Spectral properties of integrable Schrödinger operators with singular potentials

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A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy in Mathematics of Loughborough University

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September 2015

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Abstract

The integrable Schrödinger operators often have a singularity on the real line, which creates problems for their spectral analysis. A classical example is the Lamé operator

$$L = -\frac{d^2}{dx^2} + m(m+1)\wp(x),$$

where $\wp(z)$ is the classical Weierstrass elliptic function. We study the spectral properties of its complex regularisations of the form

$$L = -\frac{d^2}{dx^2} + m(m+1)\omega^2\wp(\omega x + z_0), \quad z_0 \in \mathbb{C},$$

where ω is one of the half-periods of $\wp(z)$. In several particular cases we show that all closed gaps lie on the infinite spectral arc.

In the second part we develop a theory of complex exceptional orthogonal polynomials corresponding to integrable rational and trigonometric Schrödinger operators, which may have a singularity on the real line. In particular, we study the properties of the corresponding complex exceptional Hermite polynomials related to Darboux transformations of the harmonic oscillator, and exceptional Laurent orthogonal polynomials related to trigonometric monodromy-free operators.

Key words: Complex Lamé operators, monodromy-free Schrödinger operators, exceptional orthogonal polynomials

Acknowledgements

I would like to thank the following people:

My supervisors Sasha Veselov and Martin Hallnäs, for always being available when I needed their advice, and for dedicating countless long evenings to our meetings. I quite literally would not have managed without either of you.

The academic and support staff of the Department of Mathematical Sciences, for creating a friendly and supportive environment.

The friends I have made during my time in Loughborough.

David and James, for their cross-disciplinary input.

Joachim, for his patience and understanding.

My family, especially my parents, for their unwavering love and support.

And finally, my partner Frances, for being there from the very start of my mathematical journey, and for being my rock.

Contents

Introduction

The Lamé equation

$$-\psi'' + m(m+1)\wp(x)\psi = \lambda\psi, \tag{1}$$

where $\wp(x)$ is Weierstrass' elliptic function, satisfying

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

was a classical object of study of 19$^{\text{th}}$ century mathematics.

Its solutions have remarkable properties in the complex domain, and can be described explicitly (Hermite, Halphen, see [48]). Note that for x on the real line, the potential $\wp(x)$ has singularities. However, for real e_1, e_2, e_3, one can make a pure imaginary half-period shift $z_0 = \omega_3$ and consider the Lamé operator

$$L = -D^2 + m(m+1)\wp(x + z_0), \quad D = \frac{d}{dx}, \tag{2}$$

with a potential which is real, periodic and regular on the whole of $\mathbb{R}$. This means that one can apply Bloch-Floquet theory, which states that in that case, the spectrum should have a band structure [41].

Generically, the spectrum of a periodic Schrödinger operator (or Hill operator) on the line consists of an infinite number of bands. It was Ince who in 1940 first pointed out a remarkable fact that for $m \in \mathbb{N}$, the spectrum of L has a band structure with not more than m gaps [26].

Nowadays, this example is just the simplest one in a large class of finite-gap operators discovered in the 1970s [36, 9, 15, 28, 31, 35]. It turns out that all such operators can be described explicitly in terms of hyperelliptic Riemann theta functions, and the Lamé operator (2) corresponds to the elliptic case. However, the

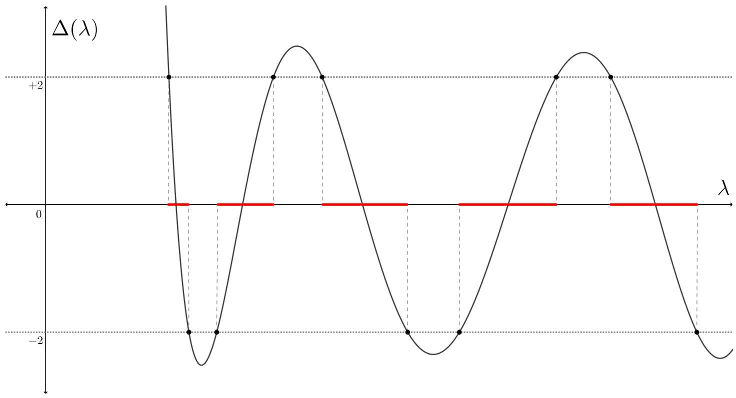


Figure 1: Band structure of a generic Hill operator's spectrum (in red).

question of exactly which gaps in the spectrum are open seems to be not explicitly discussed in the literature even in this case.

The first result of this thesis, dealt with in Chapter 1 of Part I, is to show that it is precisely the *first* m gaps which are open. For two linearly independent solutions ψ_1, ψ_2 of the time-independent Schrödinger equation $L\psi = \lambda\psi$ with a regular, periodic potential (known as Hill's equation), we can consider the *monodromy matrix*

$$M(\lambda) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

defined by

$$\psi_1(x+T) = a\psi_1(x) + b\psi_2(x),$$
$$\psi_2(x+T) = c\psi_1(x) + d\psi_2(x),$$

where T is a period. If ρ is an eigenvalue of M, then one can consider the Bloch-Floquet eigenfunction satisfying

$$\psi(x+T) = \rho\psi(x). \tag{3}$$

We refer to ρ as the *Floquet multiplier*. The spectrum corresponds to the bounded solutions, which means that $|\rho| = 1$. The points corresponding to $\rho = \pm 1$, such that $\psi(x)$ is periodic or anti-periodic with period T, occur at the edges of the spectral bands. The importance of these "band edges" lies in the fact that once they are known, the location of the rest of the spectrum is apparent. In the finite-gap case there are however infinitely many values of λ corresponding to a coexistence of *two* linearly independent periodic (or anti-periodic) solutions, embedded within the spectrum. These are known as "closed gaps"; as though two of the band edges, with one linearly independent solution each, collide.

The spectral bands are defined by the condition $-2 \le \Delta(\lambda) \le 2$, where $\Delta(\lambda) := \mathrm{tr}(M(\lambda))$ is usually called Hill's discriminant (see Figure 1).

To determine the location of the infinite *closed gaps*, we will follow the approach taken by Magnus and Winkler [33, 34] to prove Ince's result that there are exactly m open gaps. In addition, we prove that it is exactly the first m gaps which are open. The analysis of Chapter 1 can be summarized in the following theorem:

Theorem 1. *The Lamé operator* (2) *has all gaps of its spectrum open unless* $m \in \mathbb{Z}$*, in which case all gaps are closed except for the* first m*.*

Although this fact may not be surprising for the experts, we could not find a rigorous proof in the literature. For $m = 1$ this fact is demonstrated in Figure 2, showing that all the closed gaps are indeed in the infinite band.

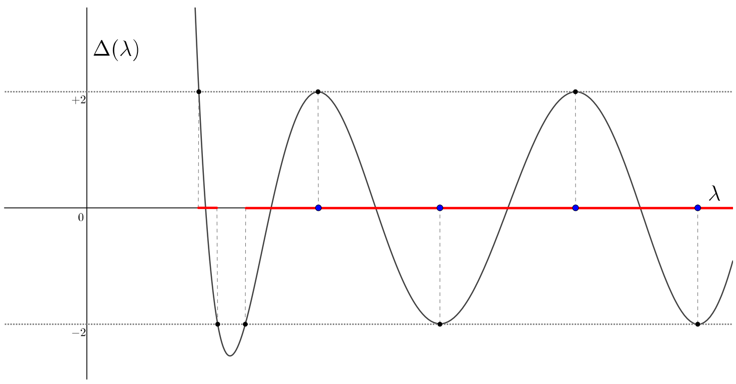


Figure 2: Band structure of Lamé operator in $m = 1$ case, where blue points are "closed gaps."

In Chapter 2, we consider the Lamé operator with a complex-valued periodic potential

$$V(x) = m(m+1)\omega^2\wp(\omega x + z_0),$$

where $\omega = \omega_i$, $i = 1, 2, 3$, are the real, complex and pure-imaginary (respectively) half-periods of $\wp$, and the only assumption on $z_0 \in \mathbb{C}$ is that the corresponding potential is non-singular.

The spectral theory of Schrödinger operators with a complex periodic potential has been studied in [42, 16, 47]. The *spectrum* of a Schrödinger operator

$$L = -D^2 + u(x),$$

with periodic, regular, but complex-valued potential $u(x)$ can be defined as the set of $\lambda \in \mathbb{C}$ such that all solutions of the equation $L\psi = \lambda\psi$ are bounded on the whole line. Equivalently, the corresponding Floquet multipliers $\rho(\lambda)$ should lie on a unit circle:

$$|\rho(\lambda)| = 1.$$

In the case of the Lamé operator, it follows from Rofe-Beketov [42] and Weikard [47] that the spectrum consists of finitely many regular analytic arcs of the stability set:

$$\mathscr{S}(L) = \{\lambda \in \mathbb{C} : -2 \leq \Delta(\lambda) \leq 2\},$$

with at most one additional arc within $\mathscr{S}(L)$ tending to infinity.

Note that if the shift is different from $z_0 = \omega_3$ in the Lamé operator (2), we in general have a periodic, regular, but complex-valued potential. It is easy to see that the Floquet multipliers do not depend on z_0, so we have the same spectrum as in the self-adjoint case.

For $\omega = \omega_1, \omega_3$ we have essentially Ince's result, since the corresponding operator is equivalent to the previous case.

In Chapter 2 we consider the first new and true complex case of $\omega = \omega_2$ assuming that $m = 1$. The solutions of the Lamé equation

$$-\frac{d^2\psi}{dz^2} + 2\wp(z)\psi = \lambda\psi, \quad \lambda = -\wp(k),$$

were found explicitly by Hermite:

$$\psi(z;k) = \frac{\sigma(z+k)}{\sigma(z)} \exp(-\zeta(k)z), \tag{4}$$

where $k \in \mathbb{C}$ and $\sigma(z), \zeta(z)$ are the Weierstrass sigma and zeta functions [48]. They have the Floquet property

$$\psi(z + 2\omega) = \exp(2\eta k - 2\zeta(k)\omega)\psi(z),$$

where $\eta = \zeta(\omega)$. The solutions remain bounded on the line $z = \omega x + z_0$, $x \in \mathbb{R}$ when

$$\mathrm{Re}[\eta k - \zeta(k)\omega] = 0, \tag{5}$$

which describe the corresponding spectral values of k for the Lamé operator

$$L = -\frac{d^2}{dx^2} + 2\omega^2 \wp(x\omega + z_0) \tag{6}$$

We use (5) to study the geometry of the spectral arcs of L. Figure 3 shows the *Mathematica* plots of the solutions of the system for $g_2 = 4$, $g_3 = 1$ and the corresponding values of $\lambda = -\wp(k)$.

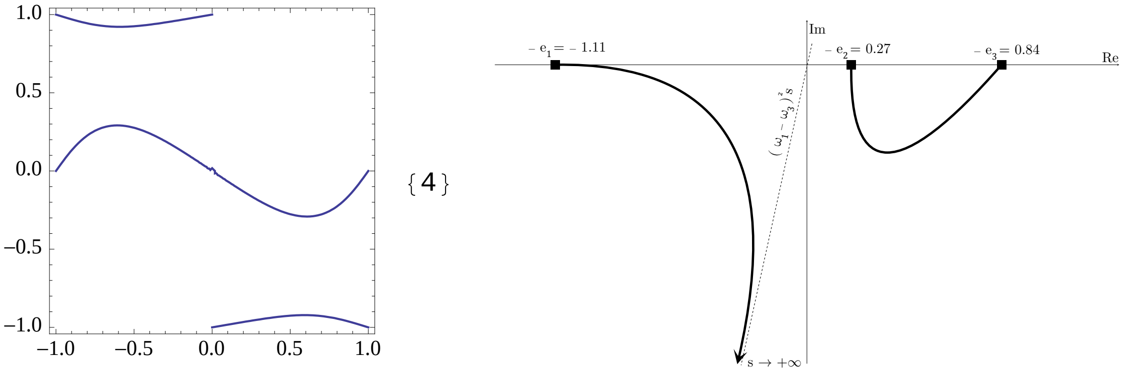


Figure 3: Solutions of (5) for k and corresponding spectrum for $g_2 = 4$, $g_3 = 1$.

In agreement with Weikard [47] we see two arcs, one of them is infinite. On the left figure this corresponds to the middle curve passing through $k = 0$. We show that the infinite spectral arc has the asymptote $\bar{\omega}_2^2 s$, where $s \in \mathbb{R}$ (see the right figure in Figure 3).

Consider now the (anti-)periodic solutions of $L\psi = \lambda\psi$ with k satisfying:

$$\underbrace{\zeta(\omega)k - \zeta(k)\omega}_{f(k)} = \pm p\frac{\pi i}{2}, \quad p \in \mathbb{Z}_{\geq 0} \tag{7}$$

One can check that the solutions for $p = 0, 1$ are exactly half-periods $k = \pm\omega_j$, $j = 1, 2, 3$ and correspond to the edges of spectral arcs. The other solutions must correspond to the "closed gaps". The question is where are they located.

The main result of Chapter 2 is the following:

Theorem 2. *All closed gaps of the complex Lamé operator (6) are contained on the infinite spectral arc.*

To prove this we consider first the lemniscatic case with $\omega_3 = i\omega_1$, when a part of the spectrum can be found explicitly (see Figure 4) and show that the closed gaps must be on the vertical line below the intersection point. Then apply continuity arguments for the eigenvalues to handle the general case.

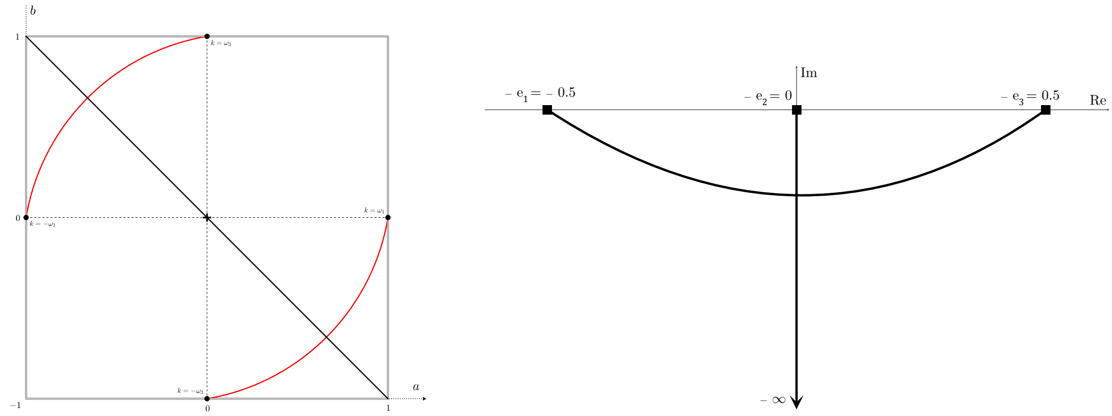


Figure 4: Solutions of (5) for k and corresponding spectrum in the lemniscatic case.

Part II of the thesis deals with rational and trigonometric monodromy-free Schrödinger operators, and the related theory of the complex exceptional orthogonal polynomials.

Consider polynomials $p_n(x) \in \mathbb{R}[x]$ of degrees $n = 0, 1, \ldots$, satisfying the orthogonality relation

$$(p_m, p_n) = \delta_{mn} g_n,$$

where the inner product of polynomials is defined by a real integral

$$(p, q) := \int_a^b p(x)q(x)w(x)dx \tag{8}$$

for some positive weight function w. Suppose that there exists a second order differential operator

$$T = A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx} + C(x)$$

having these polynomials as eigenvectors:

$$Tp_n(x) = E_n p_n(x), \quad n = 0, 1, \ldots.$$

A classical result due to Bochner [5] says that in that case the sequence of polynomials $p_n(x)$, $n \in \mathbb{Z}_{\geq 0}$, must coincide (up to a linear change of x) with one of the systems of classical orthogonal polynomials of Hermite, Laguerre or Jacobi.

Gómez-Ullate, Kamran and Milson [19] considered the following variation of Bochner's question. Let us assume now that in the previous considerations n belongs to a certain proper subset $S \subset \mathbb{Z}_{\geq 0}$ such that $\mathbb{Z}_{\geq 0} \setminus S$ is finite. To make this non-trivial they added the following *density condition*: the linear span $U = \langle p_n : n \in S \rangle$ of the corresponding polynomials must be dense in $\mathbb{R}[x]$ in the sense that if $(p, p_n) = 0$ for all $n \in S$ then $p \equiv 0$. In that case the sequence $p_n(x)$, $n \in S$ is called a system of *exceptional orthogonal polynomials.*

The main example of such polynomials are exceptional Hermite polynomials [18] having the Wronskian form

$$H_{\lambda,l}(x) := \mathrm{Wr}(H_l(x), H_{k_1}(x) \dots, H_{k_n}(x)), \quad l \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \dots, k_n\}, \tag{9}$$

where $H_l(x)$ are classical Hermite polynomials, $\lambda = (\lambda_1, \dots, \lambda_n)$ is a double partition and

$$k_i = \lambda_i + n - i, \;\; i = 1, \dots, n.$$

The double partitions have the very special form

$$\lambda = \mu^2 = (\mu_1, \mu_1, \mu_2, \mu_2, \dots, \mu_k, \mu_k),$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ is another partition with $n = 2k$ (see [13]). According to Krein and Adler [1] this guarantees that the corresponding Wronskian

$$W_\lambda(x) = \mathrm{Wr}(H_{k_1}(x) \dots, H_{k_n}(x))$$

has no zeroes on the real line and thus determines a non-singular weight function

$$w(x) = W_\lambda^{-2}(x) e^{-x^2}. \tag{10}$$

The geometry of the complex zeroes of the corresponding Wronskians is quite interesting and was studied by Felder et al. in [13].

The simplest example is given by

$$L = -D^2 + x^2 + \frac{2}{x^2} \tag{11}$$

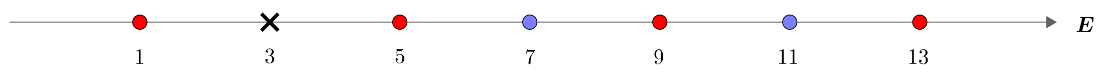


Figure 5: The eigenvalues for the first few eigenfunctions when $\lambda = (1)$. Red and blue points correspond to singular, non-singular eigenfunctions, respectively.

which has eigenfunctions

$$\psi_n = \frac{P_n(x)}{x} e^{-x^2/2},$$

where $P_n(x) = \mathrm{Wr}(H_n, x) = H_{(1),n}(x), n \neq 1$. Regarding the spectrum of eigenvalues E_n corresponding to these exceptional Hermite polynomials, there exists now a subset of n-values at which ψ_n is in fact singular. We can interpret these as "gaps" in the spectrum (see Figure 5).

One of the goals of the chapter is to find a proper interpretation of the exceptional Hermite polynomials (9) for all partitions λ. As we will see, this will naturally lead us to the notion of *quasi-invariance*, which appeared in the theory of *monodromy-free Schrödinger operators*, going back to Picard and Darboux and more recently revisited by Duistermaat and Grünbaum [10]. In certain classes such operators were explicitly described in terms of Wronskians in [10, 6, 37, 17]. Grinevich and Novikov studied the spectral properties of these and more general singular finite-gap operators and emphasized the important link with the theory of Pontrjagin spaces (see [20] and references therein). This chapter can be considered as dealing with the implications of all these results for the theory of exceptional orthogonal polynomials.

More precisely, we first complexify the picture by considering the vector space $V = \mathbb{C}[z]$ and replace the inner product (8) by a Hermitian product of the form

$$\langle p, q \rangle := \int_C p(z)\bar{q}(z)w(z)dz,$$

where $\bar{q}(z) := \overline{q(\bar{z})}$ is the Schwarz conjugate of the polynomial $q(z)$, $C \subset \mathbb{C}$ is a contour in the complex domain and $w(z)$ is a complex weight function. The condition that this product is Hermitian implies certain restrictions on the contour C and function $w(z)$ (see Section 3.2). It also requires certain restrictions on the set of polynomials for which the product is well defined. As it turned out, such

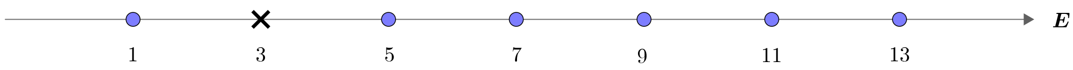


Figure 6: The eigenvalues $E_n = 2n + 1$ corresponding to the first few *complex* exceptional Hermite polynomials for $\lambda = (1)$. Comparing to Figure 5, all red points are now blue, with a single spectral “gap” remaining at $n = 1$.

polynomials form a subspace $U \subset V$ of finite codimension defined by some quasi-invariance conditions. Similarly to [19] we say that the polynomials $p_n(z)$, $n \in S$, form a system of *complex exceptional orthogonal polynomials* if their linear span is a subspace of U that is dense in U in the sense that $\langle p, p_n \rangle = 0$ for all $n \in S$, implies that $p \equiv 0$.

We will show that the Wronskians (9) satisfy this criteria for every partition λ and a suitable choice of C with w given by (10). For a double partition λ we can take as a contour C the real line with $U = V$ and recover the results of Gómez-Ullate et al. [18].

Note that the corresponding Hermitian form is positive definite only for double partitions, otherwise we always have polynomials with negative norms. The appearance of negative norms for singular potentials was first emphasized by Grinevich and Novikov [20].

Returning to the example with L as in (11), by accepting negative norms in this consideration of a general partition λ, we can “reclaim” those n-values, that had previously corresponded to singular eigenfunctions, as a part of the spectrum. Now, the spectral “gaps” are only those values $n = \mu_k$ that form the partition (see Figure 6).

We also consider the Laurent version of our approach, related to trigonometric monodromy-free Schrödinger operators. Some Laurent versions of orthogonal polynomials are already known in the literature (see e.g. [8] and references therein), but our approach is different since it is not based on the Gram-Schmidt procedure. Similarly, it does not fit into the theory of orthogonal polynomials on the unit circle initiated by Szegö [45], who considered the case of usual polynomials.

Consider the Laurent polynomials $\Lambda = \mathbb{C}[z, z^{-1}]$ and the following complex bilinear form on Λ:

$$(P, Q) = \frac{1}{2\pi i} \oint_C P(z)Q(z)\frac{dz}{z}$$

where $C = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. The standard basis z^n, $n \in \mathbb{Z}$, satisfies the *Laurent orthogonality relation*

$$(z^k, z^l) = \delta_{k+l,0}, \quad k, l \in \mathbb{Z}.$$

We consider more general forms

$$(P, Q) = \frac{1}{2\pi i} \oint_{C_\mu} P(z)Q(z)w(z)\frac{dz}{z},$$

where C_μ is the circle defined by $|z| = \mu$ and $w(z) = W(z)^{-2}$, with $W(z)$ some Laurent polynomial. For this form to be well-defined, we need to assume that P, Q belong to a suitable subspace of quasi-invariants $\mathscr{Q} \subset \Lambda$ of finite codimension.

Let $\mathcal{K}$ be a finite subset of $\mathbb{N}$. Suppose that $P_n \in \Lambda$, $n \in \mathbb{Z}$, satisfy the Laurent orthogonality relation

$$(P_k, P_l) = \delta_{k+l,0}h_k, \quad k, l \in \mathbb{Z},$$

but P_n is proportional to P_{-n} for $n \in \mathcal{K}$, which implies that the corresponding $h_n = 0$, and thus P_n is orthogonal to all P_k, $k \in \mathbb{Z}$. If the minimal complex Euclidean extension of the linear span of P_n, $n \in \mathbb{Z}$, coincides with the subspace of quasi-invariants $\mathscr{Q}$, then we call them *exceptional Laurent orthogonal polynomials.* The need to consider such an extension is the novelty of the Laurent case, which is related to the fact that the corresponding form is degenerate on the linear span of P_n, $n \in \mathbb{Z}$.

We present an example of such polynomials corresponding to the trigonometric monodromy-free Schrödinger operators [6]. Namely, for any set $\kappa = \{k_1, \ldots, k_n\}$ of distinct natural numbers $k_1 > k_2 > \cdots > k_n > 0$ and any choice of complex parameters $a = (a_1, \dots, a_n)$, $a_k \in \mathbb{C} \setminus \{0\}$, we define the Laurent polynomials

$$P_{\kappa,a;l}(z) = \begin{vmatrix} \Phi_{k_1}(a_1;z) & \Phi_{k_2}(a_2;z) & \cdots & \Phi_{k_n}(a_n;z) & z^l \\ D\Phi_{k_1}(a_1;z) & D\Phi_{k_2}(a_2;z) & \cdots & D\Phi_{k_n}(a_n;z) & Dz^l \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D^n\Phi_{k_1}(a_1;z) & D^n\Phi_{k_2}(a_2;z) & \cdots & D^n\Phi_{k_n}(a_n;z) & D^n z^l \end{vmatrix}$$

where $\Phi_k(a;z) = az^k + a^{-1}z^{-k}$, $k \in \mathbb{N}$ and $D = z\frac{d}{dz}$.

When parameters a_k satisfy the condition $|a_k| = 1$ for all $k = 1, \dots n$, we introduce a Hermitian form on a certain subspace of quasi-invariant Laurent polynomials $\mathscr{Q}_{\kappa,C}$, and show that the minimal Hermitian extension of the linear span of $P_{\kappa,a;l}$, $l \in \mathbb{Z}$, coincides with the subspace of quasi-invariants $\mathscr{Q}_\kappa$, and is dense in $\mathscr{Q}_{\kappa,C}$.

Chapters 1 and 2 of the Thesis are based on the paper in preparation [22], while Chapters 3 and 4 are based on [21].

Part I

Spectral properties of the complex Lamé operator

Chapter 1

Real spectrum

1.1 Introduction

In this chapter we study the spectrum of the Lamé operator

$$L = -\frac{d^2}{dx^2} + m(m+1)\wp(x+z_0)$$

with $m \in \mathbb{N}$ and $z_0 \in \mathbb{C}$ chosen so that $\wp(x+z_0)$ is regular for $x \in \mathbb{R}$. More specifically, following the exposition by Magnus and Winkler [33, 34], we review the general theory of the Hill equation and Ince's remarkable result that the spectrum of L has a band structure with not more than m gaps. Moreover, we prove that it is precisely the first m gaps in the spectrum that are open. As discussed in the introduction, it seems that this result has not been explicitly discussed in the literature.

We proceed to sketch a construction of Hermite's [24] solutions of the Lamé equation

$$-\frac{d^2\psi}{dx^2} + m(m+1)\wp(x+z_0)\psi = \lambda\psi, \quad x \in \mathbb{R}, \tag{1.1}$$

where λ is arbitrary. As we shall see below, the explicit form of these solutions allows us to conclude that the spectrum of L is independent of our specific choice of z_0.

A direct computation reveals that the product X of any pair of solutions of (1.1) satisfies the third order equation

$$-\frac{d^3X}{du^3} + 4\big(m(m+1)\wp(u) - \lambda\big)\frac{dX}{du} + 2m(m+1)\wp'(u)X = 0,$$

where $u = x + z_0$. Changing variable to $\xi = \wp(u)$ and utilising the chain rule, we can derive an algebraic form of this equation:

$$4(\xi - e_1)(\xi - e_2)(\xi - e_3)\frac{d^3X}{d\xi^3} + 3(6\xi^2 - \frac{1}{2}g_2)\frac{d^2X}{d\xi^2} \\ - 4\{(m^2 + m - 3)\xi + \lambda\}\frac{dX}{d\xi} - 2m(m+1)X = 0.$$

(For an explanation of the objects e_1, e_2, e_3 and g_2, see the definitions and identities associated with the Weierstrass $\wp$ function in Section 2.1.1). We can take an infinite series of the form

$$\sum_{r=0}^{\infty} c_r(\xi - e_2)^{m-r}, \quad c_0 = 1,$$

for some coefficients $c_r \in \mathbb{C}$, as a solution to this equation. From the recurrence relation for the coefficients c_r we find that it terminates at $r = m$, so that

$$X = \sum_{r=0}^{m} c_r(\xi - e_2)^{m-r}.$$

Factorising this polynomial yields the following important fact: Lamé's equation (1.1) has two solutions whose product X is of the form

$$X(u) = \prod_{r=1}^{m}(\wp(u) - \wp(k_r)),$$

for some $k_r \in \mathbb{C}$. Assuming there exist two linearly independent solutions of (1.1) $\psi_1(u), \psi_2(u)$, then their Wronskian will be constant (see Section 1.2), so that

$$\psi_1\psi_2' - \psi_2\psi_1' = 2C,$$

for some constant C. From this, and the fact that $\psi_1\psi_2 = X$, we have the following

$$\frac{d\log\psi_2}{du} - \frac{d\log\psi_1}{du} = \frac{2C}{X}, \\ \frac{d\log\psi_2}{du} + \frac{d\log\psi_1}{du} = \frac{1}{X}\frac{dX}{du},$$

from which we get

$$\psi_{1,2} := \psi^{\pm}(u) = \sqrt{X}\exp\left\{\mp C\int_0^u \frac{du}{X}\right\}. \tag{1.2}$$

We can take the Lamé equation with solutions of the form (1.2) and $u = k_r$ to find C, resulting in

$$\frac{2C}{X} = \sum_{r=1}^{m}\frac{\wp'(k_r)}{\wp(u) - \wp(k_r)}.$$

Finally, using the addition theorems for the Weierstrass σ and ζ functions:

$$\wp(u) - \wp(v) = \frac{\sigma(v+u)\sigma(v-u)}{\sigma^2(u)\sigma^2(v)},$$
$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)},$$

we are able to determine Hermite's solutions of (1.1), given by [48]:

$$\psi^{\pm}(x) = \prod_{i=1}^{m} \left\{ \frac{\sigma(x + z_0 \pm k_i)}{\sigma(x+z_0)\sigma(k_i)} \right\} e^{\mp \sum_{i=1}^{m} \zeta(k_i)(x+z_0)}. \tag{1.3}$$

They have the Floquet property

$$\psi^{\pm}(x + 2\omega) = \mu\psi^{\pm}(x),$$

with

$$\mu = \zeta(\omega)\sum_{j=1}^{m} k_j - \omega\sum_{j=1}^{m}\zeta(k_j).$$

Since μ is manifestly independent of z_0, it follows that the spectrum of L is indeed also independent of z_0 as long as $\wp(x+z_0)$ is regular for $x \in \mathbb{R}$.

Hence we can without loss of generality fix $z_0 = \omega_3$, which ensures that the potential

$$V(x) = m(m+1)\wp(x + z_0)$$

is regular and real-valued for $x \in \mathbb{R}$.

1.2 The Hill equation: General theory

Hill equations take the following form:

$$\frac{d^2y}{dx^2} + (\lambda + Q(x))y = 0, \tag{1.4}$$

where λ is a real parameter, and $Q(x)$ is a real (or complex) piecewise-continuous periodic function, with real variable x and primitive period π so that

$$Q(x+\pi) = Q(x).$$

There exist (see, e.g., [27]) two continuously differentiable solutions $y_1(x), y_2(x)$ for equation (1.4), determined uniquely by the following conditions:

$$\begin{aligned} &y_1(0) = 1, y_1'(0) = 0, \\ &y_2(0) = 0, y_2'(0) = 1. \end{aligned} \tag{1.5}$$

Due to the fact that $Q(x)$ is π-periodic, we can take $x \to x + \pi$ to get the following:

$$y''(x+\pi) + (\lambda + Q(x))y(x+\pi) = 0,$$

and so for each pair of solutions $y_1(x), y_2(x)$, we must also have the corresponding solutions $y_1(x+\pi), y_2(x+\pi)$. As $y_1(x), y_2(x)$ are linearly independent, we can use them as a basis for constructing $y_1(x+\pi), y_2(x+\pi)$:

$$\begin{aligned} y_1(x+\pi) &= \alpha_1 y_1(x) + \alpha_2 y_2(x), \\ y_2(x+\pi) &= \beta_1 y_1(x) + \beta_2 y_2(x). \end{aligned}$$

We can apply the boundary conditions (1.5) to find α_i, β_i (by taking $x = 0$):

$$\begin{aligned} \alpha_1 &= y_1(\pi), \quad & \alpha_2 &= y_1'(\pi), \\ \beta_1 &= y_2(\pi), \quad & \beta_2 &= y_2'(\pi). \end{aligned}$$

This yields the *monodromy matrix* M; an object describing the growth of solutions $y_1(x), y_2(x)$ after a shift by π in x:

$$\begin{pmatrix} y_1(x+\pi) \\ y_2(x+\pi) \end{pmatrix} = \underbrace{\begin{pmatrix} y_1(\pi) & y_1'(\pi) \\ y_2(\pi) & y_2'(\pi) \end{pmatrix}}_{M} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}. \tag{1.6}$$

The characteristic equation $\det(M - \rho I) = 0$ determining the eigenvalues ρ of M is given by:

$$\rho^2 - [y_1(\pi) + y_2'(\pi)]\,\rho + y_1(\pi)y_2'(\pi) - y_2(\pi)y_1'(\pi) = 0. \tag{1.7}$$

We define the Wronskian of $y_1(x), y_2(x)$ as follows:

$$W(y_1, y_2)(x) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x). \tag{1.8}$$

If we take the x-derivative of $W(x)$, we get the following:

$$W'(x) = y_1(x)y_2''(x) - y_2(x)y_1''(x), \tag{1.9}$$

where first-order terms have cancelled. As $y_1(x), y_2(x)$ are linearly independent solutions of (1.4), we can substitute the following for the second order terms of

(1.9):

$$y_1''(x) = -(\lambda + Q(x))y_1(x),$$
$$y_2''(x) = -(\lambda + Q(x))y_2(x).$$

It is clear that $W'(x) = 0, \forall x$, and so $W(x) = C, \forall x$, for some constant C. Taking $x = 0$ and substituting the boundary conditions (1.5) into (1.8), we see that $W(0) = 1$. Therefore, $W(y_1, y_2)(x) = 1, \forall x$. Due to this, we can simplify (1.7) as follows:

$$\rho^2 - [y_1(\pi) + y_2'(\pi)]\,\rho + 1 = 0. \tag{1.10}$$

If this equation has two distinct roots ρ_1, ρ_2, a fundamental set of solutions $y_1(x), y_2(x)$ can be found so that:

$$\begin{aligned} y_1(x+\pi) &= \rho_1 y_1(x), \\ y_2(x+\pi) &= \rho_2 y_2(x), \end{aligned} \tag{1.11}$$

whereas with repeated roots $\rho_1 = \rho_2$, we have:

$$\begin{aligned} y_1(x+\pi) &= \rho_1 y_1(x), \\ y_2(x+\pi) &= \rho_1 (y_2(x) + y_1(x)). \end{aligned} \tag{1.12}$$

Note that to ensure $y_1(x), y_2(x)$ remain bounded over all x, we must ensure $|\rho_1| = |\rho_2| = 1, \rho_1\rho_2 = 1$. For example, if $|\rho_1| > 1$, applying subsequent periods will lead to the following:

$$|y_1(x + N\pi)| = |\rho_1|^N |y_1(x)| \to \infty, \quad N \to \infty.$$

The spectrum of

$$L = -\frac{d^2}{dx^2} + Q(x)$$

consists of all $\lambda \in \mathbb{C}$ so that the solutions of (1.4) are bounded. The following theorem characterises the two forms that the solutions can take depending on whether the roots of (1.10) are repeated or distinct, and establishes the conditions that ensure boundedness.

Theorem 1.1 (Floquet's Theorem)**.** *When $\rho_1 \neq \rho_2$, equation* (1.4) *has two linearly independent solutions:*

$$f_1(x) = e^{iax} p_1(x),$$
$$f_2(x) = e^{-iax} p_2(x),$$

for some π-periodic functions $p_i(x)$, and $\rho_1 = e^{ia\pi}, \rho_2 = e^{-ia\pi}$. When $\rho_1 = \rho_2 := \rho$ (which only occurs when $\rho_1 = \rho_2 = \pm 1$), (1.4) has a non-trivial π-periodic (2π-periodic for $\rho_1 = \rho_2 = -1$) solution $g_1(x)$, as well as a linearly independent solution $g_2(x)$ satisfying:

$$g_2(x+\pi) = \rho g_2(x) + \theta g_1(x), \tag{1.13}$$

where θ is some constant.

Proof. We will split the proof into two parts. Firstly, for the case of distinct roots $\rho_1 \neq \rho_2$, we can construct two quasi-periodic solutions (1.11) to (1.4), so that:

$$\begin{aligned} y_1(x) &= e^{iax} p_1(x) =: f_1(x), \\ y_2(x) &= e^{-iax} p_2(x) =: f_2(x), \end{aligned}$$

where $p_1(x), p_2(x)$ are some π-periodic functions, and $a \in \mathbb{R}$. We can see that this satisfies (1.11) with $\rho_1 = e^{ia\pi}, \rho_2 = e^{-ia\pi}$. To show that $f_1(x), f_2(x)$ are linearly independent, we consider the contrary:

$$\begin{aligned} c_1 f_1(x) + c_2 f_2(x) &\equiv 0, \\ c_1 f_1(x+\pi) + c_2 f_2(x+\pi) = c_1 \rho_1 f_1(x) + c_2 \rho_2 f_2(x) &\equiv 0, \end{aligned}$$

for some non-vanishing constants c_1, c_2. As we know that $f_1(x), f_2(x)$ do not vanish identically, this can only be the case if $\rho_1 = \rho_2$, which is a contradiction.

For the second part of the proof, we will consider repeated roots $\rho_1 = \rho_2$. We can construct a single (anti-)periodic solution $y(x)$ from (1.12), so that:

$$y(x+\pi) = \rho_1 y(x) = \pm y(x), \tag{1.14}$$

where we have used that $|\rho_1| = |\rho_2| = 1, \rho_1 = \rho_2 \implies \rho_1 = \rho_2 = \pm 1$. We can express $y(x)$ in terms of the basis of linearly independent solutions $y_1(x), y_2(x)$:

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

for some non-vanishing constants c_1, c_2. Shifting by π, we get:

$$\begin{aligned} y(x+\pi) - \rho_1 y(x) = & \left[c_1 \left(y_1(\pi) - \rho_1\right) + c_2 y_2(\pi)\right] y_1(x) \\ & + \left[c_1 y_1'(\pi) + c_2 \left(y_2'(\pi) - \rho_1\right)\right] y_2(x) = 0, \end{aligned}$$

where we have used (1.14). As we know that $y_1(x), y_2(x)$ do not vanish identically, we get the following equations which must be satisfied for c_1, c_2:

$$c_1(y_1(\pi) - \rho_1) + c_2 y_2(\pi) = 0, \tag{1.15}$$

$$c_1 y_1'(\pi) + c_2(y_2'(\pi) - \rho_1) = 0. \tag{1.16}$$

As such, we can construct the following solution:

$$y_2(\pi)y_1(x) + [\rho_1 - y_1(\pi)]\, y_2(x) =: g_1(x), \tag{1.17}$$

where we have chosen $c_1 = y_2(\pi), c_2 = \rho_1 - y_1(\pi)$ (for (1.16), we have used the characteristic equation (1.10)). This solution (1.17) is linearly independent of the solution:

$$y_2(x) =: g_2(x),$$

as long as $y_2(\pi) \neq 0$. Shifting $g_2(x)$ by π, we get the following:

$$g_2(x+\pi) = \rho_1 g_2(x) + g_1(x), \tag{1.18}$$

where we have used $y_1(\pi) + y_2'(\pi) = 2\rho_1$ from (1.10). It is clear that due to the presence of $g_1(x)$ in (1.18), $g_2(x)$ must be unbounded. For example, applying subsequent periods gives:

$$g_2(x + N\pi) = \pm g_2(x) + N g_1(x) \to \pm\infty, \quad N \to \pm\infty.$$

If instead $y_2(\pi) = 0$, we can construct the following linearly independent solutions:

$$g_1(x) = y_2(x), \tag{1.19}$$

$$g_2(x) = y_1(x), \tag{1.20}$$

Utilising (1.6), and shifting (1.19) and (1.20) by π, gives the following:

$$\begin{aligned} g_1(x+\pi) &= y_2(x+\pi) = y_2'(\pi)y_2(x) \\ &= \rho_1 g_1(x), \\ g_2(x+\pi) &= y_1(x+\pi) = y_1(\pi)y_1(x) + y_1'(\pi)y_2(x) \\ &= \rho_1 g_2(x) + y_1'(\pi)g_1(x), \end{aligned} \tag{1.21}$$

where we have used $W(y_1, y_2)(x) = y_1(\pi)y_2'(\pi) = 1$, and $y_1(\pi) + y_2'(\pi) = 2\rho_1$ from (1.10). Therefore, θ in (1.13) is equal to 1 when $y_2(\pi) \neq 0$, or $y_1'(\pi)$ when $y_2(\pi) = 0$. $\square$

Remark 1.1 (Stability test)**.** *Notice that if $\rho_1 = \rho_2$, it is necessary that:*

$$y_1(\pi) + y_2'(\pi) = \pm 2, \tag{1.22}$$

$$y_2(\pi) = 0, \tag{1.23}$$

$$y_1'(\pi) = 0, \tag{1.24}$$

to ensure that all solutions of (1.4) *remain bounded. Condition* (1.22) *follows from $y_1(\pi) + y_2'(\pi) = 2\rho_1$, and $\rho_1 = \pm 1$,* (1.23) *was demonstrated in* (1.18)*, and* (1.24) *was demonstrated in* (1.21)*. For $\rho_1 \neq \rho_2$, we must ensure that:*

$$|y_1(\pi) + y_2'(\pi)| < 2,$$

which occurs when $a \in \mathbb{R} \setminus \mathbb{Z}$ in (1.10)*, for all solutions to be bounded. We can see this by considering $\rho_1 + \rho_2 = 2\cos(a\pi)$, which is equal to ± 2 when $\rho_1 = \rho_2$, and belongs to $(-2, 2)$ when $\rho_1 \neq \rho_2$.*

We will call bounded solutions "stable", and unbounded solutions "unstable".

The following theorem will set out the notion of "interlacing" eigenvalues, which will play an important part in determining the structure of the Lamé spectrum in the next section. Essentially, eigenvalues corresponding to periodic and anti-periodic solutions alternate in pairs which can potentially coexist (i.e. correspond to the same eigenvalue). The proof of the theorem will consist of analysing gradients of the trace of the monodromy matrix:

$$\mathrm{tr}(M) \equiv y_1(\pi) + y_2'(\pi),$$

as a function of λ.

Theorem 1.2 (Oscillation Theorem)**.** *There exist two monotonically increasing infinite sequences of real numbers:*

$$\lambda_0, \lambda_1, \lambda_2, \ldots,$$

$$\lambda_1', \lambda_2', \lambda_3', \ldots,$$

so that the Hill equation (1.4) *has a solution of period* π *if and only if* $\lambda = \lambda_n,\ \ n \in \mathbb{N}$, *and a solution of period* 2π *if and only if* $\lambda = \lambda'_n,\ \ n \in \mathbb{N}^*$, *and:*

$$\lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda_2 < \lambda'_3 \leq \lambda'_4 < \lambda_3 \leq \lambda_4 < \ldots. \tag{1.25}$$

The solutions of (1.4) *are stable in the open intervals:*

$$(\lambda_0, \lambda'_1), (\lambda'_2, \lambda_1), (\lambda_2, \lambda'_3), (\lambda'_4, \lambda_3), \ldots, (\lambda_{2n}, \lambda'_{2n+1}), (\lambda'_{2n+2}, \lambda_{2n+1}), \ldots.$$

Solutions at the endpoints of an "interval of stability" will be unstable (i.e. $y'_1(\pi) \neq 0$ *or* $y_2(\pi) \neq 0$*) unless* $\lambda_{2n+1} = \lambda_{2n+2}$, $\lambda'_{2n+1} = \lambda'_{2n+2}$, *for any* $n \in \mathbb{N}$. *Solutions at* λ_0 *are always unstable.*

Figure 1.1 provides a visual demonstration of the intertwining eigenvalues.

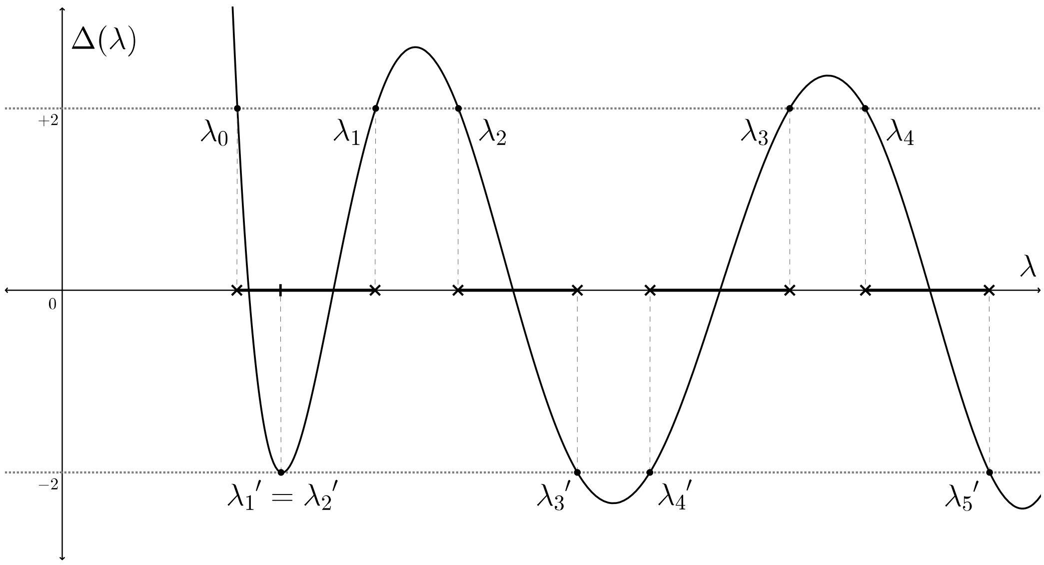


Figure 1.1: A hypothetical $\Delta(\lambda)$ with roots at ± 2. Intervals of instability (or "gaps") are seen at $[\lambda_1, \lambda_2], [\lambda'_3, \lambda'_4], [\lambda_3, \lambda_4]$. $\lambda'_1 = \lambda'_2$ is what we will call a "closed gap".

In order to prove Theorem 1.2, we will need to first prove three lemmas.

Lemma 1.1. *There exists some* $\lambda_0 \in \mathbb{R}$ *so that for* $\lambda \leq \lambda_0$ *all solutions of* (1.4) *are unstable.*

Proof. We choose λ_0 so that, $\forall x$:

$$\lambda_0 + Q(x) < 0,$$

which is possible due to $Q(x)$ being periodic in x, and therefore bounded $\forall x$. This ensures that (1.4) becomes:

$$\frac{d^2y}{dx^2} = D(x)y, \tag{1.26}$$

with $D(x) > 0, \forall x$, where $D(x) = -(\lambda_0 + Q(x))$. We will show that if $\lambda \leq \lambda_0$, $y_1(x,\lambda) \to \infty$ as $x \to \infty$, so that $y_1(x,\lambda)$ is unstable. Substituting $x = 0$ into (1.26) and using the boundary conditions (1.5) for $y = y_1(x,\lambda)$, we find that $y_1''(0,\lambda) > 0$, $y_1'(0,\lambda) = 0$. The second-derivative test tells us that $y_1(x,\lambda)$ has a local minimum at $x = 0$. Therefore, either $y_1'(x,\lambda) > 0, \forall x > 0$ (i.e. $y_1(x,\lambda)$ is monotonically increasing for $x > 0$), or there exists some positive ϵ so that $y_1'(\epsilon,\lambda) = 0$, and $y_1'(x,\lambda) > 0$ for $x \in (0,\epsilon)$. We will take the latter case and prove that it leads to a contradiction, so that we must instead have the former.

By substituting $y_1''(x,\lambda) = \frac{1}{2y_1'(x,\lambda)}\frac{d}{dx}\left(y_1'^2(x,\lambda)\right)$ into (1.26) we get the following when integrating over $0 \leq x \leq \epsilon$:

$$y_1'^2(\epsilon,\lambda) = 2\int_0^\epsilon D(x)y_1(x,\lambda)y_1'(x,\lambda)\,dx.$$

Now, the LHS will equal zero due to $y_1'(\epsilon,\lambda) = 0$, but $D(x) > 0, \forall x$, $y_1'(x,\lambda) \geq 0$ for $0 \leq x \leq \epsilon$ and, as $y_1(0,\lambda) = 1$ from (1.5), $y_1(x,\lambda) \geq 1$ for $0 \leq x \leq \epsilon$. The RHS is therefore positive, and we have a contradiction. This implies that $y_1'(x,\lambda) > 0, \forall x > 0$. Let us choose some $x = x_0 > 0$, so that $\delta = y_1'(x_0,\lambda) > 0$. Then, from the Taylor expansion of $y_1(x,\lambda)$ about x_0, we have:

$$\begin{aligned} y_1(x,\lambda) &= y_1(x_0,\lambda) + (x - x_0)y_1'(x_0,\lambda) + \ldots \\ &\geq 1 + (x - x_0)\delta, \quad x \geq x_0 \end{aligned}$$

and so $y_1(x,\lambda) \to \infty$ as $x \to \infty$, as required.

Using the same method as above, we can show that $y_2'(x,\lambda) > 1$ for $x > 0$. When we also take into consideration that $y_1(x,\lambda) \geq 1$ for $x \geq 0$ from (1.5), we subsequently find that:

$$\Delta(\lambda) \equiv y_1(\pi,\lambda) + y_2'(\pi,\lambda) > 2, \tag{1.27}$$

for $\lambda \leq \lambda_0$, where $\Delta(\lambda)$ will be termed Hill's discriminant. □

Lemma 1.2. *Let* $\mu,\ \mu'$ *be roots of the equations* $\Delta(\lambda) = 2,\ \Delta(\lambda) = -2$, *so that* $\Delta'(\mu) \leq 0,\ \Delta'(\mu') \geq 0$, *respectively. Then* $\Delta'(\lambda) < 0$ *in any open interval* $\mu < \lambda < \mu_1$, *so that* $\Delta(\lambda) > -2$, *and* $\Delta'(\lambda) > 0$ *in any open interval* $\mu' < \lambda < \mu_1'$, *so that* $\Delta(\lambda) < 2$.

The assertions of Lemma 1.2 are demonstrated in Figure 1.1.

Proof of Lemma 1.2. We will begin by attempting to construct a convenient representation of $\Delta'(\lambda)$. Let us introduce the following:

$$\begin{aligned} z_i(x,\lambda) &= \frac{d}{d\lambda} y_i(x,\lambda), \\ z_i'(x,\lambda) &= \frac{d}{d\lambda} y_i'(x,\lambda), \end{aligned}$$

for $i = 1, 2$, where the prime represents differentiation with respect to x. This allows us to write the following:

$$\Delta'(\lambda) = z_1(\pi,\lambda) + z_2'(\pi,\lambda), \tag{1.28}$$

where $\Delta'(\lambda)$ implies differentiation with respect to λ. We can express $z_i(x,\lambda)$, $z_i'(x,\lambda)$ in integral form by differentiating (1.4) with respect to λ:

$$z_i''(x,\lambda) + (\lambda + Q(x))z_i(x,\lambda) = -y_i(x,\lambda),$$

and solving using the variation of constants method for $z_i(x,\lambda)$ (see [27]). We obtain the following linearly independent solutions, and their derivatives:

$$\begin{aligned} z_i(x,\lambda) &= y_1(x,\lambda)\int_0^x y_2(t,\lambda)y_i(t,\lambda)\,dt - y_2(x,\lambda)\int_0^x y_1(t,\lambda)y_i(t,\lambda)\,dt, \\ z_i'(x,\lambda) &= y_1'(x,\lambda)\int_0^x y_2(t,\lambda)y_i(t,\lambda)\,dt - y_2'(x,\lambda)\int_0^x y_1(t,\lambda)y_i(t,\lambda)\,dt, \end{aligned} \tag{1.29}$$

where we have used the boundary conditions $z_i(0) = z_i'(0) = 0$ as a direct result of (1.5). Taking $x = \pi$ and substituting (1.29) into (1.28), we get the following:

$$\begin{aligned} \Delta'(\lambda) = \int_0^\pi &\{(y_1(\pi,\lambda) - y_2'(\pi,\lambda))y_1(t,\lambda)y_2(t,\lambda) \\ &-y_2(\pi,\lambda)y_1^2(t,\lambda) + y_1'(\pi,\lambda)y_2^2(t,\lambda)\}\,dt. \end{aligned} \tag{1.30}$$

As the integrand is quadratic, we can complete the square to form the following:

$$\begin{aligned} \Delta'(\lambda) = \pm &\int_0^\pi \left(\sqrt{|y_1'(\pi,\lambda)|}y_2(t,\lambda) \pm \frac{y_1(\pi,\lambda) - y_2'(\pi,\lambda)}{2\sqrt{|y_1'(\pi,\lambda)|}}y_1(t,\lambda)\right)^2 dt \\ &\mp \frac{\Delta^2(\lambda) - 4}{4|y_1'(\pi,\lambda)|}\int_0^\pi y_1^2(t,\lambda)\,dt, \end{aligned} \tag{1.31}$$

where the choice of sign depends on whether $y_1'(\pi,\lambda) > 0$ or $y_1'(\pi,\lambda) < 0$, respectively, and it is assumed that $y_1'(\pi,\lambda) \neq 0$. We have also made use of the identity $W(y_1, y_2)(t) = 1$, $\forall t$, and subsequently that:

$$\Delta^2(\lambda) - 4 = (y_1(\pi,\lambda) - y_2'(\pi,\lambda))^2 + 4y_1'(\pi,\lambda)y_2(\pi,\lambda). \tag{1.32}$$

Notice from Theorem 1.1 that $y_1'(\pi,\lambda) \neq 0$ corresponds to there being unstable solutions when $\Delta(\lambda) = \pm 2$. We can see from (1.31) that $\Delta'(\lambda)$ shares the sign of $y_1'(\pi,\lambda)$ as long as $\Delta^2(\lambda) \leq 4$, and also that:

$$y_1'(\pi,\lambda) = 0 \implies \Delta'(\lambda) = 0. \tag{1.33}$$

Returning now to the statement of the lemma, we want to show that when taking $\lambda = \mu$ so that $\Delta(\mu) = 2$, $\Delta'(\mu) \leq 0$, we will find $2 > \Delta(\lambda) > -2$, $\Delta'(\lambda) < 0$ for $\mu < \lambda < \mu_1$, where $\mu_1 - \lambda$ is sufficiently small. Obviously, this is the case when $\Delta'(\mu) < 0$, so let us consider the case where $\Delta'(\mu) = 0$, which occurs when $y_1'(\pi,\mu) = 0$. We can make use of (1.32), where

$$\Delta(\mu) = 2 \implies \Delta^2(\mu) - 4 = 0,$$

and $W(y_1, y_2)(t) = 1$, $\forall t$, to find that:

$$y_1(\pi,\mu) = y_2'(\pi,\mu) = 1,$$

and so, with $y_1'(\pi,\mu) = 0$, (1.30) becomes:

$$\Delta'(\mu) = -y_2(\pi,\mu)\int_0^\pi y_1^2(t,\mu)\,dt = 0,$$

which implies that:

$$y_2(\pi,\mu) = 0.$$

We clearly can't determine whether $\Delta(\lambda)$ is decreasing just from $\Delta'(\mu) = 0$, so we will need to take the second derivative of $\Delta(\lambda)$ at $\lambda = \mu$ and check that it is negative when:

$$y_1'(\pi,\mu) = y_2(\pi,\mu) = 0,$$
$$y_1(\pi,\mu) = y_2'(\pi,\mu) = 1.$$

Differentiating (1.30) with respect to λ, and substituting the relations (1.29) with $\lambda = \mu$, we find:

$$\Delta''(\mu) = 2\left\{\int_0^\pi y_1(t,\mu)y_2(t,\mu)\,dt\right\}^2 - 2\int_0^\pi y_1^2(t,\mu)dt\int_0^\pi y_2^2(t,\mu)\,dt.$$

We can see using Schwarz's Inequality that, because $y_1(t,\mu), y_2(t,\mu)$ are linearly independent, we get:

$$\Delta''(\mu) < 0,$$

as required.

Let us assume, contrary to the statement of the lemma, that there exists some $\mu^* > \mu$, so that $\Delta'(\lambda) < 0$ for $\mu < \lambda < \mu^*$, but $\Delta'(\mu^*) = 0$ for $\Delta(\mu^*) > -2$ (i.e. there is a turning point within the region $|\Delta(\lambda)| < 2$ in Figure 1.1). From (1.32) with $\lambda = \mu^*$ we find that $y_1'(\pi,\mu^*)y_2(\pi,\mu^*) < 0$. But because $\Delta'(\mu^*) = 0$, we also find from (1.31) that $y_1'(\pi,\mu^*) = 0$, which implies that $y_1'(\pi,\mu^*)y_2(\pi,\mu^*) = 0$, and so we have a contradiction. This proves the Lemma in the case where $\Delta(\lambda) = 2$. For $\Delta(\lambda) = -2$, we can apply the same method for some $\lambda = \mu'$ to prove the Lemma completely. □

Remark 1.2. *By considering the equation:*

$$\Delta^2(\lambda) - 4 = 0, \tag{1.34}$$

in Lemma 1.2 for a particular $\lambda = \mu$, we see that $\Delta'(\mu) = 0$ implies a double root, so that:

$$\Delta''(\mu) < 0, \quad \text{if} \quad \Delta(\mu) = 2,$$

$$\Delta''(\mu) > 0, \quad \text{if} \quad \Delta(\mu) = -2,$$

whereas $\Delta'(\mu) \neq 0$ implies a simple root.

Lemma 1.3. *Let λ_0 be the smallest root of $\Delta^2(\lambda) - 4 = 0$. Then λ_0 is a simple root so that $\Delta'(\lambda_0) < 0$.*

Proof. We have already proved that $\lambda < \lambda_0$ implies $\Delta(\lambda) > 2$ in (1.27). Therefore, if $\Delta(\lambda_0) = 2$, we must have either that $\Delta'(\lambda_0) = 0$ or $\Delta'(\lambda_0) < 0$. If we take the former case, Remark 1.2 highlights that $\Delta''(\lambda_0) < 0$ (i.e. a maximum), which is a contradiction due to $\Delta(\lambda) > 2$ for $\lambda < \lambda_0$. Therefore, we must have the latter case; that $\Delta'(\lambda_0) < 0$ and λ_0 is a simple root. □

We now have everything we need to prove Theorem 1.2.

Proof of Theorem 1.2. It is clear from considering Remarks 1.1 and 1.2 that the roots corresponding to the equation (1.34), which we shall call $\lambda = \lambda_{2n+1}$ for $\Delta(\lambda) = 2$ and $\lambda = \lambda'_{2n+1}$ for $\Delta(\lambda) = -2$, are simple iff $\Delta'(\lambda) \neq 0$, so that $y_1'(\pi, \lambda) \neq 0$ from (1.33), which implies that solutions are unstable. Whereas, they are double iff $\Delta'(\lambda) = 0$, so that $y_1'(\pi, \lambda) = y_2(\pi, \lambda) = 0$, which implies that solutions are stable, and in turn that $\lambda_{2n+1} = \lambda_{2n+2}$, $\lambda'_{2n+1} = \lambda'_{2n+2}$, as required. We can see from Theorem 1.1 that $\rho_1 = \rho_2 = \pm 1$ corresponds to $\Delta(\lambda) = \pm 2$, and so solutions corresponding to these roots will be π-periodic, 2π-periodic, respectively. This completes the proof of Theorem 1.2. □

It was subsequently proved in [33], by analysing $\Delta(\lambda)$ as an entire analytic function of a complex variable λ, that the functions $\Delta(\lambda) \pm 2$ have infinitely many zeroes. This implies that there are infinitely many points of coexistence of (anti-)periodic solutions (i.e. “closed gaps”) in the spectrum of (1.4).

In the next section, we will aim to utilise the theory of this section to investigate the spectrum of the Jacobi Lamé equation, which is of Hill type.

1.3 Spectral gaps of the Lamé operator: Jacobi form

Here, we will consider the Lamé equation in Jacobi form:

$$\frac{d^2 y}{dx^2} + \left[\lambda - m(m+1)k^2 \mathrm{sn}^2(x)\right] y = 0, \tag{1.35}$$

where $0 < k < 1$, $m \in \mathbb{R}$, and $\mathrm{sn}(x)$ is one of the Jacobi elliptic functions (see [48]). As $\mathrm{sn}(x)$ is periodic in x with real period $4K$, it is clear that (1.35) is an equation of Hill type (1.4), and therefore subject to the findings of the preceding section. We can use this to help us prove the following theorem (and accompanying lemmas) due to Magnus and Winkler, which will be the focus of the entire section.

Theorem 1.3. *[34] Periodic (resp. anti-periodic) solutions of period* $4K$ *of* (1.35) *coexist (i.e. correspond to a double root of* $\Delta^2(\lambda) - 4 = 0$*) iff* $m \in \mathbb{Z}$*. If* l *is defined* *by:*

$$l = \begin{cases} m, & m \in \mathbb{Z}_+, \\ -m-1, & m \in \mathbb{Z}_-, \end{cases}$$

then (1.35) *will have at most* $l+1$ *intervals of instability, including* $(-\infty, \lambda_0]$.

To prove Theorem 1.3, we will need to prove three lemmas and a corollary. First, it will be necessary to transform (1.35) into Ince's equation:

$$(1 + a\cos 2\phi)\frac{d^2y}{d\phi^2} + b(\sin 2\phi)\frac{dy}{d\phi} + (c + d\cos 2\phi)y = 0, \tag{1.36}$$

where a, b, c, d are real constants. We define $\mathrm{sn}(x)$ as the solution to:

$$\left(\frac{\partial y}{\partial x}\right)^2 = (1-y^2)(1-k^2y^2).$$

We can reformulate this to get the following integral equation:

$$x = \int_0^{\mathrm{sn}(x)} (1-y^2)^{-\frac{1}{2}}(1-k^2y^2)^{-\frac{1}{2}}\, dy,$$

which, with the substitution of $y = \sin t$, becomes:

$$x = \int_0^{\phi} (1 - k^2\sin^2 t)^{-\frac{1}{2}}\, dt = F(\phi, k), \quad \text{the elliptic integral of the first kind,}$$

where $\phi = \mathrm{am}(x,k)$ is known as the Jacobi amplitude, which obeys the following:

$$\begin{aligned} \sin\phi &= \mathrm{sn}(x,k), \\ \frac{d\phi}{dx} &= \mathrm{dn}(x,k) = (1 - k^2\sin^2\phi)^{\frac{1}{2}}. \end{aligned}$$

Now, using the chain rule we can convert (1.35) to the following:

$$(1 - k^2\sin^2\phi)\frac{d^2y}{d\phi^2} - \frac{k^2}{2}(\sin 2\phi)\frac{dy}{d\phi} + \left[\lambda - m(m+1)k^2\sin^2\phi\right]y = 0, \tag{1.37}$$

which is in the form of Ince's equation (1.36), where a, b, c, d are as follows:

$$\begin{aligned} a &= \frac{k^2}{2-k^2} = -b, \\ c &= \frac{2\lambda - m(m+1)k^2}{2-k^2}, \\ d &= \frac{m(m+1)k^2}{2-k^2}. \end{aligned}$$

The real period $4K$ now corresponds to π in (1.37).

We now state the first lemma, which characterises the solutions to (1.36).

Lemma 1.4. *If Ince's equation* (1.36) *has two linearly independent solutions of period π or 2π, then two solutions y_1, y_2 can be found so that:*

$$y_1 = \sum_{n=0}^{\infty} A_{2n} \cos 2n\phi, \quad y_2 = \sum_{n=1}^{\infty} B_{2n} \sin 2n\phi, \tag{1.38}$$

for period π, or:

$$y_1 = \sum_{n=0}^{\infty} A_{2n+1} \cos(2n+1)\phi, \quad y_2 = \sum_{n=0}^{\infty} B_{2n+1} \sin(2n+1)\phi, \tag{1.39}$$

for period 2π, so that $N^p A_N, N^p B_N \to 0$ as $N \to \infty$ for every $p > 0$.

Proof. We can convert Ince's equation (1.36) into an equation of Hill type (1.4) by applying the following transformation:

$$y = (1 + a\cos 2\phi)^{\frac{b}{4a}} z, \tag{1.40}$$

where a, b are defined as before (where $a \neq 0$), so that we get the following:

$$\frac{d^2 z}{d\phi^2} + \underbrace{\frac{\alpha + \beta\cos 2\phi + \gamma\cos 4\phi}{(1 + a\cos 2\phi)^2}}_{Q(\phi)} z = 0, \tag{1.41}$$

where α, β, γ are as follows:

$$\alpha = c - ab - \frac{b^2}{8} + \frac{ad}{2},$$
$$\beta = d + ac - b,$$
$$\gamma = \frac{ad}{2} + \frac{b^2}{8}.$$

Clearly, the coefficient $Q(\phi)$ of (1.41) is π-periodic, and therefore the equation is of Hill form (1.4). From Theorem 1.2, we know that an equation of Hill type cannot have both a π-periodic and 2π-periodic solution corresponding to the same double root of $\Delta(\lambda)$. Therefore, since y in (1.40) is π-periodic iff z is, the same must be true for Ince's equation. Due to the boundary conditions for linearly independent solutions of Hill equations:

$$z_1(0) = 1, z_1'(0) = 0,$$
$$z_2(0) = 0, z_2'(0) = 1,$$

we see that $z_1(\phi)$ must be even, while $z_2(\phi)$ must be odd. On this basis, (1.36) will have an even and an odd solution, both with the same period (being either π or 2π).

Solutions y_1, y_2 in (1.38),(1.39) are π-periodic (2π-periodic, resp.), solve (1.36) for some fixed a, b, c, d, and are linearly independent. Also, y_1 is even, while y_2 is odd, as required.

To show that $A_N, B_N \to 0$ as $N \to \infty$, we can invoke the Riemann-Lebesgue lemma for Fourier series. In particular, because y_1, y_2 are infinitely differentiable:

$$|A_n| \leq \frac{M}{|n|^p}, \quad |B_n| \leq \frac{N}{|n|^p}, \tag{1.42}$$

for some constants M, N and $\forall p \in \mathbb{N}$. This proves the lemma. $\square$

Starting with the π-periodic case, we can plug (1.38) into (1.37) and compare coefficients of $\cos 2n\phi$ and $\sin 2n\phi$ respectively, to get the following recurrence relations:

$$\left.\begin{aligned} \Lambda_0 A_0 + P_{-1} A_2 &= 0, \\ 2P_0 A_0 + \Lambda_1 A_2 + P_{-2} A_4 &= 0, \\ P_{n-1} A_{2n-2} + \Lambda_n A_{2n} + P_{-n-1} A_{2n+2} &= 0, \text{ for } n \geq 2, \end{aligned}\right\} \tag{1.43}$$

$$\left.\begin{aligned} \Lambda_1 B_2 + P_{-2} B_4 &= 0, \\ P_{n-1} B_{2n-2} + \Lambda_n B_{2n} + P_{-n-1} B_{2n+2} &= 0, \text{ for } n \geq 2, \end{aligned}\right\} \tag{1.44}$$

where we have used the following:

$$\begin{aligned} P_n &= \frac{m(m+1)k^2}{4} - \frac{nk^2}{2} - n^2 k^2, \\ \Lambda_n &= \lambda - \frac{m(m+1)k^2}{2} - \left(1 - \frac{k^2}{2}\right) 4n^2. \end{aligned} \tag{1.45}$$

For the 2π-periodic solutions we repeat the same procedure with (1.39), comparing coefficients of $\cos(2n+1)\phi$ and $\sin(2n+1)\phi$ respectively, to get the following recurrence relations:

$$\left.\begin{aligned} \left(P_0^* + \Lambda_0^*\right) A_1 + P_{-1}^* A_3 &= 0, \\ P_n^* A_{2n-1} + \Lambda_n^* A_{2n+1} + P_{-n-1}^* A_{2n+3} &= 0, \text{ for } n \geq 1, \end{aligned}\right\} \tag{1.46}$$

$$\left.\begin{aligned} \left(-P_0^* + \Lambda_0^*\right) B_1 + P_{-1}^* B_3 &= 0, \\ P_n^* B_{2n-1} + \Lambda_n^* B_{2n+1} + P_{-n-1}^* B_{2n+3} &= 0, \text{ for } n \geq 1, \end{aligned}\right\} \tag{1.47}$$

where we have used the following:

$$\begin{aligned} P_n^* &= \frac{m(m+1)k^2}{4} - \frac{(2n-1)k^2}{4} - \frac{(2n-1)^2k^2}{4}, \\ \Lambda_n^* &= \lambda - \frac{m(m+1)k^2}{2} - \left(1 - \frac{k^2}{2}\right)(2n+1)^2. \end{aligned} \tag{1.48}$$

Noticing that the coefficients of (1.43) and (1.44) are equal for general n (as are the coefficients of (1.46) and (1.47)), we can multiply (1.43) by B_{2n}, (1.44) by A_{2n} (multiply (1.46) by B_{2n+1}, (1.47) by A_{2n+1}, respectively), and subtract to get the following:

$$\left.\begin{aligned} P_nD_n &= P_{-n-2}D_{n+1}, \text{ for } n \geq 1, \\ 2P_0D_0 &= P_{-2}D_1, \end{aligned}\right\} \tag{1.49}$$

$$\left.\begin{aligned} P_n^*D_n^* &= P_{-n-1}^*D_{n+1}^*, \text{ for } n \geq 1, \\ 2P_0^*D_0^* &= P_{-1}^*D_1^*, \end{aligned}\right\} \tag{1.50}$$

where we have used the following:

$$\left.\begin{aligned} D_n &= A_{2n}B_{2n+2} - B_{2n}A_{2n+2}, \text{ for } n = 1, 2, \ldots, \\ D_0 &= A_0B_2, \end{aligned}\right\} \tag{1.51}$$

$$\left.\begin{aligned} D_n^* &= A_{2n-1}B_{2n+1} - B_{2n-1}A_{2n+1}, \text{ for } n = 1, 2, \ldots, \\ D_0^* &= A_1B_1, \end{aligned}\right\}$$

which will satisfy the following conditions due to the vanishing nature of the coefficients (see Lemma 1.4):

$$\begin{aligned} \lim_{t\to\infty} t^pD_t &= 0, \\ \lim_{t\to\infty} t^pD_t^* &= 0, \end{aligned} \tag{1.52}$$

for any positive integer p.

If the recurrence relations (1.49) and (1.50) are satisfied, then the functions y_1, y_2 from (1.38), (1.39) solve Ince's equation (1.36). The following lemma states what is required to ensure that this is the case.

Lemma 1.5. *For the recurrence relation* (1.49) *(resp.* (1.50)*) to be satisfied, either all D_n (resp. D_n^*) vanish or P_n (resp. P_n^*) has an integral root when viewed as a function of n.*

As all D_n (resp. D_n^*) vanishing implies trivial solutions, we will concern ourselves with the case of P_n (resp. P_n^*) having an integral root. Returning to (1.45) and (1.48), we see that P_n, P_n^* have roots when:

$$\begin{aligned} n &= \frac{m}{2}, -\frac{m+1}{2}, \\ n^* &= \frac{m+1}{2}, -\frac{m}{2}, \end{aligned} \tag{1.53}$$

respectively, and these are only integers when m is. For fixed integer m, if P_n has a positive integer root, P_n^* must have a negative one (and vice versa). Both cases will be explored in the following lemma and corollary.

Lemma 1.6. *If P_n (P_n^*, resp.) has a non-negative integral root, with n_0 (n_0^*, resp.) being the largest such root, then* (1.37) *will have two linearly independent solutions of period π (2π, resp.) provided that one solution exists that is either "infinite", in the sense that $\forall n_1 > n_0$ ($n_1 > n_0^*$, resp.), there exists $n \geq n_1$, so that*

$$\begin{aligned} A_{2n}, B_{2n} &\neq 0, \\ A_{2n+1}, B_{2n+1} &\neq 0, \end{aligned}$$

or "finite of order n_1", in the sense that for some $n_1 > n_0$ ($n_1^* > n_0^*$, resp.):

$$\begin{aligned} A_{2n} = B_{2n} &= 0, \ \ \forall n > n_1, \\ A_{2n+1} = B_{2n+1} &= 0, \ \ \forall n > n_1^*, \end{aligned}$$

whereas there exist at most n_0+1 (n_0^*+1, resp.) values of λ so that (1.37) *has only one linearly independent periodic solution.*

Corollary 1.1. *If P_n (P_n^*, resp.) has a negative root, so that $-n_0 - 1$ for $n_0 = 0, 1, \ldots$ ($-n_0^* - 1$ for $n_0^* = 0, 1, \ldots$, resp.) is the smallest, then* (1.37) *will have two "infinite" linearly indepedent solutions of period π (2π, resp.), except for at most n_0+1 (n_0^*+1, resp.) values of λ for which* (1.37) *has only one linearly independent periodic solution.*

Proof of Lemma 1.5. Notice in (1.49) that if P_n does not have integral roots, none of P_n for $n = 1, 2, \ldots$ will vanish, so either all D_n are zero, or none of them are. Let us assume that none of D_n vanish for $n = 0, 1, \ldots$ (which we require to ensure non-trivial solutions). Taking some fixed integer $j > 0$, we can iterate (1.49) to get the following:

$$D_j = \frac{P_{-j-2}P_{-j-3}\dots P_{-j-r-2}}{P_j P_{j+1}\dots P_{j+r}} D_{j+r+1}, \quad r = 0, 1, 2, \dots, \tag{1.54}$$

As P_n is a polynomial in n of degree two, we can express it in the following way:

$$P_n = A(n - \mu_1)(n - \mu_2),$$

where A is some constant. Now (1.54) becomes:

$$\frac{\Gamma(j + 2 + \mu_1)\Gamma(j + 2 + \mu_2)}{\Gamma(j - \mu_1)\Gamma(j - \mu_2)} D_j = \frac{\Gamma(j + 3 + \mu_1 + r)\Gamma(j + 3 + \mu_2 + r)}{\Gamma(j - \mu_1 + 1 + r)\Gamma(j - \mu_2 + 1 + r)} D_{j+r+1}, \tag{1.55}$$

where $\Gamma(n)$ is the classical Gamma function defined by (see [3]):

$$\Gamma(n) = (n - 1)!, \tag{1.56}$$

$$\Gamma(x) = \lim_{k\to\infty} \frac{k!k^{x-1}}{(x)_k}, \tag{1.57}$$

for $n \in \mathbb{N}$, $x \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, while $(x)_k$ is the Pochhammer symbol defined by:

$$(x)_k = x(x - 1)(x - 2)\dots(x - n + 1),$$

so that from (1.56) we get

$$(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)}. \tag{1.58}$$

Setting $t = j + 2 + r$ and $\rho_\alpha = \mu_\alpha + 1$ for $\alpha = 1, 2$, the RHS of (1.55) becomes the following:

$$\frac{\Gamma(t + \rho_1)}{\Gamma(t - \rho_1)} t^{-2\rho_1} \frac{\Gamma(t + \rho_2)}{\Gamma(t - \rho_2)} t^{-2\rho_2} t^{2(\rho_1+\rho_2)} D_t,$$

where we have included the t factors for what follows. Now, taking the limit $r \to \infty$ in (1.55) is equivalent to taking $t \to \infty$, so this becomes:

$$\lim_{t\to\infty} \left| \frac{\Gamma(\rho_1)}{\Gamma(-\rho_1)} \frac{(\rho_1)_t}{(-\rho_1)_t} t^{-2\rho_1} \frac{\Gamma(\rho_2)}{\Gamma(-\rho_2)} \frac{(\rho_2)_t}{(-\rho_2)_t} t^{-2\rho_2} t^{2(\rho_1+\rho_2)} D_{t-1} \right| \tag{1.59}$$

$$= \frac{\Gamma(\rho_1)}{\Gamma(-\rho_1)} \frac{\Gamma(\rho_2)}{\Gamma(-\rho_2)} \lim_{t\to\infty} \left| \frac{(\rho_1)_t}{(-\rho_1)_t} t^{-2\rho_1} \frac{(\rho_2)_t}{(-\rho_2)_t} t^{-2\rho_2} t^{2(\rho_1+\rho_2)} D_{t-1} \right|$$

$$= \frac{\Gamma(\rho_1)}{\Gamma(-\rho_1)} \frac{\Gamma(\rho_2)}{\Gamma(-\rho_2)} \frac{\Gamma(-\rho_1)}{\Gamma(\rho_1)} \frac{\Gamma(-\rho_2)}{\Gamma(\rho_2)} \lim_{t\to\infty} \left| t^{2(\rho_1+\rho_2)} D_{t-1} \right| \tag{1.60}$$

$$= \lim_{t\to\infty} \left| t^{2(\rho_1+\rho_2)} D_{t-1} \right| = 0, \tag{1.61}$$

where we have used (1.58) for (1.59), (1.57) for (1.60), and (1.52) for (1.61). Therefore, the LHS of (1.55), which has no dependence on r, must be equal to zero as follows:

$$\frac{\Gamma(j+2+\mu_1)\Gamma(j+2+\mu_2)}{\Gamma(j-\mu_1)\Gamma(j-\mu_2)}D_j = 0. \tag{1.62}$$

As none of D_j vanish for integer $j > 0$ in our assumption, their coefficient must be equal to zero, or rather:

$$(j+\mu_1+1)(j+\mu_1)\dots(j-\mu_1+1)(j-\mu_1)(j+\mu_2+1)(j+\mu_2)\dots(j-\mu_2+1)(j-\mu_2) = 0,$$

but this can only occur if μ_1 or μ_2 are equal to an integer so that one of the brackets disappears. Therefore, P_n must have an integral root. The proof for P_n^* follows the same method. □

Remark 1.3. *Notice that all D_n (resp. D_n^*) vanish only if P_n (resp. P_n^*) has no integral root. If it has a positive integral root n_0 (resp. n_0^*), then according to* (1.49) *(resp.* (1.50)*) $D_n = 0$ for $n > n_0$ (resp. $D_n^* = 0$ for $n > n_0^*$). If it has a negative root $-n_0 - 1$ (resp. $-n_0^* - 1$), then all $D_n = 0$ for $n < n_0$ (resp. $D_n^* = 0$ for $n < n_0^*$).*

Proof of Lemma 1.6. Consider that (1.37) has the following solution:

$$y_2 = \sum_{n=1}^{\infty} B_{2n} \sin 2n\phi,$$

so that $B_{2n_1} \neq 0$ for some $n_1 > n_0$. Notice that this solution can be either infinite or finite of order n_1. Now, we can consider a second solution:

$$y_1 = \sum_{n=0}^{\infty} A_{2n} \cos 2n\phi,$$

so that $A_{2n} = B_{2n}$ for $n > n_0$, to ensure that the solution is not identically zero. For y_1 to be a solution, it must satisfy the recurrence relations (1.43) for all n. Clearly for $n > n_0$ the recurrence relations are already satisfied, as $A_{2n} = B_{2n}$ and (1.44) (which we know to be satisfied due to y_2 being a solution) is identical to (1.43). For $n \leq n_0$, we construct the following matrix representation of the recurrence relations

(1.43), where we utilise the fact that $P_{n_0} = 0$ and $A_{2n} = B_{2n}$ for $n > n_0$:

$$\begin{pmatrix} \Lambda_0 & P_{-1} & 0 & \cdots & 0 \\ 2P_0 & \Lambda_1 & P_{-2} & \ddots & \vdots \\ 0 & P_1 & \Lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P_{-n_0} \\ 0 & \cdots & 0 & P_{n_0-1} & \Lambda_{n_0} \end{pmatrix} \begin{pmatrix} A_0 \\ A_2 \\ \vdots \\ \vdots \\ A_{2n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -P_{-n_0-1}B_{2n_0+2} \end{pmatrix}. \tag{1.63}$$

Reducing an infinite tri-diagonal matrix in this way is only possible by setting one of the off-diagonal elements to zero. In other words, ensuring P_n has a non-negative or negative integral root will split an infinite matrix into one finite and one infinite tri-diagonal block.

Recall that Λ_n contains the eigenvalue λ of (1.37) linearly. As the determinant of the LHS of (1.63) is not equal to zero due to $B_{2n_0+2} \neq 0$, by Cramer's rule there exists a unique solution for $A_0, \ldots, A_{2n_0}$. Therefore, we have two linearly independent solutions y_1, y_2 to (1.36) as long as y_2 is "infinite" or "finite of order n_1", as required.

However, if instead we define y_1 as before but with $B_{2n_0+2} = A_{2n_0+2} = 0$, we can construct the following matrix equation for its coefficients:

$$\underbrace{\begin{pmatrix} \Lambda_0 & P_{-1} & 0 & \cdots & 0 \\ 2P_0 & \Lambda_1 & P_{-2} & \ddots & \vdots \\ 0 & P_1 & \Lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P_{-n_0} \\ 0 & \cdots & 0 & P_{n_0-1} & \Lambda_{n_0} \end{pmatrix}}_{M} \begin{pmatrix} A_0 \\ A_2 \\ \vdots \\ \vdots \\ A_{2n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \tag{1.64}$$

On the basis that $P_{n_0} = 0$, (1.43) with $n = n_0 + 1$ will become the following:

$$P_{-n_0-2}A_{2n_0+4} = 0,$$

and as P_{-n_0-2} is not equal to zero, we must have that $A_{2n_0+4} = 0$, which ensures that all $A_{2n} = 0, n > n_0$ (i.e. y_1 is finite of order n_0). Therefore, taking the determinant of the LHS of (1.64) will form a polynomial in λ of degree $\leq n_0+1$ (being the rank of the matrix). The roots of this polynomial correspond to the points of the spectrum where there exists one linearly independent solution to (1.37) that is finite of order n_0. The same result applies had we chosen some y_2 of finite order k_0.

The proof for non-negative roots of P_n^* follows the same procedure. □

Remark 1.4. *The proof to Lemma 1.6 highlights the procedure by which we can locate the finite simple roots of* $\Delta(\lambda) = \pm 2$ *corresponding to a single bounded linearly independent solution: By solving det*$(M) = 0$ *for* M *in* (1.64) *(as well as similar determinants for periodic* y_2 *and the anti-periodic solutions) for some* n_0 *(which depends on fixed integer* m*). The two linearly independent (anti-)periodic solutions must occur instead at the double roots of* $\Delta(\lambda) = \pm 2$*, of which there are an infinite number.*

Proof of Corollary 1.1. The proof follows on from the fact that if P_n (P_n^*, resp.) has no non-negative roots, the recurrence relations (1.43) and (1.44) demonstrate that all A_{2n},B_{2n} (A_{2n+1}, B_{2n+1}, resp.) will vanish for $n \leq n_0$ if they also vanish for $n_0 + 1, n_0 + 2$. Therefore, periodic solutions can not be of finite order (as all the coefficients would vanish).

Now, if we assume that P_n has a negative root at $-n_0 - 1$ and no positive root, (1.49) shows that all D_n will vanish for $n < n_0$ (see Remark 1.3). So considering $D_0 = A_0 B_2 = 0$, we have two sets of conditions. Either $A_0 = 0$, in which case (1.43) and (1.51) show us that:

$$A_0 = A_2 = \ldots = A_{2n_0} = 0, B_{2n_0} = 0, \tag{1.65}$$

where we assume A_{2n_0+2} is the first coefficient not to vanish, or $B_2 = 0$, in which case (1.44) and (1.51) show us that:

$$B_2 = \ldots = B_{2n_0} = 0, A_{2n_0} = 0, \tag{1.66}$$

where we assume B_{2n_0+2} is the first coefficient not to vanish. This ensures that there exist two linearly independent (anti-)periodic solutions of infinite order, as required.

For example, taking the first set of conditions (1.65) and assuming a solution y_2 from (1.38) exists so that B_{2n} satisfy (1.44). The fact that B_{2n_0} is zero implies that all B_{2n} for $n < n_0$ also equal zero by (1.44), with $P_{-n_0-1} = 0$. Therefore we can construct two infinite order linearly independent solutions as follows:

$$y_1 = \sum_{n=n_0+1}^{\infty} B_{2n} \cos 2n\phi,$$
$$y_2 = \sum_{n=n_0+1}^{\infty} B_{2n} \sin 2n\phi,$$

where $A_{2n} = B_{2n}$ for $n > n_0$ to ensure the solution y_1 exists.

However, if say we take the second set of conditions (1.66) but assume that B_{2n_0+2} vanishes this time, rather than A_{2n_0}, we see that due to $P_{-n_0-1} = 0$ and (1.44), all B_{2n} will vanish. Therefore, we can construct a solution y_1 from (1.38), and form the following matrix equation from the recurrence relation of its coefficients (1.43):

$$\begin{pmatrix} \Lambda_0 & P_{-1} & 0 & \cdots & 0 \\ 2P_0 & \Lambda_1 & P_{-2} & \ddots & \vdots \\ 0 & P_1 & \Lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P_{-n_0} \\ 0 & \cdots & 0 & P_{n_0-1} & \Lambda_{n_0} \end{pmatrix} \begin{pmatrix} A_0 \\ A_2 \\ \vdots \\ \vdots \\ A_{2n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

which is reduced from the infinite tri-diagonal form due to P_{-n_0-1}. As in Lemma 1.6, taking the determinant will form a polynomial equation in λ of degree $\leq n_0+1$. Therefore there can be at most $n_0 + 1$ values of λ that ensure we only have one linearly independent periodic solution y_1. If we had taken the first set of conditions (1.65) with $A_{2n_0+2} = 0$ instead of $B_{2n_0} = 0$, we can find the same result for a solution of the form y_2 in (1.38).

The proof for negative roots of P_n^* follows using the same procedure. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. For two linearly independent (anti-)periodic solutions of (1.37) to exist, Lemma 1.5 implies that P_n (P_n^*, resp.) must have an integral root. Lemma 1.6 and Corollary 1.1 state the conditions required to ensure this is the case for non-negative and negative integer values of n, respectively. From (1.53), $n \in \mathbb{Z}$ iff $m \in \mathbb{Z}$. This completes the first part of the proof.

For the second part, we must consider the integral roots (1.53) of P_n and P_n^* once more. When m is even, so that $m = 2l'$ for some non-negative integer l', we have roots when:

$$n = l', \quad n^* = -l',$$

so from Lemma 1.6 and Corollary 1.1, respectively, we find that for (1.37) to have only one linearly independent solution of period π there can be at most $l'+1$ values of λ ($l' = n_0$ the largest non-negative root), while for only one linearly independent solution of period 2π there can be at most l' values of λ ($-l' = -n_0 - 1$ the smallest negative root). Therefore, by summing these two sets of eigenvalues, there can be at most $2l' + 1 = m + 1$ intervals of instability in the spectrum of (1.37) (and therefore also that of (1.35)), when m is non-negative and even. For non-negative, odd m, we get exactly the same result.

To complete the proof of Theorem 1.3, we note that the Lamé equation (1.35) is invariant when m is replaced by $-m-1$, and therefore we only need to consider non-negative m. □

The following case studies will explore the spectrum of the Lamé operator for fixed values of m, utilising the machinery highlighted in Remark 1.4 to find explicitly the location of the simple roots of $\Delta(\lambda) = \pm 2$ that correspond to one linearly independent solution. Subsequently, we will state the equations that must be solved to find these simple roots for the general $m \in \mathbb{N}$ case.

1.3.1 Case study: $m = 1$

For $m = 1$, it follows from (1.53) that the integral roots of P_n, P_n^* are:

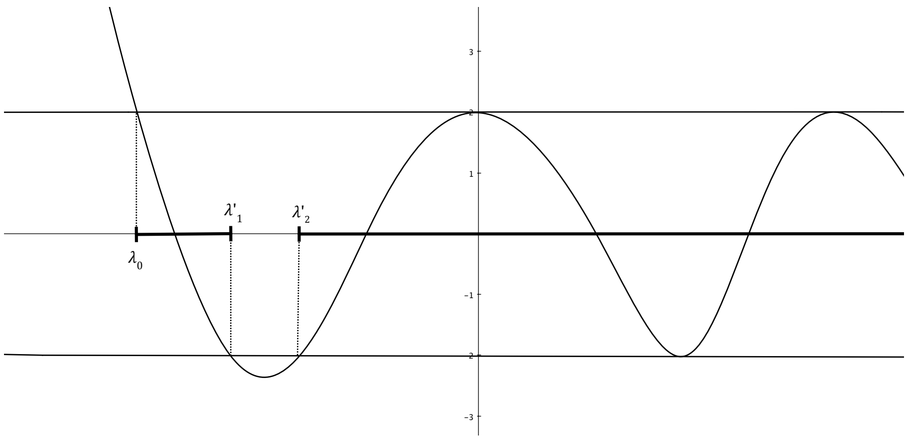
$$-n_0 - 1 = -1,$$

$$n_0^* = 1,$$

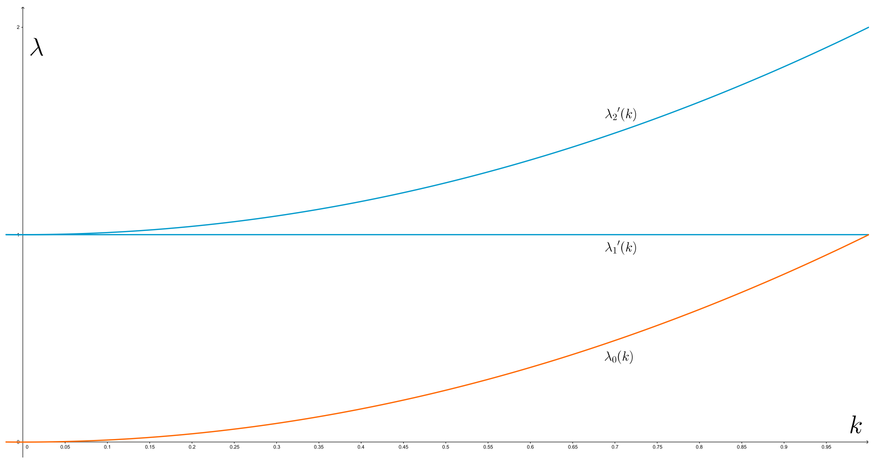
where we have used (1.45),(1.48), respectively. For the positive root, we utilise the fact that $P_1^* = 0$ to solve (1.46),(1.47) for $n < 1$. In so doing, we are able to determine the finite number of eigenvalues corresponding to a single linearly independent 2π-periodic solution of (1.37):

$$P_0^* + \Lambda_0^* = 0 \qquad \rightarrow \qquad \lambda_1' = 1,$$

$$-P_0^* + \Lambda_0^* = 0 \qquad \rightarrow \qquad \lambda_2' = 1 + k^2,$$



(a) A hypothetical spectrum corresponding to $m = 1$, for some $k \in (0, 1)$. The curve is defined by $\Delta(\lambda)$.



(b) Eigenvalues plotted over k for $m = 1$, demonstrating that at no point does the spectrum become multi-valued. Notice that there is apparent closing of gaps at $k = 1$, also.

Figure 1.2

which correspond to solutions of the form:

$$y_1 = \sum_{n=0}^{\infty} A_{2n+1} \cos(2n+1)\phi,$$
$$y_2 = \sum_{n=0}^{\infty} B_{2n+1} \sin(2n+1)\phi,$$

respectively, so that $A_{2n+1} = B_{2n+1} = 0, \forall n \geq 1$. For the negative root we utilise the fact that $P_{-1} = 0$ to solve (1.43), and determine the eigenvalue corresponding to a single linearly independent π-periodic solution of (1.37):

$$\Lambda_0 = \lambda - k^2 = 0 \qquad \rightarrow \qquad \lambda_0 = k^2,$$

which corresponds to a solution of the form:

$$y_1 = \sum_{n=0}^{\infty} A_{2n} \cos(2n)\phi,$$

with $A_{2n} = 0, \forall n \geq 1$. As we can see in Figure 1.2a, the eigenvalues obtained by solving the equations correspond to the edges of the spectral bands (with a dependence only on parameter k), or in terms of Theorem 1.2, simple roots of the equation $\Delta^2(\lambda) - 4 = 0$.

1.3.2 Case study: $m = 2$

Using (1.53), (1.45), and (1.48) once more, integral roots for P_n, P_n^* are:

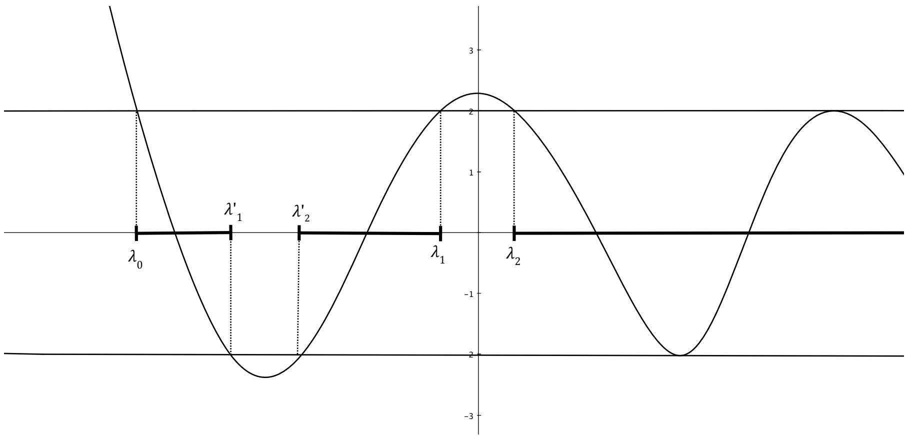
$$n_0 = 1,$$
$$-n_0^* - 1 = -1,$$

respectively. Therefore, we solve the following determinant equation for the positive first root:

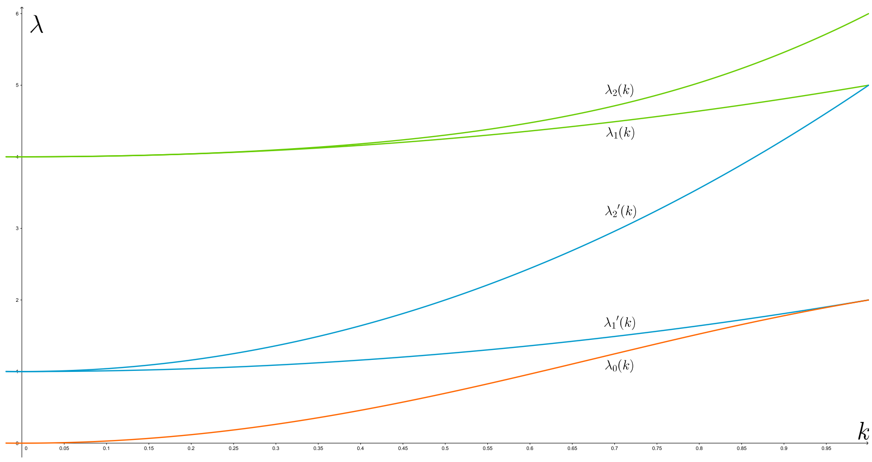
$$\begin{vmatrix} \Lambda_0 & P_{-1} \\ 2P_0 & \Lambda_1 \end{vmatrix} = 0,$$

which provides eigenvalues corresponding to solutions of the form y_1 in (1.38)

$$\lambda_0 = 2\left(1 + k^2 - \sqrt{1 - k^2 + k^4}\right),$$
$$\lambda_2 = 2\left(1 + k^2 + \sqrt{1 - k^2 + k^4}\right).$$



(a) A hypothetical spectrum corresponding to $m = 2$, for some $k \in (0, 1)$



(b) Eigenvalues plotted over k for $m = 2$

Figure 1.3

We can also solve the following equation:

$$\Lambda_1 = 0,$$

to find the single eigenvalue which corresponds to a solution of the form y_2 in (1.38) (notice that the index of the solution coefficient starts at $n = 1$, which explains the reduction by one in the rank of the matrix):

$$\lambda_1 = 4 + k^2.$$

It is clear that as $k \to 0$, the gap between λ_1 and λ_2 (as illustrated in Figure 1.3a) will close, so that $\lambda_1 = \lambda_2 = 4$. In such a case we will have coexistence of two linearly independent solutions. For the anti-periodic solutions, P_n^* has a root at $n = -1$, therefore we again find the eigenvalues using the equations from (1.46), (1.47):

$$\begin{aligned} P_0^* + \Lambda_0^* = 0 \qquad &\to \qquad \lambda_1' = 1 + k^2, \\ -P_0^* + \Lambda_0^* = 0 \qquad &\to \qquad \lambda_2' = 1 + 4k^2. \end{aligned}$$

The corresponding gap (as illustrated in Figure 1.3a) will again close, so that $\lambda_1' = \lambda_2' = 1$, as $k \to 0$. We can see how the eigenvalues change when altering k in Figure 1.3b.

1.3.3 Case study: $m = 3$

Following the same procedure, we solve the following equations:

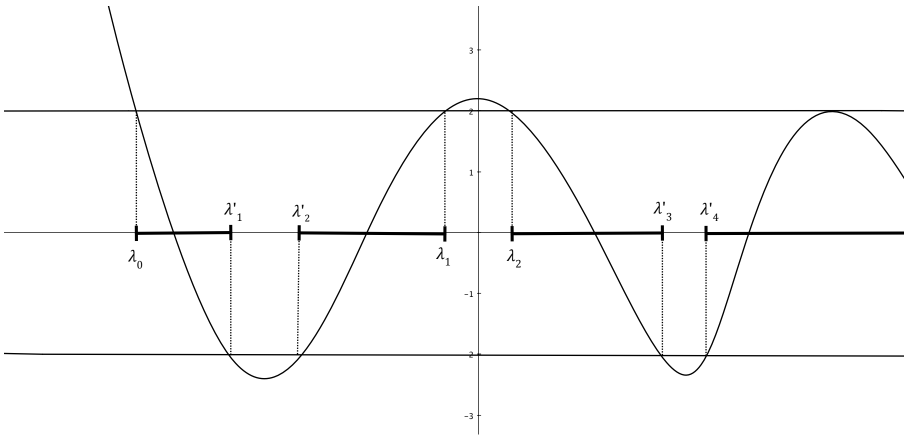
$$\Lambda_1 = 0,$$

$$\begin{vmatrix} \Lambda_0 & P_{-1} \\ 2P_0 & \Lambda_1 \end{vmatrix} = 0,$$

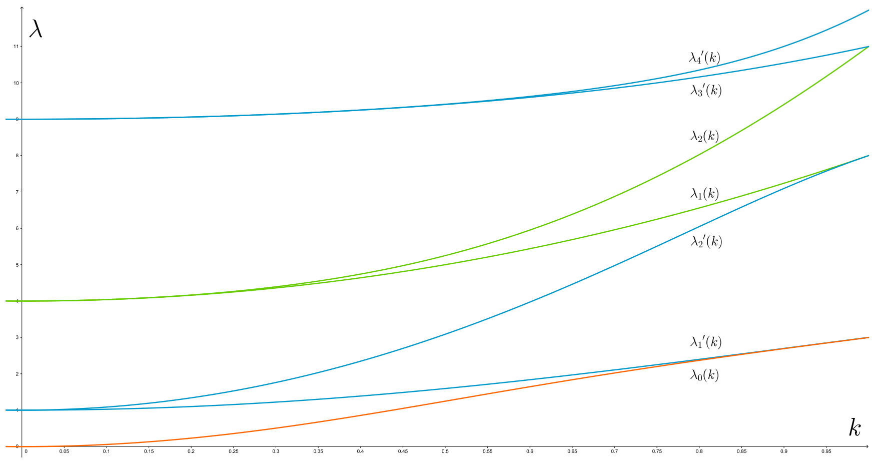
$$\begin{vmatrix} \pm P_0^* + \Lambda_0^* & P_{-1}^* \\ P_1^* & \Lambda_1^* \end{vmatrix} = 0,$$

which yield the following eigenvalues for the band edges, respectively:

$$\begin{aligned} \lambda_1 &= 4(1 + k^2), \\ \lambda_{0,2} &= 2 + 5k^2 \mp 2\sqrt{1 - k^2 + 4k^4}, \\ \lambda_{1,3}' &= 5 + 2k^2 \mp 2\sqrt{4 - k^2 + k^4}, \\ \lambda_{2,4}' &= 5 + 5k^2 \mp 2\sqrt{4 - 7k^2 + 4k^4}, \end{aligned}$$



(a) A hypothetical spectrum corresponding to $m = 3$, for some $k \in (0, 1)$



(b) Eigenvalues plotted over k for $m = 3$

Figure 1.4

corresponding to periodic, anti-periodic solutions for λ_i, λ_i', respectively. The spectrum is shown in Figure 1.4a, while the eigenvalues as a function of k are displayed in Figure 1.4b.

It is clear that we can manually continue this process for ever higher values of m showing the same pattern every time. For this reason, we will next consider the case of general m.

1.3.4 General m

For even $m = 2j$, where $j = 1, 2, \ldots$, we must solve the following equations:

$$\begin{matrix} P_{n_0} = 0, \\ n_0 = j, \end{matrix} \left\{ \begin{aligned} \det M_j^0 &\equiv \begin{vmatrix} \Lambda_0 & P_{-1} & 0 & \cdots & 0 \\ 2P_0 & \Lambda_1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P_{-j} \\ 0 & \cdots & 0 & P_{j-1} & \Lambda_j \end{vmatrix} = 0, \\ \det M_j^1 &\equiv \begin{vmatrix} \Lambda_1 & P_{-2} & 0 & \cdots & 0 \\ P_1 & \Lambda_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P_{-j} \\ 0 & \cdots & 0 & P_{j-1} & \Lambda_j \end{vmatrix} = 0, \end{aligned} \right.$$

$$\begin{matrix} P^*_{-n_0^*-1} = 0, \\ n_0^* = j-1, \end{matrix} \left\{ \begin{aligned} \det N_{j-1}^+ &\equiv \begin{vmatrix} P_0^* + \Lambda_0^* & P_{-1}^* & 0 & \cdots & 0 \\ P_1^* & \Lambda_1^* & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P^*_{-(j-1)} \\ 0 & \cdots & 0 & P^*_{j-1} & \Lambda^*_{j-1} \end{vmatrix} = 0, \\ \det N_{j-1}^- &\equiv \begin{vmatrix} -P_0^* + \Lambda_0^* & P_{-1}^* & 0 & \cdots & 0 \\ P_1^* & \Lambda_1^* & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P^*_{-(j-1)} \\ 0 & \cdots & 0 & P^*_{j-1} & \Lambda^*_{j-1} \end{vmatrix} = 0. \end{aligned} \right.$$

For odd $m = 2j - 1$, where $j = 1, 2, \ldots$, we must solve the following equations:

$$\begin{array}{r} P_{-n_0-1} = 0, \\ n_0 = j - 1, \end{array} \left\{ \begin{array}{l} \det M^0_{j-1} \equiv \begin{vmatrix} \Lambda_0 & P_{-1} & 0 & \cdots & 0 \\ 2P_0 & \Lambda_1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P_{-(j-1)} \\ 0 & \cdots & 0 & P_{j-2} & \Lambda_{j-1} \end{vmatrix} = 0, \\ \det M^1_{j-1} \equiv \begin{vmatrix} \Lambda_1 & P_{-2} & 0 & \cdots & 0 \\ P_1 & \Lambda_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P_{-(j-1)} \\ 0 & \cdots & 0 & P_{j-2} & \Lambda_{j-1} \end{vmatrix} = 0, \end{array} \right.$$

$$\begin{array}{r} P^*_{n^*_0} = 0, \\ n^*_0 = j, \end{array} \left\{ \begin{array}{l} \det N^+_{j-1} \equiv \begin{vmatrix} P^*_0 + \Lambda^*_0 & P^*_{-1} & 0 & \cdots & 0 \\ P^*_1 & \Lambda^*_1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P^*_{-(j-1)} \\ 0 & \cdots & 0 & P^*_{j-1} & \Lambda^*_{j-1} \end{vmatrix} = 0, \\ \det N^-_{j-1} \equiv \begin{vmatrix} -P^*_0 + \Lambda^*_0 & P^*_{-1} & 0 & \cdots & 0 \\ P^*_1 & \Lambda^*_1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & P^*_{-(j-1)} \\ 0 & \cdots & 0 & P^*_{j-1} & \Lambda^*_{j-1} \end{vmatrix} = 0. \end{array} \right.$$

These equations will need to be solved simultaneously (for even or odd m) in order to find the explicit location of the eigenvalues for the edges of the spectral bands. We now present the main result of the chapter.

Theorem 1.4. *The Lamé operator:*

$$-\frac{d^2}{dx^2} + m(m+1)k^2 sn^2(x) \quad k \in (0,1), x \in \mathbb{R}, \tag{1.67}$$

has all gaps of its spectrum open unless $m \in \mathbb{Z}$, in which case all gaps are closed except for the first m.

Proof. As we proved in Theorem 1.3, coexistence of solutions only occurs when m is an integer. This coexistence also corresponds to a closed gap of the corresponding spectrum (a double root of the equation $\Delta^2(\lambda) - 4 = 0$). Therefore if $m \notin \mathbb{Z}$, we can only have open gaps in the spectrum.

In Theorem 1.3 it was proved that the Lamé operator (1.67) will contain no *more* than m gaps in its spectrum, while Erdélyi showed [11, 12] that there will be no *fewer* than m gaps, for non-negative m. Therefore we can assert that the number of gaps in the spectrum will be exactly m, for fixed $m \in \mathbb{N}$.

In order to prove the assertion that only the *first* m gaps are open if $m \in \mathbb{Z}$, implying that no "closed gaps" (or double roots of $\Delta(\lambda) = \pm 2$) appear before open ones in the spectrum, we will need to prove the following lemma:

Lemma 1.7. *The last open gap of the spectrum will close at $\lambda = m^2$, as $k \to 0$.*

Proof. For a generic tri-diagonal matrix K_n^0:

$$K_n^0 = \begin{pmatrix} \lambda_0 & a_1 & 0 & \cdots & 0 \\ b_1 & \lambda_1 & a_2 & \ddots & \vdots \\ 0 & b_2 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_n \\ 0 & \cdots & 0 & b_n & \lambda_n \end{pmatrix},$$

we can utilise the following recurrence relation to express the determinant:

$$\det K_n^0 = \lambda_0 \det K_n^1 - a_1 b_1 \det K_n^2,$$

where K_n^j are the sub-matrices with λ_j in the upper left and λ_n in the lower right, for $j \leq n$. Applying this to our determinants from Section 1.3.4, and taking $k \to 0$

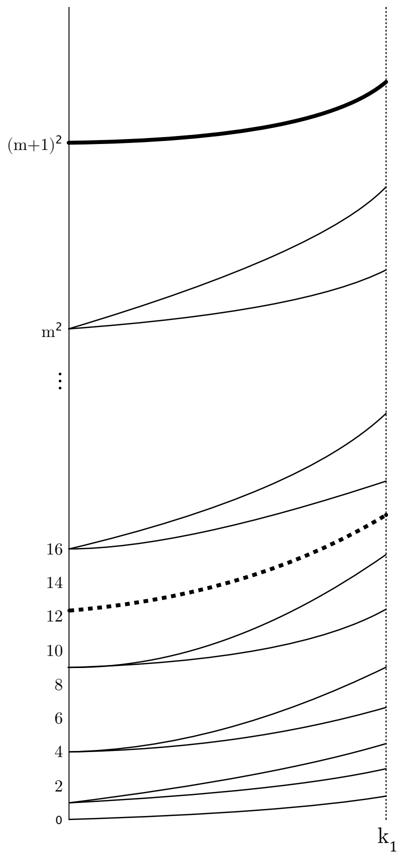


Figure 1.5: m gaps expressed over the range $0 < k \leq k_1$. The curve in bold represents a closed gap $\forall k$, which we shall call an "always-closed gap". The existence of a potential always-closed gap amongst the open gaps is represented by the dashed curve with a question mark.

(so that $P_i, P_i^* \to 0$), we can derive the following:

$$\begin{aligned}
\det M_j^0 &= \Lambda_0 \det M_j^1 - 2P_0P_{-1} \det M_j^2 \\
&= \Lambda_0\Lambda_1 \dots \Lambda_j - \mathcal{O}(k^4) \\
&= \prod_{i=0}^{j} \left[\lambda - (2i)^2\right] - \mathcal{O}(k^2), \\
\det M_j^1 &= \prod_{i=1}^{j} \left[\lambda - (2i)^2\right] - \mathcal{O}(k^2), \\
\det N_{j-1}^{\pm} &= \prod_{i=0}^{j-1} \left[\lambda - (2i+1)^2\right] - \mathcal{O}(k^2),
\end{aligned}$$

for the even $m = 2j$ determinants (the odd $m = 2j-1$ determinants are found in a similar way). At the limit $k = 0$, the roots corresponding to even (odd, respectively) solutions will be equal, other than the first root $\lambda_0 = 0$. The fact that the eigenvalues are analytic, and therefore continuous in k (see Theorem XIII.89 in [41]), implies that the gaps will close as $k \to 0$. Therefore, it is clear that whether we take $m = 2j$ or $m = 2j - 1$ for $j = 1, 2, \ldots$, for some small $k = k_1 > 0$, the last open gap will close at the point $\lambda = m^2$ (as illustrated in Figure 1.5) as $k \to 0$:

$$\begin{aligned}
\det M_j^0 &= \lambda(\lambda - 4)\ldots(\lambda - m^2) - \mathcal{O}(k^2),\\
\det M_j^1 &= (\lambda - 4)\ldots(\lambda - m^2) - \mathcal{O}(k^2),\\
\det N_{j-1}^+ &= (\lambda - 1)\ldots(\lambda - (m-1)^2) - \mathcal{O}(k^2),\\
\det N_{j-1}^- &= (\lambda - 1)\ldots(\lambda - (m-1)^2) - \mathcal{O}(k^2),
\end{aligned}$$

where we have shown this for the even $m = 2j$ determinants. □

Now, if we assume that an always-closed gap appears amongst the open gaps (as illustrated in Figure 1.5), it would have to approach one of the points $\lambda = n^2, n \in \mathbb{Z}$, for $k \to 0$. As all of these points, up to and including $\lambda = m^2$, are associated with gaps that open for $k > 0$ (from Lemma 1.7), we can not have always-closed gaps within this range. Therefore, there exists some $k_1 > 0$ so that the open gaps are the first m gaps for all $k \leq k_1$.

In order to complete the proof of Theorem 1.4, we will need to address three potential scenarios for $k > k_1$, as illustrated in Figure 1.6. Scenario (a) considers the possibility that for $k > k_1$, a previously always-closed gap becomes open. Scenario (b) considers the possibility that for $k > k_2$, a previously open gap becomes always-closed. Scenario (c) considers the possibility that for some $k > k_1$, an always-closed gap intersects an open gap.

Scenario (a)'s occurrence can be disproved by the following lemma:

Lemma 1.8. *A gap which is closed for $0 < k < k_1$, for some $k_1 > 0$, will remain closed for all $k \in (0, 1)$.*

Proof. Let $f(k) = \lambda_i(k) - \lambda_{i-1}(k)$. As $\lambda_n(k)$ are analytic $\forall k$, so must be $f(k)$,

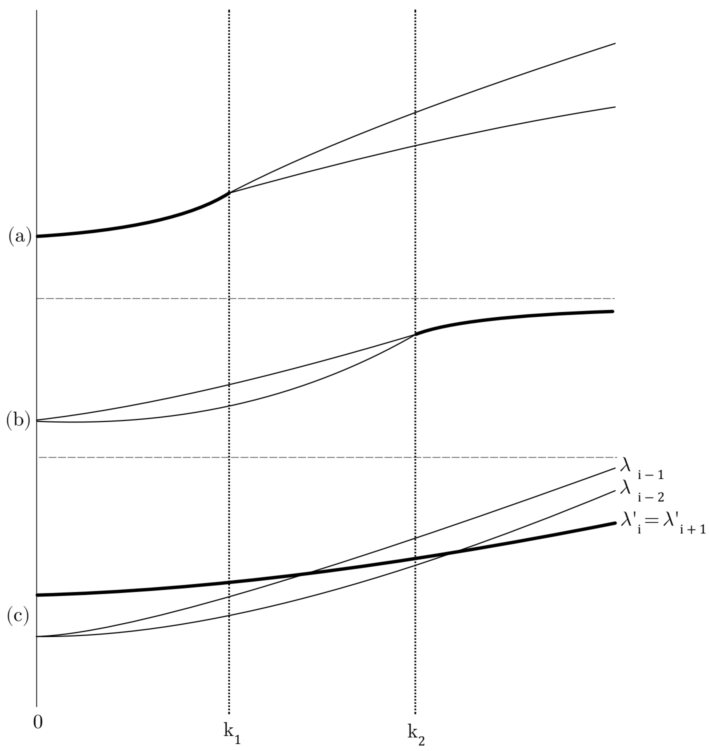


Figure 1.6: Three potential scenarios for $k > k_1$ that must be disproved.

therefore we can express it in the following way:

$$f(k) = \sum_{n=0}^{\infty} c_n (k - k_0)^n,$$

where $k_0 \in (0, k_1)$ is a zero of $f(k)$. From the Taylor Series expansion about $k = k_0$, we can express c_n in the following way:

$$c_n = \frac{f^{(n)}(k_0)}{n!}.$$

Now, either all $c_n = 0$, or $\exists j > 0$ so that $c_j \neq 0$. In the first case, $f(k) \equiv 0, \forall k \in (0, 1)$, which implies that the always-closed gaps will remain closed, as required. In the second case:

$$f(k_0) = f'(k_0) = \ldots = f^{(j-1)}(k_0) = 0, \quad f^{(j)}(k_0) \neq 0,$$

and so $f(k)$ can be rewritten as the following:

$$\begin{aligned} f(k) &= \sum_{n=j}^{\infty} \frac{f^{(n)}(k_0)}{n!}(k-k_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n+j)}(k_0)}{(n+j)!}(k-k_0)^{n+j} \\ &= (k-k_0)^j \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n+j)}(k_0)}{(n+j)!}}_{d_n}(k-k_0)^n \\ &= (k-k_0)^j g(k), \end{aligned}$$

where $g(k)$ is analytic in k, by definition. At k_0, $g(k_0) = d_0 \neq 0$, and as $g(k)$ is continuous at k_0 (owing to it being analytic here), there must exist a disk centred at k_0 with radius greater than zero, in which $g(k)$ is also non-zero. Therefore, $f(k)$ must only have isolated zeroes (at discrete points such as k_0). However, as $f(k) = 0$ for $k \in (0, k_1)$, we know that this can not be the case.

On this basis, only the first case is viable, and so $f(k) = 0, \forall k$, corresponding to an always-closed gap remaining closed everywhere in its domain, as required. $\square$

For scenario (b) we refer back to the assertion that the number of gaps is always fixed at m. When taken in conjunction with the fact that gaps can not "split" open (from Lemma 1.8), we know that, were any particular gap to close at some k, we would have only $m-1$ open gaps in the spectrum (as opposed to the required m). Therefore, this can not happen.

Finally, for scenario (c), the interlacing property (1.25) from Theorem 1.2, combined with the continuity of the eigenvalues in k, ensures that two gaps can not "swap places". Indeed, the interlacing property states that

$$\ldots < \lambda_{i-2}(k) \leq \lambda_{i-1}(k) < \lambda'_i(k) \leq \lambda'_{i+1}(k) < \ldots,$$

and so what is shown in Figure 1.6 can not occur.

We have shown that the first m gaps, all of which are open for some small $k = k_1 > 0$, must remain open $\forall k \in (0, 1)$, while the subsequent always-closed gaps in the spectrum will remain closed $\forall k \in (0, 1)$. This concludes the proof of Theorem 1.4. $\square$

1.4 Discussion

The approach used in formulating Theorem 1.4 was taken in a recent paper by Hemery and Veselov [23], where the terminology and results of Magnus and Winkler [33, 34] were utilised in a similar way as that presented above. In their case, the concern involved extending another theorem presented in [34] regarding the Whittaker-Hill operator:

$$L = -\frac{d^2}{dx^2} - \left[4\alpha s \cos 2x + 2\alpha^2 \cos 4x\right], \tag{1.68}$$

where $\alpha, s \in \mathbb{R}$. The operator (1.68) is in Hill form, with period π, and can be transformed to an operator of Ince type (1.36) by the substitution:

$$\psi = y e^{\alpha \cos 2x},$$

where $\psi(x)$ is an eigenfunction of (1.68). The theorem in question stated that if $s = 2m$, then all even gaps will be open, while all but m odd gaps will be closed. Likewise, with $s = 2m + 1$, all odd gaps will be open, while all but m even gaps will be closed. The operator (1.68) was extended using Darboux transformations in [23], and it was proved that the spectra of these transformed operators would have the same structure as that of (1.68).

This notion of Darboux transformed Schrödinger operators and their spectra will be explored in Part II. In the next chapter, we will investigate whether Lamé equations with complex eigenvalues obey an analogous result to that of Theorem 1.4.

Chapter 2

Complex spectrum

2.1 Introduction

Consider the Lamé operator with complex potential

$$L = -\frac{d^2}{dx^2} + m(m+1)\wp(\omega x + z_0), \tag{2.1}$$

where $x \in \mathbb{R}$, ω is one of the half-periods $\omega_1, \omega_2, \omega_3$. We assume also that $z_0 \in \mathbb{C}$ is in a generic position so that the real line $z = \omega x + z_0$ does not contain any lattice points $z = 2k_1\omega_1 + 2k_3\omega_3,\ k_1, k_3 \in \mathbb{Z}$.

By the spectrum of this operator we mean the set of $\lambda \in \mathbb{C}$ such that all the solutions of the corresponding Schrödinger equation $L\psi = \lambda\psi$ are bounded. Equivalently, the corresponding Floquet multipliers $\mu(\lambda)$ defined by

$$\psi(x + 2\omega) = \mu\psi(x)$$

must have the absolute value $|\mu| = 1$.

In the present chapter, we will investigate the location of the complex spectrum λ in the case of $m = 1$. From the results of Weikard [47] we have that for $m \in \mathbb{N}$, the spectrum consists of finitely many analytic arcs and up to one arc tending to infinity. In analogy with Theorem 1.4 we show that all "closed gaps" of the spectrum (corresponding to coexistence of two bounded eigenfunctions) are contained only on the infinite spectral arc.

First, we will outline some general theory of the Weierstrass functions.

2.1.1 The Weierstrass $\wp$ function, and its associated differential equation

We define the Weierstrass elliptic function $\wp(z)$ as

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z}^*} \left\{ \frac{1}{(z - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2} \right\}, \tag{2.2}$$

where $z \in \mathbb{C}$ and $\frac{\omega_3}{\omega_1} = \tau \in i\mathbb{R} \neq 0$.

Now, to derive the differential equation satisfied by $\wp(z)$, we take Taylor series expansions of (2.2) and its derivative close to $z = 0$ to get the following:

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \dots, \\ \wp'(z) &= -\frac{2}{z^3} + \frac{g_2}{10}z + \frac{g_3}{7}z^3 + \dots, \end{aligned} \tag{2.3}$$

where the *elliptic invariants* g_2, g_3, defined with respect to ω_1, ω_3, are:

$$\begin{aligned} g_2 &= 60 \sum_{m,n\in\mathbb{Z}^*} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, \\ g_3 &= 140 \sum_{m,n\in\mathbb{Z}^*} \frac{1}{(2m\omega_1 + 2n\omega_3)^6}. \end{aligned} \tag{2.4}$$

We can now construct the following:

$$F(\wp(z), \wp'(z)) := [\wp'(z)]^2 - 4\wp^3(z) + g_2\wp(z) + g_3 = O(z^2).$$

The elliptic function $F(\wp(z), \wp'(z))$ is analytic, and therefore bounded, for all z. Therefore, by Liouville's Theorem (see [2]), $F(\wp(z), \wp'(z))$ is constant. Taking $z \to 0$ we can see that this constant is zero, revealing to us that $\wp(z)$ satisfies the following differential equation:

$$[\wp'(z)]^2 = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{2.5}$$

As $\wp(z)$ is even, $\wp'(z)$ will be odd, and so substituting $z = -\omega$ into

$$\wp'(z + 2\omega) = \wp'(z)$$

reveals that $\wp'(\omega) = 0$. Therefore we can consider the following polynomial equation for $y = \wp(\omega)$:

$$\begin{aligned} 4y^3 - g_2y - g_3 &= 0, \\ (y - e_1)(y - e_2)(y - e_3) &= 0, \end{aligned} \tag{2.6}$$

where the roots of the equation e_1, e_2, e_3 satisfy the following:

$$\begin{gathered} e_1 + e_2 + e_3 = 0, \\ \wp(\omega_1) = e_1, \quad \wp(\omega_2) = e_2, \quad \wp(\omega_3) = e_3, \end{gathered} \tag{2.7}$$

and are related to g_2, g_3 in the following way:

$$g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1), \quad g_3 = 4e_1e_2e_3. \tag{2.8}$$

As (2.6) is a third order polynomial equation, we can analyse its discriminant to determine the nature of the roots e_1, e_2, e_3:

$$\frac{\Delta}{16} = g_2^3 - 27g_3^2, \tag{2.9}$$

so that:

$$\begin{aligned} &\Delta > 0: \quad e_1 > e_2 > e_3, \quad e_1, e_2, e_3 \in \mathbb{R}, \\ &\Delta = 0: \quad e_1 = e_2, e_3 = -2e_1, \\ &\Delta < 0: \quad \begin{cases} e_1 = -\alpha + i\beta, \\ e_2 = 2\alpha, & \text{for } \alpha, \beta \in \mathbb{R}. \\ e_3 = \bar{e_1} = -\alpha - i\beta, \end{cases} \end{aligned} \tag{2.10}$$

$\wp(z)$ is homogeneous, so that:

$$\wp(\lambda z; g_2, g_3) = \lambda^{-2}\wp(z; \lambda^4 g_2, \lambda^6 g_3). \tag{2.11}$$

2.1.2 The Weierstrass zeta and sigma functions: ζ and σ

We define $\zeta(z)$ with respect to $\wp$ [48]

$$\zeta(z) = \frac{1}{z} - \int_0^z \left\{\wp(v) - \frac{1}{v^2}\right\} dv, \tag{2.12}$$

or as a series

$$\zeta(z) = \frac{1}{z} + \sum_{m,n\in\mathbb{Z}^*} \left\{\frac{1}{z - 2m\omega_1 - 2n\omega_3} + \frac{1}{2m\omega_1 + 2n\omega_3} + \frac{z}{(2m\omega_1 + 2n\omega_3)^2}\right\}, \tag{2.13}$$

so that:

$$\zeta'(z) = -\wp(z). \tag{2.14}$$

From (2.13), it is clear that ζ is an odd function, so that:

$$\zeta(-z) = -\zeta(z). \tag{2.15}$$

When shifting by 2ω in relation (2.14), we find

$$\begin{aligned} \zeta'(z+2\omega) &= \zeta'(z), \\ \zeta(z+2\omega) &= \zeta(z) + 2\eta, \end{aligned} \tag{2.16}$$

and exploiting (2.15), we can substitute $z = -\omega$ into this equation to find η:

$$\eta = \zeta(\omega), \tag{2.17}$$

for $i = 1, 2, 3$. The objects η_i, ω_j satisfy the following identity:

$$\eta_i\omega_j - \eta_j\omega_i = \rho_{ij}\frac{\pi i}{2}, \tag{2.18}$$

where $i, j \in (1\ 2\ 3)$ is a cycle so that:

$$\begin{aligned} \rho_{i,i} = \rho_{i,i+3} &= 0, \\ \rho_{i,i+1} &= -1, \\ \rho_{i,i+2} &= 1. \end{aligned}$$

The addition theorem for ζ is

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}. \tag{2.19}$$

We define the Weierstrass sigma function as [48]

$$\sigma(z) = z \prod_{m,n\in\mathbb{Z}^*} \left(1 - \frac{z}{\Omega_{m,n}}\right) \exp\left(\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2}\right),$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$, so that

$$\frac{d}{dz}\log\sigma(z) = \zeta(z).$$

Shifting by 2ω yields

$$\sigma(z+2\omega) = -\exp(2\eta(z+\omega))\sigma(z).$$

2.2 Conditions for bounded solutions

In the case $m = 1$, the complex Lamé equation

$$-\frac{d^2\psi}{dz^2} + 2\wp(z)\psi = \lambda\psi, \quad \lambda = -\wp(k), \tag{2.20}$$

has the explicit solutions [48]:

$$\psi(z;k) = \frac{\sigma(z+k)}{\sigma(z)} \exp(-\zeta(k)z), \tag{2.21}$$

where $k \in \mathbb{C}$. These solutions have the following periodicity property:

$$\psi(z + 2\omega) = \exp(2\zeta(\omega)k - 2\zeta(k)\omega)\psi(z),$$

To ensure that our solutions remain bounded on the line $z = \omega x + z_0$, $x \in \mathbb{R}$, we require that the following condition is satisfied:

$$|\exp(2\zeta(\omega)k - 2\zeta(k)\omega)| = 1. \tag{2.22}$$

Therefore, values of $k \in \mathbb{C}$ satisfying this condition will constitute the spectrum. Equivalently we have the following condition for k to be in the spectrum:

$$\mathrm{Re}\,[\zeta(\omega)k - \zeta(k)\omega] = 0, \tag{2.23}$$

The periodic and anti-periodic solutions correspond to

$$\exp(2\zeta(\omega)k - 2\zeta(k)\omega) = \pm 1$$

or, equivalently,

$$\begin{aligned} \text{periodic} : \zeta(\omega)k - \zeta(k)\omega &= 2p\frac{i\pi}{2}, \\ \text{anti-periodic} : \zeta(\omega)k - \zeta(k)\omega &= (2q+1)\frac{i\pi}{2}, \end{aligned} \tag{2.24}$$

for some $p, q \in \mathbb{Z}$. From the relations (2.18):

$$\zeta(\omega)\omega_j - \zeta(\omega_j)\omega = \rho_j \frac{\pi i}{2}, \tag{2.25}$$

where $\rho_j = 0$ or ± 1 depending on the choice of half-periods (see [2]), it follows that that $k = \omega_1, \omega_2, \omega_3$ are solutions of (2.24) with $q = 0$, $p = 0$ and $q = -1$, which correspond to the edges of the spectral arcs.

Now we are going to study the geometry of these arcs in more detail.

2.3 Investigation of the complex spectrum

We start with the special lemniscatic case when $\omega_3 = i\omega_1$, or equivalently when $g_3 = 0$.

2.3.1 The lemniscatic case

Without loss of generality we can assume that $g_2 = 1$, so e_1, e_2, e_3 are solutions of

$$y(4y^2 - 1) = 0,$$

which are $-e_1 = -0.5, -e_2 = 0, -e_3 = 0.5$. We can write k from a period parallelogram as

$$k = a\omega_1 + b\omega_3, \quad a, b \in [-1, 1].$$

Using (2.19), we have for $\omega = \omega_1$:

$$\mathrm{Re}\left[\zeta(\omega_1)(a\omega_1 + b\omega_3) - \zeta(a\omega_1 + b\omega_3)\omega_1\right] = 0, \qquad (2.26)$$

$$\mathrm{Re}\left[a\omega_1\zeta(\omega_1) + \underbrace{b\omega_3\zeta(\omega_1)}_{\in i\mathbb{R}} - \left\{\zeta(a\omega_1) + \underbrace{\zeta(b\omega_3)}_{\in i\mathbb{R}} + \frac{1}{2}\frac{\wp'(a\omega_1)}{\wp(a\omega_1) - \wp(b\omega_3)} - \underbrace{\frac{1}{2}\frac{\wp'(b\omega_3)}{\wp(a\omega_1) - \wp(b\omega_3)}}_{\in i\mathbb{R}}\right\}\omega_1\right] = 0,$$

which gives us the following condition for the spectrum:

$$a\omega_1\zeta(\omega_1) - \omega_1\zeta(a\omega_1) - \frac{1}{2}\frac{\omega_1\wp'(a\omega_1)}{\wp(a\omega_1) - \wp(b\omega_3)} = 0.$$

If $a = \pm 1$ then from (2.16) we see that the first two terms cancel, while the third will be equal to zero, and thus $a = \pm 1$ is a solution for all b. If instead we take $a = 0$ in (2.26), we get the following:

$$\mathrm{Re}\left[\underbrace{b\omega_3\zeta(\omega_1)}_{\in i\mathbb{R}} - \underbrace{\omega_1\zeta(b\omega_3)}_{\in i\mathbb{R}}\right] = 0,$$

which is satisfied as long as $b \neq 0$ (to avoid poles). From this, we can deduce that $k = a\omega_1 + b\omega_3$ satisfies (2.26) iff

$$b \in \begin{cases} [-1, 1], & \text{for } a = \pm 1, \\ [-1, 0) \cup (0, 1], & \text{for } a = 0. \end{cases}$$

For $\omega = \omega_3$ we have the following condition:

$$\mathrm{Re}\left[\zeta(\omega_3)(2m\omega_1 + b\omega_3) - \zeta(2m\omega_1 + b\omega_3)\omega_3\right] = 0.$$

Using the same method we come to the following set of solutions:

$$a \in \begin{cases} [-1,1], & \text{for } b = \pm 1, \\ [-1,0) \cup (0,1], & \text{for } b = 0. \end{cases}$$

The corresponding values of k are represented in Figure 2.1. Since $\omega_3 = i\omega_1$ they differ by multiplication by i. The associated spectral bands with gaps are represented in Figure 2.2. As expected, we can see two gaps in the spectrum: $(-\infty, -e_1), (-e_2, -e_3)$ for $\omega = \omega_1$ and $(-e_1, -e_2), (-e_3, +\infty)$ for $\omega = \omega_3$, in agreement with the results of the previous chapter.

The case $\omega = \omega_2$ represents more of a challenge. However, in the lemniscatic case, we can use that $\omega_3 = i\omega_1$ and $\zeta(ik) = -i\zeta(k)$ to find that $k = a\bar{\omega}_2, a \in [-1,0)\cup(0,1]$ satisfies the condition (2.23). Indeed,

$$\begin{aligned}
&\mathrm{Re}\left[\zeta(\omega_2)(a\bar{\omega}_2) - \zeta(a\bar{\omega}_2)\omega_2\right] \\
&= \mathrm{Re}\left[a\zeta(\omega_1 + i\omega_1)(\omega_1 - i\omega_1) - \zeta(a(\omega_1 - i\omega_1))(\omega_1 + i\omega_1)\right] \\
&= -\frac{\omega_1}{2}\frac{\wp'(a\omega_1) + i\wp'(ia\omega_1)}{\wp(a\omega_1) - \wp(ia\omega_1)} = 0,
\end{aligned}$$

where we have used the property of the lemniscatic $\wp$-function

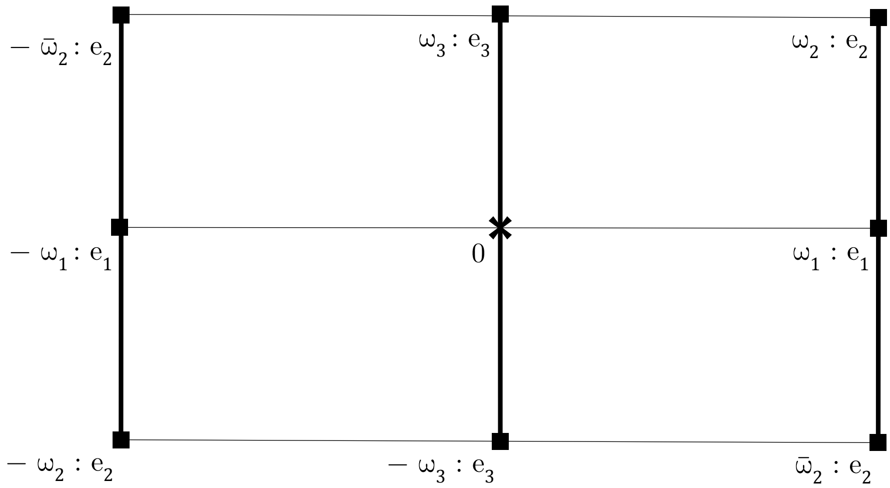
$$\wp(iz) = -\wp(z), \quad \wp'(iz) = i\wp'(z).$$

We use *Mathematica* to find the remaining solutions of (2.23) numerically and plot the subsequent eligible values of k, as shown in Figure 2.3.

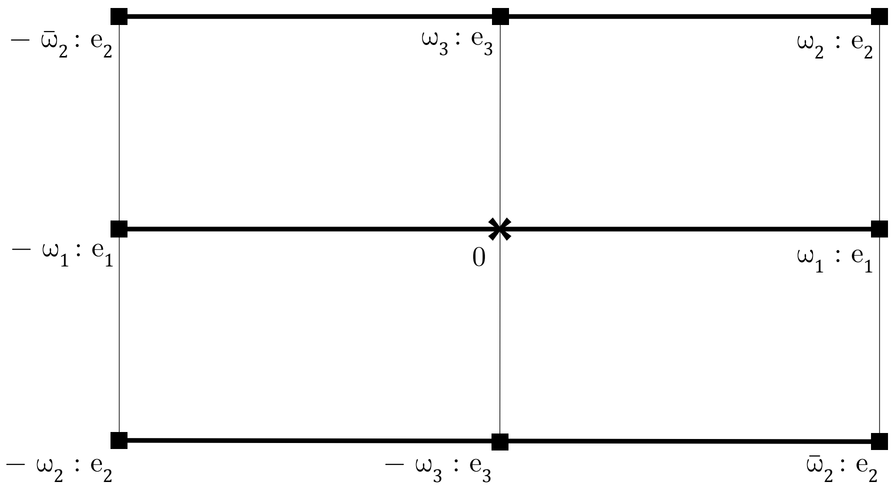
We can map the coordinates of k that analytically solve (2.23) (as well as an approximation of the rest from Figure 2.3) to $\lambda = -\wp(k)$, and construct a visual representation of the spectrum, as seen in Figure 2.4.

2.3.2 Generic case

The cases $\omega = \omega_1$ and $\omega = \omega_3$ can be analysed in the same way as in the lemniscatic case.



(a) $\omega = \omega_1$



(b) $\omega = \omega_3$

Figure 2.1: Spectral values of k.

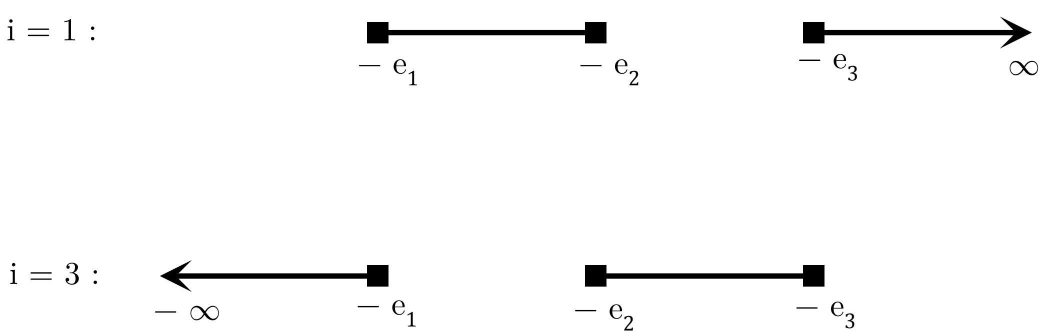


Figure 2.2: Spectral bands, corresponding to $\omega = \omega_i$.

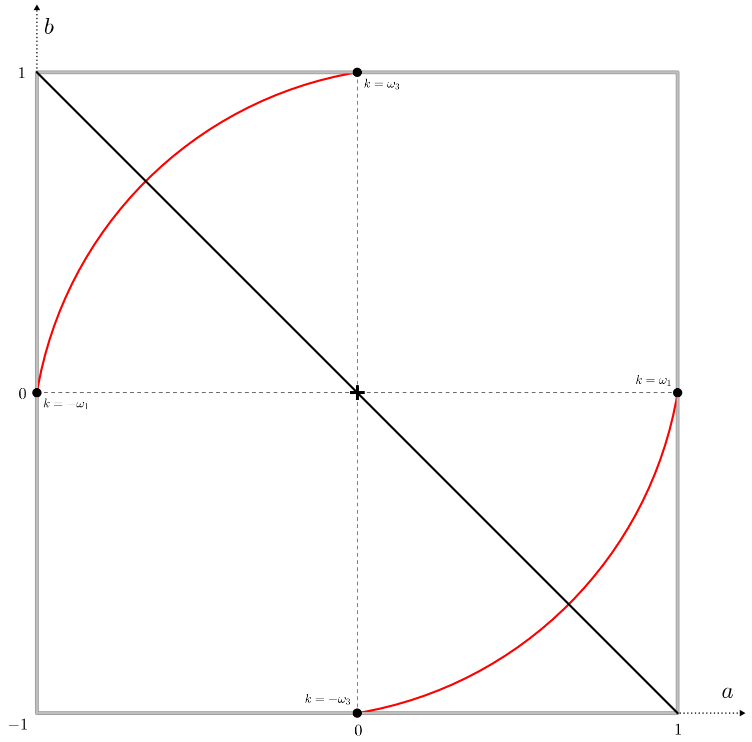


Figure 2.3: Spectral values of for $k = a\omega_1 + b\omega_3$ corresponding $\omega = \omega_2$. Black points and diagonal line are found analytically, red arcs are from numerical solution of (2.23) in *Mathematica*.

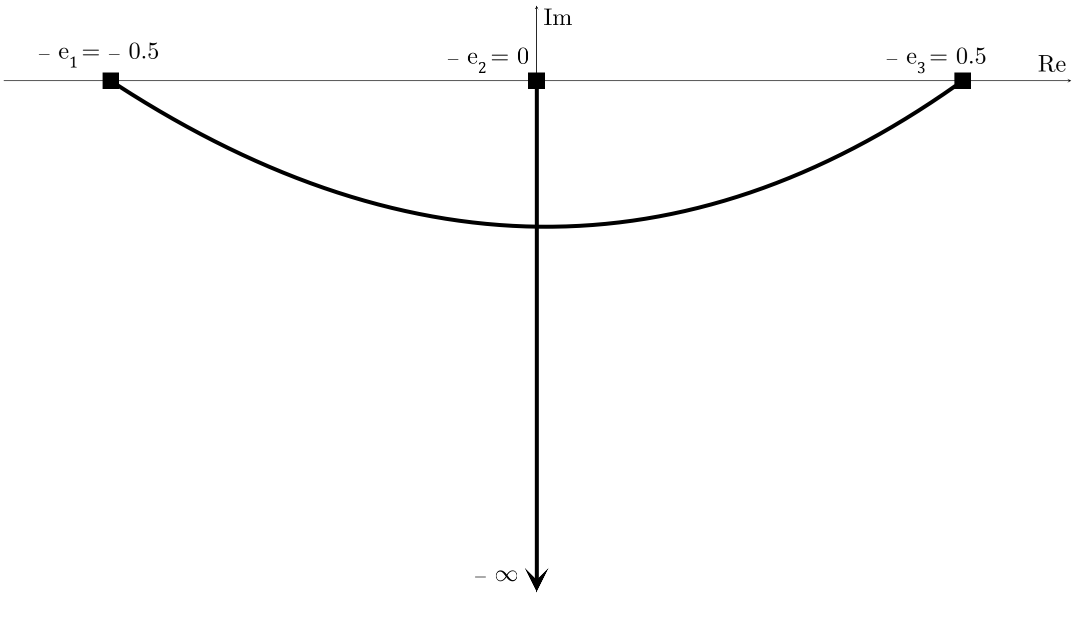


Figure 2.4: Complex spectrum corresponding to Figure 2.3.

For $\omega = \omega_2$ we can again use *Mathematica* to solve numerically the spectral condition (2.23). Figure 2.5 shows the corresponding plots for $g_3 = 1$ and different values of g_2.

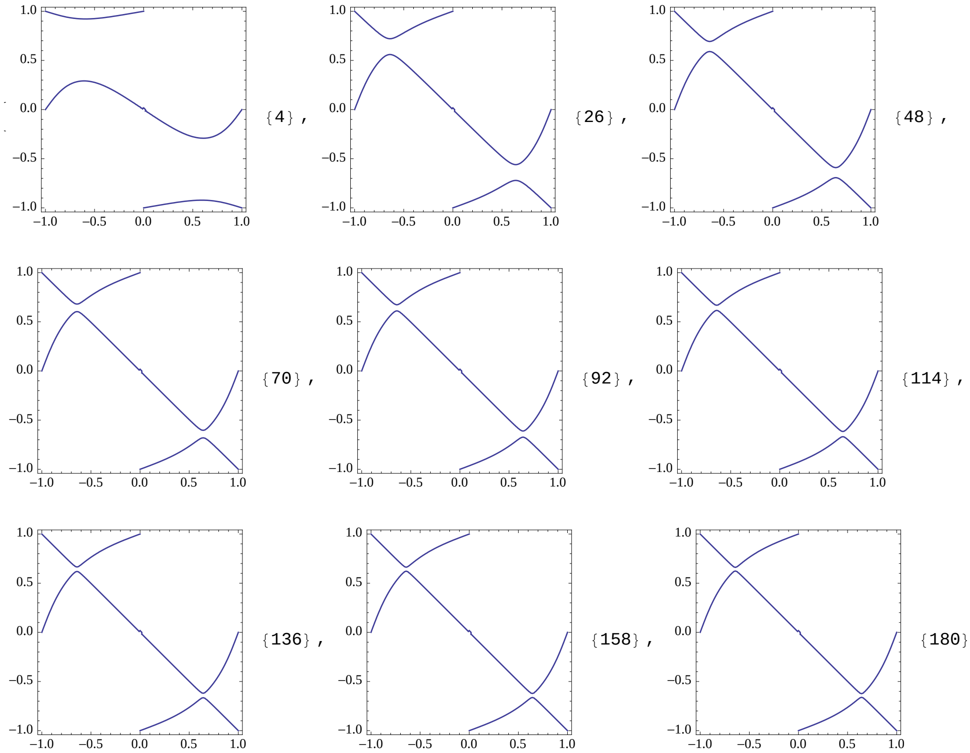


Figure 2.5: Spectral values of k corresponding to $\omega = \omega_2$. For each image, $g_3 = 1$, while g_2 is equal to the bracketed number.

It is clear that for large g_2 and fixed g_3, we have a picture similar to the lemniscatic $g_2 = 1, g_3 = 0$ case (as expected).

Note that all these curves are passing through $k = 0$, which corresponds to the limit $\lambda \to \infty$. We can use this to study the asymptotic behaviour of the infinite spectral arc at infinity.

Proposition 2.1. *The infinite spectral arc has the asymptote $\bar{\omega}_2^2 s$, where $s \in \mathbb{R}$.*

Proof. Assume that

$$k = a\omega_1 + b\omega_3, \quad a, b \ll 1$$

and substitute this into (2.23) with $\omega = \omega_2$. Using the expansion of $\zeta(k)$ for $k \approx 0$

$$\zeta(k) = \frac{1}{k} - \frac{g_2}{60}k^3 + \dots$$

we have

$$\begin{aligned}
&\mathrm{Re}\left[\left(\frac{1}{\omega_2}+\dots\right)(a\omega_1+b\omega_3)-\left(\frac{1}{a\omega_1+b\omega_3}+\dots\right)\omega_2\right]\\
&=\underbrace{\frac{1}{a^2\omega_1^2-b^2\omega_3^2}}_{\in\mathbb{R},\neq 0}\mathrm{Re}\left[-\omega_2(a\omega_1-b\omega_3)+\frac{a\omega_1+b\omega_3}{\omega_2}\left(a^2\omega_1^2-b^2\omega_3^2\right)+\dots\right]\\
&\approx -\frac{a\omega_1^2-b\omega_3^2}{a^2\omega_1^2-b^2\omega_3^2}=0.
\end{aligned}$$

This implies that

$$a\approx\tau^2 b,$$

where $\tau = {}^{\omega_3}/_{\omega_1}$. Therefore, k will be parametrised by b, in a neighbourhood close to zero, in the following way:

$$k=b(\tau^2\omega_1+\omega_3)=b\tau(\omega_1+\omega_3)=b\tau\omega_2=itb\omega_2,$$

where $\tau=it$, $t\in\mathbb{R}_+$.

Using the expansion of $\wp$ at zero we have

$$\lambda=-\wp(k)=-\wp(itb\omega_2)=\frac{1}{(tb\omega_2)^2}+\frac{g_2}{20}(tb\omega_2)^2+\dots\approx\frac{1}{\omega_2^2}s\sim\bar{\omega}_2^2 s$$

where $s=(tb)^{-2}\to+\infty$ as $b\to 0$. □

The special case $g_2=4, g_3=1$ is plotted in Figure 2.6.

2.3.3 Location of closed gaps in the spectrum

Consider the (anti-)periodic solutions of (2.24) and the corresponding conditions on k:

$$\underbrace{\zeta(\omega)k-\zeta(k)\omega}_{f(k)}=\pm p\frac{\pi i}{2},\quad p\in\mathbb{Z}_{\geq 0}. \tag{2.27}$$

We know that the solutions of this condition for $p=0,1$ are $k=\pm\omega_j$, $j=1,2,3$, which correspond to the edges of arcs. The question is what can we say about other solutions, which must correspond to the "closed gaps."

The main result of this chapter is the following theorem.

Theorem 2.1. *All closed gaps of the complex Lamé operator (2.1) with $m=1$ and $\omega=\omega_2$ are contained on the infinite spectral arc.*

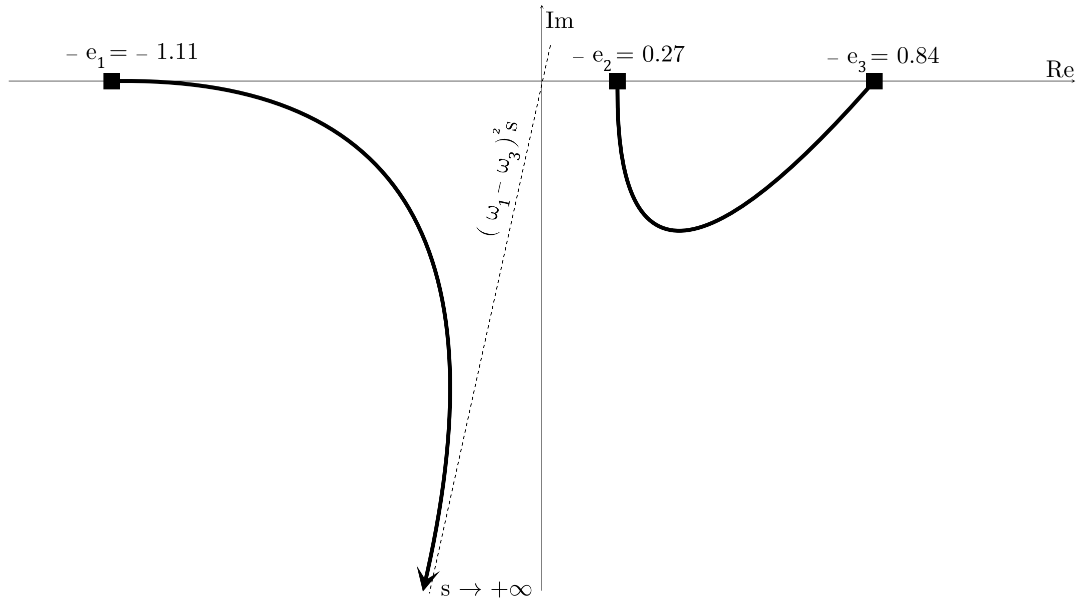


Figure 2.6: Complex spectrum corresponding to $g_2 = 4, g_3 = 1$ in Figure 2.5.

Proof. The location of closed gaps are the solutions of the following systems

$$\begin{cases} \mathrm{Re}[\zeta(\omega_2)k - \zeta(k)\omega_2] = 0, \\ \mathrm{Im}[\zeta(\omega_2)k - \zeta(k)\omega_2] = \pm p\dfrac{\pi}{2}, \end{cases} \tag{2.28}$$

with integer $p \geq 2$.

Numerical analysis of the case $g_2 = 4, g_3 = 1$ for different values of p are shown in Figure 2.7.

We prove first analytically that the closed gaps appear on the infinite arc in the lemniscatic case, and then using continuity arguments demonstrate that it must be true for all g_2, g_3.

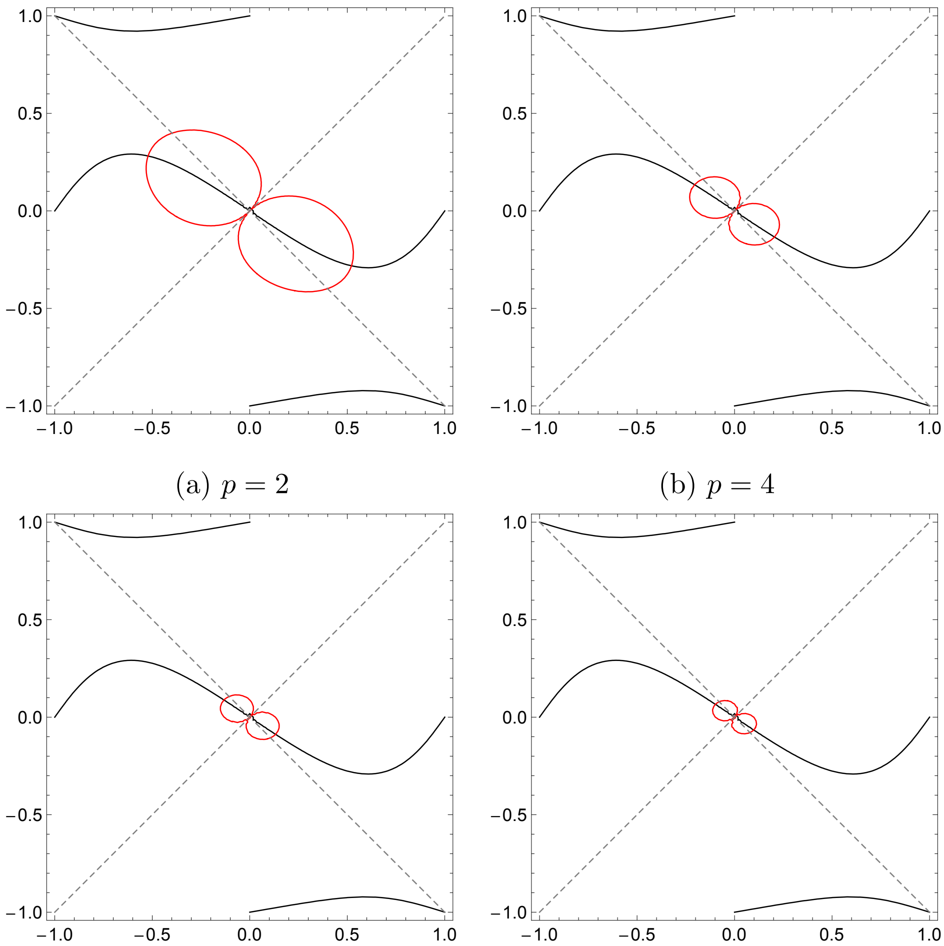
Consider the function

$$f(k) = \zeta(\omega_2)k - \zeta(k)\omega_2.$$

We know that in the lemniscatic case the arc edges are at $k = \pm\omega_1, \pm\omega_3, \pm\bar{\omega}_2$ and the line $k = a(\omega_3 - \omega_1), a \in [-1, 1]$ is part of the spectrum.

To find the point of intersection of the arcs in this case write $f(k)$ as

$$f(k) = u(k) + iv(k), \quad k = x + iy.$$



(a) $p = 2$

(b) $p = 4$

(c) $p = 6$

(d) $p = 8$

Figure 2.7: Solutions of equations (2.28) for $g_2 = 4, g_3 = 1$. Intersections of curves represent the k values of closed gaps. It is evident that the closed gaps tend towards infinity along the spectral arc as $p \to \infty$.

On the curves in Figure 2.3 we have $\mathrm{Re}[f(k)] = u(k) = 0$. At the intersection point k_* we must have additionally that $u_x(k_*) = u_y(x_*) \equiv 0$.

From the Cauchy-Riemann relations we have

$$u_x(k_*) = v_y(k_*) = 0,$$
$$u_y(k_*) = -v_x(k_*) = 0,$$

which implies that the complex derivative $f'(k_*) = 0$ at this intersection point:

$$f'(k_*) = \zeta(\omega_2) + \wp(k_*)\omega_2 = 0 \implies -\wp(k_*) = \frac{\zeta(\omega_2)}{\omega_2}.$$

As $\wp$ is even and of order 2 we have two intersection points at $\pm k_*$.

Now, as $f(k)$ is holomorphic at k_*, we can consider the Taylor expansion for k close to k_*:

$$f(k) = a_0 + a_1(k - k_*) + \frac{a_2}{2}(k - k_*)^2 + \ldots = c(k - k_*)^2 + \ldots,$$

where $a_0 \equiv f(k_*) = 0$, $a_1 \equiv f'(k_*) = 0$, and c is some complex number: $c = a + ib$. Let us substitute $z = k - k_*$, where $z = x + iy$, and take the leading order

$$\begin{aligned} f(z) &\approx (a + ib)(x + iy)^2 \\ &= a(x^2 - y^2) - 2bxy + i\left\{b(x^2 - y^2) + 2axy\right\}. \end{aligned}$$

We are interested in the imaginary part of $f(z)$. The Hessian matrix $H(\mathrm{Im}[f(z)])$ is

$$H(\mathrm{Im}[f(z)]) = \begin{bmatrix} 2b & 2a \\ 2a & -2b \end{bmatrix}.$$

It is easy to see that H is indefinite (i.e. it has both positive and negative eigenvalues), therefore k_* is a saddle point of the function $\mathrm{Im}[f(z)]$.

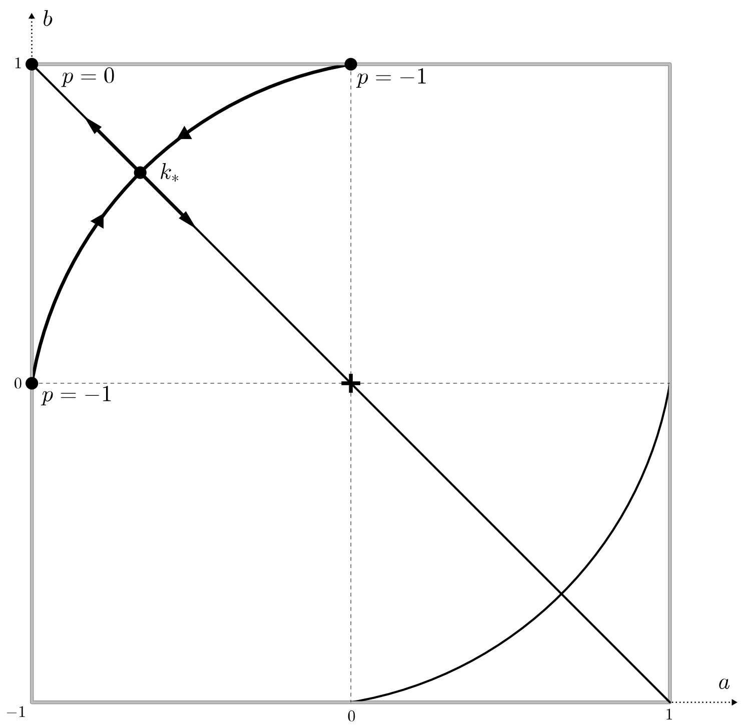


Figure 2.8: Direction of increase of $\mathrm{Im}[f(k)]$ along the curve $\mathrm{Re}[f(k)] = 0$. Values of p at the end-points correspond to $\mathrm{Im}[f(k)] = \pm p\frac{\pi}{2}$.

Now, recalling (2.28), we know that closed gap locations correspond to the values of k that simultaneously solve:

$$\begin{cases} \mathrm{Re}[\zeta(\omega_2)k - \zeta(k)\omega_2] = 0, \\ \mathrm{Im}[\zeta(\omega_2)k - \zeta(k)\omega_2] = \pm p\dfrac{\pi}{2}. \end{cases}$$

We also know that $\mathrm{Im}[f(k)]$ close to k_* is a saddle surface, which implies that in one direction $\mathrm{Im}[f(k)]$ will be increasing, whereas in another it will be decreasing.

We claim that on each of the intersecting curves from Figure 2.8 the function $\mathrm{Im}[f(k)]$ is monotonic. Indeed, we know that $\pm k_*$ are the only zeros of $f'(z)$, which means that $\mathrm{Im}[f(k)]$ can not contain any critical points on the curves $\mathrm{Im}[f(k)] = 0$.

Thus we can conclude that for any k along the finite band, as well as from $k = k_*$ to $k = \omega_3 - \omega_1$, the associated value of p must be between -1 and 0.

Since p must be an integer this is impossible. This proves the theorem in the lemniscatic case.

The general case follows from the continuity arguments. Indeed, if the arcs continue to intersect when we change τ from $\tau = i$ then we can repeat the arguments of the lemniscatic case. If the curves break up as in Figure 2.7 we will have two disconnected curves and the closed gaps can not change the curve by continuity. $\square$

We believe that the theorem is true for all $m \in \mathbb{N}$.

2.4 Discussion

We also considered a difference version of the Lamé operator, which appeared in relation with representations of the so-called Sklyanin algebra [44]. This algebra is generated by S_a, $a = 0, \ldots, 3$, with the relations

$$[S_\alpha, S_0]_- = iJ_{\beta\gamma}[S_\beta, S_\gamma]_+,$$
$$[S_\alpha, S_0]_- = i[S_0, S_\gamma]_+,$$

where $[A, B]_\pm = AB \pm BA$. The difference analogue of the Lamé operator corresponds to S_0 in Sklyanin's representation of this algebra

$$S_0 = \frac{\theta_1(x - m\eta)}{\theta_1(x)}T^\eta + \frac{\theta_1(x + m\eta)}{\theta_1(x)}T^{-\eta}, \quad x \in \mathbb{C}, \tag{2.29}$$

where T is the shift operator defined by $T^{\eta}\psi(x) = \psi(x+\eta)$, and θ_1 are the Jacobi theta functions (see e.g. [30]). The case of $\eta \in \mathbb{C}$ has been studied by Ruijsenaars [43].

We can show that S_0 reduces to the differential Lamé operator (2) at the non-relativistic limit $\eta \to 0$. For this, we can rewrite $S_0\psi = \lambda\psi$ using

$$\psi(x) = \Psi(x)\prod_{j=1}^{m}\theta_1(x - j\eta),$$

as (see [29])

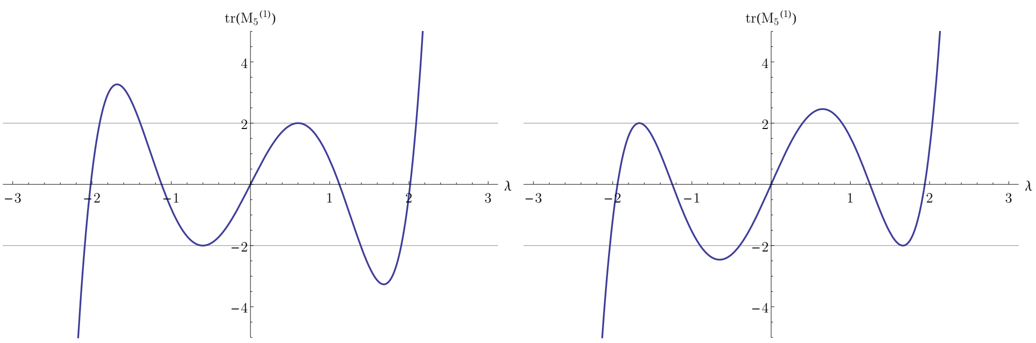
$$\tilde{S}_0\Psi \equiv \Psi(x+\eta) + \frac{\theta_1(x+m\eta)\theta_1(x-(m+1)\eta)}{\theta_1(x)\theta_1(x-\eta)}\Psi(x-\eta) = \lambda\Psi(x).$$

Now, taking Taylor series expansions for $\eta \to 0$, and using the fact that $\wp(x) = D^2(\log\theta_1(x))$, we get:

$$\tilde{S}_0\Big|_{\eta\to 0} = 2 + \eta^2\left(D^2 - m(m+1)\wp(x)\right) + \mathcal{O}\left(\eta^3\right).$$

Krichever and Zabrodin [29, 49, 50] studied the spectrum of (2.29), for arbitrary generic $\eta \in \mathbb{R}$, and derived the explicit formulas for the band edges.

We considered the case of rational $\eta = {}^{P}/_{Q}$, $P, Q \in \mathbb{N}$. While it is known that the spectrum will contain exactly Q bands [49], we wanted to explore the role that P played in determining the location of closed gaps (see Figure 2.9, as an example).



(a) $P = 1, 4$ (b) $P = 2, 3$

Figure 2.9: Spectral gaps for $l = 1, Q = 5$, with different P values.

This yielded some interesting patterns, which warrant further study.

Part II

Integrable Schrödinger operators and exceptional orthogonal polynomials

Chapter 3

Complex exceptional Hermite polynomials

3.1 Darboux transformations and exceptional Hermite polynomials

The following operator defines the harmonic oscillator in quantum mechanics:

$$\mathscr{L} = -D^2 + x^2, \tag{3.1}$$

where $D = \frac{d}{dx}$. It is well known that

$$\psi_k = H_k(x) e^{-x^2/2}, \tag{3.2}$$

where $H_k(x)$ are the classical Hermite polynomials, are the eigenfunctions:

$$\mathscr{L}\psi_k(x) = (2k+1)\psi_k(x), \ k \in \mathbb{Z}_{\geq 0},$$

of $\mathscr{L}$, which can be used as a definition of Hermite polynomials. We choose the normalisation of Hermite polynomials such that the highest coefficient of $H_l(z)$ is 2^l and all the coefficients are integer:

$$H_0 = 1, \ H_1 = 2z, \ H_2 = 4z^2 - 2, \ H_3 = 8z^3 - 12z, \ H_4 = 16z^4 - 48z^2 + 12, \ldots.$$

Alternatively Hermite polynomials can be defined as orthogonal polynomials with respect to the Gaussian measure

$$d\mu(x) = e^{-x^2} dx.$$

The *Darboux transformation* (DT) can be defined for any Schrödinger operator $\mathscr{L}$ with potential $u(x)$ with known eigenfunction $\psi_k(x)$ by

$$\tilde{u}(x) = u(x) - 2D^2 \log(\psi_k(x)). \tag{3.3}$$

If we know all the eigenfunctions ψ_l of $\mathscr{L}$ then one can check that the corresponding Schrödinger equation

$$-D^2\tilde{\psi}_l + \tilde{u}(x)\tilde{\psi}_l = (2l+1)\tilde{\psi}_l$$

has solutions

$$\tilde{\psi}_l = \left(D - \frac{\psi_k'}{\psi_k}\right)\psi_l, \tag{3.4}$$

for $l \neq k$. These transformations can be iterated a number of times [7]. Eigenfunctions for an operator iterated n times can be written in the form:

$$\tilde{\psi}_l^{(n)} = \frac{\mathrm{Wr}(\psi_l, \psi_{k_1}, \psi_{k_2}, \ldots, \psi_{k_n})}{\mathrm{Wr}(\psi_{k_1}, \psi_{k_2}, \ldots, \psi_{k_n})}, \tag{3.5}$$

or alternatively:

$$\tilde{\psi}_l^{(n)} = \left(D - D\log\left(\tilde{\psi}_{k_n}^{(n-1)}\right)\right)\tilde{\psi}_l^{(n-1)}, \tag{3.6}$$

for some increasing sequence of integers $k_1, \ldots, k_n$, where $l \notin k_1, \ldots, k_n$, Wr is the Wronskian determinant, and where the transformed potential takes the form:

$$\tilde{u}^{(n)}(x) = u(x) - 2D^2 \log \mathrm{Wr}(\psi_{k_1}, \ldots, \psi_{k_n}). \tag{3.7}$$

In our case of the harmonic oscillator, all DT are described by partitions as follows [13]. Let $0 \leq k_1 < \ldots < k_n$ be the level at which we applied the DT and define the partition λ by

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad 0 \leq \lambda_1 \leq \ldots \leq \lambda_n, \quad \lambda_i = k_i - i + 1.$$

The resulting potential of these DT (up to adding a constant) has the form

$$\tilde{u}^{(n)}(x) = x^2 - 2D^2 \log W_\lambda,$$

where

$$W_\lambda = \mathrm{Wr}(H_{k_1}, \ldots, H_{k_n}). \tag{3.8}$$

Felder et al [13] showed that these Wronskians have some remarkable properties, in particular

$$\deg W_\lambda = |\lambda| \equiv \sum_{i=1}^{n} \lambda_i. \tag{3.9}$$

Note that in order to have a regular potential, we need that $W_\lambda \neq 0$ for any $x \in \mathbb{R}$. It is known after M. Krein and V. Adler [1] that this is the case as long as the sequence $\{k_1, \dots, k_n\}$ is composed of a finite number of blocks of even length, preceded by an arbitrary length block of integers starting from 0. In the language of Felder et al. [13], this corresponds to the case of *double partitions* of the form

$$\lambda = \mu^2 = (\mu_1, \mu_1, \mu_2, \mu_2, \dots, \mu_p, \mu_p),$$

where $\mu = (\mu_1, \dots, \mu_p)$ is another partition with $n = 2p$.

In that case, Gomez-Ullate et al. [18] defined *exceptional Hermite polynomials* by

$$H_{\lambda,l} \coloneqq \mathrm{Wr}(H_{k_1} \dots, H_{k_n}, H_l), \quad l \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \dots, k_n\}.$$

They have shown that they are orthogonal and, in spite of the fact that some of the degrees are missing, they are dense in $L_2(\mathbb{R})$ with density measure

$$w(x) = W_\lambda^{-2}(x) e^{-x^2}. \tag{3.10}$$

Note that after the gauge transformation using the weight function:

$$-\left(e^{x^2/2} W_\lambda\right) \circ \tilde{\mathscr{L}}^{(n)} \circ \left(e^{-x^2/2} W_\lambda^{-1}\right), \tag{3.11}$$

where

$$\tilde{\mathscr{L}}^{(n)} = -D^2 + \tilde{u}^{(n)}(x)$$

we derive the following operators:

$$T_\lambda = D^2 - 2\left(x + \frac{W'_\lambda}{W_\lambda}\right) D + \left(\frac{W''_\lambda}{W_\lambda} + 2x \frac{W'_\lambda}{W_\lambda}\right) \tag{3.12}$$

$$T = D^2 - 2xD \tag{3.13}$$

where $'$ denotes differentiation with respect to x. T is the classical Hermite operator, while T_λ is solved by $H_{\lambda,j}$ with eigenvalue $2n - 2j$. For a double partition λ, T_λ is non-singular in $\mathbb{R}$.

Now we are going to understand the situation for general partition λ.

3.2 Complex exceptional Hermite polynomials

Let us define now the *complex exceptional Hermite polynomials* (CEHPs) by the same formula:

$$H_{\lambda,l} := \mathrm{Wr}(H_{k_1} \dots, H_{k_n}, H_l), \quad l \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \dots, k_n\}, \tag{3.14}$$

but now partition λ is arbitrary. This means that the corresponding measure has singularities on the real line. From the previous section, the functions

$$\psi_{\lambda,l} = H_{\lambda,l} \frac{e^{-z^2/2}}{W_\lambda}, \tag{3.15}$$

have the eigenfunction property

$$\mathscr{L}_\lambda \psi_{\lambda,l} = (2l+1)\psi_{\lambda,l},$$

where

$$\mathscr{L}_\lambda = -\frac{d^2}{dz^2} - 2\frac{d^2}{dz^2}\big(\log \mathrm{Wr}(\psi_{k_1}, \dots, \psi_{k_n})\big) + z^2, \tag{3.16}$$

and $\psi_l = H_l(z)e^{-z^2/2}$. By a direct computation, it is readily inferred that $H_{\lambda,l}$ is a formal eigenfunction of the singular operator

$$T_\lambda = \frac{d^2}{dz^2} - 2\left(z + \frac{W_\lambda'}{W_\lambda}\right)\frac{d}{dz} + \left(\frac{W_\lambda''}{W_\lambda} + 2z\frac{W_\lambda'}{W_\lambda}\right)$$

with eigenvalue $2n - 2l$.

Example 3.1. Consider the special case $\lambda = (1)$, which corresponds to the Schrödinger operator

$$\begin{aligned}\mathscr{L}_{(1)} &= -\frac{d^2}{dz^2} - 2\frac{d^2}{dz^2}\big(\log(2ze^{-z^2/2})\big) + z^2 \\ &= -\frac{d^2}{dz^2} + z^2 + \frac{2}{z^2} + 2.\end{aligned}$$

Already in this simple example, we obtain eigenfunctions

$$\psi_{\lambda,l} = \frac{\mathrm{Wr}(\psi_l, \psi_{k_1}, \dots, \psi_{k_n})}{\mathrm{Wr}(\psi_{k_1}, \dots, \psi_{k_n})}, \quad l \notin k_1, \dots, k_n, \tag{3.17}$$

with a singularity on the real line (at $z = 0$). Indeed, this can be seen explicitly by writing out the first exceptional Hermite polynomials $H_{(1),k} = \mathrm{Wr}(\psi_k, \psi_1)$ and the corresponding few eigenfunctions $\psi_{(1),k} = \frac{\mathrm{Wr}(\psi_k,\psi_1)}{\psi_1}$:

$$H_{(1),0} = 1, \qquad \psi_{(1),0} = \tfrac{1}{z}e^{-z^2/2},$$

$$H_{(1),2} = -(2+4z^2), \qquad \psi_{(1),2} = -\tfrac{2+4z^2}{z}e^{-z^2/2},$$

$$H_{(1),3} = -16z^3, \qquad \psi_{(1),3} = -16z^2e^{-z^2/2},$$

$$H_{(1),4} = 12(1+4z^2-4z^4), \qquad \psi_{(1),4} = \tfrac{12(1+4z^2-4z^4)}{z}e^{-z^2/2},$$

$$H_{(1),5} = 64z^3(5-2z^2), \qquad \psi_{(1),5} = 64z^2(5-2z^2)e^{-z^2/2},$$

$$H_{(1),6} = 40(3+18z^2-36z^4+8z^6), \qquad \psi_{(1),6} = -\tfrac{40(3+18z^2-36z^4+8z^6)}{z}e^{-z^2/2}.$$

More generally, using the fact that

$$\mathrm{Wr}(\psi_{2l+1},\psi_1)(-z) = -\mathrm{Wr}(\psi_{2l+1},\psi_1)(z), \quad l \in \mathbb{N},$$

as well as the fact that each classical Hermite polynomial $H_{2l}(z)$, $l \in \mathbb{Z}_{\geq 0}$, has a nonzero constant term, it is readily seen that $\psi_{(1),l}(x)$ is regular on the whole real line if and only if $l \in \mathbb{Z}_{\geq 0} \setminus \{1\}$ is odd. The eigenvalues of the first few eigenfunctions are given in Figure 3.1, where open and filled circles indicate that the corresponding eigenfunctions are singular and non-singular, respectively. In addition, the cross represents the eigenvalue removed by the Darboux transformation.

Note that in the theory of quantum Calogero-Moser systems (of which this example is the simplest case) only non-singular solutions are considered (see e.g. [39]).

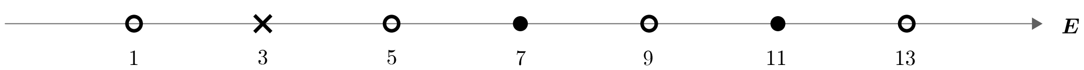


Figure 3.1: The eigenvalues of the first few eigenfunctions for $\lambda = (1)$.

The Schrödinger operator $\mathscr{L}_\lambda$ satisfies the intertwining relation

$$\mathscr{D}_\lambda \circ \mathscr{L} = \mathscr{L}_\lambda \circ \mathscr{D}_\lambda, \tag{3.18}$$

where $\mathscr{L} = -d^2/dz^2 + z^2$, and where the intertwining operator $\mathscr{D}_\lambda$ acts according to

$$\mathscr{D}_\lambda \psi = \frac{\mathrm{Wr}(\psi, \psi_{k_1}, \dots, \psi_{k_n})}{\mathrm{Wr}(\psi_{k_1}, \dots, \psi_{k_n})},$$

see e.g. [1, 7]. We will now use the fact that $\mathscr{D}_\lambda$ is obtained as the composition of first order intertwining operators. To be more specific, let us introduce the short hand notation

$$\mathscr{W}_m = \mathrm{Wr}(\psi_{k_m}, \dots, \psi_{k_n}), \quad \mathscr{W}_m(\psi) = \mathrm{Wr}(\psi, \psi_{k_m}, \dots, \psi_{k_n}),$$

(where it is convenient to allow $m = n+1$ and set $\mathscr{W}_{n+1} = 1$, $\mathscr{W}_{n+1}(\psi) = \psi$), and recall the standard identity

$$\mathscr{W}_{m-1}\mathscr{W}_m(\psi) = \mathscr{W}_m \frac{d}{dx}\mathscr{W}_{m-1}(\psi) - \mathscr{W}_{m-1}(\psi)\frac{d}{dx}\mathscr{W}_m, \quad m \geq 1.$$

Then it is readily verified that

$$\mathscr{D}_\lambda = D_1 \circ \cdots \circ D_m \circ \cdots \circ D_n, \tag{3.19}$$

with

$$D_m = \frac{d}{dz} - \frac{d}{dz}\left(\log \frac{\mathscr{W}_m}{\mathscr{W}_{m+1}}\right). \tag{3.20}$$

For our purposes, a key notion is that of trivial monodromy, see e.g. [46]. A Schrödinger operator $\mathscr{L} = -d^2/dz^2 + u(z)$, whose potential u is a meromorphic function of z, is said to have *trivial monodromy* if all solutions of its eigenvalue equation

$$\mathscr{L}\psi(z) = E\psi(z) \tag{3.21}$$

are meromorphic in z for all E.

We recall that every monodromy-free Schrödinger operator $\mathscr{L}$ with a quadratically increasing rational potential is of the form (3.16) for some partition λ. The fact that each Schrödinger operator $\mathscr{L}_\lambda$ has trivial monodromy is easily seen. Indeed, in the special case $u(z) = z^2$ all eigenfunctions are entire, and trivial monodromy is preserved under (rational) Darboux transformations. The converse result is due to Oblomkov [37].

Duistermaat and Grünbaum [10] obtained local conditions for trivial monodromy. Specifically, in a neighbourhood of a pole $z = z_i$ the potential $u(z)$ must have a Laurent series expansion of the form

$$u(z) = \sum_{r \geq -2} c_r (z - z_i)^r,$$

with

$$c_{-2} = m_i(m_i + 1) \quad \text{for some} \quad m_i \in \mathbb{N},$$

and

$$c_{2j-1} = 0, \quad \forall j = 0, 1, \ldots, m_i.$$

In addition, every eigenfunction ψ has a Laurent series expansion of the form

$$\psi(z) = (z - z_i)^{-m_i} \sum_{r=0}^{\infty} d_r (z - z_i)^r,$$

with

$$d_{2j-1} = 0, \quad \forall j = 1, \ldots, m_i.$$

We proceed to consider the implications for the CEHPs $H_{\lambda,l}$. Let Z_λ be the set of zeros $z_i \in \mathbb{C}$ of the Wronskian $W_\lambda(z)$ with multiplicities $m_i \in \mathbb{N}$. In addition, we need the subset $Z_\lambda^{\mathbb{R}} \subset Z_\lambda$ obtained by restriction to $z_i \in \mathbb{R}$. We say that a meromorphic function $\psi(z)$ is *quasi-invariant* at the point $z = z_i$ with multiplicity m_i if it satisfies the following two conditions:

1. $\psi(z)(z - z_i)^{m_i}$ is analytic at $z = z_i$,

2. $(\psi(z)(z - z_i)^{m_i})^{(2j-1)}|_{z=z_i} = 0$, for all $j = 1, \ldots, m_i$.

The second condition can be rewritten as

$$\psi(\sigma_i(z)) = (-1)^{m_i} \psi(z) + O((z - z_i)^{m_i}),$$

where $\sigma_i(z) = 2z_i - z$ is the reflection with respect to z_i. This explains the terminology.

Introducing the subspace

$$\mathcal{Q}_\lambda = \left\{ p \in \mathbb{C}[z] : \psi(z) := p(z)\frac{e^{-z^2/2}}{W_\lambda(z)} \text{ is quasi-invariant at } z = z_i, \forall z_i \in Z_\lambda \right\},$$

it follows from the above that the $\mathbb{C}$-linear span

$$\mathcal{U}_\lambda = \langle H_{\lambda,l} : l \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \dots, k_n\} \rangle$$

belongs to $\mathcal{Q}_\lambda$. From Proposition 5.3 in [18], we recall that the codimension of $\mathcal{U}_\lambda$ in $\mathbb{C}[z]$ is equal to $|\lambda|$. On the other hand, $|\lambda|$ is the degree of $W_\lambda(z)$, and therefore the number of quasi-invariance conditions that any $p \in \mathcal{Q}_\lambda$ should satisfy. This yields the converse inclusion, and thus the following result.

Proposition 3.1. *The $\mathbb{C}$-linear span of CEHPs coincides with polynomial quasi-invariants:*

$$\langle H_{\lambda,l} : l \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \dots, k_n\} \rangle = \mathcal{Q}_\lambda.$$

Whenever λ is not a double partition, the Wronskian $W_\lambda(z)$ will have one or more real zeros [1], so that the weight function is no longer non-singular on the real line. To resolve this problem, we replace the standard contour $\mathbb{R}$ by a shifted contour $C = i\xi + \mathbb{R}$ and consider a corresponding Hermitian product

$$\langle p, q \rangle := \int_C p(z)\bar{q}(z)w(z)dz.$$

where $\bar{q}(z) := \overline{q(\bar{z})}$ is the Schwarz conjugate of the polynomial $q(z)$, and $w(z)$ is a complex weight function. As will become clear below, to ensure that the product is Hermitian we need to restrict attention to the following subspace of quasi-invariant polynomials:

$$\mathcal{Q}_{\lambda,\mathbb{R}} = \left\{ p \in \mathbb{C}[z] : \psi(z) := p(z)\frac{e^{-z^2/2}}{W_\lambda(z)} \text{ is quasi-invariant at } z = z_i, \forall z_i \in Z_\lambda^{\mathbb{R}} \right\}.$$

By counting quasi-invariance conditions, we obtain the next proposition.

Proposition 3.2. *The codimension of $\mathcal{Q}_\lambda$ in $\mathcal{Q}_{\lambda,\mathbb{R}}$ is $|\lambda| - \sum_{z_i \in Z_\lambda^{\mathbb{R}}} m_i$.*

We are now ready for the main definition of this section.

Definition 3.1. Let $\xi \in \mathbb{R}$ be such that

$$0 < |\xi| < |\mathrm{Im}\ z_i|, \quad \forall z_i \in Z_\lambda \setminus Z_\lambda^{\mathbb{R}}. \tag{3.22}$$

Then, we define a sesquilinear product $\langle \cdot, \cdot \rangle$ on $\mathcal{Q}_{\lambda,\mathbb{R}}$ by setting

$$\langle p, q \rangle = \int_{i\xi+\mathbb{R}} p(z)\bar{q}(z)\frac{e^{-z^2}}{W_\lambda^2(z)}dz, \quad p, q \in \mathcal{Q}_{\lambda,\mathbb{R}}. \tag{3.23}$$

Now we will show that the product does not depend on the specific choice of ξ. We find it worth stressing that this important property relies on our restriction to the subspace $\mathcal{Q}_{\lambda,\mathbb{R}}$.

Proposition 3.3. *For any $p, q \in \mathcal{Q}_{\lambda,\mathbb{R}}$, the value of $\langle p, q \rangle$ is independent of $\xi \in \mathbb{R}$ provided the condition* (3.22) *is satisfied.*

Proof. Let I_ξ denote the integral in the right-hand side of (3.23). By Cauchy's theorem, it suffices to show that $I_\xi - I_{-\xi} = 0$ for some ξ satisfying (3.22). From the residue theorem, we deduce that the difference between the two integrals is proportional to

$$\sum_{z_i \in Z_\lambda^{\mathbb{R}}} \underset{z=z_i}{Res} \left(p(z)\bar{q}(z)\frac{e^{-z^2}}{W_\lambda^2(z)} \right).$$

We claim that each of these residues vanish. In fact, we have the following more general result.

Lemma 3.1. *If ψ, ϕ are quasi-invariant at $z = z_i$ with multiplicity m_i, then*

$$\underset{z=z_i}{Res} \big(\psi(z)\phi(z)\big) = 0.$$

Indeed, it follows from Condition (2) above that

$$\begin{aligned}
&\big(\psi(z)\phi(z)(z-z_i)^{2m_i}\big)^{(2m_i-1)}\big|_{z=z_i} \\
&\quad = \sum_{j=0}^{2m_i-1} \binom{2m_i-1}{j} \big(\psi(z)(z-z_i)^{m_i}\big)^{(2m_i-1-j)}\big|_{z=z_i} \big(\phi(z)(z-z_i)^{m_i}\big)^{(j)}\big|_{z=z_i} \\
&\quad = 0.
\end{aligned}$$

□

It is now straightforward to show that Definition 3.1 yields a Hermitian product.

Proposition 3.4. *The sesquilinear product $\langle\cdot,\cdot\rangle$ is Hermitian:*

$$\langle p, q\rangle = \overline{\langle q, p\rangle}, \quad \forall p, q \in \mathcal{Q}_{\lambda,\mathbb{R}}.$$

Proof. In what follows, we find it convenient to use the notation

$$w(z) = \frac{e^{-z^2}}{W_\lambda^2(z)},$$

and use a subscript to indicate the choice of ξ in (3.23). Since the classical Hermite polynomials have real coefficients, it is evident from (3.8) that $\bar{w}(z) = w(z)$. Hence, we have the following equalities:

$$\begin{aligned}\langle p, q\rangle_\xi &= \int_{\mathbb{R}} p(i\xi + x)\bar{q}(i\xi + x)w(i\xi + x)dx \\ &= \overline{\int_{\mathbb{R}} \bar{p}(-i\xi + x)q(-i\xi + x)w(-i\xi + x)dx} \\ &= \overline{\langle q, p\rangle_{-\xi}}.\end{aligned}$$

Combined with Proposition 3.3, this yields the asserted Hermiticity property. □

We recall that the classical Hermite polynomials $H_l(x)$ satisfy the orthogonality relation

$$\int_{\mathbb{R}} H_j(x)H_l(x)e^{-x^2}dx = \delta_{jl}2^l l!\sqrt{\pi}, \quad j, l \in \mathbb{Z}_{\geq 0}. \tag{3.24}$$

Combining this fact with the factorisation (3.19) of the intertwining operator $\mathscr{D}_\lambda$, it is now readily established by induction on the length n of λ that the CEHPs $H_{\lambda,l}(x)$ are orthogonal with respect to the Hermitian form $\langle\cdot,\cdot\rangle$ (cf. [18]).

Theorem 3.1. *The CEHPs $H_{\lambda,l}$ satisfy the orthogonality relation*

$$\langle H_{\lambda,j}, H_{\lambda,l}\rangle = \delta_{jl}\sqrt{\pi}2^l l! \prod_{m=1}^{n} 2(l - k_m), \quad j, l \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \ldots, k_n\}. \tag{3.25}$$

Proof. The assertion clearly holds true for $n = 0$, with the empty product taken to be equal to one. Introducing the partition

$$\hat{\lambda} = (\lambda_2, \ldots, \lambda_n),$$

we have

$$\langle H_{\lambda,j}, H_{\lambda,l}\rangle = \int_{i\xi+\mathbb{R}} (D_1\psi_{\hat{\lambda},j})(z)(\overline{D_1\psi_{\hat{\lambda},l}})(z)dz.$$

Since $\overline{\mathscr{W}_m} = \mathscr{W}_m$, the (formal) adjoint of D_1 is given by

$$D_1^* = -\frac{d}{dx} - \frac{d}{dx}\left(\log \frac{\mathscr{W}_1}{\mathscr{W}_2}\right).$$

The factorisation

$$D_1^* D_1 = \mathscr{L}_{\hat{\lambda}} - 2k_1 - 1$$

thus entails that

$$\langle H_{\lambda,j}, H_{\lambda,l}\rangle = 2(l-k_1)\langle H_{\hat{\lambda},j}, H_{\hat{\lambda},l}\rangle.$$

This completes the induction step, and the theorem is proved. □

Remark 3.1. *Since $\langle\cdot,\cdot\rangle$ is Hermitian, each squared norm $\langle p,p\rangle$, $p \in \mathcal{Q}_{\lambda,\mathbb{R}}$, is real, but need not be positive. In fact, if the partition is not double, there is always a finite number of polynomials with negative squared norm, which can be easily identified using formula (3.25). For example, setting $\lambda = (1)$ in (3.25), we see that $\langle H_{(1),l}, H_{(1),l}\rangle < 0$ if and only if $l = 0$. Grinevich and Novikov [20] pointed out a similar fact in a finite-gap case.*

We conclude this chapter by showing that the linear span of the CEHPs $H_{\lambda,l}$, $l \in \mathbb{Z}\setminus\{k_1,\dots,k_n\}$, is dense in $\mathcal{Q}_{\lambda,\mathbb{R}}$ in the sense that

$$\langle p, H_{\lambda,l}\rangle = 0, \ \forall l \in \mathbb{Z}_{\geq 0}\setminus\{k_1,\dots,k_n\} \implies p \equiv 0.$$

By Proposition 3.1, we can formulate the result as follows.

Theorem 3.2. *The subspace $\mathcal{Q}_\lambda$ is dense in $\mathcal{Q}_{\lambda,\mathbb{R}}$.*

Proof. Suppose that $p \in \mathcal{Q}_{\lambda,\mathbb{R}}$ is such that

$$\langle p, q\rangle = 0, \quad \forall q \in \mathcal{Q}_\lambda.$$

Introducing the polynomials

$$q_{\lambda,l}(z) = W_\lambda^2(z) H_l(z), \quad l \in \mathbb{Z}_{\geq 0},$$

which clearly belong to the subspace $\mathcal{Q}_\lambda$, we obtain

$$0 = \langle p, q_{\lambda,l}\rangle = \int_{i\xi+\mathbb{R}} p(z)\bar{H}_l(z) e^{-z^2} dz, \quad \forall l \in \mathbb{Z}_{\geq 0}.$$

Since the integrand is entire, we can take the limit $\xi \to 0$. Then expanding p in terms of the classical Hermite polynomials H_l, it follows immediately from (3.24) that $p \equiv 0$. □

Remark 3.2. *If we assume that* λ *is a double partition, then we recover orthogonality and completeness results from [18] (see Propositions 5.7–5.8). Indeed, to recover the former it is enough to note that the weight function* (3.10) *is guaranteed to be non-singular on the real line, so that we can take the limit* $\xi \to 0$ *in* (3.23)*; and the latter follows from the observation that we have* $\mathcal{Q}_{\lambda,\mathbb{R}} = \mathbb{C}[z]$.

Chapter 4

Exceptional Laurent orthogonal polynomials

In this chapter we generalise our approach to the space of Laurent polynomials $\Lambda = \mathbb{C}[z, z^{-1}]$ using the trigonometric monodromy-free Schrödinger operators [6], which play an important role in the theory of Huygens' principle [4].

More specifically, we consider the Laurent polynomials $P_{\kappa,a;l}$:

$$P_{\kappa,a;l}(z) = \begin{vmatrix} \Phi_{k_1}(a_1;z) & \Phi_{k_2}(a_2;z) & \cdots & \Phi_{k_n}(a_n;z) & z^l \\ D\Phi_{k_1}(a_1;z) & D\Phi_{k_2}(a_2;z) & \cdots & D\Phi_{k_n}(a_n;z) & Dz^l \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D^n\Phi_{k_1}(a_1;z) & D^n\Phi_{k_2}(a_2;z) & \cdots & D^n\Phi_{k_n}(a_n;z) & D^n z^l \end{vmatrix}, \tag{4.1}$$

where $\Phi_k(a;z) = az^k + a^{-1}z^{-k}$, $k \in \mathbb{N}$ and $D = z\frac{d}{dz}$. Due to the results of Theorem 4.1 and Proposition 4.2 we call $P_{\kappa,a;l}$, $l \in \mathbb{Z}$, exceptional Laurent orthogonal polynomials (ELOPs).

4.1 The general case

In this first section we allow any choice of complex parameters $a = (a_1, \ldots, a_n)$, $a_k \in \mathbb{C} \setminus \{0\}$.

We start from the elementary fact that the exponential functions

$$e_l(x) = \exp(ilx), \quad l \in \mathbb{Z},$$

have the eigenfunction property

$$\mathscr{L} e_l \equiv -\frac{d^2 e_l}{dx^2} = l^2 e_l, \quad x \in \mathbb{C}/2\pi\mathbb{Z}.$$

Note that, instead of the usual unit circle $\mathbb{R}/2\pi\mathbb{Z}$, we consider its complex version: the cylinder $\mathbb{C}/2\pi\mathbb{Z}$. This is natural from the trivial mondromy point of view, see [6].

Sequences of Darboux transformations at the levels $0 < k_n < k_{n-1} < \cdots < k_1$ are now parametrised by n complex parameters $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_k \in \mathbb{C}$. Specifically, introducing the functions

$$\phi_{k_j}(\theta_j, x) = 2\cos(k_j x + \theta_j), \quad j = 1, \ldots, n, \tag{4.2}$$

the resulting Schrödinger operator takes the form

$$\mathscr{L}_\kappa = -\frac{d^2}{dx^2} - 2\frac{d^2}{dx^2}\big(\log \mathrm{Wr}(\phi_{k_1}, \ldots, \phi_{k_n})\big), \tag{4.3}$$

where $\kappa = \{k_1, \ldots, k_n\}$. Furthermore, letting $\mathscr{D}_\kappa$ act by

$$\mathscr{D}_\kappa \phi = \frac{\mathrm{Wr}(\phi, \phi_{k_1}, \ldots, \phi_{k_n})}{\mathrm{Wr}(\phi_{k_1}, \ldots, \phi_{k_n})},$$

the intertwining relation (3.18) holds true, and the functions

$$\phi_{\kappa,\theta;l} = \frac{\mathrm{Wr}(e_l, \phi_{k_1}, \ldots, \phi_{k_n})}{\mathrm{Wr}(\phi_{k_1}, \ldots, \phi_{k_n})}, \quad l \in \mathbb{Z}, \tag{4.4}$$

have the eigenfunction property

$$\mathscr{L}_\kappa \phi_{\kappa,l} = l^2 \phi_{\kappa,l}.$$

We note that at each level k_j, $j = 1, \ldots, n$, the multiplicity is reduced from two to one. Indeed, by (4.2)–(4.4) and linearity of the Wronskian, we have the relation

$$\exp(i\theta_j)\phi_{\kappa,k_j}(\theta_j; x) + \exp(-i\theta_j)\phi_{\kappa,-k_j}(\theta_j; x) \equiv 0, \quad j = 1, \ldots, n.$$

To establish the precise connection between the functions $\phi_{\kappa,l}$ and the ELOPs $P_{\kappa,a;l}$ given by (4.1), we change variable to

$$z = \exp(ix)$$

and fix the values of the parameters $a = (a_1, \ldots, a_n)$ according to

$$a_k = \exp(i\theta_k) \in \mathbb{C} \setminus \{0\}, \quad k = 1, \ldots, n.$$

Then, it is readily seen that

$$\phi_{\kappa,\theta;l}(\theta, x) = P_{\kappa,a;l}(z)\mathcal{W}_{\kappa,a}(z)^{-1},$$

with $P_{\kappa,a;l}(z)$ given by (4.1) and

$$\mathcal{W}_{\kappa,a}(z) = \begin{vmatrix} \Phi_{k_1}(a_1; z) & \Phi_{k_2}(a_2; z) & \cdots & \Phi_{k_n}(a_n; z) \\ D\Phi_{k_1}(a_1; z) & D\Phi_{k_2}(a_2; z) & \cdots & D\Phi_{k_n}(a_n; z) \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}\Phi_{k_1}(a_1; z) & D^{n-1}\Phi_{k_2}(a_2; z) & \cdots & D^{n-1}\Phi_{k_n}(a_n; z) \end{vmatrix}, \tag{4.5}$$

where $D = zd/dz$ and

$$\Phi_k(a; z) = az^k + a^{-1}z^{-k}.$$

Furthermore, a direct computation reveals that $P_{\kappa,a;l}$ is an eigenfunction of the operator

$$T_\kappa = -D^2 + 2\frac{D\mathcal{W}_{\kappa,a}}{\mathcal{W}_{\kappa,a}}D - \frac{D^2\mathcal{W}_{\kappa,a}}{\mathcal{W}_{\kappa,a}}$$

with eigenvalue l^2.

Example 4.1. In the particular case $\kappa = \{1\}$ the corresponding Schrödinger operator is given by

$$\begin{aligned} \mathscr{L}_{\{1\}} &= -\frac{d^2}{dx^2} - 2\frac{d^2}{dx^2}\big(\log(2\cos(x + \theta_1))\big) \\ &= -\frac{d^2}{dx^2} + \frac{2}{\cos^2(x + \theta_1)}. \end{aligned}$$

When expressed in terms of the variable z and the parameter a_1, the first few exceptional Laurent polynomials $P_{\{1\},a;l}$ defined by (4.1) and the corresponding eigenfunctions $\Phi_{\{1\},a;l} = P_{\{1\},a;l}/\Phi_1$, $l \in \mathbb{Z}$ are given by

$$
\begin{aligned}
P_{\{1\},a;0} &= a_1 z - a_1^{-1} z^{-1}, & \Phi_{\{1\},a;0} &= \tfrac{a_1 z - a_1^{-1} z^{-1}}{a_1 z + a_1^{-1} z^{-1}},\\
P_{\{1\},a;-1} &= 2a_1, & \Phi_{\{1\},a;-1} &= \tfrac{2a_1}{a_1 z + a_1^{-1} z^{-1}},\\
P_{\{1\},a;1} &= -2a_1^{-1}, & \Phi_{\{1\},a;1} &= -\tfrac{2a_1^{-1}}{a_1 z + a_1^{-1} z^{-1}},\\
P_{\{1\},a;-2} &= a_1^{-1} z^{-3} + 3a_1 z^{-1}, & \Phi_{\{1\},a;-2} &= \tfrac{a_1^{-1} z^{-3} + 3a_1 z^{-1}}{a_1 z + a_1^{-1} z^{-1}},\\
P_{\{1\},a;2} &= a_1 z^3 + 3a_1^{-1} z, & \Phi_{\{1\},a;2} &= -\tfrac{a_1 z^3 + 3a_1^{-1} z}{a_1 z + a_1^{-1} z^{-1}}.
\end{aligned}
$$

From these explicit formulae, it is manifest that both $P_{\{1\},a;\pm 1}$ and $\Phi_{\{1\},a;\pm 1}$ are linearly dependent and that each eigenfunction is singular at $z = \pm i/a_1$. For general $l \in \mathbb{Z}$, the latter fact can be easily seen from the definition of $P_{\{1\},a;l}$.

We note that, upon setting

$$\mathscr{W}_m = \mathrm{Wr}(\phi_{k_m}, \dots, \phi_{k_n}),$$

the intertwining operator $\mathscr{D}_\kappa$ factorises according to (3.19)–(3.20). Just as in the Hermite case, it follows that each Schrödinger operator $\mathscr{L}_\kappa$ has trivial monodromy. Moreover, every monodromy-free trigonometric Schrödinger operator is of the form (4.3), see [6].

Let Z_κ be the set of zeros $z_i \in \mathbb{C}$ of the function $\mathcal{W}_{\kappa,a}(z)$ with multiplicities $m_i \in \mathbb{N}$ and X_κ be the corresponding set consisting of x_j such that $\exp(ix_j) = z_j,\ z_j \in Z_\kappa$ (we drop the dependence on a in the notations for brevity in the rest of this section).

Introduce the subspace

$$\mathscr{Q}_\kappa = \left\{ P \in \Lambda : \Phi(x) := (P/\mathcal{W}_\kappa)(\exp(ix)) \text{ is quasi-invariant at all } x_j \in X_\kappa \right\}.$$

It follows from trivial monodromy property that

$$\mathcal{U}_\kappa := \langle P_{\kappa,l} : l \in \mathbb{Z} \rangle \subset \mathscr{Q}_\kappa.$$

However, in contrast to Hermite case (see Proposition 3.1), the converse inclusion does not hold. Instead, we have the following result.

Proposition 4.1. *The codimension of $\mathcal{U}_\kappa$ in $\mathcal{Q}_\kappa$ is n.*

Proof. From (4.1), we deduce that

$$P_{\kappa,l}(z) = z^{l+|\kappa|} \det V(l, k_1, \dots, k_n) \prod_{j=1}^{n} k_j + \text{l.d.},$$

where

$$|\kappa| = \sum_{i=1}^{n} k_i,$$

l.d. stands for terms of lower degree and V is the Vandermonde matrix

$$V(\alpha_1, \dots, \alpha_m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_m^{m-1} \end{bmatrix}.$$

Since $\det V(l, k_1, \dots, k_n) = 0$ if and only if $l = k_1, \dots, k_n$, it follows that the degree sequence

$$I_\kappa^+ = \{\deg P(z) : P \in \mathcal{U}_\kappa\}$$

stabilises at $k_1 + |\kappa| + 1$ in the sense that $l \in I_\kappa^+$ for all $l \geq k_1 + |\kappa| + 1$. Applying the same line of reasoning to the Laurent polynomials $P_{\kappa,-l}(1/z)$, we find that the same statement holds true for

$$I_\kappa^- = \{\deg P(z^{-1}) : P \in \mathcal{U}_\kappa\}.$$

Among the ELOPs $P_{\kappa,l}$ with $|l| < k_1 + |\kappa| + 1$, a maximal set of linearly independent Laurent polynomials is given by

$$l \in \{k_1, \dots, k_n\} \cup \{0, \pm 1, \dots, \pm(k_n - 1)\} \cup \{\pm(k_n + 1), \dots, \pm(k_{n-1} - 1)\} \\ \cup \cdots \cup \{\pm(k_2 + 1), \dots, \pm(k_1 - 1)\}.$$

The cardinality of this index set equals

$$n + 2k_n - 1 + 2(k_{n-1} - k_n - 1) + \cdots + 2(k_1 - k_2 - 1) = 2k_1 - n + 1.$$

Observing that

$$2k_1 + 2|\kappa| + 1 - (2k_1 - n + 1) = 2|\kappa| + n,$$

we conclude that the codimension of $\mathcal{U}_\kappa$ in Λ is $2|\kappa| + n$.

On the other hand, counting quasi-invariance conditions, we find that the codimension of $\mathcal{Q}_\kappa$ in Λ equals $2|\kappa|$ and so the assertion follows. $\square$

Remark 4.1. *In contrast to the case of usual polynomials there are several definitions of the degree of a Laurent polynomial, but none of them are convenient for our purposes. Let us define the L-degree* $Ldeg P$ *of a Laurent polynomial* $P = \sum_{i=p}^{q} c_i z^i$ *with* $c_p \neq 0, c_q \neq 0$ *as* q *if* $q > -p$*, and* p *if* $q < -p$*. If* $q = -p$ *the L-degree is not well-defined since it could be both* p *and* q*. Under these assumptions*

$$Ldeg\, P_{\kappa,l} = |\kappa| + l,\ l \in \mathbb{Z}_+ \setminus \kappa, \qquad Ldeg\, P_{\kappa,l} = -|\kappa| + l,\ -l \in \mathbb{Z}_+ \setminus \kappa,$$

otherwise it is not well-defined. Note that the polynomials P_{κ,k_j} *and* $P_{\kappa,-k_j}$ *with undefined L-degrees are linearly dependent.*

Next, we consider a particular complex bilinear form on $\mathcal{Q}_\kappa$, given by

$$(P, Q) = \frac{1}{2\pi i} \oint_{C_\mu} P(z)Q(z)w(z)\frac{dz}{z}, \tag{4.6}$$

where C_μ is the circle defined by $|z| = \mu$ and $w(z) = W(z)^{-2}$, with $W(z) = \mathcal{W}_\kappa$. We establish the corresponding Laurent orthogonality relations. A related Fourier theory for more general algebro-geometric operators was studied by Grinevich and Novikov in [20].

Definition 4.1. Let $\mu \in \mathbb{R}_{>0}$ be such that

$$\mu \neq |z_i|, \quad \forall z_i \in Z_\kappa. \tag{4.7}$$

Then, we define a complex bilinear form $(\cdot, \cdot)$ on $\mathcal{Q}_\kappa$ by setting

$$(P, Q) = \frac{1}{2\pi i} \oint_{C_\mu} P(z)Q(z)\mathcal{W}_\kappa^{-2}\frac{dz}{z}, \quad P, Q \in \mathcal{Q}_\kappa, \tag{4.8}$$

where

$$C_\mu = \{z \in \mathbb{C} : |z| = \mu\}. \tag{4.9}$$

Substituting $z = \exp(ix)$ and following the line of reasoning used in the proof of Lemma 3.3, we readily find that the product is well-defined in the sense that it does not depend on the choice of μ. More precisely, we have the following lemma.

Lemma 4.1. *For any $P, Q \in \mathscr{Q}_\kappa$, the value of (P, Q) is independent of $\mu \in \mathbb{R}_{>0}$ provided (4.7) is satisfied.*

We are now ready to state and prove the first of the main results in this section, which may be viewed as a natural analogue of Theorem 3.1.

Theorem 4.1. *The ELOPs $P_{\kappa,l}$ satisfy the Laurent orthogonality relation*

$$(P_{\kappa,j}, P_{\kappa,l}) = \delta_{j+l,0} \prod_{m=1}^{n} (l^2 - k_m^2), \quad j, l \in \mathbb{Z}.$$

Proof. Just as in the proof of Theorem 3.1, we note that the assertion holds true for $n = 0$, and proceed by induction on the length n of κ. Letting $\hat{\kappa} = (k_2, \ldots, k_n)$, we have

$$(P_{\kappa,j}, P_{\kappa,l}) = \frac{1}{2\pi} \int_0^{2\pi} (D_1 \phi_{\hat{\kappa},j})(x)(D_1 \phi_{\hat{\kappa},l})(x) dx.$$

Making use of the factorisation

$$D_1^* D_1 = \mathscr{L}_{\hat{\kappa}} - k_1^2, \tag{4.10}$$

with

$$D_1^* = -\frac{d}{dx} - \frac{d}{dx}\left(\log \frac{\mathscr{W}_1}{\mathscr{W}_2}\right)$$

the (formal) adjoint of D_1, we deduce

$$(P_{\kappa,j}, P_{\kappa,l}) = (l^2 - k_1^2)(P_{\hat{\kappa},j}, P_{\hat{\kappa},l}),$$

which completes the induction step. ☐

Remark 4.2. *Having started from an eigenvalue problem with doubly degenerate eigenvalues, we have that $(P_{\kappa,l}, P_{\kappa,-l}) = 0$ for some of the ELOPs $P_{\kappa,l}$. More specifically, it is evident from the theorem that this is the case if and only if $l = \pm k_m$, $m = 1, \ldots, n$.*

Expanding on the result of Proposition 4.1, we proceed to establish the precise relationship between $\mathcal{U}_\kappa$ and $\mathscr{Q}_\kappa$. We begin with a general definition.

Let V be a vector space over $\mathbb{C}$. Then V is called *complex Euclidean space* if it is equipped with a non-degenerate bilinear form $B : V \otimes V \to \mathbb{C}$.

Definition 4.2. Let $W \subset V$ be a subspace of complex Euclidean space V. We say that V is a minimal complex Euclidean extension of W if

$$\dim \big(\ker B|_W\big) = \mathrm{codim}_V W.$$

For any linear space W and bilinear form B with non-trivial kernel

$$K := \ker B,$$

it is readily verified that there is a unique (up to isomorphisms) minimal complex Euclidean extension $V \supset W$. Letting K^* denote the dual space of K, it can be realised as follows:

$$V = K \oplus K^* \oplus W/K,$$

with the extension of B determined by

$$(k_1 + \hat{k}_1 + w_1, k_2 + \hat{k}_2 + w_2) \mapsto \hat{k}_2(k_1) + \hat{k}_1(k_2) + B(w_1, w_2),$$

where $k_1, k_2 \in K$, $\hat{k}_1, \hat{k}_2 \in K^*$ and $w_1, w_2 \in W$. Moreover, for each basis $k_1, \dots, k_n \in K$, there is a unique basis $\hat{k}_1, \dots, \hat{k}_n \in K^*$ such that $(k_j, \hat{k}_l) = \delta_{jl}$.

Example 4.2. Suppose that $B|_W = 0$, so that each vector $w \in W$ is isotropic. Then we have

$$V \cong W \oplus W^*,$$

with

$$B(w_1 + \hat{w}_1, w_2 + \hat{w}_2) = \hat{w}_2(w_1) + \hat{w}_1(w_2), \quad w_1, w_2 \in W, \quad \hat{w}_1, \hat{w}_2 \in W^*.$$

As demonstrated by the following proposition, the inclusion $\mathcal{U}_\kappa \subset \mathscr{Q}_\kappa$ provides a concrete example of a minimal complex Euclidean extension in the sense of Definition 4.2.

Proposition 4.2. *$\mathscr{Q}_\kappa$ is the minimal complex Euclidean extension of $\mathcal{U}_\kappa$.*

Proof. From Theorem 4.1 we infer that

$$\ker(\cdot,\cdot)|_{\mathcal{U}_\kappa} = \langle P_{\kappa,k_j} : j = 1, \dots, n \rangle.$$

(Note the linear relations $a_j P_{\kappa,k_j} + a_j^{-1} P_{\kappa,-k_j} = 0$.) Since ELOPs $P_{\kappa,l}$ corresponding to different values of l^2, and hence different eigenvalues, are linearly independent, it follows that

$$\dim\left(\ker(\cdot,\cdot)|_{\mathcal{U}_\kappa}\right) = n.$$

Recalling Proposition 4.1, we see that it remains only to verify that $(\cdot,\cdot)$ is non-degenerate on $\mathscr{Q}_\kappa$. Observing that

$$\mathcal{W}_\kappa^2(z) z^j \in \mathscr{Q}_\kappa, \quad \forall j \in \mathbb{Z},$$

this follows, e.g., from the computation

$$\begin{aligned}\left(P_{\kappa,l}, \mathcal{W}_\kappa^2(z) z^{-l-|\kappa|}\right) &= \frac{1}{2\pi i}\oint_{C_\mu} P_{\kappa,l}(z) z^{-l-|\kappa|}\frac{dz}{z} \\ &= \det V(l, k_1, \dots, k_n)\prod_{j=1}^n k_j,\end{aligned}$$

which is non-zero as long as $l \neq \pm k_j$, cf. the proof of Proposition 4.1. □

4.2 The Hermitian case

In the case when all θ_k are real or, equivalently, when parameters $a = (a_1, \dots, a_n)$ satisfy

$$|a_k| = 1, \quad k = 1, \dots, n, \tag{4.11}$$

we can introduce the Hermitian structure as follows.

Note that in this case the weight function $w(z) = \mathcal{W}_\kappa(z)^{-2}$ is invariant under the antilinear involution

$$P^\dagger(z) := \overline{P(1/\bar{z})}, \quad P \in \Lambda, \tag{4.12}$$

which will play much the same role as the Schwartz conjugate did in the Hermite case. In fact, observing that $(DP)^\dagger = -DP^\dagger$ and that $\Phi_k^\dagger = \Phi_k$, we can deduce from (4.5) that

$$\mathcal{W}_\kappa^\dagger(z) = (-1)^{n(n-1)/2}\mathcal{W}_\kappa(z), \quad \kappa = \{k_1, \dots, k_n\}. \tag{4.13}$$

In addition, the zero set Z_κ becomes invariant under the involution $z \to 1/\bar{z}$, i.e.

$$z_i \in Z_\kappa \implies 1/\bar{z}_i \in Z_\kappa,$$

and, since $z = 1/\bar{z}$ whenever $|z| = 1$, we have that

$$\mathcal{W}_\kappa(z)\mathcal{W}_\kappa^\dagger(z) = |\mathcal{W}_\kappa|^2, \quad |z| = 1. \tag{4.14}$$

Letting $Z_\kappa^C = \{z_i \in Z_\kappa : |z| = 1\}$ and $X_\kappa^\mathbb{R} = \{x_j : \exp(ix_j) = z_j,\ z_j \in Z_\kappa^C\} \subset \mathbb{R}$, we introduce the following subspace of quasi-invariant Laurent polynomials:

$$\mathscr{Q}_{\kappa,C} = \left\{P \in \Lambda : \Phi(x) := (P/\mathcal{W}_\kappa)(\exp(ix)) \text{ is quasi-invariant at all } \ x_j \in X_\kappa^\mathbb{R}\right\}.$$

From (4.13), it is straightforward to infer that

$$\mathscr{Q}_\kappa^\dagger = \mathscr{Q}_\kappa, \quad \mathscr{Q}_{\kappa,C}^\dagger = \mathscr{Q}_{\kappa,C},$$

which allows us to define a natural sesquilinear product on $\mathscr{Q}_{\kappa,C}$.

Definition 4.3. Assuming that (4.11) holds true, we introduce

$$\nu = \min_{\substack{z_i \in Z_\kappa \\ |z_i| > 1}} |z_i|,$$

and let $\mu \in \mathbb{R}_{>0}$ be such that

$$1 < \max(\mu, 1/\mu) < \nu. \tag{4.15}$$

Then, we define a sesquilinear product $\langle\cdot,\cdot\rangle_L$ on $\mathscr{Q}_{\kappa,C}$ by setting

$$\langle P, Q\rangle_L = \frac{1}{2\pi i}\oint_{C_\mu} P(z)Q^\dagger(z)\big(\mathcal{W}_\kappa(z)\mathcal{W}_\kappa^\dagger(z)\big)^{-1}\frac{dz}{z}, \quad P, Q \in \mathscr{Q}_{\kappa,C}. \tag{4.16}$$

Again, the product does not depend on the specific choice of μ.

Lemma 4.2. *For any $P, Q \in \mathscr{Q}_{\kappa,C}$, the value of $(P, Q)_L$ is independent of $\mu \in \mathbb{R}_{>0}$ provided* (4.15) *is satisfied.*

By adapting the proof of Proposition 3.4, we can use the lemma to show that Definition 4.3 yields a Hermitian product.

Proposition 4.3. *The sesquilinear product $\langle\cdot,\cdot\rangle_L$ is Hermitian:*

$$\langle P, Q\rangle_L = \overline{\langle Q, P\rangle_L}, \quad \forall P, Q \in \mathscr{Q}_{\kappa,C}.$$

Proof. Using the notation

$$w(z) = 1/\mathcal{W}_\kappa(z)\mathcal{W}_\kappa^\dagger(z)$$

and using a subscript to indicate the choice of μ in (4.16), we deduce the following equalities:

$$\begin{aligned}\langle P, Q\rangle_{L,\mu} &= \frac{1}{2\pi}\int_0^{2\pi} P(\mu e^{i\varphi})\bar{Q}(\mu^{-1}e^{i\varphi})w(\mu e^{i\varphi})d\varphi \\ &= \overline{\frac{1}{2\pi}\int_0^{2\pi} \bar{P}(\mu e^{-i\varphi})Q(\mu^{-1}e^{i\varphi})w(\mu^{-1}e^{i\varphi})d\varphi} \\ &= \overline{\langle Q, P\rangle_{L,\mu^{-1}}},\end{aligned}$$

and so hermiticity follows from Lemma 4.2. □

Moreover, the proof of Theorem 4.1 is readily adapted to yield the following orthogonality result.

Theorem 4.2. *Assuming that* (4.11) *holds true, the ELOPs* $P_{\kappa,l}$ *satisfy the orthogonality relation*

$$\langle P_{\kappa,j}, P_{\kappa,l}\rangle_L = \delta_{jl}\prod_{m=1}^{n}(k_m^2 - l^2), \quad j, l \in \mathbb{Z}. \tag{4.17}$$

Proof. Taking $z = \exp(ix)$ in the integral in (4.16) and observing that (cf. (4.13))

$$\overline{\mathscr{W}_m}(-x) = (-1)^{(n-m)(n-m+1)/2}\mathscr{W}_m(x),$$

we establish the equalities

$$\begin{aligned}\langle P_{\kappa,j}, P_{\kappa,l}\rangle_L &= \frac{1}{2\pi}\int_0^{2\pi}(D_1\phi_{\hat{\kappa},j})(x)\overline{(D_1\phi_{\hat{\kappa},l})}(-x)dx \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\phi_{\hat{\kappa},j}(x)\overline{(D_1^*D_1\phi_{\hat{\kappa},l})}(-x)dx.\end{aligned}$$

Appealing to the factorisation (4.10), we thus obtain the relation

$$\langle P_{\kappa,j}, P_{\kappa,l}\rangle_L = (k_1^2 - l^2)\langle P_{\hat{\kappa},j}, P_{\hat{\kappa},l}\rangle_L,$$

and the assertion follows by induction on n. □

After replacing the bilinear form B by a Hermitian sesquilinear form h, Definition 4.2 as well as the succeeding discussion applies with minor changes also in the present situation. Specifically, we say that V is a *minimal Hermitian extension* of W if

$$\dim\big(\ker h|_W\big) = \mathrm{codim}_V W.$$

Then, we have the following analogue of Theorem 3.2.

Theorem 4.3. *The subspace $\mathscr{Q}_\kappa$, which is the minimal Hermitian extension of $\mathcal{U}_\kappa$, is dense in $\mathscr{Q}_{\kappa,C}$.*

Proof. Suppose that $P \in \mathscr{Q}_{\kappa,C}$ is such that

$$\langle P, Q\rangle_L = 0, \quad \forall Q \in \mathscr{Q}_\kappa.$$

Since the Laurent polynomials

$$Q_{\kappa,l} = \mathcal{W}_\kappa(z)\mathcal{W}_\kappa^\dagger(z)z^l, \quad l \in \mathbb{Z},$$

clearly are contained in $\mathscr{Q}_\kappa$, we have that

$$0 = \langle P, Q_{\kappa,l}\rangle_L = \frac{1}{2\pi i}\oint_{C_\mu} P(z)z^l\frac{dz}{z}, \quad \forall l \in \mathbb{Z}.$$

Taking the limit $\mu \to 1$ and using the property that

$$\frac{1}{2\pi i}\oint_{|z|=1} z^k z^l \frac{dz}{z} = \delta_{k+l,0}, \quad k, l \in \mathbb{Z}$$

we conclude that $P \equiv 0$. □

Remark 4.3. *It is known from the soliton theory that for every non-empty set κ and any choice of real θ_k the corresponding potential always has singularities on the real line. This means that in the Laurent case we do not have non-trivial regular examples (unlike the Hermite case with double partitions).*

Conclusion

In Part I, we studied the spectral properties of the complex Lamé operator

$$L = -\frac{d^2}{dx^2} + m(m+1)\omega^2\wp(\omega x + z_0), \quad z_0 \in \mathbb{C},$$

with ω one of the half-periods of $\wp(z)$. In particular, when ω is real and z_0 is any complex number such that L is regular for all real x, the spectrum is independent of z_0 and coincides with the classical self-adjoint case $z_0 = \omega_3$. In that case we showed that all closed gaps belong to the infinite spectral band for all $m \in \mathbb{N}$.

When $\omega = \omega_2$ we considered the first non-trivial case $m = 1$, and studied the geometry of the corresponding spectral arcs. In this case we also showed that all closed gaps belong to the infinite spectral arc. It remains an open problem whether the same is true for all $m \in \mathbb{N}$ and ω.

Another interesting direction for future research is the difference case, where we believe the answer will depend on the arithmetic properties of the shift $\eta = P/Q$. In the limit $Q \to \infty$ we should recover the previous results.

In Part II, we have discussed two complex versions of the exceptional orthogonal polynomials, related to two classes of monodromy-free Schrödinger operators. It would be interesting to explore whether a similar procedure could be applied to other classes of exceptional orthogonal polynomials, like the Jacobi and Laguerre families [25, 40, 32].

We would like to emphasize two novelties compared to the original approach of Gómez-Ullate et al. [19, 18].

First, in order to define the inner product in general we have to reduce the space of polynomials to the subspace of quasi-invariants, which has a finite codimension. The only exception is the Hermite case with double partitions considered in [18].

Second, in the Laurent case the space of quasi-invariants is not generated by the corresponding exceptional Laurent polynomials, so we need to consider the minimal complex Euclidean extension.

The scheme for generating extended recurrence relations of exceptional orthogonal polynomials from the three-term recurrence relations of the corresponding classical orthogonal polynomial families was established by Odake in [38]. This was adapted by Gomez-Ullate et al. for the exceptional Hermite polynomials [18]. It would be interesting to investigate whether the same method could be applied to the ELPQs, as well as any subsequently found exceptional orthogonal polynomials with quasi-invariance, to develop a full picture of their characterisation.

In the rational case with sextic growth at infinity there are some partial results [17], which lead to finite sets of orthogonal polynomials of the same degree. It would be interesting to analyse this situation in the view of a very interesting recent paper by Felder and Willwacher [14].

It would be interesting also to see what happens with exceptional orthogonal polynomials in the multidimensional case. One can use the monodromy-free generalised Calogero-Moser operators, playing an important role in the theory of Huygens principle [6].

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