# Symmetric and Optimality of Disjoint Hamilton Cycles in Star Graphs 

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#### Abstract

Multiple edge-disjoint Hamilton cycles have been obtained in labelled star graphs $S t_{n}$ of degree n-1, using number-theoretic means, as images of a known base 2-labelled Hamilton cycle under label-mapping automorphisms of $S t_{n}$. However, no optimum bounds for producing such edge-disjoint Hamilton cycles have been given, and no positive or negative results exist on whether Hamilton decompositions can be produced by such constructions other than a positive result for $S t_{5}$. We show that for all even $n$ there exist such collections, here called symmetric collections, of $\varphi(n) / 2$ edge-disjoint Hamilton cycles, where $\varphi$ is Euler's totient function, and that this bound cannot be improved for any even or odd $n$. Thus, $S t_{n}$ is not symmetrically Hamilton decomposable if n is not prime. Our method improves on the known bounds for numbers of any kind of edge-disjoint Hamilton cycles in star graphs.


Keywords: Star graph, Hamilton cycle, Hamilton decomposition, Automorphism.

## 1 Introduction

In this paper, we are interested in general symmetric properties of edge-disjoint Hamilton cycles in star graphs $S t_{n}$ [1] for the purposes of designing better fault tolerant interconnection network topologies. Star graphs are Cayley graphs over the symmetric group and not much was known about disjoint Hamilton cycles in star graphs until recently, with much of the work on Hamilton decompositions of Cayley graphs revolving around Alspach's longstanding conjecture for Cayley graphs over Abelian groups [2]. The breakthrough came when a Hamilton decomposition for the star graph $S t_{5}$ of dimension 5 was constructed in [4] and multiple edge-disjoint Hamilton cycles for the n-dimensional star graph $S t_{n}$ in

[^0][6]. Surprisingly, the constructions were symmetric in the sense that (the edges of) any two Hamilton cycles were images of each other under automorphisms of labelled versions of $S t_{n}$, mapping labels consistently, and all of them were automorphic to a base 2-labelled Hamilton cycle constructed in [6]. Although asymptotic bounds for the number of disjoint Hamilton cycles in $S t_{n}$ are given in [6], and the stated $\varphi(n) / 10$ bounds for all n in [6] have been improved to $\varphi(n) / 5$ for odd n in [3], it was not known what the optimum bounds are for obtaining Hamilton cycles in this way and, indeed, whether or not $S t_{n}$ is Hamilton decomposable by these means for any n other than 5 . In this paper, we define symmetric collections of disjoint Hamilton cycles for labelled versions $S t_{n}$ to be those for which, given a Hamilton cycle in the collection, there is an automorphism mapping labels consistently such that the chosen Hamilton cycle is the image of the base 2-labelled Hamilton cycle in [6]. We show in the remainder of the report that there are at most $\varphi(n) / 2$ symmetric disjoint Hamilton cycles, where $\varphi$ is Euler's totient function, and that this bound is sharp for all even n .

Throughout this report, whenever we refer to 'disjoint' Hamilton cycles, we will mean edge-disjoint Hamilton cycles. If $G$ is a graph, $H$ is a subgraph of $G$, and $\Phi$ an automorphism of $G, \Phi(H)$ will refer to the subgraph of $G$ that is the image of the vertices and edges of $H$ under $\Phi$. Equality of subgraphs $H$ and $H^{\prime}, H=H^{\prime}$, will mean equality of both the sets of vertices and edges.

## 2 Symmetry

In this section we work with edge-labelled undirected star graphs. We define an edge labelling for star graphs $S t_{n}$ and label automorphisms which are automorphisms that map these labels consistently. We show that $S t_{n}$ cannot have symmetric collections of greater than $\varphi(n) / 2$ disjoint Hamilton cycles in Theorem 2.16 and that therefore $S t_{n}$ is not symmetrically Hamilton decomposable for non-prime n (Corollary 2.17). If n is even, we show that $S t_{n}$ does have a symmetric collection of $\varphi(n) / 2$ Hamilton cycles in Theorem 2.20 and that such a collection cannot be enlarged to include further non-symmetric 2-labelled edgedisjoint Hamilton cycles (Theorem 2.21).

### 2.1 Labelled star graphs and label automorphisms

Definition 2.1 The n-dimensional labelled star graph $S t_{n}=(V, E, L)$ is the ( $n$ -1)-regular graph of order $\left|S_{n}\right|$, where $S_{n}$ is the symmetric group of permutations of order $n$, with a set $V$ of vertices, $E$ of edges and a mapping of edges to integer labels $L: E \mapsto\{1, \ldots,\lfloor n / 2\rfloor\}$, given by:

$$
\begin{aligned}
& V\left(S t_{n}\right)=\left\{a_{\rho(1)} \cdots a_{\rho(n)} \mid \rho \in S_{n}\right\}, \\
& E\left(S t_{n}\right)=\left\{e \mid e=\left\{a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)},\right.\right. \\
&\left.\left.a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}\right\}, \rho \in S_{n}\right\} \\
& L\left(\left\{a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}\right\}\right)
\end{aligned}
$$

$$
=\delta\left(a_{\rho(1)}, a_{\rho(i)}\right)
$$

where

$$
\delta\left(a_{i}, a_{j}\right)=\min \{|i-j|, n-|i-j|\} \quad(1 \leq i, j \leq n)
$$

is the distance between $a_{i}$ and $a_{j}$ on the cyclic graph whose vertices are $a_{1}, \ldots, a_{n}$ in which $a_{n}$ is adjacent to $a_{n-1}$ and $a_{1}$.

Definition 2.2 A Hamilton cycle in a spanning subgraph $G$ of $S t_{n}$ with a set of edges $E(G)$, is a pair of sequences $(\bar{v}, \bar{e})$ of vertices $\bar{v}=v_{1} \ldots v_{n!+1}$ and edges $\bar{e}=e_{1} \ldots e_{n!}$ such that:
(i) $e_{i}=\left\{v_{i}, v_{i+1}\right\} \in E(G)(1 \leq i \leq n!)$,
(ii) $\left\{v_{1}, \ldots, v_{n!+1}\right\}=V\left(S t_{n}\right)$,
(iii) $v_{1}=v_{n!+1}$.

The class of automorphisms of $S t_{n}$ of interest are those which map labels consistently.

Definition 2.3 A label map for $S t_{n}$ is a bijection

$$
\phi^{l}:\{1, \ldots,\lfloor n / 2\rfloor\} \mapsto\{1, \ldots,\lfloor n / 2\rfloor\}
$$

of labels. An automorphism is a mapping

$$
\Phi: V\left(S t_{n}\right) \mapsto V\left(S t_{n}\right)
$$

such that:
(i) $\Phi$ is bijective
(ii) for all $v_{1}, v_{2} \in V\left(S t_{n}\right),\left\{v_{1}, v_{2}\right\} \in E\left(S t_{n}\right)$ if and only if $\left\{\Phi\left(v_{1}\right), \Phi\left(v_{2}\right)\right\} \in$ $E\left(S t_{n}\right)$

It is a label automorphism if, in addition, there exists a label map $\phi^{l}$ such that:
(iii) for all $v_{1}, v_{2} \in V\left(S t_{n}\right), L\left(\left\{\Phi\left(v_{1}\right), \Phi\left(v_{2}\right)\right\}\right)=\phi^{l}\left(L\left\{v_{1}, v_{2}\right\}\right)$

### 2.2 Pointwise maps and distance maps

We will generate automorphism 'pointwise' by means of a bijection of the elements $\left\{a_{1}, \ldots, a_{n}\right\}$.

Lemma 2.4 ([3]) Let $\phi:\left\{a_{1}, \ldots, a_{n}\right\} \mapsto\left\{a_{1}, \ldots, a_{n}\right\}$ be a bijection. Then:
(i) $\Phi: V\left(S t_{n}\right) \mapsto V\left(S t_{n}\right)$, given by $\Phi\left(a_{\rho(1)} \ldots a_{\rho(n)}\right)=\phi\left(a_{\rho(1)}\right) \ldots \phi\left(a_{\rho(n)}\right)$, is an automorphism of the graph $S t_{n}$,
(ii) if $\bar{v}=v_{1}, \ldots, v_{n!+1}, \bar{e}=\left\{v_{1}, v_{2}\right\}, \ldots,\left\{v_{n!}, v_{n!+1}\right\}$ and $(\bar{v}, \bar{e})$ is a Hamilton cycle in $S t_{n}$, then the pair of sequences of vertices and edges $\Phi_{H}(\bar{v}, \bar{e})$ defined by
$\Phi_{H}(\bar{v}, \bar{e})=\left(\Phi\left(v_{1}\right), \ldots, \Phi\left(v_{n!+1}\right), \quad\left\{\Phi\left(v_{1}\right), \Phi\left(v_{2}\right)\right\}, \ldots,\left\{\Phi\left(v_{n!}\right), \Phi\left(v_{n!+1}\right)\right\}\right)$
is also a Hamilton cycle,
(iii) if a spanning subgraph $G$ of $S t_{n}$ is a Hamilton graph, then so is the spanning subgraph that is its image $\Phi(G)$.

Definition 2.5 A pointwise map for $S t_{n}$ is a bijection $\phi$ as in Lemma 2.4. The corresponding automorphism is the automorphism $\Phi$ as defined in Lemma 2.4. If $\phi$ is such that there exists a bijection

$$
\phi^{d}:\{1, \ldots,\lfloor n / 2\rfloor\} \mapsto\{1, \ldots,\lfloor n / 2\rfloor\}
$$

satisfying, for all $a_{i}, a_{j} \in\left\{a_{1}, \ldots, a_{n}\right\}$,

$$
\begin{equation*}
\delta\left(\phi\left(a_{i}\right), \phi\left(a_{j}\right)\right)=\phi^{d}\left(\delta\left(a_{i}, a_{j}\right)\right) \tag{1}
\end{equation*}
$$

then $\Phi$ is trivially a label automorphism with $\phi^{l}=\phi^{d}$ in Definition 2.3 (iii). We shall call $\phi^{d}$ the corresponding distance map.

Distance maps allude to distances in the cyclic graph of the elements $\left\{a_{1}, \ldots, a_{n}\right\}$, and not to distances in $S t_{n}$. The class of label automorphisms generated by a pointwise map and with a distance map as in Definition 2.5 will be denoted by $\mathcal{A}_{n}$.

### 2.3 Symmetry

Our definition of symmetry is with respect to this class of automorphisms and the Hamilton cycle with edge labels 1 and 2 constructed in [6] as the base Hamilton cycle with which all Hamilton cycles have to be symmetric via an automorphism $\Phi \in \mathcal{A}_{n}$. First of all, we introduce some notation.

Definition 2.6 $A$ vertex $v \in V\left(S t_{n}\right)$ of the form $a_{i} \ldots$ (respectively $\ldots a_{i}$ ), where $a_{i} \in\left\{a_{1}, \ldots a_{n}\right\}$ will be denoted by $\vec{a}_{i}$ (respectively $\overleftarrow{a}_{i}$ ) or $\vec{a}_{i}^{k}$ (respectively $\overleftarrow{a}_{i}^{k}$ ) for some subscript $k$ if several such vertices are under consideration. For a vertex $v=\vec{a}_{i}=\overleftarrow{a}_{j}$ we define head $(v)=\operatorname{head}\left(\vec{a}_{i}\right)=a_{i}$ and $\operatorname{last}(v)=\operatorname{last}\left(\overleftarrow{a}_{j}\right)=a_{j}$.

Definition 2.7 The base Hamilton cycle $H_{12}(n)$ in $S t_{n}$ is the Hamilton cycle constructed in [6] consisting of alternate paths of $n(n-1)-1$ edges with label 1 and single edges with label 2:

where the total number of edges with label 1 in $H_{12}(n)$ is $n!-(n-2)$ ! which is greater than the number of remaining edges with label $1(=n!-(n!-(n-2)!)=(n-$ 2)!) in $S t_{n}$, and such that

$$
\operatorname{last}(v)=a_{n}
$$

for all vertices $v$ in $H_{12}(n)$ of edges with label 2.
A collection of edge-disjoint Hamilton cycles in $S t_{n}$ are 'symmetric' if any Hamilton cycle in the collection is the image of $H_{12}(n)$ under an automorphism in $\mathcal{A}_{n}$.

Definition 2.8 A collection $\widetilde{H}$ of edge-disjoint Hamilton cycles in $S t_{n}$ is symmetric if $H_{12}(n) \in \widetilde{H}$ and if, for all $H^{e}, H^{f} \in \widetilde{H}$, there is a label automorphism $\Phi_{\text {ef }} \in \mathcal{A}_{n}$ such that

$$
\begin{equation*}
\Phi_{e f}\left(H^{e}\right)=H^{f} \tag{2}
\end{equation*}
$$

Hamilton cycles that are the image of automorphisms in $\mathcal{A}_{n}$ have a similar structure.

Lemma 2.9 Let $\Phi \in \mathcal{A}_{n}$ be a label automorphism with corresponding distance map $\phi^{d}$. Then, $\Phi\left(H_{12}(n)\right)$ is a Hamilton cycle consisting of alternate paths of $n(n-1)-1$ edges with label $\phi^{d}(1)$ and single edges with label $\phi^{d}(2)$ :


Proof Follows from Definitions 2.5 and 2.7.
From Lemma 2.9, we see that a Hamilton cycle which is the image of $H_{12}(n)$ under a label automorphism in $\mathcal{A}_{n}$, is a succession of edges the majority of which share the same label and the remaining minority of which share the same second label. This leads to the following definition.

Definition 2.10 A Hamilton cycle which is the image of $H_{12}(n)$ under an automorphism as in Lemma 2.9, will be denoted by $H_{i j}(n)$ (or just $H_{i j}$ if $n$ is clear from the context) where the subscript $i=\phi^{d}(1)$ is the label for the majority of the edges and the subscript $j=\phi^{d}(2)$ is the label for the minority of the edges. We shall call these two sets of edges the majority and minority edges of $H_{i j}$ and shall denote them by $E_{m a j}\left(H_{i j}\right)$ and $E_{m i n}\left(H_{i j}\right)$ respectively.

### 2.4 Upper bounds for symmetric collections

Not all labels can be majority or minority labels of images of $H_{12}$ under label automorphisms from $\mathcal{A}_{n}$. The underlying reason for this is the difference in the length of cycles of different labels.

Definition 2.11 The spanning subgraph of $S t_{n}$ comprising edges with labels $i$ and $j$ where $i, j \in\{1, \ldots,\lfloor n / 2\rfloor\}$ will be denoted by $C_{i j}(n)$ and the spanning
subgraph comprising only edges with label $i$ will be denoted $C_{i}(n)$. Each $C_{i}(n)$ is a union of disjoint cycles $B_{i}^{x}(n)$ of edges with label $i$

$$
E\left(C_{i}(n)\right)=\bigcup_{x \in X} E\left(B_{i}^{x}(n)\right) \quad(X \text { is some index set })
$$

We shall call a cycle $B_{i}^{x}(n)$ an $i$-ball. Again, we will abbreviate our notation to $C_{i j}, C_{i}$ and $B_{i}^{x}$ when $n$ is clear from the context and will drop the $x$ index in $B_{i}^{x}$ when only one $i$-ball is under consideration. For an $i$-ball $B_{i},\left|B_{i}\right|$ will denote the number of edges in $B_{i}$.

Lemma 2.12 Let $B_{i}$ be an $i$-ball in $S t_{n}$, where $i \in\{1, \ldots,\lfloor n / 2\rfloor\}$. Then,
(i) $\left|B_{i}\right|=n(n-1)$ if $i$ is coprime to $n$, and
(ii) $\left|B_{i}\right|<n(n-1)$ if $i$ is not coprime to $n$.

Proof Let $n=d q_{1}$ and $i=d q_{2}$ where $d=\operatorname{gcd}(n, i)$ and $\operatorname{gcd}\left(q_{1}, q_{2}\right)=1$. Without loss of generality, assume that the vertex

$$
a_{1} \ldots a_{n} \in B_{i}
$$

Now, the elements

$$
a_{1}, a_{1+i}, \ldots, a_{1+\left(q_{1}-1\right) i}
$$

are distinct (else, for some $r, s$ such that $0 \leq r<s \leq\left(q_{1}-1\right)$ and $K \in \mathbb{N}$, $K n+(1+r i)=(1+s i)$ and so $K d q_{1}=(s-r) d q_{2}$ and as $g c d\left(q_{1}, q_{2}\right)=1, q_{1}$ divides $(s-r)$ which is a contradiction as $\left.(s-r) \leq\left(q_{1}-1\right)\right)$. The path in $B_{i}$ of the form

$$
\vec{a}_{1}, \vec{a}_{1+i}, \ldots, \vec{a}_{1+\left(q_{1}-1\right) i},
$$

where $\vec{a}_{1}=a_{1} \ldots a_{n}$, rotates the elements $a_{1}, \ldots, a_{1+\left(q_{1}-1\right) i}$ within the vertex $a_{1} \ldots a_{n}$ thus:

$$
a_{1} \rightarrow a_{1+i} \rightarrow \ldots a_{1+\left(q_{1}-1\right) i} \rightarrow a_{1}
$$

After $q_{1}-1$ such rotations, the starting vertex $a_{1} \ldots a_{n}$ is reached again, i.e. there is a path in $B_{i}$ of $\left(q_{1}-1\right)$ sets of $q_{1}$ vertices

$$
\underbrace{\vec{a}_{1}, \vec{a}_{1+i}, \ldots \vec{a}_{1+\left(q_{1}-1\right) i}}_{q_{1} \text { vertices }}, \underbrace{\ldots \ldots \ldots,}_{q_{1} \text { vertices }}, \quad, \quad \underbrace{\ldots \ldots \ldots}_{q_{1} \text { vertices }}, \vec{a}_{1}
$$

separated by edges with label $i$, and returning to $\vec{a}_{1}$ after $q_{1}\left(q_{1}-1\right)$ steps. If $i$ is coprime to $n, q_{1}=n$ and ( $i$ ) follows. If $i$ is not coprime to $n$, then $q_{1}<n$ and (ii) follows.

Lemma 2.13 Let $\Phi \in \mathcal{A}_{n}$ and let $B_{i}^{x}$ be an $i$-ball in $S t_{n}$, where $1 \leq i \leq\lfloor n / 2\rfloor$. Then, there exists an $i^{\prime}$-ball $B_{i^{\prime}}^{x^{\prime}}$ in $S t_{n}$, for some $i^{\prime}$ with $1 \leq i^{\prime} \leq\lfloor n / 2\rfloor$, such that

$$
\Phi\left(B_{i}^{x}\right)=B_{i^{\prime}}^{x^{\prime}} \text { and } \operatorname{gcd}(i, n)=1 \text { iff } \operatorname{gcd}\left(i^{\prime}, n\right)=1
$$

Proof As $\Phi$ is an automorphism, $\Phi\left(B_{i}^{x}\right)$ is a cycle such that $\left|\Phi\left(B_{i}^{x}\right)\right|$ equals $\left|B_{i}^{x}\right|$. Also, as $\Phi$ is a label automorphism all edges of $\Phi\left(B_{i}^{x}\right)$ must have the same label, and thus $\Phi\left(B_{i}^{x}\right)$ must be an $i^{\prime}$-ball, $B_{i^{\prime}}^{x^{\prime}}$ say, for some $i^{\prime}$ where $1 \leq i^{\prime} \leq\lfloor n / 2\rfloor$. Then, by Lemma 2.12,

$$
\operatorname{gcd}(i, n)=1 \text { iff }\left|B_{i}^{x}\right|=n(n-1)=\left|B_{i^{\prime}}^{x^{\prime}}\right| \text { iff } \operatorname{gcd}\left(i^{\prime}, n\right)=1
$$

As a result of Lemma 2.13, we are able to give constraints on how automorphisms $\Phi \in \mathcal{A}_{n}$ map labels. Indeed, we can characterize the pointwise maps $\phi$ that generate label automorphisms $\Phi \in \mathcal{A}_{n}$.

Lemma 2.14 Let $\Phi \in \mathcal{A}_{n}$ be a label automorphism with corresponding pointwise and distance maps $\phi$ and $\phi^{d}$ respectively, as in Definition 2.5. Then:
(i) for all labels $l \in\{1, \ldots,\lfloor n / 2\rfloor\}$,

$$
\operatorname{gcd}(l, n)=1 \quad \text { iff } \quad \operatorname{gcd}\left(\phi^{d}(l), n\right)=1
$$

(ii) there exist $i_{0}, j \in\{1, \ldots, n\}$, where $j$ is coprime to $n$, such that

$$
\phi\left(a_{i}\right)=a_{i_{0}+j i} \quad(1 \leq i \leq n)
$$

Proof For (i), let $B_{l}^{x}$ be a $l$-ball in $S t_{n}$. As $\Phi$ is a label automorphism with distance map $\phi^{d}, \Phi\left(B_{l}^{x}\right)$ is a $\phi^{d}(l)$-ball, $B_{\phi^{d}(l)}^{x^{\prime}}$ in $S t_{n}$. By Lemma 2.13, $\operatorname{gcd}(l, n)=1$ iff $\operatorname{gcd}\left(\phi^{d}(l), n\right)=1$.

For (ii), let $i_{0}, i_{1} \in\{1, \ldots, n\}$ be such that

$$
\phi\left(a_{n}\right)=a_{i_{0}} \text { and } \phi\left(a_{1}\right)=a_{i_{1}}
$$

where $\phi$ is the pointwise map of $\Phi$. Put

$$
j_{p}=\delta\left(\phi\left(a_{n}\right), \phi\left(a_{1}\right)\right)=\min \left\{\left|i_{0}-i_{1}\right|, n-\left|i_{0}-i_{1}\right|\right\}
$$

As $\delta\left(a_{n}, a_{1}\right)=1$ and $\delta\left(\phi\left(a_{n}\right), \phi\left(a_{1}\right)\right)=j_{p}$, it follows that

$$
\begin{equation*}
\phi^{d}(1)=j_{p} \tag{3}
\end{equation*}
$$

Let $a_{i} \in\left\{a_{1}, \ldots, a_{n}\right\}$ and consider the $a_{g}, a_{h} \in\left\{a_{1}, \ldots, a_{n}\right\}$ such that

$$
\phi\left(a_{i}\right)=a_{g} \text { and } \phi\left(a_{i+1}\right)=a_{h}
$$

As $\delta\left(a_{i}, a_{i+1}\right)=1$, by (1) of Definition 2.5 and (4) we have that

$$
\delta\left(a_{g}, a_{h}\right)=j_{p}
$$

Therefore,

$$
g-h=j_{p} \bmod n \quad \text { or } \quad g-h=-j_{p} \bmod n
$$

and so

$$
h=g-j_{p} \bmod n \quad \text { or } \quad h=g+j_{p} \bmod n
$$

As $\Phi\left(a_{n}\right)=a_{i_{0}}$ and $\phi$ is injective it is clear that either

$$
\begin{equation*}
\Phi\left(a_{n}\right)=a_{i_{0}}, \Phi\left(a_{1}\right)=a_{i_{0}-j_{p}}, \ldots, \Phi\left(a_{n-1}\right)=a_{i_{0}-(n-1) j_{p}} \tag{4}
\end{equation*}
$$

or

$$
\begin{equation*}
\Phi\left(a_{n}\right)=a_{i_{0}}, \Phi\left(a_{1}\right)=a_{i_{0}+j_{p}}, \ldots, \Phi\left(a_{n-1}\right)=a_{i_{0}+(n-1) j_{p}} \tag{5}
\end{equation*}
$$

hold. If (5) is the case put $j=-j_{p}$ and if (6) is the case put $j=j_{p}$ and the proof of (ii) is complete.

Definition 2.15 Given a label automorphism $\Phi \in \mathcal{A}_{n}$ and corresponding pointwise map $\phi\left(a_{i}\right)=a_{i_{0}+j i}, i_{0}$ is called the offset and $j$ the generator of $\phi$.

The constraints of label automorphisms in turn impose limits on the number of edge-disjoint Hamilton cycles in symmetric collections.
Theorem 2.16 Let $\widetilde{H}$ be a symmetric collection of disjoint Hamilton cycles in $S t_{n}$. Then $|\widetilde{H}| \leq \varphi(n) / 2$, where $|\widetilde{H}|$ is the number of Hamilton cycles in $\widetilde{H}$.

Proof By Definition 2.8, as $\widetilde{H}$ is symmetric, any Hamilton cycle in $\widetilde{H}$ is the image of $H_{12}$ under a label automorphism and thus, by Lemma 2.9 and Definition 2.10, is of the form $H_{i j}$ with majority edge labels $i$ and minority edge labels $j$. By Lemma 2.14 (i) with $l=1, \operatorname{gcd}(i, n)=1$. Thus, the disjoint Hamilton cycles in $\widetilde{H}$ can be listed as

$$
H_{i_{1} j_{1}}, H_{i_{2} j_{2}}, \ldots, H_{i_{s} j_{s}}
$$

with majority edges with labels $i_{1}, \ldots, i_{s}$ respectively and minority edges with labels $j_{1}, \ldots, j_{s}$ respectively, and

$$
\operatorname{gcd}\left(i_{r}, n\right)=1 \quad(\text { for all } r \text { with } 1 \leq r \leq s)
$$

Therefore, $\left\{i_{1}, \ldots, i_{s}\right\} \subseteq\{1, \ldots,\lfloor n / 2\rfloor\}$ is a set of edge labels coprime to $n$, and there are at most $\varphi(n) / 2$ such integer labels.

An important corollary to Theorem 2.16 is that, if $n$ is not a prime number, $S t_{n}$ is not symmetrically Hamilton decomposable.

Corollary 2.17 If $n \geq 5$ is not a prime number, then there is no symmetric collection of disjoint Hamilton cycles $\widetilde{H}$ such that

$$
E\left(S t_{n}\right)=\bigcup_{H \in \widetilde{H}} E(H)
$$

where $E(H)$ denotes the set of edges in Hamilton cycle $H$.
Proof If the edges $E\left(S t_{n}\right)$ of $S t_{n}$ are partitioned into a collection $\widetilde{H}$ of disjoint Hamilton cycles, $\widetilde{H}$ will have $\lfloor n / 2\rfloor$ such cycles if $n$ is odd and $n / 2-1$ such cycles if $n$ is even. However, if the non-prime $n$ is odd then $\varphi(n)<n-1$ and if $n$ is even $\varphi(n) \leq n / 2$. By Theorem 2.16, $\widetilde{H}$ cannot be symmetric.

### 2.5 Lower bounds in even dimensions

Although $S t_{n}$ is not symmetrically Hamilton decomposable for any even integer $n$, we will find an optimal symmetric collection of disjoint Hamilton cycles, i.e. a collection with $\varphi(n) / 2$ Hamilton cycles, in Theorem 2.20 below. Constructing a symmetric collection involves finding a collection of label automorphisms which, when applied to $H_{12}$, generate disjoint Hamilton cycles as the images of $H_{12}$. Lemma 2.14 (ii) characterizes the pointwise maps of label automorphisms to be of the form $\phi\left(a_{i}\right)=a_{i_{0}+j i}$. In the following Lemma 2.18 (i) and (ii), the converse is given, i.e. that any pointwise map of the form $\phi\left(a_{i}\right)=a_{i_{0}+j i}$ consistently defines a distance map of edge labels

$$
\phi^{d}:\{1, \ldots,\lfloor n / 2\rfloor\} \mapsto\{1, \ldots,\lfloor n / 2\rfloor\}
$$

and therefore a label automorphism.
Lemma 2.18 Let $n$ be odd or even and $i_{0}, j \in\{1, \ldots, n\}$ be such that $j$ is coprime to $n$. If the bijection $\phi_{j}:\left\{a_{1}, \ldots, a_{n}\right\} \mapsto\left\{a_{1}, \ldots, a_{n}\right\}$ is defined by

$$
\phi_{j}\left(a_{i}\right)=a_{i_{0}+j i} \quad(1 \leq i \leq n)
$$

then the following hold:
(i) for all $a_{g}, a_{h} \in\left\{a_{1}, \ldots, a_{n}\right\}$,

$$
\delta\left(\phi_{j}\left(a_{g}\right), \phi_{j}\left(a_{h}\right)\right)=\min \{|j(g-h) \bmod n|, n-|j(g-h) \bmod n|\},
$$

(ii) there exists a bijection $\phi_{j}^{d}:\{1, \ldots,\lfloor n / 2\rfloor\} \mapsto\{1, \ldots,\lfloor n / 2\rfloor\}$ such that, for all $a_{g}, a_{h} \in\left\{a_{1}, \ldots, a_{n}\right\}$,

$$
\delta\left(\phi_{j}\left(a_{g}\right), \phi_{j}\left(a_{h}\right)\right)=\phi_{j}^{d}\left(\delta\left(a_{g}\right), \delta\left(a_{h}\right)\right)
$$

(iii) if $i_{0}=n$, i.e. $\phi_{j}\left(a_{i}\right)=a_{j i}$, then for the label automorphism $\Phi_{j}$ corresponding to $\phi_{j}$ as in Definition 2.5, we have that, for all $\overleftarrow{a}_{n} \in V\left(S t_{n}\right)$, there exists $\overleftarrow{a}^{\prime}{ }_{n} \in V\left(S t_{n}\right)$ such that

$$
\Phi_{j}\left(\overleftarrow{a}_{n}\right)=\overleftarrow{a}_{n}^{\prime}
$$

i.e. vertices ending in $a_{n}$ are mapped to vertices ending in $a_{n}$ by $\Phi_{j}$.

Proof For (i), we have that (arithmetic expressions are evaluated modulo n):

$$
\begin{aligned}
\delta\left(\phi_{j}\left(a_{g}\right), \phi_{j}\left(a_{h}\right)\right) & =\min \left\{\left|\left(i_{0}+j g\right)-\left(i_{0}+j h\right)\right|, n-\left|\left(i_{0}+j g\right)-\left(i_{0}+j h\right)\right|\right\} \\
& =\min \{|j(g-h)|, n-|j(g-h)|\}
\end{aligned}
$$

To prove (ii), we need to show that if $a_{g}, a_{h}, a_{g^{\prime}}, a_{h^{\prime}} \in\left\{a_{1}, \ldots, a_{n}\right\}$, then $\delta\left(a_{g}, a_{h}\right)=\delta\left(a_{g^{\prime}}, a_{h^{\prime}}\right)$ implies that $\delta\left(\phi_{j}\left(a_{g}\right), \phi_{j}\left(a_{h}\right)\right)=\delta\left(\phi_{j}\left(a_{g^{\prime}}\right), \phi_{j}\left(a_{h^{\prime}}\right)\right)$. We
have that:

$$
\begin{aligned}
\delta\left(a_{g}, a_{h}\right)=\delta\left(a_{g^{\prime}}, a_{h^{\prime}}\right) & \Rightarrow \min \{|g-h|, n-|g-h|\} \\
& =\min \left\{\left|g^{\prime}-h^{\prime}\right|, n-\left|g^{\prime}-h^{\prime}\right|\right\} \\
& \Rightarrow|g-h|=\left|g^{\prime}-h^{\prime}\right| \text { or }\left|g^{\prime}-h^{\prime}\right|=n-|g-h| \\
& \Rightarrow\{|g-h|, n-|g-h|\}=\left\{\left|g^{\prime}-h^{\prime}\right|, n-\left|g^{\prime}-h^{\prime}\right|\right\} \\
& \Rightarrow\{|j(g-h)|, n-|j(g-h)|\} \\
& =\left\{\left|j\left(g^{\prime}-h^{\prime}\right)\right|, n-\left|j\left(g^{\prime}-h^{\prime}\right)\right|\right\} \\
& \left.\Rightarrow \delta\left(\phi_{j}\left(a_{g}\right), \phi_{j}\left(a_{h}\right)\right)=\delta\left(\phi_{j}\left(a_{g^{\prime}}\right), \phi_{j}\left(a_{h^{\prime}}\right)\right) \quad \text { (by (i) }\right)
\end{aligned}
$$

Condition (iii) follows immediately from the definition of the corresponding label automorphism $\Phi_{j}$, Lemma 2.4, and the fact that $\phi_{j}\left(a_{n}\right)=a_{n}$ if $i_{0}=n$.

The offset $i_{0}$ in pointwise maps $\phi\left(a_{i}\right)=a_{i_{0}+j i}$ is important for ensuring that there is no clash of minority edges. Lemma 2.18 (iii) above shows that, if $i_{0}$ is not used, then vertices ending in $a_{n}$ are mapped to vertices ending in $a_{n}$. As, by Definition 2.7, minority edges have vertices ending in $a_{n}$, any collection of disjoint Hamilton cycles which use exclusively pointwise maps without $i_{0}$, would have all minority edges in the collection with vertices ending in $a_{n}$. This would lead to the possibility of the same edges belonging to different Hamilton cycles in the collection, as a clash of edge labels of minority edges in unavoidable for all even $n$. By use of $i_{0}$, we can ensure that even though different Hamilton cycles may share the same minority edge labels, different Hamilton cycles will not share the same edges as their vertices will end in a different $a_{i} \in\left\{a_{1}, \ldots, a_{n}\right\}$. The next lemma, Lemma 2.19, introduces the pointwise map $\phi_{+1}$ which just replaces $a_{i}$ by $a_{i+1}$.

Lemma 2.19 Let $\phi_{+1}:\left\{a_{1}, \ldots, a_{n}\right\} \mapsto\left\{a_{1}, \ldots, a_{n}\right\}$ be the pointwise map defined by:

$$
\phi_{+1}\left(a_{i}\right)=a_{i+1} \quad(1 \leq i \leq n)
$$

Then:
(i) $\phi_{+1}$ defines a corresponding distance map

$$
\phi_{+1}^{d}:\{1, \ldots,\lfloor n / 2\rfloor\} \mapsto\{1, \ldots,\lfloor n / 2\rfloor\}
$$

such that, for all $l \in\{1, \ldots,\lfloor n / 2\rfloor\}$,

$$
\phi_{+1}^{d}(l)=l
$$

(ii) if $\Phi_{+1}$ is the label automorphism corresponding to $\phi_{+1}$ then, for all $\overleftarrow{a}_{n} \in V\left(S t_{n}\right)$, there exists $\overleftarrow{a}_{1} \in V\left(S t_{n}\right)$ such that

$$
\Phi_{+1}\left(\overleftarrow{a}_{n}\right)=\overleftarrow{a}_{1}
$$

i.e. vertices ending in $a_{n}$ are mapped to vertices ending in $a_{1}$ by $\Phi_{+1}$.

Proof If $a_{g}, a_{h} \in\left\{a_{1}, \ldots, a_{n}\right\}$ then (with arithmetic being modulo n)

$$
\begin{aligned}
\delta\left(\phi_{+1}\left(a_{g}\right), \phi_{+1}\left(a_{h}\right)\right) & =\min \{|(g+1)-(h+1)|, n-|(g+1)-(h+1)|\} \\
& =\min \{|g-h|, n-|g-h|\} \\
& =\delta\left(a_{g}, a_{h}\right)
\end{aligned}
$$

Thus, $\phi_{+1}$ defines the identity distance $\operatorname{map} \phi_{+1}^{d}: L \mapsto L$. For (ii), we have that:

$$
\begin{aligned}
\Phi_{+1}\left(a_{g_{1}} \ldots a_{g_{n-1}} a_{n}\right)= & \phi_{+1}\left(a_{g_{1}}\right) \ldots \phi_{+1}\left(a_{g_{n-1}}\right) \phi_{+1}\left(a_{n}\right) \\
& =a_{g_{1}+1} \ldots a_{g_{n-1}+1} a_{1}
\end{aligned}
$$

We now prove that, for all even $n$, there are $\phi(n) / 2$ symmetric disjoint Hamilton cycles. The Hamilton cycles are generated by the label automorphisms of chosen pointwise maps, and make additional use of the pointwise map $\phi_{+1}$ of Lemma 2.19 to resolve any possible clashes of minority edges.

Theorem 2.20 For all even $n, S t_{n}$ has a symmetric collection of $\varphi(n) / 2$ disjoint Hamilton cycles $\widetilde{H}$.

Proof Let

$$
i_{1}, \ldots, i_{\varphi(n) / 2}
$$

be the $\varphi(n) / 2$ integers less than $n / 2$ which are coprime to $n$. First of all, for all $j \in\left\{i_{1}, \ldots, i_{\varphi(n) / 2}\right\}$ define $\phi_{j}:\left\{a_{1}, \ldots, a_{n}\right\} \mapsto\left\{a_{1}, \ldots, a_{n}\right\}$ by

$$
\phi_{j}\left(a_{i}\right)=a_{j i}
$$

Then, by Lemma 2.18 (ii), $\phi_{j}$ defines a distance map $\phi_{j}^{d}$ and corresponding label automorphism $\Phi_{j}$ as in Definition 2.5. Consider the image of $H_{12}$ under $\Phi_{j}$. From Lemma 2.18 (i) and as $j<n / 2$, we have that:

$$
\delta\left(a_{2}, a_{1}\right)=1 \text { and } \delta\left(\phi_{j}\left(a_{2}\right), \phi_{j}\left(a_{1}\right)\right)=\min \{|j|, n-|j|\}=j
$$

and

$$
\delta\left(a_{3}, a_{1}\right)=2 \text { and } \delta\left(\phi_{j}\left(a_{3}\right), \phi_{j}\left(a_{1}\right)\right)=\min \{|2 j|, n-|2 j|\}
$$

Thus, $\phi_{j}^{d}(1)=j$ and $\phi_{j}^{d}(2)= \pm 2 j \bmod n$. Taking the image $\Phi_{j}\left(H_{12}\right)$ for each $j \in\left\{i_{1}, \ldots, i_{\varphi(n) / 2}\right\}$ we produce a list of Hamilton cycles (with the majority and minority edge labels indicated in the subscripts):

$$
\begin{equation*}
H_{i_{1} \pm 2 i_{1}}, \ldots, H_{i_{\varphi(n / 2)} \pm 2 i_{\varphi(n / 2)}} \tag{6}
\end{equation*}
$$

as in Definition 2.10. As $i_{1}, \ldots, i_{\varphi(n) / 2}$ are distinct odd integers coprime to $n$, each majority edge in any Hamilton cycle in (7) only occurs in that Hamilton cycle as no other Hamilton cycle has the same edge label. However, it is possible that different Hamilton cycles in (7) share the same minority edge labels. We may have, for some distinct $i_{r}, i_{s} \in\left\{i_{1}, \ldots, i_{\varphi(n) / 2}\right\}$,

$$
\min \left\{\left|2 i_{r} \bmod n\right|, n-\left|2 i_{r} \bmod n\right|\right\}=\min \left\{\left|2 i_{s} \bmod n\right|, n-\left|2 i_{s} \bmod n\right|\right\}
$$

when $2 i_{r}=-2 i_{s} \bmod n$, i.e.

$$
\begin{equation*}
2 i_{s}=n-2 i_{r} \text { and so } i_{s}=n / 2-i_{r} \tag{7}
\end{equation*}
$$

From (8), it is clear that any minority edge label may be common to at most two Hamilton cycles in (7). To resolve this clash of minority edge labels, we replace one of the Hamilton cycles involved by one with the same labels but different vertices for minority edges. Suppose that the minority edges of $H_{i_{r} \pm 2 i_{r}}$ and $H_{i_{s} \pm 2 i_{s}}$ clash, so that $i_{s}=n / 2-i_{r}$. Consider the Hamilton cycles:

$$
\begin{equation*}
H_{i_{r} \pm 2 i_{r}}^{\prime}=\Phi_{i_{r}}\left(H_{12}\right) \text { and } H_{i_{s} \pm 2 i_{s}}^{\prime}=\Phi_{+1}\left(H_{i_{s} \pm 2 i_{s}}\right)=\Phi_{+1}\left(\Phi_{i_{s}}\left(H_{12}\right)\right) \tag{8}
\end{equation*}
$$

By Definitions 2.7 and 2.10, all vertices of minority edges of $H_{12}$ are of the form $\overleftarrow{a}_{n}$, and so, by Lemma 2.18 (iii), all vertices of minority edges of $\Phi_{i_{r}}\left(H_{12}\right)$ and $\Phi_{i_{s}}\left(H_{12}\right)$ are also of the form $\overleftarrow{a}_{n}$. From the latter it follows, by Lemma 2.19 (ii), that all vertices of minority edges of $\Phi_{+1}\left(\Phi_{i_{s}}\left(H_{12}\right)\right)$ are of the form $\overleftarrow{a}_{1}$. Thus, as the vertices of minority edges of $H_{i_{r} \pm 2 i_{r}}$ are of the form $\overleftarrow{a}_{n}$ and those of $H_{i_{s} \pm 2 i_{s}}^{\prime}$ are of the form $\overleftarrow{a}_{1}, H_{i_{r} \pm 2 i_{r}}$ and $H_{i_{s} \pm 2 i_{s}}$ are edge disjoint despite having the same minority edge labels. By resolving all pairs of clashes in this way in (7) we produce a collection of $\varphi(n) / 2$ symmetric and edge-disjoint cycles as required.

Theorem 2.20 shows that, for all even $n$, there is a symmetric collection of $\varphi(n) / 2$ disjoint Hamilton cycles $\widetilde{H}$ and Theorem 2.16 shows that this is the best that can be achieved for symmetric collections. Can this $\varphi(n) / 2$ bound be improved by adding non-symmetric disjoint Hamilton cycles to the collection $\widetilde{H}$ in Theorem 2.20? The answer is negative for 2-labelled Hamilton cycles sharing labels with Hamilton cycles in $\widetilde{H}$. If an extra disjoint Hamilton cycle $H_{j i}^{\prime}$ could be added, such that there is some Hamilton cycle $H_{i j} \in \widetilde{H}$, then the label automorphism that maps $H_{12}$ to $H_{i j}$ would also map $H_{21}^{\prime}$ to $H_{j i}^{\prime}$, where

$$
H_{21}^{\prime}=C_{12}-H_{12}
$$

is the spanning subgraph of $S t_{n}$ comprising the edges with labels 1 and 2 that are not in $H_{12}$, and $H_{21}^{\prime}$ would be also be hamiltonian. If $H_{21}^{\prime}$ is hamiltonian then, even though it is not symmetric to $H_{12}$ (as there is no distance map of $\left\{a_{1}, \ldots, a_{n}\right\}$ mapping distances 1 to distances 2 and distances 2 to distances 1 for all n greater than 5 ) the symmetric collection of $\varphi(n) / 2$ disjoint Hamilton cycles in Theorem 2.20 could be doubled in size to produce a non-symmetric collection of $\varphi(n)$ Hamilton cycles that are still edge-disjoint. Unfortunately, $H_{21}^{\prime}$ is not hamiltonian as the following theorem shows.

Theorem 2.21 The spanning subgraph $H_{21}^{\prime}$ of $S t_{n}$, comprising the edges of labels 1 and 2 that are not in $H_{12}$, is not a Hamilton cycle if $n$ is even.

Proof It is clear from Definition 2.7 that the number of edges with label 2 in $H_{12}$ is (n-2)!. Therefore, $H_{12}$ meets at most (n-2)! 2-balls. The total number
of 2-balls in $C_{12}$ is the number of vertices in $C_{12}(=\mathrm{n}!)$ divided by the number of vertices in a 2 -ball:

$$
\begin{equation*}
\left|C_{12}\right| /\left|B_{2}\right| \tag{9}
\end{equation*}
$$

As n is even and hence 2 is not coprime to n , by Lemma 2.12(ii) the number of vertices in a 2 -ball is less than $\mathrm{n}(\mathrm{n}-1)$ and so, by (10), the number of 2 balls exceeds (n-2)!. Hence, there is some 2-ball $B_{2}^{k}$ which $H_{12}$ does not meet. Clearly, the edges of this 2-ball $B_{2}^{k}$ must belong to $H_{21}^{\prime}$ which then cannot be hamiltonian as it contains a cycle with fewer than $n$ ! vertices.

### 2.6 Symmetric collections in odd dimensions

Whilst the $\varphi(n) / 2$ upper bound, on the number of Hamilton cycles in a symmetric collection also holds for $S t_{n}$ if n is odd, it is not clear that this bound can be achieved for any odd $n$ other than $n$ equals 5 [4]. In the case of even $n$, the number of Hamilton cycles in a symmetric collection $\widetilde{H}$ is limited to $\varphi(n) / 2$ because every majority edge label in $\widetilde{H}$ has to be coprime to n as the majority edge label 1 of the base Hamilton cycle $H_{12}$ is coprime to $n$. However, in the case of odd n, both the majority and minority edge labels of Hamilton cycles in symmetric collections have to be coprime to n as both the majority and minority edge labels of $H_{12}$, i.e. 1 and 2, are coprime to n. For this reason, it would appear that the upper bound for symmetric collections in the case of odd n should be $\varphi(n) / 4$. To exceed this bound would require a symmetric collection of Hamilton cycles $\widetilde{H}$ containing Hamilton cycles

$$
H_{i l}, H_{l j} \in \widetilde{H}
$$

such that the minority edges of $H_{i l}$ are exactly the edges with label l that are not present as majority edges in $H_{l j}$. This is a very tight restriction which is satisfied for n equals 5 [4] where there is a distance map which maps labels 1 to 2 , and therefore 2 to 1 as there are no other labels, such that the 2 Hamilton cycles produced automorphically map minority edges with label 2 in one Hamilton cycle to the unused edges with label 1 as minority edges in the second Hamilton cycle. It seems unlikely that the same majority and minority edge labels can occur in symmetric collections for odd n if n is greater than 5 and labels 1 and 2 cannot map to each other, though this remains an open problem. However, if $\varphi(n) / 4$ is the true bound, this is nearly achieved for all but one odd n by the construction in [3].

Theorem 2.22 For all odd $n \neq 127, S t_{n}$ has a symmetric collection of $2 \varphi(n) / 9$ disjoint Hamilton cycles $\widetilde{H}$.

Proof See [3].

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