The economic use of time and effort in the teaching and learning of mathematics

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I start with two statements:

1. The learning of very young children before they enter school is impressive.
2. The learning of those same children, later on when they are in secondary school, is less impressive.

With respect to the first, newly born children cannot walk, speak in their first language, control their bowel movements, feed themselves, throw and catch things,…, etc. The list goes on.

For the second, I look at the mathematics curriculum at the end of primary school and compare this with the end of high school and the difference does not seem so profound. Of course, there are many other subjects as well but overall I find myself far more impressed with the learning which takes place in a child's first few years (see Hewitt, 2009, for how observation of this has helped me reflect upon my practice).

I also note that:

1. Children are not taught formally how to do the things they achieve in their pre-school years.
2. The students in secondary school are formally taught in their subject lessons in school.

These statements raise the issue of how we are asking students to work in school and how this relates to the way they worked as younger children before entering school.

During my talk I showed two videos, each available on YouTube, which used Cuisenaire rods to help teach the addition of fractions. The first is a clip[1] which lasts for 5 minutes 48 seconds where there is a lot of verbal explanation. It uses

[1] http://www.youtube.com/watch?v=QuJayqMsXE0 [Accessed 3rd November 2014]

the rods to explain how to work out $\frac{2}{3}+\frac{5}{6}$. The second[2] lasts for 1 minute 32 seconds and has no spoken words as it works on $\frac{1}{2}+\frac{1}{3}$. What explanation there is comes mainly from the way the rods are arranged and some pointing gestures. The dynamics in what might be learned through watching each of these is complex but an oversimplified impression I have is that with the first I am listening to explanations and with the second I am trying to generate explanations. These are examples of two quite different ways in which a learner is being asked to work. There is a temptation here to offer a constructivist perspective, however, in both cases students have to construct their own knowledge, from a radial constructivist viewpoint (von Glasersfeld, 1987) . So, for me the language of construction does not help me to work on the difference I experience when viewing these two videos. Instead I turn to what Gattegno (1971) calls *powers of the mind*.

For each *power of the mind* I will offer an activity to try to help a reader gain a sense of that power from within. Mason (1987) makes reference to the Rig Veda which talks of two birds, one eating the sweet fruit whilst the other looks on without eating. In this spirit, I ask for you to engage in each activity, which you also observe yourself whilst doing so. There are nine powers I will introduce within three separate headings: Guiding; Working with 'material'; and Holding information.

Powers of the mind

I would like to start by asking you to remember the word *pimolitel*.

The powers of the mind are exactly that. This means, since we all have minds, that we all have these powers of the mind. As such, there is nothing profound about these activities. They are designed just to help get in touch with those things which are ordinary and which we use moment by moment, every day of our lives.

[2] http://www.youtube.com/watch?v=1_E_SrpyPvU&list=UUOE7NqEwBhF-bhN7Sh77_Ag [Accessed 3rd November 2014]

Guiding

Activity 1:

Read the following and do as it says:

- If you can read this please put your left hand on your head
- If you can read this the please use your right hand to point to your nose
- If you can read this then say “I am sorry but I cannot read this”
- If you can read this then try to whistle

As you try to read the above note the effort which you are putting into your eyes and the straining involved. The wanting to read is an act of will, and the *Will* places energy to where it is needed in order to try to do what you want to do. As Dewey (1975, p.8) remarked “The exercise of will is manifest in the direction of attention”. Of course, you will only experience this presence of the *Will* if you really tried to read the above. Instead you may have taken one look and decided either not to engage in the activity or started engaging and then quickly decided that the text was too small and so not bothered trying to read it. In such circumstances your *Will* did not place energy into the act of reading and you are unlikely to have noticed anything. If you go back and try to read each line, then note how the energy placed in your sight increases. You may also note the moving of your head forwards. The *Will* is the first *power of the mind* and one which controls the placement of energy within your internal system. It is at the heart of everything we do. As such it is also an indicator of the nature of all the *powers of the mind*; they are within us, no matter what gender, race, socio-economic class or disability we may have. It is an attribute of the mind (Gattegno, 1971) no matter what our circumstances.

A consequence of the *Will* is that energy is channelled somewhere in particular and this results in some things being stressed whilst others, as a consequence, are ignored. Stressing and ignoring is the result of an act of the *Will*.

Activity 2:

Take out a pen and a pencil and hold them both in one hand.

Close your eyes and keep them closed.

Drop both the pen and the pencil so that they fall on the floor.

With your eyes remaining closed, bend down and pick up the pen, not the pencil.

As you bent down, there is an issue about how much you have to bend. This will be judged partly from a bodily ‘memory’ of bending down many times in the past but also from the sense of touch you experience when your hand touches the floor. You may well have found that you bent down a certain amount but your hand had not touched the floor yet and so more bending was needed. Alternatively, you may have bent down too much and found that your hand ‘hit’ the floor and so you came back up a little so that your body was in a position such that the height of your shoulder above the floor was just a little less than the length of your arm. So the amount of bending was informed by the sensations gained from your hand being in contact with the floor. Following this, you may have moved your hand over an area until you felt it touch something. You may then have moved your hand so that your fingertips could come in contact with the object and you had to decide whether this was the shape of something which might be a pen and judge whether it was the pen or the pencil. If that tactile sensation did not ‘feel right’ you would have let go and continued moving your hand along the floor. It might have been the case that your body position needed to change as only a certain area of the floor could be covered from one body position and the pen may have bounced further away. Eventually, you felt that the touch sensations from your fingers were consistent with that of felling a pen, rather than a pencil, and that was when you picked it up with a degree of confidence that you had the pen in your hand.

During this activity, it is possible to gain a sense of how your body position changed so that the height of your shoulder ‘felt right’ in order for you to explore an area of the floor with your hand. You then made use of the tactile sensations in your fingertips until those sensations were consistent with what you would expect from feeling a pen. Your actions are guided by what feels right and consistent with your expectations. This *sense of truth* is a power of the mind which guides your actions.

So there are two *powers of the mind* which are concerned with guiding:

- *Will*
- *A sense of truth*

Working with ‘material’

Let me first address the word ‘material’ (Hewitt, 1997). A common usage of this word relates to the substance with which we might work with in order to create something. For example, curtains are made from material, or the materials with which a shelf is made might include wood, metal brackets and screws. Materials are the things with which we work in order to produce or make something. In a similar way, I can work with ideas in order to produce something, which may or may not be physical. For example, within the sphere of history there are scripts which contain written comments and ideas which a historian may use to argue for a particular perspective upon someone’s life or about a series of events which happened in the past. Although the scripts may be physical, it is the ideas and information which come from the written texts which are the real ‘material’ with which such a historian may work. A mathematician works with certain ideas, theorems and images in order to create a line or argument which can result in a proof. The material with which they work are the awarnesses they have of certain mathematical properties and relationships. A politician may work with statistical information on a particular issue and data on popular opinion about that issue, in order to offer an argument for why a particular policy should be adopted. All of these use ideas, information, images from our senses, etc. We work with those things in order to make our actions and decisions. It is these things with which we work, that I describe as *material*.

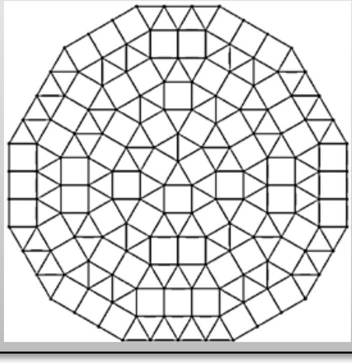
Activity 3:

Write down a ‘sum’ equalling -12 which involves all of these operations:

- Add
- Subtract
- Square
- Cube root

I have provided some material within the box above and you have provided other material from your own knowledge and awareness. However, you have not been told how to use that material in order to succeed with the challenge. You have tried this and tried that, perhaps finding that a little adjustment needs to be made to your initial ideas. You may have used an awareness that certain numbers might be helpful within your calculation, such as cube numbers, and you have made decisions about what to try out through the knowledge and awareness you have and worked within the constraints stated within the task. You had to come up with numbers and ideas of how you might get -12 whilst meeting the constraints stated. Another power of the mind is *creativity*. I am not talking here about exceptional creative talent, but about the everyday ability to generate ideas and ways to proceed with the material at hand given certain constraints. Indeed, the constraints are part of the material with which one works. As such constraints are a necessary aspect of creativity. Whether someone else gets the same expression or not does not change the fact that someone has been creative in order to produce their expression. Creativity, in this sense, has nothing to do with the uniqueness of the final product. The creation by two people of the ‘same’ final expression, will inevitably have involved unique ways in which each person used their creativity to arrive at what looks the same in terms of an expression on paper. The final articulation can never reflect all that has been involved in producing it.

Activity 4: Look at this and say something that you can see.



This image is one for which there is no common name. As such it is difficult to express the whole in words. Instead it is likely that you are driven to attend to the parts which make the whole. There are many things which can be noticed, of which a few are:

- There are squares/triangles;
- There is a triangle in the centre… or is it a square at the centre?;
- If I move out horizontally from the centre, then I see a collection of one, three and then five triangles;
- I see squares with a triangle on each side (but not every square has this);
- I see a triangle with a square on each side (are there more like this?).

Another power of the mind is that of *extraction*. We can extract parts from a whole. Indeed to do otherwise would make our lives almost impossible. There is so much potential within our field of vision alone that to act in any way will require stressing part of what is available. Those people who suffer from a little deafness and wear a hearing aid, talk about finding it difficult to hear in crowded, noisy, situations. It is not because they cannot hear; it is because the hearing aid magnifies all the sounds and is not discriminating. What we use in hearing something is not only the volume level but the ability to stress one particular set of sounds over all the sound waves which enter our ears. Indeed, there are times when you might not have heard something because your attention was elsewhere. The sound may well have been loud enough, it was because your *Will* directed your attention elsewhere. To *hear* is not about sounds being loud enough but about an ability to stress and ignore.

To attend to something has a consequence that some things are stressed whilst others are ignored. This can result in us becoming aware of something in particular. This is the power of *extraction* and something which we all possess. We can, do, and must, extract parts from the whole.

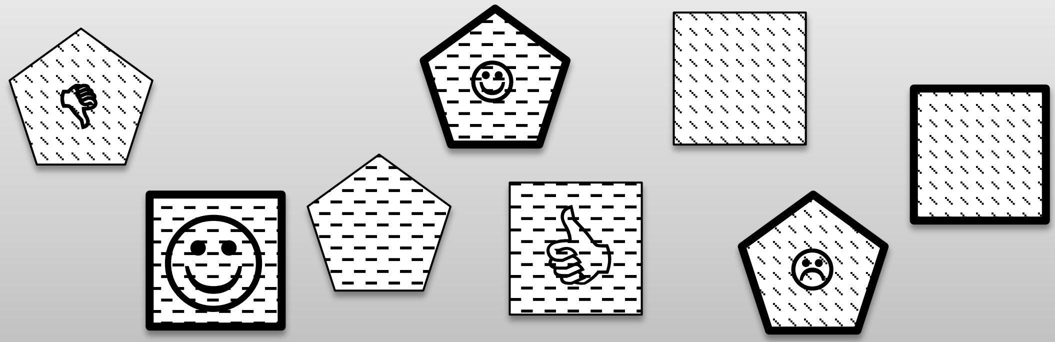
Activity 5: (a) say the following out loud: 1, 2, 3, 4, 5, 6, 7, 8, 9

(b) what does this sign mean?

The symbols 1, 2, 3, etc. are just squiggles on paper. You have associated a sound with each particular squiggle. There is nothing within the squiggle ‘2’ which means you have to say *two*. Indeed, you may well have said *deux*. For someone new to this squiggle, there is nothing about it which someone can ‘work out’ as to how the squiggle is to be said. It is about associating a sound with a squiggle. Likewise the sign is representing a gesture of a finger being placed vertically by the lips, a gesture which we associate with being quiet or not talking. However, there is no reason why such a gesture must mean this. We make associations with signs happening at the same time as the context in which they appear. *Association* is another power of the mind. We have been exposed to two pieces of material – the squiggle ‘2’ and the word ‘two’ – and we are able to associate one with the other.

The next activity I offer is slightly different to the one I used in the talk. This is because the original activity made use of time in the way which cannot be done within just this text.

Activity 6: What sort of thumb or face should be in each of the three remaining shapes?



I have not told you any rules behind which symbol appears within which shapes, except saying that it will be a thumb or a face. Instead I offer examples and a question which implies that there are particular symbols which *should* be in the remaining three shapes. You are likely to have looked at what is the same and what is different in the given shapes, such as there are a number of squares, there are different shadings and some have a bold boundary whilst others do not. Each of these awarenesses come from extracting some things from the whole. I suggest you began to see whether you can associate a face, for example, with some attributes of the shapes. When is there a face and when is there a thumb? You also need to consider the range of different possibilities, there are thumbs and faces, but what kind of thumb and what kind of face? Some are larger than others, some have thumb up and a smiley face, and others have thumb down and a sad face. Within all these variations I suggest that you were looking for what attributes within the shapes are associated with which of these variations. A consistent association can then lead to a sense of spotting rules and from there you might apply those rules to the three remaining shapes. *Abstraction* of patterns and rules from examples is another power of the mind and, as with all powers of the mind, is something which we use on a daily basis.

We have not run our lives by only those things which we are told to do. I can say this as it is not possible for other people to tell us everything that is involved in speaking our first language or knowing how to manage our way around a city we have not visited before. The shear variety of what is involved in such activities means that there is simply too much for us to be told everything. The amount we are told is minute in comparison to the amount we have worked out for ourselves. In our first language, we looked for patterns and rules in how the language seemed to behave and we applied those patterns to similar situations. An example in English is how verbs tend to change when shifting from present to past tense. Instead of having to be told how each separate verb behaves, we apply an observed pattern of adding *–ed* on the end. The evidence is seen every day with children saying sentences such as *I goed to the park yesterday*. They have not been told to say this and so it comes from their ability to abstract a rule from the examples they have heard and apply the perceived rule with other verbs (Ginsburg, 1977). Most often, this results in them saying the verbs correctly, but they then learn that there are exceptions, and these do have to be learned on a more individual basis. Abstraction allows us to deal with new situations based upon what we have noticed and learned from the experiences we have had up to this point in time.

The powers of the mind which relate to working with material are:

- Creativity
- Extraction
- Association
- Abstraction

These are ways in which we work with material to select, link and take forward what we have noticed, into new situations.

Holding information

As well as working with material, we also need to retain information which is of significance to us.

Activity 7:

Several pages ago, I asked you to remember a word.

Without looking back, say that word and write it down.

I have asked people to do some something equivalent to this on many occasions. On each occasion, only about half the people managed to write down the word correctly spelt. This is just one word and, within a relatively short space of time, so many failed to remember it correctly. We all have the power of *memory*, but sometimes as educators, we do not acknowledge the inefficiency of this power. I do not know whether you, the reader, did try to remember this word when I asked you to do so several pages back. If so, I suggest that you spent a certain amount of energy trying to memorise it at the time of reading. You may have used association to try to link it with another word or words that you knew already. Those that were successful at remembering the word, reported shifting attention back to that word on several occasions whilst engaged in the rest of the tasks and talk (or text, in the case of you reading this now).

Remembering and forgetting exist alongside each other. To remember successfully requires significant effort, both at the time of being asked to memorise, and also at intervals thereafter. That is why *practice* has played such a significant role in many classrooms, because it is memory which is called upon so often. There *are* times when we need to memorise but as Gattegno (1986, p.126) said "memory should be relegated to a limited area in education - that it should be used only for that which we cannot invent".

The next activity is again slightly adapted from that which I gave in the talk.

Activity 8:

Consider somewhere you have visited in the last month which is not a place you spend a lot of time on a frequent basis.

Imagine that place now and say out loud two things about the surroundings there.

When you visited that place you did not try to memorise the surroundings just in case you were going to be asked about it when you came to read this article. You have not spent your time in between time to check whether you still remembered it. Indeed, this is something quite different to memorisation. Here, you put no effort at all whilst you were at that particular place and you have not needed to do so since either. It is only now, when asked to recall some things about the surroundings, that you have used a little energy to do so. Another way we hold information is through *imagery* and this is very different in nature to memory.

Activity 9:

- Get a pen or pencil.
- Hold out your hand, palm upwards and place the pen so that the nib is pointing away from you.
- Throw it from one hand to another so that it rotates 360^{o} with nib of the pen ending up pointing in the same direction as it did at the start.
- Do this again, going back to the original hand.
- Continue.
- Now stop. Do not throw it again.

Whether you were successful with this is not important. I am about to ask you a question but before doing so I would like you to put any free hand you have (you can continue to hold this book with one or both hands) on your lap, palm down. Now, without moving your hands, what provided the twist of the pen: fingers, wrist, or something else?

Without the freedom to repeat the physical movements, you are likely to use your imagery in trying to answer this question. What is of significance is that I suggest you were not able to answer this question immediately without recall to some imagery to try to run through doing this movement again. Yet, at the time of doing the activity, I conjecture that you did manage to make the pen twist and so within you, at some level, you knew what to do. Yet now, when asked about it, the answer is not immediately available despite you doing this only a matter of seconds previously. We hold a lot of information at a deeper level than that of which we are consciously aware. Here the information is of a functional nature, it is available as and when we need it with no or little cost in terms of energy. Knowing how to walk, or scratching an itch, are other examples. We have not always been able to do these things, so they are learned activities. I suggest that there was a time in our lives when we were very conscious of what was involved with such things. However, now we are so skilled that we often do not even notice what we are doing. The same can be said about counting. Watching young children learn the complexities of counting (and counting *is* a complex activity) can help us realise how much we can do this with such little attention; so much so that we might find it difficult to answer the question “how did you manage to count those objects?” other than to say “I just did”. The ability to make some things seemingly automatic frees us to give our attention to new things and learn more. Wood (1988, p.175) expressed the significance of automaticity:

> “developing ‘automaticity’ means that the child no longer has to consciously *attend* to the practised elements of her task activity. 'Automated' actions may be performed without the need for constant monitoring or awareness. As some aspect of the developing skill is automated, the learner is left free to pay attention to some other aspect of the task at hand.”

So, the powers of the mind are:

Guiding:

- Will
- A sense of truth

Working with material:

- Creativity
- Extraction
- Association
- Abstraction

Holding information:

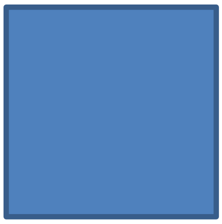
- Memory
- Imagery
- Automaticity

It seems to me that common practice in many mathematics classrooms means that of all the powers which can be called upon, it is memory which is called upon most. Yet, as demonstrated by my little activity, it is one which requires significant energy and is often accompanied with forgetting. If learning is then based upon memory, and something has been forgotten, then it is hard for that to become known again. So, the task for us is to consider ways in which we can call upon more of the available powers and restrict memory to its rightful place. To do so I will consider four frameworks:

- Arbitrary and necessary;
- Practise through progress;
- Subordination;
- Direct Access.

Arbitrary and necessary

I have discussed this framework in greater detail elsewhere (Hewitt, 1999). The basis of the framework is one of viewing each part of the mathematics curriculum and asking the question *is it possible for someone to come to know this for sure without being informed of it?* If the answer is no, then that aspect is arbitrary; if yes, then it is necessary. For example, what is the name of the shape below?



I cannot stare at this shape and know for sure it is called *a square*. If I am in a French speaking area then it is not called a square, it is called *un carré*. In other languages it has a different name. There is nothing about the shape which means a particular collection of sounds, in the form of a word, must be associated with it. Consequently, if I am trying to learn the name of this shape then I will not know for sure what it is called within a certain language unless I am informed. Of course, I can also invent a name. However, if we all did this then we would find that we are unlikely to have come to the same decision. So, for agreement to happen, for new learners to use the same names as other people do, then they need to be informed. So names are arbitrary and I use arbitrary as this describes how it might feel for a learner. As a learner might ask, *why is it called that?* A question for which there is no mathematical reason.

As well as names, conventions are also arbitrary, in the sense that I am using this word. Why is the *x* co-ordinate written before the *y* co-ordinate when a position on a graph is described? Why is there a comma in-between the two? And why are brackets involved? I class all socially agreed conventions as arbitrary. No matter how often I might turn round or stare at a circle, I cannot know for sure that there are 360 degrees in whole turn. Why 360? Why not 100?

There are historical reasons, as the Babylonians were working in base 60 and there were mathematical conveniences with having a number with many factors. However, this is still about *choice*; there is nothing that means a learner would know it *has* to be 360 and could not be anything else. Indeed, for other mathematical reasons, radians are preferred. Even though there are reasons, it is still not necessary, it is only convenient. Thus I still class this as an arbitrary aspect of the curriculum.

Not everything on the curriculum is arbitrary. For example, in Euclidean geometry all triangles tessellate. This is not a matter of choice, it is something which can be worked out and argued that it *must* be true. Given the conventions (arbitrary) regarding names, symbols and number system (base 10), then 3+4=7. This is something which everyone will agree upon. It is the necessary where mathematics lies. It is here where things must be how they are. It is not a matter of choice, instead is it a matter of justification and proof. It is with the necessary that the question *why?* is appropriate, unlike with the arbitrary.

The arbitrary is about acceptance and memory as there is no mathematical reason why something is how it is. Without reasons, a learner is left only with memory. The necessary is about questioning and awareness as here there are always reasons. The fact that there are reasons means that a learner can use and educate their awareness to come to know these things. This dichotomy has significant implications for the teacher as well as a learner. To teach something which is arbitrary involves assisting memory, whereas I suggest teaching something which is necessary involves the very different task of educating awareness.

With respect to the arbitrary, I know as a teacher that I need to inform my students of what is arbitrary. More detail can be found elsewhere (Hewitt, 2001) but there are many ways in which I can go about telling someone something. This is, in itself, something worthy of careful consideration. The *when* and *how* to tell is important. I will offer one example here relating to the order in which a co-ordinate is written. One way is to say that you write the *x* co-ordinate first and then the *y* co-ordinate and leave it there. A minor addition would make use of the fact that *x* comes before *y* in the alphabet to help them know which comes first. This small addition does, at least, try to make use of the power of association in their attempt to memorise this.

Practise of the arbitrary is important since it is about memorisation, and students need to be helped in their attempts to memorise. One example of a way to practise the convention of co-ordinates is for two pairs of students to play a game on a co-ordinate grid. The game is secondary to the way in which the game is played. The rules of playing the game involve each pair having one person who decides which co-ordinate they should go for next (in whatever game it is they are playing) and s*aying out loud (or writing down)* the co-ordinate, whilst the other person has to *listen to (or read)* this and put a cross at that co-ordinate (the first person not being allowed to say "no not there", etc.). This means that each person is practising the convention of which comes first, one saying/writing and the other listening/reading. The game itself could be one of many. An example would be taking turns between the teams to put a cross in their colour on a co-ordinate grid in order to have four crosses of their colour which would lie on the corners of a square. One point for each square created. The size of the grid could be decided and whether it involved all positive numbers in the co-ordinates or a mix of positive and negative numbers.

The arbitrary is the rightful place to call upon memory and a teacher's role is to acknowledge this and assist students in their task of memorising.

The necessary can be known without students being informed. As such, I suggest there is a very different job to be done. Figure 1 represents two aspects to consider, the awareness of students and the desired mathematical property or relationship which you might want them to come to know.

Figure 1

The first challenge of a teacher is to design or choose and activity which can (a) be meaningfully engaged with the awareness the students already possess and (b) engagement with that activity can lead to an awareness of the desired properties or relationships (see Figure 2).

Figure 2

However, the choice of the activity is not the end of a teacher's role, of course. What is important is the way in which that teacher works with the students whilst the students are working on the activity. This may involve a series of questions which help challenge and focus students on particular aspects whilst they are working (see Figure 3).

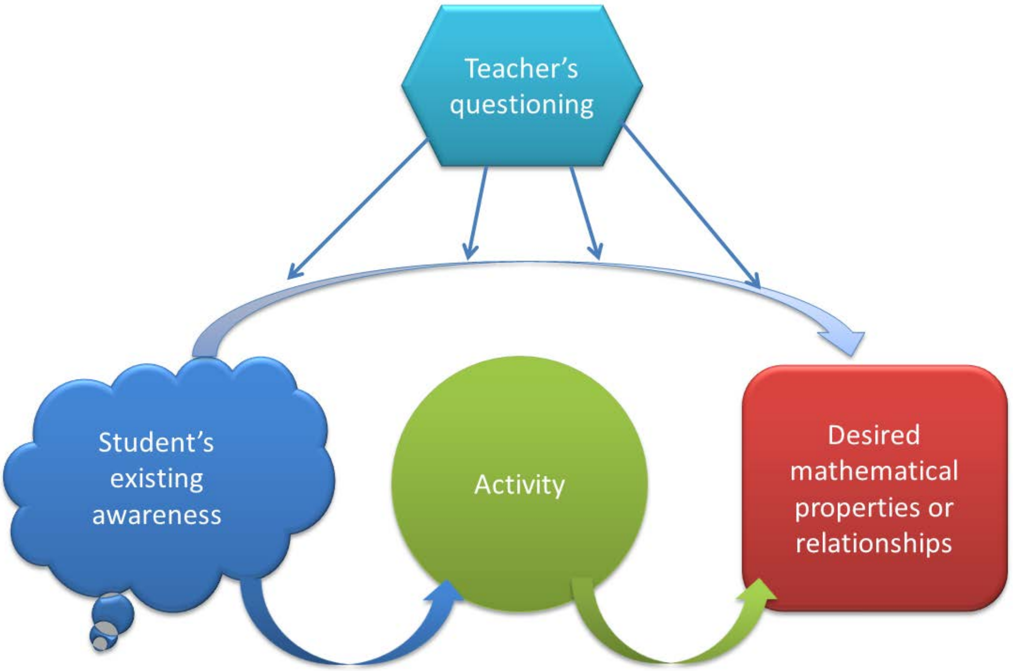


Figure 3

An example of such an activity might be where students are given the information that 100% is $360 and asked what other percentages they could work out. A gradual development of something like Figure 4 can come from students starting to say they can work out 50% by halving, then 25% by halving that; 10% is often stated quite quickly and this can lead to other percentages, such as 5% and 20%. Then someone might realise that if they know 10% and

20%, they can also work out 30%. This can go on for some time until 1% is known and someone realises that they can then find any percentage at all from the 1%. This can lead to an awareness of how any percentage can be found.

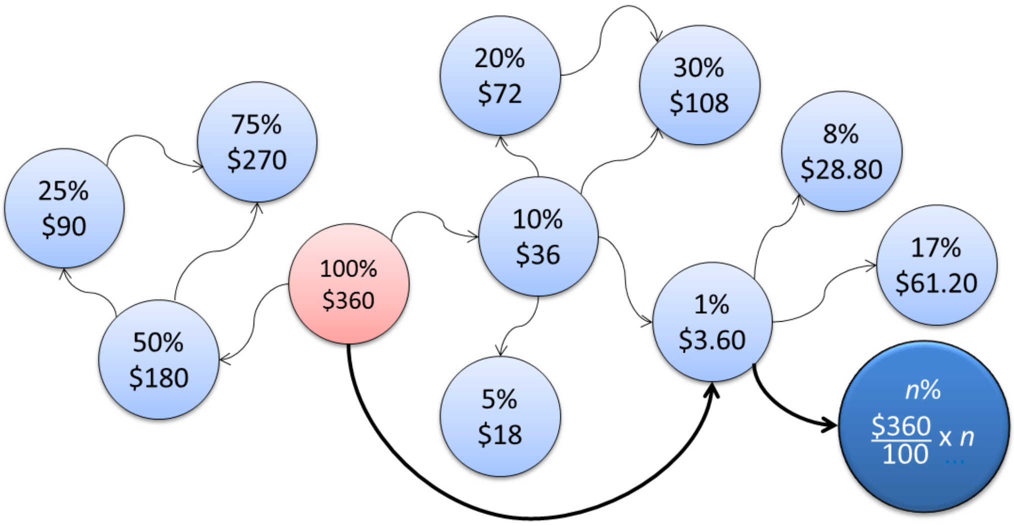


Figure 4

Such an activity does not call upon students being told a rule, which they then have to memorise. Instead they have to use a variety of powers, such as will, a sense of truth, creativity and abstraction. Memory is kept in its rightful place and not called upon explicitly.

Practice through progress

I will only briefly discuss this aspect. However, the nature of practice is important for successful learning. Firstly, I will make use of the two topics mentioned above: co-ordinates and percentages. The suggested activity above for practising the convention of saying and writing co-ordinates gives the possibility of becoming aware of something new. It is sometimes a while into such an activity before one team realises that squares do not have to have horizontal or vertical sides. In fact, going for such obvious placements of your team's crosses are more likely to be thwarted by the other team as they are more obvious. Consequently, students are coming to learn about different orientations of a square and how they can be sure whether any four particular crosses of the same colour are positioned at the corners of a 'squiffy' square. The practice of co-ordinate conventions does not just result on students standing still. Instead they can continue to progress in terms of educating their awareness whilst practising a particular convention. This contracts to a tradition form of practice in the form of an exercise, where they have to plot given co-ordinates, or write down the co-ordinates of given points (and are never doing anything with those answers). In such a traditional exercise the best that can be hoped for is that someone does not 'go backwards'.

With respect to percentages, rather than doing a traditional exercise with questions such as *find 40% of 260*, a challenge such as that in figure 5 will have students carrying out a lot of practice of find percentages but also this practice is carried out with a purpose in order to succeed with the given challenge. Finding more examples which have one or two steps to make a 50% increase overall can result in educating their awareness in how percentages behave and that attention needs to be placed on what a percentage is *of*, as much as the numerical value of the percentage.

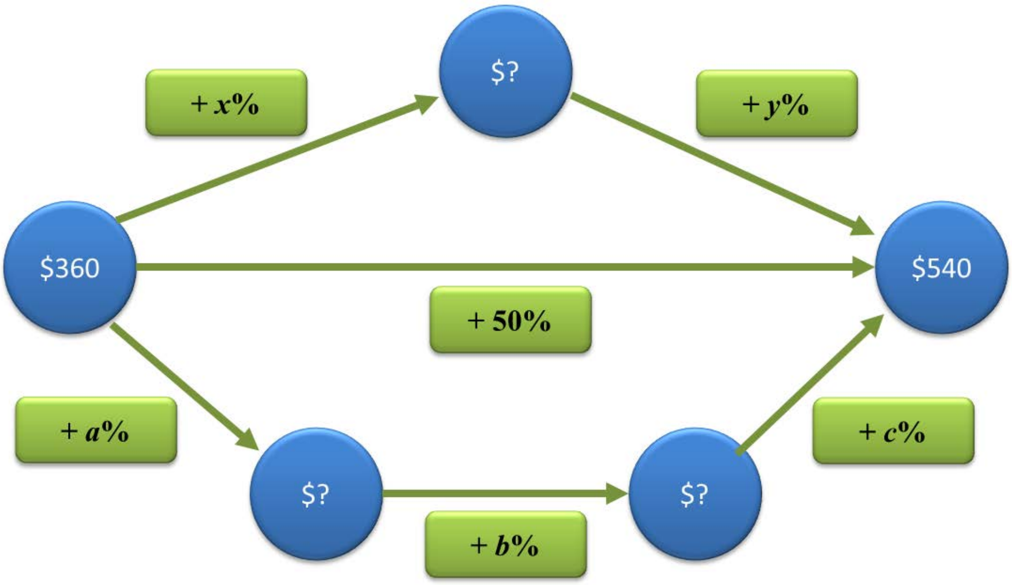


Figure 5

Subordination

Subordination has some of the features of *practise through progress* but with significant shifts in some of the components. With practise through progress, something has already been learned and it is a matter of finding an activity which practises what is already known whilst simultaneously allowing opportunities to progress in other areas. Subordination turns this on its head, by having the activity clearly understood whilst the thing which the activity practises is *not* known to the student. The activity calls upon the practice of something of which the students does not yet know. So the learning takes place with what is being practised rather than what comes out from the activity itself. So, as a teacher I will appear, as far as the students are concerned, to be interested in the outcome of the activity. However, my real agenda is not that at all, but whether what is required to be practised by the activity has been acquired or not.

The example I offer is based upon the computer program *Grid Algebra*[3]. This is based upon a grid a numbers in multiplication tables (see Figure 6).

1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25
6	6	12	18	24	30

Figure 6

There are many ways activities and features of the software, but here I concentrate mainly on just one. Initially students become familiar with the structure of the grid through several activities built into the software. Then it is revealed that any number can be picked up and dragged to a cell either horizontally or vertically (see Figure 7). When such movements are made, the software shows the notational consequence of such a movement. So, the number 3 is dragged one cell to the right (addition), and this results in 3+1 appearing in the cell which previous had shown 4. The 3+1 then becomes an object in its own right and can be dragged down (from the one times table down to the six times table: multiplication) to show 6(3+1), which in turn is dragged to the left

[3] *Grid Algebra* is available from the Association of Teachers of Mathematics (ATM): http://www.atm.org.uk/Shop/Primary-Education/Software-Media/Grid-Algebra---Single-User-Licence/sof071

(subtraction) to show 6(3+1)-12. The ‘peeled back corners’ in certain cells indicate that there is more than one expression in those cells. For example, the number 4 is still in the cell which now has 3+1 showing and can be seen again by clicking on the peeled back corner.

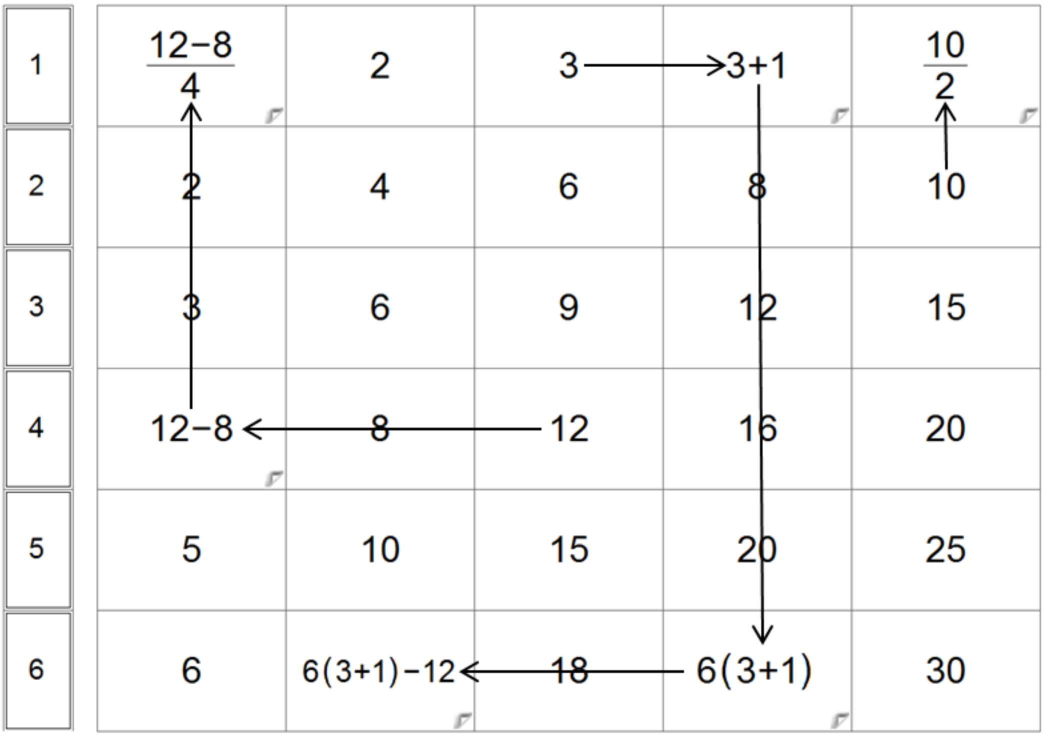


Figure 7

A key activity can then be set up where the grid is empty apart from one number and the result of that number being dragged around the grid with all intermediate expressions rubbed out. Thus, only the number and the final expression can be seen (such as in Figure 8).

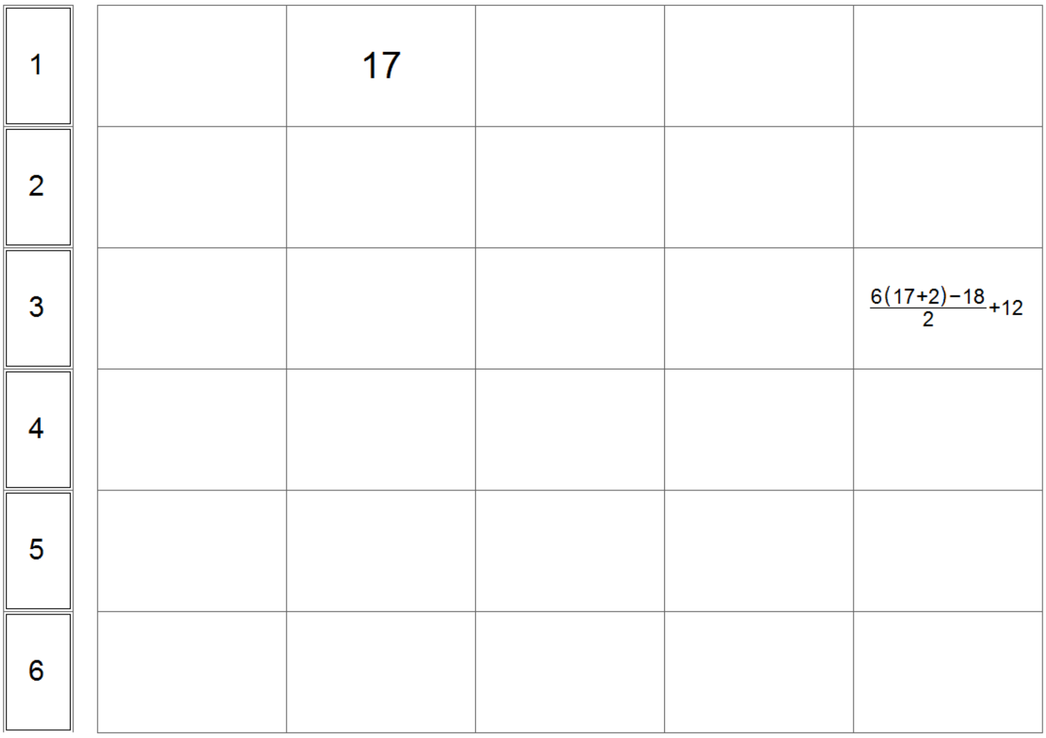


Figure 8

The task for the students is to re-create that journey by picking up the 17 and dragging it to various positions until they produce the expression $\frac{6(17+2)-18}{2}+12$. The formal notation of this expression might be something of which students are not familiar and they may not know about order of operations either. Yet these are the things which need to be used in order to carry out the activity. This is a situation which involves subordination. The students are familiar with the idea of physical journeys and so can understand the nature of the challenge. They know they have to make movements on the grid and this is something they can do (whether or not the movements are correct!). Yet this activity requires practice of interpreting formal notation and knowing order of operations – something, let us assume, they do not know about. So the desired learning is in what is required to be practised rather than the result of the activity.

A key factor with subordination is that someone needs to be able to see the consequences of their actions and be able to understand those in relation to their success or otherwise in achieving the challenge. So, in the case of the Grid Algebra activity, students will make a decision to move 17 somewhere and the software will feed back the consequence of their movement in the form of notation. The students can then see whether this notation is beginning to build up to the desired final expression $\frac{6(17+2)-18}{2}+12$. Suppose, for example, they started off with the correct order of operations by adding two (moving to the right) and then multiplying by six (moving down). However, after they then subtracted 18 (moved to the left), they felt that they should add 12 next and then divide by two, they would find the software would show $\frac{6(17+2)-18+12}{2}$ which looks different to the target expression. Hence, they can tell that they have done something wrong simply because it does not look the same. They would also end up in a different place (see Figure 9).

1		17		17+2	
2					
3			$\frac{6(17+2)-18+12}{2}$		$\frac{6(17+2)-18}{2}+12$
4					
5					
6	6(17+2)−18		6(17+2)−18+12	6(17+2)	

Figure 9

Thus, the feedback is understandable in terms of the achievement or otherwise of the task and so they can become aware from this feedback whether they have interpreted the notation correctly or not and make adjustments accordingly. My experience working with students is that it does not take long for them to learn how the four operations are written in formal notation and the order of operations. Furthermore, they become very fluent with this quite quickly (Hewitt, 2012).

When something has become fluent, we hardly know we are doing what we do. This includes a whole range of knowings and skills, such as walking, counting, spelling of many words and, for many of us, correct use of algebraic notation. Every person has a long list of things which have become automated and little or no attention is given to these things. Instead attention is placed on some other goal for which one or more of these are required. Subordination attempts to mimic this relative imbalance of where attention is placed. As I teacher I focus students' attention on the goal, rather than the means of achieving that goal. So, in the Grid Algebra example above, the goal is to re-produce a particular expression through movements on the grid. The means of how to achieve that – being able to interpret formal notation and know order of operations – is not explicitly mentioned and certainly not 'taught' beforehand. The learning comes from noticing the effect of movements on the 'look' of the expression generated compared with the target expression. Plenty of incorrect movements are made initially, but after many of these tasks, students become very adept at interpreting notation and can begin to articulate the 'rules' of what the notation means in terms of operations and order of operations. The fact that their attention is deliberately placed in the goal rather than the means, allows the means to be learned in a way more akin to much of their early pre-school learning, where little was explicitly explained and yet they had to find the means to achieve what they wanted to achieve.

Direct access

The phrase 'direct access' comes from Laurinda Brown with whom I spent many an evening discussing our classrooms when we were both teaching in Bristol in the UK. Too often the mathematics curriculum is structured in a way where small steps are made and one piece of the curriculum is built upon another which is, itself, built upon another, which, in turn, is built upon another, etc… Figure 10 gives a sense of a typical situation where, in order to learn Y, you need to know C, which in turn requires you to know B, which is built upon A. The problem with this is two-fold: it takes time to come to know A, B and C; and by that time, there is a chance that at least one of A, B or C have been 'forgotten'. So trying to teach Y becomes a problem.

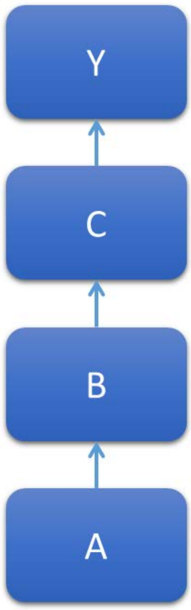


Figure 10

Instead, a pedagogic challenge is to analyse Y to find its fundamental essence and structure, and consider what is the least which needs to be used to engage in a meaningful way with Y (see Figure 11).

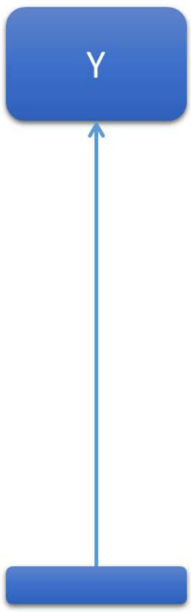


Figure 11

This has connections with Bruner's (1960, p.52) statement that "any subject can be taught to any child in some honest form". This statement becomes possible if we do not call upon a list of previous learning. Instead, the powers of the mind can be called upon since we all possess these. So the challenge is to use as little prior learning as possible in order to engage in the mathematical essence of what Y is really about. What can be used is the expectation that a student will engage in a way where their powers of the mind are being utilised. An example is the image in Figure 12, of a dot moving round a circle.

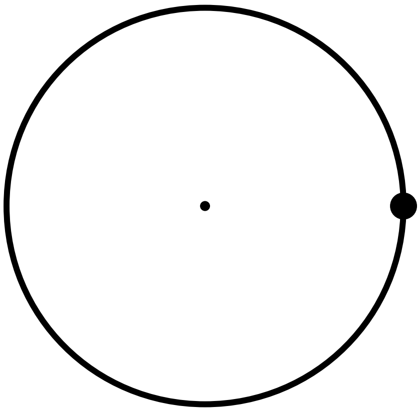


Figure 12

I will briefly give a sense of how I might work with a class. The dot starts as in Figure 12 and rotates anti-clockwise. I ask a class to say “now” when the dot is at its highest position and I tell them that at this point the height of the dot is one. I do likewise with the place where it is lowest and tell them the height is negative one. I then ask them to say “now” when the height is zero. We establish that this gets said twice within one revolution. I introduce the radius and the angle the radius has turned through from its start position (see Figure 13).

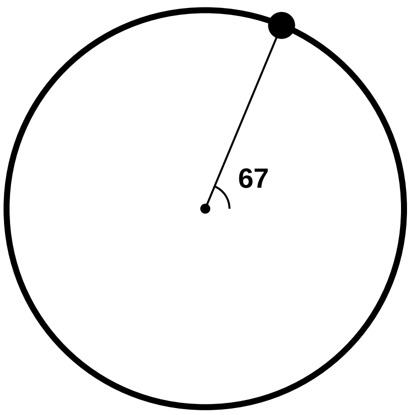


Figure 13

The students articulate that the angle says 90 when the height is one, 270 when it is negative one, and 0 and 180 when it is zero height. A discussion often arises as to whether one of those occurrences of zero height is at 0 or 360 and I suggest it is both and ask what happens after the 360 as the point continues turning. We establish a sequence of 0, 180, 360, 540, 720,… I rotate my hand many times round the circle quickly and then move another 90 degrees. I begin to introduce some notation and the awareness that a height of one can be obtained from lots of 360 degrees followed by another 90 degrees is written as

$$\text{height}(360n+90) = 1$$

This, later on, becomes:

$$\sin(360n+90) = 1$$

I will not go into detail but I have a way of addressing the issue of which angle produces a height of 0.5 and the following is established:

$$\text{If } \sin x = 0.5 \text{ then } x = 360n+30 \text{ or } x = 360n+180-30 \ \forall n \in \mathbb{Z}$$

The notation comes only as a form of expressing the awareness they have already revealed. Whether the values *n* can take is expressed as $\forall n \in \mathbb{Z}$ or in words is not of importance. However, sometimes I find students can be excited by a bit of ‘weird’ notation being used as a way of expressing an awareness as long as they are feeling quite comfortable with that awareness.

A different stressing of the dot leads to establishing cosine and tangent and further work leads to working on equations such as cos x = sin x. At some point the graphs of these functions are produced (see Figure 14).

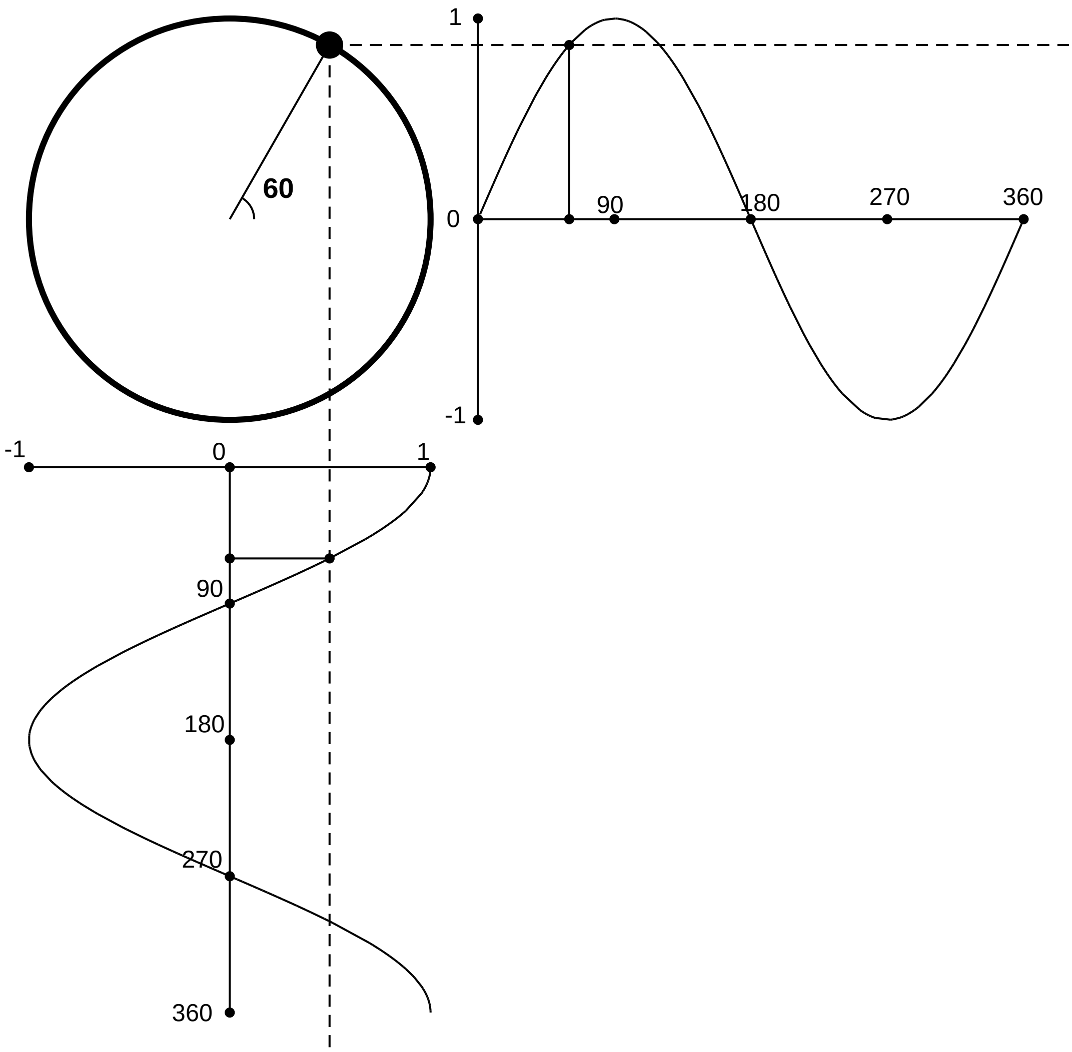


Figure 14

This image is quite a well-known one. The significance is in the way of working with students so that they are not being asked to remember anything particular from their previous mathematics lessons. Instead they are just being asked to watch, observe and comment upon what they see. They will use most if not all of the powers of the mind listed earlier. The labelling and notation is something that I will look after as teacher. So, I provide only that which is arbitrary – the notation. The students provide that which is necessary – the properties and relationships. I simply label any awareness which becomes established. Gradually the students begin to adopt the notation as part of the way they communicate any further awareness they have. As little previous knowledge is called upon, such work can take place with relatively young students. I have worked with 11-12 years olds in a mixed attainment class in such ways and it may be that this is just as possible with younger students. However, the aim is not to play a game of seeing how young students can be to engage with this particular idea but to note that it is possible to work on a topic such as finding general solutions to some trigonometric equations without the need to have a long series of previous mathematics work to prepare them for this. Note also that this image is not something to be remembered (as in memorising) but is something which is recalled. The significance of recalling rather than remembering is that it is not necessary to ask students to memorise this image; instead the process of working with the image on activities over time means that this is an image which can be recalled in the same way as you recalled a place in activity 8 above.

Concluding remarks

The framework of arbitrary and necessary offers a way to distinguish between the areas of the mathematics curriculum which necessitate the use of memory and those areas where the use of memory is best avoided. Most significantly the aspects of the curriculum which are necessary are where the real mathematics lies. This is the area where the other powers of the mind can be called upon so that awareness is educated rather than the inappropriate and inefficient use of memorisation. Whether practising an arbitrary name/convention, or the use of a necessary property/relationship, there are ways to practise which can call upon a range of powers of the mind where progress is made alongside the desired practice; and the practice is seen as purposeful.

Subordination offers a way in new things can be met for the first time as the vehicle through which a certain challenge can be achieved. The notion of immediately having to use and practise something which is unfamiliar may seem strange, yet I argue that this is what we all did as young children in learning language in order to say what we wish to communicate, and in developing the skills of walking in order to get to objects we find desirable. Subordination mirrors the behaviour of when something is already automatised. In such circumstances, we do not place our attention on what it is we are using; instead we place our attention on the effect its use has upon a desired goal. I suggest that this form of practice not only helps what is used to be learned relatively quickly, but also drives it into becoming automatised.

The notion of direct access takes away the need to have remembered earlier mathematics content; instead a carefully designed activity utilises the powers of the mind which can result in educating awareness, which can take the form of items on the mathematics curriculum through direction of attention and carefully timed notating and formalising.

A final note is that I have not specifically focused on the use of one power of the mind over others. This is due to the fact that they do not come singly. For example, association and imagery are frequently used when someone is trying to memorise. In fact, it is impossible to stop any of these powers being at work. The argument I am making is about the degree to which each is stressed when working with students. In some classrooms, students can come to know that it is memory which is stressed over other powers and come to expect this within mathematics lessons. As a consequence, when a greater use of the other powers is suddenly expected, students can respond in a way which makes it appear that they do not have them! All students do, of course, have all of these powers, and use them on a daily basis. However, there can be a culture established where the guiding powers of *will* and *a sense of truth* can result in a student directing energy into seeking what it is that needs to be memorised (since this is the norm). Something different to this can mean their *sense of truth* provides a feeling that this is not what mathematics lessons are about; and this can result in a lack of engagement. Shifting the culture is required which means that a student's *will* and *sense of truth* are aligned with the expectation that the full range of powers of the mind are utilised rather than seeking only to memorise. To change that culture is a teacher's responsibility so that students can learn more, faster and in a deeper way.

References

Bruner, J. S. (1960). *The process of education.* Cambridge, Massachusetts: Harvard University Press.

Dewey, J. (1975). *Interest and Effort in Education.* Carbondale: Southern Illinois University Press.

Gattegno, C. (1971). *What we owe children. The subordination of teaching to learning.* London: Routledge and Kegan Paul Ltd.

Gattegno, C. (1986). *The Generation of Wealth.* New York: Educational Solutions.

Ginsburg, H. (1977). *Children's Arithmetic. The Learning Process.* D. Van Nostrand Company.

Hewitt, D. (1997). Teacher as amplifier, teacher as editor: a metaphor on some dynamics in communication. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, Helsinki: Lahti Research and Training Centre, University of Helsinki, pp. 73-80.

Hewitt, D. (1999). Arbitrary and Necessary: Part 1 a Way of Viewing the Mathematics Curriculum. *For the Learning of Mathematics 19*(3)**,** pp. 2-9.

Hewitt, D. (2001). Arbitrary and Necessary: Part 2 Assisting Memory. *For the Learning of Mathematics 21*(1)**,** pp. 44-51.
Hewitt, D. (2009). From before birth to beginning school. In J. Houssart and J. Mason (Eds), *Listening Counts: listening to young learners of mathematics*, Stoke-on-Trent: Trentham Books, pp. 1-15.
Hewitt, D. (2012). Young students learning formal algebraic notation and solving linear equations: are commonly experienced difficulties avoidable? *Educational Studies in Mathematics 81*(2)**,** pp. 139-159.
Mason, J. (1987). Only awareness is educable. *Mathematics Teaching 120***,** pp. 30-31.
von Glasersfeld, E. (1987). Learning as a Constructive Activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*, New Jersey USA: Lawrence Erlbaum Associates, pp. 3-18.
Wood, D. (1988). *How Children Think and Learn: The Social Contexts of Cognitive Development.* Oxford: Basil Blackwell.