Chapter 18:
Developing mathematics teaching:
what can we learn from the literature?

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Abstract

In this chapter we address the extensive literature which can inform the teaching of mathematics drawing on our own experience of using and finding value in the literature to enhance our own knowledge and practice in teaching mathematics at university level. Three areas of literature are recognised and addressed: *professional literature*, in which we gain insights into the ways in which other teachers/lecturers have thought about their teaching and the approaches/strategies and frameworks they have used; *research literature* which offers *what is known*, findings from research that can enable more informed approaches to teaching; and *pedagogical literature* that deals overtly with developing and enhancing teaching, through the lecturer engaging with new ideas (for example those offered in the professional literature), attending to research findings or using specific tactics or teaching approaches recommended by the authors. Many examples are provided, both of particular sources in the literature and of approaches to learning and teaching in mathematics, with an extensive reference list. This approach is intended to be informative to mathematics teachers at university level who look for knowledge and ideas to inform their teaching and support its development.

Introduction

There is a wealth of literature focusing upon learning and teaching in mathematics. Much of it focuses at school level, although there is a growing literature base focused upon higher education. Some of the latter has resulted from research which has been conducted into a range of issues in learning and teaching in areas of mathematics at higher levels such as in calculus, linear algebra, or abstract algebra. There are more studies focusing on issues in students' learning than on issues in teaching. Research studies are conducted to address

clearly defined research questions and according to recognised methodological principles. We refer to this as *'the research literature'*. In other cases we see the reflections of people who are teaching at this level, often mathematicians, who write to communicate their thinking about teaching and their students' learning, and to share aspects of their practice. We refer to this as *'the professional literature'*. Both kinds of literature are informative to people teaching at this level who are interested in developing their practice.

There is a third kind of literature which we refer to as *'the pedagogical literature'* in mathematics. This refers to writing in which the focus is on *approaches* to mathematics teaching and the learning of students, with some analyses of the ways in which approaches can contribute to effective teaching and learning. The writers synthesise from the research and professional literature to offer a commentary on teaching and learning in mathematics and on ways of developing teaching. This book in its entirety is an example of such literature.

In this chapter, we focus on these kinds of literature and the ways in which they can be used to help practitioners – those who work with students and teach mathematics at higher levels – to become more informed about learning and teaching and to develop their practice. We are ourselves teachers of mathematics at this level and are interested in developing our own practice. We have also conducted research into mathematics teaching in a university. As researchers we have needed to read what others have written in both the professional and research literature and to base our research on what is already known. Here, we draw on this experience. We refer to research in which we have studied the teaching of linear algebra, and therefore provide examples of the literature we have read relating to linear algebra learning and teaching, and ways in which it can inform practice in the teaching of linear algebra. We also look beyond linear algebra to literature relating to other areas of mathematics and to the pedagogical literature which discusses the learning and teaching of mathematics and its development.

The issues we are addressing

Why should we want to consult the literature anyway? Why might it be important?

In 1982, the authors of the Cockcroft report into the teaching of mathematics in schools claimed *"Mathematics is a difficult subject both to teach and to learn"* (Cockcroft, 1982:67). They reasoned that mathematics is a *hierarchical subject* and that *"ability to proceed to new work is very often dependent on a sufficient understanding of one or more pieces of work which have gone before"*. However, learning mathematics is important because *"mathematics provides a means of communication which is powerful, concise and unambiguous"* (Cockcroft, 1982:1). We see these quotations to be just as relevant to the learning and teaching of mathematics in higher education as they are to school mathematics. In 2000, Savage and Hawkes addressed *"The Mathematics Problem"* relating to mathematics in higher education. Their report states:

> *"Evidence is presented of a serious decline in students' mastery of basic mathematical skills and level of preparation for mathematics-based degree courses. This decline is well established and affects students at all levels. As a result, acute problems now confront those teaching mathematics and mathematics-based modules across the full range of universities."*
>
> Savage & Hawkes (2000:ii)

Perkin, Bamforth and Robinson (2010:1) acknowledge that *"Since the 1990's there has been well documented concern and evidence regarding the declining mathematical ability of students entering numerate courses at university, see: LMS, IMA and RSS (1995), Sutherland and Pozzi (1995) and, Savage, Kitchen, and Sutherland et. al., (2000)"*. More recent evidence shows that students entering university with A grades at A-level achieve these grades through a highly instrumentalised learning and teaching approach in which the questions addressed in examinations can be tackled through memory and rote learning with little demand for problem solving and higher level thinking skills (Minards, 2012). Thus, those teaching mathematics in higher education have to work with students who are ill-prepared for university-level study.

So, how do we approach teaching mathematics for such students? In what ways do teaching approaches take into account the prior experiences and levels of preparedness of the students to whom they are addressed? University lecturers acknowledge, and research shows, that traditional lecturing based on mathematical abstraction and formalism causes severe difficulty and alienates some students who experience it (Alsina, 2001; Artique, Batanero & Kent 2007; Solomon, 2007; Nardi, 2008). Yet, the traditional lecture is the most common approach to teaching mathematics in higher education as new lecturers typically teach the way they were taught (Felder, 1993; and see also Chapter 5).

In many universities now, new lecturers are inducted to university teaching via some pedagogical course taught by a university teaching centre. Often the pedagogy of such a course is rather general since it applies to lecturers in all subjects. We know mathematics lecturers who disdain general pedagogy because they cannot see how it applies to teaching mathematics. The mathematics discipline requires students to learn the nature of mathematics with its characteristics of abstraction and formalism and the centrality of proof, however, it is not clear what are the best ways of supporting this learning. Often lecturers cannot see beyond the formal traditional ways – the ways they were taught themselves – of presenting their subject in a lecture theatre. Those who are keen to find alternative ways of teaching their students sometimes look for advice from the literature. We were fortunate enough to work with a mathematician who was in this position.

Mathematician *M*, in the early years of his teaching career was asked to teach the first semester of a two-semester module in linear algebra to a first year cohort of (200+) mathematics students. In his first delivery of the module, he followed very closely the way the module had been taught by his predecessor, but was very dissatisfied with the way it went and the outcomes for his students. He had read an article in the professional literature (Uhlig, 2003) which inspired him to redesign his teaching in order to achieve better outcomes for his students. At this point we asked if we could work with him to study his teaching approach, and he accepted.

A research study of the teaching of linear algebra

The research was qualitative, interpretative and developmental. We were outsider researchers and *M* was an insider researcher exploring his own practice and learning from his inquiry. We observed (and audio-recorded) all of *M*'s teaching sessions (lectures and tutorials) and recorded extensive conversations with him before, after and between teaching

sessions. We collected and analysed data formally and rigorously, and developed theory (see Jaworski, Treffert-Thomas & Bartsch, 2009; Thomas, 2012). *M* reflected on his teaching in conversation with us and considered modifications to his teaching as a consequence of his reflection. During this process, we recommended some articles from the literature that could be useful to him in reflecting upon and (re)designing his teaching (for example Dorier, 1998, on the role of formalism in the teaching of linear algebra). On several occasions he read and later commented on these articles in our research meetings.

As a normal part of a research study we explored the relevant literature in both research and professional domains. We discuss here, briefly, key items from the literature and explain the ways in which they informed our study of the teaching and learning of linear algebra.

There were some recurring themes within the body of the literature that we looked at. Some articles focused on the content matter that students ought to learn and hence lecturers needed to teach (Carlson, 1993). Some authors discussed students' difficulties with linear algebra (Dorier, 2000) and what could be done to address these (Hillel, 2000). Some authors presented ideas for developing resources, in particular incorporating technological tools and devices in teaching (Sierpinska, 2005; Berry, Lapp & Nyman, 2008), while others discussed different approaches to teaching (Dorier, Robert, Robinet & Rogalski, 2000a; Uhlig, 2002, 2003) including more student-centred approaches (Uhl, 1999).

We discuss each of these aspects in more detail. Since these were situated in the subject area of linear algebra, this discussion is by nature subject specific to linear algebra. However, we hope that the reader can see that this discussion has relevance in other areas of mathematics and might adapt our approach to research in their own field.

Content

When teaching any area of mathematics lecturers will consider what their students will bring into the classroom by way of previous knowledge. Any changes in the post-16 school curriculum will therefore impact directly on a first year course at university. Since linear algebra is not usually taught at pre-university level (at least not in the UK where this study took place) the subject matter should be new to all students. Having said this there are some aspects such as matrix operations, calculating determinants and eigenvalues and eigenvectors that some students in the UK may have studied at school or sixth form college when taking an additional A-level course in mathematics. At our university this additional knowledge was not assumed and hence linear algebra was designed as a course to be taught 'from scratch' by lecturer *M*. It is fair to say that most mathematics undergraduate courses will offer new content for students as well as new ways of looking at and working with mathematics.

There are many examples in the literature where authors discuss the subject matter to be taught in a first year linear algebra course. Our study of the literature included an early publication by the LACSG, the Linear Algebra Curriculum Study Group, set in a US context. The group drew together the concerns raised over students' ability to cope with an undergraduate course in linear algebra as a result of changes in the school curriculum (Carlson, Johnson, Lay & Porter, 1993). The group suggested a core syllabus for a linear algebra course that consisted of matrix algebra, the solution of linear equation systems, determinants, and eigenvalues and eigenvectors, as well as the fundamental concepts of

subspace, basis, span, linear dependence, and so on, in the context of the vector space $\mathbb{R}^n$. They also stated explicitly that students should be introduced to problem-solving situations.

This list represented what mathematician M himself considered important. Furthermore this list of topics is shared by many authors in the literature and appears as the content page in many textbooks on linear algebra (for example Poole, 2006). Thus we concluded that linear algebra was one of those topics that had a universally accepted character in terms of content. This is not necessarily the case in other areas of mathematics.

The main disagreement in the literature, if we can call it that, surrounded the context in which linear algebra concepts were to be first introduced. Some authors preferred the most general and abstract vector spaces while others preferred to reduce the context to the vector space $\mathbb{R}^3$ only. The reasons lay with the target audience: engineering students, for example, might not require the more abstract theory so that teaching it may not only be unnecessary but also hinder students' progression and understanding (Stewart & Thomas, 2009).

One important source of information when first teaching an undergraduate course at university are literature reviews. These are articles that synthesise research in an area of mathematics teaching and learning, and not only present research outcomes and conclusions but invariably contain information regarding content (for example Britton & Henderson, 2009, for linear algebra).

Students' difficulties

When considering students' learning of linear algebra almost all authors agreed that students find linear algebra a difficult subject to study at university (Carlson, 1993; Hillel & Sierpinska, 1994; Dorier, Robert, Robinet & Rogalski, 2000b; Berry et al., 2008). Authors also agreed on the sources and the kind of difficulties students experience with linear algebra. This may not be the case with all mathematical topics but the difficulties mentioned were identical to the ones that mathematician M identified in his own students.

Students typically are not accustomed to working with a theoretical framework that is as highly conceptual in nature as linear algebra. The body of formal and interrelated knowledge requires abstract ways of thinking and behaving that M's students had not met before they arrived at university. It is the first time that students must learn about axioms and algebraic structures such as vector spaces. It does not fit well with students' previous experience of mathematics at school where knowledge of computations and procedures often sufficed to solve the problems presented. It is also a time when students are required to learn a whole new language to express the ideas in linear algebra, *"comparable with learning a foreign language"* (Artique, Chartier and Dorier, 2000:256).

Dorier (2000) presented a comprehensive account of students' cognitive difficulties with linear algebra and included a historical overview of the development of the subject. Linear algebra is a tool and a body of knowledge that arose from considering problems of linearity. Dorier argued that students encountered the same difficulties that mathematicians had in creating linear algebra as a unifying theory. He argued that what took mathematicians years to develop, students were expected to accomplish in the period of a few months, but the intellectual leaps required were the same.

Sometimes lecturers notice in their students the kinds of difficulties described above. Sometimes the brevity of lectures and the brevity of contact make it difficult to get to know students. Literature can make the lecturer or teacher of mathematics aware of the kinds of difficulties that can arise so that he or she may be more prepared and more able to respond in a manner that helps students to move forward.

Mathematician *M* noticed these difficulties with students that he taught in a small group (usually a maximum of six to eight students), a particular feature of the first year structure at our university. Here he noticed some severe difficulties that students seemed to have with linear algebra. He concluded that his teaching approach, a deductive style, stating definitions and theorems first, followed by a proof, hindered students' progress and decided to change his style.

He discussed this with us, the researchers, in our weekly meetings. As time went by he became well versed in the literature into students' difficulties with linear algebra and used this knowledge to good effect in lectures, for example spending more time discussing carefully chosen examples, both with individuals and collectively.

How to overcome students' difficulties

There were many examples in the literature that suggested ways of addressing students' difficulties with linear algebra.

Dorier and Sierpinska (2001) commented on lecturers' use of multiple representations such as geometric explanations of subspace alongside algebraic understandings. They argued, as did Hillel (2000), that students were not able to follow the switching between representations, often losing the thread of the explanation altogether. They suggested that lecturers should make their students aware when switching between representations.

Geometric reasoning and visualisation in linear algebra were at the heart of many articles in the literature. This is hardly surprising as linear algebra and geometry can be seen as two sides of the same coin. However, research with students found that geometric insights can both help and hinder students' developing understanding (Gueudet-Chartier, 2004). If studying linear algebra involved a context up to $\mathbb{R}^3$ only, then students benefited greatly from geometric insights (Stewart & Thomas, 2009). When abstraction beyond $\mathbb{R}^3$ became necessary, geometry and visualisations hindered weaker students who took the geometric language of linear algebra too literally (Hillel, 2000). At the same time it helped stronger students who became more secure in their knowledge through making the connections between $\mathbb{R}^3$ and abstract vector spaces (Sierpinska, 2005).

Our discussion in the paragraph above provides an example where the literature points to ways of teaching that depend on the body of students and the course to be taught. It requires the lecturer to make an informed decision on the most appropriate action in light of the students that he or she will be teaching.

The examples in the literature vary greatly and suggest many possibilities beyond mathematical content and delivery. Some authors emphasise everyday language and encourage journal writing for students as a means of engaging with the concepts in linear algebra (Hamdan, 2005). Another avenue suggested was to develop teaching that involved the use of technology and technological devices. Hillel (2001) reported on the use of a

computer algebra system for exploring the concepts of linear algebra. Using software often involved changing the learning context from whole-class teaching to lab-style tutorials, with students working on computers. This is closely related to teaching styles and teaching approaches which we discuss in more detail below.

Teaching style

One more resource in dealing with students' difficulties lies with approaches to teaching and teaching style. The traditional way to present mathematics in lectures is what we call 'top-down' – definitions and theorems are presented first, followed by examples and exercises. There is little opportunity for students to build own understandings since all theory is presented to them and for them. Some authors advocate this style as their preferred method. For example, Wu (1999) compares lecturing to a performance on stage aimed at capturing students' imagination. Pritchard (2010) also favours teaching via lectures. He states that lectures, when delivered well, present effective communication of mathematical ideas and furthermore, that students like them.

Students at university represent a diverse body and are probably more diverse now than has ever been the case. It is unlikely that one approach will fit all. We find examples in the literature that describe alternatives to the traditional lecturing style. These include 'bottom-up' approaches that stress intuition and exploration before the formal theory of linear algebra is introduced. Mathematician *M* fell into this category. He redesigned his teaching in order to engage students more, and more actively in lectures. As we have said, he was inspired by the article written by Frank Uhlig (2003), a mathematician who advocated an essentially 'bottom-up' approach to teaching linear algebra based on posing questions such as 'What happens if…?' 'Why does it happen?' 'How do different cases occur?' Uhlig used these questions to make students think about the mathematics and recognise generalities for themselves leading to a more gradual understanding of the generalisation and abstraction processes in mathematics. Uhlig's approach relied heavily on matrix algebra which was not a central feature in mathematician *M*'s approach. For his own teaching *M* developed a structure that used *examples* as starting points from which to introduce concepts and theorems in linear algebra. This involved students working on the examples first by themselves or in small groups within a lecture in order to 'develop a feel for what is going on' as *M* often used to say (see Thomas, 2012).

Our aim in presenting these examples of teaching is to encourage the reader to reflect on their own practice and to open up possibilities for acting in the classroom, as indeed we do ourselves. The suggestions are not intended to be prescriptive. They are certainly not exhaustive. But these themes and suggestions recurred as themes in the literature in relation to other undergraduate mathematics courses (more of this later). We hope that the reader may take inspiration from some of them.

In following up on some of the ideas in the literature, *M* came to understand students' difficulties better. He spent more time repeating explanations or using different words as Uhlig had done. This was probably the first and most immediate result from his reading. Understanding of this kind can motivate a teacher to find different explanations or approaches to teaching a topic.

The literature in other areas of mathematics and its learning/teaching

As well as literature relating to linear algebra, we have found literature regarding the learning and teaching of students in calculus, abstract algebra, group theory, differential equations, vector calculus, analytic geometry, functions in two variables, and proofs. Much of this literature is professional literature, although there is some that reports research. Some literature is relatively old, showing that mathematicians have been addressing teaching and learning issues for many years. For example, Freedman (1983:641) described a way of teaching *abstract algebra* that included making students *"play a more active part in their own education"* echoing statements made by the lecturer in our linear algebra study.

Several examples show that issues raised in relation to linear algebra are issues more generally. Staying within the context of abstract algebra, Cullinane (2005:339) expressed a desire *"to connect what [students] are learning to their prior experiences … in high school"*, recognising a lack of connection between school mathematics and mathematics in higher education. Other issues concern students' engagement with abstract algebra and the value of communication on a number of levels. Miller and Madore (2004) gave examples of in-class group projects while Warrington (2009) provided students with a practical out-of-class assignment as an introduction to abstract algebra. Leganza (1995) described the use of three specific writing assignments in an abstract algebra course. Several authors considered the use of technology in their teaching. Perry (2004) described his teaching using software entirely throughout the course while Leron and Dubinsky (1995) used interactive programming activities with students. Blyth and Rainbolt (2010:217) described three examples that were designed to help students *"discover classical theorems"* (in abstract algebra) *"with the help of leading questions and the software"*. Comparing these sources with those we have found relating to linear algebra, we see that the themes are the same: connecting with previous knowledge, engaging with interactive or group work, use of technology, written assignments, use of questions and questioning.

In relation to set theory, Padraig (Padraig & McLoughlin, 2010) discussed an inquiry-based approach to his teaching combined with a method called the Moore Method involving questions and questioning (which we discuss further below). He showed why and how he used this approach to teach students the beginnings of set theory. He argued that his pedagogic approach was equivalent to the approach that he took in his own research in mathematics. This echoes sentiments expressed by Uhlig (2002, 2003) in relation to linear algebra.

Some literature is more mathematically focused with discussions centering around the mathematics. Quinn and Rai (2010), for example, discussed the mathematics behind the form of the solution to differential equations and provided some reasoning for the 'method of variation of parameter'. Several articles in *The American Mathematical Monthly* presented discussions of the *mathematics* in an approach rather than the *pedagogy* of the approach. For example, Ebanks (2012) about the Mean Value Theorem, Thomson (2007) about the Riemann and Lesbegue integral (and how another integral altogether would make more sense for teaching and for students), O'Bryant, Reznick and Serbinowska (2006) on the limit of a sum, Cobb and Moore (1997) about the mathematics of statistics. Some articles

include the historical development of a piece of mathematics such as a theorem or a conjecture which provides a rich starting point for thinking about teaching.

Most of the examples referenced above form part of what we have called the professional literature relating to mathematics; they have been published in just a few journals such as *Primus*. There are other publications and periodicals that have *mathematics* as their main focus but will from time to time include articles that relate to teaching and learning issues in mathematics. Examples include publications by the Mathematical Association of America, the European Mathematical Society and The Institute of Mathematics and Its Applications.

We turn now to literature in the domain of mathematics education which, largely, draws upon research to discuss issues in and approaches to learning and teaching mathematics and develops associated theory. A seminal work, published in 1991, draws together key elements of the work of researchers in *Advanced Mathematical Thinking* (AMT) (Tall, 1991). This book is special in a number of ways. It emerged from a working group at the annual PME (International Group for the Psychology of Mathematics Education) Conference focusing on AMT, it had a strong emphasis on psychological theory linked to mathematical learning, and it reported research in areas of AMT. The cognitive theory on which 'mathematical thinking' is based, derives from the work of Jean Piaget (for example Piaget, 1950; Piaget & Inhelder, 1973) and underpins psychological constructs such as *concept image*, *concept definition* and *reflective abstraction* which provide insights into ways in which learners construct mathematical concepts. It takes seriously the difficulties that students face in learning mathematics, with discussion of *cognitive obstacles* which impede a conceptual appreciation of mathematics: these are mathematical concepts which challenge the learner's intuition, for example the notion of *limit* which (some) students see as an approximate process in which the limit is never attained. It discusses research into students' mathematical thinking in areas such as functions, limits and analysis, recognising the centrality of generality, abstraction and proof. For example, taking linear algebra as an example Tall (1991:11) writes:

> *"The terms "generalization" and "abstraction" are used in mathematics both to denote processes in which concepts are seen in a broader context and also the products of those processes. For instance we <u>generalize</u> the solution of linear equations in two and three dimensions to n dimensions and we <u>abstract</u> from this context the notion of a vector space. In doing so two very different mental objects are produced: the <u>generalization</u> $\mathbb{R}^n$ and the <u>abstraction</u>, a vector space V over a field F."*

<div align="right">Tall (1990:11)</div>

Because of the fundamental nature of the mathematical concepts and the psychological theory, most of this book is still relevant today for those who teach mathematics at this level.

The importance of mathematics learning and teaching in higher education was further recognised at the end of the 1990s with an international study conference sponsored by the International Commission on Mathematical Instruction (ICMI). Following the conference a 'study volume' was published entitled *The Teaching and Learning of Mathematics at University Level* (Holton, 2001). Papers in the volume were based on presentations from the conference and included both professional and research contributions. In the opening chapter, Claudi Alsina addresses, *"Why the professor must be a stimulating teacher"*. The author critiques a

range of *"myths"* about teaching, such as the top down approach, deductive organisation, 'researchers make the best lecturers', and goes on to discuss *"a new paradigm of teaching mathematics at university level"* (Alsina, 2001:7). He writes that university teaching of mathematics should be taken as seriously as is research in mathematics, that teachers can benefit from efficient training and that innovative teaching approaches, including use of technological tools, will have positive effects on learning. The book is wide ranging, written by mathematics educators and mathematicians, and addressing research, teaching, assessment, technology, teacher education, and a range of mathematical topics such as linear algebra and calculus. As well as teaching future mathematicians, it addresses also the teaching of mathematics to future engineers and scientists.

We end this section with reference to two important areas of pedagogy in the learning and teaching of mathematics in higher education. They are important because they have been influential among some mathematicians in offering different ways of thinking about encouraging students' conceptual engagement with mathematics.

The work of Robert Burn focusing on approaches to teaching related to promoting students' learning of the concepts in focus

Burn was a mathematician and mathematics educator in Cambridge and then in Exeter. He proposed *"A Pathway"* into an area of mathematics in focus, for example, *"A pathway into number theory"* (Burn, 1982). A 'pathway' is a bottom up (rather than top down) approach; it consists of a sequence of problems with which students would engage and through which a student would climb towards the standard theorems in the mathematical topic. Burn (1982:1) writes, *"Time and again, it was the exploration of special cases which illuminated the generalities for me. … [the pathway] follows more closely than usual the natural process of discovery and puts logic in its proper place"*. Burn quotes Hadamard as having said, *"The purpose of rigour is to legitimate the conquests of the intuition, and it never had any other purpose"* (Burn, 1982:1).

In Burn's book on number theory, the first chapter consists of 67 problems followed by 'Notes and answers' and a 'Historical note' and this sets the pattern for future chapters. The problems of Chapter 1 take the learner through the division algorithm, the Euclidean algorithm, unique prime factorisation, the infinity of primes and Mersenne primes, proving theorems as part of the building process. Lest the reader thinks that there is something special about number theory, for which this approach might work differently than for other areas of mathematics, Burn's two further volumes (1985, 1992) offer *"A Pathway to Geometry"* and *"Steps into Analysis"*.

The Moore Method

This section is written with the help of a personal communication from Chris Sangwin, who has used the Moore Method in his teaching and written about its use (for example Sangwin, 2011). It is adapted from an unpublished document (Sangwin, 2013) entitled *"Choosing Problems for a Moore Method Course"*[1].

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1   A copy of which can be obtained from the author via email: C.J.Sangwin@lboro.ac.uk.

The Moore Method is a type of enquiry based learning (EBL) developed by the influential Texan topologist Robert Lee Moore (1882–1974) for university mathematics courses. The central tenet of the Method is learning by problem solving, where students answer questions that accumulate into theorems and lemmas. The essence of the Method is to engage students via the following process:

1. Mathematical problems are posed by the lecturer to the whole class.

2. Students solve the problems independently of each other.

3. Students present their solutions to the class, on the board.

4. Students discuss solutions to decide whether they are correct and complete.

Solutions are not imposed by the lecturer, who acts as facilitator by offering comment and guiding discussion. Moore is quoted as saying (Parker, 2004:vii) *"That student is taught the best who is told the least"*. As Mahavier (1999) said of Moore: *"Moore helped his students a lot but did it in such a way that they did not feel that the help detracted from the satisfaction they received from having solved a problem. He was a master at saying the right thing to the right student at the right time."*

Colleagues varied aspects of the Method, with some encouraging students to work as a group, both answering questions and formulating research topics of their own. Such teachers encouraged alternative solutions to be presented and discussed, helping students refine their sense of aesthetics, and providing other strategies for subsequent problems. In all forms, a key aspect is that it is the students' responsibility to solve the problems for themselves. And, in all versions, *the group* criticises these solutions and ultimately, together with the teacher, decides if a solution is complete and correct.

The pedagogical literature

In what we have written above, we have provided examples of literature which addresses pedagogy in mathematics learning and teaching in higher education. What we are calling *pedagogical literature* goes beyond the exposition of approaches to teaching and learning, or research findings in teaching and learning, to discuss the nature of teaching and learning and to analyse ways in which teaching can be designed and improved to enhance mathematical learning opportunities for students. This literature draws on both professional and research literature and there are elements of it in the sources discussed above (Burn, 1982; Tall, 1991; Holton, 2001; Sangwin, 2011).

Fundamentally, pedagogic literature analyses approaches to teaching mathematics in higher education, discusses how such approaches do or can contribute to students' learning of advanced mathematics and provides guidance to those teaching at this level. The authors of such books or articles are often mathematics educators who teach advanced mathematics and who reflect critically on the nature of their teaching and how it develops. Often they work closely with mathematicians and are able to take mathematicians' views into account in their analyses. The following are some examples of what is available, not a comprehensive list.

In *Fundamentals of Teaching Mathematics at University Level*, Baumslag (2000) presents his perspective on what it means to teach at university level and ways in which those teaching at this level can/should think about teaching. The book ranges over issues related to systems of education, approaches to teaching, organisation and management of teaching and assessment, and advice for the lecturer. Thus it deals with the practice and processes of teaching within a general framework of the demands of the system and principles of education.

On the other hand, Khan's (2001) *Studying Mathematics and its Applications* is a study guide for undergraduate students studying mathematics as part of a degree in mathematics, science or engineering. The author writes, *"We will focus here exclusively on the distinctive approaches needed to study mathematics and its applications. What you will not find here is more general advice on time management, positive thinking and so on"* (Khan, 2001:xiii). Since this book is written for students, it might be asked why we consider it as pedagogic literature; the reason is that as teachers we are fundamentally concerned about the learning of our students and any approaches by which students can learn more effectively are likely to be of interest to us. For example, in a section labelled *"Skills"* the author addresses, using examples, thinking visually, and coping with symbols; under *"Tasks"*, he focuses on solving problems, applying mathematics and constructing proofs. We suggest that all of these constitute areas in which we should like our students to gain understanding and fluency and therefore it is valuable to be better informed on how to encourage students in these areas of their study.

In a book entitled *Mathematics Teaching Practice* Mason (2002) writes explicitly for lecturers at university or college level. He takes the perspective that *"mathematics has to be learned by actively engaging with it"* (Mason, 2002:v). For this purpose he recommends a set of *"tactics"*, specific acts on the part of the lecturer designed to *"stimulate students to take the initiative, to act upon the mathematical ideas and make sense of them and not just attempt to master a succession of techniques"* (Mason, 2002:1-2). Mason encourages readers to reflect on their own teaching and identify what they do, how they do it and in particular what they value about the teaching process. The book is organised around aspects of teaching, such as lecturing, tutoring, constructing tasks, marking and commenting, offering and discussing tactics in all of these areas. However, the first chapter relates to the students: it focuses on *"Student Difficulties with Mathematics"* highlighting the kinds of difficulties that lecturers typically notice and offering tactics to address them. Tactics include *using common errors; specialising, generalising and counter examples; boundary examples; say what you see; reconstructing definitions;* and many more. In the chapter on *Lecturing*, tactics include *talking in pairs; pausing; presenting the essence of a theorem; invoking mental imagery; being human;* and many more. Tactics are explained and discussed and readers are drawn into thinking about how they might themselves use such tactics with their own students. The book as a whole is a pedagogic resource which might well form the basis of a training course for new lecturers in mathematics.

Ideas from Mathematics Education (Alcock & Simpson, 2009) is also written explicitly for those teaching undergraduate mathematics with *"an accessible introduction to some ideas from mathematics education research"* (Alcock & Simpson, 2009:1). The authors focus explicitly on Definitions, Mathematics Objects, and Reasoning Strategies, relating these to findings from research and related areas of theory such as Concept Image and Definition,

APOS (Action-Process-Object-Schema) theory, and Semantic and Syntactic Strategies. Explaining research findings and exemplifying theoretical perspectives, the authors address issues in learning and teaching with guidance as to how these inform those teaching at this level.

Nardi's (2008) *Amongst Mathematicians, Teaching and Learning mathematics at University Level* is a rather different sort of book from those mentioned above. It presents the results of several years of research in which the author has worked closely with and studied the practices of mathematicians as teachers of mathematics along with the learning of their students. The book is presented as a series of dialogues between a *mathematician* and a *researcher in mathematics education* (RME). The two together focus on data from mathematics learning and teaching sessions and dialogues between students and their tutor. For example, on page 172, the focus is on the duality of *function as a process* and *function as an object*. They focus on a piece of dialogue which addresses the importance of *domain* when constructing the *inverse* of a function, and the inverse function theorem. The mathematician presents the mathematical thinking behind the theorem while RME plays a 'devil's advocate' role representing the student. *"So here I am, solving a differential equation and, until now whenever I solve an equation, the answer is a number. And all of a sudden, you are asking me to solve an equation and the answer is a function, or a family of functions. I wouldn't be sure what you mean! Functions are formulas I use to evaluate y in terms of x. They are not solutions to equations"* (Nardi, 2008:172). The reader's attention is drawn to the student perspective within the formality of mathematical exposition and thinking; this enables a broadening of the teacher's perspective, beyond the mathematics, to the leaps of conceptualisation required of students.

We end this section with reference to some activity and research at our own university focusing on the teaching of mathematics and on developing a discourse on mathematics teaching. Based on the idea that we learn from each other in discussion of mathematics and the ways we teach it (the focus of many of the books mentioned above) we instituted a series of seminars called the 'How we Teach' seminars. In each seminar, one teacher of mathematics (in some cases a mathematician, in others a mathematics educator) gave a presentation of some aspect of his or her teaching, and the thinking associated with it. Those attending listened, raised questions and discussed issues arising from the presentation. Through this activity, there were two important outcomes: first, we all gained from hearing a colleague's experiences and discussing issues, enhancing our own thinking and possibilities for practice; second, we were brought closer as a community, sharing perspectives, trusting each other, and supporting less experienced colleagues. The seminar series was accepted within the University's New Lecturers Course, as a medium through which new mathematics lecturers could demonstrate their growth of understanding about teaching – preparing and offering a seminar and reflecting on its outcomes, as well as reflecting on other seminars attended. The first 20 of these seminars were video recorded and 10 of them analysed to elicit aspects of what was presented and what was discussed. Several papers resulted from this analysis (for example Jaworski & Matthews, 2011; Matthews & Jaworski, 2012).

In conclusion

This chapter has been about the literature that is available to lecturers and others teaching mathematics in higher education and has suggested ways in which readers might use this literature to inform their own teaching. We have distinguished particularly the professional, research and pedagogic literature. Each of these has something specific to offer. From the professional literature, we gain insights into the ways in which other teachers/lecturers have thought about their teaching and the approaches/strategies and frameworks they have used. The research literature offers us *what is known*, findings from research that can enable more informed approaches to teaching. The pedagogical literature deals overtly with developing and enhancing teaching, through the lecturer engaging with new ideas (for example those offered in the professional literature), attending to research findings or using specific tactics or teaching approaches recommended by the authors. We have also referred to some of our own experiences in using the literature in a research study with a mathematician teaching linear algebra, and in working with colleagues to generate a pedagogical discourse and increase awareness of teaching processes. In the end, we hope to have shown that there is a wealth of literature on which teachers/lecturers can draw to guide and inform their teaching. While this can be an individual initiative, it can be especially fruitful to join with colleagues to read and discuss what is offered.

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