

WHAT IS ALGEBRAIC ACTIVITY? CONSIDERATION OF 9-10 YEAR OLDS LEARNING TO SOLVE LINEAR EQUATIONS

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This paper tries to raise issues about what constitutes algebraic activity through looking at number of episodes from a series of three lessons taught to 9-10 year olds using the software Grid Algebra[1]. From different viewpoints the work the students achieved could be viewed as anything from impressive algebraic activity after relatively short time of teaching, to feeling as if no algebraic activity took place at all. The aim of the paper is to raise issues rather than come to a particular position. It ends up highlighting the fact that such viewpoints are the result of us considering certain things are important and to encourage pursuit of what those are and why we give them that significance.

SOME DIFFICULTIES AND SUCCESSES WITH ALGEBRA

There has been much research reporting difficulties students have with algebra. Küchemann (1981) highlighted the fact that many students had considerable difficulty in developing meaning for letters. Difficulties students experience are not restricted to letters as Collis (1974; 1975) identified a tendency for 6 to 10 year olds to want to replace two numbers connected by an operation with a single number. He described this as students struggling with a *lack of closure*. Sfard and Linchevski (1994) talked about the need for students to be able to see an expression as an object as well as a process to be carried out. The equals sign has also been shown to have meanings for students where certain correct mathematical forms of statements are deemed to be unacceptable. Behr et al. (1980) showed that 6-7 year olds viewed the equals sign as a *do something signal*. Kieran (1981) pointed out that this was not just an issue with younger students but something which carried on throughout elementary school, into high school and even college as well. These issues with the equals sign still persist as shown in more recent studies (Knuth et al., 2005; Linsell and Allan, 2010).

Over the last 10 years there has been a number of reports on what students are able to do, rather than what they are not able to do. Younger children in primary schools have been shown to be able to work with algebraic ideas, use letters as unknowns and operate on letters without having to know their values (Schliemann et al., 2003). An example of this is students of 8-9 years of age being able to explain why $N + 3 - 5 + 4$ must be equal to $N + 2$ whatever the value of N (Carragher et al., 2001). Projects based on the ideas of Davydov have engaged 6 year old students with relational ideas using letters before formal work on numbers (Dougherty and Zilliox, 2003).

There is an interesting contrast between studies showing the difficulties that students have with algebra, and unquestionably continue to have with algebra in many secondary mathematics classrooms in particular, and the increasing evidence that

young children are able to engage with algebraic ideas and begin to work with more formal notation. Within this contrast I feel are questions about what we actually perceive algebra to be along with questions about pedagogic approaches which might be adopted as a consequence. The terms *pre-algebra* or *early algebra* have often been used, maybe as a way of being able to avoid contention as to whether something a student does might be deemed as algebraic or not. This is something I will now pursue by considering some different ways of viewing what it is to work algebraically.

WHAT IS ALGEBRA?

In jest, algebra has been described as the study of the 24th letter of the alphabet. If algebra is not as simple as the appearance of a letter then the issue of when does algebra begin is one which has been debated over some time. Mary Boole (1931, p. 1231) described the move from arithmetic to algebra in terms of acknowledging “the fact of our own ignorance” which leads to explicitly labelling an unknown. Filloy and Rojano (1989) talked about a *didactic cut* between arithmetic and algebra, this occurring when a letter appears on both sides of an equation. However, Herscovics and Linchevski (1994) argued that it was not about the form of the equation but about when a student begins to work with the unknown. For example, the equation $2x+4=19-3x$ could be solved by trying different numbers for the letter x whereas someone might change the equation into the form $5x=15$ in which case they have worked with the unknown. For Herscovics and Linchevski, the issue was more about the human activity of how someone worked on an equation rather than the form the equation took. They talked about a *cognitive gap* with many students having difficulty with working spontaneously with or on the unknown. The shift away from symbols themselves onto human activity is one which Radford has followed, looking at algebraic activity in terms of semiotics where “mathematical cognition is not only mediated by written symbols, but that it is also mediated, in a genuine sense, by actions, gestures and other types of signs” (Radford, 2009, p. 112). He has recently argued for attention to shift from our obsession with mathematical symbolism and onto what he calls the *zone of emergence of algebraic thinking* (Radford, 2010a) where the expression of general rules can take place with the use of words, actions and gestures. Mason has for a long time considered algebraic activity in terms of expressing generality (Mason, 1996) and seeing the general in the particular and the particular in the general, where the existence of symbols is not central to the consideration of when algebraic activity takes place. He has talked about three pairs of powers which students bring with them into the classroom: imagining and expressing; specialising and generalising; and conjecturing and reasoning (Mason, 2002) and challenges us as teachers to consider whether we are stimulating these powers or trying to do the work for the students. The notion of *powers* which students bring with them has its roots with Gattegno (1971) who argued that we all possess powers of the mind, which are attributes of being human. These powers are used by very young children in their early learning before they ever enter a school and remain in daily use throughout all our lives. Gattegno (1988) argued that it is helpful to

consider algebraic activity in a wider sense than only in a mathematical context. He spoke of algebra as operations upon operations which can be manifested within the learning of language (such as noticing a rule in the way verbs change tense) as much as within mathematics. As such, algebraic activity is an attribute of the mind and so everyone has already worked algebraically and continues to do so. The issue is then more concerned with Mason's challenge and whether a student's powers of working algebraically are called upon within a mathematics classroom when working on the topic of algebra!

I have offered a brief summary of different ways in which algebra might be viewed and in particular I will analyse students' work during a series of three lessons in terms of the following perspectives:

- Algebra as appearance of letters
- Algebra as working with equations with a letter on both sides of the equation
- Algebra as working with or on the unknown
- Algebra as an expression of generality using actions, words and gestures
- Algebra as seeing the general in the particular and the particular in the general
- Algebra as an attribute of the mind: operations upon operations

THE STUDY

I carried out a series of three lessons with a mixed ability group of 21 9-10 year olds in an inner city primary school. These students had never met the use of letters formally within their lessons and had never been introduced to formal algebraic notation. The students' attainment levels were based by their teachers on the UK National Curriculum levels where most 6-7 year olds are expected to achieve a level 2 and most 10-11 year olds are expected to achieve a level 4. The range of teacher assessed levels for these students was as follows: 2 (level 2); 13 (level 3); 3 (level 4); and 3 (level 5). The lessons were taught by myself and nearly all of the lesson time was spent either using the computer software *Grid Algebra* with occasional time spent on pen and paper activities related to the software. It is important for the reader to be aware that I wore three hats during this study; researcher, teacher and also the person who had developed the software. As such my comments and analysis have to be read with this in mind. One significant aspect about the way of working with the students was that at no time was anything explained to the students, including the particular appearance of formal notation. Instead, certain actions were carried out using the software on an interactive whiteboard, challenges were given to the students and questions were asked.

The teaching sessions were video recorded along with times when individual students worked in a computer room with computer generated tasks. Some pairs of students' work on computers was captured using *Camtasia* software, which records everything they are doing on the computer. Written work was also collected in. These were all analysed through a coding process which was based upon themes and links which developed through the analysis process.

GRID ALGEBRA

The software is based upon a multiplication grid with the one times table in the top row, the two times table underneath that, the three times table underneath that and so on (see Figure 1 where only the first two rows are shown).

1	1	2	3	4	5
2	2	4	6	8	10

1	$\frac{6-4}{2}$	2 → 2+1	5-1 ← 5
2	6-4	← 4 ← 6	↓ 2(5-1) 10

Figure 1: the first two rows of the grid

Figure 2: some movements on the grid

A key feature of the software concerns the relationship between numbers in this multiplication grid. For example, moving from one number to the next number in the one times table would involve adding one and a number such as 2 can be picked up and dragged to the next cell and it would show in notation $2+1$ (see Figure 2). There would now be a peeled back corner in that cell showing that there is also another expression in the cell, in this case the original number 3. On each click of the peeled back corner the expressions in that cell would be revealed one at a time in a cycle of all expressions which have been entered into that cell. Likewise other movements are possible with any movement to the right resulting in addition, to the left would result in subtraction, a movement down would produce multiplication and up division (see Figure 2). Once a movement has taken place the resultant expression becomes an object which can be moved once again. Thus in Figure 2 the number 5 in row one has been moved one cell to the left, producing $5 - 1$, and then that has then been picked up and dragged down from the one to the two times table to produce $2(5 - 1)$. Likewise the 6 in the two times table (row 2) has been moved twice to produce $\frac{6-4}{2}$. There are a large number of other features to the software but only those relevant to particular incidents below will be mentioned.

I will now describe a number of incidents which happened over the three lessons and later I will look at these in terms of the different views about what might constitute working algebraically. These incidents are chosen so as to get a general sense of the development of activities which took place over the three lessons, although it should be noted that there were several additional activities to these which took place.

Episode 1

At the beginning of the first lesson, students were shown the grid with the times tables shown as in Figure 1. After two minutes of them describing what they saw and which numbers might come next in each row (the grid could be scrolled so that they could see which numbers do appear next), a pre-prepared grid was loaded which showed the same grid but with some of the numbers rubbed out. Below the grid was a 'number box' which, when scrolled through, contained the numbers from 1 to 200. The students were asked to come up and drag an appropriate number from the number box into one of the empty cells in the grid. If it was correct the number would

stay in the cell. If it was wrong a sign would indicate this and the number would drift off into a ‘bin’. Chris (pseudonyms are used for all the students) dragged the number 12 into the shaded cell in Figure 3 and I asked how he worked out that it was 12.

1			14		
2	24		28		

Figure 3: which number should go into the shaded cell and why?

He said “If it’s the one times table it’s going to be plotting one up or one down so I just counted two down from 14 which is 12.” As he did this he pointed from 14 back to 12. Abbas said that he could explain it differently and said that he halved the 24. He came up and pointed from the 24 up to the cell which now had 12 in it. Such activities continued with grids having more rows, fewer numbers given and with a greater ‘space’ between any number given and the highlighted cell.

Episode 2

Here I will describe a series of incidents where the class were all together using the Interactive Whiteboard (IWB). Towards the end of the first lesson I had placed the number 15 into a cell in an otherwise empty grid. I made a journey with 15 as indicated by the arrows in Figure 4, and rubbed out all the expressions in the cells along the route except for the final expression. Note that the arrows did not appear on the IWB. They only appear here for clarity of description.

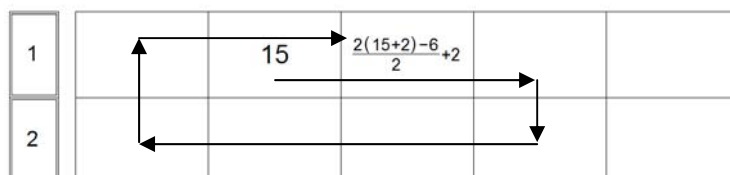


Figure 4: A journey made with the number 15.

After rubbing out the middle stages I then announced that I had forgotten what I did to make that expression and asked them to re-create the journey I had made. During this task a student came up to the board and was successfully given the directions of how to re-create it by fellow students. However, I noted these directions were in the form of *across*, *down*, etc. Mathematical operations were not mentioned. The second lesson I repeated this task with other journeys but worked on the language so that mathematical operations were being used to describe what operations were carried out with increasingly complex expressions such as $2 \left(\frac{2 \left(\frac{2(33-2)+2}{2} + 3 \right) - 2}{2} - 1 \right) - 4$.

Collectively the students were successful at re-creating these expressions with very few, if any, incorrect movements on the grid. Each time I began a new journey I talked about starting with my “favourite number” which changed every time I did this

activity. Then I continued with a same activity but started with a letter and took that letter on a journey rather than a number. So, for example, they were able to tell me the order of operations with the expression $\frac{2\left(\frac{r}{2}-2\right)-2}{2} + 2$.

Episode 3

Julie (a level 5 student) was working on a computer generated task from the software where she was told that $x=4$ and had to drag the correct number from the number box into the cell which had within it the expression $2(x-2+2)+8$. She said: *Four. Ex equals four. Four take away two plus... that's just the same as saying four [pointing to $x-2+2$] times two equals eight, plus eight equals sixteen.* She talked through her thinking with several of these tasks and she got many of them incorrect as her arithmetic was often faulty even though she correctly said what operations had to be carried out.

Episode 4

Abbas (level 3) was working on a paper exercise where an expression involving a letter was written on one cell on a grid and the task was to find which cell the letter must have come from. A colleague of mine asked him a question of why he chose to undo the dividing first with $\frac{2w-4}{2}$ but did not do so with the expression $\frac{n}{2} - 2$ from the previous question (see Figure 5, note that both these are correct).

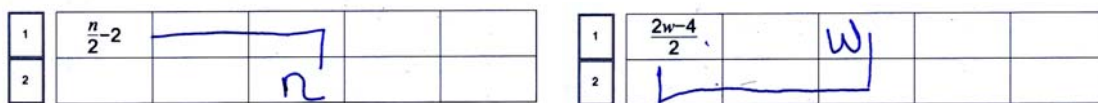


Figure 5: Inverse journey task to find where the letter was originally

In his explanation he moved his pen rapidly horizontally between $\frac{n}{2}$ and 2 in the expression $\frac{n}{2} - 2$ saying “*These two are together so it just tells me that I need to do these two first. That’s why I had to do that last because these two had to so I, so I knew I had to do that, that, that, um, first.*” He struggled to express himself in words and the action seemed to hold more meaning than the words did.

Episode 5

At the end of the final lesson, Julie was working on a sheet of equations to solve. She was talking through solving $\frac{f-8}{2} + 6 = 57$ and was able to express clearly the operations she carried out to solve this (no working was written on the paper, just the answer). Having described taking away the 6 she went on to say *Now you times by two* and at the same time she used her pen in a downwards stroking gesture from the division line to the 2 underneath it.

Other students were also able to solve equations, some with support of the grid and others without, like Julie. This included one of the level 2 students who could solve

equations with the aid of the grid and write the solutions in correct notation, such as $\frac{24}{2} - 3$ for the equation $24 = 2(p + 3)$.

ANALYSIS

All students were working with confidence with formal algebraic expressions of reasonable complexity and the fact that students of this age were doing so could be viewed as impressive after only three lessons. Many of the students were able to solve linear equations, albeit some still needing the support of the grid, and express their answers in formal notation. However, it is another matter to consider whether they were doing any algebra or not, and if so, when that algebraic work started. I will now consider each of the six ways of viewing algebra mentioned earlier with reference to these episodes.

Algebra as appearance of letters

This would mean there was a shift from arithmetic to algebra when I introduced a letter in Episode 2. The interesting thing with regard to this viewpoint is that the students really did not meet a conceptual difficulty in this transition. There was an initial reaction to the idea of a letter but as I did not react to that and moved on to the activity quickly, they found they could do the activity just as well as they had done earlier when a letter was not present.

Algebra as working with equations with a letter on both sides of the equation

This never happened throughout the three lessons and so that would imply that the students never began working algebraically and stayed in the realm of arithmetic.

Algebra as working with or on the unknown

The students in this study did not manipulate an unknown from one side of an equation to the other. However, they did work with the idea of an unknown and this was manifested in the particular example in Episode 3 of Julie recognising that $x - 2 + 2$ did not change the value of x . Even though she was substituting in a particular value for x I argue that her awareness was of the generality of $-2 + 2$ and not the particularity of x being 4 in this case. So from this viewpoint she might be working algebraically even though she struggled with the arithmetic. This is similar to Carraher et al. (2001) reporting that their 8-9 year olds were able to articulate why $N + 3 - 5 + 4$ was equal to $N + 2$. They could account for this irrespective of the value of N . Other students in my study were also able to work successfully with the unknown by working out solutions for the linear equations given at the end of the third lesson (Episode 5).

Algebra as an expression of generality using actions, words and gestures

Here I would like to discuss Episode 4 where Abbas was rapidly moving his pen from $\frac{n}{2}$ to 2 in the expression $\frac{n}{2} - 2$ when trying to explain why he did not start by undoing division in this case. The rapid speed of the pen movement was striking and it seemed to be expressing what he was struggling to express in words. I would like to argue

that he had a sense of generality of which operation he would undo first and that this was expressed with a gesture more effectively than words. However, this generality concerned a notational convention and as such might be considered qualitatively different to Radford's (2010b) example of a rule for the number of squares in a geometrically arranged sequence. I argue that such a geometric sequence is also arbitrary in its nature since there are not reasons why the squares must have been arranged how they were. It was a human construct in a similar way to mathematical notation. So did Abbas reveal algebraic thinking within that gesture? In Episode 5, Julie used a downwards stroke of her pen from the division line to the 2 below whilst saying "multiply by two". The combination of this gesture on the division line whilst saying "multiplication" revealed that she could see one operation and think of its inverse at the same time.

Algebra as seeing the general in the particular and the particular in the general

In Episode 2 the nature of the activity of re-creating journeys was such that attention was placed on the mathematical operations rather than any particular starting number I used. This focus of attention allowed a letter to be introduced without causing too many issues for the students, since they often never paid attention to the start number anyway. I deliberately varied the start number to try to develop a sense of variation and also irrelevance. Fujii and Stephens (2001) talked about the idea of a *quasi-variable* where numbers were used to demonstrate a mathematical relationship which would be true irrespective of the numbers, such as $78+49-49=78$. The particularity of the number in both cases is irrelevant. In this way the general can be seen through the particular and indeed as the teacher I tried to judge when this was the case for most of the students so that I introduced a letter when that sense of generality was already present.

Algebra as an attribute of the mind

Tahta (1981) has talked about inner and outer meanings of activities. In Episode 1 an outer meaning might be to place the correct number into the highlighted cell, whereas the inner meaning in the design of such a task for myself was for students to begin to form mathematical connections between different cells on the grid (in preparation for the later activities involving movements). The students' explicit attention might have been with the numbers, whereas the work they had to do to achieve placing a correct number was to work out relationships between different cells on the grid. I argue that students were working with operations in order to carry out these tasks and the awareness of equivalence of different sets of operations was certainly operating upon operations. So with this view of algebra, the students were working algebraically already with the initial 'number' activity.

CONCLUDING THOUGHTS

How we view algebraic activity changes when we feel students have started such activity. It might be argued anything from the students in this study not doing any algebra at all over the three lessons to them working algebraically from the very first

activity. Naming is an act to label that which is deemed to be significant and what is important is what someone wishes to stress. So the fruits of a discussion about what constitutes algebraic activity can come from what each person reveals to be particularly significant for them in the developing process students make within their work towards algebra and within the algebra curriculum. Not only what is significant but why it is significant. That is what I feel is particularly useful in considering the question *what is algebra?*

NOTES

1. *Grid Algebra* is available from the Association of Teachers of Mathematics at <http://www.atm.org.uk/shop/products/sof071.html>

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