Jordan–Kronecker invariants of finite-dimensional Lie algebras

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Abstract

For any finite-dimensional Lie algebra we introduce the notion of Jordan-Kronecker invariants, study their properties and discuss examples. These invariants naturally appear in the framework of the bi-Hamiltonian approach to integrable systems on Lie algebras and are closely related to Mischenko-Fomenko's argument shift method. We also state a generalised argument shift conjecture and prove it for many series of Lie algebras.

1 Motivation and historical remarks

A Lie algebra $\mathfrak{g}$ is defined by its structure tensor c_{ij}^k. The invariants of $\mathfrak{g}$ are, in essence, those of c_{ij}^k. This tensor is quite complicated to study and it is natural to try somehow to simplify it first. The classical method is to consider, instead of this tensor, a simpler object, namely, the operator $\mathrm{ad}_\xi = \left(\sum c_{ij}^k \xi^i\right)$ for a generic vector $\xi \in \mathfrak{g}$. This operator defines the decomposition of $\mathfrak{g}$ into generalised eigenspaces: the generalised 0-eigenspace is known as a Cartan subalgebra, the others are root subspaces. Using this approach systematically leads, in particular, to the classification of semisimple Lie algebras.

We are going to do a similar thing but instead of the operator ad_ξ, we suggest to consider the bilinear form $\mathcal{A}_x = \left(\sum c_{ij}^k x_k\right)$ for a regular covector $x \in \mathfrak{g}^*$. This form does not give any non-trivial invariants (except for its corank called the *index* of $\mathfrak{g}$). However, non-trivial invariants immediately appear as soon as we consider a pair of forms $\mathcal{A}_x$ and $\mathcal{A}_a$ for $x, a \in \mathfrak{g}^*$. From the algebraic viewpoint these invariants look quite natural, and their systematic analysis seems to be an interesting mathematical problem. The *Jordan–Kronecker invariants* of $\mathfrak{g}$ are defined to be the invariants of the pair of forms $\mathcal{A}_x$ and $\mathcal{A}_a$ related to a generic pair $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$.

Some already known results become more transparent and receive a new interpretation if we look at them from the viewpoint of Jordan–Kronecker invariants. Besides useful reformulations, in this way one can get new non-trivial results (for example, Theorems 5, 6 and 7 below). We expect that these techniques will be useful in the study of the coadjoint representation of non-semisimple Lie algebras. Moreover, the idea of JK invariants[1] can be naturally transferred to arbitrary finite-dimensional representations [8].

However, the main reason why we have been involved in this area is the generalised "argument shift conjecture" discussed below. Apparently, to prove or disprove it will necessarily require the concept of JK invariants. This conjecture itself seems to be important as the argument shift method is one of few indeed universal constructions which are worth being treated in detail.

The idea of Jordan–Kronecker invariants is based on the results, methods and constructions invented and developed by different mathematicians in different years and sometimes even not related to each other.

[1]Sometimes we use *JK* as abbreviation of *Jordan-Kronecker*.

The main point for us is, no doubt, the argument shift method suggested in 1976 by A.S. Mischenko and A.T. Fomenko [30] as a generalisation of S.V.Manakov's construction [29]. This concept has been analysed, developed and generalised by participants of the seminar "Modern geometric methods" at Moscow State University in the 80s (V.V. Trofimov, A.V. Brailov, Dao Trong Tkhi, M.V. Mescherjakov and others) and many of their results have been extremely important to us.

In the late 80s, I.M. Gelfand and I. Zakharevich discovered an interesting relationship between compatible Poisson brackets, Veronese webs and the Jordan–Kronecker decomposition theorem for a pair of skew-symmetric forms. This observation then played an important role in a series or papers by I.M. Gelfand and I. Zakharevich [16, 17, 18, 54] devoted, in particular, to Kronecker pencils and their applications to the theory of integrable systems.

The Jordan–Kronecker decomposition theorem in full generality is presented in the paper [43] by R. Thompson together with other results on pencils of bilinear forms. The author refers to them as a kind of folkloric results and say that his paper "may be regarded as a supplement to Gantmacher's chapters on pencils of matrices". We do not know who was the first to state and prove this theorem in the form we need, the earliest reference we could find with the help of Yu. Neretin is the paper by G.B. Gurevich [21]. However, Gantmacher's book [15] indeed contains all necessary ingredients for this theorem going back to classical works by K. Weierstrass [50] and L. Kronecker [28] and also a simple explanation of how to deduce the classification of pencils of forms (symmetric of skew-symmetric) from the classification of pencils of linear maps.

In the symplectic case, when one of two skew-symmetric forms is non-degenerate, a transition from the algebraic canonical form of a pair of skew-symmetric matrices to the differential-geometric normal form of a pair of compatible Poisson structures has been carried out by F.-J. Turiel [45]. That was a crucial step in understanding local structure of compatible Poisson structures. However, the description of their normal forms in the general case still remains an open and difficult problem, see [47], [46] for recent development in this area.

In implicit form, the concept of JK invariants can be found in many papers devoted to integrable systems on Lie algebras. Besides the above mentioned papers, first of all we would like to refer to the series of papers by A. Panasyuk [34, 35, 36] where the JK decomposition has been effectively used. Quite explicitly, these techniques have been used in a series of recent papers [7, 10, 8, 23, 49, 56].

Although all these ideas based on the JK decomposition seem to be very useful, they still remain widely unknown. The present paper can be considered as an attempt to summarise them in a unified and systematic way by putting into focus the JK invariants as a natural algebraic object.

The structure of the paper is as follows. Sections 2, 3, 4 can be viewed as an introduction to the main subject of the paper. In Section 2, we recall some basic notions and notation to be used throughout the paper. Section 3 is devoted to the argument shift method, Mischenko–Fomenko conjecture and its generalisation which we consider as the main motivation for our work. In Section 4, we formulate the Jordan–Kronecker decomposition theorem for a pair of skew-symmetric forms and discuss some linear algebraic corollaries from this result. These quite elementary facts will then be "translated" into the language of Lie algebras and will lead us (surprisingly easily) to some not at all obvious results.

This programme will be realised in Sections 6–10 in the context of JK invariants which are introduced in Section 5. The final section is devoted to examples.

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This paper is a revised version of [12].

2 Background: basic notions and notation

Here we recall some basic notions and introduce notation we use throughout the paper. In what follows, we consider vector spaces, Lie algebras and other algebraic objects over $\mathbb{C}$ unless otherwise specified. The transition to the real case is usually straightforward.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{g}^*$ be its dual space. The *Lie–Poisson bracket* on $\mathfrak{g}^*$ is defined as follows:

$$\{f,g\}(x) = \langle x, [df(x), dg(x)]\rangle, \qquad x \in \mathfrak{g}^*, \quad f,g : \mathfrak{g}^* \to \mathbb{C}. \tag{1}$$

The corresponding Poisson tensor is given by the skew-symmetric matrix $\mathcal{A}_x = \left(c_{ij}^k x_k\right)$, i. e., depends linearly on coordinates. The algebra $P(\mathfrak{g})$ of polynomials on $\mathfrak{g}^*$ endowed with this bracket is called the *Lie-Poisson algebra* (associated with $\mathfrak{g}$).

The coadjoint orbits are *symplectic leaves* of the Lie-Poisson bracket. The *Casimir functions* (i.e., functions f satisfying $\{f,g\} = 0$ for all g) are exactly the invariants of the coadjoint representation. Notice that in general we can only guarantee existence of sufficiently many local analytic Casimir functions in a neighborhood of a generic point. But even local Casimirs will be sufficient for our purposes. From the algebraic viewpoint, however, Ad^*-invariant polynomials are much more natural and we denote by $P(\mathfrak{g})^{\mathfrak{g}}$ the subalgebra of polynomial coadjoint invariants or, equivalently, the Poisson centre of $P(\mathfrak{g})$.

The *annihilator* of an element $a \in \mathfrak{g}^*$ is, by definition, the stationary subalgebra of a in the sense of the coadjoint representation:

$$\mathrm{Ann}\, a = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi a = 0\}.$$

In terms of the Lie-Poisson structure, the annihilator of $a \in \mathfrak{g}^*$ is the kernel of the form $\mathcal{A}_a$. We can also characterise $\mathrm{Ann}\, a$ as the "orthogonal complement" of the tangent space of the coadjoint orbit $\mathcal{O}(a)$ at point $a \in \mathfrak{g}^*$:

$$\mathrm{Ann}\, a = \{\xi \in \mathfrak{g} \mid \langle \xi, T_a\mathcal{O}(a)\rangle = 0\}.$$

We will say that $a \in \mathfrak{g}^*$ is *regular*, if its annihilator $\mathrm{Ann}\, a$ has the least possible dimension. In this case $\mathrm{Ann}\, a$ is generated by the differentials $df(a)$ of local analytic coadjoint invariants. Otherwise, they span a certain subspace in $\mathrm{Ann}\, a$.

The *index* of a Lie algebra $\mathfrak{g}$ is the codimension of a regular coadjoint orbit. Equivalently,

$$\mathrm{ind}\, \mathfrak{g} = \min_{x \in \mathfrak{g}^*} \dim \mathrm{Ann}\ x$$

The index can also be characterised as the number of functionally independent local analytic coadjoint invariants, i.e., Casimirs. If $\mathrm{ind}\, \mathfrak{g} = 0$, then the Lie algebra $\mathfrak{g}$ is said to be *Frobenius.*

The *singular set* $\mathsf{Sing} \subset \mathfrak{g}^*$ consists, by definition, of those points $y \in \mathfrak{g}^*$ for which $\mathrm{corank}\, \mathcal{A}_y > \mathrm{ind}\, \mathfrak{g}$, where $\mathcal{A}_y$ is the Lie-Poisson tensor at the point y. In other words, Sing is the union of all coadjoint orbits of non-maximal dimension. Equivalently,

$$\mathsf{Sing} = \{y \in \mathfrak{g}^* \mid \dim \mathrm{Ann}\,(y) > \mathrm{ind}\, \mathfrak{g}\}.$$

3 Generalised argument shift conjecture

We first notice that the formula (1) for the Lie-Poisson bracket on $\mathfrak{g}^*$ can be rewritten in the form

$$\{f,g\}(x) = \mathcal{A}_x\big(df(x), dg(x)\big) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

To each fixed element $a \in \mathfrak{g}^*$, one can assign another well-known Poisson bracket on $\mathfrak{g}^*$ by setting:

$$\{f,g\}_a(x) = \mathcal{A}_a\big(df(x), dg(x)\big) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}. \tag{2}$$

Here we assume $a \in \mathfrak{g}^*$ to be regular although this formula makes sense for an arbitrary a.

The matrices of these Poisson brackets, $\mathcal{A}_x = \left(\sum c_{ij}^k x_k\right)$ and $\mathcal{A}_a = \left(\sum c_{ij}^k a_k\right)$, look similar and are related to the skew-symmetric forms on $\mathfrak{g}^*$ mentioned in Section 1. However the essential difference is that in $\mathcal{A}_x$ we consider x as a variable, whereas $a \in \mathfrak{g}^*$ in $\mathcal{A}_a$ is a fixed element, so that $\{\ ,\ \}_a$ is a constant bracket on $\mathfrak{g}^*$ in contrast to $\{\ ,\ \}$ which is linear.

These two brackets are *compatible* in the sense that each linear combination $\mu\{\ ,\ \} + \lambda\{\ ,\ \}_a$ with constant coefficients μ, λ is a Poisson bracket too.

Recall that from the algebraic viewpoint, a completely integrable system on $\mathfrak{g}^*$ is a complete commutative family (subalgebra) $\mathcal{F} \subset P(\mathfrak{g})$. *Completeness* means that $\mathcal{F}$ contains $\frac{1}{2}(\dim\mathfrak{g} + \operatorname{ind}\mathfrak{g})$ algebraically independent polynomials. Compatible Poisson brackets can be used as a good tool for constructing such families. In this context, the brackets (1) and (2) are related to the argument shift method suggested by A.S.Mischenko and A.T.Fomenko in [30], which is based on the following observation. Let f and g be coadjoint invariants. Notice that the shifts $f(x+\lambda a)$ are exactly Casimir functions for the linear combination $\{\ ,\ \} + \lambda\{\ ,\ \}_a$. Hence $f(x+\lambda a)$ and $g(x+\mu a)$ commute with respect to the both brackets (1) and (2). Using the shifts $f(x+\lambda a)$ as generators one can often construct a big commutative subalgebra of $P(\mathfrak{g})$. Since the polynomial coadjoint invariants of a Lie algebra do not necessarily separate generic orbits, it is convenient to modify this construction.

To that end, consider local analytic invariants $f_1, \dots, f_s$, $s = \operatorname{ind}\mathfrak{g}$ defined in a neighbourhood of $a \in \mathfrak{g}^*$ such that their differentials $df_i(a)$ form a basis of $\operatorname{Ann} a$ (recall that a is regular so that such invariants do exist). Take the Taylor expansions of f_i at a:

$$f_i(a+\lambda x) = f_i^{(0)} + \lambda f_i^{(1)}(x) + \lambda^2 f_i^{(2)}(x) + \lambda^3 f_i^{(3)}(x) + \dots \tag{3}$$

where $f_i^{(k)}(x)$ is a homogeneous polynomial in x of degree k and λ is considered as a formal parameter which will be useful later.

It is not hard to see that the collection of $f_i^{(k)}$'s is somehow equivalent to the family of classical shifts $f(x+\lambda a)$: in the simplest case, for example, when f_i are homogeneous polynomials, $f_i^{(k)}$'s form a spanning set of the family of shifts $f_i(x+\lambda a)$. That is why, in what follows, we replace the classical shifts by the subalgebra $\mathcal{F}_a \subset P(\mathfrak{g})$ generated by the homogeneous polynomials

$$f_i^{(k)}(x), \quad i = 1, \dots, \operatorname{ind}\mathfrak{g},\ k > 0. \tag{4}$$

We call $\mathcal{F}_a$ the *algebra of (polynomial) shifts.* Of course, we could confine ourselves with polynomial Ad^*-invariants from the very beginning and consider, generally speaking, a smaller subalgebra $\mathcal{Y}_a \subset P(\mathfrak{g})$, called a *Mischenko-Fomenko subalgebra* (see Section 9 for details). Many authors prefer this approach, but we believe that our modification is useful at least for the following reason (see [6, 11] for details).

The point is that for constructing generators of $\mathcal{F}_a$ we don't need to know and even to mention Ad^*-invariants of $\mathfrak{g}$ (no matter polynomial or local analytic). These generators can be found explicitly by solving relatively simple systems of linear equations, step by step, starting from linear generators, then quadratic, cubic and so on. We briefly describe this procedure as it is closely related to some algebraic properties of pencils (e.g., see the definition of minimal row and column indices of pencils in [15] and Corollary 2 below).

If f is a local analytic Ad^*-invariant function at a regular point $a \in \mathfrak{g}^*$, then the right hand side of its Taylor expansion (3) satisfies the formal relation

$$\mathcal{A}_{a+\lambda x}\big(df^{(0)} + \lambda df^{(1)}(x) + \lambda^2 df^{(2)}(x) + \lambda^3 df^{(3)}(x) + \dots\big) = 0,$$

which, if we use the fact that $f^{(0)}$ is constant, amounts to the following system of linear recurrence relations[2]:

$$\begin{aligned} \mathcal{A}_a df^{(1)} &= 0 \\ \mathcal{A}_a df^{(2)} &= -\mathcal{A}_x df^{(1)} \\ \mathcal{A}_a df^{(3)} &= -\mathcal{A}_x df^{(2)} \\ &\dots \end{aligned} \tag{5}$$

We now forget about the Ad^*-invariant function f we started with and consider the right hand side of (3) as a formal power series satisfying these relations. The first equation simply means that $df^{(1)} \in \mathrm{Ann}\, a$ and since $f^{(1)}$ is linear, we set $f^{(1)}(x) = \langle x, \xi\rangle$ for some $\xi \in \mathrm{Ann}\, a$. Using this function as "initial condition", we can solve step by step the chain of the above equations to find consecutively $f^{(2)}$, $f^{(3)}$ and so on.

Although the solution is not unique, the system of linear equations we obtain on each next step will be consistent independently of the choice we made on the previous step and this recurrent procedure can always be continued up to infinity (see [11]). As a result, we get a formal series $\sum \lambda^k f^{(k)}(x)$ satisfying (5). We may think of it as a *formal* Ad^*-invariant at the point $a \in \mathfrak{g}^*$. Starting with a basis of $\mathrm{Ann}\, a$, we can find in this way all the generators of $\mathcal{F}_a$ of any fixed degree. This procedure is canonical in the sense that the algebra so obtained will not depend on the choice of formal invariants. Of course, one may equally use local analytic or polynomial invariants $f_1, \dots, f_s$, if their differentials at $a \in \mathfrak{g}^*$ generate $\mathrm{Ann}\, a$, the resulting algebra will be the same. We refer to [6, 11] for further discussion on the relationship between the algebras $\mathcal{F}_a$ and $\mathcal{Y}_a$.

In terms of the algebra $\mathcal{F}_a$ of polynomial shifts, the main result of [30] can be formulated as follows.

Theorem 1 (A.S. Mischenko, A.T. Fomenko [30])**.**
1) *The functions from $\mathcal{F}_a$ pairwise commute with respect to the both brackets $\{\ ,\ \}$ and $\{\ ,\ \}_a$.*
2) *If $\mathfrak{g}$ is semisimple, then $\mathcal{F}_a$ is complete, i.e. contains $\frac{1}{2}(\dim\mathfrak{g} + \mathrm{ind}\,\mathfrak{g})$ algebraically independent polynomials.*

Although in general $\mathcal{F}_a$ is not necessarily complete, A.S. Mischenko and A.T. Fomenko stated the following well known conjecture.

Mischenko–Fomenko conjecture. *On the dual space $\mathfrak{g}^*$ of an arbitrary Lie algebra $\mathfrak{g}$ there exists a complete family $\mathcal{F}$ of commuting polynomials.*

In other words, for each $\mathfrak{g}$ one can construct a completely integrable (polynomial) system on $\mathfrak{g}^*$ or, speaking in algebraic terms, the Lie-Poisson algebra $P(\mathfrak{g})$ always contains a complete commutative subalgebra.

This conjecture was proved in 2004 by S.T.Sadetov [42], see also [5],[48]. However, Sadetov's family $\mathcal{F} \subset P(\mathfrak{g})$ is essentially different from the algebra $\mathcal{F}_a$ of shifts. Thus, it is still an open question whether or not one can modify the argument shift method to construct a complete family

[2]This chain of equations was used by A. Mischenko and A. Fomenko in [30]. Similar relations are known in the theory of integrable systems as *Magri-Lenard scheme*.

of polynomials in bi-involution, that is, commuting with respect to the both brackets (1) and (2). In all the examples we know, the answer is positive which allows us to propose the following bi-Hamiltonian version of the Mischenko–Fomenko conjecture.

Generalised argument shift conjecture. *Let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie algebra. Then for every regular element $a \in \mathfrak{g}^*$, there exists a complete family $\mathcal{G}_a \subset P(\mathfrak{g})$ of polynomials in bi-involution, i.e. in involution w.r.t. the two brackets $\{\ ,\ \}$ and $\{\ ,\ \}_a$.*

In fact, our conjecture can be reformulated in the following equivalent way (see discussion at the end of Section 4): *the algebra $\mathcal{F}_a$ of polynomial shifts can always be extended up to a complete subalgebra $\mathcal{G}_a \subset P(\mathfrak{g})$ of polynomials in bi-involution.*

4 Jordan–Kronecker decomposition theorem

The below theorem gives the classification of pairs of skew-symmetric bilinear forms $\mathcal{A}, \mathcal{B}$ by reducing them simultaneously to an elegant canonical block-diagonal form. We refer to this result as a Jordan–Kronecker decomposition as this canonical form consist of two kinds of blocks, Jordan and Kronecker. This theorem goes back to Weierstrass and Kronecker (see the introduction). A proof of it can be found in [43].

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two skew-symmetric bilinear forms on a complex vector space V. Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:*

$$\mathcal{A} \mapsto \begin{pmatrix} \mathcal{A}_1 & & & \\ & \mathcal{A}_2 & & \\ & & \ddots & \\ & & & \mathcal{A}_k \end{pmatrix}, \qquad \mathcal{B} \mapsto \begin{pmatrix} \mathcal{B}_1 & & & \\ & \mathcal{B}_2 & & \\ & & \ddots & \\ & & & \mathcal{B}_k \end{pmatrix}$$

where the pairs of the corresponding blocks $\mathcal{A}_i$ and $\mathcal{B}_i$ can be of the following three types:

	$\mathcal{A}_i$	$\mathcal{B}_i$
Jordan block ($\lambda_i \in \mathbb{C}$)	$\begin{pmatrix} & J(\lambda_i) \\ -J^\top(\lambda_i) & \end{pmatrix}$	$\begin{pmatrix} & -\mathrm{Id} \\ \mathrm{Id} & \end{pmatrix}$
Jordan block ($\lambda_i = \infty$)	$\begin{pmatrix} & -\mathrm{Id} \\ \mathrm{Id} & \end{pmatrix}$	$\begin{pmatrix} & J(0) \\ -J^\top(0) & \end{pmatrix}$
Kronecker block	$\left(\begin{array}{c\|c} & \begin{array}{cccc} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{array} \\ \hline \begin{array}{ccc} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{array} & \end{array}\right)$	$\left(\begin{array}{c\|c} & \begin{array}{cccc} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{array} \\ \hline \begin{array}{ccc} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{array} & \end{array}\right)$

where $J(\lambda_i)$ denotes the standard Jordan block

$$J(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

As a special case in this theorem, we consider the pair of trivial 1×1 blocks $\mathcal{A}_i = 0$ and $\mathcal{B}_i = 0$. We refer to such a pair as a *trivial* Kronecker block.

Notice that the choice of a canonical basis is not unique. Equivalently, one can say that the automorphism group of the pair $(\mathcal{A}, \mathcal{B})$ is not trivial (this group has been described and studied in [55, 56]). However, the blocks $\mathcal{A}_i$ and $\mathcal{B}_i$ are defined uniquely up to permutation.

For the linear combination $\mathcal{A} + \lambda\mathcal{B}$ we will sometimes use the notation $\mathcal{A}_\lambda$. Besides, we will formally set $\mathcal{A}_\infty = \mathcal{B}$ having in mind that we are interested in these forms up to proportionality so that the parameter λ of the pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$ generated by $\mathcal{A}$ and $\mathcal{B}$ belongs, in fact, to the projective line $\mathbb{C}P^1$.

The rank of the pencil $\mathcal{P}$ is naturally defined as $\operatorname{rank} \mathcal{P} = \max_\lambda \operatorname{rank} \mathcal{A}_\lambda$. The numbers λ_i that appear in the Jordan blocks $\mathcal{A}_i$ of the Jordan–Kronecker canonical form given in Theorem 2 are called *characteristic numbers* of the pencil $\mathcal{P}$. They play the same role as eigenvalues in the case of linear operators. More precisely, λ_i are those numbers for which the rank of $\mathcal{A}_\lambda$ with $\lambda = \lambda_i$ is not maximal, i.e., $\operatorname{rank} \mathcal{A}_{\lambda_i} < \operatorname{rank} \mathcal{P}$.

Instead of two particular forms $\mathcal{A}$ and $\mathcal{B}$, from the geometric viewpoint it is more natural to consider the whole pencil $\mathcal{P}$ generated by them. If we accept this point of view, then $\mathcal{A} = \mathcal{A}_0$ and $\mathcal{B} = \mathcal{A}_\infty$ are just two basis elements of $\mathcal{P}$, which can be replaced by any other pair $\mathcal{A}_\lambda$, $\mathcal{A}_\mu$, $\lambda \neq \mu$. After such a "change of basis", the JK decomposition remains essentially the same but the characteristic numbers change by means of the transformation $\lambda_i \mapsto \dfrac{\lambda_i - \lambda}{\mu - \lambda_i}$. In particular, the case of Jordan blocks with $\lambda_i = \infty$ can always be avoided by replacing $\mathcal{B}$ with $\mathcal{B}' = \mathcal{B} + \mu\mathcal{A}$ for a suitable μ. So from now on, unless otherwise stated, we shall assume that ∞ is not a characteristic number, so that no Jordan block with "infinite eigenvalue" appears.

There is a natural analog of the characteristic polynomial $\mathsf{p}(\lambda)$ whose roots are exactly the characteristic numbers with multiplicities. In order to define $\mathsf{p}(\lambda)$ in invariant terms, we consider all diagonal minors of the matrix $\mathcal{A} + \lambda\mathcal{B}$ of order $\operatorname{rank} \mathcal{P}$ and take the Pfaffians, i.e. square roots, for each of them. They are obviously polynomial in λ. Then $\mathsf{p}(\lambda)$ is the greatest common divisor of all these Pfaffians.

If $\mu \neq \lambda_i$, then we call the form $\mathcal{A}_\mu$ *regular* (in the pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$). The set of characteristic numbers λ_i of the pencil $\mathcal{P}$ will be denoted by Λ.

The size of each Kronecker block is an odd number $2k_i - 1$, $i = 1, \dots, s$. As we shall see below, the numbers k_i have a natural algebraic interpretation and we shall call them the *Kronecker indices*[3] of the pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$. Notice, by the way, that the number of Kronecker blocks s is equal to $\operatorname{corank} \mathcal{P}$. Also we have the following obvious formula:

$$\sum_{i=1}^{s} k_i + \deg \mathsf{p}(\lambda) = \frac{1}{2}(\dim V + \operatorname{corank} \mathcal{P}). \tag{6}$$

The Jordan–Kronecker decomposition theorem immediately implies several important facts. First of all, we can always find a large subspace which is isotropic simultaneously for all forms

[3] If we consider $\mathcal{A} + \lambda\mathcal{B}$ as a pencil of linear operators from V to V^* then, in terminology of [15], $k_i - 1$ are *minimal indices for columns and rows* (in our case, due to skew symmetry, the column and row indices are the same).

from a given pencil $\mathcal{P}$. Speaking more formally, we call a subspace $U \subset V$ *bi-Lagrangian* w.r.t. a pencil $\mathcal{P}$, if U is isotropic for all $\mathcal{A}_\lambda \in \mathcal{P}$ and $\dim U = \frac{1}{2}(\dim V + \operatorname{corank} \mathcal{P})$. In other words, U is a common maximal isotropic subspace for all regular forms $A_\lambda \in \mathcal{P}$.

Corollary 1. *For every pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$, there is a bi-Lagrangian subspace $U \subset V$.*

Proof. The proof is evident: as such a subspace U one can take the direct sum of the subspaces related to the right lower zero blocks of the submatrices $\mathcal{A}_i$ and $\mathcal{B}_i$ in the JK decomposition. □

In fact, this result gives an algebraic explanation of that role which compatible Poisson brackets play in the theory of completely integrable systems: an analog of a bi-Lagrangian subspace is just a complete family of integrals in bi-involution. In particular, Corollary 1 can be understood as an algebraic counterpart for the generalised argument shift conjecture. By using the results of F.-J. Turiel [45], [46] on the local classification of compatible Poisson brackets, one can show that a local version of this conjecture holds true if we replace polynomials by local analytic functions (see also paper by P. Olver [31]). The problem is to show that these local analytic functions can be chosen as polynomials. Turiel's construction uses arguments from local differential geometry which do not guarantee any kind of "polynomiality".

Let us list some more corollaries of Theorem 2 having important applications in the theory of bi-Hamiltonian systems.

Let $U \subset V$ be a bi-Lagrangian subspace. By definition, U is maximal isotropic with respect to each regular form A_λ, $\lambda \notin \Lambda$. This implies that U contains $\operatorname{Ker} A_\lambda$ for all $\lambda \notin \Lambda$. Hence, it makes sense to consider the subspace

$$L = \sum_{\lambda \notin \Lambda} \operatorname{Ker} \mathcal{A}_\lambda \subset V. \tag{7}$$

In terms of the JK decomposition, L can be characterised in a very natural way. Namely, for each Kronecker block consider the isotropic subspace that corresponds to the right lower zero block. Then L is just the direct sum of these isotropic subspaces (over all Kronecker blocks).

The subspace L admits another useful description. Assume that $\mathcal{B}$ is regular in $\mathcal{P} = \{\mathcal{A} + \lambda\mathcal{B}\}$. The first observation is that for every $v^{(0)} \in \operatorname{Ker} \mathcal{B}$ there exists a sequence of vectors $\{v^{(k)} \in V\}$, finite or infinite, such that the expression $v(\lambda) = \sum_{k=0}^{r} v^{(k)} \lambda^k$ is a formal solution of the equation

$$(\mathcal{B} + \lambda\mathcal{A}) v(\lambda) = 0, \tag{8}$$

with λ being a formal variable. For an infinite sequence we set $r = \infty$. The following statement easily follows from Theorem 2.

Corollary 2. *Let $k_1 \le k_2 \le \dots \le k_s$ be the Kronecker indices of $\mathcal{P} = \{\mathcal{A} + \lambda\mathcal{B}\}$ and $\mathcal{B}$ be regular. Suppose the expressions*

$$v_i(\lambda) = \sum_{k=0}^{m_i} v_i^{(k)} \lambda^k, \quad \textit{where } v_i^{(k)} \in V, \ i = 1, \dots, s = \operatorname{corank} \mathcal{P},$$

are formal solutions of (8) *such that their initial vectors $v_i(0) = v_i^{(0)}$ form a basis of* $\operatorname{Ker} \mathcal{B}$, *and the numbers $m_i = \deg v_i(\lambda)$ are ordered so that $m_1 \le m_2 \le \dots \le m_s$. Then*

1) *$m_i \ge k_i - 1$ for $i = 1, \dots, s$,*

2) *the linear span of all $v_i^{(k)}$ coincides with the subspace $L \subset V$.*

In fact, by considering each Kronecker block separately, one can easily find a set of polynomial solutions $u_1(\lambda), \dots, u_s(\lambda)$ of (8) with $\deg u_i(\lambda) = k_i - 1$. Such a set satisfies the following natural property: any other polynomial solution $v(\lambda)$ of (8) can be uniquely represented as $v(\lambda) = \sum c_i(\lambda) u_i(\lambda)$ where $c_i(\lambda)$ are some polynomials. Another property of such a *basis set* is the following algebraic formula.

Corollary 3. *Let $u_1(\lambda),\dots,u_s(\lambda)$ be solutions of* (8) *with* $\deg u_i(\lambda)=k_i-1$ *such that* $u_1(0),\dots,u_s(0)$ *form a basis of* $\operatorname{Ker}\mathcal{B}$*, then*

$$\underbrace{(\mathcal{B}+\lambda\mathcal{A})\wedge\dots\wedge(\mathcal{B}+\lambda\mathcal{A})}_{k\ \text{times}}=c\cdot\mathsf{p}(\lambda)\cdot\star\big(u_1(\lambda)\wedge u_2(\lambda)\wedge\dots\wedge u_s(\lambda)\big)\tag{9}$$

where $c\neq 0$ is a constant, $2k=\dim V-s$, $\mathsf{p}(\lambda)$ is the characteristic polynomial of the pencil $\mathcal{B}+\lambda\mathcal{A}$ and $\star:\wedge^s V\to\wedge^{n-s}V^*$ denotes the operator (isomorphism) acting by

$$\star(\xi_1\wedge\dots\wedge\xi_s)(\eta_1,\dots,\eta_{n-s})=\sigma(\xi_1,\dots,\xi_s,\eta_1,\dots,\eta_{n-s}),\quad \xi_i,\eta_j\in V,\ n=\dim V,$$

where σ is a volume form on V. The form $v_1(\lambda)\wedge v_2(\lambda)\wedge\dots\wedge v_s(\lambda)$ is not zero for all $\lambda\in\mathbb{C}$, i.e., the vectors $v_1(\lambda),\dots,v_s(\lambda)$ are linearly independent.

The next statement summarises the properties of L.

Corollary 4.

1. *The subspace $L\subset V$ is bi-isotropic, i.e., isotropic w.r.t. all forms $\mathcal{A}_\lambda\in\mathcal{P}$.*
2. *L is contained in every bi-Lagrangian subspace $U\subset V$. Moreover, L can be characterised as the intersection of all bi-Lagrangian subspaces.*
3. $\dim L=\sum_{i=1}^s k_i$*, where $k_1,\dots,k_s$ are the Kronecker indices of $\mathcal{P}$.*

A characterisation of Kronecker pencils (i.e., with no Jordan blocks) is given by

Corollary 5. *The following statements are equivalent:*

1. *$\mathcal{P}$ is of Kronecker type, i.e., the JK decomposition of $\mathcal{P}$ has no Jordan blocks;*
2. $\operatorname{rank}\mathcal{A}_\lambda=\operatorname{rank}\mathcal{P}$ *for all $\lambda\in\bar{\mathbb{C}}$, i.e., $\Lambda=\varnothing$;*
3. *the characteristic polynomial of $\mathcal{P}$ is trivial, i.e., $\mathsf{p}(\lambda)=1$;*
4. *the subspace $L=\sum_{\lambda\notin\Lambda}\operatorname{Ker}\mathcal{A}_\lambda$ is bi-Lagrangian;*
5. *a bi-Lagrangian subspace is unique.*

The following statement allows us to compute the number of Jordan blocks (both trivial, i.e., of size 2×2, and non-trivial) for each characteristic number.

Corollary 6. *Let $\mathcal{P}=\{\mathcal{A}+\lambda\mathcal{B}\}$ with B regular. Then for any $\mu\in\mathbb{C}$*

1. $\operatorname{corank}\big(\mathcal{B}|_{\operatorname{Ker}(\mathcal{A}+\mu\mathcal{B})}\big)\geq\operatorname{corank}\mathcal{P}$*;*
2. $\operatorname{corank}\big(\mathcal{B}|_{\operatorname{Ker}(\mathcal{A}+\mu\mathcal{B})}\big)=\operatorname{corank}\mathcal{P}$ *iff the Jordan μ-blocks are all trivial;*
3. *the number of all Jordan μ-blocks is equal to*
$$\frac{1}{2}\big(\dim\operatorname{Ker}(\mathcal{A}+\mu\mathcal{B})-\operatorname{corank}\mathcal{P}\big);$$
4. *the number of non-trivial Jordan μ-blocks is equal to*
$$\frac{1}{2}\left(\operatorname{corank}\big(\mathcal{B}|_{\operatorname{Ker}(\mathcal{A}+\mu\mathcal{B})}\big)-\operatorname{corank}\mathcal{P}\right).$$

These purely algebraic and elementary results have natural analogs (in fact, direct implications) in the theory of integrable systems. Here is a kind of dictionary that allows one to translate “linear algebra” to “Poisson geometry”:

skew-symmetric form	$\longleftrightarrow$	Poisson structure
kernel of a skew-symmetric form	$\longleftrightarrow$	Casimir functions
pencil of skew-symmetric forms	$\longleftrightarrow$	compatible Poisson brackets
isotropic subspace	$\longleftrightarrow$	family of commuting functions
maximal isotropic subspace	$\longleftrightarrow$	integrable system
bi-Lagrangian subspace	$\longleftrightarrow$	complete family of functions in bi-involution

Understanding this relationship allows us not only to interpret, but also to prove many important facts related to compatible Poisson structures and bi-Hamiltonian systems. For example, the direct linear-algebraic analog of the algebra $\mathcal{F}_a$ is the subspace $L \subset V$, see (7), so that the argument shift method (part 1 of Theorem 1) is just a reformulation of item 1 of Corollary 4 in terms of compatible Poisson brackets (1) and (2) on $\mathfrak{g}^*$. The passage from the “classical shifts” $f(x+\lambda a)$ to the algebra of polynomial shifts $\mathcal{F}_a$ in Section 3 is equivalent to the interpretation of L given by Corollary 2.

As another example of this relationship, let us show that the reformulation of the generalised argument shift conjecture given at the end of Section 3 can be understood as a “translation” of item 2 of Corollary 4. Indeed, consider two skew-symmetric forms $\mathcal{A}$ and $\mathcal{B}$ on a vector space V and the subspace $L \subset V$ defined by (7). If $U \subset V$ is a bi-Lagrangian subspace w.r.t. $\mathcal{P} = \{\mathcal{A} + \lambda \mathcal{B}\}$, then according to Corollary 4, L is contained in U and moreover L in the intersection of all bi-Lagrangian subspaces.

In the context of the generalised argument shift conjecture, $\mathcal{F}_a$ and $\mathcal{G}_a$ are analogs of L and U respectively in the sense that at a generic point $x \in \mathfrak{g}^*$ (here “generic” means “on a Zariski open non-empty set”) the subspaces of $V = T^*_x\mathfrak{g}^* \simeq \mathfrak{g}$ generated by the differentials

$$d\mathcal{F}_a(x) = \operatorname{span}\{df(x),\ f \in \mathcal{F}_a\} \quad \text{and} \quad d\mathcal{G}_a(x) = \operatorname{span}\{dg(x),\ g \in \mathcal{G}_a\}$$

are exactly L and U from the above algebraic statement, if we consider on $V \simeq \mathfrak{g}$ the pencil generated by $\mathcal{A}_x$ and $\mathcal{A}_a$ (cf. proof of Theorem 3).

Thus, at a generic point we have inclusion $d\mathcal{F}_a(x) \subset d\mathcal{G}_a(x)$. This immediately implies that $\mathcal{F}_a$ is “in essence contained” in $\mathcal{G}_a$. More precisely, every polynomial $f \in \mathcal{F}_a$ is algebraic over $\mathcal{G}_a$ so that f is automatically in bi-involution with $\mathcal{G}_a$. Therefore if $\mathcal{G}_a$ (complete and in bi-involution) exists then we can always take a larger algebra generated by both $\mathcal{G}_a$ and $\mathcal{F}_a$, which will be an extension of $\mathcal{F}_a$ up to a complete algebra of polynomials in bi-involution.

In fact, the main idea of this paper is just to use this relationship between “linear algebra” and “Poisson geometry” in a systematic way for compatible Poisson brackets (1) and (2) on the dual space $\mathfrak{g}^*$ in order to get some information about the Lie algebra $\mathfrak{g}$ itself and its coadjoint representation.

5 Definiton of Jordan–Kronecker invariants

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ be its dual space. For a pair of points $x, a \in \mathfrak{g}^*$, consider the skew-symmetric forms $\mathcal{A}_x = \left(\sum c^k_{ij} x_k\right)$ and $\mathcal{A}_a = \left(\sum c^k_{ij} a_k\right)$ and the pencil $\{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ generated by them. We say that two pencils have the same *algebraic type*, if they have the same Kronecker blocks and there is one-to-one correspondence between their spectra such that the sizes of Jordan blocks for any corresponding characteristic numbers are the same. Clearly, the algebraic type of $\{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ essentially depends on the choice of x and a. However, it remains “constant” almost everywhere.

Proposition 1. *There exists a non-empty Zariski open subset $U \subset \mathfrak{g}^* \times \mathfrak{g}^*$ such that the algebraic type of the pencil $\{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ is the same for all $(x, a) \in U$.*

Proof. First we consider the vector space $W = \Lambda^2(V^*) \times \Lambda^2(V^*)$ of all pairs of skew symmetric forms $(\mathcal{A}, \mathcal{B})$ with the natural action of the general linear group $\mathrm{GL}(V)$. The Jordan–Kronecker decomposition theorem gives a natural partition of W into finitely many subsets:

$$W = \cup_{\alpha \in I} W_\alpha, \quad W_\alpha \cap W_\beta = \varnothing \text{ if } \alpha \neq \beta,$$

where I denotes the (finite!) set of all possible algebraic types and $W_\alpha \subset W$ is the set of pairs $(\mathcal{A}, \mathcal{B})$ related to a fixed algebraic type $\alpha \in I$.

It is easy to see that W_α is a constructible subset of W. Indeed, consider the set $W_{\alpha,\mathsf{can}}$ of canonical pairs $(\mathcal{A}_{\mathsf{can}}, \mathcal{B}_{\mathsf{can}})$ of a fixed type α. According to Theorem 2, the matrix $\mathcal{A}_{\mathsf{can}}$ is fixed, whereas $\mathcal{B}_{\mathsf{can}} = \mathcal{B}_{\mathsf{can}}(\lambda_1, \dots, \lambda_s)$ depends on characteristic numbers $\lambda_1, \dots, \lambda_s$ so that $W_{\alpha,\mathsf{can}}$ is an affine subspace of W of dimension s from which all the hyperplanes $\lambda_i = \lambda_j$ are removed. In particular, each $W_{\alpha,\mathsf{can}}$ is constructible. Now the subset W_α is just the union of orbits of $\mathrm{GL}(V)$ intersecting $W_{\alpha,\mathsf{can}}$. In other words, W_α is the image of $W_{\alpha,\mathsf{can}} \times \mathrm{GL}(V)$ under the map

$$W \times \mathrm{GL}(V) \to W, \quad \big((\mathcal{A}, \mathcal{B}), P\big) \mapsto (P\mathcal{A}P^\top, P\mathcal{B}P^\top).$$

Hence, W_α is constructible as the image of a constructible set.

Take $V = \mathfrak{g}$. The linear map $\phi : (x, a) \mapsto (\mathcal{A}_x, \mathcal{A}_a)$ induces a partition of $\mathfrak{g}^* \times \mathfrak{g}^*$ into constructible subsets $\phi^{-1}(W_\alpha)$. Since the number of such subsets is finite, one of them must contain a non-empty Zariski open subset, as required. $\square$

Let U be a non-empty Zariski open subset from Proposition 1. We will say that $(x, a) \in U$ is a *generic pair*. The corresponding pencil $\{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ is called *generic* too.

Definition 1. The algebraic type of a generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ is called the *Jordan–Kronecker invariant* of $\mathfrak{g}$.

In particular, we will say that a Lie algebra $\mathfrak{g}$ is of

- *Kronecker* type,
- *Jordan* (symplectic) type,
- *mixed* type,

if the Jordan–Kronecker decomposition for the generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ consists of

- only Kronecker blocks,
- only Jordan blocks,
- both Jordan and Kronecker blocks,

respectively.

Definition 2. The Kronecker indices of a generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ are called the *Kronecker indices* of $\mathfrak{g}$.

Similarly, the characteristic numbers λ_i of a generic pencil $\mathcal{P} = \{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ can be thought of as *characteristic numbers* of $\mathfrak{g}$. However, these are not "numbers" but functions $\lambda_i = \lambda_i(x, a)$ that are well defined (up to ordering) and analytic in a neighborhood of any generic pair (x, a) with a regular (if not, we may still consider λ_i as an analytic map to $\mathbb{C}P^1 = \overline{\mathbb{C}}$). To each of them we can assign the sequence of sizes $m_1(\lambda_i), \dots, m_{s_i}(\lambda_i)$ of the Jordan λ_i-blocks in the canonical JK decomposition of $\mathcal{P}$. These numbers do not depend of (x, a) and can be called *Jordan indices* of $\mathfrak{g}$ relative to λ_i.

6 Basic properties of JK invariants

The next two theorems easily follow from the definition of JK invariants and give characterisation of Lie algebras of Kronecker and Jordan types respectively.

Theorem 3. *The following properties of a Lie algebra* $\mathfrak{g}$ *are equivalent:*

1. $\mathfrak{g}$ *is of Kronecker type,*
2. $\operatorname{codim} \mathsf{Sing} \geq 2$,
3. *the algebra* $\mathcal{F}_a$ *is complete.*

Proof. This theorem is, in fact, the main result of [4]. We give a sketch of proof (see details in [4] and, in a more general case, [11]). A generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ is Kronecker, if and only if the rank of $\mathcal{A}_x + \lambda \mathcal{A}_a = \mathcal{A}_{x+\lambda a}$ is maximal for all λ (Corollary 5), i.e., a generic line $x + \lambda a$ does not intersect the singular set Sing. This is obviously equivalent to the condition $\operatorname{codim} \mathsf{Sing} \geq 2$. The equivalence of 1 and 3 follows directly from Corollary 5 (see items 1 and 3). Indeed, the generators $f_i^{(k)}$ of the algebra $\mathcal{F}_a$ are, by definition, the coefficients of the expansion

$$f_i(a + \lambda x) = f_i^{(0)} + \lambda f_i^{(1)}(x) + \lambda^2 f_i^{(2)}(x) + \lambda^3 f_i^{(3)}(x) + \dots,$$

where f_i is a local Ad^*-invariant of $\mathfrak{g}$ in a neighborhood of $a \in \mathfrak{g}^*$. The differential of $f_i(a + \lambda x)$ satisfies the relation

$$\mathcal{A}_{a+\lambda x} df_i(a + \lambda x) = (\mathcal{A}_a + \lambda \mathcal{A}_x) \sum_{k=1}^{\infty} \lambda^k df_i^{(k)}(x) = 0,$$

which is a particular case of (8). Therefore, by Corollary 2, the subspace $d\mathcal{F}_a(x) \subset \mathfrak{g}$ spanned by the differentials of the generators $f_i^{(k)}$ at $x \in \mathfrak{g}^*$ admits a purely algebraic description in terms of the pencil $\mathcal{A}_{x+\lambda a}$, namely $d\mathcal{F}_a(x)$ coincides with the subspace $L(x, a) \subset \mathfrak{g}$ defined by means of (7):

$$d\mathcal{F}_a(x) = L(x, a) \stackrel{\text{def}}{=} \sum_{\lambda \notin \Lambda} \operatorname{Ker}(\mathcal{A}_x + \lambda \mathcal{A}_a) = \sum_{x+\lambda a \notin \mathsf{Sing}} \operatorname{Ann}(x + \lambda a).$$

The completeness of $\mathcal{F}_a$ means that $L(x, a)$ is bi-Lagrangian for generic $x \in \mathfrak{g}^*$. According to Corollary 5, this condition is equivalent to the property that $\mathfrak{g}$ is of Kronecker type. $\square$

Notice that for Lie algebras of Kronecker type, the generalised argument shift conjecture holds true automatically as the family of shifts $\mathcal{F}_a$ itself is complete and in bi-involution. Examples of such Lie algebras include, first of all, semisimple Lie algebras [30] and semiderect sums $\mathfrak{g} +_\rho V$, where $\mathfrak{g}$ is simple, V is Abelian and $\rho : \mathfrak{g} \to \mathrm{gl}(V)$ is irreducible [3], [41], [25] (see Section 11).

Remark 1. The proof of Theorem 3 contains the following statement which is important on its own. Let $a \in \mathfrak{g}^*$ be regular, then the subspace $d\mathcal{F}_a(x) = \operatorname{span}\{df(x) \mid f \in \mathcal{F}_a\}$ can be characterised in terms of the pencil $\mathcal{A}_{x+\lambda a}$ as $L(x, a) = \sum_{\lambda \notin \Lambda} \operatorname{Ker}(\mathcal{A}_x + \lambda \mathcal{A}_a)$. In particular, according to Corollary 4, item 3 and formula (6)

$$\dim L(x, a) = \sum_{i=1}^{s} k_i(x, a) = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{p}_{x,a}(\lambda), \tag{10}$$

where $k_1(x, a), \dots, k_s(x, a)$ are the Kronecker indices and $\mathsf{p}_{x,a}(\lambda)$ is the characteristic polynomial of $\mathcal{A}_{x+\lambda a}$.

The next theorem is obvious and can be viewed as an interpretation of the notion of a Frobenius Lie algebra ([14], [32]) in terms of JK invariants.

Theorem 4. *The following properties of a Lie algebra* $\mathfrak{g}$ *are equivalent:*

1. $\mathfrak{g}$ *is of Jordan type,*
2. *a generic form* $\mathcal{A}_x$ *is non-degenerate, i.e.,* $\mathfrak{g}$ *is Frobenius,*
3. *the algebra* $\mathcal{F}_a$ *is trivial, i.e.,* $\mathcal{F}_a = \mathbb{C}$.

7 Kronecker blocks and Kronecker indices

Here we focus on Kronecker blocks and discuss some elementary results to illustrate a relationship between Kronecker indices and properties of a Lie algebra $\mathfrak{g}$.

Proposition 2. *Let* $\mathcal{P} = \{\mathcal{A}_{x+\lambda a}\}$ *be a generic pencil,* $x, a \in \mathfrak{g}^*$. *Then:*

1. *the number of Kronecker blocks in the JK decomposition for* $\mathcal{P}$ *equals the index of* $\mathfrak{g}$*;*
2. *the number of trivial Kronecker blocks is greater than or equal to the dimension of the centre of* $\mathfrak{g}$*;*
3. *the number of algebraically independent polynomials in the algebra* $\mathcal{F}_a$ *of shifts equals* $\sum_{i=1}^{s} k_i$*, where* $k_1, \dots, k_s$ *are Kronecker indices of* $\mathfrak{g}$*,* $s = \operatorname{ind}\mathfrak{g}$*, i.e.,*

$$\text{tr.deg.}\, \mathcal{F}_a = \sum_{i=1}^{s} k_i. \tag{11}$$

Proof. Items 1 and 2 are obvious. The third statement follows from Remark 1. □

It is interesting to notice that Kronecker indices give a simple and natural estimate for the degrees of polynomial coadjoint invariants. This result has been recently obtained by A. Vorontsov.

Theorem 5 (A. Vorontsov [49])**.** *Let* $f_1(x), f_2(x), \dots, f_s(x) \in P(\mathfrak{g})$ *be algebraically independent* Ad^**-invariant polynomials,* $s = \operatorname{ind}\mathfrak{g}$*, and* $m_1 \le m_2 \le \dots \le m_s$ *be their degrees,* $m_i = \deg f_i$*. Then*

$$m_i \ge k_i. \tag{12}$$

where $k_1 \le k_2 \le \dots \le k_s$ *are Kronecker indices of* $\mathfrak{g}$.

Proof. This statement follows directly from Corollary 2. Indeed, choose a generic $a \in \mathfrak{g}^*$ such that $df_1(a), \dots, df_s(a)$ generate $\operatorname{Ker}\mathcal{A}_a$ and let (x, a) be a generic pair. Since f_i are Ad^*-invariant, we have the relations similar to (8):

$$(\mathcal{A}_a + \lambda\mathcal{A}_x) \sum_{k=1}^{m_i} \lambda^k df_i^{(k)}(x) = 0,$$

where $f_i(a + \lambda x) = \sum_{k=0}^{m_i} \lambda^k f_i^{(k)}(x)$, and we may apply Corollary 2. Since $f_i^{(0)}$ is constant, after differentiation this term disappears and we may divide the left hand side by λ. As a result, the estimate from Corollary 2 becomes $m_i \ge k_i$, as required. □

Remark 2. Theorem 5 still holds if the number of algebraically independent Ad^*-invariant polynomials is smaller than $\mathrm{ind}\,\mathfrak{g}$. In other words, if $f_1(x), \dots, f_q(x) \in P(\mathfrak{g})^{\mathfrak{g}}$, $q < s = \mathrm{ind}\,\mathfrak{g}$, are algebraically independent and $\deg f_1 \le \dots \le \deg f_q$, then $\deg f_i \ge k_i$, where $k_1 \le \dots \le k_s$ are the Kronecker indices of $\mathfrak{g}$.

Remark 3. This proof gives, in fact, a stronger result. Let $f_1, f_2, \dots, f_s \in P(\mathfrak{g})$, $s = \mathrm{ind}\,\mathfrak{g}$, be algebraically independent Ad^*-invariant polynomials such that their differentials are independent at a regular point a. Then for any $x \in \mathfrak{g}^*$ we have

$$\deg f_i \ge k_i(x, a)$$

where $k_1(x, a) \le \dots \le k_s(x, a)$ are the Kronecker indices of the pencil $\mathcal{A}_{x+\lambda a}$.

Corollary 7. *Let $f_1, \dots, f_s$, $s = \mathrm{ind}\, g$ be algebraically independent Ad^*-invariant polynomials such that*

$$\sum_{i=1}^{s} \deg f_i = \sum_{i=1}^{s} k_i, \tag{13}$$

then

$$\deg f_i = k_i.$$

This observation can sometimes be used to compute Kronecker indices for Lie algebras. For example, if $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is of Kronecker type and, therefore, $\sum k_i = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g})$. On the other hand, the algebra of Ad^*-invariant polynomials is freely generated and its generators $f_1, \dots, f_s$ satisfy $\sum \deg f_i = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g})$ (the numbers $e_i = \deg f_i - 1$ are known as *exponents* of a semisimple Lie algebra $\mathfrak{g}$). Hence $k_i = \deg f_i = e_i + 1$ (A. Panasyuk [34]). See more examples in Section 11.

Theorem 5 is related to the case when $\mathrm{tr.deg.} P(\mathfrak{g})^{\mathfrak{g}} = \mathrm{ind}\,\mathfrak{g}$, i.e., $\mathfrak{g}$ admits a "complete set" of independent polynomial Ad^*-invariants. However, in general, this is not true. Nevertheless, a similar estimate still holds true. We only need to replace the degree of f by another characteristic of an analytic function. Namely, let f be local analytic in a neighborhood of a generic point $a \in \mathfrak{g}^*$ and consider its Taylor expansion:

$$f(a + \lambda x) = g^{(0)} + \lambda g^{(1)}(x) + \lambda^2 g^{(2)}(x) + \lambda^3 g^{(3)}(x) + \dots$$

where $g^{(k)}$ is a homogeneous polynomial in x of degree k. Denote by $m(f)$ the number of algebraically independent polynomials among $g^{(i)}$'s. If a is generic, then $m(f)$ does not depend on a. It is clear that if f is a polynomial, then $m(f) \le \deg f$. Similarly, if $f = \dfrac{p}{q}$ is a rational function, then $m(f) \le \deg p + \deg q$.

Now let $f_1, \dots, f_s$ be independent local analytic Ad^*-invariants, $s = \mathrm{ind}\,\mathfrak{g}$, and $m(f_1) \le m(f_2) \le \dots \le m(f_s)$, then we still have the same estimate

$$m(f_i) \ge k_i, \quad i = 1, \dots, s = \mathrm{ind}\,\mathfrak{g}.$$

Moreover, if $\sum_{i=1}^{s} m(f_i) = \sum_{i=1}^{s} k_i$, then $m(f_i) = k_i$. The proof given in [49] works in this case without any changes.

8 Singular set, fundamental semi-invariant and characteristic numbers

The singular set $\mathsf{Sing} \subset \mathfrak{g}^*$ plays an important role in our construction. Here we briefly discuss some of its elementary properties.

As a subset of $\mathfrak{g}^*$, the singular set Sing is an algebraic variety given by the system of homogeneous polynomial equations of the form:

$$\operatorname{Pf} C_{i_1 i_2 \dots i_{2k}} = 0, \quad 1 \le i_1 < i_2 < \dots < i_{2k} \le \dim \mathfrak{g} \tag{14}$$

where Pf denotes the Pffafian, and $C_{i_1 i_2 \dots i_{2k}}$ is the diagonal submatrix of the skew-symmetric matrix $\mathcal{A}_x = (c_{ij}^k x_k)$, related to the rows and columns with numbers $i_1, i_2, \dots, i_{2k}$, $2k = \dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}$. The case of Abelian Lie algebra should, perhaps, be considered as an exception: in this case $\mathsf{Sing} = \varnothing$. Otherwise, Sing is not empty and contains at least the zero element.

Sing may consist of several irreducible components which, in general, may have different dimensions. One of the simplest examples is the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the singular (non-empty) sets $\mathsf{Sing}_i \subset \mathfrak{g}_i^*$ $(i = 1, 2)$ are irreducible and have different codimensions. Then the singular set for $\mathfrak{g}$ is $\mathsf{Sing} = (\mathsf{Sing}_1 \times \mathfrak{g}_2^*) \cup (\mathfrak{g}_1^* \times \mathsf{Sing}_2)$, i.e., consists of two components with different dimensions.

The codimension of Sing can be arbitrarily large. As an example, consider the semidirect sum of a one-dimensional Lie algebra $\mathfrak{h}$ and an n-dimensional vector space V, where a generator $h \in \mathfrak{h}$ acts on V as a regular semisimple operator. It is easy to check that $\mathsf{Sing} \subset (\mathfrak{h} + V)^*$ is one-dimensional, i.e., $\operatorname{codim} \mathsf{Sing} = n$.

If $\mathfrak{g}$ is a semisimple Lie algebra, then $\operatorname{codim} \mathsf{Sing} = 3$.

The structure of the singular set Sing becomes very important in the case when it has codimension 1. Let us denote by $\mathsf{f}_{\mathfrak{g}}(x)$ the *fundamental semi-invariant* of $\mathfrak{g}$, i.e. the greatest common divisor of all the Pfaffians $\operatorname{Pf} C_{i_1 i_2 \dots i_{2k}}(x)$. Then the singular set Sing can be represented as the union of two subsets

$$\mathsf{Sing}_0 = \{\mathsf{f}_{\mathfrak{g}} = 0\} \quad \text{and} \quad \mathsf{Sing}_1 = \{h_{i_1 i_2 \dots i_{2k}} = 0, \quad 1 \le i_1 < i_2 < \dots < i_{2k} \le \dim \mathfrak{g}\}, \tag{15}$$

where $\operatorname{Pf} C_{i_1 i_2 \dots i_{2k}}(x) = \mathsf{f}_{\mathfrak{g}}(x) \cdot h_{i_1 i_2 \dots i_{2k}}(x)$. Notice that $\operatorname{codim} \mathsf{Sing}_1 \ge 2$ as the polynomials $h_{i_1 i_2 \dots i_{2k}}(x)$ do not have any nontrivial common divisor and $\mathsf{Sing} = \mathsf{Sing}_1$, if $\mathsf{f}_{\mathfrak{g}} = 1$.

Clearly, the characteristic polynomial $\mathsf{p}_{x,a}(\lambda)$ of a generic pencil $\mathcal{A}_{x+\lambda a}$ is just $\mathsf{p}_{x,a}(\lambda) = \mathsf{f}_{\mathfrak{g}}(x + \lambda a)$. In particular, from (6) and (11) we get

$$\text{tr.deg.}\, \mathcal{F}_a = \sum_{i=1}^{\operatorname{ind} \mathfrak{g}} k_i = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}. \tag{16}$$

Also, in the context of the JK invariants, we can characterise Sing_1 as follows. As above, given a pencil $\mathcal{A}_{x+\lambda a}$, we use $k_1(x,a), \dots, k_s(x,a)$ for its Kronecker indices, $\mathsf{p}_{a,x}(\lambda)$ for its characteristic polynomial and $L(x,a) = \sum_{\lambda \notin \Lambda} \operatorname{Ker} \mathcal{A}_{x+\lambda a}$.

Proposition 3. *Let $a \in \mathfrak{g}^*$ be regular. Then the following properties are equivalent:*

1. *the line $x + \lambda a$, $\lambda \in \mathbb{C}$, does not intersect* Sing_1*,*
2. $\mathsf{p}_{a,x}(\lambda)$ *coincides with* $\mathsf{f}_{\mathfrak{g}}(x + \lambda a)$*,*
3. $\dim L(x,a) = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}$*,*
4. $\sum k_i(x,a) = \sum k_i$*, where $k_1, \dots, k_s$ are the Kronecker indices of* $\mathfrak{g}$.

Proof. In view of (6) or, more specifically (10) and (16), items 2, 3 and 4 are obviously equivalent. Thus, we only need to give a geometric interpretation of the condition $\mathsf{p}_{a,x}(\lambda) = \mathsf{f}_{\mathfrak{g}}(x + \lambda a)$.

By definition, $\mathsf{p}_{x,a}(\lambda)$ is the greatest common divisor of all the Pfaffians $\operatorname{Pf} C_{i_1 i_2 \dots i_{2k}}(x + \lambda a)$ considered as polynomials in λ with x and a fixed, see (14). Obviously, $\mathsf{p}_{x,a}(\lambda)$ is divisible by $\mathsf{f}_{\mathfrak{g}}(x + \lambda a)$ but not necessarily coincides with it. In other words, in general $\mathsf{p}_{x,a}(\lambda) = \mathsf{f}_{\mathfrak{g}}(x + \lambda a) \cdot h(\lambda)$ where $h(\lambda)$ can be characterised as the greatest common divisor of the polynomials $h_{i_1 i_2 \dots i_{2k}}(x + \lambda a)$ that define the subset $\mathsf{Sing}_1 \subset \mathsf{Sing}$, see (15).

Thus, the condition $\mathsf{p}_{x,a}(\lambda) = \mathsf{f}_{\mathfrak{g}}(x + \lambda a)$ can be rephrased by saying that $h_{i_1 i_2 \dots i_{2k}}(x + \lambda a)$, as polynomials in λ, have no common divisor. Geometrically, this condition simply means that the line $x + \lambda a$ does not intersect Sing_1, as needed. $\square$

Let $\mathsf{f}_{\mathfrak{g}}$ be non-trivial and

$$\mathsf{f}_{\mathfrak{g}}(x) = \underbrace{f_1(x) \cdot \dots \cdot f_1(x)}_{s_1 \text{ times}} \cdot \dots \cdot \underbrace{f_k(x) \cdot \dots \cdot f_k(x)}_{s_k \text{ times}} \tag{17}$$

be its decomposition into irreducible polynomials so that Sing_0 is the union k irreducible components of codimension 1:

$$\mathsf{Sing}_0 = \mathsf{S}_1 \cup \dots \cup \mathsf{S}_k, \qquad \mathsf{S}_i = \{f_i(x) = 0\}.$$

Along with the polynomial $\mathsf{f}_{\mathfrak{g}}$, we will consider its *reduced* (or *square free*) version:

$$\mathsf{f}_{\mathfrak{g},\mathrm{red}}(x) = f_1(x) \cdot \dots \cdot f_k(x), \tag{18}$$

i.e., each irreducible component appears with multiplicity one. Clearly, $\mathsf{f}_{\mathfrak{g},\mathrm{red}}(x) = 0$ still defines the codimension one singular set Sing_0.

The set Sing_0 and polynomials $\mathsf{f}_{\mathfrak{g}}$, $\mathsf{f}_{\mathfrak{g},\mathrm{red}}$ are closely related to the characteristic numbers of the Lie algebra $\mathfrak{g}$. Indeed, the characteristic numbers $\lambda_\alpha = \lambda_\alpha(x, a)$ of a generic pencil $\mathcal{P} = \{\mathcal{A}_{x+\lambda a}\}$ can be characterised by the simple algebraic condition that $x + \lambda_\alpha a \in \mathsf{Sing}$. Since the pair (x, a) is generic, Sing can be replaced by its codimension one part Sing_0 and we come to the following natural conclusion.

Proposition 4. *Characteristic numbers of $\mathfrak{g}$ exist if and only if* $\operatorname{codim} \mathsf{Sing} = 1$ *and they are the the roots of the characteristic polynomial $\mathsf{p}_{x,a}(\lambda) = \mathsf{f}_{\mathfrak{g}}(x + \lambda a)$.*

According to (17) or (18), the characteristic numbers can be partitioned into k groups $\Lambda_1, \dots, \Lambda_k$ each of which naturally corresponds to one of these irreducible polynomials $f_1(x), \dots, f_k(x)$, namely Λ_i is the set of roots of $p_i(\lambda) = f_i(x + \lambda a)$. Hence we immediately obtain

Proposition 5.

1. *The number of distinct characteristic numbers λ_α of $\mathfrak{g}$ equals the degree of $\mathsf{f}_{\mathfrak{g},\mathrm{red}}$. Similarly, the degree of $\mathsf{f}_{\mathfrak{g}}$ is the number of characteristic numbers with multiplicities.*

2. *More precisely, the number of characteristic numbers in each group Λ_i is equal to the degree of f_i. The multiplicity of a characteristic number $\lambda_\alpha \in \Lambda_i$ is equal to the multiplicity s_i of f_i in the decomposition* (14)*. In particular, all characteristic numbers within a group Λ_i have the same multiplicity.*

3. *If some of the characteristic numbers have different multiplicities, then Sing_0 is reducible.*

By using "general position" argument, it is not hard to show that for characteristic numbers from a fixed group Λ_i, the structure of Jordan blocks is the same too.

Recall that speaking of characteristic numbers λ_α of $\mathfrak{g}$, we consider them as local analytic functions $\lambda_\alpha(x, a)$ defined in a neighbourhood of a generic pair $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$. For applications, however, we need globally defined invariants of the pencil $\mathcal{A}_{x+\lambda a}$. They can be easily constructed by means of Viète's theorem.

Proposition 6. *The symmetric polynomials of characteristic numbers are rational functions of x and a. Moreover, if $a \in \mathfrak{g}^*$ is regular and fixed, then they are polynomial in x.*

In this statement, we can consider all distinct characteristic numbers, or all characteristic numbers with multiplicities, or all characteristic numbers from a certain group Λ_i. The conclusion of this proposition holds true in each of these cases.

In the contest of the generalised argument shift conjecture, the role of characteristic numbers and symmetric polynomials of them is explained by the following statement which is a particular case of the "shift of semi-invariants" method suggested by A.A. Arkhangelskii [2] and then developed by V.V. Trofimov [44], see also [23] for an algebraic proof.

Proposition 7. *Let us consider the fundamental semi-invariant* $\mathsf{f}_\mathfrak{g}(x)$ *and take its Taylor expansion at a point* $a \in \mathfrak{g}^*$*:*

$$\mathsf{f}_\mathfrak{g}(a + \lambda x) = g^{(0)} + \lambda g^{(1)}(x) + \lambda^2 g^{(2)}(x) + \cdots + \lambda^m g^{(m)}(x).$$

Then the homogeneous polynomials $g^{(1)}(x), \dots, g^{(m)}(x)$ *are in bi-involution w.r.t. brackets* (1) *and* (2)*. Moreover, they are in bi-involution with the algebra* $\mathcal{F}_a$*.*

Remark 4. In this proposition, the fundamental semi-invariant $\mathsf{f}_\mathfrak{g}$ can, of course, be replaced by $\mathsf{f}_{\mathfrak{g},\mathrm{red}}$. Then we obtain a fewer number of functions, say, $g^{(1)}_{\mathrm{red}}(x), \dots, g^{(m')}_{\mathrm{red}}(x)$, $m' = \deg \mathsf{f}_{\mathfrak{g},\mathrm{red}} \le m = \deg \mathsf{f}_\mathfrak{g}$, but $g^{(i)}$ will polynomially depend on $g^{(i)}_{\mathrm{red}}$'s. In particular, the maximal number of independent polynomials that we might expect to get in this way is $\deg \mathsf{f}_{\mathfrak{g},\mathrm{red}}$ but not $\deg \mathsf{f}_\mathfrak{g}$.

Proof. Clearly, the polynomials $g^{(1)}, \dots, g^{(m)}$ up to a certain constant (that depends on a) are exactly the symmetric polynomials of characteristic numbers. So this proposition is just a particular case of a well-known statement from the theory of bi-Hamiltonian systems: for any pencil of compatible Poisson structures $\mathcal{P} = \{\mathcal{A} + \lambda \mathcal{B}\}$, its characteristic numbers are in bi-involution. It is also well known that the characteristic numbers are in bi-involution with the Casimirs of every regular Poisson structure $\mathcal{A}_\mu \in \mathcal{P}$, which immediately implies the second statement of the proposition. ∎

Thus, if $\operatorname{codim} \mathsf{Sing} = 1$ and therefore the algebra $\mathcal{F}_a$ of polynomial shifts is not complete, we can always construct a "bigger" subalgebra $\widetilde{\mathcal{F}}_a \subset P(\mathfrak{g})$, still in bi-involution, by adding to $\mathcal{F}_a$ the "shifts" of the fundamental semi-invariant $\mathsf{f}_\mathfrak{g}$, i.e. the polynomials $g^{(1)}, \dots, g^{(m)}$ as additional generators. Is this extended subalgebra $\widetilde{\mathcal{F}}_a$ complete?

First of all, formula (16) shows that to make $\mathcal{F}_a$ complete we need exactly $m = \deg \mathsf{f}_\mathfrak{g}$ additional polynomials. However, our new generators $g^{(1)}, \dots, g^{(m)}$ must be not only algebraically independent over the ground field, but they also must be algebraically independent over $\mathcal{F}_a$. An obvious necessary condition is that $\mathsf{f}_\mathfrak{g} = \mathsf{f}_{\mathfrak{g},\mathrm{red}}$ which is equivalent to the fact that each characteristic number has multiplicity one. The three dimensional Heisenberg Lie algebra shows that this condition, in general, is not sufficient. Nevertheless, the completeness problem for $\widetilde{\mathcal{F}}_a$ admits a quite natural algebraic solution.

First we consider the case when the Lie algebra $\mathfrak{g}$ is Frobenius, i.e. its index is zero. Then Sing is defined by one single polynomial, namely: $\mathsf{f}_\mathfrak{g}(x) = \operatorname{Pf} \mathcal{A}_x = \sqrt{\det\left(c^k_{ij} x_k\right)}$. Assume that this polynomial is square free, i.e., in its decomposition (14) into irreducibles polynomials, all s_i equal 1. This is equivalent to the fact that its degree $\deg \mathsf{f} = \frac{1}{2} \dim \mathfrak{g}$ coincides with the (geometric) degree of the singular set Sing which can be understood as the number of distinct intersection points of a generic line $x + \lambda a$ with Sing. Under this assumption we have the following

Theorem 6. *Let* $\mathfrak{g}$ *be a Frobenius Lie algebra, and the (geometric) degree of* $\mathsf{Sing} \subset \mathfrak{g}^*$ *be equal to* $k = \frac{1}{2} \dim \mathfrak{g}$*. Then a generic pencil* $\mathcal{A}_x + \lambda \mathcal{A}_a$ *is diagonalisable (i.e. has no Jordan blocks of size greater than* 2×2*), all characteristic numbers are distinct, and the coefficients of the characteristic polynomial* $\mathsf{p}_{x,a}(\lambda) = \operatorname{Pf} \mathcal{A}_{x+\lambda a}$ *form a complete family of polynomials in bi-involution.*

Proof. The diagonalisability of $\mathcal{A}_x + \lambda\mathcal{A}_a$ is obvious as all characteristic numbers are distinct. The second statement of the theorem contains one non-trivial ingredient: from the existence of k *distinct* characteristic numbers[4] we can immediately conclude that they are *functionally independent* (by the way, it is for this reason that we need the Jacobi identity). The explanations of this "miracle" comes from the theory of bi-Hamiltonian systems and compatible Poisson brackets. If we consider the so-called recursion operator $R = \mathcal{A}_x\mathcal{A}_a^{-1}$, then the compatibility condition for the Poisson structures $\mathcal{A}_x$ and $\mathcal{A}_a$ immediately implies vanishing the Nijenhuis tensor for R. It is a well-known fact from local differential geometry that non-constant eigenvalues of such operators have to be functionally independent. The point is that R (with zero Nijenhuis tensor) can locally be reduced to a block-diagonal form where each block possesses exactly one eigenvalue and, moreover, this eigenvalue depends only of the coordinates related to the block (see [24], or [9] for the general case)[5]. Thus, the purely algebraic fact (algebraic independence of the coefficients of $\mathsf{p}_{x,a}(\lambda) = \mathrm{Pf}\,\mathcal{A}_{x+\lambda a}$) which would probably be not so easy to prove by algebraic means, turns out to be almost obvious from the viewpoint of bi-Poisson geometry. □

Two examples of Lie algebras satisfying the assumptions of Theorem 6 are given in Section 11.6.

Notice that if the geometric degree of Sing is smaller than $\frac{1}{2}\dim\mathfrak{g}$, then in the case of a Frobenius Lie algebra $\mathfrak{g}$ we can still assert that the coefficients of the reduced polynomial $\mathsf{p}_{\mathrm{red}}(\lambda) = \mathsf{f}_{\mathfrak{g},\mathrm{red}}(x + \lambda a)$ are algebraically independent, i.e., in any case we obtain k independent functions in bi-involution, where k is the geometric degree of Sing.

If $\mathfrak{g}$ is not Frobenius, this statement, in general, fails. As an example, consider a two-step nilpotent Lie algebra with basis $e_1, \dots, e_8$ and relations

$$[e_1, e_2] = e_7, \quad [e_3, e_4] = e_8, \quad [e_5, e_6] = e_7 + e_8.$$

The JK decomposition of a generic pencil consists of 2 trivial Kronecker blocks and 3 trivial Jordan blocks with distinct characteristic numbers $\lambda_1 = \frac{x_7}{a_7}$, $\lambda_2 = \frac{x_8}{a_8}$, $\lambda_3 = \frac{x_7+x_8}{a_7+a_8}$. The fundamental semi-invariant $\mathsf{f}_{\mathfrak{g}} = x_7x_8(x_7 + x_8)$ gives only two independent shifts, but not three.

In the general case, a completeness criterion for $\widetilde{\mathcal{F}}_a$ follows from a beautiful construction due to A. Izosimov [23]. As we noticed, a necessary condition for the completeness of $\widetilde{\mathcal{F}}_a$ is that each characteristic number λ_i (of a generic pencil $\mathcal{A}_{x+\lambda a}$) has multiplicity one. This means that the JK decomposition $\mathcal{A}_{x+\lambda a}$ contains just one trivial 2×2 Jordan λ_i-block and therefore $\dim \mathrm{Ann}\,(x + \lambda_i a) = \mathrm{ind}\,\mathfrak{g} + 2$, moreover, $\mathrm{ind}\,\mathrm{Ann}\,(x+\lambda_i a) = \mathrm{ind}\,\mathfrak{g}$ (see e.g. Proposition 12 below). If in addition, the singular set Sing is smooth at the point $x + \lambda_i a$, then it can be shown that there are only two possibilities: $\mathrm{Ann}\,(x + \lambda_i a)$ is isomorphic to either $\mathfrak{b}_2 \oplus \mathbb{C}^s$ or $\mathfrak{h}_3 \oplus \mathbb{C}^{s-1}$ where $\mathfrak{b}_2$ is a non-Abelian two-dimensional Lie algebra, $\mathfrak{h}_3$ is a Heisenberg Lie algebra of dimension 3 and $s = \mathrm{ind}\,\mathfrak{g}$.

The difference between these two cases can be understood if we look at the differential of λ_i considered as a function of x. It is a simple fact from bi-Poisson geometry that $d\lambda_i(x) \in \mathrm{Ann}\,(x + \lambda_i a)$. In both cases, the centre $\mathfrak{z}$ of $\mathrm{Ann}\,(x + \lambda_i a)$ has codimension two. A. Izosimov observed that $d\lambda_i \in \mathfrak{z}$ if $\mathrm{Ann}\,(x + \lambda a) \simeq \mathfrak{h}_3 \oplus \mathbb{C}^{s-1}$ and $d\lambda_i \notin \mathfrak{z}$ if $\mathrm{Ann}\,(x + \lambda a) \simeq \mathfrak{b}_2 \oplus \mathbb{C}^s$. From the viewpoint of differential geometry, the first case means that λ_i is functionally dependent on the generators of $\mathcal{F}_a$ and therefore does not give any non-trivial contribution to $\widetilde{\mathcal{F}}_a$. On the contrary, the case $\mathrm{Ann}\,(x+\lambda_i a) \simeq \mathfrak{b}_2 \oplus \mathbb{C}^s$ guarantees the independence of λ_i of $\mathcal{F}_a$ and, moreover, independence of all λ_i's modulo $\mathcal{F}_a$. This leads us to the conclusion that a generic singular point $y \in \mathsf{Sing}$ should satisfy the condition $\mathrm{Ann}\,(y) \simeq \mathfrak{b}_2 \oplus \mathbb{C}^s$ and this condition is also sufficient.

To formulate this result rigorously, let us introduce the subset

$$\mathsf{Sing}_{\mathfrak{b}} = \{y \in \mathsf{Sing}_0 \mid \mathrm{Ann}\,(y) \simeq \mathfrak{b}_2 \oplus \mathbb{C}^s\} \subset \mathsf{Sing}_0.$$

[4]To prove the theorem we can obviously pass from coefficients of $\mathsf{f}_{\mathfrak{g}}(x + \lambda a)$ to its roots, i.e., to the characteristic numbers.

[5]Alternatively, one can use the normal form theorem for non-degenerate compatible Poisson structures by F.-J.Turiel [45] from which the desired result immediately follows.

In can be shown that $\mathsf{Sing}_{\mathfrak{b}}$ is always Zariski open in Sing_0, but might be empty.

Theorem 7 (A.Izosimov [23])**.** *The extended Mischenko-Fomenko subalgebra $\widetilde{\mathcal{F}}_a$ is complete if and only if $\mathsf{Sing}_{\mathfrak{b}}$ is dense in Sing_0.*

9 Kronecker indices, Mischenko-Fomenko subalgebras and polynomiality

As we have already noticed, the number of algebraically independent polynomials in $\mathcal{F}_a$ can be easily found with the help of JK invariants, see (16). (Recall that speaking of $\mathcal{F}_a$ we always assume that $a \in \mathfrak{g}^*$ is regular.)

In many cases, however, we need to estimate the dimension of the subspace $d\mathcal{F}_a(x) \subset \mathfrak{g}$ generated by the differentials of polynomials $f \in \mathcal{F}_a$ at a certain point $x \in \mathfrak{g}^*$ without assuming that $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is generic. For example, this question becomes important if we want to describe the set of critical points of $\mathcal{F}_a$, i.e. those points where the dimension of $d\mathcal{F}_a(x)$ drops.

Using JK invariants makes the answer very natural. We just reformulate Proposition 3 by using the interpretation of the subspace $d\mathcal{F}_a(x) = L(x, a)$ in terms of the pencil $\mathcal{A}_{x+\lambda a}$ from the proof of Theorem 3 (see also Remark 1).

Proposition 8. *Let $a \in \mathfrak{g}^*$ be regular and $d\mathcal{F}_a(x) = \mathrm{span}\{df(x),\ f \in \mathcal{F}_a\} \subset \mathfrak{g}$. Then*

$$\dim d\mathcal{F}_a(x) \le \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\, \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}} \tag{19}$$

with equality if and only if the line $x + \lambda a$, $\lambda \in \mathbb{C}$ does not intersect the subset Sing_1.

The statement of Proposition 8 is similar to the Joseph–Shafrir formula [20] for the number of algebraically independent polynomials in the classical Mischenko–Fomenko subalgebra $\mathcal{Y}_a \subset P(\mathfrak{g})$. In many cases $\mathcal{Y}_a$ coincides with our algebra $\mathcal{F}_a$ of polynomial shifts. If $\mathrm{tr.deg.} P(\mathfrak{g})^{\mathfrak{g}} = \mathrm{ind}\, \mathfrak{g}$, the difference between $\mathcal{F}_a$ and $\mathcal{Y}_a$ becomes subtle and is discussed in [6].

Unlike $\mathcal{F}_a$, to define $\mathcal{Y}_a$ one uses only polynomial Ad^*–invariants. Namely, let $f \in P(\mathfrak{g})^{\mathfrak{g}}$ and consider the expansion

$$f(x + \lambda a) = \sum_{k=0}^{\deg f} \lambda^k f^{(k)}(x).$$

The Mischenko–Fomenko subalgebra $\mathcal{Y}_a$ is defined as a subalgebra of $P(\mathfrak{g})$ generated by the polynomials $f^{(k)}(x)$, where f runs over $P(\mathfrak{g})^{\mathfrak{g}}$ (or over the set of its generators). Notice that this definition makes sense for all $a \in \mathfrak{g}$, both regular and singular.

On a formal level, the difference between $\mathcal{Y}_a$ and $\mathcal{F}_a$ is that x and a are interchanged. In both cases we consider the expansions of Ad^*-invariant functions into powers of λ but using two different substitutions: $x + \lambda a$ for $\mathcal{Y}_a$ and $a + \lambda x$ for $\mathcal{F}_a$. (Notice, however, that the pencils $\mathcal{A}_{x+\lambda a}$ and $\mathcal{A}_{a+\lambda x}$ are just two "simplified versions" of the same "full pencil" $\mathcal{A}_{\lambda_1 a + \lambda_2 x}$ and, of course, they have the same algebraic type.)

Interchanging x and a in the proof of Theorem 3 immediately gives the following result which, in different versions, was used by many authors.

Proposition 9. *Let $\mathrm{tr.deg.}\, P(\mathfrak{g})^{\mathfrak{g}} = \mathrm{ind}\, \mathfrak{g}$ and $x \in \mathfrak{g}^*$ be a regular element such that the differentials $df(x)$, $f \in P(\mathfrak{g})^{\mathfrak{g}}$ generate $\mathrm{Ann}\, x$. Consider the subspace $d\mathcal{Y}_a(x) = \mathrm{span}\{df(x),\ f \in \mathcal{Y}_a\} \subset \mathfrak{g}$. Then*

$$d\mathcal{Y}_a(x) = L(a, x) = \sum_{\lambda \notin \Lambda} \mathrm{Ker}\,(\mathcal{A}_a + \lambda \mathcal{A}_x) = d\mathcal{F}_x(a)$$

Hence, Proposition 8 can be reformulated, in terms of $\mathcal{Y}_a$, as follows. If $x \in \mathfrak{g}^*$ is a regular element such that the differentials $df(x)$, $f \in P(\mathfrak{g})^{\mathfrak{g}}$ generate $\operatorname{Ann} x$, then

$$\dim d\mathcal{Y}_a(x) \le \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}$$

with equality if and only if the line $a + \lambda x$, $\lambda \in \mathbb{C}$ does not intersect the subset Sing_1. Since $\operatorname{codim} \mathsf{Sing}_1 \ge 2$, such a line $a + \lambda x$ exists if and only if the element $a \in \mathfrak{g}^*$ itself does not belong to Sing_1 and we get the Joseph–Shafrir result (Theorem 7.2 in [20]) in a slightly different form.

Theorem 8 (Joseph, Shafrir [20])**.** *Let* $\operatorname{tr.deg.} P(\mathfrak{g})^{\mathfrak{g}} = \operatorname{ind} \mathfrak{g}$ *and* $a \in \mathfrak{g}^*$ *(not necessarily regular). Then*

$$\operatorname{tr.deg.} \mathcal{Y}_a(\mathfrak{g}) \le \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}$$

with equality if and only if $a \notin \mathsf{Sing}_1$.

Remark 5. As mentioned above, Theorem 8 is formally different from the original Theorem 7.2 in [20]. Instead of Sing_1, explicitly defined by (15), Joseph and Shafrir consider the complement to a certain subset $\mathfrak{g}^*_{\mathrm{wreg}}$ whose definition is different from ours. Of course, these two results imply that $\mathsf{Sing}_1 = \mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{wreg}}$, but this fact does not seem to be obvious.

Let $f_1, \dots, f_s$, $s = \operatorname{ind} \mathfrak{g}$, be algebraically independent Ad^*-invariant polynomials. Then we always have the estimate $\sum_{i=1}^s \deg f_i \ge \sum_{i=1}^s k_i = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}$ (one can use Theorem 5 (Vorontsov) or easily derive this from Theorem 8 (Joseph–Shafrir)).

For many classes of Lie algebras, this estimate becomes an equality, i.e.,

$$\sum_{i=1}^{s} \deg f_i = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}, \tag{20}$$

known as a *sum rule* which is related to important algebraic properties of $\mathfrak{g}$ (see [38], [33], [20] and Theorem 5). Here we want to look at some of them from the viewpoint of JK invariants. First of all, we notice that according to Corollary 7, the sum rule (20) implies $k_i = \deg f_i$.

We also have the following result that resembles, in the case $\operatorname{codim} \mathsf{Sing} \ge 2$, one very interesting result by D. Panyushev [38, Theorem 1.2].

Proposition 10. *Let* $f_1, \dots, f_s$*,* $s = \operatorname{ind} \mathfrak{g}$*, be algebraically independent homogeneous* Ad^**-invariant polynomials satisfying* (20)*. Then the differentials* $df_1, \dots, df_s$ *are linearly independent at* $x \in \mathfrak{g}^*$ *if and only if* $x \notin \mathsf{Sing}_1$*.*

Proof. As noticed above, (20) implies that $\deg f_i = k_i$. Let (x, a) be a generic pair such that x is regular and the differentials of f_i are linearly independent at x. Consider the vectors $u_i(\lambda) = df_i(x + \lambda a)$. These are polynomial expressions in λ of degree $k_i - 1$ (the degree drops by one after differentiation), satisfying $(A_x + \lambda \mathcal{A}_a) u_i(\lambda) = 0$ and such that $u_1(0), \dots, u_s(0)$ form a basis of $\operatorname{Ker} \mathcal{A}_x$. From Corollary 3, we get

$$\underbrace{\mathcal{A}_{x+\lambda a} \wedge \dots \wedge \mathcal{A}_{x+\lambda a}}_{k \text{ times}} = c \cdot \mathsf{f}_{\mathfrak{g}}(x + \lambda a) \cdot \star\big(df_1(x + \lambda a) \wedge \dots \wedge df_s(x + \lambda a)\big) \tag{21}$$

where $k = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g})$ and $c \ne 0$ is a multiplier which, in general, might depend on both x and a but not on λ. By substituting $\lambda = 0$, we see that c does not depend on a. But then c does not depend on x either as, in fact, x and a are involved into this formula in a symmetric way. Hence c is just a constant. Since (21) holds for generic pairs (x, a) (i.e., on a Zariski open set), then this is an identify for all x, a and $\lambda \in \bar{\mathbb{C}}$.

Now let $x \notin \mathsf{Sing}_1$, then we can find a regular $a \in \mathfrak{g}^*$ such that the line $x + \lambda a$ does not intersect Sing_1. The characteristic polynomial for $A_{x+\lambda a}$ is still $\mathsf{f}_{\mathfrak{g}}(x + \lambda a)$ (see Proposition 3) and applying Corollary 3 to this pencil, we see that the form $df_1(x + \lambda a) \wedge \cdots \wedge df_s(x + \lambda a) \neq 0$ for all λ and in particular for $\lambda = 0$, i.e. $df_1(x), \ldots, df_s(x)$ are linearly independent.

Now consider a non-generic pencil $\mathcal{A}_{x+\lambda a}$ (with $a \in \mathfrak{g}^*$ regular) such that the line $x + \lambda a$ intersects Sing_1. In this case, the characteristic polynomial of $A_{x+\lambda a}$ becomes "bigger" and can be written as $\mathsf{p}_{x,a}(\lambda) = \mathsf{f}_{\mathfrak{g}}(x + \lambda a)\, h(\lambda)$ (see the proof of Proposition 3). Moreover, if the point x itself belongs to Sing_1, then $h(0) = 0$. Hence, formula (9) applied to $\mathcal{A}_{x+\lambda a}$ gives

$$\mathcal{A}_{x+\lambda a} \wedge \cdots \wedge \mathcal{A}_{x+\lambda a} = \mathsf{f}_{\mathfrak{g}}(x + \lambda a)\, h(\lambda) \cdot \beta \tag{22}$$

where β is some exterior form which depends on λ polynomially. Comparing (21) and (22), we get

$$h(\lambda) \cdot \beta = c \cdot \star\big(df_1(x + \lambda a) \wedge \cdots \wedge df_s(x + \lambda a)\big).$$

Substituting $\lambda = 0$ gives $df_1(x) \wedge \cdots \wedge df_s(x) = 0$, i.e., the differentials of $f_1, \ldots, f_s$ are linearly dependent at $x \in \mathsf{Sing}_1$. □

Taking into account [39, Theorem 1.1] by Premet, Panyushev and Yakimova, and the fact that $\operatorname{codim} \mathsf{Sing}_1 \geq 2$, we conclude (just in the same way as it was done in [38]) that under the assumptions of Proposition 10, the algebra $P(\mathfrak{g})^{\mathfrak{g}}$ is polynomial on $f_1, \ldots, f_s$. The same observation, by using a different argument, has been obtained by A. Joseph and D. Shafrir as a remark to the main result of [20] stating that for unimodular Lie algebras with $\mathsf{f}_{\mathfrak{g}}$ an invariant, the polynomiality of $P(\mathfrak{g})^{\mathfrak{g}}$ on $s = \operatorname{ind} \mathfrak{g}$ generators $f_1, \ldots, f_s$ implies (20).

Thus, summarising these facts we come to the following conclusion (communicated to us by an anonymous referee as a conjecture).

Theorem 9. *Let $k_1 \leq \cdots \leq k_s$ be the Kronecker indices of $\mathfrak{g}$ and $f_1, \ldots, f_s \in P(\mathfrak{g})^{\mathfrak{g}}$ be algebraically independent* Ad^**-invariant polynomials with $\deg f_1 \leq \deg f_2 \leq \cdots \leq \deg f_s$, $s = \operatorname{ind} \mathfrak{g}$. Assume that $\mathfrak{g}$ is unimodular and $\mathsf{f}_{\mathfrak{g}} \in P(\mathfrak{g})^{\mathfrak{g}}$. Then the following conditions are equivalent:*

1. *$k_i = \deg f_i$, $i = 1 \ldots, s$;*
2. *$\sum_{i=1}^{s} \deg f_i = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}$;*
3. *$P(\mathfrak{g})^{\mathfrak{g}}$ is polynomial on $f_1, \ldots, f_s$.*

Remark 6. To be more precise, ($1 \Leftrightarrow 2 \Rightarrow 3$) holds for an arbitrary Lie algebra $\mathfrak{g}$, and the additional assumptions on $\mathfrak{g}$ are essential for the implication ($3 \Rightarrow 2$) only. The relationship between (2) and (3) is due to Panyushev [38], Ooms and Van der Bergh [33] and Joseph and Shafrir [20].

Remark 7. In this context, the following observation might be interesting. Let $\mathfrak{g}$ satisfy condition 1 (or, equivalently, 2) from Theorem 9. Then for any pencil $\mathcal{A}_{x+\lambda a}$ such that a is regular and the line $x + \lambda a$ does not intersect Sing_1 (but not necessarily generic), we have $k_i(x, a) = k_i$. In other words, all such pencils have equal Kronecker indices. This follows immediately from Remark 3 and Proposition 3. For example, if $\mathfrak{g}$ is semisimple, then for all pencils $\mathcal{A}_{x+\lambda a}$ such that $x + \lambda a \notin \mathsf{Sing}$, $\lambda \in \bar{\mathbb{C}}$, the Kronecker indices are the same and such pencils are automatically generic.

On the other hand, if we can find two such pairs (x_1, a_1) and (x_2, a_2) (with a_i regular and $x_i + \lambda a_i \notin \mathsf{Sing}_1$, $\lambda \in \mathbb{C}$) having different Kronecker indices (i.e., $k_i(x_1, a_1) \neq k_i(x_2, a_2)$ for some i), then for any collection of algebraically independent polynomials $f_1, \ldots, f_s \in P(\mathfrak{g})^{\mathfrak{g}}$, $s = \operatorname{ind} \mathfrak{g}$, we have the strict inequality $\sum_{i=1}^{s} \deg f_i > \sum_{i=1}^{s} k_i = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{f}_{\mathfrak{g}}$.

10 Elashvili conjecture from the viewpoint of JK invariants

In this section, instead of a generic pair $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$ we consider a pair (y, a) where $y \in \mathfrak{g}^*$ is singular and fixed, whereas $a \in \mathfrak{g}^*$ is still generic.

Let $\operatorname{Ann} y = \{\xi \in \mathfrak{g} \mid \operatorname{ad}^*_\xi y = 0\}$ be the stationary subalgebra of $y \in \mathfrak{g}^*$ with respect to the coadjoint representation. The following estimate is well-known (see, for example, Chapter 2, Section 3.3 in [1]):

$$\operatorname{ind} \operatorname{Ann} y \geq \operatorname{ind} \mathfrak{g}. \tag{23}$$

Notice that in the context of the Jordan-Kronecker decomposition theorem, this estimate is just a particular case of item 1 of Corollary 6 (see below the proof of Proposition 11).

The Elashvili conjecture [6] states that if $\mathfrak{g}$ is semisimple then for any $y \in \mathfrak{g}^* = \mathfrak{g}$ we have the equality

$$\operatorname{ind} \operatorname{Ann} y = \operatorname{ind} \mathfrak{g}. \tag{24}$$

This conjecture was recently proved by J–Y. Charbonnel and A. Moreau [13], see also discussion in [19, 37, 51].

Here is the reformulation of (24) in terms of JK decomposition:

Proposition 11. *Let $y \in \mathfrak{g}^*$ be fixed and $a \in \mathfrak{g}^*$ be generic in the sense that the algebraic type of the pencil $\mathcal{A}_y + \lambda \mathcal{A}_a$ remains unchanged under small perturbation of a. Then*

$$\operatorname{ind} \operatorname{Ann} y = \operatorname{ind} \mathfrak{g}$$

if and only if the JK decomposition of $\mathcal{A}_y + \lambda \mathcal{A}_a$ does not contain non-trivial Jordan blocks with $\lambda_i = 0$. Otherwise, i.e. if there are non-trivial Jordan 0-blocks, we have strong inequality:

$$\operatorname{ind} \operatorname{Ann} y > \operatorname{ind} \mathfrak{g}.$$

Proof. This result is a reformulation of item 2 of Corollary 6 for the pencil $\mathcal{P} = \{\mathcal{A}_y + \lambda \mathcal{A}_a\}$. Indeed, (23) is a particular case of item 1 (when $\mu = 0$). For our pencil, $\operatorname{corank} \mathcal{P} = \operatorname{ind} \mathfrak{g}$, $\operatorname{Ker} \mathcal{A} = \operatorname{Ann} y$ and $\mathcal{B}|_{\operatorname{Ker} \mathcal{A}}$ is just the skew-symmetric form on $\operatorname{Ann} y$ related to the element $\pi(a) \in (\operatorname{Ann} y)^*$ where $\pi : \mathfrak{g}^* \to (\operatorname{Ann} y)^*$ is the natural projection. In particular, $\operatorname{Ker}\,(\mathcal{B}|_{\operatorname{Ker} \mathcal{A}}) = \operatorname{Ann}_{\operatorname{Ann} y} \pi(a)$ and if a is generic, then we have

$$\operatorname{corank}\,(\mathcal{B}|_{\operatorname{Ker} \mathcal{A}}) = \dim \operatorname{Ker}\,(\mathcal{B}|_{\operatorname{Ker} \mathcal{A}}) = \operatorname{ind} \operatorname{Ann} y.$$

Item 2 of Corollary 6 is then equivalent to the desired conclusion. □

Notice that Corollary 6 also says that the difference $\operatorname{ind} \operatorname{Ann} y - \operatorname{ind} \mathfrak{g}$ is twice the number of non-trivial Jordan blocks. An example of a strict inequality in (23) is given in the next section where we discuss the Lie algebra $\mathrm{gl}(n) + \mathbb{R}^{n^2}$.

It would be interesting to understand if the observation made in Proposition 11 could lead to another proof of the Elashvili conjecture and/or to its generalisation to other classes of Lie algebras (not necessarily semisimple).

The above discussion can be helpful to answer the following question. Let $\lambda_\alpha = \lambda_\alpha(x, a)$ be a characteristic number of a generic pencil $\mathcal{A}_{x+\lambda a}$, i.e. $x + \lambda_\alpha a \in \mathsf{Sing}$. What can we say about the number and sizes of the corresponding Jordan λ-blocks?

[6] This conjecture has its origin in the theory of integrable systems on Lie algebras. Namely, in [4] it was proved that the condition $\operatorname{ind} \operatorname{Ann} y = \operatorname{ind} \mathfrak{g}$ is equivalent to the completeness of the family of shifts on the singular coadjoint orbit $O(y)$ and it was pointed out that this equality holds for all singular elements $y \in \mathrm{sl}(n)$. While preparing [4] for publication, the first author discussed his observation with A. G. Elashvili, which resulted in this conjecture, briefly mentioned in [4] too.

Proposition 12.

1) *The number of Jordan λ_α-blocks is equal to* $\frac{1}{2}(\dim \mathrm{Ann}\,(x+\lambda_\alpha a) - \mathrm{ind}\,\mathfrak{g})$.

2) *The number of non-trivial λ_α-blocks (i.e. of size greater than 2×2) is equal to* $\frac{1}{2}(\mathrm{ind}\,\mathrm{Ann}\,(x+\lambda_\alpha a) - \mathrm{ind}\,\mathfrak{g})$.

Proof. See items 3 and 4 of Corollary 6. □

Recall that the factorisation (17) of the fundamental semi-invariant $\mathsf{f}_\mathfrak{g}$ defines a decomposition of the singular set Sing_0 into irreducible components $\mathsf{S}_i = \{f_i(x) = 0\}$, and induces a partition of the set of characteristic numbers Λ into subsets Λ_i, $i = 1, \dots, k$. Namely, a characteristic number λ_α of a generic pencil $\mathcal{A}_{x+\lambda a}$ belongs to Λ_i if $x + \lambda_\alpha a \in \mathsf{S}_i$ or, equivalently, λ_α is a root of the polynomial $f_i(x+\lambda a) = 0$. As we pointed out after Proposition 5, the structure of Jordan blocks for all λ_α's within a group Λ_i is the same. Here is a reformulation of Proposition 12 for each individual group Λ_i of characteristic numbers.

Consider a non-empty Zariski open subset of $U_i \subset \mathsf{S}_i$ which is characterised by the property that $\dim \mathrm{Ann}\, y$ and $\mathrm{ind}\,\mathrm{Ann}\, y$ are constant for all $y \in U_i$. We may call such points *generic in* S_i.

Proposition 13. *Let $\lambda_\alpha \in \Lambda_i$ be a characteristic number of a generic pencil $\mathcal{A}_{x+\lambda a}$ and $y \in U_i$ be a generic point of S_i. Then*

1) *the number of Jordan λ_α-blocks is equal to* $\frac{1}{2}(\dim \mathrm{Ann}\, y - \mathrm{ind}\,\mathfrak{g})$*;*

2) *the number of non-trivial λ_α-blocks is equal to* $\frac{1}{2}(\mathrm{ind}\,\mathrm{Ann}\, y - \mathrm{ind}\,\mathfrak{g})$.

To derive one more fact from Corollary 6, we introduce a subset $V_i \subset \mathsf{S}_i$ of (in some sense also generic) points $y \in \mathsf{S}_i$ satisfying the following conditions:

1. y is a smooth point of S_i, that is $df_i(y) \neq 0$;
2. $y \notin \mathsf{S}_j$ for $j \neq i$;
3. $y \notin \mathsf{Sing}_1$.

These conditions guarantee that for a generic (regular) a, the characteristic polynomial of the pencil $\mathcal{A}_{y+\lambda a}$ is $\mathsf{f}_\mathfrak{g}(y+\lambda a)$ (see Proposition 3) and, moreover, the multiplicity of $\lambda_0 = 0$ as a characteristic number this pencil is exactly s_i.

Then for any $y \in V_i$ we have

$$\dim \mathrm{Ann}\, y \leq \mathrm{ind}\,\mathfrak{g} + 2s_i$$

with equality if and only if $\mathrm{ind}\,\mathrm{Ann}\, y = \mathrm{ind}\,\mathfrak{g}$, where s_i is the multiplicity of f_i in (17). Indeed, the number of Jordan 0-blocks cannot exceed the multiplicity of 0 and coincides with it in the case of absence of non-trivial 0-blocks.

It is interesting to notice (see Section 11.4) that for a Frobenius Lie algebra $\mathfrak{g}$, a stronger result holds: let $y \in V_i$, then $\mathrm{ind}\,\mathrm{Ann}\, y = \mathrm{ind}\,\mathfrak{g}$ if and only if $s_i = 1$.

11 Examples

There are only a few examples where JK invariants have been explicitly described. In this section we discuss some types of Lie algebras for which this can be done. Notice that for all these Lie algebras the generalised argument shift conjecture holds.

11.1 Semisimple case

As was already mentioned, a semisimple Lie algebra $\mathfrak{g}$ is of Kronecker type and its Kronecker indices $k_1, \dots, k_s$, $s = \operatorname{ind}\mathfrak{g} = \operatorname{rank}\mathfrak{g}$ coincide with the degrees of basis invariant polynomials of $\mathfrak{g}$. Equivalently, $k_i = e_i + 1$, where $e_1, \dots, e_s$ are exponents of $\mathfrak{g}$.

For simple Lie algebras, the Kronecker indices are as follows:

- A_n: $2, 3, 4, \dots, n+1$;
- B_n: $2, 4, 6, \dots, 2n$;
- C_n: $2, 4, 6, \dots, 2n$;
- D_n: $2, 4, 6, \dots, 2n-2$ and n;
- G_2: $2, 6$;
- F_4: $2, 6, 8, 12$;
- E_6: $2, 5, 6, 8, 9, 12$;
- E_7: $2, 6, 8, 10, 12, 14, 18$;
- E_8: $2, 8, 12, 14, 18, 20, 24, 30$.

11.2 Semidirect sums

As an example, consider first the Lie algebra $\mathrm{e}(n) = \mathrm{so}(n) + \mathbb{R}^n$ of the group of affine orthogonal transformations. We know that the algebra $\mathcal{F}_a$ of shifts for this Lie algebra is complete [3]. This means that $\mathrm{e}(n)$ is of Kronecker type. To determine the Kronecker indices k_i of $\mathrm{e}(n)$, we may apply Theorem 9. It is well known that the basis coadjoint invariants of $\mathrm{e}(n)$ have the same degrees m_i as those of $\mathrm{so}(n+1)$ (in fact, there is a natural relationship between the invariants of $\mathrm{so}(n+1)$ and $\mathrm{e}(n)$ based on the fact that $\mathrm{e}(n)$ can be obtained from $\mathrm{so}(n+1)$ by the so-called $\mathbb{Z}_2$-contraction). Since in this case we have the exact equality

$$\sum m_i = \frac{1}{2}\big(\dim \mathrm{e}(n) + \operatorname{ind} \mathrm{e}(n)\big) = \frac{1}{2}\big(\dim \mathrm{so}(n+1) + \operatorname{ind} \mathrm{so}(n+1)\big),$$

then the Kronecker indices of $\mathrm{e}(n)$ are exactly $k_i = m_i$. In other words, the JK invariants of the Lie algebras $\mathrm{e}(n)$ and $\mathrm{so}(n+1)$ coincide.

More generally, let $\mathfrak{g}$ be a semisimple Lie algebra with $\mathbb{Z}_2$-grading:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Then we can construct a new Lie algebra $\tilde{\mathfrak{g}}$ that coincides with $\mathfrak{g}$ as vector space, but $\mathfrak{p}$ becomes a commutative ideal (whereas the commutation relations within $\mathfrak{k}$ and between $\mathfrak{k}$ and $\mathfrak{p}$ remain the same as in $\mathfrak{g}$). In such a situation, one says that $\tilde{\mathfrak{g}}$ is obtained from $\mathfrak{g}$ by $\mathbb{Z}_2$-contraction. In the above example, $\mathrm{e}(n)$ and $\mathrm{so}(n+1)$ are related exactly in this way. Is there any natural relationship between the Jordan-Kronecker invariants of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$?

In the context of polynomiality of the algebra of coadjoint invariants $P(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$, a similar question was studied in [38] by D. Panyushev who conjectured that the polynomiality is preserved under $\mathbb{Z}_2$-contractions and proved this property for the majority of pairs $(\mathfrak{g}, \mathfrak{k})$. His approach was based on the implication ($2 \Rightarrow 3$) from Theorem 9 (proved in [38] in the case $\mathsf{f}_{\mathfrak{g}} = 1$) and, roughly speaking, he showed that, in "good" cases, some appropriately chosen generators of $P(\mathfrak{g})^{\mathfrak{g}}$ remain independent under the $\mathbb{Z}_2$-contraction and preserve their degrees.

However, he observed that at least in 4 cases, namely

$$(\mathrm{E}_6, \mathrm{F}_4), \quad (\mathrm{E}_6, \mathrm{D}_5 \oplus \mathbb{C}), \quad (\mathrm{E}_7, \mathrm{E}_6 \oplus \mathbb{C}) \quad \text{and} \quad (\mathrm{E}_8, \mathrm{E}_7 \oplus \mathrm{A}_1),$$

the degrees of free generators, if they at all exist for $\tilde{\mathfrak{g}}$, must change when passing from $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$. Recently, O.Yakimova [52] proved that these four pairs could be the only possible exceptions: in all other cases $P(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ remains polynomial and, moreover, the degrees of generators for $\tilde{\mathfrak{g}}$ coincides with those for $\mathfrak{g}$. In view of Theorem 9, the same is true for the Kronecker indices of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. The situation with the exceptional four cases remained unclear for some time, but finally O.Yakimova has succeeded to show that all of them are indeed counterexamples, i.e. the corresponding algebras $P(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ are not polynomial [53]. For these $\mathbb{Z}_2$-contractions, the question about Kronecker indices remains open.

Another interesting example is the semidirect sum $\mathfrak{g} = \mathrm{sl}(n) + \mathbb{R}^n$ with the natural action of $\mathrm{sl}(n)$ on $\mathbb{R}^n$. In this case, the algebra of Ad^*-invariant polynomials has only one generator. Its degree m is exactly equal to $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) = \frac{1}{2}(n^2 + n)$. We also know that $\mathcal{F}_a$ is complete [3]. Hence we conclude that $\mathfrak{g} = \mathrm{sl}(n) + \mathbb{R}^n$ is a Lie algebra of Kronecker type with one Kronecker block whose size, therefore, equals to $\dim \mathfrak{g}$. Notice, however, that for this conclusion the information about the degree of the co-adjoint invariant is not essential.

More generally, consider the semidirect sum $\mathfrak{g} +_\phi V$, where $\mathfrak{g}$ is simple and $\phi : \mathfrak{g} \to \mathrm{End}(V)$ is irreducible. Such Lie algebras are all of Kronecker type. This fact amounts to the condition $\operatorname{codim} \mathsf{Sing} \geq 2$ which is not obvious at all and follows from three papers [25], [3], [41]. In particular, for these Lie algebras the algebra $\mathcal{F}_a$ of shifts is complete and in bi-involution. For some of them the Kronecker indices can be found by using Theorem 5, but in general the question is open.

11.3 Lie algebra of upper triangular matrices

Let $\mathfrak{t}_n$ be the Lie algebra of upper triangular $n \times n$ matrices. The description of Jordan-Kronecker invariants for $\mathfrak{t}_n$ easily follows from a very interesting paper [2] by A.Arkhangelskii. The main result of [2] is a proof of the generalised argument shift conjecture for $\mathfrak{t}_n$ (the bracket $\{\ ,\ \}_a$ was not discussed in [2], but the complete family of commuting polynomials constructed by A.Arkhangelskii is, in fact, in bi-involution).

If n is even, then $\mathfrak{t}_n$ is of mixed type, i.e., the JK decomposition of a generic pencil $\{\mathcal{A}_{x+\lambda a}\}$ contains both Kronecker and Jordan blocks. The Kronecker indices are closely related to the coadjoint invariants of $\mathfrak{t}_n$ explicitly described in [2]. These invariants are rational functions $f_k = \frac{P_k}{Q_k}$, $k = 1, \dots, \frac{n}{2}$ with $\deg P_k = k + 1$ and $\deg Q_k = k$. The Kronecker indices are exactly $\deg P_k + \deg Q_k$ (cf. discussion after Theorem 5), namely

$$1, 3, 5, \dots, n - 1.$$

The singular set $\mathsf{Sing}_0 \subset \mathfrak{t}_n^*$ is defined by an irreducible polynomial $\mathsf{f}_\mathfrak{g}$ of degree $\frac{n}{2}$. Therefore, $\mathfrak{t}_n$ possesses $\frac{n}{2}$ distinct characteristic numbers, each of multiplicity one. In particular, the Jordan part of a generic pencil $\mathcal{A}_{x+\lambda a}$ is diagonalisable.

A complete family of polynomials in bi-involution is formed by the "shifts" $P_k(x+\lambda a)$, $Q_k(x+\lambda a)$ and $\mathsf{f}_\mathfrak{g}(x + \lambda a)$. Equivalently, we can say that the complete family of polynomials in bi-involution for $\mathfrak{t}_n$ (n is even) is given by Proposition 7, i.e., we need to take the algebra $\mathcal{F}_a$ of shifts and complete it with $\frac{n}{2}$ homogeneous polynomials $g^{(j)}(x)$, $j = 1, \dots, \frac{n}{2}$, defined as coefficients of the Taylor expansion

$$\mathsf{f}_\mathfrak{g}(a + \lambda x) = g^{(0)} + \lambda g^{(1)}(x) + \lambda^2 g^{(2)}(x) + \dots + \lambda^{\frac{n}{2}} g^{(\frac{n}{2})}(x),$$

where $\mathsf{f}_\mathfrak{g}$ is the fundamental semi-invariant.

If n is odd, then $\mathfrak{t}_n$ is of Kronecker type and the Kronecker indices are $1, 3, 5, \dots, n$.

11.4 Lie algebras with arbitrarily given JK invariants

Let $\mathcal{P} = \{\mathcal{A} + \lambda\mathcal{B}\}$ be an arbitrary pencil of skew-symmetric bilinear forms. A natural question to ask is whether $\mathcal{P}$ can be realised as a generic pencil $\mathcal{A}_{x+\lambda a}$ for a suitable Lie algebra $\mathfrak{g}$? In other words, we want to describe all admissible JK invariants of finite dimensional Lie algebras.

First of all, notice that the JK invariants of a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can naturally be obtained from those of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by "summation". In particular, the set of characteristic numbers for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can be understood as the *disjoint* union of the corresponding sets for $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Thus, first it is natural to study the realisation problem for the following simplest cases:

- a single Kronecker block,
- a single λ-block which consists of several Jordan blocks.

Examples of such Lie algebras were constructed and communicated to us by I. Kozlov [27].

The first case is realised for the Lie algebra $\mathfrak{g}$ with the basis $e_1, \dots e_k$, $f_1, \dots, f_{k+1}$ and commutation relations:

$$[e_i, f_i] = f_i, \quad [e_i, f_{i+1}] = -f_{i+1}, \quad i = 1, \dots, k \quad \text{(all the other commutators equal 0)}.$$

This Lie algebra admits the following matrix representation

$$\begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \in \mathrm{gl}(k+2, \mathbb{C}),$$

where A denotes the matrix $\mathrm{diag}(a_1, a_2 - a_1, a_3 - a_2, \dots, a_k - a_{k-1}, -a_k)$, i.e., an arbitrary diagonal matrix with zero trace, and b is a column of length $k+1$ with arbitrary entries.

The index of $\mathfrak{g}$ equals 1. The singular set Sing consists of several connected components each of which has codimension 2 and is defined by two linear equations $f_i = 0, f_j = 0$, $i \neq j$. The Casimir function of the Lie-Poisson bracket on $\mathfrak{g}^*$ is $f_1 f_2 \cdot \ldots \cdot f_{k+1}$.

The second case (a single λ-block) is realised for the following matrix Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} a_0 & x_1 & x_2 & \dots & x_m & b_0 \\ & A_1 & 0 & \dots & 0 & y_1 \\ & & A_2 & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & y_{m-1} \\ & & & & A_m & y_m \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \right\}.$$

Here x_k is an arbitrary row of length n_k, y_k is an arbitrary column of length n_k, and A_k is the $n_k \times n_k$-matrix related to the row $x_k = (x_k^1, \dots, x_k^{n_k})$ in the following way:

$$A_k = \begin{pmatrix} a_0 & x_k^1 & x_k^2 & \dots & x_k^{n_k-2} & x_k^{n_k-1} \\ & a_0 & x_k^1 & \ddots & & x_k^{n_k-2} \\ & & a_0 & \ddots & & \vdots \\ & & & \ddots & x_k^1 & x_k^2 \\ & & & & a_0 & x_k^1 \\ & & & & & a_0 \end{pmatrix}$$

This Lie algebra is Frobenius, its singular set $\mathsf{Sing} \subset \mathfrak{g}^*$ is defined by the linear equation f_0=0, where $f_0 \in \mathfrak{g}$ is the matrix whose entries are all zero except for $b_0 = 1$ in the upper right corner.

Let $n_1 = \max_{k=1,\dots,m} n_k$. Then the JK decomposition of a generic pencil $\mathcal{A}_{x+\lambda a}$ consists of Jordan blocks of sizes $2(n_1+1), 2n_2, \dots, 2n_m$.

Notice that the sizes of these Jordan blocks can be arbitrary with the only restriction that the largest Jordan block is unique, as by construction $n_1 + 1 > n_k$. This restriction turns out to be a general property of non-degenerate Poisson pencils with non-constant characteristic numbers (see [45]) and, therefore, is unavoidable. In particular, there is no Frobenius Lie algebra with diagonalisable λ-blocks if the multiplicity of λ is greater than 1. (It follows from this, by the way, that if a characteristic number of a Frobenius Lie algebra $\mathfrak{g}$ is multiple, then $\operatorname{ind} \operatorname{Ann} y > \operatorname{ind} \mathfrak{g}$ for any generic singular y from the corresponding irreducible component of Sing).

However this restriction disappears if we allow Kronecker blocks. The simplest example which illustrates this phenomenon is the Heisenberg algebra with the basis e_i, f_i, h $(i = 1, \dots, n)$ and relations $[e_i, f_j] = \delta_{ij} h$. A generic pencil $\mathcal{A}_{x+\lambda a}$ consists of one trivial Kronecker block and n Jordan 2×2 blocks with the same characteristic number $\lambda(x, a) = -\frac{\langle h,x\rangle}{\langle h,a\rangle}$.

We hope that these observations will help to solve the realisation problem completely, but so far this problem remains open. The difficulty consists in non-trivial relations between Casimir functions and characteristic numbers. By "non-trivial" we mean that the characteristic numbers can, in general, be functionally dependent of the Casimir functions. If it is not the case, then the splitting theorem recently proved by F.-J.Turiel [46] implies that the JK invariants of a finite-dimensional Lie algebra $\mathfrak{g}$ obey the restriction described above: for each characteristic number, the largest Jordan block is unique.

11.5 Lie algebras of low dimension

The Jordan–Kronecker invariants for Lie algebras of low dimension ≤ 5 have been explicitly computed by Pumei Zhang [55] (the list of such Lie algebras with some additional useful information can be found in [40] and [26]). The complete description of JK invariants can be found in [12] together with complete sets $\mathcal{G}_a$ of polynomials in bi-involution.

11.6 Two examples of Frobenius Lie algebras

The first example is the Lie algebra $\mathfrak{aff}(n) = \mathrm{gl}(n) + \mathbb{R}^n$ of the group of affine transformations. This Lie algebra is Frobenius and, therefore, $\mathfrak{aff}(n)$ is of Jordan type. To determine the sizes of Jordan blocks, we need to describe the structure of the singular set. It can be shown that Sing is defined by one irreducible polynomial $\mathsf{f}_\mathfrak{g}$ of degree $\frac{1}{2} \dim \mathfrak{aff}(n)$. This polynomial is exactly the Pfaffian of the form $\mathcal{A}_x = \left(\sum c_{ij}^k x_k\right)$ which can be rewritten in a much nicer form (see [55] for details). To that end, we use the standard matrix realisation of $\mathfrak{aff}(n)$:

$$\begin{pmatrix} M & v \\ 0_n & 0 \end{pmatrix}$$

where M is an arbitrary $n \times n$ matrix, v is a column vector of length n and 0_n denotes the zero row vector of length n. If we identify this Lie algebra $\mathfrak{aff}(n)$ with its dual space $\mathfrak{aff}(n)^*$ by means of (non-invariant) pairing

$$\langle (M_1, v_1), (M_2, v_2) \rangle = \operatorname{Tr} M_1 M_2 + \operatorname{Tr} v_1^\top v_2$$

then Sing can be defined by the equation $\mathsf{f}_\mathfrak{g}(x) = 0$, where

$$\mathsf{f}_\mathfrak{g}(x) = \det(v, Mv, M^2 v, \dots, M^{n-1} v), \quad x = (M, v) \in \mathfrak{aff}^*(n). \tag{25}$$

Hence, by Theorem 6, this Lie algebra has $\frac{1}{2} \dim \mathfrak{aff}(n)$ distinct characteristic numbers. Each of them has multiplicity 1, i.e., a generic pencil $\mathcal{A}_{x+\lambda a}$ is diagonalisable, and the size of each Jordan block in the JK decomposition is 2×2.

Also, by using Theorem 6, we get

Proposition 14. *For the Lie algebra* $\mathfrak{aff}(n)$*, the generalised argument shift conjecture holds true. As a complete family of polynomials in bi-involution we can take the coefficients of the expansion of* $\mathsf{f}_{\mathfrak{g}}(a+\lambda x)$ *into powers of* λ*, where* $\mathsf{f}_{\mathfrak{g}}$ *is given by* (25).

Another interesting example is $\mathfrak{g} = \mathrm{gl}(n) + \mathbb{R}^{n^2}$, where the vector space $\mathbb{R}^{n^2}$ is realised by $n\times n$ matrices, and $\mathrm{gl}(n)$ acts on it by left multiplication. The matrix realisation of $\mathfrak{g}$ is as follows:

$$\begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix},$$

where all entries are $n\times n$ blocks, and A and C are arbitrary. The index of $\mathfrak{g}$ is zero and, therefore, this Lie algebra is of Jordan type. The set of singular elements is defined (after natural, but not invariant identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the pairing $\langle (A_1, C_1), (A_2, C_2)\rangle = \mathrm{Tr}\, A_1A_2 + \mathrm{Tr}\, C_1C_2$, $(A_i, C_i) \in \mathfrak{g}$) by the equation[7]

$$\mathsf{f}_{\mathfrak{g},\mathrm{red}}(x) = \det C = 0, \quad x = (A, C) \in \mathfrak{g}^*.$$

Since the (geometric) degree of Sing is n, there are n distinct characteristic numbers $\lambda_1, \dots, \lambda_n$. Moreover, the irreducibility of Sing implies (Proposition 5, item 3) that all of them have the same multiplicity n and the sizes of Jordan blocks are the same for each λ_i.

To compute the sizes of Jordan blocks, it is sufficient to describe the annihilator of a generic singular point $y \in \mathsf{Sing} \subset \mathfrak{g}^*$. Straightforward computation shows that $\dim \mathrm{Ann}\, y = 2n - 2$. Hence (see Proposition 12) we have $n - 1$ Jordan blocks and there is only one possibility for their sizes, namely[8]:

$$\underbrace{2,\, 2,\, \dots,\, 2}_{n-2\,\text{times}},\, 4.$$

Theorem 6 does not help to verify the generalised argument shift conjecture in this case, as characteristic numbers have non-trivial multiplicity $n > 1$. But in this case the ideal $\mathfrak{h} = \mathbb{R}^{n^2} \subset \mathrm{gl}(n) + \mathbb{R}^{n^2}$ is commutative and therefore $P(\mathfrak{h}) \subset P(\mathfrak{g})$ can be taken as the desired algebra $\mathcal{G}_a$ of polynomials in bi-involution. The completeness is obvious as n^2 is exactly $\frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\, \mathfrak{g})$.

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