R_x + \epsilon^2(a_1 R_{xxx} + a_2 R_{xx} R_x + a_3 R_x^3) + O(\epsilon^4),$$

$$R_t = \lambda(R)\ R_x + \epsilon^2(b_1 R_{xxx} + b_2 R_{xx} R_x + b_3 R_x^3) + O(\epsilon^4),$$

where

$$\begin{aligned} a_1 =& \frac{1}{12}\left(\mu^2 - 1\right)\mu', \\ b_1 =& \frac{(\mu^2-1)\, e^R \left(\mu^2 + 2\mu\mu' e^R + 2\mu\mu' - 1\right)}{24\left(e^R+1\right)^2 \lambda}, \end{aligned}$$

etc. The remaining coefficients a_i, b_i have a far more complicated structure, however, all of them are rational expressions in μ and its derivatives. Note that higher powers of λ can be eliminated via the dispersion relation $\lambda^2 = \frac{1}{e^R+1} + \frac{e^R}{e^R+1}\mu^2$.

5.3 Nondegeneracy conditions

We have already mentioned that the method of dispersive deformations applies to 3D equations with a nondegenerate dispersionless limit. In general, this means that, see [14, 33],

- the principal symbol of the dispersionless equation defines an *irreducible* algebraic curve, and
- the dispersionless equation is *not linearly degenerate*.

To be more specific, let us restrict to quasilinear PDEs of the form

$$\sum_{i,j=1}^{3} f_{ij}(u_k) u_{ij} = 0, \tag{5.6}$$

that arise as dispersionless limits for most of the examples discussed in this chapter; here the coefficients f_{ij} depend on first-order derivatives u_k only. In this case the first nondegeneracy condition is equivalent to $\det f_{ij} \neq 0$ (it is required for the applicability of the method of hydrodynamic reductions). This is because, upon setting $u_1 = a, u_2 = b, u_3 = c$, from their compatibility conditions we have

$$a_2 = b_1, \qquad a_3 = c_1, \qquad b_3 = c_2, \tag{5.7}$$

and equation (5.6) takes the form

$$f_{11}a_1 + f_{22}b_2 + f_{33}c_3 + f_{12}a_2 + f_{13}a_3 + f_{23}b_3 = 0.$$

Recall that for the method of hydrodynamic reductions, we seek solutions $a = a(R), b = b(R), c = c(R)$, where R satisfies the pair $R_2 = \mu(R)R_1, R_3 = \lambda(R)R_1$. Substituting these solutions in the compatibility conditions (5.7), we obtain

$$b'(R) = a'(R)\mu(R), \qquad c'(R) = a'(R)\lambda(R),$$

and from the equation one obtains the dispertion relation

$$D(\lambda, \mu) = f_{11} + f_{22}\mu^2 + f_{33}\lambda^2 + f_{12}\mu + f_{13}\lambda + f_{23}\lambda\ \mu = 0,$$

which is irreducible, when the determinant of the coefficient matrix is nonzero.
To define the second nondegeneracy condition for this class of equations, recall the concept of linearly degenerate equations. These are characterised by the identity

$$\partial_{(k}f_{ij)} = \varphi_{(k}f_{ij)},$$

where $\partial_k = \partial_{u_k}$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is a covector, and brackets denote complete symmetrisation in $i, j, k \in \{1, 2, 3\}$. Explicitly, this gives ten relations:

$$\partial_1 f_{11} = \varphi_1 f_{11}, \quad \partial_2 f_{22} = \varphi_2 f_{22}, \quad \partial_3 f_{33} = \varphi_3 f_{33},$$

$$\partial_2 f_{11} + 2\partial_1 f_{12} = \varphi_2 f_{11} + 2\varphi_1 f_{12}, \quad \partial_1 f_{22} + 2\partial_2 f_{12} = \varphi_1 f_{22} + 2\varphi_2 f_{12},$$

$$\partial_3 f_{11} + 2\partial_1 f_{13} = \varphi_3 f_{11} + 2\varphi_1 f_{13}, \quad \partial_1 f_{33} + 2\partial_3 f_{13} = \varphi_1 f_{33} + 2\varphi_3 f_{13},$$

$$\partial_2 f_{33} + 2\partial_3 f_{23} = \varphi_2 f_{33} + 2\varphi_3 f_{23}, \quad \partial_3 f_{22} + 2\partial_2 f_{23} = \varphi_3 f_{22} + 2\varphi_2 f_{23},$$

$$\partial_1 f_{23} + \partial_2 f_{13} + \partial_3 f_{12} = \varphi_1 f_{23} + \varphi_2 f_{13} + \varphi_3 f_{12}.$$

On elimination of φ's, these conditions give rise to seven first-order differential constraints for f_{ij} alone. The conditions of linear degeneracy appear as denominators in the computation of dispersive corrections (to be precise, the denominator is a polynomial whose coefficients are conditions of linear degeneracy; it vanishes identically if and only if the equation is linearly degenerate).

In order to obtain these conditions, one proceeds as follows: Taking travelling wave reductions, of the form $u(x_1, x_2, x_3) = u(\xi, \eta) + \alpha x_1 + \beta x_2 + \gamma x_3$ where $\xi = x_1 + \lambda x_3,\ \eta = x_2 + \mu x_3$, and $\alpha, \beta, \gamma, \lambda, \mu$ are arbitrary constants, we can reduce equation (5.6) to a 2D equation,

$$\sum_{i,j=1}^{2} f_{ij}(u_1, u_2) u_{ij} = 0.$$

Setting $a = u_1,\ b = u_2$, we can rewrite this equation in a hydrodynamic form and, hence, following the same procedure as the one described in chapter 3. The linearly degeneracy relations obtained, are required to be satisfied for *any* travelling wave reduction. Also, both nondegeneracy conditions are satisfied (possibly, after a change of variables) for all known examples of integrable PDEs in 3D.

Remark. Linearly degenerate PDEs are quite exceptional from the point of view of solvability of the Cauchy problem: for these PDEs the gradient catastrophe, typical for genuinely nonlinear equations, does not occur, which implies global existence results for an open set of initial data. The reason for this is that linear degeneracy is closely related to the null conditions of Klainerman known in the theory of second-order quasilinear PDEs.

5.4 Discrete conservation laws

In this section we classify integrable equations of the form

$$\triangle_1 f + \triangle_2 g + \triangle_3 h = 0, \tag{5.8}$$

where f, g, h are functions of $\triangle_1 u, \triangle_2 u, \triangle_3 u$ only. The dispersionless limit,

$$\sum_{i,j=1}^{3} f_{ij}(u_k)\ u_{ij} = 0,$$

is assumed to be nondegenerate, as described in the previous section, 5.3. The classification is performed modulo transformations of the form $u \to \alpha u + \alpha_i x^i$, as well as relabelling of the independent variables x^i.

Theorem 5.1 *Integrable discrete conservation laws are naturally grouped into seven three-parameter families,*

$$aI + \beta J + \gamma K = 0,$$

where a, β, γ are arbitrary constants, while I, J, K denote left hand sides of three linearly independent discrete conservation laws of the seven octahedron-type equations listed below. In each case we give explicit forms of I, J, K, as well as the underlying octahedron equation.

Case 1.

Conservation Laws	*Octahedron equation*
$I = \triangle_1 e^{\triangle_2 u} + \triangle_3 \left(e^{\triangle_2 u - \triangle_1 u} - e^{\triangle_2 u}\right) = 0$ $J = \triangle_1 e^{-\triangle_3 u} + \triangle_2 \left(e^{\triangle_1 u - \triangle_3 u} - e^{-\triangle_3 u}\right) = 0$ $K = \triangle_2 \left(\triangle_3 u - \ln(1 - e^{\triangle_1 u})\right) +$ $+\triangle_3 \left(\ln(1 - e^{\triangle_1 u}) - \triangle_1 u\right) = 0$	$\frac{T_2\tau - T_{12}\tau}{T_{23}\tau} = T_1\tau \left(\frac{1}{T_{13}\tau} - \frac{1}{T_3\tau}\right)$ *(setting $\tau = e^{u/\epsilon}$)*

Case 2.

Conservation Laws	*Octahedron equation*
$I = \triangle_2 \ln \triangle_1 u + \triangle_3 \ln \left(1 - \frac{\triangle_2 u}{\triangle_1 u}\right) = 0$ $J = \triangle_1 \ln \triangle_2 u + \triangle_3 \ln \left(\frac{\triangle_1 u}{\triangle_2 u} - 1\right) = 0$ $K = \triangle_1 \left(\frac{(\triangle_2 u)^2}{2} - \triangle_2 u \triangle_3 u\right) +$ $+\triangle_2 \left(\triangle_1 u \triangle_3 u - \frac{(\triangle_1 u)^2}{2}\right) = 0$	$T_{12}uT_{13}u + T_2uT_{23}u + T_1uT_3u$ $= T_{12}uT_{23}u + T_1uT_{13}u + T_2uT_3u$

Case 3. Generalised lattice Toda *(depending on a parameter α)*

Conservation Laws	*Octahedron equation*
subcase $\alpha \neq 0$ $I = \triangle_1(e^{\triangle_2 u - \triangle_3 u} + \alpha e^{-\triangle_3 u}) - \triangle_2(e^{\triangle_1 u - \triangle_3 u} + \alpha e^{-\triangle_3 u}) = 0$ $J = \triangle_2 \ln\left(e^{\triangle_1 u} + \alpha\right) + \triangle_3 \left(\ln \frac{e^{\triangle_1 u} - e^{\triangle_2 u}}{e^{\triangle_1 u} + \alpha} - \triangle_2 u\right) = 0$ $K = \triangle_1 \ln\left(e^{\triangle_2 u} + \alpha\right) + \triangle_3 \left(\ln \frac{e^{\triangle_1 u} - e^{\triangle_2 u}}{e^{\triangle_2 u} + \alpha} - \triangle_1 u\right) = 0$	$\frac{T_{23}\tau}{T_3\tau} + \frac{T_{12}\tau}{T_2\tau} + \alpha \frac{T_{12}\tau T_{23}\tau}{T_2\tau T_3\tau} =$ $\frac{T_{12}\tau}{T_1\tau} + \frac{T_{13}\tau}{T_3\tau} + \alpha \frac{T_{12}\tau T_{13}\tau}{T_1\tau T_3\tau}$ *(setting $\tau = e^{-u/\epsilon}$)*

subcase $\alpha = 0$	*lattice Toda equation*
$I = \triangle_1 e^{\triangle_2 u - \triangle_3 u} - \triangle_2 e^{\triangle_1 u - \triangle_3 u} = 0$	$(T_1 - T_3)\frac{T_2\tau}{\tau} = (T_2 - T_3)\frac{T_1\tau}{\tau}$
$J = \triangle_2\triangle_1 u + \triangle_3(\ln(1 - e^{\triangle_2 u - \triangle_1 u}) - \triangle_2 u) = 0$	*(setting* $\tau = e^{-u/\epsilon}$*)*
$K = \triangle_1 e^{-\triangle_2 u} - \triangle_2 e^{-\triangle_1 u} + \triangle_3(e^{-\triangle_1 u} - e^{-\triangle_2 u}) = 0$	

Case 4. Lattice KP

Conservation Laws	*Octahedron equation*
$I = \triangle_1((\triangle_3 u)^2 - (\triangle_2 u)^2) + \triangle_2((\triangle_1 u)^2$	$(T_1 u - T_2 u)T_{12}u + (T_3 u - T_1 u)T_{13}u$
$- (\triangle_3 u)^2) + \triangle_3((\triangle_2 u)^2 - (\triangle_1 u)^2) = 0$	$+(T_2 u - T_3 u)T_{23}u = 0$
$J = \triangle_1 \ln(\triangle_3 u - \triangle_2 u) - \triangle_3 \ln(\triangle_2 u - \triangle_1 u) = 0$	
$K = \triangle_2 \ln(\triangle_1 u - \triangle_3 u) - \triangle_3 \ln(\triangle_2 u - \triangle_1 u) = 0$	

Case 5. Lattice mKP

Conservation Laws	*Octahedron equation*
$I = \triangle_1(e^{\triangle_2 u} - e^{\triangle_3 u}) + \triangle_2(e^{\triangle_3 u} - e^{\triangle_1 u}) + \triangle_3(e^{\triangle_1 u} - e^{\triangle_2 u}) = 0$	$\frac{T_{13}\tau - T_{12}\tau}{T_1\tau} + \frac{T_{12}\tau - T_{23}\tau}{T_2\tau}$
$J = \triangle_1 \ln\left(e^{\triangle_3 u} - e^{\triangle_2 u}\right) - \triangle_2 \ln\left(e^{\triangle_3 u} - e^{\triangle_1 u}\right) = 0$	$+\frac{T_{23}\tau - T_{13}\tau}{T_3\tau} = 0$
$K = \triangle_2 \ln\left(e^{\triangle_3 u} - e^{\triangle_1 u}\right) - \triangle_3 \ln\left(e^{\triangle_2 u} - e^{\triangle_1 u}\right) = 0$	*(setting* $\tau = e^{u/\epsilon}$*)*

Case 6. Schwarzian KP

Conservation Laws	*Octahedron equation*
$I = \triangle_2 \ln\left(1 - \frac{\triangle_3 u}{\triangle_1 u}\right) - \triangle_3 \ln\left(\frac{\triangle_2 u}{\triangle_1 u} - 1\right) = 0$	$(T_2\triangle_1 u)(T_3\triangle_2 u)(T_1\triangle_3 u)$
$J = \triangle_3 \ln\left(1 - \frac{\triangle_1 u}{\triangle_2 u}\right) - \triangle_1 \ln\left(\frac{\triangle_3 u}{\triangle_2 u} - 1\right) = 0$	$= (T_2\triangle_3 u)(T_3\triangle_1 u)(T_1\triangle_2 u)$
$K = \triangle_1 \ln\left(1 - \frac{\triangle_2 u}{\triangle_3 u}\right) - \triangle_2 \ln\left(\frac{\triangle_1 u}{\triangle_3 u} - 1\right) = 0$	

Case 7. Lattice spin

Conservation Laws	*Octahedron equation*
Hyperbolic version	*lattice-spin equation*
$I = \triangle_1 \ln \frac{\sinh \triangle_3 u}{\sinh \triangle_2 u} + \triangle_2 \ln \frac{\sinh \triangle_1 u}{\sinh \triangle_3 u} + \triangle_3 \ln \frac{\sinh \triangle_2 u}{\sinh \triangle_1 u} = 0$	$\left(\frac{T_{12}\tau}{T_2\tau} - 1\right)\left(\frac{T_{13}\tau}{T_1\tau} - 1\right)\left(\frac{T_{23}\tau}{T_3\tau} - 1\right)$
$J = \triangle_1 \ln \frac{\sinh(\triangle_2 u - \triangle_3 u)}{\sinh \triangle_2 u} - \triangle_3 \ln \frac{\sinh(\triangle_1 u - \triangle_2 u)}{\sinh \triangle_2 u} = 0$	$= \left(\frac{T_{12}\tau}{T_1\tau} - 1\right)\left(\frac{T_{13}\tau}{T_3\tau} - 1\right)\left(\frac{T_{23}\tau}{T_2\tau} - 1\right)$
$K = \triangle_2 \ln \frac{\sinh(\triangle_3 u - \triangle_1 u)}{\sinh \triangle_1 u} - \triangle_3 \ln \frac{\sinh(\triangle_1 u - \triangle_2 u)}{\sinh \triangle_1 u} = 0$	*(setting* $\tau = e^{2u/\epsilon}$*)*

Trigonometric version	*Sine-Gordon equation*
$I=\triangle_1\ln\frac{\sin\triangle_3 u}{\sin\triangle_2 u}+\triangle_2\ln\frac{\sin\triangle_1 u}{\sin\triangle_3 u}+\triangle_3\ln\frac{\sin\triangle_2 u}{\sin\triangle_1 u}=0$	$(T_2\sin\triangle_1 u)(T_3\sin\triangle_2 u)(T_1\sin\triangle_3 u)$
$J=\triangle_1\ln\frac{\sin(\triangle_2 u-\triangle_3 u)}{\sin\triangle_2 u}-\triangle_3\ln\frac{\sin(\triangle_1 u-\triangle_2 u)}{\sin\triangle_2 u}=0$	$=(T_2\sin\triangle_3 u)(T_3\sin\triangle_1 u)(T_1\sin\triangle_2 u)$
$K=\triangle_2\ln\frac{\sin(\triangle_3 u-\triangle_1 u)}{\sin\triangle_1 u}-\triangle_3\ln\frac{\sin(\triangle_1 u-\triangle_2 u)}{\sin\triangle_1 u}=0$	

Remark. Although cases 1, 2 do not bear any special name, the corresponding equations can be obtained as degenerations from cases 3-7. Furthermore, they are contained in the classification of [5].

Proof of Theorem 5.1:

The dispersionless limit of (5.8) is a quasilinear conservation law

$$\partial_1 f+\partial_2 g+\partial_3 h=0, \tag{5.9}$$

where f, g, h are functions of the variables $a=u_1,\ b=u_2,\ c=u_3$. Requiring that all one-phase reductions of the dispersionless equation (5.9) are inherited by the discrete equation (5.8) we obtain a set of differential constraints for f, g, h, which are the necessary conditions for integrability. Thus, at the order ϵ we get

$$f_a=g_b=h_c=0, \quad f_b+g_a+f_c+h_a+g_c+h_b=0. \tag{5.10}$$

The first set of these relations implies that the dispersionless limit is equivalent to the second order PDE

$$Fu_{12}+Gu_{13}+Hu_{23}=0, \tag{5.11}$$

where $F=f_b+g_a,\ G=f_c+h_a,\ H=g_c+h_b$. Note that, by virtue of (5.10), the coefficients F, G, H satisfy the additional constraint $F+G+H=0$. It follows from [14] that, up to a non-zero factor, any integrable equation of this type is equivalent to

$$[p(u_1)-q(u_2)]u_{12}+[r(u_3)-p(u_1)]u_{13}+[q(u_2)-r(u_3)]u_{23}=0, \tag{5.12}$$

where the functions $p(a), q(b), r(c)$ satisfy the integrability conditions

$$\begin{aligned} p'' &= p'\left(\tfrac{p'-q'}{p-q} + \tfrac{p'-r'}{p-r} - \tfrac{q'-r'}{q-r}\right), \\ q'' &= q'\left(\tfrac{q'-p'}{q-p} + \tfrac{q'-r'}{q-r} - \tfrac{p'-r'}{p-r}\right), \\ r'' &= r'\left(\tfrac{r'-p'}{r-p} + \tfrac{r'-q'}{r-q} - \tfrac{p'-q'}{p-q}\right). \end{aligned} \tag{5.13}$$

Our further strategy can be summarised as follows:

Step 1. First, we solve equations (5.13). Modulo unessential translations and rescalings this leads to seven quasilinear integrable equations of the form (5.12), see the details below.

Step 2. Next, for all of the seven equations found at step 1, we calculate first order conservation laws. It was demonstrated in [14] that any integrable second order quasilinear PDE possesses exactly four conservation laws of the form (5.9) (the converse statement is not true).

Step 3. Taking linear combinations of the four conservation laws in each of the above seven cases, and replacing partial derivatives u_1, u_2, u_3 by discrete derivatives $\triangle_1 u, \triangle_2 u, \triangle_3 u$, we obtain discrete equations (5.8) which, at this stage, are the *candidates* for integrability.

Step 4. Applying the ϵ^2-integrability conditions (see Appendix C), we obtain constraints for the coefficients of linear combinations. It turns out that only linear combinations of three (out of four) conservation laws pass the integrability test. In what follows, we present conservation laws in such a way that the first three are the ones that pass the integrability test, while the fourth one doesn't. Each triplet of conservation laws corresponds to one and the same discrete integrable equation of octahedron type. In other words, there are overall seven discrete integrable equations of octahedron type, each of them possesses three conservation laws, and linear combinations thereof give all integrable examples of the form (5.8).

Let us proceed to the solution of the system (5.13). There are three essentially different cases to consider, depending on how many functions among p, q, r are constant (the case when all of them are constant corresponds to linear equations). Some of these cases have additional subcases. These correspond to the seven cases of Theorem 5.1, in the same order as they appear below (note that the labelling below is different, dictated by the logic of the classification procedure).

Case 1: q and r are distinct constants. Without any loss of generality one can set $q = 1,\ r = -1$. In this case the equations for q and r will be satisfied identically, while the equation for p takes the form $p'' = 2pp'^2/(p^2-1)$. Modulo unessential scaling parameters this gives $p = (1+e^{u_1})/(1-e^{u_1})$, resulting in the PDE

$$e^{u_1}u_{12} - u_{13} + (1-e^{u_1})u_{23} = 0.$$

This equation possesses four conservation laws:

$$\partial_1 e^{u_2} + \partial_3\left(e^{u_2-u_1} - e^{u_2}\right) = 0,$$

$$\partial_1 e^{-u_3} + \partial_2\left(e^{u_1-u_3} - e^{-u_3}\right) = 0,$$

$$\partial_2\left(u_3 - \ln(1-e^{u_1})\right) + \partial_3\left(\ln(1-e^{u_1}) - u_1\right) = 0,$$

$$\partial_1\left(\frac{u_2u_3}{2}\right) - \partial_2\left(\frac{u_1u_3}{2} - u_1\ln(1-e^{u_1}) - Li_2(e^{u_1})\right) + \\ \partial_3\left(\frac{u_1^2}{2} - \frac{u_1u_2}{2} - u_1\ln(1-e^{u_1}) - Li_2(e^{u_1})\right) = 0,$$

where Li_2 is the dilogarithm function, $Li_2(z) = -\int \frac{\ln(1-z)}{z}\,dz$. Applying steps 3 and 4, one can show that discrete versions of the first three conservation laws correspond to the discrete equation

$$e^{(T_1u-T_{13}u)/\epsilon} + e^{(T_{12}u-T_{23}u)/\epsilon} = e^{(T_1u-T_3u)/\epsilon} + e^{(T_2u-T_{23}u)/\epsilon}.$$

Setting $\tau = e^{u/\epsilon}$ it can be rewritten as

$$\frac{T_2\tau - T_{12}\tau}{T_{23}\tau} = T_1\tau\left(\frac{1}{T_{13}\tau} - \frac{1}{T_3\tau}\right).$$

Case 2: r is constant. Without any loss of generality one can set $r = 0$. In this case the above system of ODEs for p and q takes the form

$$\tfrac{p''}{p'} = \tfrac{p'-q'}{p-q} + \tfrac{p'}{p} - \tfrac{q'}{q}, \qquad \tfrac{q''}{q'} = \tfrac{p'-q'}{p-q} + \tfrac{q'}{q} - \tfrac{p'}{p}.$$

Subtraction of these equations and the separation of variables leads, modulo unessential rescalings, to the two different subcases.

subcase 2a: $p = 1/u_1,\ q = 1/u_2$. The corresponding PDE is

$$(u_2 - u_1)u_{12} - u_2 u_{13} + u_1 u_{23} = 0.$$

It possesses four conservation laws:

$$\partial_2 \ln u_1 + \partial_3 \ln \left(1 - \frac{u_2}{u_1}\right) = 0,$$

$$\partial_1 \ln u_2 + \partial_3 \ln \left(\frac{u_1}{u_2} - 1\right) = 0,$$

$$\partial_1 \left(u_2^2 - 2u_2u_3\right) + \partial_2 \left(2u_1u_3 - u_1^2\right) = 0,$$

$$\partial_1 \left(-\frac{2u_2^3}{9} + u_2^2u_3 - u_2u_3^2\right) + \partial_2 \left(\frac{2u_1^3}{9} - u_1^2u_3 + u_1u_3^2\right) + \partial_3 \left(\frac{u_1^2u_2 - u_1u_2^2}{3}\right) = 0.$$

Applying steps 3 and 4, one can show that discrete versions of the first three conservation laws correspond to the discrete equation

$$T_{12}uT_{13}u + T_2uT_{23}u + T_1uT_3u = T_{12}uT_{23}u + T_1uT_{13}u + T_2uT_3u.$$

subcase 2b: $p = 1/(e^{u_1} + \alpha),\ q = 1/(e^{u_2} + \alpha)$, $\alpha = const$. The corresponding PDE is

$$(e^{u_2} - e^{u_1})u_{12} - (e^{u_2} + \alpha)u_{13} + (e^{u_1} + \alpha)u_{23} = 0.$$

If $\alpha \neq 0$ it possesses the following four conservation laws:

$$\partial_1(e^{u_2-u_3} + \alpha e^{-u_3}) - \partial_2(e^{u_1-u_3} + \alpha e^{-u_3}) = 0,$$

$$\partial_2 \ln (e^{u_1} + \alpha) + \partial_3 \left(\ln \frac{e^{u_1} - e^{u_2}}{e^{u_1} + \alpha} - u_2\right) = 0,$$

$$\partial_1 \ln (e^{u_2} + \alpha) + \partial_3 \left(\ln \frac{e^{u_1} - e^{u_2}}{e^{u_2} + \alpha} - u_1\right) = 0,$$

$$\partial_1 \left(2u_2 \ln\left(\frac{e^{u_2}+\alpha}{\alpha}\right) + 2Li_2\left(-\frac{e^{u_2}}{\alpha}\right) - u_2u_3\right) + \partial_2\left(u_1u_3 - 2u_1\ln\left(\frac{e^{u_1}+\alpha}{\alpha}\right) - 2Li_2\left(-\frac{e^{u_1}}{\alpha}\right)\right)$$
$$+\partial_3\left(u_2^2 - u_1u_2 + 2(u_2-u_1)\ln\left(1-e^{u_1-u_2}\right) + 2u_1\ln\left(\frac{e^{u_1}+\alpha}{\alpha}\right) - 2u_2\ln\left(\frac{e^{u_2}+\alpha}{\alpha}\right)\right.$$
$$\left. + 2Li_2\left(-\frac{e^{u_1}}{\alpha}\right) - 2Li_2\left(-\frac{e^{u_2}}{\alpha}\right) - 2Li_2\left(e^{u_1-u_2}\right)\right) = 0,$$

while when $\alpha = 0$ the conservation laws take the form:

$$\partial_1 e^{u_2-u_3} - \partial_2 e^{u_1-u_3} = 0,$$

$$\partial_2 u_1 + \partial_3(\ln(1-e^{u_2-u_1}) - u_2) = 0,$$

$$\partial_1 e^{-u_2} - \partial_2 e^{-u_1} + \partial_3(e^{-u_1} - e^{-u_2}) = 0,$$

$$\partial_1(u_2^2 - u_2u_3) + \partial_2(u_1u_3 - u_1^2) + \partial_3\left(u_1^2 - u_1u_2 + 2(u_2-u_1)\ln\left(1-e^{u_1-u_2}\right) - 2Li_2\left(e^{u_1-u_2}\right)\right) = 0.$$

Applying steps 3 and 4 to the subcase $\alpha \neq 0$, one can show that discrete versions of the first three conservation laws correspond to the discrete equation

$$e^{(T_3u - T_{23}u)/\epsilon} + e^{(T_2u - T_{12}u)/\epsilon} + \alpha e^{(T_2u + T_3u - T_{12}u - T_{23}u)/\epsilon} =$$
$$e^{(T_3u - T_{13}u)/\epsilon} + e^{(T_1u - T_{12}u)/\epsilon} + \alpha e^{(T_1u + T_3u - T_{12}u - T_{13}u}.$$

Setting $\tau = e^{-u/\epsilon}$, this equation can be rewritten as

$$\frac{T_{23}\tau}{T_3\tau} + \frac{T_{12}\tau}{T_2\tau} + \alpha\frac{T_{12}\tau T_{23}\tau}{T_2\tau T_3\tau} = \frac{T_{12}\tau}{T_1\tau} + \frac{T_{13}\tau}{T_3\tau} + \alpha\frac{T_{12}\tau T_{13}\tau}{T_1\tau T_3\tau}.$$

The special case $\alpha = 0$ leads to the lattice Toda equation,

$$(T_1 - T_3)\frac{T_2\tau}{\tau} = (T_2 - T_3)\frac{T_1\tau}{\tau},$$

see section 5.1.

Case 3: none of p, q, r are constant. In this case we can separate the variables in (5.13) as follows. Dividing equations (5.13) by p', q', r' respectively, and adding the first two of them we obtain

$$p''/p' + q''/q' = 2(p' - q')/(p - q).$$

Multiplying both sides by $p - q$ and applying the operator $\partial_a\partial_b$ we obtain $(p''/p')' = 2\alpha p',\ (q''/q')' = 2\alpha q',\ \alpha = const$. Thus, $p''/p' = 2\alpha p + \beta_1,\ q''/q' = 2\alpha q + \beta_2$. Substituting these expressions back into the above relation we obtain that p' and q' must be (the same) quadratic polynomials in p and q, respectively. Ultimately,

$$p' = \alpha p^2 + \beta p + \gamma, \quad q' = \alpha q^2 + \beta q + \gamma, \quad r' = \alpha r^2 + \beta r + \gamma.$$

Modulo unessential translations and rescalings, this leads to the four subcases.

subcase 3a: $p = u_1,\ q = u_2,\ r = u_3$. The corresponding PDE is

$$(u_2 - u_1)u_{12} + (u_1 - u_3)u_{13} + (u_3 - u_2)u_{23} = 0.$$

It possesses four conservation laws:

$$\partial_1(u_3^2 - u_2^2) + \partial_2(u_1^2 - u_3^2) + \partial_3(u_2^2 - u_1^2) = 0,$$

$$\alpha_1\partial_1 \ln(u_3 - u_2) + \alpha_2\partial_2 \ln(u_1 - u_3) + \alpha_3\partial_3 \ln(u_2 - u_1) = 0,$$

$$\partial_1\left(\frac{u_3^3 - u_2^3}{3} + \frac{u_2u_3^2 - u_2^2u_3}{2}\right) + \partial_2\left(\frac{u_1^3 - u_3^3}{3} + \frac{u_3u_1^2 - u_3^2u_1}{2}\right) + \partial_3\left(\frac{u_2^3 - u_1^3}{3} + \frac{u_1u_2^2 - u_1^2u_2}{2}\right) = 0,$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants satisfying $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Applying steps 3 and 4, one can show that discrete versions of the first three conservation laws correspond to the discrete equation

$$(T_1u - T_2u)T_{12}u + (T_3u - T_1u)T_{13}u + (T_2u - T_3u)T_{23}u = 0,$$

which is known as the lattice KP equation (see section 5.1).

subcase 3b: $p = e^{u_1},\ q = e^{u_2},\ r = e^{u_3}$. The corresponding PDE is

$$(e^{u_1} - e^{u_2})u_{12} + (e^{u_3} - e^{u_1})u_{13} + (e^{u_2} - e^{u_3})u_{23} = 0.$$

It possesses four conservation laws:

$$\partial_1(e^{u_2} - e^{u_3}) + \partial_2(e^{u_3} - e^{u_1}) + \partial_3(e^{u_1} - e^{u_2}) = 0,$$

$$\partial_1 \ln\left(e^{u_3} - e^{u_2}\right) - \partial_2 \ln\left(e^{u_3} - e^{u_1}\right) = 0,$$

$$\partial_2 \ln\left(e^{u_3} - e^{u_1}\right) - \partial_3 \ln\left(e^{u_2} - e^{u_1}\right) = 0,$$

$$\begin{gathered}\partial_1\left(u_2u_3 - u_3^2 + 2\left(u_2 - u_3 - 1\right)\ln\left(1 - e^{u_2 - u_3}\right) + 2Li_2\left(e^{u_2 - u_3}\right)\right) + \\ \partial_2\left(u_3^2 - u_1u_3 + 2\left(u_3 - u_1 + 1\right)\ln\left(1 - e^{u_1 - u_3}\right) - 2Li_2\left(e^{u_1 - u_3}\right)\right) + \\ \partial_3\left(u_1u_2 - u_2^2 - 2\left(u_1 - u_2\right) + 2\left(u_1 - u_2\right)\ln\left(1 - e^{u_1 - u_2}\right) + 2Li_2\left(e^{u_1 - u_2}\right)\right) = 0.\end{gathered}$$

Again, applying steps 3 and 4, one can show that discrete versions of the first three conservation laws correspond to the discrete equation

$$e^{-\frac{T_1 u}{\epsilon}}(e^{\frac{T_{13}u}{\epsilon}} - e^{\frac{T_{12}u}{\epsilon}}) + e^{-\frac{T_2 u}{\epsilon}}(e^{\frac{T_{12}u}{\epsilon}} - e^{\frac{T_{23}u}{\epsilon}}) + e^{-\frac{T_3 u}{\epsilon}}(e^{\frac{T_{23}u}{\epsilon}} - e^{\frac{T_{13}u}{\epsilon}}) = 0.$$

Setting $\tau = e^{u/\epsilon}$, this takes the form

$$\frac{T_{13}\tau - T_{12}\tau}{T_1\tau} + \frac{T_{12}\tau - T_{23}\tau}{T_2\tau} + \frac{T_{23}\tau - T_{13}\tau}{T_3\tau} = 0,$$

which is known as the lattice mKP equation (see section 5.1).

subcase 3c: $p = 1/u_1,\ q = 1/u_2,\ r = 1/u_3$. The corresponding PDE is

$$u_3(u_2 - u_1)u_{12} + u_2(u_1 - u_3)u_{13} + u_1(u_3 - u_2)u_{23} = 0.$$

It possesses four conservation laws:

$$\partial_2 \ln\left(1 - \frac{u_3}{u_1}\right) - \partial_3 \ln\left(\frac{u_2}{u_1} - 1\right) = 0,$$
$$\partial_3 \ln\left(1 - \frac{u_1}{u_2}\right) - \partial_1 \ln\left(\frac{u_3}{u_2} - 1\right) = 0,$$
$$\partial_1 \ln\left(1 - \frac{u_2}{u_3}\right) - \partial_2 \ln\left(\frac{u_1}{u_3} - 1\right) = 0,$$
$$\partial_1\left(u_2^2 u_3 - u_2 u_3^2\right) + \partial_2\left(u_3^2 u_1 - u_3 u_1^2\right) + \partial_3\left(u_1^2 u_2 - u_1 u_2^2\right) = 0.$$

Applying steps 3 and 4, one can show that discrete versions of the first three conservation laws correspond to the discrete equation

$$(T_2\triangle_1 u)(T_3\triangle_2 u)(T_1\triangle_3 u) = (T_2\triangle_3 u)(T_3\triangle_1 u)(T_1\triangle_2 u),$$

known as the Schwarzian KP equation (see section 5.1).

subcase 3d: $p = \coth u_1,\ q = \coth u_2,\ r = \coth u_3$ (one can also take the trigonometric version $\coth \to \cot$). The corresponding PDE is

$$(\coth u_2 - \coth u_1)u_{12} + (\coth u_1 - \coth u_3)u_{13} + (\coth u_3 - \coth u_2)u_{23} = 0.$$

It possesses four conservation laws:

$$\partial_1 \ln\frac{\sinh u_3}{\sinh u_2} + \partial_2 \ln\frac{\sinh u_1}{\sinh u_3} + \partial_3 \ln\frac{\sinh u_2}{\sinh u_1} = 0,$$

$$\partial_1 \ln \frac{\sinh(u_2-u_3)}{\sinh u_2} - \partial_3 \ln \frac{\sinh(u_1-u_2)}{\sinh u_2} = 0,$$
$$\partial_2 \ln \frac{\sinh(u_3-u_1)}{\sinh u_1} - \partial_3 \ln \frac{\sinh(u_1-u_2)}{\sinh u_1} = 0,$$

$$\partial_1 \Big(-2u_3^2 + 2u_2u_3 - 2u_2 \ln \frac{\sinh(u_2-u_3)}{\sinh u_2} + (2u_3-1) \ln \frac{\sinh(u_2-u_3)}{\sinh u_3} +$$
$$Li_2(e^{2u_2}) - Li_2(e^{2u_3}) - Li_2(e^{2(u_2-u_3)}) \Big) +$$
$$\partial_2 \Big(2u_3^2 - 2u_1u_3 + (2u_1-1) \ln \frac{\sinh(u_3-u_1)}{\sinh u_1} + (1-2u_3) \ln \frac{\sinh(u_3-u_1)}{\sinh u_3} -$$
$$Li_2(e^{2u_1}) + Li_2(e^{2u_3}) + Li_2(e^{2(u_1-u_3)}) \Big) +$$
$$\partial_3 \Big(-2u_2^2 + 2u_1u_2 + 2u_2 \ln \frac{\sinh(u_1-u_2)}{\sinh u_2} + (1-2u_1) \ln \frac{\sinh(u_1-u_2)}{\sinh u_1} +$$
$$Li_2(e^{2u_1}) - Li_2(e^{2u_2}) - Li_2(e^{2(u_1-u_2)}) \Big) = 0.$$

Applying steps 3 and 4, one can show that discrete versions of the first three conservation laws correspond to the discrete equation

$$(e^{2(T_{12}u-T_2u)/\epsilon}-1)(e^{2(T_{13}u-T_1u)/\epsilon}-1)(e^{2(T_{23}u-T_3u)/\epsilon}-1) =$$
$$(e^{2(T_{12}u-T_1u)/\epsilon}-1)(e^{2(T_{13}u-T_3u)/\epsilon}-1)(e^{2(T_{23}u-T_2u)/\epsilon}-1).$$

Setting $\tau = e^{2u/\epsilon}$, it can be rewritten as

$$\left(\frac{T_{12}\tau}{T_2\tau}-1\right)\left(\frac{T_{13}\tau}{T_1\tau}-1\right)\left(\frac{T_{23}\tau}{T_3\tau}-1\right) = \left(\frac{T_{12}\tau}{T_1\tau}-1\right)\left(\frac{T_{13}\tau}{T_3\tau}-1\right)\left(\frac{T_{23}\tau}{T_2\tau}-1\right),$$

which is known as the lattice spin equation (see section 5.1). In the trigonometric case, one can show that discrete versions of the conservation laws

$$\partial_1 \ln \frac{\sin u_3}{\sin u_2} + \partial_2 \ln \frac{\sin u_1}{\sin u_3} + \partial_3 \ln \frac{\sin u_2}{\sin u_1} = 0,$$
$$\partial_1 \ln \frac{\sin(u_2-u_3)}{\sin u_2} - \partial_3 \ln \frac{\sin(u_1-u_2)}{\sin u_2} = 0,$$
$$\partial_2 \ln \frac{\sin(u_3-u_1)}{\sin u_1} - \partial_3 \ln \frac{\sin(u_1-u_2)}{\sin u_1} = 0,$$

correspond to the discrete Sine-Gordon equation,

$$(T_2 \sin \triangle_1 u)(T_3 \sin \triangle_2 u)(T_1 \sin \triangle_3 u) = (T_2 \sin \triangle_3 u)(T_3 \sin \triangle_1 u)(T_1 \sin \triangle_2 u).$$

This finishes the proof of Theorem 5.1. ■

Remark. It was observed in [59] that the Lagrangians $L(u, T_1u, T_2u; \alpha_1, \alpha_2)$ of 2D discrete integrable equations of the ABS type [4] satisfy the closure relations

$$\triangle_1 L(u, T_2u, T_3u; \alpha_2, \alpha_3) + \triangle_2 L(u, T_3u, T_1u; \alpha_3, \alpha_1) + \triangle_3 L(u, T_1u, T_2u; \alpha_1, \alpha_2) = 0, \quad (5.14)$$

which can be interpreted as 3D discrete conservation laws. For instance, the Q_1 case corresponds to the Lagrangian

$$L(u, T_1u, T_2u; \alpha_1, \alpha_2) = \alpha_2 \ln\left(1 - \frac{\triangle_1 u}{\triangle_2 u}\right) - \alpha_1 \ln\left(\frac{\triangle_2 u}{\triangle_1 u} - 1\right).$$

Remarkably, the corresponding closure relation (5.14), viewed as a single 3D equation, turns out to be integrable (subcase 6 of Theorem 5.1). Note that the constraint $\alpha_1 = \alpha_2 = \alpha_3$ reduces (5.14) to the Schwarzian KP equation,

$$\triangle_1\left(\ln\frac{\triangle_3 u}{\triangle_2 u}\right) + \triangle_2\left(\ln\frac{\triangle_1 u}{\triangle_3 u}\right) + \triangle_3\left(\ln\frac{\triangle_2 u}{\triangle_1 u}\right) = 0.$$

On the contrary, closure relations corresponding to the Lagrangians containing the dilogarithm Li_2 fail the ϵ^2 integrability test. We refer to [5] for further connections between ABS equations and 3D integrable equations of octahedron type.

5.4.1 Two discrete and one continuous variables.

In this subsection we classify conservative equations of the form

$$\triangle_1 f + \triangle_2 g + \partial_3 h = 0, \quad (5.15)$$

where f, g, h are functions of $\triangle_1 u, \triangle_2 u, u_3$. Again, nondegeneracy of the dispersionless limit is assumed. Our classification result is as follows:

Theorem 5.2 *Integrable equations of the form (5.15) are grouped into seven three-parameter families,*

$$\alpha I + \beta J + \gamma K = 0,$$

where α, β, γ are arbitrary constants, while I, J, K denote left hand sides of three linearly independent semi-discrete conservation laws of the seven differential-difference equations listed below. In each case we give explicit forms of I, J, K, as well as the underlying differential-difference equation.

Case 1.

Conservation Laws	*Differential-difference eqn*
$I = \triangle_1 e^{\triangle_2 u} - \partial_3 e^{\triangle_2 u - \triangle_1 u} = 0$ $J = \triangle_1 u_3 + \triangle_2 \left(e^{\triangle_1 u} - u_3\right) = 0$ $K = \triangle_1 u_3^2 + \triangle_2 \left(2e^{\triangle_1 u} u_3 - e^{2\triangle_1 u} - u_3^2\right) - \partial_3 \left(2e^{\triangle_1 u}\right) = 0$	$\frac{T_{12}v}{T_2 v} + \frac{T_1 v_3}{T_1 v} = \frac{T_1 v}{v} + \frac{T_2 v_3}{T_2 v}$ *(setting* $v = e^{u/\epsilon}, \partial_3 \to \frac{1}{\epsilon}\partial_3$*)*

Case 2.

Conservation Laws	*Differential-difference equation*
$I = \triangle_1(e^{\triangle_2 u} - u_3) + \partial_3 \ln\left(e^{\triangle_1 u} - e^{\triangle_2 u}\right) = 0$ $J = \triangle_2(e^{\triangle_1 u} - u_3) + \partial_3 \ln\left(e^{\triangle_1 u} - e^{\triangle_2 u}\right) = 0$ $K = \triangle_1(e^{2\triangle_2 u} - 2e^{\triangle_2 u} u_3 + u_3^2) + \triangle_2(2e^{\triangle_1 u} u_3 -$ $e^{2\triangle_1 u} - u_3^2) + \partial_3(2e^{\triangle_2 u} - 2e^{\triangle_1 u}) = 0$	$T_{12}v = \frac{T_1 v T_2 v}{v} + \frac{T_2 v T_1 v_3 - T_1 v T_2 v_3}{T_2 v - T_1 v}$ *(setting* $v = e^{u/\epsilon}, \partial_3 \to \frac{1}{\epsilon}\partial_3$*)*

Case 3.

Conservation Laws	*Differential-difference equation*
$I = \triangle_1(e^{\triangle_2 u} u_3) - \partial_3 e^{\triangle_2 u} = 0$ $J = \triangle_2(e^{-\triangle_1 u} u_3) + \partial_3 e^{-\triangle_1 u} = 0$ $K = \triangle_1(\triangle_2 u + \ln u_3) - \triangle_2 \ln u_3 = 0$	$\frac{vT_{12}v}{T_1 v} = \frac{T_1 v T_2 v_3}{T_1 v_3}$ *(setting* $v = e^{u/\epsilon}$*)*

Case 4.

Conservation Laws	*Differential-difference equation*
$I = \triangle_2\left(\frac{u_3}{\triangle_1 u}\right) - \partial_3 \ln(\triangle_1 u) = 0$ $J = \triangle_1 \ln u_3 + \triangle_2 \ln\left(\frac{\triangle_1 u}{u_3}\right) = 0$ $K = \triangle_1(2u_3 \triangle_2 u) + \partial_3\left((\triangle_1 u)^2 - 2\triangle_1 u \triangle_2 u\right) = 0$	$(T_{12}u - T_2 u)T_1 u_3 = (T_1 u - u)T_2 u_3$

Case 5.

Conservation Laws	*Differential-difference equation*
$I=\triangle_1(e^{\triangle_2 u}u_3)+\partial_3(e^{\triangle_2 u-\triangle_1 u}-e^{\triangle_2 u})=0$ $J=\triangle_1\ln u_3+\triangle_2\ln\left(\frac{1-e^{\triangle_1 u}}{u_3}\right)=0$ $K=\triangle_2\left(\frac{u_3}{1-e^{\triangle_1 u}}\right)+\partial_3\left(\ln(1-e^{\triangle_1 u})-\triangle_1 u\right)=0$	$v(T_{12}v-T_2v)T_1v_3=$ $T_1v(T_1v-v)T_2v_3$ *(setting $v=e^{u/\epsilon}$)*

Case 6.

Conservation Laws	*Differential-difference equation*
$I=\triangle_1\ln\left(\frac{\triangle_2 u}{u_3}\right)+\triangle_2\ln\left(\frac{u_3}{\triangle_1 u}\right)=0$ $J=\triangle_1\left(\frac{u_3}{\triangle_2 u}\right)+\partial_3\ln\left(1-\frac{\triangle_1 u}{\triangle_2 u}\right)=0$ $K=\triangle_2\left(\frac{u_3}{\triangle_1 u}\right)+\partial_3\ln\left(1-\frac{\triangle_2 u}{\triangle_1 u}\right)=0$	$(T_2\triangle_1 u)(\triangle_2 u)T_1u_3=$ $(T_1\triangle_2 u)(\triangle_1 u)T_2u_3$

Case 7.

Conservation Laws	*Differential-difference equation*
$I=\triangle_1\ln\left(\frac{\sinh\triangle_2 u}{u_3}\right)-\triangle_2\ln\left(\frac{\sinh\triangle_1 u}{u_3}\right)=0$ $J=\triangle_1\left(u_3\coth\triangle_2 u\right)+\triangle_2\ln\left(\frac{\sinh(\triangle_1 u-\triangle_2 u)}{\sinh\triangle_2 u}\right)=0$ $K=\triangle_2\left(u_3\coth\triangle_1 u\right)+\partial_3\ln\left(\frac{\sinh(\triangle_1 u-\triangle_2 u)}{\sinh\triangle_1 u}\right)=0$	$(T_2\sinh\triangle_1 u)(\sinh\triangle_2 u)T_1u_3=$ $(T_1\sinh\triangle_2 u)(\sinh\triangle_1 u)T_2u_3$

Remark. See the proof below for Lax pairs of the above differential-difference equations.

Proof of Theorem 5.2:

The proof is parallel to that of Theorem 5.1. The dispersionless limit of (5.15) is again a quasilinear conservation law of the form (5.9),

$$\partial_1 f+\partial_2 g+\partial_3 h=0,$$

where f,g,h are functions of the variables $a=u_1,\ b=u_2,\ c=u_3$. Requiring that all one-phase reductions of the dispersionless equation are inherited by the differential-difference equation (5.15), we obtain a set of differential constraints for f,g,h, which are the necessary conditions for integrability. Thus, at the order ϵ we get

$$f_a=g_b=h_c=0,\quad f_c+h_a+g_c+h_b=0, \tag{5.16}$$

note the difference with Theorem 5.1. The first set of these relations implies that the quasilinear conservation law is equivalent to the second order equation

$$Fu_{12} + Gu_{13} + Hu_{23} = 0,$$

where $F = f_b + g_a,\ G = f_c + h_a,\ H = g_c + h_b$. Note that, by virtue of (5.16), the coefficients F, G, H satisfy the additional constraint $G + H = 0$. It follows from [14] that, up to a non-zero factor, any integrable equation of this type is equivalent to

$$[p(u_1) - q(u_2)]u_{12} + r(u_3)u_{13} - r(u_3)u_{23} = 0, \tag{5.17}$$

where the functions $p(a), q(b), r(c)$ satisfy the integrability conditions

$$\begin{aligned} p'' &= p'\left(\tfrac{p'-q'}{p-q} + (p-q)\tfrac{r'}{r^2}\right), \\ q'' &= q'\left(\tfrac{p'-q'}{p-q} - (p-q)\tfrac{r'}{r^2}\right), \\ r'' &= 2\tfrac{r'^2}{r}. \end{aligned} \tag{5.18}$$

Our further strategy is the same as in Theorem 5.1, namely:

Step 1. First, we solve equations (5.18). Modulo unessential translations and rescalings this leads to seven quasilinear integrable equations of the form (5.17).

Step 2. For all of the seven equations found at step 1, we calculate first order conservation laws (there will be four of them in each case).

Step 3. Taking linear combinations of the four conservation laws, and replacing u_1, u_2 by $\triangle_1 u, \triangle_2 u$ (keeping u_3 as it is), we obtain differential-difference equations (5.15) which are the *candidates* for integrability.

Step 4. Applying the ϵ^2-integrability test, we find that only linear combinations of three conservation laws (out of four) pass the integrability test. Below we list conservation laws in such a way that the first three are the ones that pass the integrability test, while the fourth one doesn't. Moreover, each triplet of conservation laws corresponds to one and the same differential-difference equation.

Let us begin with the solution of system (5.18). The analysis leads to seven essentially different cases, which correspond to cases 1-7 of Theorem 5.2 in the same order as they appear below. First of all, the equation for r implies that there are two essentially different cases: $r = 1$ and $r = 1/c$.

Case 1: $r = 1$. Then equations (5.18) simplify to

$$p'' = p'\frac{p'-q'}{p-q}, \quad q'' = q'\frac{p'-q'}{p-q}.$$

There are two subcases depending on how many functions among p, q are constant.

subcase 1a: q is constant (the case p=const is similar). Without any loss of generality one can set $q = 0$. Modulo unessential translations and rescalings this leads to $p = e^a$, resulting in the PDE

$$e^{u_1}u_{12} + u_{13} - u_{23} = 0.$$

This equation possesses four conservation laws:

$$\partial_1 e^{u_2} - \partial_3 e^{u_2-u_1} = 0,$$

$$\partial_1 u_3 + \partial_2\left(e^{u_1} - u_3\right) = 0,$$

$$\partial_1 u_3^2 + \partial_2\left(2u_3e^{u_1} - e^{2u_1} - u_3^2\right) - \partial_3\left(2e^{u_1}\right) = 0,$$

$$\partial_1\left(u_2u_3\right) + \partial_2\left(2u_1e^{u_1} - 2e^{u_1} - u_1u_3\right) + \partial_3\left(u_1^2 - u_1u_2\right) = 0.$$

Applying steps 3 and 4, we can show that semi-discrete versions of the first three conservation laws correspond to the differential-difference equation

$$e^{(T_{12}u-T_2u)/\epsilon} - e^{(T_1u-u)/\epsilon} + T_1u_3 - T_2u_3 = 0, \tag{5.19}$$

which possesses the Lax pair

$$T_2\psi = e^{(T_1u-T_2u)/\epsilon}\,(T_1\psi + \psi), \qquad \epsilon\psi_3 = -e^{(T_1u-u)/\epsilon}\,(T_1\psi + \psi).$$

Setting $v = e^{u/\epsilon}$ and $\partial_3 \to \frac{1}{\epsilon}\partial_3$, we can rewrite (5.19) in the form

$$\frac{T_{12}v}{T_2v} + \frac{T_1v_3}{T_1v} = \frac{T_1v}{v} + \frac{T_2v_3}{T_2v}.$$

subcase 1b: both p and q are non-constant. Modulo unessential translations and rescalings, the elementary separation of variables gives $p = e^a, q = e^b$. The corresponding PDE is

$$(e^{u_1} - e^{u_2})u_{12} + u_{13} - u_{23} = 0.$$

It possesses four conservation laws:

$$\partial_1(e^{u_2} - u_3) + \partial_3 \ln (e^{u_1} - e^{u_2}) = 0,$$

$$\partial_2(e^{u_1} - u_3) + \partial_3 \ln (e^{u_1} - e^{u_2}) = 0,$$

$$\partial_1(e^{2u_2} - 2e^{u_2}u_3 + u_3^2) + \partial_2(2e^{u_1}u_3 - e^{2u_1} - u_3^2) + \partial_3(2e^{u_2} - 2e^{u_1}) = 0,$$

$$\partial_1 \left(-2e^{u_2}u_2 + u_2u_3 + 2e^{u_2}\right) + \partial_2 \left(2e^{u_1}u_1 - u_1u_3 - 2e^{u_1}\right) +$$
$$\partial_3 \left(u_1u_2 - u_2^2 + 2\left(u_1 - u_2\right) \ln \left(1 - e^{u_1 - u_2}\right) + 2Li_2 \left(e^{u_1 - u_2}\right)\right) = 0.$$

Applying steps 3 and 4, we can show that semi-discrete versions of the first three conservation laws correspond to the differential-difference equation

$$e^{(T_{12}u - T_2u)/\epsilon} - e^{(T_{12}u - T_1u)/\epsilon} + e^{(T_2u - u)/\epsilon} - e^{(T_1u - u)/\epsilon} + T_1u_3 - T_2u_3 = 0. \tag{5.20}$$

Equation (5.20) possesses the Lax pair

$$T_2\psi = e^{(T_1u - T_2u)/\epsilon}\, T_1\psi + (1 - e^{(T_1u - T_2u)/\epsilon})\psi, \qquad \epsilon\psi_3 = e^{(T_1u - u)/\epsilon}(T_1\psi - \psi).$$

Note that this case has been recorded before. Setting $v = e^{u/\epsilon}$ and $\partial_3 \to \frac{1}{\epsilon}\partial_3$, we obtain the equation

$$T_{12}v = \frac{T_1vT_2v}{v} + \frac{T_2vT_1v_3 - T_1vT_2v_3}{T_2v - T_1v},$$

which has appeared in the context of discrete evolutions of plane curves [3].

Case 2: $r = 1/c$. In this case the equations for p and q simplify to

$$p'' = p'\left(\frac{p' - q'}{p - q} - (p - q)\right), \quad q'' = q'\left(\frac{p' - q'}{p - q} + (p - q)\right).$$

There are several subcases depending on how many functions among p, q are constant.

subcase 2a: both p and q are constant. The corresponding PDE is

$$u_{12} + \frac{1}{u_3}(u_{13} - u_{23}) = 0.$$

It possesses four conservation laws:

$$\partial_1(e^{u_2}u_3) - \partial_3 e^{u_2} = 0,$$

$$\partial_2(e^{-u_1}u_3) + \partial_3 e^{-u_1} = 0,$$

$$\partial_1(u_2 + \ln u_3) - \partial_2 \ln u_3 = 0,$$

$$\partial_1\left(u_2u_3 + 2u_3\right) + \partial_2\left(u_1u_3 - 2u_3\right) - \partial_3\left(u_1u_2\right) = 0.$$

Applying steps 3 and 4, we can show that semi-discrete versions of the first three conservation laws correspond to the differential-difference equation

$$\frac{T_2u_3}{T_1u_3} = e^{(T_{12}u - T_1u - T_2u + u)/\epsilon}. \tag{5.21}$$

This equation possesses the Lax pair

$$T_1\psi = -e^{(T_1u-u)/\epsilon}\,(T_2\psi - \psi), \qquad \epsilon\psi_3 = -u_3(T_2\psi - \psi).$$

Setting $v = e^{u/\epsilon}$ we can rewrite (5.21) as

$$\frac{vT_{12}v}{T_1v} = \frac{T_1vT_2v_3}{T_1v_3}.$$

subcase 2b: q is constant (the case p=const is similar). Without any loss of generality one can set $q = 0$. The equation for p takes the form $p'' = p'^2/p - pp'$, which integrates to $p'/p + p = \alpha$. There are further subcases depending on the value of the integration constant α.

subcase 2b(i): $\alpha = 0$. Then one can take $p = 1/a$, which results in the PDE

$$\frac{1}{u_1}u_{12} + \frac{1}{u_3}(u_{13} - u_{23}) = 0.$$

It possesses four conservation laws:

$$\partial_2(u_3/u_1) - \partial_3 \ln u_1 = 0,$$

$$\partial_1 \ln u_3 + \partial_2 \ln (u_1/u_3) = 0,$$

$$\partial_1(2u_2u_3) + \partial_3 \left(u_1^2 - 2u_1u_2\right) = 0,$$

$$\partial_1 \left(u_2^2u_3\right) - \partial_2 \left(\frac{u_1^2u_3}{3}\right) + \partial_3 \left(u_1^2u_2 - u_2^2u_1 - \frac{2u_1^3}{9}\right) = 0.$$

Applying steps 3 and 4, we can show that semi-discrete versions of the first three conservation laws correspond to the differential-difference equation

$$(T_{12}u - T_2u)T_1u_3 = (T_1u - u)T_2u_3. \tag{5.22}$$

This equation possesses the Lax pair

$$T_1\psi = -\frac{(T_1u - u)}{\epsilon}\, T_2\psi + \psi, \qquad \epsilon\psi_3 = -u_3T_2\psi.$$

subcase 2b(ii): $\alpha \neq 0$ (without any loss of generality one can set $\alpha = 1$). Then one has $p = e^a/(e^a - 1)$, which corresponds to the PDE

$$\frac{e^{u_1}}{e^{u_1} - 1}u_{12} + \frac{1}{u_3}(u_{13} - u_{23}) = 0.$$

It possesses four conservation laws:

$$\partial_1(u_3e^{u_2}) + \partial_3(e^{u_2-u_1} - e^{u_2}) = 0,$$

$$\partial_1 \ln u_3 + \partial_2 \ln \left(\frac{1 - e^{u_1}}{u_3}\right) = 0,$$

$$\partial_2 \left(\frac{u_3}{1 - e^{u_1}}\right) + \partial_3 \left(\ln(1 - e^{u_1}) - u_1\right) = 0,$$

$$\partial_1 \left(\frac{u_2u_3}{2} + u_3\right) + \partial_2 \left(\frac{u_1u_3\,(e^{u_1} + 1)}{2\,(e^{u_1} - 1)} - u_3\right) + \partial_3 \left(\frac{u_1^2 - u_1u_2}{2} - u_1 \ln (1 - e^{u_1}) - Li_2\,(e^{u_1})\right) = 0.$$

Applying steps 3 and 4, we can show that semi-discrete versions of the first three conservation laws correspond to the differential-difference equation

$$(1 - e^{(T_{12}u - T_2u)/\epsilon})T_1u_3 = (1 - e^{(T_1u - u)/\epsilon})T_2u_3, \tag{5.23}$$

which possesses the Lax pair

$$T_1\psi = (1 - e^{(T_1u-u)/\epsilon})T_2\psi - e^{(T_1u-u)/\epsilon}\psi, \qquad \epsilon\psi_3 = u_3T_2\psi + u_3\psi.$$

Setting $v = e^{u/\epsilon}$ we can rewrite equation (5.23) in the form

$$v(T_{12}v - T_2 v)T_1 v_3 = T_1 v(T_1 v - v)T_2 v_3.$$

subcase 2c: both p and q are non-constant. Subtracting the ODEs for p and q from each other and separating the variables gives $p' = \alpha - p^2,\ q' = \alpha - q^2$. There are further subcases depending on the value of the integration constant α.

subcase 2c(i): $\alpha = 0$. Then one can take $p = 1/a,\ q = 1/b$, which results in the PDE

$$\left(\frac{1}{u_1} - \frac{1}{u_2}\right) u_{12} + \frac{1}{u_3}(u_{13} - u_{23}) = 0.$$

It possesses four conservation laws:

$$\partial_1 \ln\left(\frac{u_2}{u_3}\right) + \partial_2 \ln\left(\frac{u_3}{u_1}\right) = 0,$$
$$\partial_1 \left(\frac{u_3}{u_2}\right) + \partial_3 \ln\left(1 - \frac{u_1}{u_2}\right) = 0,$$
$$\partial_2 \left(\frac{u_3}{u_1}\right) + \partial_3 \ln\left(1 - \frac{u_2}{u_1}\right) = 0,$$
$$\partial_1 \left(u_2^2 u_3\right) - \partial_2 \left(u_1^2 u_3\right) + \partial_3 \left(u_1^2 u_2 - u_2^2 u_1\right) = 0.$$

Applying steps 3 and 4, we can show that semi-discrete versions of the first three conservation laws correspond to the differential-difference equation

$$(T_2\triangle_1 u)(\triangle_2 u)T_1 u_3 = (T_1\triangle_2 u)(\triangle_1 u)T_2 u_3, \tag{5.24}$$

which appeared in [12]. Equation (5.24) possesses the Lax pair

$$T_1\psi = \frac{\triangle_1 u}{\triangle_2 u}T_2\psi + \left(1 - \frac{\triangle_1 u}{\triangle_2 u}\right)\psi, \qquad \epsilon\psi_3 = \frac{u_3}{\triangle_2 u}(T_2\psi - \psi).$$

subcase 2c(ii): $\alpha \neq 0$ (we will consider the hyperbolic case $\alpha = 1$; the trigonometric case $\alpha = -1$ is similar). Then one can take $p = \coth a,\ q = \coth b$, which results in the PDE

$$(\coth u_1 - \coth u_2)\, u_{12} + \frac{1}{u_3}(u_{13} - u_{23}) = 0.$$

It possesses four conservation laws:

$$\partial_1 \ln\left(\frac{\sinh u_2}{u_3}\right) - \partial_2 \ln\left(\frac{\sinh u_1}{u_3}\right) = 0,$$

$$\partial_1 \left(u_3 \coth u_2\right) + \partial_2 \ln \left(\frac{\sinh(u_1 - u_2)}{\sinh u_2}\right) = 0,$$

$$\partial_2 \left(u_3 \coth u_1\right) + \partial_3 \ln \left(\frac{\sinh(u_1 - u_2)}{\sinh u_1}\right) = 0,$$

$$\partial_1 \left(4u_3(1 - \coth u_2 - u_2 \coth u_2)\right) + \partial_2(4u_1 u_3 \coth u_1) +$$
$$\partial_3 \Big(4u_1 - 2u_2^2 - 12u_2 + 2(u_1 - u_2 - 2) \ln \left(1 - e^{2u_1 - 2u_2}\right) + 2(u_1 - u_2) \ln \left(1 - e^{2u_2 - 2u_1}\right) +$$
$$4(u_2 + 1) \ln \left(1 - e^{2u_2}\right) - 2u_1 \ln \left((1 - e^{-2u_1})(1 - e^{2u_1})\right) + Li_2 \left(e^{-2u_1}\right) - Li_2 \left(e^{2u_1}\right) +$$
$$Li_2 \left(e^{2u_1 - 2u_2}\right) + 2Li_2 \left(e^{2u_2}\right) - Li_2 \left(e^{2u_2 - 2u_1}\right)\Big) = 0.$$

Applying steps 3 and 4, we can show that semi-discrete versions of the first three conservation laws correspond to the differential-difference equation

$$(T_2 \sinh \triangle_1 u)(\sinh \triangle_2 u) T_1 u_3 = (T_1 \sinh \triangle_2 u)(\sinh \triangle_1 u) T_2 u_3, \tag{5.25}$$

which possesses the following Lax pair:

$$T_2 \psi = \frac{e^{2\triangle_2 u} - 1}{e^{2\triangle_1 u} - 1} T_1 \psi + \frac{e^{2\triangle_1 u} - e^{2\triangle_2 u}}{e^{2\triangle_1 u} - 1} \psi, \qquad \epsilon \psi_3 = \frac{2u_3}{e^{2\triangle_1 u} - 1}(T_1 \psi - \psi).$$

This finishes the proof of Theorem 5.2. ■

5.4.2 One discrete and two continuous variables.

One can show that there exist no nondegenerate integrable equations of the form

$$\triangle_1 f + \partial_2 g + \partial_3 h = 0,$$

where f, g, h are functions of $\triangle_1 u, u_2, u_3$.

5.5 Discrete second order quasilinear equations

Here we present the result of classification of integrable equations of the form

$$\sum_{i,j=1}^{3} f_{ij}(\triangle u) \triangle_{ij} u = 0,$$

where f_{ij} are functions of $\triangle_1 u, \triangle_2 u, \triangle_3 u$ only. These equations can be viewed as discretisations of second order quasilinear PDEs

$$\sum_{i,j=1}^{3} f_{ij}(u_k) u_{ij} = 0,$$

whose integrability was investigated in [14].

Theorem 5.3 *There exists a unique nondegenerate discrete second order quasilinear equation in 3D, known as the lattice KP equation,*

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u + (\triangle_3 u - \triangle_1 u)\triangle_{13} u + (\triangle_2 u - \triangle_3 u)\triangle_{23} u = 0.$$

In different contexts and equivalent forms, it has appeared in [16, 68, 67]. The proof is similar to that of Theorem 5.1, and will be omitted.

5.5.1 Two discrete and one continuous variables

The classification of semi-discrete integrable equations of the form

$$f_{11}\triangle_{11} u + f_{12}\triangle_{12} u + f_{22}\triangle_{22} u + f_{13}\triangle_1 u_3 + f_{23}\triangle_2 u_3 + f_{33} u_{33} = 0,$$

where the coefficients f_{ij} are functions of $\triangle_1 u, \triangle_2 u, u_3$, gives the following result:

Theorem 5.4 *There exists a unique nondegenerate second order equation of the above type, known as the semi-discrete Toda lattice,*

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u - \triangle_1 u_3 + \triangle_2 u_3 = 0.$$

It has appeared before in [3, 58]. Again, we skip the details of calculations.

5.5.2 One discrete and two continuous variables

One can show that there exist no nondegenerate semi-discrete integrable equations of the form

$$f_{11}\triangle_{11} u + f_{12}\triangle_1 u_2 + f_{22} u_{22} + f_{13}\triangle_1 u_3 + f_{23} u_{23} + f_{33} u_{33} = 0,$$

where the coefficients f_{ij} are functions of $\triangle_1 u, u_2, u_3$.

5.6 Numerical simulations

In this section we present some results of numerical simulations comparing solutions for the gauge-invariant form of Hirota equation (5.4),

$$\triangle_{t\bar{t}}\, u - \triangle_{x\bar{x}}\, [u - \ln(e^u + 1)] - \triangle_{y\bar{y}}\, [\ln(e^u + 1)] = 0,$$

and its dispersionless limit (5.5),

$$u_{tt} - [u - \ln(e^u + 1)]_{xx} - [\ln(e^u + 1)]_{yy} = 0,$$

obtained using Mathematica. We choose the following Cauchy data:

Disrete equation (5.4): $u(x, y, 0) = 3e^{-(x^2+y^2)},\ u(x, y, -\epsilon) = 3e^{-(x^2+y^2)}$.

Dispersionless equation (5.5): $u(x, y, 0) = 3e^{-(x^2+y^2)},\ u_t(x, y, 0) = 0$.

According to Klainerman's theory [49], there exists a number of conditions, the so-called *null conditions*, which, when satisfied, establish global existence results for nonlinear wave PDEs. In fact, these conditions are automatically satisfied in the case of linearly degenerate systems (which are excluded from our study). Moreover, the smaller the initial (Cauchy) data that one chooses for a given equation, the longer the solution remains smooth, (see also [6, 15, 47]).

In Figure 5.1 we plot the numerical solution of the dispersionless equation (5.5) for $t = 0, 4, 8$. As equation (5.5) does not satisfy the null conditions of Klainerman [49], according to the general theory this solution is expected to break down in finite time.

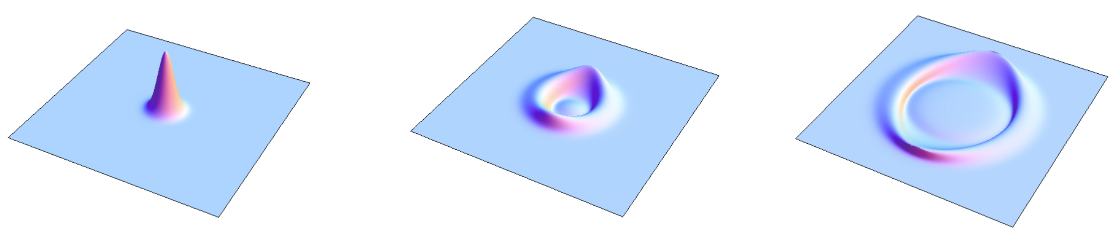

Figure 5.1: Numerical solution of the dispersionless equation (5.5) for $t = 0, 4, 8$, showing the onset of breaking.

On the contrary, solutions to the dispersive regularisation (5.4) (which can be viewed as a difference scheme) do not break down. Indeed, (5.4) can be rewritten in the form

$$u(t+\epsilon) = -u(t-\epsilon) + (T_x + T_{\bar{x}})(u - \ln(e^u + 1)) + (T_y + T_{\bar{y}})\ln(e^u + 1),$$

which allows the computation of $u(t+\epsilon)$ once u and $u(t-\epsilon)$ are known. Figures 5.2, 5.3 and 5.4 illustrate the solution for different values of ϵ at $t = 0, 4, 8$. As ϵ becomes smaller, one can see the formation of a dispersive shock wave in Figure 5.5 (see [50] for a detailed numerical study of this phenomenon for the KP equation).

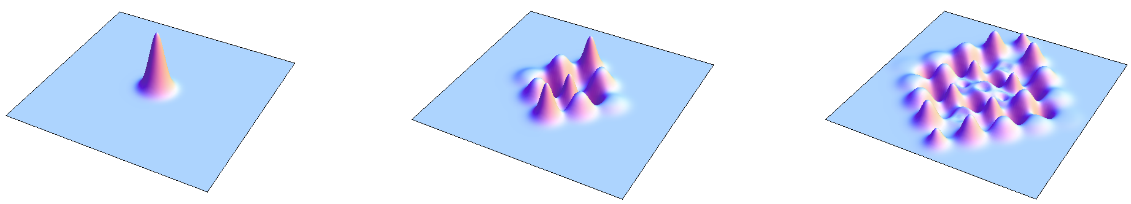

Figure 5.2: Numerical solution of the discrete equation (5.4) for $\epsilon = 2$ and $t = 0, 4, 8$.

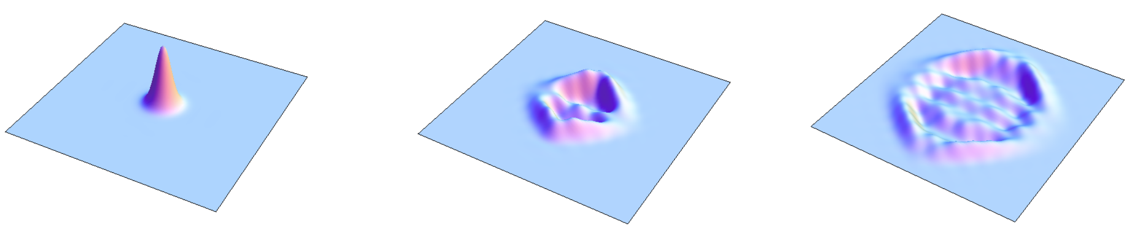

Figure 5.3: Numerical solution of the discrete equation (5.4) for $\epsilon = 1$ and $t = 0, 4, 8$.

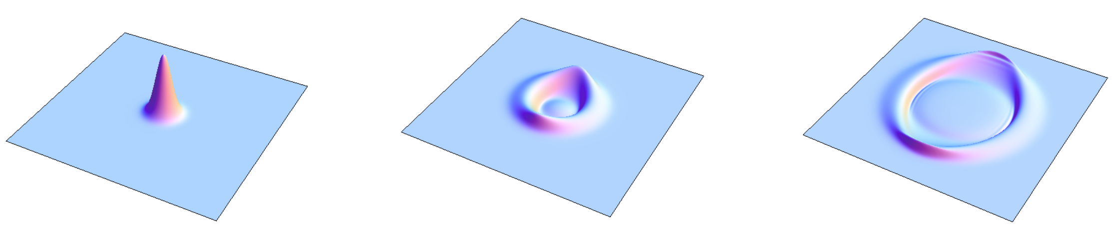

Figure 5.4: Numerical solution of the discrete equation (5.4) for $\epsilon = 1/8$ and $t = 0, 4, 8$.

As $\epsilon \to 0$, solutions of the discrete equation tend to solutions of the dispersionless limit until the breakdown occurs. At the breaking point, one can see the formation of a dispersive shock wave, see Figure 5.4.

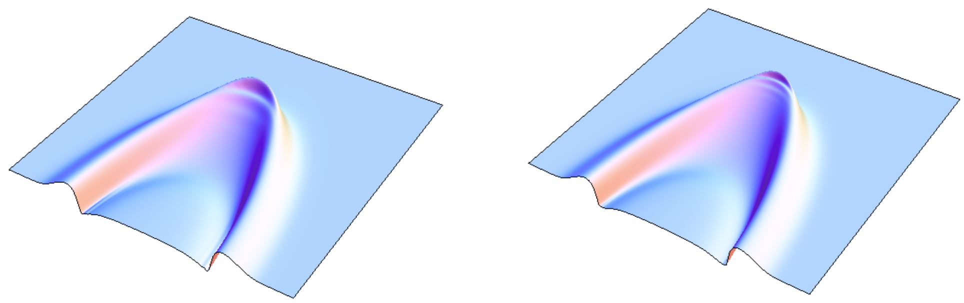

Figure 5.5: Formation of a dispersive shock wave in the numerical solution of the discrete equation (5.4) for $\epsilon = 1/8$ (left) and $\epsilon = 1/16$ (right), at $t = 8$.

There are very few results on dispersive shock waves in 2+1 dimensions (see [50, 51] for a detailed numerical investigation of this phenomenon for the KP and DS equations). This is primarily due to the computational complexity of problems involving rapid oscillations. On the contrary, in the discrete example discussed here one does not require dedicated numerical methods to observe the formation of a dispersive shock wave: this is achieved by simply iterating an explicit recurrence relation.

An example from 2D.

We now perform the same computation for a simpler example in two dimensions. Consider the Hopf equation,

$$u_t + uu_x = 0, \tag{5.26}$$

and the following discretisation

$$T_t u - T_{\bar{t}} u + u(T_x u - T_{\bar{x}} u) = 0. \tag{5.27}$$

Note that in order to obtain a dispersive equation we consider the naive discretisation $\partial_i \to \triangle_i - \triangle_{\bar{i}}$, rather than $\partial_i \to \triangle_i$. The latter would lead to a dissipative equation, where the phenomenon of formation of a dispersive shock wave would not be observed. Equation (5.27) is not integrable anymore, but exhibits the same numerical features as the gauge-invariant form of Hirota equation (5.4).

We choose the following Cauchy data:

Disrete equation (5.27): $u(x,0) = 0.5e^{-x^2},\ u(x,-\epsilon) = 0.5e^{-x^2}$.

Dispersionless equation (5.26): $u(x,0) = 0.5e^{-x^2}$.

In Figure 5.6 we plot the numerical solution of the dispersionless equation (5.26) for $t = 0, 1, 2, 2.5$. Breakdown occurs between $2 \leq t \leq 2,5$ and at this point Mathematica's built in programme becomes unreliable.

Figure 5.6: Numerical solution of the dispersionless equation (5.26) for $t = 0, 1, 2, 2.5$, showing the onset of breaking.

On the other hand, solutions to the dispersive regularisation (5.27) do not break down. Indeed, the discrete equation can be rewritten in the form

$$u(x, t+\epsilon) = u(x, t-\epsilon) - u(x,t)(T_x + T_{\bar{x}})u(x,t),$$

which allows the computation of $u(x, t+\epsilon)$ once $u(x,t)$ and $u(x, t-\epsilon)$ are known. The next series of Figures, 5.7−5.11, illustrate the solution for different values of ϵ at $t = 0, 1, 2, 2.5$. As ϵ becomes smaller, one can observe that solutions of the discrete equation tend to solutions of the dispersionless equation until the breakdown occurs. At the breaking point, one can see the formation of a dispersive shock wave.

Figure 5.7: Numerical solution of the discrete equation (5.27) for $\epsilon = 1/2$ and $t = 0, 1, 2, 2.5$.

Figure 5.8: Numerical solution of the discrete equation (5.27) for $\epsilon = 1/4$ and $t = 0, 1, 2, 2.5$.

Figure 5.9: Numerical solution of the discrete equation (5.27) for $\epsilon = 1/8$ and $t = 0, 1, 2, 2.5$.

Figure 5.10: Numerical solution of the discrete equation (5.27) for $\epsilon = 1/20$ and $t = 0, 1, 2, 2.5$.

Figure 5.11: Numerical solution of the discrete equation (5.27) for $\epsilon = 1/40$ and $t = 0, 1, 2, 2.5$. Before the breakdown, solutions tend to the solutions of the dispersionless equation.

A detailed study of the Hopf equation and its semi-discrete analogue is given in [39]. The authors discretise the equation in the x-variable, keeping t continuous, and using the Runge-Kutta method they perform numerical experiments, that illustrate oscillatory behaviour of the semi-discrete equation.

Chapter 6

Concluding remarks

In the theory of multidimensional integrable systems, we restricted ourselves to three dimensions. We used a novel approach to the integrability of $(2+1)$-dimensional differential equations, which is basically an effective perturbative technique, based on the method of dispersive deformations of hydrodynamic reductions. We reviewed the method for a variety of quasilinear PDEs. Then, we successfully extended the method to the case of differential-difference equations in $2+1$ dimensions, obtaining classification results for classes of equations generalising the Intermediate long wave, and Toda type equations. We also considered fully discrete equations in 3D, providing a new approach to the problem of classification of such equations which, until now, were treated via the multidimensional consistency approach. Finally, we illustrated the formation of dispersive shock waves, through a numerical study of the gauge-invariant form of the Hirota equation.

The study of three-dimensional systems can be continued in several directions. Among others, the following problems are of interest:

- prove the existence of $\epsilon-$deformations at any order. This is the theoretical justification that the perturbative approach works at all orders of the deformation parameter;
- our classification method requires the study of the corresponding dispersionless equation first. This means that the equation is initially put in a continuous frame, and then, by deforming the hydrodynamic reductions, it is lifted to the discrete level.

Translation of our approach to a purely discrete language would be of interest and would enable us to relate it with the multidimensional consistency approach;

- our observation of dispersive shock waves in 3D should be further investigated. Results in this direction have been obtained by Klein, [50, 51]. Constructing analytic solutions of systems, that tend to break down in finite time at a point, combined with the numerical study, can lead to the development of the general theory of dispersive shock waves in higher dimensions;

- linearly degenerate systems are excluded from our studies. Our theory of hydrodynamic reductions does not apply to those equations and they need to be treated in a different way;

- we would like to construct exact solutions, like N-soliton, elliptic solutions, Bäcklund transformations, of discrete equations in 3D, as it was done for Hirota equation by Hirota, Nimmo, Kuniba, Zabrodin, etc. (see for example [45, 79, 56, 89]);

- it would also be interesting to investigate discrete Painlevé reductions of integrable discrete (or dispersive) equations. Steps in this direction, in the case of dispersionless systems, have been made in the works by Dunajski and Tod [24], and Ferapontov, Huard and Zhang in [26]. We would like to explore whether something similar can be done for discrete equations;

Finally, the fact that the method of hydrodynamic reductions can be applied directly to a given equation, enables us to study the integrability of a variety of (semi-)discrete models in 3D, and produce classification lists. Thus, one only needs to detect suitable classes of equations for this purpose.

Appendix A

Classification Program

Most of the theorems of this thesis are proved using the method of hydrodynamic reductions and dispersive deformations of dispersionless systems. For the application of the method, we use computer algebra. Here we give the Mathematica program that was used to perform classification and hence produce the lists of integrable equations. This is the general program, and minor modifications may be required, depending on the initial form of the equation.

Define ϵ. Start running the programme for $\epsilon = 1$.

```
Quit
```

```
emax = 1;
```

```
Unprotect[Power];
ϵ^n_ := 0 /; n > emax
Protect[Power];
```

- **(A) Preliminaries**

Define derivatives.

```
imax = 20;
```

```
DX[exp_] := Block[{i, res}, res = 0;
   Do[res = res + R[i + 1] D[exp, R[i]], {i, 0, imax}]; res];
DXN[exp_, n_] := Block[{i, res}, res = exp;
   If[n > 0, Do[res = DX[res], {i, 1, n}], None]; res];
Off[General::spell1]; Off[General::spell];
```

```
Dy[exp_] := Block[{i, res = 0}, Do[If[ToString[D[exp, R[i]]] ≠ "0",
      res = res + D[exp, R[i]] DXN[Ry, i], None], {i, 0, imax}]; res];
DT[exp_] := Block[{i, res = 0}, Do[If[ToString[D[exp, R[i]]] ≠ "0",
      res = res + D[exp, R[i]] DXN[Rt, i], None], {i, 0, imax}]; res];
```

Define $u(x, y, t) = R$, the expressions for $R_y(x, y, t)$, $R_t(x, y, t)$ and $w(x, y, t)$, if necessary.

```
r = R[0];

Ry = μ[r] R[1] + ϵ (a1[r] R[2] + a2[r] R[1]^2) +
    ϵ^2 (b1[r] R[3] + b2[r] R[1] R[2] + b3[r] R[1]^3) + ϵ^3 (c1[r] R[4] +
      c2[r] R[1] R[3] + c3[r] R[2]^2 + c4[r] R[1]^2 R[2] + c5[r] R[1]^4) +
    ϵ^4 (d1[r] R[5] + d2[r] R[1] R[4] + d3[r] R[2] R[3] + d4[r] R[1]^2 R[3] +
      d5[r] R[1] R[2]^2 + d6[r] R[1]^3 R[2] + d7[r] R[1]^5) +
    ϵ^5 (e1[r] R[6] + e2[r] R[1] R[5] + e3[r] R[2] R[4] + e4[r] R[3]^2 +
      e5[r] R[1]^2 R[4] + e6[r] R[1] R[2] R[3] + e7[r] R[2]^3 + e8[r] R[1]^3 R[3] +
      e9[r] R[1]^2 R[2]^2 + e10[r] R[1]^4 R[2] + e11[r] R[1]^6) +
    ϵ^6 (f1[r] R[7] + f2[r] R[1] R[6] + f3[r] R[2] R[5] + f4[r] R[3] R[4] +
      f5[r] R[1]^2 R[5] + f6[r] R[1] R[2] R[4] + f7[r] R[1] R[3]^2 +
      f8[r] R[2]^2 R[3] + f9[r] R[1]^3 R[4] + f10[r] R[1]^2 R[2] R[3] +
      f11[r] R[1] R[2]^3 + f12[r] R[1]^4 R[3] +
      f13[r] R[1]^3 R[2]^2 + f14[r] R[1]^5 R[2] + f15[r] R[1]^7);


Rt = λ[r] R[1] + ϵ (A1[r] R[2] + A2[r] R[1]^2) +
    ϵ^2 (B1[r] R[3] + B2[r] R[1] R[2] + B3[r] R[1]^3) + ϵ^3 (C1[r] R[4] +
      C2[r] R[1] R[3] + C3[r] R[2]^2 + C4[r] R[1]^2 R[2] + C5[r] R[1]^4) +
    ϵ^4 (D1[r] R[5] + D2[r] R[1] R[4] + D3[r] R[2] R[3] + D4[r] R[1]^2 R[3] +
      D5[r] R[1] R[2]^2 + D6[r] R[1]^3 R[2] + D7[r] R[1]^5) +
    ϵ^5 (E1[r] R[6] + E2[r] R[1] R[5] + E3[r] R[2] R[4] + E4[r] R[3]^2 +
      E5[r] R[1]^2 R[4] + E6[r] R[1] R[2] R[3] + E7[r] R[2]^3 + E8[r] R[1]^3 R[3] +
      E9[r] R[1]^2 R[2]^2 + E10[r] R[1]^4 R[2] + E11[r] R[1]^6) +
    ϵ^6 (F1[r] R[7] + F2[r] R[1] R[6] + F3[r] R[2] R[5] + F4[r] R[3] R[4] +
      F5[r] R[1]^2 R[5] + F6[r] R[1] R[2] R[4] + F7[r] R[1] R[3]^2 +
      F8[r] R[2]^2 R[3] + F9[r] R[1]^3 R[4] + F10[r] R[1]^2 R[2] R[3] +
      F11[r] R[1] R[2]^3 + F12[r] R[1]^4 R[3] +
      F13[r] R[1]^3 R[2]^2 + F14[r] R[1]^5 R[2] + F15[r] R[1]^7);

u = r;
w = W[r] + ϵ w1[r] R[1] + ϵ^2 (w2[r] R[2] + w3[r] R[1]^2) +
    ϵ^3 (w4[r] R[3] + w5[r] R[1] R[2] + w6[r] R[1]^3) + ϵ^4 (w7[r] R[4] +
      w8[r] R[1] R[3] + w9[r] R[2]^2 + w10[r] R[1]^2 R[2] + w11[r] R[1]^4) +
    ϵ^5 (w12[r] R[5] + w13[r] R[1] R[4] + w14[r] R[2] R[3] + w15[r] R[1]^2 R[3] +
      w16[r] R[1] R[2]^2 + w17[r] R[1]^3 R[2] + w18[r] R[1]^5) +
    ϵ^6 (w19[r] R[6] + w20[r] R[1] R[5] + w21[r] R[2] R[4] + w22[r] R[3]^2 +
      w23[r] R[1]^2 R[4] + w24[r] R[1] R[2] R[3] + w25[r] R[2]^3 + w26[r] R[1]^3
       R[3] + w27[r] R[1]^2 R[2]^2 + w28[r] R[1]^4 R[2] + w29[r] R[1]^6);
```

Define $\mu(R)$, and the dispersion relation $\lambda(R)$. Since the nonlocalities considered, in the limit $\epsilon \to 0$ go to $w_x = u_y$, we have $\mu(R) = W'(R)$, and the formula of $\lambda(R)$ depends on the initial equation.

```
μ[x_] := W′[x]

λ[x_] := Expand["input dispersion relation here"]
```

Define all terms that appear in the equation. For example w_y, u_y, u_{yy}, u_t, f_x, ψ_y, ψ_{yy}, etc.

```
wy = Expand[Normal[Series[Dy[w], {ϵ, 0, emax}]]];
uy = Dy[u];
uyy = Expand[Dy[uy]];
ut = Expand[Normal[Series[DT[u], {ϵ, 0, emax}]]];

fx = Expand[Normal[Series[DX[f[u]], {ϵ, 0, emax}]]];
ψy = Expand[Normal[Series[Dy[ψ[u, w]], {ϵ, 0, emax}]]];
ψyy = Expand[Normal[Series[Dy[ψy], {ϵ, 0, emax}]]];
```

Input the formula for the equation (eq), the nonlocality (nonloc) and the compatibility condition $R_{yt} = R_{ty}$ (comp)

```
nonloc = Expand[

   "input formula of nonlocality here. E.g (wx-uy-ϵ/2 uyy-ϵ^2/6 uyyy-...)"];

eq = Expand[Normal[
      Series["input the equation here. E.g (ut- ϕ[u,w]ux - ψ[u,w] wy-...)",
       {ϵ, 0, emax}]]] // Factor;

comp = Expand[DT[Ry] - Dy[Rt]] // Factor;
```

■ (B) Terms at ϵ

Collect coefficients of $R[2]$, $R[1]^2$ (order 2 in the derivatives) in the nonlocality, set them equal to 0 and then solve for a2(R), w1(R)

```
Factor[nonloc]

Solve[Coefficient[nonloc, R[1]^2] == 0, a2[r]] /. {R[0] → x}
Solve[Coefficient[nonloc, R[2]] == 0, w1[r]] /. {R[0] → x}

a2[x_] := "write down the result for a2"
w1[x_] := "write down the result for w1"
```

Collect coefficients of $R[2]$, $R[1]^2$ (order 2) in the equation, set them equal to 0 and then solve for A2(R), A1(R)

```
Factor[eq]

Solve[Coefficient[eq, R[1]^2] == 0, A2[r]] /. {R[0] → x}
Solve[Coefficient[eq, R[2]] == 0, A1[r]] /. {R[0] → x}

A2[x_] := "write down the result for A2"
A1[x_] := "write down the result for A1"
```

Collect coefficients of $R[3]$, $R[1]R[2]$, $R[1]^3$ (order 3), set them equal to 0 and then solve wrt a1(R).

```
Factor[comp];

kk0 = Factor[Coefficient[comp, R[3]]]
kk1 = Factor[Coefficient[comp, R[1] R[2]]]
kk2 = Factor[Coefficient[comp, R[1]^3]]

Solve[kk1 == 0, a1[r]] // Factor

a1[x_] := "write down the result for a1"
```

From the remaining relation (kk2) we obtain a system (sys). We collect coefficients of $W'[R[0]]$, $W''[R[0]]$, and so on, and set them equal to 0 (since μ(R) is an arbitrary function for the method). This way we create the resulting system of integrability conditions (IC), we count and sort the equations of this system from the simplest to the most complicated, and eventually solve them.

```
sys = Numerator[Factor[kk2]];

SetAttributes[killfactor, Listable];
killfactor[a_^_] := a;
killfactor[a_^_] := 1 /; MemberQ[factorlist, a];
killfactor[a_] := 1 /; MemberQ[factorlist, a];
killfactor[a_] := a;
killfactor[a_ b_] := killfactor[a] killfactor[b];
killfactor[a_] := 1 /; IntegerQ[a];
killfactor[0] := 0;

IC = Complement[Flatten[CoefficientList[Numerator[sys],
      {W'[R[0]], W''[R[0]], W^(3)[R[0]], W^(4)[R[0]], W^(5)[R[0]], W^(6)[R[0]]}]],
    {0}] /. {W[R[0]] → y} /. {R[0] → x};


IC = Union[killfactor[Complement[Factor[IC], {0}]]];
IC = Sort[IC, LeafCount[#1] < LeafCount[#2] &];
Length[IC]

factorlist =
 {"list here quantities that cannot be zero due to dispersion relation"}

jj = killfactor[Factor[IC[[1]]]]
```

Write down the results, and then go back to the beginning and repeat the procedure for emax = 2.
Namely, go back to (A) and run preliminaries. From (B) run only the results for a1, a2, w1, A1, A2. Then go to (C).

■ (C) Terms at ϵ^2

Collect coefficients of $R[3]$, $R[1]R[2]$, $R[1]^3$ (order 3) in the nonlocality, set them equal to 0 and then solve for b3(R), w3(R), w2(R).

```
Factor[nonloc]

b3[x_] := "write down the result for b3"
w2[x_] := "write down the result for w2"
w3[x_] := "write down the result for w3"
```

Collect coefficients of $R[3]$, $R[1]R[2]$, $R[1]^3$ (order 3) in the equation, set them equal to 0 and then solve for Bi(R), $i = 1, 2, 3$.

```
Factor[eq]

B1[x_] := "write down the result for B1"
B2[x_] := "write down the result for B2"
B3[x_] := "write down the result for B3"
```

Collect coefficients of $R[4]$, $R[2]^2$, $R[1]^2 R[2]$, $R[1]^4$ (order 4), set them equal to 0 and then solve wrt b1(R), b2(R).

```
Factor[comp];

kk1 = Factor[Coefficient[comp, R[1]^4]];
kk2 = Factor[Coefficient[comp, R[1]^2 R[2]]];
kk3 = Factor[Coefficient[comp, R[2]^2]];
kk4 = Factor[Coefficient[comp, R[1] R[3]]];
kk5 = Factor[Coefficient[comp, R[4]]];

b1[x_] := "write down the result for b1"
b2[x_] := "write down the result for b2"
```

From the remaining nonzero relation (kk4), obtain a system (sys) and the integrability conditions (IC) repeating the procedure described in (B).

Write down the results, and then go back to the beginning and run the programme for emax = 3, 4, etc , following the same procedure.

Appendix B

Computation of Lax pairs

Here we illustrate the computation of dispersionless and dispersive Lax pairs, using a particular example. Consider the integrable equation (4.32)

$$u_t = (\alpha u + \beta)\triangle_{\bar{y}} e^w, \qquad w_x = \triangle_y u, \tag{B.1}$$

which was obtained in theorem 4.5 in chapter 4. Its dispersionless limit is

$$u_t = (\alpha u + \beta)e^w w_y, \qquad w_x = u_y.$$

In order to find the Lax pair of the differential-difference equation, we first need to derive the corresponding dispersionless Lax pair. This Lax pair is of the form

$$S_y = F(u, w, S_x),$$
$$S_t = G(u, w, S_x).$$

Calculating the compatibility condition $S_{yt} = S_{ty}$ results in

$$F_u u_t + F_w w_t + F_\xi(G_u u_x + G_w w_x) = G_u u_y + G_w w_y + G_\xi F_u u_x, \qquad \xi = S_x,$$

or

$$F_w = 0,$$
$$F_\xi G_u = G_\xi F_u,$$
$$G_u = F_\xi G_w,$$
$$G_w = (\alpha u + \beta)e^w F_u.$$

The solution of the system above gives

$$F = \ln\left(\frac{\alpha u + \beta}{\xi + u}\right),$$
$$G = \frac{e^w}{\xi + u}(\alpha\xi - \beta),$$

where the extra constants that appear in these formulas have been scaled. Thus

$$S_y = \ln\left(\tfrac{\alpha u + \beta}{S_x + u}\right)$$
$$S_t = \tfrac{e^w}{S_x + u}(\alpha S_x - \beta)$$

or

$$S_x e^{S_y} = \alpha u + \beta - u e^{S_y},$$
$$S_t = \alpha e^w - e^w e^{S_y}.$$

Now, there is no algorithmic way to quantise the Lax pair. We know that $\psi = e^{S/\epsilon}$, which means that we can quantise the terms containing S, but normally we have to guess the form of the coefficients that appear in front. Then, these coefficients can be specified by requiring that the compatibility condition is satisfied, modulo the original equation. Let us try the following quantisation

$$\begin{aligned} \epsilon T_y \psi_x &= (\alpha u + \beta)\psi - u T_y \psi, \\ \epsilon \psi_t &= \alpha e^w \psi - e^w T_y \psi. \end{aligned} \tag{B.2}$$

Computing the compatibility condition $\partial_x T_y(\epsilon\psi_t) = \partial_t(\epsilon T y \psi_x)$, modulo the relations (B.2), we obtain

$$w_x = \frac{1}{\epsilon}(u - T_{\bar{y}} u)$$
$$u_t = (\alpha u + \beta)\frac{T_y - 1}{\epsilon} e^w,$$

which is similar to equation (B.1), but not the same. Making the change $u \to T_y u$, we obtain exactly the initial differential-difference equation we are interested in. Therefore, the Lax pair is of the form

$$\epsilon T_y \psi_x = (\alpha T_y u + \beta)\psi - (T_y u) T_y \psi,$$
$$\epsilon \psi_t = \alpha e^w \psi - e^w T_y \psi.$$

Appendix C

ϵ^2−integrability conditions

The integrability conditions, at order ϵ^2, of equation (5.8) are

$$\frac{h_{aa}+h_{ab}+g_{ac}-f_{bb}}{g_a+f_b}+\frac{-h_{aa}+g_{ac}+f_{bc}+g_{cc}}{h_a+f_c}+\frac{h_{aaa}+h_{aab}}{h_{aa}+h_{ab}}+ \\ +\frac{2\left(h_{ab}+g_{ac}\right)-h_{ab}-f_{bb}-f_{bc}-g_{cc}}{h_b+g_c}=0,$$

$$\frac{-h_{aa}+g_{ac}+f_{bc}+g_{cc}}{h_a+f_c}+\frac{h_{ab}+f_{bb}+f_{bc}-g_{cc}}{h_b+g_c}+\frac{g_{acc}+g_{ccc}}{g_{ac}+g_{cc}}+ \\ +\frac{2\left(g_{ac}+f_{bc}\right)-h_{aa}-h_{ab}-g_{ac}-f_{bb}}{g_a+f_b}=0,$$

$$\frac{h_{\mathrm{aa}}+h_{ab}+g_{ac}-f_{bb}}{g_a+f_b}+\frac{h_{ab}+f_{bb}+f_{bc}-g_{cc}}{h_b+g_c}+\frac{f_{bbb}+f_{bbc}}{f_{bb}+f_{bc}}+ \\ +\frac{2\left(h_{ab}+f_{bc}\right)-h_{aa}-g_{ac}-f_{bc}-g_{cc}}{h_a+f_c}=0,$$

$$-\frac{h_{aa}+h_{ab}+g_{ac}-f_{bb}}{g_a+f_b}+\frac{-h_{aa}+g_{ac}+f_{bc}+g_{cc}}{h_a+f_c}+\frac{h_{aab}}{h_{ab}}-\frac{h_{ab}+f_{bb}+f_{bc}+g_{cc}}{h_b+g_c}=0,$$

$$\frac{h_{aa}+h_{ab}+g_{ac}-f_{bb}}{g_a+f_b}+\frac{h_{aa}+g_{ac}+f_{bc}+g_{cc}}{h_a+f_c}+\frac{g_{aac}}{g_{ac}}-\frac{h_{ab}+f_{bb}+f_{bc}+g_{cc}}{h_b+g_c}=0,$$

$$\frac{h_{aa}+h_{ab}+g_{ac}-f_{bb}}{g_a+f_b}-\frac{h_{aa}+g_{ac}+f_{bc}+g_{cc}}{h_a+f_c}-\frac{h_{ab}+f_{bb}+f_{bc}-g_{cc}}{h_b+g_c}+\frac{f_{bbc}}{f_{bc}}=0.$$

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