Geometrisation of Chaplygin's reducing multiplier theorem

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Abstract. We develop the reducing multiplier theory for a special class of nonholonomic dynamical systems and show that the non-linear Poisson brackets naturally obtained in the framework of this approach are all isomorphic to the Lie-Poisson $e(3)$-bracket. As two model examples, we consider the Chaplygin ball problem on the plane and the Veselova system. In particular, we obtain an integrable gyrostatic generalisation of the Veselova system.

Introduction

In [13] S. A. Chaplygin found a special class of systems with two degrees of freedom which can be reduced to a Lagrangian and thus Hamiltonian form by a suitable change of time $dt = \rho\, d\tau$, where ρ is a reducing multiplier depending on the coordinates. As an illustration, he considered the problem of motion of the so-called Chaplygin sleigh, which can be integrated by the Hamilton–Jacobi method using the *reducing multiplier method* proposed by himself. Afterwards it was shown that a number of systems in nonholonomic mechanics can also be represented in the form of *Chaplygin systems* or *generalised Chaplygin systems* [8], and thereby are conformally Hamiltonian [16, 8, 25, 4, 10]. Thus, the reducing multiplier method is one of the most effective methods for explicit *Hamiltonisation* of dynamical systems.

From today's perspective, the reducing multiplier theory is a method for finding one of the most important *tensor invariants* [20] of a dynamical system — the *Poisson structure* [8]. At the same time, the application of this method requires rewriting the equations of motion in local coordinates, which usually involves extremely cumbersome calculations. In this paper we develop the Chaplygin method for one class of systems frequently discussed in nonholonomic mechanics, which allows to achieve

their Hamiltonisation in a much simpler way. We shall not dwell here on the derivation of equations of motion for nonholonomic mechanics. A fairly detailed treatment of this can be found in [6]. In conclusion, we note that the possibility of representing the system in conformally Hamiltonian form essentially affects its dynamical behavior. For a more detailed discussion of these issues, see [6], [8], [9].

1. Generalised Chaplygin systems

We recall that according to [8], a *generalised Chaplygin system* is a mechanical system with two degrees of freedom whose equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \dot{q}_2 S, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = -\dot{q}_1 S,$$
$$S = a_1(\boldsymbol{q})\dot{q}_1 + a_2(\boldsymbol{q})\dot{q}_2 + b(\boldsymbol{q}),$$
(1)

where L is a function of generalised coordinates $\boldsymbol{q} = (q_1, q_2)$ and velocities $\dot{\boldsymbol{q}} = (\dot{q}_1, \dot{q}_2)$, which we may call the Lagrangian of the system. It is straightforward to verify that this system admits an energy integral of standard form

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L.$$
(2)

Remark. A usual Chaplygin system can be obtained by a special choice of the function S (a fortiori $b(\boldsymbol{q}) = 0$) [13]. A somewhat different generalisation of the Chaplygin systems is proposed in [12, 19].

If there is an invariant measure with density depending only on the coordinates, the system can be represented in conformally Hamiltonian form [8] (for $b(\boldsymbol{q}) = 0$ this was shown by S. A. Chaplygin [13]). To show this, we use the Legendre transform for the initial system (1):

$$P_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad H = \sum_i P_i \dot{q}_i - L\Big|_{\dot{q}_i \to P_i}.$$

Then the equations of motion (1) can be recast as

$$\dot{q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_1 = -\frac{\partial H}{\partial q_1} + \frac{\partial H}{\partial P_2}S, \qquad \dot{P}_2 = -\frac{\partial H}{\partial q_2} - \frac{\partial H}{\partial P_1}S,$$
$$S = a_1(\boldsymbol{q})\dot{q}_1 + a_2(\boldsymbol{q})\dot{q}_2 + b(\boldsymbol{q}) = A_1(\boldsymbol{q})P_1 + A_2(\boldsymbol{q})P_2 + B(\boldsymbol{q}).$$
(3)

Here H coincides with the energy integral (2) expressed in terms of the new variables.

Now assume that the system admits an invariant measure with density depending only on the coordinates:

$$\mu = \mathcal{N}(\boldsymbol{q})\, dP_1\, dP_2\, dq_1\, dq_2.$$
(4)

In this case the Liouville equation for $\mathcal{N}(\boldsymbol{q})$ reduces to

$$\dot{q}_1\left(\frac{1}{\mathcal{N}}\frac{\partial \mathcal{N}}{\partial q_1} - A_2(q)\right) + \dot{q}_2\left(\frac{1}{\mathcal{N}}\frac{\partial \mathcal{N}}{\partial q_2} + A_1(q)\right) = 0,$$

and since $\mathcal{N}$ depends only on the coordinates, each of the brackets must vanish separately:

$$\frac{1}{\mathcal{N}}\frac{\partial \mathcal{N}}{\partial q_1} - A_2(\boldsymbol{q}) = 0, \qquad \frac{1}{\mathcal{N}}\frac{\partial \mathcal{N}}{\partial q_2} + A_1(\boldsymbol{q}) = 0. \tag{5}$$

Let us now make the change of variables

$$P_i = \frac{p_i}{\mathcal{N}(\boldsymbol{q})}, \qquad i = 1, 2.$$

Denote the Hamiltonian in the new variables as $\overline{H}(\boldsymbol{q}, \boldsymbol{p}) = H(\boldsymbol{q}, \boldsymbol{P}(\boldsymbol{q}, \boldsymbol{p}))$. Then the following relations hold for the derivatives

$$\frac{\partial H}{\partial P_i} = \mathcal{N}\frac{\partial \overline{H}}{\partial p_i}, \qquad \frac{\partial H}{\partial q_i} = \frac{\partial \overline{H}}{\partial q_i} + \frac{1}{\mathcal{N}}\frac{\partial N}{\partial q_i}\left(\frac{\partial \overline{H}}{\partial p_1}p_1 + \frac{\partial \overline{H}}{\partial p_2}p_2\right).$$

Substituting them into (3) and using (5), we obtain

$$\dot{q}_i = \mathcal{N}(\boldsymbol{q})\frac{\partial \overline{H}}{\partial p_i},$$

$$\dot{p}_1 = \mathcal{N}(\boldsymbol{q})\left(-\frac{\partial \overline{H}}{\partial q_1} + \mathcal{N}(\boldsymbol{q})B(\boldsymbol{q})\frac{\partial \overline{H}}{\partial p_2}\right), \qquad \dot{p}_2 = \mathcal{N}(\boldsymbol{q})\left(-\frac{\partial \overline{H}}{\partial q_2} - \mathcal{N}(\boldsymbol{q})B(\boldsymbol{q})\frac{\partial \overline{H}}{\partial p_1}\right).$$

Thus, the following result holds.

Theorem 1 *If the system (3) admits an invariant measure of the form (4), it can be represented in conformally Hamiltonian form*

$$\dot{q}_i = \mathcal{N}(\boldsymbol{q})\{q_i, \overline{H}\}, \qquad \dot{p}_i = \mathcal{N}(\boldsymbol{q})\{p_i, \overline{H}\}, \qquad i = 1, 2,$$

where the Poisson brackets are given by

$$\{q_i, p_j\} = \delta_{ij}, \qquad \{q_i, q_j\} = 0, \qquad \{p_1, p_2\} = \mathcal{N}(\boldsymbol{q})B(\boldsymbol{q}).$$

Proof. The proof is a straightforward verification of the Jacobi identity. ∎

2. The Chaplygin system on T^*S^2

We now consider a system which is described by means of two three-dimensional vectors $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ and whose equations of motion are

$$\dot{\boldsymbol{M}} = (\boldsymbol{M} - S\boldsymbol{\gamma}) \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}}, \tag{6}$$

where the "Hamiltonian" $H(\boldsymbol{M}, \boldsymbol{\gamma})$ is an arbitrary function (quadratic and non-degenerate in $\boldsymbol{M}$) and $S(\boldsymbol{M}, \boldsymbol{\gamma})$ is a function linear in $\boldsymbol{M}$:

$$S = (\boldsymbol{K}(\boldsymbol{\gamma}), \boldsymbol{M}) = K_1(\boldsymbol{\gamma})M_1 + K_2(\boldsymbol{\gamma})M_2 + K_3(\boldsymbol{\gamma})M_3.$$

It can be proved by a straightforward verification that the system (6) always admits three integrals of motion:

$$F_1 = \boldsymbol{\gamma}^2, \qquad F_2 = (\boldsymbol{M}, \boldsymbol{\gamma}), \qquad F_3 = H(\boldsymbol{M}, \boldsymbol{\gamma}).$$

Without loss of generality we can set $\boldsymbol{\gamma}^2 = 1$, so that equations (5) govern the dynamical system on the family of four-dimensional manifolds

$$\mathcal{M}_c^4 = \{\boldsymbol{M}, \boldsymbol{\gamma} \mid \boldsymbol{\gamma}^2 = 1, (\boldsymbol{M}, \boldsymbol{\gamma}) = c\},$$

each of which is diffeomorphic to TS^2.

If the entire set of variables is denoted as $\boldsymbol{x} = (\boldsymbol{M}, \boldsymbol{\gamma})$, then equations (6) can be represented in the skew-symmetric form

$$\dot{\boldsymbol{x}} = \mathbf{P}_0 \frac{\partial H}{\partial \boldsymbol{x}},$$

$$\mathbf{P}_0 = \begin{pmatrix} \mathbf{M} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma} & 0 \end{pmatrix} - S(\boldsymbol{x}) \begin{pmatrix} \boldsymbol{\Gamma} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} 0 & -M_3 & M_2 \\ M_3 & 0 & -M_1 \\ -M_2 & M_1 & 0 \end{pmatrix}, \qquad \boldsymbol{\Gamma} = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}.$$

Here the first term is a standard Poisson structure corresponding to the Lie algebra $e(3)$. Moreover, $\mathbf{P}_0$ additionally satisfies the equations

$$\mathbf{P}_0 \frac{\partial F_1}{\partial \boldsymbol{x}} = 0, \qquad \mathbf{P}_0 \frac{\partial F_2}{\partial \boldsymbol{x}} = 0.$$

As above, assume that (6) admits an invariant measure with density depending only on $\boldsymbol{\gamma}$:

$$\mu = \rho(\boldsymbol{\gamma}) \, d\boldsymbol{M} \, d\boldsymbol{\gamma}. \tag{7}$$

In this case the Liouville equation for the vector field $\boldsymbol{V}(\boldsymbol{M}, \boldsymbol{\gamma})$ defined by the system (6) can be represented as

$$\operatorname{div} \rho \boldsymbol{V} = \left(\frac{\partial H}{\partial \boldsymbol{M}}, \rho \boldsymbol{\gamma} \times \boldsymbol{K} - \boldsymbol{\gamma} \times \frac{\partial \rho}{\partial \boldsymbol{\gamma}} \right) = 0.$$

Hence, owing to non-degeneracy of the Hamiltonian in $\boldsymbol{M}$, we obtain the vector equation

$$\left(\frac{1}{\rho} \frac{\partial \rho}{\partial \boldsymbol{\gamma}} - \boldsymbol{K} \right) \times \boldsymbol{\gamma} = 0. \tag{8}$$

Using this relation, we can prove by direct computation

Proposition 1 *If $\rho(\boldsymbol{\gamma})$ satisfies equation (8), then the tensor $\mathbf{P} = \frac{1}{\rho(\boldsymbol{\gamma})} \mathbf{P}_0$ satisfies the Jacobi identity and therefore is a Poisson structure on $\mathbb{R}^6(\boldsymbol{M}, \boldsymbol{\gamma})$.*

Thus, we finally obtain

Theorem 2 *If the system (6) admits an invariant measure (7) with density depending only on $\boldsymbol{\gamma}$, it can be represented in the conformally Hamiltonian form*

$$\dot{\boldsymbol{x}} = \rho(\boldsymbol{\gamma}) \mathbf{P}(\boldsymbol{x}) \frac{\partial H}{\partial \boldsymbol{x}},$$

where $\mathbf{P}(\boldsymbol{x}) = \rho^{-1} \mathbf{P}_0(\boldsymbol{x})$ is a Poisson structure of rank 4 with the Casimir functions

$$F_1 = \boldsymbol{\gamma}^2, \qquad F_2 = (\boldsymbol{M}, \boldsymbol{\gamma}).$$

Equation (8) can be solved for the vector $\boldsymbol{K}$ as follows:

$$\boldsymbol{K} = \rho f(\boldsymbol{\gamma})\boldsymbol{\gamma} + \frac{1}{\rho}\frac{\partial \rho}{\partial \boldsymbol{\gamma}},$$

where $f(\boldsymbol{\gamma})$ is an arbitrary function. Thus, we have naturally obtained a special class of Poisson structures on the space $\mathbb{R}^6(\boldsymbol{M}, \boldsymbol{\gamma})$, which can be written as

$$\mathbf{P} = \frac{1}{\rho}\begin{pmatrix} \mathbf{M} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma} & 0 \end{pmatrix} - \left(\frac{1}{\rho^2}\frac{\partial \rho}{\partial \boldsymbol{\gamma}} + f(\boldsymbol{\gamma})\boldsymbol{\gamma}, \boldsymbol{M}\right)\begin{pmatrix} \boldsymbol{\Gamma} & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Remark. Another approach to treating such Poisson structures was introduced in [1], [2], [17].

Remark. If we add a term of the form

$$\Phi(\boldsymbol{\gamma})\begin{pmatrix} \boldsymbol{\Gamma} & 0 \\ 0 & 0 \end{pmatrix}$$

to the bracket (9), where $\Phi(\boldsymbol{\gamma})$ is an arbitrary function, then the Jacobi identity will still hold. We use such a modification of (9) below, see (20).

We give two examples.

The problem of the Chaplygin ball on a plane [14] describing the rolling of a balanced dynamically asymmetric ball without slipping on a horizontal plane.

In appropriate variables the equations of motion can be represented in the form (6), see [11, 5, 6] with

$$H_1 = \frac{1}{2}\left((\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + \frac{(\mathbf{A}\boldsymbol{M}, \boldsymbol{\gamma})^2}{\mathcal{D}^{-1} - (\mathbf{A}\boldsymbol{\gamma}, \boldsymbol{\gamma})}\right) + U_1(\boldsymbol{\gamma}), \qquad S = \frac{(\mathbf{A}\boldsymbol{M}, \boldsymbol{\gamma})}{\mathcal{D}^{-1} - (\mathbf{A}\boldsymbol{\gamma}, \boldsymbol{\gamma})}, \tag{10}$$

where $\mathcal{D} = \text{const}$, $\mathbf{A}$ is a constant diagonal matrix (notice that the largest eigenvalue of the matrix $\mathbf{A}$ is smaller than $\mathcal{D}^{-1}$ so that the denominator $\mathcal{D}^{-1} - (\mathbf{A}\boldsymbol{\gamma}, \boldsymbol{\gamma})$ does not vanish). The ball's angular momentum $\boldsymbol{M}$ relative to the point of contact is expressed in terms of the physical variable $\boldsymbol{\omega}$, angular velocity, by the formula

$$\boldsymbol{M} = \mathbf{A}^{-1}\boldsymbol{\omega} - \mathcal{D}(\boldsymbol{\omega}, \boldsymbol{\gamma})\boldsymbol{\gamma}, \qquad \boldsymbol{\omega} = \mathbf{A}(\boldsymbol{M} + S\boldsymbol{\gamma}), \tag{11}$$

and the integral $(\boldsymbol{M}, \boldsymbol{\gamma})$ can take arbitrary values.

The density of the invariant measure (7) of the system and the function $f(\boldsymbol{\gamma})$ for the bracket (9) has the form

$$\rho = \frac{1}{\sqrt{\mathcal{D}^{-1} - (\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})}}, \qquad f(\boldsymbol{\gamma}) = 0.$$

The Veselova system [28, 27, 15] governing the dynamics of a body with a fixed point subject to the nonholonomic constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = b = \text{const}$, where $\boldsymbol{\omega}$ is the angular velocity of the ball and $\boldsymbol{\gamma}$ is a unit vector fixed in space.

In the body-fixed frame, the equations of motion can be represented in the form (6), see [9] with

$$H_2 = \frac{1}{2}\left((\boldsymbol{M}, \widehat{\mathbf{A}}\boldsymbol{M}) - \frac{((\widehat{\mathbf{A}} - \mathbf{E})\boldsymbol{M}, \boldsymbol{\gamma})^2}{(\widehat{\mathbf{A}}\boldsymbol{\gamma}, \boldsymbol{\gamma})}\right) + U_2(\boldsymbol{\gamma}), \qquad S = -\frac{((\widehat{\mathbf{A}} - \mathbf{E})\boldsymbol{M}, \boldsymbol{\gamma})}{(\widehat{\mathbf{A}}\boldsymbol{\gamma}, \boldsymbol{\gamma})}, \tag{12}$$

where $\widehat{\mathbf{A}} = \mathbf{I}^{-1}$ is the constant matrix inverse to the tensor of inertia, $\mathbf{E}$ is the identity matrix and the generalized angular momentum $\boldsymbol{M}$ is expressed in terms of the angular velocity $\boldsymbol{\omega}$ of the body as follows

$$\boldsymbol{M} = \widehat{\mathbf{A}}^{-1}\boldsymbol{\omega} - ((\widehat{\mathbf{A}}^{-1} - \mathbf{E})\boldsymbol{\omega}, \boldsymbol{\gamma})\boldsymbol{\gamma}, \qquad \boldsymbol{\omega} = \widehat{\mathbf{A}}(\boldsymbol{M} - S\boldsymbol{\gamma}), \tag{13}$$

where the area integral coincides with the constraint equation:

$$(\boldsymbol{M}, \boldsymbol{\gamma}) = (\boldsymbol{\omega}, \boldsymbol{\gamma}) = b.$$

The density of the invariant measure (7) and the function $f(\boldsymbol{\gamma})$ coincide in this case:

$$\rho(\boldsymbol{\gamma}) = f(\boldsymbol{\gamma}) = \frac{1}{\sqrt{(\boldsymbol{\gamma}, \widehat{\mathbf{A}}\boldsymbol{\gamma})}}.$$

A Lagrangian representation for $b = 0$ after a change of time was obtained in [16], the corresponding conformally Hamiltonian representation in [9], and another conformally Hamiltonian representation was found in [8]. Using another method, the Hamiltonian representation for $b \neq 0$ was found in [18].

If the potential $U(\boldsymbol{\gamma})$ for these systems is not zero, then, as a rule, the corresponding equations of motion turn out to be nonintegrable so that this Hamiltonisation method is essentially different from that used in [3], where the existence of a complete set of first integrals was required.

We also note that if one makes a change of the parameters and the potential in the Chaplygin ball problem:

$$\mathbf{A} = \mathcal{D}^{-1}(\mathbf{E} - \widehat{\mathbf{A}}), \qquad U_1(\boldsymbol{\gamma}) = \mathcal{D}^{-1}U_2(\boldsymbol{\gamma}),$$

then we find that the Hamiltonian (10) becomes

$$H_1 = \frac{\mathcal{D}^{-1}}{2}(\boldsymbol{M}, \boldsymbol{M}) - \mathcal{D}^{-1}H_2, \tag{14}$$

and the Poisson structure of the Chaplygin ball is transformed into a Poisson structure of the Veselova system. Consequently, these two systems are defined on the same Poisson manifold [26], and their Hamiltonians are related by (14). If $U_i(\boldsymbol{\gamma}) = 0$, $i = 1, 2$, then the function $F = \boldsymbol{M}^2$ is an integral for the both systems, which implies that their trajectories turn out to be rectilinear windings (transverse to each other) on the same invariant tori [15].

3. Reduction to the $e(3)$-bracket

Introducing new notation $g = \rho^{-1}$, we can rewrite the Poisson structure (9) in a shorter form that is more convenient for further analysis

$$\mathbf{P} = g \begin{pmatrix} \mathbf{M} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma} & 0 \end{pmatrix} + \left(\frac{\partial g}{\partial \boldsymbol{\gamma}} - f(\boldsymbol{\gamma})\boldsymbol{\gamma}, \boldsymbol{M}\right) \begin{pmatrix} \boldsymbol{\Gamma} & 0 \\ 0 & 0 \end{pmatrix}. \tag{15}$$

Let us examine the family of such Poisson structures in more detail. First of all, we see that this family is parametrised by two arbitrary functions $g(\boldsymbol{\gamma}) > 0$ and $f(\boldsymbol{\gamma})$

and we will denote the corresponding Poisson structures by $\mathbf{P}_{g,f}$. Notice that all $\mathbf{P}_{g,f}$ possess the same Casimir functions $(\boldsymbol{M}, \boldsymbol{\gamma})$ and $(\boldsymbol{\gamma}, \boldsymbol{\gamma})$.

For simplicity we confine our attention to the physical case $\boldsymbol{\gamma}^2 = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1$, that is, we restrict all the objects to the five-dimensional (Poisson) manifold $S^2(\boldsymbol{\gamma}) \times \mathbb{R}^3(\boldsymbol{M})$.

One of our goals is to find out to what canonical form these Poisson structures can be reduced. First of all, we note that the symplectic leaves of $\mathbf{P}_{g,f}$ are all diffeomorphic to the cotangent bundle to the sphere T^*S^2. From the explicit form (15) of the Poisson structure it may be inferred that the symplectic structure on each leaf T^*S^2 will be the sum of the canonical form $dp \wedge dq$ and a magnetic term, that is, a closed 2-form ω_{magn} on the sphere. By the Moser theorem [24], such forms ω_{magn} are parametrised up to a symplectomorphism by one single number, namely $\displaystyle\int_{S^2} \omega_{\mathrm{magn}}$. Thus, for each Poisson structure we have a one-parameter family of symplectic leaves whose type is also defined by exactly one parameter. This observation leads us to the conjecture that by "redistributing", if necessary, the symplectic leaves and then by applying a certain symplectomorphism to each single symplectic leaf, we can transform any Poisson structure $\mathbf{P}_{g,f}$ to any other $\mathbf{P}_{\widetilde{g},\widetilde{f}}$.

Remark. On the zero level $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$, the Poisson structure (15) is reduced to the canonical $e(3)$-bracket by a very simple transformation [8]:

$$(\boldsymbol{M}, \boldsymbol{\gamma}) \mapsto (g^{-1}(\boldsymbol{\gamma})\boldsymbol{M}, \boldsymbol{\gamma}).$$

Thus, for the Chaplygin ball we have:

$$(\boldsymbol{M}, \boldsymbol{\gamma}) \mapsto \left((\mathcal{D}^{-1} - (\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma}))^{-1/2}\boldsymbol{M}, \boldsymbol{\gamma}\right),$$

and for the Veselova system:

$$(\boldsymbol{M}, \boldsymbol{\gamma}) \mapsto ((\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})^{-1/2}\boldsymbol{M}, \boldsymbol{\gamma}).$$

We start by describing a class of natural transformations which preserve the form of $\mathbf{P}_{g,f}$, but change the parameters g and f. Consider the transformations of the form

$$(\boldsymbol{M}, \boldsymbol{\gamma}) \mapsto (\widetilde{\boldsymbol{M}}, \boldsymbol{\gamma}), \qquad \widetilde{\boldsymbol{M}} = \mathbf{A}(\boldsymbol{\gamma})\boldsymbol{M}, \tag{16}$$

where $\mathbf{A}(\boldsymbol{\gamma})$ is a linear operator in $\mathbb{R}^3$ whose components depend on $\boldsymbol{\gamma}$.

Proposition 2 *For each point $\boldsymbol{\gamma} \in S^2$, consider the orthogonal decomposition $\boldsymbol{M} = \boldsymbol{M}' + \boldsymbol{M}''$, where $\boldsymbol{M}'' = (\boldsymbol{M}, \boldsymbol{\gamma})\boldsymbol{\gamma}$ is the projection of $\boldsymbol{M}$ onto the vector $\boldsymbol{\gamma}$, and $\boldsymbol{M}' = \boldsymbol{M} - \boldsymbol{M}''$ is the projection of $\boldsymbol{M}$ onto the plane perpendicular to $\boldsymbol{\gamma}$, i.e., the tangent plane $T_\gamma S^2$. Let*

$$\widetilde{\boldsymbol{M}} = \alpha(\boldsymbol{\gamma})\boldsymbol{M}' + c\boldsymbol{M}'' + \boldsymbol{M}'' \times \boldsymbol{h}(\boldsymbol{\gamma}),$$

where $c \neq 0$ is a constant, $\alpha(\boldsymbol{\gamma}) > 0$ is an arbitrary scalar function, and $\boldsymbol{h}(\boldsymbol{\gamma})$ is an arbitrary vector function of $\boldsymbol{\gamma}$. Then the transformation (16) sends $\mathbf{P}_{g,f}$ to a Poisson structure $\mathbf{P}_{\widetilde{g},\widetilde{f}}$ of the same kind with parameters

$$\widetilde{g} = \alpha g, \qquad \widetilde{f} = \frac{\alpha^2}{c}f + \left(\frac{\alpha}{c} - 1\right)\left(\widetilde{g} - \left(\boldsymbol{\gamma}, \frac{\partial \widetilde{g}}{\partial \boldsymbol{\gamma}}\right)\right) + \frac{1}{c}\left(\boldsymbol{\gamma}, \widetilde{g}\frac{\partial \alpha}{\partial \boldsymbol{\gamma}} + \widetilde{g}^2 \operatorname{curl}\left(\frac{\boldsymbol{h}}{\widetilde{g}}\right)\right). \tag{17}$$

The proof of Proposition 2 is a straightforward verification and we confine ourselves to commenting on the geometric meaning of the transformation $\boldsymbol{M} \mapsto \widetilde{\boldsymbol{M}} = \mathbf{A}(\boldsymbol{\gamma})\boldsymbol{M}$ used in this proposition. Consider the orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ related to the vector $\boldsymbol{\gamma}$ in the space $\mathbb{R}^3(\boldsymbol{M})$. Namely, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two orthonormal vectors lying in a tangent plane to the unit sphere at point $\boldsymbol{\gamma}$, and $\boldsymbol{e}_3$ is the normal vector to this sphere at the same point, i.e., $e_3 = \boldsymbol{\gamma}$. In this basis the matrix of $\mathbf{A} = \mathbf{A}(\boldsymbol{\gamma})$ has the form

$$\mathbf{A} = \begin{pmatrix} \alpha & 0 & a \\ 0 & \alpha & b \\ 0 & 0 & c \end{pmatrix}$$

where α, a and b depend on $\boldsymbol{\gamma}$, and c is constant.

This is exactly the general form of the transformation $\mathbf{A}$ which satisfies our requirements. Indeed, the Casimir function $(\boldsymbol{M}, \boldsymbol{\gamma})$ should be mapped to itself with possible multiplication by some constant c. Therefore, the plane defined by the equation $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ is sent to itself, and in the orthogonal direction the transformation is a dilation with factor c independent of $\boldsymbol{\gamma}$. These conditions completely define the last row of the matrix $\mathbf{A}$.

Furthermore, the relations $\{M_i, \gamma_j\} = -g\,\varepsilon_{ijk}\gamma_k$ can be rewritten in the following vector form $\{c_1 M_1 + c_2 M_2 + c_3 M_3, \boldsymbol{\gamma}\} = -g\,\mathbf{c} \times \boldsymbol{\gamma}$, where $\mathbf{c} = (c_1, c_2, c_3)$. Since their form must remain the same, we obtain the condition

$$g\,(\mathbf{A}^\top(\boldsymbol{\gamma})\mathbf{c}) \times \boldsymbol{\gamma} = \tilde{g}\,\mathbf{c} \times \boldsymbol{\gamma}$$

for any $\mathbf{c} \in \mathbb{R}^3$.

This means that on the tangent plane $T_\gamma S^2$ the operator $\mathbf{A}$ must act as multiplication by some number α (depending on $\boldsymbol{\gamma}$). There are no restrictions on the elements a and b, they are given by the vector function $\boldsymbol{h}$ (this function itself has 3 components, but only two of them are significant, since nothing is changed by adding to $\boldsymbol{h}$ any vector proportional to $\boldsymbol{\gamma}$).

Notice that the set of transformations described in Proposition 2 forms a group (which is, of course, infinite-dimensional, since its parameters contain arbitrary functions α and $\boldsymbol{h}$). It is easily verified that performing successively two transformations with parameters $(\alpha_1, c_1, \boldsymbol{h}_1)$ and $(\alpha_2, c_2, \boldsymbol{h}_2)$ is equivalent to the transformation with parameters $(\alpha_1\alpha_2, c_1 c_2, \boldsymbol{h}_1\alpha_2 + \boldsymbol{h}_2 c_1)$. The above-mentioned rule specifies a group binary operation, which simply copies the matrix multiplication:

$$\begin{pmatrix} \alpha_2 & \boldsymbol{h}_2 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \boldsymbol{h}_1 \\ 0 & c_1 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 & \boldsymbol{h}_1\alpha_2 + \boldsymbol{h}_2 c_1 \\ 0 & c_1 c_2 \end{pmatrix}$$

This group acts in a natural way on the family of Poisson structures $\{\mathbf{P}_{g,f}\}$ or, which is the same, on the space of parameters g, f. The above relations (17) can be understood as explicit formulae for this action. If the action is formally denoted by $(\tilde{g}, \tilde{f}) = \Psi_{(\alpha, c, \boldsymbol{h})}(g, f)$, then, as is easily verified by successively performing two transformations, it satisfies the standard action rule. Namely, if

$$(\widetilde{g}, \widetilde{f}) = \Psi_{(\alpha_1, c_1, \boldsymbol{h}_1)}(g, f) \qquad \text{and} \qquad (\widetilde{\widetilde{g}}, \widetilde{\widetilde{f}}) = \Psi_{(\alpha_2, c_2, \boldsymbol{h}_2)}(\widetilde{g}, \widetilde{f}),$$

then

$$(\widetilde{g}, \widetilde{f}) = \Psi_{(\alpha_1\alpha_2, c_1c_2, \boldsymbol{h}_1\alpha_2 + \boldsymbol{h}_2 c_1)}(g, f).$$

For an explicit verification of this fact it is convenient to rewrite (17) as

$$\widetilde{g} = \alpha g, \qquad \widetilde{f} = \frac{\alpha^2}{c}\left(f + g - \left(\boldsymbol{\gamma}, \frac{\partial g}{\partial \boldsymbol{\gamma}}\right)\right) - \left(\widetilde{g} - \left(\boldsymbol{\gamma}, \frac{\partial \widetilde{g}}{\partial \boldsymbol{\gamma}}\right)\right) + \frac{\widetilde{g}^2}{c}\left(\boldsymbol{\gamma}, \operatorname{curl}\left(\frac{\boldsymbol{h}}{\widetilde{g}}\right)\right).$$

Now the verification presents no difficulty.

From the viewpoint of group theory it would now be natural to ask the question: *what are the orbits of this action?* In other words, we want to understand which Poisson structures may be transferred to each other by the above-mentioned transformations. The answer turns out to be very simple: *the action described above has one single orbit*, i.e., all Poisson structures in this family are equivalent to each other. In particular, the following theorem holds:

Theorem 3 *Every Poisson structure* $\mathbf{P}_{g,f}$ *of the form (15) on the level* $\boldsymbol{\gamma}^2 = 1$ *is isomorphic to the standard Lie-Poisson structure* $\mathbf{P}_{1,0}$ *related to the Lie algebra* $e(3)$.

Proof. It is sufficient to choose parameters $(\alpha, c, \boldsymbol{h})$ in (17) in such a way that $\widetilde{g} = 1$ and $\widetilde{f} = 0$. The first condition immediately defines the function α, namely, $\alpha = g^{-1}$. After that the second condition reduces to

$$\frac{\alpha^2}{c}f + \left(\frac{\alpha}{c} - 1\right) + \frac{1}{c}\left(\boldsymbol{\gamma}, \frac{\partial \alpha}{\partial \boldsymbol{\gamma}}\right) + \frac{1}{c}(\boldsymbol{\gamma}, \operatorname{curl} \boldsymbol{h}) = 0$$

or, equivalently,

$$\alpha^2 f + \alpha + \left(\boldsymbol{\gamma}, \frac{\partial \alpha}{\partial \boldsymbol{\gamma}}\right) - c + (\boldsymbol{\gamma}, \operatorname{curl} \boldsymbol{h}) = 0,$$

where the constant c and the vector function $\boldsymbol{h}$ are the unknowns. This equation can now be rewritten as

$$(\boldsymbol{\gamma}, \operatorname{curl} \boldsymbol{h}) = F(\boldsymbol{\gamma}) + c, \tag{18}$$

where $F(\boldsymbol{\gamma})$ is a given function. Notice that (18) has to be fulfilled only on the unit sphere $\boldsymbol{\gamma}^2 = 1$. The conditions for solving the equations of this form are well known. In the differential-geometric sense this equation simply means that we are looking for an antiderivative of the 2-form $(F + c)\, d\sigma$ on the unit sphere, where $d\sigma$ is the standard area form. Such a 1-form can be found if and only if $\int_{S^2}(F + c)\, d\sigma = 0$. This condition can always be achieved by choosing a constant c. $\blacksquare$

Remark. In a similar manner, the bracket $\mathbf{P}_{g,f}$ can be reduced to the standard form on the whole space $\mathbb{R}^6(\boldsymbol{M}, \boldsymbol{\gamma}) \simeq e^*(3)$, i.e., without the additional restriction $\boldsymbol{\gamma}^2 = 1$. To that end, we have to extend the class of transformations by assuming that c depends on $\boldsymbol{\gamma}^2$. Since $\boldsymbol{\gamma}^2$ is a Casimir function, $c(\boldsymbol{\gamma}^2)$ may be treated, as before, as a constant and hence the formulae do not essentially change. The conditions for solvability of the equation $(\boldsymbol{\gamma}, \operatorname{curl} \boldsymbol{h}) = F(\boldsymbol{\gamma}) + c(\boldsymbol{\gamma}^2)$ remain the same, but now they have to be verified on the spheres of all radii. As before, we are able to ensure that they

are satisfied, since the necessary constants can now be chosen depending on the square γ^2 of the radius $|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}$.

For the Chaplygin ball problem, the function $F(\gamma)$ in Eq. (18) has the form

$$F(\gamma) = -\frac{\mathcal{D}^{-1}}{\left(\mathcal{D}^{-1} - (\gamma, \mathbf{A}\gamma)\right)^{3/2}}.$$

The solutions of Eq. (18) for the unknowns c and $\boldsymbol{h}$ can be expressed in this case in terms of complete and incomplete elliptic integrals. Thus, although theoretically it is not difficult to prove reducibility of $\mathbf{P}_{g,f}$ to the $e(3)$-bracket, in practice the resulting transformation can be transcendental.

4. Generalisation to the case of a gyrostat

In this section we consider the dynamical systems obtained by adding a rotor with constant gyroscopic momentum $\boldsymbol{k}$ to the Chaplygin ball and a rigid body in the Veselova problem. A detailed derivation of the equations of motion for compound bodies can be found in books [7, 29, 21].

The new equations with gyrostatic terms take the following form

$$\dot{\boldsymbol{M}} = (\boldsymbol{M} + \boldsymbol{k} - S\gamma) \times \frac{\partial H}{\partial \boldsymbol{M}} + \gamma \times \frac{\partial H}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \frac{\partial H}{\partial \boldsymbol{M}}, \tag{19}$$

where the new "Hamiltonian" H and function S may now depend on the gyrostatic momentum $\boldsymbol{k}$ as a parameter, but S preserves its original structure as in Section 2. In particular,

$$S = \frac{1}{g}\left(-\frac{\partial g}{\partial \gamma} + f(\gamma)\gamma, \boldsymbol{M}\right) + \frac{1}{g}\Phi(\gamma) \tag{20}$$

for some smooth functions $g(\gamma)$, $f(\gamma)$ and $\Phi(\gamma)$.

A direct calculation shows that this system remains conformally Hamiltonian. Namely, (19) can be rewritten as

$$\dot{\boldsymbol{x}} = g^{-1}\mathbf{P}_{\boldsymbol{k}}(\boldsymbol{x})\frac{\partial H}{\partial \boldsymbol{x}},$$

with the Poisson structure $\mathbf{P}_{\boldsymbol{k}}$ of a more general form

$$\mathbf{P}_{\boldsymbol{k}}(\boldsymbol{x}) = g\begin{pmatrix} \mathbf{M}_{\boldsymbol{k}} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma} & 0 \end{pmatrix} - gS\begin{pmatrix} \boldsymbol{\Gamma} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{M}_{\boldsymbol{k}} = \begin{pmatrix} 0 & -M_3 - k_3 & M_2 + k_2 \\ M_3 + k_3 & 0 & -M_1 - k_1 \\ -M_2 - k_2 & M_1 + k_1 & 0 \end{pmatrix}, \tag{21}$$

where $\boldsymbol{x} = (\boldsymbol{M}, \gamma)$ is a complete set of variables.

The Jacobi identity for $\mathbf{P}_{\boldsymbol{k}}$ is fulfilled, and the Casimir functions are

$$F_1 = \gamma^2, \qquad F_2 = (\boldsymbol{M} + \boldsymbol{k}, \gamma).$$

The new expressions for H and S presented below can be obtained by using the methods developed in [7, 29, 21]. We omit this computation.

For the *Chaplygin ball*, the vector $\boldsymbol{M}$ is still expressed in terms of the angular velocity $\boldsymbol{\omega}$ by means of (11), and the Hamiltonian (10) also remains the same. For the bracket (21) we set

$$g = \sqrt{\mathcal{D}^{-1} - (\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})}, \qquad f(\boldsymbol{\gamma}) = 0, \qquad \Phi(\boldsymbol{\gamma}) = 0.$$

In other words, all the ingredients remain unchanged except for the additional terms involving $\boldsymbol{k}$ in the bracket (21). Thus, to obtain the gyrostatic generalisation of the Chaplygin ball we simply need to replace $\mathbf{M}$ by $\mathbf{M_k}$ in (15).

For the *Veselova system*, when a gyrostat is added, the situation becomes less trivial and the relations (13) as well as H_2 and S given by (12) need to be modified. As before, we shall assume that $\boldsymbol{M} = \widehat{\mathbf{A}}^{-1}\boldsymbol{\omega} + \lambda\boldsymbol{\gamma}$, where the coefficient λ can be found from the condition

$$(\boldsymbol{M} + \boldsymbol{k}, \boldsymbol{\gamma}) = (\boldsymbol{\omega}, \boldsymbol{\gamma}).$$

We obtain

$$\boldsymbol{M} = \widehat{\mathbf{A}}^{-1}\boldsymbol{\omega} - ((\widehat{\mathbf{A}}^{-1} - \mathbf{E})\boldsymbol{\omega} + \boldsymbol{k}, \boldsymbol{\gamma})\boldsymbol{\gamma}, \qquad \boldsymbol{\omega} = \widehat{\mathbf{A}}(\boldsymbol{M} - S\boldsymbol{\gamma}),$$

$$S = \frac{(\widehat{\mathbf{A}}\boldsymbol{M} - \boldsymbol{M} - \boldsymbol{k}, \boldsymbol{\gamma})}{(\widehat{\mathbf{A}}\boldsymbol{\gamma}, \boldsymbol{\gamma})}.$$

Here S coincides with the corresponding function in the bracket (21) provided that g is given as

$$g = \sqrt{(\widehat{\mathbf{A}}\boldsymbol{\gamma}, \boldsymbol{\gamma})}.$$

In this case the Hamiltonian reads

$$H = \frac{1}{2}\Big((\widehat{\mathbf{A}}\boldsymbol{M}, \boldsymbol{M}) + \frac{(\widehat{\mathbf{A}}\boldsymbol{M} - \boldsymbol{M} - \boldsymbol{k}, \boldsymbol{\gamma})^2}{(\widehat{\mathbf{A}}\boldsymbol{\gamma}, \boldsymbol{\gamma})}\Big).$$

It turns out that this modified Veselova system with gyrostatic terms still admits one additional integral of the form

$$F_3 = (\boldsymbol{M} + \boldsymbol{k}, \boldsymbol{M} + \boldsymbol{k}).$$

Thus, this new system is conformally Hamiltonian and integrable. Its dynamics can be further analysed by the standard methods.

Conclusion and discussion

We have obtained an invariant (independent of the choice of local coordinates on S^2) conformally Hamiltonian representation of generalised Chaplygin systems on T^*S^2 using a degenerate Poisson structure of rank 4 in the six-dimensional space $\mathbb{R}^6(\boldsymbol{M}, \boldsymbol{\gamma})$ and have shown that this structure is a deformation of the standard Lie–Poisson bracket in $\mathbb{R}^6(\boldsymbol{M}, \boldsymbol{\gamma})$ corresponding to the Lie algebra $e(3)$.

As applications, we have considered two nonholonomic systems: the Chaplygin ball and Veselova problem. In this approach (after a suitable change of parameters) they turn out to be integrable conformally Hamiltonian systems on the same Poisson manifold with the same set of first integrals. The above conformally Hamiltonian representation has been generalised to the case of adding a gyrostat (although in this case there is no analogy between these systems any more).

The conformally Hamiltonian description obtained in this paper for the Veselova system with $(\boldsymbol{\omega}, \boldsymbol{\gamma}) \neq 0$ is the same (up to an appropriate change of variables) as that presented previously in [6], but the integrability of its gyrostatic generalisation was unknown before and is presented in this paper, to the best of our knowledge, for the first time.

This paper poses a number of questions related primarily to nonholonomic systems.

1. Can the above approach be used to obtain a conformally Hamiltonian description for an integrable generalisation of the Chaplygin ball rolling on a spherical base (BMF-system) found in [4, 10, 26]?

2. Poisson brackets of a quite similar type are encountered in examples but with a Casimir function linear in $\boldsymbol{M}$ different from $(\boldsymbol{M}, \boldsymbol{\gamma})$ [3]. It would be interesting to find out whether such brackets can be reduced to the standard Poisson-Lie bracket on $e^*(3)$ using the technique described above.

3. Since the Chaplygin ball problem without potential (i.e., $U(\boldsymbol{\gamma}) = 0$) is integrable on the whole space $\mathbb{R}^6(\boldsymbol{M}, \boldsymbol{\gamma})$, Theorem 3 allows us to obtain a globally integrable Hamiltonian system on $e^*(3)$, i.e., for all values of the area constant $(\boldsymbol{M}, \boldsymbol{\gamma})$. As is well known, this circumstance may be interpreted as integrability of a natural system with a magnetic field whose additional integral is quadratic in momenta. The issue of description of all such systems was actively discussed in the literature. It would be interesting to interpret the system thus obtained in the context of recent classification results by V. Marikhin and V. Sokolov [22, 23].

Acknowledgments

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