Partially ample line bundles on toric varieties

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Ample line bundles are a fundamental concept in algebraic geometry, encapsulating the central notion of positivity. A natural extension of the notion of ampleness is that of q-ampleness, for nonnegative integers q. Roughly speaking, q-ample line bundles on a variety are those which "kill cohomology in degrees above q." Line bundles of this kind have been studied by a number of authors, including Andreotti–Grauert [AG], Sommese [Som], Demailly–Peternell–Schneider [DPS], and Totaro [Tot].

In this note, we verify some basic properties of q-ample line bundles on toric varieties. We begin by reviewing basic facts about q-ampleness. Then in Section 2 we study the structure of the set of all q-ample line bundles on a simplicial toric variety. We show that the cone of q-ample line bundles is the interior of a finite union of rational polyhedral cones, and that it is defined by the vanishing of asymptotic cohomological functions. As an illustration, in Section 3 we give examples of explicit calculations of q-ample cones of two families of toric varieties.

In Sections 4 and 5 we prove that q-ampleness of big line bundles on toric varieties is detected by restriction to torus-invariant divisors, and use this fact to study q-ampleness of the anticanonical bundle: in particular, we give an example showing that 1-ampleness of $-K$ is not preserved by flips, answering a question of Demailly–Peternell–Schneider. We finish in Section 6 by proving a Kodaira-type vanishing theorem for q-ample bundles on toric varieties.

1 q-ample line bundles

Throughout the paper we stick to toric varieties over algebraically closed fields of characteristic zero. We switch between additive and multiplicative notation for line bundles wherever convenient, and freely identify line bundles and the corresponding divisors.

In the 1950s Serre gave a cohomological characterisation of ample line bundles: a line bundle is ample if and only if some sufficiently high power of it kills cohomology of any coherent sheaf in degrees above zero. This characterisation suggests the following generalisation of ampleness, introduced by Sommese [Som]. (Note that Sommese's definition requires that some power of the line bundle be globally generated, but we drop that hypothesis here.)

Definition 1.1. *Let X be a projective variety. A line bundle L on X is called q-ample (for some integer $q \geq 0$) if for any coherent sheaf F on X, there exists a natural number m_0 (depending on F) such that*

$$H^i(X, mL \otimes F) = 0 \text{ for all } i > q \text{ and } m \geq m_0. \tag{1}$$

Any line bundle on a variety of dimension n is n-ample; by Serre, 0-ample is the same as ample.

At first sight the q-ample condition seems hard to check, since it involves tensoring with an arbitrary coherent sheaf. The following result reduces this to a much simpler condition:

Proposition 1.2 ([Ott], Lemma 2.1). *Let X be a projective variety over a field of characteristic 0, and fix an ample line bundle $\mathcal{O}(1)$ on X. A line bundle L on X is q-ample if and only if for each $r \geq 0$, we have $H^i(X, mL \otimes \mathcal{O}(-r)) = 0$ for m sufficiently large and all $i > q$. In particular, condition* (1) *need only be checked for locally free sheaves.*

In fact, as shown in [Tot], a much stronger statement is true: L is q-ample provided it kills higher cohomology of a fixed finite set of line bundles $\mathcal{O}, \ldots, \mathcal{O}(-r)$.

For $q > 0$, it remains an open problem to give a simple numerical or geometric condition, in the spirit of Kleiman's criterion, for q-ampleness.

2 The q-ample cone of a toric variety

In this section we recall some basic facts about cohomology of line bundles on toric varieties, together with Hering–Küronya–Payne's formula for calculating dimensions of cohomology groups. We then use to describe the structure of the cone of q-ample line bundles on a toric variety, and to show that it is characterised by the vanishing of asymptotic cohomological functions.

Let $X = X(\Delta)$ be a simplicial projective n-dimensional toric variety, corresponding to some complete fan Δ in a lattice $N \cong \mathbf{Z}^n$. We denote by $\Delta(1)$ the set of rays of Δ, and write v_i for the primitive generators of the ray $i \in \Delta(1)$. There is a one-to-one correspondence between prime torus-invariant divisors and rays [Fu, Chapter 3]. We denote these divisors by $\{D_i \mid i \in \Delta(1)\}$ and the free group generated by them by $\mathbf{Z}^{\Delta(1)}$. The dual space $\mathbf{Z}_{\Delta(1)}$ is generated by the dual basis $\{e_i \mid i \in \Delta(1)\}$.

Let $M := \operatorname{Hom}(N, \mathbf{Z}) \cong \mathbf{Z}^n$ be the dual lattice to N, with pairing $\langle \cdot, \cdot \rangle$. We have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \operatorname{Div}_T(X) & \longrightarrow & \operatorname{Pic}(X) & \longrightarrow & 0 \\
& & \| & & \cap & & \cap & & \\
0 & \longrightarrow & M & \longrightarrow & \mathbf{Z}^{\Delta(1)} & \xrightarrow{[-]} & Cl(X) & \longrightarrow & 0
\end{array}
\tag{2}
$$

where $\operatorname{Div}_T(X)$ is the group of torus-invariant Cartier divisors, and $Cl(X)$ is the class group. Applying the functor $- \otimes_{\mathbf{Z}} \mathbf{R}$ and using the fact that simplicial toric varieties are $\mathbf{Q}$-factorial, we obtain the following exact sequence

$$
0 \longrightarrow M_{\mathbf{R}} \longrightarrow \mathbf{R}^{\Delta(1)} \xrightarrow{[-]} N^1(X) \longrightarrow 0
\tag{3}
$$

where $N^1(X)$ denotes $\operatorname{Pic}(X) \otimes \mathbf{R}$.

For $I \subset \Delta(1)$, we define Δ_I to be the subfan of Δ consisting of cones whose rays belong to I. For a torus-invariant divisor $D = \sum_\rho a_\rho D_\rho$, we define the polyhedral region

$$
P_{D,I} = \{u \in M_{\mathbf{R}} \mid \langle u, v_\rho \rangle \geq -a_\rho \Leftrightarrow \rho \in I\}.
\tag{4}
$$

Note that for a positive integer m, we have $P_{mD,I} = mP_{D,I}$.

Hering–Küronya–Payne [HKP] gave a description of the cohomology of the divisor D in terms of local cohomology groups and lattice points in $P_{D,I}$. For a subfan Δ_I, we denote the dimension of the (topological) local cohomology group $H^i_{|\Delta_I|}(N_\mathbf{R})$ with support in the subspace $|\Delta_I|$ by $h^i_{|\Delta_I|}$.

Proposition 2.1 (Hering–Küronya–Payne). *For a line bundle D on a simplicial projective toric variety we have*

$$h^i(X, D) = \sum_{I \subset \Delta(1)} h^i_{|\Delta_I|} \cdot \# \left(P_{D,I} \cap M \right)$$

We also recall Küronya's definition of *asymptotic cohomological functions*. For an n-dimensional projective variety X and a line bundle L on X, we define

$$\widehat{h}^i(L) = n! \lim_{m \to \infty} \frac{h^i(X, mL)}{m^n}$$

Note that $\widehat{h}^0$ is just the usual volume function. Küronya showed that the functions $\widehat{h}^i$ give well-defined homogeneous continuous functions on the space $N^1(X)$. By Serre it is clear that for each $i > 0$ the function h^i vanishes identically on the nef cone; de Fernex–Küronya–Lazarsfeld [dFKL] showed that in fact this characterises the nef cone.

For a bounded polyhedron $P \in M_\mathbf{R}$, let $\mathrm{Vol}(P)$ denote the volume of P, normalised so that the smallest lattice simplex has unit volume. Then

$$\mathrm{Vol}(P) = n! \lim_{m \to \infty} \frac{\# mP \cap M}{m^n}.$$

Combining the three displayed equations above then gives

$$\widehat{h}^i(D) = \sum_{I \subset \Delta(1)} h^i_{|\Delta_I|} \cdot \mathrm{Vol}(P_{D,I}). \tag{5}$$

Finally we need the following elementary result.

Lemma 2.2. *Let A be a real $m \times n$ matrix with $m \leq n$, and let $b \in \mathbf{R}^m$. If the inequality $Ax \leq b$ has a solution, then there exists a vector $v \in \mathbf{R}^m$ such that for any $\epsilon > 0$ the set $\{x \in \mathbf{R}^n \mid Ax \leq b + \epsilon v\}$ is an n-dimensional polyhedron.*

Now we can state the main result of this section.

Theorem 2.3. *Let X be a simplicial projective toric variety and let $D = \sum_\rho a_\rho D_\rho$ be a torus-invariant divisor on X. Then the following are equivalent:*

(a) *D is q-ample;*

(b) *For every ample divisor A, we have $H^i(X, mD - A) = 0$ for $i > q$ and $m \gg 0$;*

(c) *There exists an open neighbourhood U of $[D] \in N^1(X)$ such that $\widehat{h}^i(x) = 0$ for all $i > q$ and all $x \in U$.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ is Proposition 1.2.

Next, if D is q-ample, then immediately from the definition we get $\widehat{h}^i(D) = 0$ for all $i > q$. Moreover, for each q, the q-ample cone is open in $N^1(X)$ [Tot, Theorem 8.3]. This proves the implication $(a) \Rightarrow (c)$.

So it suffices to prove the implication $(c) \Rightarrow (b)$. Fix $i > q$. Let $I \subset \Delta(1)$ be a subset such that $h^i_{|\Delta_I|} > 0$: that is, a subset which could contribute nonzero terms to $\widehat{h}^i$. Then for any $D' \in U$, the polyhedron $P_{D',I}$ must be the empty set: if it were not, by Lemma 2.2 we could choose a class E such that for all $0 < \epsilon \ll 1$, the perturbed class $D' + \epsilon E \in U$ has the property that $P_{D'+\epsilon E, J}$ has positive volume. But then formula (5) implies that $\widehat{h}^i(X, D' + \epsilon E)$ is nonzero, contradicting our choice. Since for large m we know that $D - \frac{1}{m}A$ lies in U, we must have that $P_{D - \frac{1}{m}A, I} = \emptyset$. By our earlier remark, this implies that $P_{mD - A, I} = \emptyset$ too. Since this is true for all I which could contribute to H^i, we get that $H^i(X, mD - A) = 0$ as required. $\square$

The theorem allows us to describe the q-ample cone of a toric variety. By definition, an **R**-divisor D is q-ample if it is numerically equivalent to a sum $cL + A$ where L is a q-ample divisor, c a positive real number, and A is an ample R-divisor. The set of all q-ample **R**-divisors defines an open cone in $N^1(X)$ whose integer points are exactly the q-ample divisors [Tot, Theorem 8.3]. Chen–Lazarsfeld asked if, for Fano varieties, these cone are always the interior of a finite union of rational polyhedral cones. (This is the simplest possible generalisation of the Cone Theorem in this context, since these cones are known not to be convex in general.) Here we give a positive answer to the analogue of Chen–Lazarsfeld's question for toric varieties.

Corollary 2.4. *If X is a simplicial projective toric variety, then the closure of the q-ample cone $\overline{Amp_q}(X)$ is a union of rational polyhedral cones, for each $q \geq 0$.*

Proof. Theorem 2.3 says that $\overline{Amp_q}(X) \subset N^1(X)$ is the common vanishing locus of the functions $\widehat{h}^i$ for $i > q$. Formula (5) shows that $\widehat{h}^i(D) = 0$ for all $i > q$ if and only if:

$$\text{Vol } P_{D,I} = 0 \text{ for all } I \subset \Delta(1) \text{ such that } h^i_{|\Delta|_I} > 0 \text{ for some } i > q.$$

The basic point is that for each appropriate I, the subset in $\mathbf{R}^{\Delta(1)}$ of D satisfying the above condition is cut out by a collection of rational hyperplanes, and the images of these hyperplanes in $N^1(X)$ then define the cone $\overline{Amp_q}(X)$.

In more detail, let I be any subset of $\Delta(1)$, and consider a divisor $D = \sum d_\rho D_\rho$. First, if $d_\rho > 0$ for $\rho \in I$ and $d_\rho < 0$ for $\rho \notin I$, then clearly $P_{D,I}$ contains a small ball around the origin: in particular, $\text{Vol } P_{D,I} > 0$. Note that if we replace D with a linearly equivalent divisor $D' = D + \text{div}(u)$ (for some rational function $u \in M$), the new polytope $P_{D',I}$ is just the translate $P_{D,I} - U$, so it also has positive volume.

Conversely, if $P_{D,I}$ contains a ball around the origin, it is clear we must have $d_\rho > 0$ for $\rho \in I$ and $d_\rho < 0$ for $\rho \notin I$. Now if D is any divisor such that $\text{Vol } P_{D,I} > 0$, then (perhaps after scaling D) there exists a rational function u such that for the divisor $D' = D + \text{div}(u)$, the polytope $P_{D',I}$ contains a small ball around the origin, and so D' has the property stated.

To summarise, we have shown that $\text{Vol } P_{D,I} > 0$ if and only if D is linearly equivalent to a divisor $D' = \sum d_\rho D_\rho$ with $d_\rho > 0$ if and only if $\rho \in I$. The set of such divisors D' forms an (open) orthant O_I in $\mathbf{R}^{\Delta(1)}$, and so the closure of its image $\overline{[O_I]}$ in $N^1(X)$ is a rational polyhedral cone. Theorem 2.3 shows that $Amp_q(X)$ is the complement of the union (over a finite set of I) of the cones $\overline{[O_I]}$, and hence is the interior of a union of rational polyhedral cones. $\square$

3 Examples

In this section, we illustrate Theorem 2.3 by calculating the q-ample cones of two families of examples: projective bundles over $\mathbf{P}^1$, and toric $\mathbf{P}^1$-bundles over projective spaces.

To make the calculations easier, we find it convenient to reformulate our earlier conditions on cohomology vanishing in terms of the polytope of our toric variety X. Recall that the polytope P_X of X is defined exactly as in formula (4) in Section 2, where D is chosen to be any ample divisor on X, and $J = \Delta(1)$. Broomhead [B] showed how to calculate cohomology of line bundles on X in terms of the topology of certain subspaces of P_X. For this statement, given a subset $I \subset \Delta(1)$, let P_X^I denote the subset of P_X consisting of the union of all top-dimensional faces corresponding to rays in I. The statement we need is the following:

Proposition 3.1. *Let X be a simplicial projective toric variety and $D = \sum a_\rho D_\rho$ a torus-invariant divisor. Then $\widehat{h}^i(D) \neq 0$ if and only if the following is true: there exists $I \subset \Delta(1)$ such that $\widetilde{H}^{i-1}(P_X^I) \neq 0$, and a divisor $D' = \sum d_\rho D_\rho$, linearly equivalent to D, such that $d_\rho < 0$ if and only if $\rho \in I$.*

Proof. According to formula (5), $\widehat{h}^i(D) \neq 0$ if and only if there exists a subset $J \subseteq \Delta(1)$ such that $H^i_{|\Delta_J|} \neq 0$ and $\mathrm{Vol}\, P_{D,J} \neq 0$. Fix any such subset J and let $I = \Delta(1) \setminus J$.

In the proof of Corollary 2.4 we saw that $\mathrm{Vol}\, P_{D,J} > 0$ if and only if there exists D' linearly equivalent to D satisfying the stated condition. So it remains to prove that $H^i_{|\Delta_J|} \cong \widetilde{H}^{i-1}(P_X^I)$.

For this, denote by S the unit sphere in the vector space $N_{\mathbf{R}}$. Then there is an isomorphism

$$H^i_{|\Delta_J|} \cong \widetilde{H}^{i-1}(S \setminus S \cap |\Delta_J|).$$

The fan of X induces the structure of a simplicial complex on S, and the polytope P_X can be viewed as the dual complex. Then $S \setminus S \cap |\Delta_J|$ retracts onto P_X^I. Combining with the previous displayed isomorphism, this gives an isomorphism

$$H^i_{|\Delta_J|} \cong \widetilde{H}^{i-1}(P_X^I).$$

$\square$

We emphasise that this gives a method for computing q-ample cones in practice. Given X, we take its polytope P_X. For each i, we enumerate the subsets $I \subset \Delta(1)$ such that $\widetilde{H}^i(P_X^I)$ is nontrivial. Each such subset I defines an orthant O_I in $\mathbf{R}^{\Delta(1)}$ consisting of line bundles whose i-th asymptotic cohomology has a nonzero contribution from I. Let $[O_I]$ denote the image of this orthant in $N^1(X)$. Then the proposition shows that $Amp_q(X)$ is the complement of $\bigcup_I \overline{[O_I]}$, where the union is over all $I \subset \Delta(1)$ such that $\widetilde{H}^i(P_X^I) \neq 0$ for some $i \geq q$. The next two subsections will illustrate this algorithm.

3.1 Bundles over $\mathbf{P}^1$

We follow the notation for projective bundles from [CLS, Chapter 7]. Let $X = \mathbf{P}(\mathcal{V})$, where $\mathcal{V}$ is a vector bundle over $\mathbf{P}^1$ of rank $n+1$. By the Birkhoff theorem $\mathcal{V}$ is a direct sum of line bundles; after twisting, we can assume without loss of generality that $\mathcal{V} = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_n)$, where $0 \leq a_1 \leq \cdots \leq a_n$.

The fan of X is described as follows. Let $\mathbf{R} \times \mathbf{R}^n$ have basis $v_1, e_1, \ldots, e_n$, and set

$$e_0 = -\sum_{i=1}^n e_i; \quad v_0 = -v_1 + \sum_{i=1}^n a_i e_i.$$

Then the vectors $e_0, \ldots, e_n, v_0, v_1$ span the rays of the fan of X, and the top-dimensional cones are of the following form:

$$\langle v_i, e_0, \ldots, \widehat{e_j}, \ldots, e_n \rangle \quad i \in \{0, 1\}, j \in \{0, \ldots, n\}.$$

All of these cones are simplicial, so the codimension-1 cones in the fan, corresponding to the torus-invariant curves on X, are obtained by omitting one spanning vector from one of the cones above. Using the intersection formulas from [Fu, Section 5.1] it is straightforward to calculate intersections between torus-invariant curves and divisors. This allows us to identify ample divisors on X: in particular, we find that the divisor

$$A := \sum_{i=0}^n E_i + \left(\sum_{i=1}^n a_i + 1 \right) (V_0 + V_1)$$

is ample. (Here the E_i and V_j are the torus-invariant divisors corresponding to the vectors e_i and v_j generating rays of the fan: geometrically, E_i is the sub-bundle of X obtained by quotienting V by the summand $\mathcal{O}(a_i)$, and V_j is the fibre over one of the torus-invariant points of $\mathbf{P}^1$.)

Given the ample divisor A, we have the corresponding polytope P_X as described above:

$$P_X = \left\{ u \in M_{\mathbf{R}} : \langle u, e_i \rangle \geq -1, \langle u, v_j \rangle \geq -\sum a_i - 1 \right\}.$$

The inequalities involving the e_i cut out a polyhedron of the form $\mathbf{R} \times \Delta^n$; the inequalities with v_0 and v_1 then bound this in the direction of the $\mathbf{R}$-factor. It is straightforward to check that the faces of the polytope corresponding to v_0 and v_1 are disjoint, irrespective of the values of the a_i, so that P_X is combintorially equivalent to the polytope $[0, 1] \times \Delta^n$. The homology groups of unions of faces of this polytope are described in the following lemma.

Lemma 3.2. *Label the the faces of the polytope P_X so that the unique pair of disjoint faces are labelled F_{n+2} and F_{n+3}. Let Y be a union of closed top-dimensional faces of P_X. Then its reduced homology groups are*

$$\widetilde{H}_k(Y) = \begin{cases} \mathbf{k} & \text{if } Y = \partial P_X, k = n \\ \mathbf{k} & \text{if } Y = \partial P_X \setminus \{F_{n+2} \cup F_{n+3}\}, k = n - 1 \\ \mathbf{k} & \text{if } Y = F_{n+2} \cup F_{n+3}, k = 0 \\ \mathbf{k} & \text{if } Y = \emptyset, k = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof comes from the long exact sequence of reduced homology groups associated to a sequence $A \hookrightarrow B \to B/A$, where A is a (reasonable) closed subspace of a topological space B. Applying this with $B = Y$, a union of top-dimensional faces of P_X, and $A = Y \cap F_{n+2}$ we reduce the problem to calculating the reduced homology of either a union of faces of a simplex or the disjoint union of a point with a union of faces of a simplex. Using the fact that the union of any proper subset of faces of a simplex has no reduced homology, the result follows. $\square$

This lemma immediately identifies the index sets I_α which give nonzero contributions to cohomology of a line bundle, as described above. We have

$$I_0 = \left\{ \alpha \subseteq \Delta(1) = \{1, \ldots, n+3\} \mid \widetilde{H}^0(Z_\alpha, \mathbf{k}) \neq 0 \right\} = \{\{n+2, n+3\}\}$$

$$I_1 = \cdots = I_{n-2} = \emptyset$$

$$I_{n-1} = \left\{ \alpha \subseteq \Delta(1) \mid \widetilde{H}^{n-1}(Z_\alpha, \mathbf{k}) \neq 0 \right\} = \{\{1, \ldots, n+1\}\}$$

$$I_n = \left\{ \alpha \subseteq \Delta(1) \mid \widetilde{H}^n(Z_\alpha, \mathbf{k}) \neq 0 \right\} = \{\{1, \ldots, n+3\}\}.$$

The corresponding orthants in $\mathbf{R}^{\Delta(1)} \cong \mathbf{R}^{n+3}$ are then

$$O_{I_0} = \left\{ (d_1, \ldots, d_{n+3}) \in \mathbf{R}^{n+3} \mid d_{n+2} < 0, d_{n+3} < 0, d_i \geq 0 \text{ for all } i = 1, \ldots, n+1 \right\}$$

$$O_{I_1} = \cdots = O_{I_{n-2}} = \emptyset$$

$$O_{I_{n-1}} = \left\{ (d_1, \ldots, d_{n+3}) \in \mathbf{R}^{n+3} \mid d_{n+2} \geq 0, d_{n+3} \geq 0, d_i < 0 \text{ for all } i = 1, \ldots, n+1 \right\}$$

$$O_{I_n} = \left\{ (d_1, \ldots, d_{n+3}) \in \mathbf{R}^{n+3} \mid d_i < 0 \text{ for all } i = 1, \ldots, n+3 \right\}$$

Now we can calculate the q-ample cones $Amp_q(X)$. Proposition 3.1 says that $Amp_q(X)$ is the complement in $\operatorname{Pic}(X)$ of the union of the images of all the closed orthants $\overline{O_{I_i}}$ for $i \geq q$. The map $\mathbf{R}^{n+3} \to \operatorname{Pic}(X)$ has kernel equal to the column space of the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_n & -1 \end{pmatrix}$$

whose rows are the primitive vectors of the rays of the fan of X, expressed in the basis $\langle e_1, \ldots, e_n, v_1 \rangle$. So in $\operatorname{Pic}(X)$ we have $E_i = E_0 - a_i V_0$ and $V_0 = V_1$. Let us denote the latter linear equivalence class by V. The images of the closed orthants $\overline{O_{I_i}}$ above are then

$$\overline{O_{I_0}} \mapsto \langle E_0, E_0 - a_1 V, \ldots, E_0 - a_n V, -V, -V \rangle = \langle E_0, -V \rangle$$

$$\overline{O_{I_{n-1}}} \mapsto \langle -E_0, -E_0 + a_1 V, \ldots, -E_0 + a_n V, V, V \rangle = \langle -E_0, V \rangle$$

$$\overline{O_{I_n}} \mapsto \langle -E_0, -E_0 + a_1 V, \ldots, -E_0 + a_n V, -V, -V \rangle = \langle -E_0 + a_n V, -V \rangle$$

where the last equality uses the fact that $a_i \leq a_n$ for all i.

Putting these regions together as described in Theorem 2.3, we get the following result:

$$\overline{Amp_0(X)} = \langle E_0, V \rangle$$

$$\overline{Amp_q(X)} = \langle V, -V \rangle, \ \ 0 < q < n$$

$$\overline{Amp_n(X)} = \operatorname{Pic}(X)_{\mathbf{R}} \setminus \langle -E_0 + a_n V, -V \rangle.$$

3.2 $\mathbf{P}^1$-bundles over projective space

In a similar way we can calculate the q-ample cones of a toric $\mathbf{P}^1$-bundle over any projective space. Let $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^n} \oplus \mathcal{O}_{\mathbf{P}^n}(a))$. Then the fan of X is the following: Let $\mathbf{R}^n \times \mathbf{R}$ have basis $v_1, \ldots, v_n, e_1$ and set

$$e_0 = -e_1 \,;\; v_0 = -\sum_{i=1}^n v_i + ae_1.$$

Then the maximal cones in the fan are of the form

$$\langle v_0, \ldots, \widehat{v_i}, \ldots, v_n, e_j \rangle \quad i \in \{0, \ldots, n\}, \, j \in \{0, 1\}.$$

Computing intersection numbers with torus-invariant curves shows that the divisor

$$A := (a+1)\,(V_0 + \cdots + V_n) + E_0 + E_1$$

is ample; as before, one finds that the polytope P_A is combinatorially equivalent to $\Delta^n \times [0, 1]$. Repeating the process above, we obtain the following result for the q-ample cones:

$$\overline{Amp_0(X)} = \langle E_0, V \rangle$$
$$\overline{Amp_q(X)} = \langle V, -V \rangle$$
$$\overline{Amp_n(X)} = \mathrm{Pic}(X)_{\mathbf{R}} \setminus \langle -E_0 + aV, -V \rangle.$$

The orthants of cohomology nonvanishing and the q-ample cones in this example are shown in the figure below.

(a) Orthants of cohomology nonvanishing (b) The q-ample cones for $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(a))$
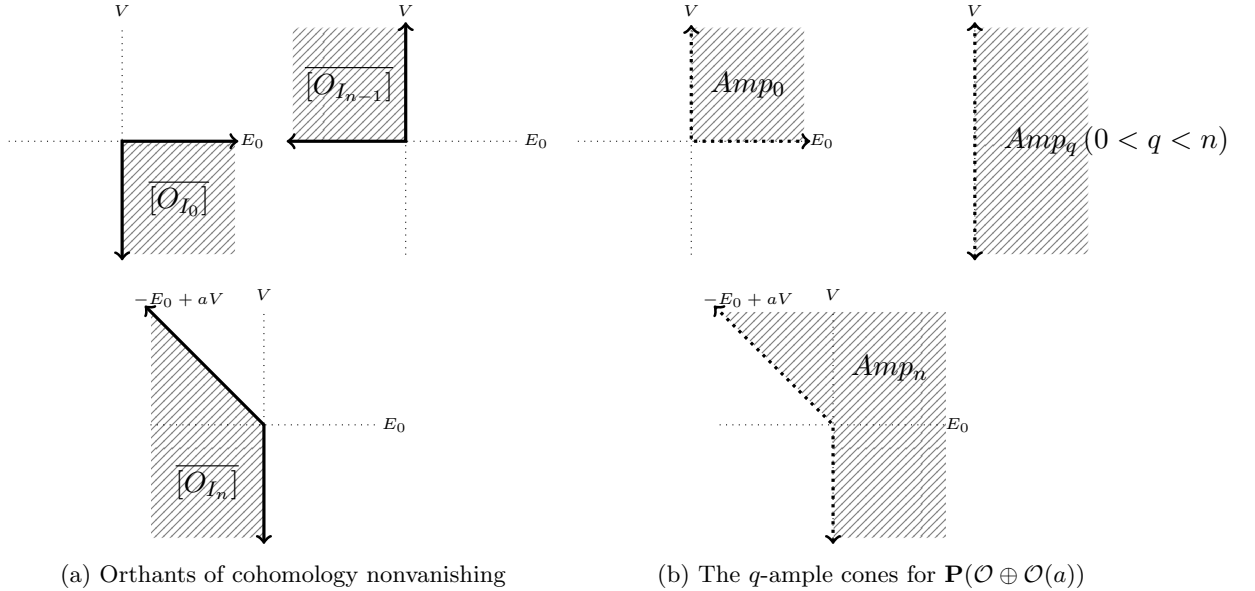
As a remark, it is well-known ([DPS, General Properties 1.5], [Tot, Theorem 9.1]) that, for any projective variety of dimension d, the $(d-1)$-ample cone is the complement in $N^1(X)$ of the negative of the pseudoeffecitve cone. This gives an easier way to calculate the cones Amp_n in the examples above. Similarly, the toric version of Kleiman's criterion gives the ample cone Amp_0.

4 Big $q-$ample line bundles

In this section we show that for big line bundles on toric varieties, q-ampleness can be detected by restriction to torus-invariant divisors. This can be deduced from a theorem of Brown [Br], who proved that for big line bundles on arbitrary projective varieties, q-ampleness can be detected by restriction to the augmented base locus. We give the proof in the toric case here since it is simple and self-contained.

Theorem 4.1. *Let X be a simplicial toric variety and L a big line bundle. Then L is q-ample if and only if $L|_E$ is $q-$ample on each torus invariant divisor E.*

Proof. In one direction, if L is q-ample, then so is its restriction to each subvariety of X from the definition of q-ampleness.

For the other direction, by Proposition 1.2 it is enough to show that for a locally free sheaf $\mathcal{E}$ on X there exists $m_0 > 0$ such that for all $m \geq m_0$ and all $i > q$, the cohomology groups $H^i(X, \mathcal{E}(mL))$ are zero.

Let $E_1, \ldots, E_r$ be the set of torus invariant divisors on X. If L is big, then there is a positive integer k such that kL has a section of the form $s = x_1^{m_r} \cdots x_r^{m_r}$, where x_i is a section that defines E_i and each m_i is strictly positive. Let $D \in |kL|$ denote the divisor of s, supported on the union of the E_i: by hypothesis, L is q-ample on each E_i, and hence on D and $D_{red} = E_1 \cup \cdots \cup E_r$ by [Ott, Proposition 2.3].

Now, if $\mathcal{E}$ is a locally free sheaf on X, we have the exact sequence

$$0 \to \mathcal{E}((m-1)D) \to \mathcal{E}(mD) \to \mathcal{E}(mD) \otimes \mathcal{O}_D \to 0$$

which, by the q-ampleness of L on D, shows that $H^i(X, \mathcal{E}((m-1)L) \to H^i(X, \mathcal{E}(mL))$ is surjective for $i > q$ and m large. It follows that there is an $m_0 > 0$ such that for each $i > q$, the canonical map

$$H^i(X, \mathcal{E}(mD)) \to \varinjlim H^i(X, \mathcal{E}(mD)) \simeq H^i(X - D, \mathcal{E})$$

is an isomorphism for each $m \geq m_0$. But the complement of D is the torus $(\mathbb{C}^*)^{\dim X} \subset X$. In particular, $X - D$ is an affine variety, and all higher cohomology groups vanish here. Hence $H^i(X, \mathcal{E}(m_0 L)) = 0$ for each $i > q$ and L is q-ample. $\qquad\square$

We remark that the above proof applies unchanged to any $\mathbf{Q}$-factorial Mori dream space X if we let the E_i denote any set of divisors whose linear equivalence classes span the effective cone of X. The fact that the complement of the union of the E_i is still affine follows from the fact that there is an embedding $X \hookrightarrow T$ into a toric variety such that the E_i are exactly the restriction of the torus-invariant divisors of T: hence $X - \cup_i E_i$ is the intersection of X with a torus in T, and so is affine.

5 q-ampleness of $-K_X$

A natural question is how to describe varieties for which $-K_X$ is q-ample, for different values of q. When $q = 0$, this means X is a Fano variety. When $q = \dim X - 1$, as mentioned in Section 3, this means that K_X is not pseduoeffective, which in turn by Boucksom–Demailly–Păun–Peternell [BDPP] means X is uniruled. The geometric meaning of q-ampleness of $-K$ remains unclear for the intermediate cases $0 < q < \dim X - 1$. For instance, if X is a threefold

with $-K_X$ 1-ample, then X need not be rationally connected: an example is $X = \mathbb{P}(\Omega_S^1)$, where S is a general quartic surface [DPS, Example 5.6].

For 3-folds, Demailly–Peternell–Schneider [DPS, Problem 5.9] asked whether 1-ampleness of $-K_X$ is preserved under flips. The following example gives a negative answer to this question.

We consider two projective toric varieties $X = X(\Delta_1)$ and $Y = Y(\Delta_2)$ whose fans have rays spanned by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}$$

and whose maximal cones are the following:

$$\Delta_1 \quad : \quad \{\langle 0,1,2 \rangle, \langle 0,2,3 \rangle, \langle 0,3,4 \rangle, \langle 0,4,1 \rangle, \langle 5,1,2 \rangle, \langle 5,2,3 \rangle, \langle 5,3,4 \rangle, \langle 5,4,1 \rangle\}$$
$$\Delta_2 \quad : \quad \{\langle 0,1,3 \rangle, \langle 1,2,3 \rangle, \langle 0,3,4 \rangle, \langle 0,4,1 \rangle, \langle 5,1,2 \rangle, \langle 5,2,3 \rangle, \langle 5,3,4 \rangle, \langle 5,4,1 \rangle\}$$

(Here an integer i denotes the ray spanned by the i-th column of the matrix.)

The variety X is smooth, while Y has exactly one singular point, which is a $\mathbf{Z}/2$-quotient singularity. The fans of X and Y are both refinements of the fan Δ_3 obtained by replacing the first two cones in either fan above by the non-simplicial cone $\langle 0,1,2,3 \rangle$. If Z is the toric variety defined by Δ_3, then both X and Y are partial resolutions of Z; in particular there is a birational map $\phi : X \dashrightarrow Y$. The indeterminacy locus of ϕ is the rational curve C corresponding to the cone $\langle 0,2 \rangle$. One calculates that $K_X \cdot C < 0$, and so $\phi : X \dashrightarrow Y$ is a flip.

Proposition 5.1. *Let X and Y be as above. Then $-K_X$ is 1-ample, while $-K_Y$ is not.*

Proof. On any simplicial toric variety, the anticanonical divisor is big, so we can apply Theorem 4.1 to the divisors $-K_X$ and $-K_Y$. One checks (e.g., using the NormalToricVarieties package in Macaulay2) that $-K_X$ restricts to a 1-ample line bundle on each torus-invariant surface in X. (Recall that the 1-ample cone of a surface is the complement of the negative of the pseudoeffective cone.) On the other hand, it is straightforward to check using the formulas of [Fu, Section 5.1] that the ($\mathbf{Q}$-Cartier) divisor $-K_Y$ is numerically trivial when restricted to the divisor D_0 corresponding to the vector $(1,0,0)$, so it cannot be 1-ample. $\square$

6 A Kodaira–Nakano type vanishing theorem

One reason for studying partial positivity comes from the possibility of vanishing theorems. Unfortunately, the analogue of the Kodaira vanishing theorem does not hold for q-ample line bundles in general: in fact, it fails already in the case of the 3-dimensional flag variety SL_3/B. In this section, we will show that the Kodaira vanishing does hold on a projective toric variety.

We mention that Greb–Küronya [GK] proved a related vanishing theorem for q-ample line bundles on smooth varieties under the additional assumption that the line bundle admits a global section with mild singularities (so that the usual proof of Kodaira's vanishing theorem using Hodge theory goes through for higher q). In our case, however, we do not require the line bundle to be effective.

Our vanishing theorem applies also for singular toric varieties. Here the sheaf of p-forms Ω^p_X may fail to be locally free, but there is a reflexive sheaf which serves as a good substitute, namely the sheaf of Zariski p-forms. This sheaf is defined by $\widehat{\Omega}^p_X = (\Omega^p_X)^{\vee\vee} = i_* \Omega^p_{X_{reg}}$. Note that this coincides with Ω^p_X when X is smooth.

The proof of Theorem 6.1 follows the standard proof of the Kodaira vanishing theorem in the toric case [CLS, §9.3] and uses the "toric Frobenius" map as its main ingredient. Let $X = X(\Delta)$ be the normal toric variety defined by a fan Δ in a lattice N. For each integer $m \geq 1$, multiplication by m gives a map $F_m : N \to N$. This induces a finite, surjective toric morphism $f : X \to X$ with the two properties

(i) $f^* L = mL$; and

(ii) there is a split injection $\widehat{\Omega}^p_X \to f_* \widehat{\Omega}^p_X$ (see [CLS, §9.3]).

With these properties, we are now ready to prove

Theorem 6.1. *Let L be a q-ample line bundle on a normal projective toric variety X. Then for all $i > q$ and $p \geq 0$, we have*

$$H^i(X, \widehat{\Omega}^p_X \otimes L) = 0. \tag{6}$$

Moreover $H^i(X, -L) = 0$ for all $i < n - q$.

Proof. Tensoring the map in (ii) above with L, we obtain a split injection

$$\widehat{\Omega}^p_X \otimes L \to f_* \widehat{\Omega}^p_X \otimes L = f_*(\widehat{\Omega}^p_X \otimes mL).$$

Since f is affine, the Leray spectral sequence shows that there is an injection of cohomology groups

$$H^i(X, \widehat{\Omega}^p_X \otimes L) \to H^i(X, \widehat{\Omega}^p_X \otimes mL).$$

However, the right-hand side vanishes for m large and $i > q$, and so we get (6).

The last statement follows in the same way, by considering the injection

$$H^i(X, -L) \to H^i(X, f_*(-mL)) \cong H^i(X, -mL) \cong H^{n-i}(X, mL \otimes \omega_X)^\vee$$

Here ω_X denotes the dualizing sheaf of X; the last isomorphism comes from the fact that Serre duality holds for any normal projective toric variety [CLS, §9.2]. $\qquad\square$

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