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# Towards the classification of homogeneous third-order Hamiltonian operators

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Let $V$ be a vector space of dimension $n+1$. We demonstrate that $n$-component third-order Hamiltonian operators of differential-geometric type are parametrised by the algebraic variety of elements of rank $n$ in $S^2(\Lambda^2 V)$ that lie in the kernel of the natural map $S^2(\Lambda^2 V) \to \Lambda^4 V$. Non-equivalent operators correspond to different orbits of the natural action of $SL(n+1)$. Based on this result, we obtain a classification of such operators for $n \leq 4$.

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## Contents



## 1 Introduction

In this paper we discuss homogeneous third-order Hamiltonian operators of differential-geometric type,

$$P = g^{ij}D^3 + b_k^{ij}u_x^k D^2 + (c_k^{ij}u_{xx}^k + c_{km}^{ij}u_x^k u_x^m)D + d_k^{ij}u_{xxx}^k + d_{km}^{ij}u_{xx}^k u_x^m + d_{kmn}^{ij}u_x^k u_x^m u_x^n. \quad (1)$$

Here $\mathbf{u} = (u^1, \dots, u^n)$ are the dependent variables, and all coefficients are functions of $\mathbf{u}$; $D = \frac{d}{dx}$. The operator $P$ is Hamiltonian if and only if the corresponding Poisson bracket,

$$\{F, G\} = \int \frac{\delta F}{\delta u} P \frac{\delta G}{\delta u} dx,$$

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is skew-symmetric and satisfies the Jacobi identity. Operators of type (1) were introduced by Dubrovin and Novikov in [6, 7], and thoroughly investigated by Potemin [24, 23], Doyle [5], Balandin and Potemin [3]. We will only consider the non-degenerate case, $\det g^{ij} \neq 0$. Under point transformations, $\mathbf{u} = \mathbf{u}(\tilde{\mathbf{u}})$, the coefficients of (1) transform as differential-geometric objects. Thus, $g^{ij}$ transforms as a $(2,0)$-tensor, so that its inverse $g_{ij}$ defines a pseudo-Riemannian metric, the expressions $-\frac{1}{3}g_{js}b^{si}_k$, $-\frac{1}{3}g_{js}c^{si}_k$, $-g_{js}d^{si}_k$ transform as Christoffel symbols of affine connections, etc. In particular, the last connection, $\Gamma^i_{jk} = -g_{js}d^{si}_k$, must be symmetric and flat [21, 5, 24]. Therefore, there exists a distinguished coordinate system (flat coordinates) such that $\Gamma^i_{jk}$ vanish. Flat coordinates are determined up to affine transformations. We will keep for them the same notation $u^i$, note that $u^i$ are nothing but the densities of Casimirs. In the flat coordinates the last three terms in (1) vanish, leading to a simplified expression [5],

$$P = D\left(g^{ij}D + c^{ij}_k u^k_x\right)D. \tag{2}$$

This operator is Hamiltonian if and only if the metric $g_{ij}$ with lower indices and the objects $c_{ijk} = g_{iq}g_{jp}c^{pq}_k$ satisfy the relations [23]:

$$g_{mn,k} = -c_{mnk} - c_{nmk}, \tag{3a}$$

$$c_{mnk} = -c_{mkn}, \tag{3b}$$

$$c_{mnk} + c_{nkm} + c_{kmn} = 0, \tag{3c}$$

$$c_{mnk,l} = -g^{pq}c_{pml}c_{qnk}. \tag{3d}$$

It was observed in [12] that equations (3) can be rewritten in terms of the metric $g$ alone: first of all, system (3) implies $c_{nkm} = \frac{1}{3}(g_{nm,k} - g_{nk,m}) = \frac{1}{3}g_{n[m,k]}$, and the elimination of $c$ results in

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0, \tag{4}$$

$$g_{m[k,n]l} = -\tfrac{1}{3}g^{pq}g_{p[l,m]}g_{q[k,n]}. \tag{5}$$

The second observation of [12] is that the generic metric $g = g_{ij}du^i du^j$ satisfying linear subsystem (4) is a quadratic form in $du^i$ and $u^j du^k - u^k du^j$, explicitly,

$$g_{ij}du^i du^j = a_{ij}du^i du^j + b_{ijk}du^i(u^j du^k - u^k du^j) + c_{ijkl}(u^i du^j - u^j du^i)(u^k du^l - u^l du^k), \tag{6}$$

where $a_{ij}$, $b_{ijk}$, $c_{ijkl}$ are arbitrary constants. Since flat coordinates are defined up to affine transformations, system (4), (5) is invariant under transformations of the form

$$\tilde{u}^i = l^i(\mathbf{u}), \quad \tilde{g} = g,$$

where $l^i$ are arbitrary linear forms in the flat coordinates, and $\tilde{g} = g$ indicates that $g$ transforms as a metric. What is less obvious is that system (4)-(5) is invariant under the bigger group of projective transformations,

$$\tilde{u}^i = \frac{l^i(\mathbf{u})}{l(\mathbf{u})}, \quad \tilde{g} = \frac{g}{l^4(\mathbf{u})},$$

where $l$ is yet another linear form in the flat coordinates. It was demonstrated in [12] that projective transformations correspond to reciprocal transformations of Hamiltonian operator (2). Note that the class of metrics (6), known in projective geometry as the Monge metrics of quadratic line complexes, is also invariant under projective transformations. Recall that a quadratic line complex is a $(2n-3)$-parameter family of lines in projective space $\mathbb{P}^n$ specified by a single quadratic equation in the Plücker coordinates. Fixing a point $p \in \mathbb{P}^n$ and taking all lines of the complex that pass through $p$ we obtain a quadratic cone with vertex at $p$. This field of cones supplies $\mathbb{P}^n$ with a conformal structure whose general form is given by (6). The key invariant of a quadratic line complex is its singular variety defined by the equation

$$\det g_{ij} = 0.$$

This is the locus where null cones of $g$ degenerate into a pair of hyperplanes; it is known to be a hypersurface in $\mathbb{P}^n$ of degree $2n-2$, see [4], Prop. 10.3.3. For $n = 2$ the singular variety is a conic in $\mathbb{P}^2$, for $n = 3$ it is the Kummer quartic in $\mathbb{P}^3$, for $n = 4$ the Segre sextic in $\mathbb{P}^4$, etc. It turns out that singular varieties of Monge metrics corresponding to homogeneous third-order Hamiltonian operators degenerate into double hypersurfaces of degree $n-1$ (equivalently, $\det g_{ij}$ is a complete square, see Theorem 4.1 of Section 4). The classification of 2- and 3-component operators can be summarised as follows.

**Two-component case [12]**: *Modulo projective transformations, every 2-component homogeneous third-order Hamiltonian operator can be reduced to constant form.*

**Three-component case [12]**: *Modulo (complex) projective transformations, the metric of every 3-component homogeneous third-order Hamiltonian operator can be reduced to one of the 6 canonical forms*:

$$g^{(1)}=\begin{pmatrix}(u^2)^2+c & -u^1u^2-u^3 & 2u^2\\ -u^1u^2-u^3 & (u^1)^2+c(u^3)^2 & -cu^2u^3-u^1\\ 2u^2 & -cu^2u^3-u^1 & c(u^2)^2+1\end{pmatrix},$$

$$g^{(2)}=\begin{pmatrix}(u^2)^2+1 & -u^1u^2-u^3 & 2u^2\\ -u^1u^2-u^3 & (u^1)^2 & -u^1\\ 2u^2 & -u^1 & 1\end{pmatrix},\quad g^{(3)}=\begin{pmatrix}(u^2)^2+1 & -u^1u^2 & 0\\ -u^1u^2 & (u^1)^2 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$g^{(4)}=\begin{pmatrix}-2u^2 & u^1 & 0\\ u^1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\quad g^{(5)}=\begin{pmatrix}-2u^2 & u^1 & 1\\ u^1 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},\quad g^{(6)}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The corresponding singular varieties, $\det g=0$, are as follows

- $g^{(1)},g^{(2)}$: double quadric;
- $g^{(3)},g^{(4)}$: two double planes, one of them at infinity;
- $g^{(5)},g^{(6)}$: quadruple plane at infinity.

Direct calculations demonstrate that the metrics $g^{(4)},g^{(5)},g^{(6)}$ are flat, while $g^{(1)},g^{(2)},g^{(3)}$ are not even conformally flat: they have non-vanishing Cotton tensor.

The structure of the paper is as follows. In Section 2 we discuss some new examples of Hamiltonian PDEs associated with third-order Hamiltonian operators. Our examples suggest that every operator (1) arises as a Hamiltonian structure of some linearly degenerate non-diagonalisable system of hydrodynamic type. In Section 3 we introduce the normal form of a quadratic complex in $\mathbb{P}^n$ that generalises the Clebsch normal form in $\mathbb{P}^3$. Theorem 4.2 of Section 4 gives a parametrisation of $n$-component third-order Hamiltonian operators by the algebraic variety of elements $Q\in S^2(\Lambda^2V)$ of rank $n$ that belong to the kernel of the natural map $S^2(\Lambda^2V)\to\Lambda^4V$. The classification results are summarised in Section 5. In particular, for $n=4$ we obtain 32 non-equivalent multi-parameter canonical forms.

All computations were performed with the software package CDE [26] of the REDUCE computer algebra system [25].

## 2 Examples

Third-order Hamiltonian operators arise in applications in the context of Monge-Ampère/WDVV equations of 2D topological field theory [9, 20, 16, 17, 12, 22]. In this Section we demonstrate that 3-component operators (2) associated with the metrics $g^{(1)}-g^{(5)}$ can be realised as Hamiltonian structures of certain linearly degenerate non-diagonalisable systems of hydrodynamic type. We emphasise that even though the corresponding Hamiltonian densities are nonlocal, the existence of *local* first-order systems with *local* third-order Hamiltonian structures is a non-trivial fact.

**Example 1: metric $g^{(1)}$.** The system

$$u^1_t=(\alpha u^2+\beta u^3)_x,$$

$$u^2_t=\left(\frac{((u^2)^2-c)(\alpha u^2+\beta u^3)+\gamma(1-c(u^2)^2)+\delta(u^1-cu^2u^3)}{u^1u^2-u^3}\right)_x,$$

$$u^3_t=\left(\frac{\alpha u^3((u^2)^2-c)+\beta u^3(u^2u^3-cu^1)+\gamma(u^1-cu^2u^3)+\delta((u^1)^2-c(u^3)^2)}{u^1u^2-u^3}\right)_x,$$

where $\alpha,\beta,\gamma,\delta$ are arbitrary constants, possesses third-order Hamiltonian structure (2) generated by the metric $g^{(1)}$ and the nonlocal Hamiltonian,

$$H=\int\Big(\tfrac12\alpha(2cxu^1D^{-1}u^2+u^3(D^{-1}u^2)^2+cx^2u^3)+\beta u^3(1-c^2)D^{-1}u^2D^{-1}u^3$$

$$+\delta(xu^1D^{-1}u^1+cu^3D^{-1}u^1D^{-1}u^2+cu^1D^{-1}u^2D^{-1}u^3+cxu^3D^{-1}u^3)$$

$$+\tfrac12\gamma(cu^1(D^{-1}u^2)^2+x^2u^1+2cxu^3D^{-1}u^2\Big)dx.$$

One can show that this system is linearly degenerate, and non-diagonalisable for generic values of parameters (the diagonalisability conditions are equivalent to $\alpha\delta-\beta\gamma=0$).

**Example 2: metric $g^{(2)}$.** The system

$$u^1_t=(\alpha u^2+\beta u^3)_x,$$

$$u^2_t=\left(\frac{((u^2)^2-1)(\alpha u^2+\beta u^3)-(\gamma+\delta u^1)}{u^1u^2-u^3}\right)_x,$$

$$u^3_t=\left(\frac{(u^2u^3-u^1)(\alpha u^2+\beta u^3)-u^1(\gamma+\delta u^1)}{u^1u^2-u^3}\right)_x,$$

where $\alpha,\beta,\gamma,\delta$ are arbitrary constants, possesses third-order Hamiltonian structure (2) generated by the metric $g^{(2)}$ and the nonlocal Hamiltonian,

$$H=\int\left(\frac{1}{2}\alpha u^3(D^{-1}u^2)^2+\beta u^3D^{-1}u^2D^{-1}u^3-\frac{1}{2}\gamma x^2u^1-\delta xu^1D^{-1}u^1\right)dx.$$

One can show that this system is linearly degenerate, and non-diagonalisable for generic values of parameters (the diagonalisability conditions are equivalent to $\alpha\delta-\beta\gamma=0$).

**Example 3: metric $g^{(3)}$.** The system

$$u^1_t=(u^2+u^3)_x,\quad u^2_t=\left(\frac{u^2(u^2+u^3)-1}{u^1}\right)_x,\quad u^3_t=u^1_x,$$

possesses third-order Hamiltonian structure (2) generated by the metric $g^{(3)}$ and the nonlocal Hamiltonian,

$$H=\int\left(-D^{-1}u^1D^{-1}u^3+xu^1D^{-1}u^2\right)dx.$$

Explicitly, $u_t=P\delta H/\delta u$ where

$$P=D\begin{pmatrix}D & D\frac{u^2}{u^1} & 0\\ \frac{u^2}{u^1}D & \frac{(u^2)^2+1}{2(u^1)^2}D+D\frac{(u^2)^2+1}{2(u^1)^2} & 0\\ 0 & 0 & D\end{pmatrix}D.$$

Setting $u^1=f_{xxt}$, $u^2=f_{xtt}-f_{xxx}$, $u^3=f_{xxx}$ we obtain $f^2_{xxt}-f_{xxx}f_{xtt}+f^2_{xtt}-f_{xxt}f_{ttt}-1=0$, which is a particular case of WDVV equation [8]; in the present form, it first appeared in [1]. This third-order Hamiltonian structure is apparently new.

**Example 4: metric $g^{(4)}$.** The system

$$u^1_t=u^2_x,\quad u^2_t=\left(\frac{(u^2)^2+u^3}{u^1}\right)_x,\quad u^3_t=u^1_x,$$

possesses third-order Hamiltonian structure (2) generated by the metric $g^{(4)}$ and the nonlocal Hamiltonian,

$$H=\int\left(u^2D^{-1}u^1D^{-1}u^2-D^{-1}u^1D^{-1}u^3\right)dx.$$

Explicitly, $u_t=P\delta H/\delta u$ where

$$P=D\begin{pmatrix}0 & D\frac{1}{u^1} & 0\\ \frac{1}{u^1}D & \frac{u^2}{(u^1)^2}D+D\frac{u^2}{(u^1)^2} & 0\\ 0 & 0 & D\end{pmatrix}D.$$

Setting $u^1=f_{xxt}$, $u^2=f_{xtt}$, $u^3=f_{xxx}$ we obtain $f_{xxx}=f_{ttt}f_{xxt}-f^2_{xtt}$, which is equivalent to the WDVV equation [8] under the interchange of $x$ and $t$. This third-order Hamiltonian representation was constructed in [16, 17].

**Example 5: metric $g^{(5)}$.** The system

$$u^1_t = u^2_x, \quad u^2_t = u^3_x, \quad u^3_t = ((u^2)^2 - u^1u^3)_x,$$

possesses third-order Hamiltonian structure (2) generated by the metric $g^{(5)}$ and the nonlocal Hamiltonian,

$$H = -\int \left( \frac{1}{2} u^1 \left( D^{-1}u^2 \right)^2 + D^{-1}u^2 D^{-1}u^3 \right) dx.$$

Explicitly, $u_t = P\delta H/\delta u$ where

$$P = D \begin{pmatrix} 0 & 0 & D \\ 0 & D & -Du^1 \\ D & -u^1 D & Du^2 + u^2 D + u^1 D u^1 \end{pmatrix} D.$$

Setting $u^1 = f_{xxx}$, $u^2 = f_{xxt}$, $u^3 = f_{xtt}$ we obtain $f_{ttt} = f^2_{xxt} - f_{xxx}f_{xtt}$, which is the simplest case of WDVV equations [8]. This third-order Hamiltonian representation was found in [9]. Note that although examples 4 and 5 are equivalent under the interchange of $x$ and $t$, the action of this elementary transformation on Hamiltonian structures is nontrivial, in particular, Hamiltonian operators from Examples 4 and 5 are not projectively-equivalent.

**Example 6.** A natural 4-component generalisation of Example 5 is the following system,

$$u^1_t = u^2_x, \quad u^2_t = u^3_x, \quad u^3_t = u^4_x, \quad u^4_t = ((u^2)^2 - u^1u^3)_x,$$

which possesses the Hamiltonian formulation $u_t = P\delta H/\delta u$ with the third-order Hamiltonian operator

$$P = D \begin{pmatrix} 0 & 0 & 0 & D \\ 0 & 0 & D & 0 \\ 0 & D & 0 & -Du^1 \\ D & 0 & -u^1 D & Du^2 + u^2 D \end{pmatrix} D,$$

and the nonlocal Hamiltonian,

$$H = -\int \left( \frac{1}{2} u^1 (D^{-1}u^2)^2 + D^{-1}u^2 D^{-1}u^4 + \frac{1}{2}(D^{-1}u^3)^2 \right).$$

Setting $u^1 = f_{xxxx}$, $u^2 = f_{xxxt}$, $u^3 = f_{xxtt}$, $u^4 = f_{xttt}$ we obtain a fourth-order Monge-Ampère equation, $f_{tttt} = f^2_{xxxt} - f_{xxxx}f_{xxtt}$. This example possesses a straightforward $n$-component generalisation,

$$u^1_t = u^2_x, \;\; u^2_t = u^3_x, \;\; \ldots, \; u^{n-1}_t = u^n_x, \;\; u^n_t = ((u^2)^2 - u^1u^3)_x,$$

with the Hamiltonian structure $u_t = P\delta H/\delta u$ where

$$P = D \begin{pmatrix} & & & D \\ & & D & \\ & \cdots & & 0 \\ & D & 0 & -Du^1 \\ D & & 0 & -u^1 D \quad Du^2 + u^2 D \end{pmatrix} D,$$

$$H = -\int \left( \frac{1}{2} u^1 (D^{-1}u^2)^2 + \frac{1}{2} \sum_{m=2}^{m} (D^{-1}u^m)(D^{-1}u^{n+2-m}) \right) dx.$$

Examples of this section make it tempting to conjecture that every third-order Hamiltonian operator (1) can be realised as a Hamiltonian structure of some linearly degenerate non-diagonalisable system of hydrodynamic type.

## 3 Canonical form of a quadratic line complex

In this section we introduce canonical form of a quadratic complex in $\mathbb{P}^n$ that can be viewed as a generalisation of the Clebsch normal form in $\mathbb{P}^3$. This form proves to be convenient for the characterisation of complexes that correspond to third-order Hamiltonian operators.

Let us recall the basics of the theory of quadratic complexes. Consider $n$-dimensional projective space $\mathbb{P}^n$ associated with $(n+1)$-dimensional vector space $V$. Given two points in $\mathbb{P}^n$ with homogeneous coordinates $u^i$ and $v^i$, $i = 1, \dots, n+1$, the Plücker coordinates $p^{ij}$ of the line through them are defined as $p^{ij} = u^i v^j - u^j v^i$ (equivalently, one can speak of Plücker coordinates of the corresponding 2-dimensional subspace in $V$). These coordinates satisfy a system of quadratic relations of the form $p^{ij}p^{kl} + p^{ki}p^{jl} + p^{jk}p^{il} = 0$ that define the Plücker embedding of the Grassmannian $Gr_2(V)$ into $\Lambda^2(V)$. For $n = 3$ one has a single quadratic relation, $p^{12}p^{34} + p^{31}p^{24} + p^{23}p^{14} = 0$, which defines the Plücker quadric in $\Lambda^2(V^4)$.

A quadratic line complex is defined by an additional quadratic relation in the Plücker coordinates. This specifies a $(2n-3)$-parameter family of lines in $\mathbb{P}^n$. Fixing a point $p \in \mathbb{P}^n$ and taking all lines of the complex that pass through $p$ one obtains a quadratic cone with vertex at $p$. This family of cones supplies $\mathbb{P}^n$ with a conformal structure (Monge metric), whose explicit form can be obtained as follows. Set $v^i = u^i + du^i$ (think of $v$ as infinitesimally close to $u$), then the Plücker coordinates take the form $p^{ij} = u^i du^j - u^j du^i$. In the affine chart $u^{n+1} = 1,\ du^{n+1} = 0$, part of the Plücker coordinates simplify to $p^{(n+1)i} = du^i$, and the equation of the complex reduces to (6).

Let $\mathbf{p} = p^{ij}$ be the vector of Plücker coordinates. Let $\mathbf{p}\Omega^\alpha\mathbf{p}^t = 0$ be the Plücker relations defining $Gr_2(V)$, and let $\mathbf{p}Q\mathbf{p}^t = 0$ be the equation of a quadratic complex (here $\Omega^\alpha$, $Q$ are symmetric matrices); note that $Q$ is defined up to transformations of the form $Q \to Q + c_\alpha\Omega^\alpha$. Remarkably, there exists a canonical choice of representative within this class. For $n = 3$ we have a unique Plücker relation defined by a $6 \times 6$ non-degenerate matrix $\Omega$, and one can fix $Q$ by the constraint $trQ\Omega^{-1} = 0$. This is known as the Clebsch normal form of a quadratic complex in $\mathbb{P}^3$ [15], p. 109. Although in higher dimensions the matrices $\Omega^\alpha$ are no longer invertible, there is nevertheless an analogue of Clebsch normal form:

**Definition 3.1.** A quadratic form $Q \in S^2(\Lambda^2 V)$ is said to be in *normal form* if $Q$ belongs to the kernel of the natural map $S^2(\Lambda^2 V) \to \Lambda^4 V$.

□

This condition, which can always be achieved via a transformation $Q \to Q + c_\alpha\Omega^\alpha$, fixes the constants $c_\alpha$ uniquely.

**Remark.** Let $Gr_2(V^*) \subset \Lambda^2(V^*)$ be the Grassmannian in the dual space, specified by quadratic relations $\mathbf{p}^*\Omega^{\alpha *}\mathbf{p}^{*t} = 0$. One can show that the canonical representative $Q$ defined above can be equivalently fixed by the apolarity conditions $trQ\Omega^{\alpha *} = 0$. Thus, every quadratic complex can be brought to a canonical form such that the corresponding quadratic form $Q$ is apolar to the Grassmannian $Gr_2(V^*) \subset \Lambda^2(V^*)$ . We refer to [4], Chapter 1 for a general discussion of apolarity in algebraic geometry.

## 4 Complexes corresponding to Hamiltonian operators

In this section we give invariant characterisation of quadratic complexes that correspond to third-order Hamiltonian operators. Let us first recall the result of Balandin and Potemin [3] according to which the general solution of system (4) - (5) is given by the formula

$$g_{ij} = \phi_{\beta\gamma}\psi_i^\beta\psi_j^\gamma, \tag{7}$$

where $\phi_{\beta\gamma}$ is a non-degenerate constant symmetric matrix, and

$$\psi_k^\gamma = \psi_{km}^\gamma u^m + \omega_k^\gamma; \tag{8}$$

here $\psi_{km}^\gamma$ and $\omega_k^\gamma$ are constants such that $\psi_{km}^\gamma = -\psi_{mk}^\gamma$, and the matrix $\psi = \psi_k^\gamma$ is non-degenerate. Furthermore, Jacobi identities imply that these constants have to satisfy a set of quadratic relations,

$$\phi_{\beta\gamma}(\psi_{is}^\beta\psi_{jk}^\gamma + \psi_{js}^\beta\psi_{ki}^\gamma + \psi_{ks}^\beta\psi_{ij}^\gamma) = 0, \tag{9}$$

$$\phi_{\beta\gamma}(\omega_i^\beta\psi_{jk}^\gamma + \omega_j^\beta\psi_{ki}^\gamma + \omega_k^\beta\psi_{ij}^\gamma) = 0. \tag{10}$$

For $n = 2$ relations (9 - 10) are vacuous. For $n = 3$ there is only 1 equation (10), while equations (9) are vacuous. For $n = 4$ we have 1 equation (9) and 4 equations (10), 5 relations altogether. In general, the total number of relations (9 - 10) equals $C_{n+1}^4$. Let us first rewrite equations (7 - 10) in invariant form. Formula (7) implies

$$g = g_{ij}du^i du^j = \phi_{\beta\gamma}(\psi_i^\beta du^i)(\psi_j^\gamma du^j),$$

where, due to the skew-symmetry conditions $\psi^\gamma_{km} = -\psi^\gamma_{mk}$, each of the expressions $\psi^\beta_i du^i$ is a linear combination of differentials $u^j du^k - u^k du^j$ and $du^j$; here $i, j, k = 1, \dots, n$. Let us introduce an auxiliary coordinate $u^{n+1}$ and consider the $(n+1) \times (n+1)$ skew-symmetric matrix $P$ with the entries $u^a du^b - u^b du^a$, where $a, b = 1, \dots, n+1$. Then $\psi^\beta_i du^i$ can be represented as $tr(A^\beta P)$ for some $(n+1) \times (n+1)$ skew-symmetric matrix $A^\beta$ (on restriction to the affine chart $u^{n+1} = 1$), so that

$$g = \phi_{\beta\gamma} tr(A^\beta P) tr(A^\gamma P)|_{u^{n+1}=1}; \tag{11}$$

one can use any other affine projection, the resulting operators will be projectively equivalent. Formula (11) involves $n$-dimensional subspace $A = span\langle A^\beta \rangle \subset \Lambda^2 V$, and an element $\phi = \phi_{\beta\gamma} A^\beta A^\gamma \in S^2 A$. Remarkably, conditions (9 - 10) simplify to

$$\phi_{\beta\gamma} A^\beta \wedge A^\gamma = 0, \tag{12}$$

that is, $\phi$ must lie in the kernel of the natural map $S^2 A \to \Lambda^4 V$. The main results of this Section are as follows.

**Theorem 4.1.** The singular variety of a quadratic complex corresponding to $n$-component third-order Hamiltonian operator (2) is a double hypersurface of degree $n-1$. □

**P**roof of Theorem 4.1. Formula (7) implies $\det g = \det \phi (\det \psi)^2$, the value $n-1$ for the degree follows from the fact that the singular variety of a quadratic complex has degree $2n-2$ [4], Prop. 10.3.3. This can also be seen directly, indeed, $\psi^\gamma_k = \psi^\gamma_{km} u^m + \omega^\gamma_k$, and it remains to note that $\det(\psi^\gamma_{km} u^m)$ vanishes identically due to the skew-symmetry condition $\psi^\gamma_{km} = -\psi^\gamma_{mk}$ (the matrix $\psi^\gamma_{km} u^m$ has zero eigenvalue corresponding to the eigenvector $u^k$). Thus, all terms of degree $n$ cancel identically, leaving an expression of degree $n-1$. ■

**Theorem 4.2.** Quadratic complexes corresponding to $n$-component third-order Hamiltonian operators (2) are in one-to-one correspondence with elements $Q \in S^2(\Lambda^2 V)$ of rank $n$ that belong to the kernel of the map $S^2(\Lambda^2 V) \to \Lambda^4 V$. □

**P**roof of Theorem 4.2. The value $n$ for the rank follows from representation (11). It remains to note that relations (9 - 10) are identical to (12). ■

To summarise, a quadratic complex corresponds to an $n$-component third-order Hamiltonian operator if and only if, in its normal form, the associated quadratic form has rank $n$.

## 5 Classification results

Based on formula (11), in this section we address the classification of homogeneous third-order Hamiltonian operators. Our strategy will be as follows:

- Classify $n$-dimensional subspaces $A = span\langle A^1, \dots, A^n \rangle$ in $\Lambda^2 V$ modulo natural action of $SL(n+1)$. Remarkably, this problem has been discussed in the context of metabelian Lie algebras [14, 13], and a complete classification is known for $n \le 4$. For $n = 1, 2, 3, 4$ the total number of non-equivalent canonical forms equals $1, 1, 5, 38$, respectively. Apparently, for $n \ge 5$ the problem becomes 'wild', and no classification is available.
- For every subspace $A$ obtained at the previous step, reconstruct non-degenerate $\phi = \phi_{\beta\gamma} A^\beta A^\gamma \in S^2(A)$ that belong to the kernel of the natural map $S^2 A \to \Lambda^4 V$, that is, for which formula (12) holds; note that this condition is *linear* in $\phi$.
  The constraint for $\phi$ can be equivalently reformulated as follows. Consider a generic element of $A$, $A(\xi) = A^\alpha \xi_\alpha$; the condition $rk A(\xi) = 2$ is given by the vanishing of the Pfaffians of all $4 \times 4$ principal minors of $A(\xi)$, in total, $C^4_{n+1}$ quadratic relations of the form $\Omega^{s\beta\gamma} \xi_\beta \xi_\gamma = 0$, $s = 1, \dots, C^4_{n+1}$. The form $\phi$ must be apolar to every $\Omega^s$: $\phi_{\beta\gamma} \Omega^{s\beta\gamma} = 0$.
  Note that in some cases the subspace $A$ may possess a non-trivial stabiliser under the action of $SL(n+1)$: this can be used to simplify the form of $\phi$.
- Reconstruct the corresponding Monge metric $g$ by formula (11).

All results below are formulated modulo (complex) projective transformations. To save space we only present canonical forms for the corresponding Monge metrics rather than Hamiltonian operators themselves.

### 5.1 1-component case

Every 1-component third-order Hamiltonian operator can be reduced to $D^3$, see [24, 23, 5]. This result goes back to [18, 2, 19].

### 5.2 **2-component case**

Similarly, every 2-component operator can be brought to constant coefficient form.

**Theorem 5.1.** [12] Modulo projective transformations, every 2-component homogeneous third-order Hamiltonian operator can be reduced to constant coefficient form. □

**P**roof of Theorem 5.1. For $n = 2$ formula (11) involves a 2-dimensional subspace $\langle A^1, A^2\rangle$ in $\Lambda^2(V^3)$. Without any loss of generality one can set $A^1 = e^1 \wedge e^3,\ A^2 = e^2 \wedge e^3$ (here and in what follows we identify $e^i \wedge e^j$ with the corresponding skew-symmetric matrix). In the affine chart $u^3 = 1$ this gives $tr(A^1P) = 2du^1,\ tr(A^2P) = 2du^2$, so that the Monge metric $g$ given by (11) is constant. ■

### 5.3 **3-component case**

In this case we have 6 canonical forms:

**Theorem 5.2.** [12] Modulo projective transformations, the metric of every 3-component homogeneous third-order Hamiltonian operator (2) can be reduced to one of the 6 canonical forms:

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + c & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2u^3 - u^1 \\ 2u^2 & -cu^2u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix},$$

$$g^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix}, \quad g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 & 0 \\ -u^1u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Here we sketch 3 proofs of this classification result. Based on different ideas, they may be of interest on their own. The first proof is based on the theory of quadratic complexes and their Segre normal forms. The second proof is based on the classification of 3-dimensional subspaces in $\Lambda^2V^4$ modulo natural action of $SL(4)$, that is, on the classification of $SL(4)$-orbits in $Gr_3(\Lambda^2V^4)$. Finally, the third proof uses explicit parametrisation of quadratic forms of rank 3 that are apolar to the Plücker quadric.

**F**irst proof of Theorem 5.2. Let us begin with the necessary information from the theory of quadratic complexes. The Plücker embedding of the Grassmannian $Gr_2(V^4)$ into $\Lambda^2(V^4)$, identified with the space of $4 \times 4$ skew-symmetric matrices $P = p^{ij}$, is the Plücker quadric, $p^{12}p^{34} + p^{31}p^{24} + p^{14}p^{23} = 0$ (the Pfaffian of $P$). Let $\Omega$ be the $6 \times 6$ symmetric matrix corresponding to the Plücker quadric. A quadratic line complex is the intersection of the Plücker quadric with another homogeneous quadratic equation in the Plücker coordinates, defined by a $6 \times 6$ symmetric matrix $Q$. The key invariant of a quadratic complex is the Jordan normal form of the matrix $C = Q\Omega^{-1}$, known as its Segre type. According to Theorem 4.2, the matrix $C$ of a quadratic complex that corresponds to a 3-component third-order Hamiltonian operator, satisfies the conditions $rkC = 3,\ trC = 0$, which impose strong constraints on the Segre type. Recall that the Segre symbol carries information about the number/sizes of Jordan blocks. Thus, the symbol [111111] indicates that the Jordan form of $C$ is diagonal; the symbol [222] indicates that the Jordan form of $C$ consists of three $2 \times 2$ Jordan blocks, etc. We will also use 'refined' Segre symbols with additional round brackets indicating coincidences among the eigenvalues of some of the Jordan blocks, e.g., [(11)(11)(11)] denotes the subcase of [111111] with three pairs of coinciding eigenvalues, the symbol [(111)(111)] denotes the subcase with two triples of coinciding eigenvalues, etc. Theorem 5.2 was proved in [12] by going through the list of 11 Segre types of quadratic complexes as listed in [15], and selecting those whose Monge metrics fulfil (5). A shorter and less computational approach is based on the remark that the only Segre types compatible with the constraints $rkC = 3,\ trC = 0$ are [(111)111], [(111)12], [11(112)], [(114)], [(123)], [(222)]. These are exactly the 6 cases of Theorem 5.2. Recall that the singular surface of a generic quadratic complex is Kummer's quartic surface. According to Theorem 4.1, for Monge metrics associated with third-order Hamiltonian operators this quartic degenerates into a double quadric (which may further split into a pair of planes). ■

**S**econd proof of Theorem 5.2. This proof is based on the classification of 3-dimensional subspaces in $\Lambda^2(V^4)$ [14, 13]. There are 5 canonical forms, that we list in the format $A = \langle A^1,\ A^2,\ A^3\rangle$:

$$\begin{gathered}\langle e^1\wedge e^2,\ e^1\wedge e^3,\ e^2\wedge e^3\rangle,\\ \langle e^1\wedge e^4,\ e^2\wedge e^4,\ e^3\wedge e^4\rangle,\\ \langle e^1\wedge e^2,\ e^2\wedge e^4,\ e^3\wedge e^4\rangle,\\ \langle e^1\wedge e^2,\ e^3\wedge e^4,\ e^1\wedge e^3+e^2\wedge e^4\rangle,\\ \langle e^1\wedge e^4+e^2\wedge e^3,\ e^2\wedge e^4,\ e^3\wedge e^4\rangle.\end{gathered}$$

Modulo permutations of indices, these are the cases 102, 99, 94, 93, 96 in Table 2 of [13], respectively. Calculating $\phi = \phi_{\beta\gamma}A^\beta A^\gamma \in S^2(A)$ that satisfy condition (12) we arrive at the corresponding Monge metrics (11); in all cases we use the affine projection $u^4 = 1$.

**Case** 1 gives a degenerate metric, and does not correspond to a non-trivial Hamiltonian operator.

**Case** 2 corresponds to the constant metric $g^{(6)}$ (after the affine projection $u^4 = 1$).

**Case** 3 gives the metric

$$g = a(p^{12})^2 + b(p^{24})^2 + c(p^{34})^2 + 2\alpha p^{12}p^{24} + 2\beta p^{24}p^{34} =$$

$$a(u^1du^2 - u^2du^1)^2 + b(du^2)^2 + c(du^3)^2 - 2\alpha(u^1du^2 - u^2du^1)du^2 + 2\beta du^2du^3.$$

Here $\det g = (abc - a\beta^2 - c\alpha^2)(u^2)^2$, so the singular variety consists of 2 double planes (one of them at infinity). The subcase $a = 0$ is affinely equivalent to $g^{(4)}$, the general case $a \neq 0$ is affinely equivalent to $g^{(3)}$.

**Case** 4 gives the metric

$$g = a(p^{12})^2 + b(p^{34})^2 + c(p^{13} + p^{24})^2 + 2cp^{12}p^{34} + 2(\alpha p^{12} + \beta p^{34})(p^{13} + p^{24}) =$$

$$\begin{gathered}a(u^1du^2 - u^2du^1)^2 + b(du^3)^2 + c(u^1du^3 - u^3du^1 - du^2)^2\\ -2c(u^1du^2 - u^2du^1)du^3 + 2(\alpha(u^1du^2 - u^2du^1) - \beta du^3)(u^1du^3 - u^3du^1 - du^2).\end{gathered}$$

We have $\det g = (abc + 2\alpha\beta c - c^3 - \alpha^2 b - \beta^2 a)(u^1u^3 + u^2)^2$, so that the singular variety is a double quadric. This leads to the cases $g^{(1)}, g^{(2)}$. Here the case of $g^{(2)}$ is distinguished by $27\mu^2 + \nu^3 = 0$ where $\mu = abc + 2\alpha\beta c - c^3 - \alpha^2 b - \beta^2 a,\ \nu = 2\alpha\beta - ab - 3c^2$.

**Case** 5 gives the metric

$$g = a(p^{24})^2 + b(p^{34})^2 + 2(\alpha p^{24} + \beta p^{34})(p^{14} + p^{23}) + 2\gamma p^{24}p^{34} =$$

$$a(du^2)^2 + b(du^3)^2 - 2(\alpha du^2 + \beta du^3)(u^2du^3 - u^3du^2 - du^1) + 2\gamma du^2du^3.$$

We have $\det g = 2\alpha\beta\gamma - \alpha^2 b - \beta^2 a = const$, so that the singular variety is a quadruple plane at infinity. This metric is affinely equivalent to $g^{(5)}$.

■

**T**hird proof of Theorem 5.2. Introducing

$$\mathbf{p} = (du^1,\ du^2,\ du^3,\ u^2du^3 - u^3du^2,\ u^3du^1 - u^1du^3,\ u^1du^2 - u^2du^1),$$

one can represent a Monge metric in the form $g = \mathbf{p}Q\mathbf{p}^t$, where $Q$ is a $6\times 6$ symmetric matrix. According to Theorem 4.2, we have $rkQ = 3,\ trQ\Omega^{-1} = 0$. Thus, $Q$ and $\Omega$ can be represented in the form

$$Q = \begin{pmatrix} A & M\\ M^t & M^tA^{-1}M\end{pmatrix},\quad \Omega = \begin{pmatrix} 0 & E\\ E & 0\end{pmatrix},$$

where $A, M$ and $E$ are $3\times 3$ matrices ($A$ is symmetric, $E$ is the identity matrix). Note that any symmetric matrix $Q$ of rank 3 can be represented in this form (one can always assume $A$ to be non-degenerate via a translation of $u^i$). The condition $trQ\Omega^{-1} = 0$ reduces to $trM = 0$. The classification of normal forms is performed modulo transformations $Q \to XQX^t$ that preserve $\Omega$: $X\Omega X^t = \Omega$. Setting

$$X = \begin{pmatrix} X_1 & X_2\\ X_3 & X_4\end{pmatrix},$$

the condition $X\Omega X^t=\Omega$ reduces to $X_2X_1^t+X_1X_2^t=0,\ X_4X_3^t+X_3X_4^t=0,\ X_2X_3^t+X_1X_4^t=E$. Our goal is to bring $Q$ to normal form by using special transformations of this type. Taking $X_1=X_4=E$, $X_2=0$, one obtains that $X_3$ must be skew-symmetric. Applying this transformation to $Q$ one obtains $A\to A,\ M\to M-AX_3$ (note that this transformation preserves the condition $trM=0$). Thus, $A^{-1}M\to A^{-1}M-X_3$, which allows one to kill the skew-symmetric part of $A^{-1}M$. Hence, one can assume that $A^{-1}M=B$ is symmetric, so that $M=AB$, and $Q$ takes the form

$$Q=\begin{pmatrix} A & AB\\ BA & BAB\end{pmatrix};$$

recall the condition $trM=trAB=0$. Applying another transformation, $X_2=X_3=0,\ X_4=(X_1^{-1})^t$, one obtains $A\to X_1AX_1^t,\ B\to (X_1^{-1})^tBX_1^{-1}$. Thus, both $A^{-1}$ and $B$ transform in the same way, and one can apply the theory of normal forms of pairs of quadratic forms. Modulo complex transformations, there are 3 cases (note that in all of them $A^{-1}=A$).

**Case 1.** In the generic (diagonal) case one has

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad B=\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix},$$

$trAB=0$ gives $a+b+c=0$. The corresponding Monge metric takes the form

$$g=(du^1+a(u^2du^3-u^3du^2))^2+(du^3+b(u^3du^1-u^1du^3))^2+(du^3+c(u^1du^2-u^2du^1))^2.$$

Here we have 3 cases: the generic case is equivalent to the metric $g^{(1)}$ of Theorem 5.2, the case $b=-a, c=0$ corresponds to $g^{(3)}$, and the case $a=b=c=0$ corresponds to $g^{(6)}$.

**Case 2.** In the case of one $2\times 2$ Jordan block one has

$$A=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad B=\begin{pmatrix}1&a&0\\a&0&0\\0&0&b\end{pmatrix},$$

$trAB=0$ gives $2a+b=0$. The corresponding Monge metric takes the form

$$g=(du^3-2a(u^1du^2-u^2du^1))^2+$$

$$2(du^2+a(u^2du^3-u^3du^2))(du^1+a(u^3du^1-u^1du^3)+u^2du^3-u^3du^2).$$

For $a\neq 0$ this is the case $g^{(2)}$, $a=0$ corresponds to $g^{(5)}$.

**Case 3.** In the case of one $3\times 3$ Jordan block one has

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&1&a\\1&a&0\\a&0&0\end{pmatrix},$$

$trAB=0$ gives $a=0$. The corresponding Monge metric takes the form

$$g=2du^3(du^1+u^3du^1-u^1du^3)+(du^2+u^2du^3-u^3du^2)^2.$$

This is the case $g^{(4)}$.

■

### 5.4 **4-component case**

For $n=4$ formula (11) involves a 4-dimensional subspace $\langle A^1,\dots,A^4\rangle$ in the space of $5\times 5$ skew-symmetric forms, equivalently, a point in the Grassmannian $Gr_4(\Lambda^2V^5)$. Modulo natural action of $SL(5)$, the classification of such subspaces was obtained in [13] in the context of metabelian Lie algebras of signature $(5,4)$. Altogether, there are 38 non-equivalent normal forms. Fixing one of the normal forms, one can reconstruct $\phi$ from condition (12). For $n=4$ this gives 5 conditions for the 10 matrix elements of $\phi$, leaving us with the freedom of at least 5 arbitrary constants. This freedom can be reduced if the subspace has a non-trivial stabiliser under the action of $SL(5)$. Note that the requirement of non-degeneracy of $\phi$ eliminates some of the 38 subcases, leaving 32 canonical forms.

As an example, let us consider the generic case of the classification [13] that corresponds to the subspace

$$\langle e^1\wedge e^2+e^4\wedge e^5,\ \ e^2\wedge e^5+e^3\wedge e^4,\ \ e^1\wedge e^5+e^2\wedge e^4,\ \ e^1\wedge e^4+e^2\wedge e^3\rangle.$$

This subspace has trivial stabiliser, and generates an open dense orbit of dimension 24 in $Gr_4(\Lambda^2V^5)$: note that $\dim SL(5)=\dim Gr_4(\Lambda^2V^5)=24$. It gives rise to the 5-parameter Monge metric

$$g=\varphi_1(p^{12}+p^{45})^2+2\varphi_2(p^{12}+p^{45})(p^{15}+p^{24})+\varphi_3(p^{15}+p^{24})^2$$

$$+\varphi_4(p^{25}+p^{34})^2+2(\varphi_1+\varphi_3)(p^{25}+p^{34})(p^{14}+p^{23})+\varphi_5(p^{14}+p^{23})^2$$

$$-2\varphi_4(p^{12}+p^{45})(p^{14}+p^{23})-2\varphi_5(p^{12}+p^{45})(p^{25}+p^{34});$$

without any loss of generality one can use the affine chart $u^5=1$. All parameters are essential. The singular variety of this metric is a double cubic,

$$u^4(u^1)^2+u^1u^2u^3-(u^2)^3-u^2-u^3u^4+(u^4)^3=0.$$

Table 1 below contains a complete list of Monge metrics/singular varieties corresponding to normal forms of 4-dimensional subspaces in $\Lambda^2V^5$; the first column contains a reference to Table 2 of [13]. The last column gives dimensions of stabilisers of these subspaces under the action of $SL(5)$. We always use the affine chart $u^5=1$.

Table 1: Monge metrics for $n=4$

| Subspace in $\Lambda^2(V^5)$ | Monge metric $g$ / singular variety | Stab |
|---|---|---|
| $e^1\wedge e^2+e^4\wedge e^5$<br>$e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>(no. 11 in [13]) | $g=\varphi_1(p^{12}+p^{45})^2+2\varphi_2(p^{12}+p^{45})(p^{15}+p^{24})$<br>$+\varphi_3(p^{15}+p^{24})^2+\varphi_4(p^{25}+p^{34})^2+2(\varphi_1+\varphi_3)(p^{25}+p^{34})(p^{14}+p^{23})$<br>$+\varphi_5(p^{14}+p^{23})^2-2\varphi_4(p^{12}+p^{45})(p^{14}+p^{23})-2\varphi_5(p^{12}+p^{45})(p^{25}+p^{34})$<br>Singular variety is a double cubic,<br>$u^4(u^1)^2+u^1u^2u^3-(u^2)^3-u^2-u^3u^4+(u^4)^3=0$ | 0 |
| $e^1\wedge e^4+e^3\wedge e^5$<br>$e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$e^1\wedge e^3$<br>(no. 15 in [13]) | $g=\varphi_1(p^{31})^2+\varphi_2(p^{42}+p^{51})^2+2\varphi_2(p^{52}+p^{43})(p^{53}+p^{41})$<br>$\varphi_3(p^{41}+p^{53})^2+2\varphi_3(p^{52}+p^{43})(p^{51}+p^{42})+\varphi_4(p^{43}+p^{52})^2$<br>$+2\varphi_4(p^{53}+p^{41})(p^{51}+p^{42})+2\varphi_5(p^{53}+p^{41})p^{31}$<br>Singular variety is a double cubic,<br>$u^1u^2u^4-(u^1)^2+u^2u^3-(u^3)^2u^4=0$ | 1 |
| $e^2\wedge e^4+e^3\wedge e^5$<br>$e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^3$<br>$e^1\wedge e^2$<br>(no. 21 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2(p^{51}+p^{32})p^{21}+\varphi_3(p^{51}+p^{32})^2-2\varphi_3(p^{53}+p^{42})p^{21}$<br>$\varphi_4(p^{53}+p^{42})^2+\varphi_4(p^{52}+p^{43})^2+2\varphi_5(p^{53}+p^{42})(p^{52}+p^{43})$<br>Singular variety is a double cubic,<br>$u^1u^2u^4-u^1u^3+(u^2)^3-u^2(u^3)^2=0$ | 2 |
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5$<br>$e^1\wedge e^3+e^2\wedge e^3$<br>$e^2\wedge e^4$<br>(no. 26 in [13]) | $g=\varphi_1(p^{42})^2+\varphi_2(p^{32}+p^{31})^2+2\varphi_3(p^{52}+p^{51}+p^{43})(p^{32}+p^{31})$<br>$+\varphi_4(p^{51})^2+2\varphi_5(p^{52}+p^{43})p^{42}$<br>Singular variety is a double cubic,<br>$u^1u^3u^4-u^1u^2-(u^2)^2=0$ | 2 |
| $e^2\wedge e^5$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$+e^3\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^3$<br>(no. 28 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2(p^{41}+p^{32})p^{31}+2\varphi_3(p^{51}+p^{43}+p^{42})p^{31}+\varphi_3(p^{41}+p^{32})^2$<br>$+2\varphi_4p^{52}p^{31}+2\varphi_4(p^{51}+p^{43}+p^{42})(p^{41}+p^{32})+\varphi_5(p^{52})^2$<br>Singular variety is a double cubic,<br>$(u^1)^2-u^1u^2u^4+u^2(u^3)^2+u^3(u^2)^2=0$ | 3 |

Continued on next page

Table 1 – continued from previous page

| Subspace in $\Lambda^2(V^5)$ | Monge metric $g$ / singular variety | Stab |
|---|---|---|
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$e^2\wedge e^3$<br>$e^1\wedge e^4$<br>(no. 31 in [13]) | $g=\varphi_1(p^{41})^2+\varphi_2(p^{32})^2+2\varphi_3(p^{51}+p^{42})p^{41}$<br>$+\varphi_4(p^{51}+p^{42})^2+2\varphi_4(p^{52}+p^{43})p^{41}+2\varphi_5(p^{52}+p^{43})p^{32}$<br>Singular variety is a double cubic,<br>$u^1u^3u^4+u^1u^2-(u^2)^2u^4=0$ | 3 |
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^3$<br>(no. 34 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2(p^{41}+p^{32})p^{31}+\varphi_3(p^{41}+p^{32})^2+2\varphi_3(p^{51}+p^{42})p^{31}$<br>$+2\varphi_4(p^{52}+p^{43})p^{31}+2\varphi_4(p^{51}+p^{42})(p^{41}+p^{32})$<br>$+\varphi_5(p^{51}+p^{42})^2+2\varphi_5(p^{52}+p^{43})(p^{41}+p^{32})$<br>Singular variety is a double cubic,<br>$u^1u^2u^4+u^1(u^3)^2-(u^1)^2-u^3(u^2)^2=0$ | 4 |
| $e^4\wedge e^5$<br>$e^1\wedge e^3+e^2\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^3$<br>$e^1\wedge e^2$<br>(no. 35 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2(p^{51}+p^{32})p^{21}+2\varphi_3(p^{42}+p^{31})p^{21}-2\varphi_4p^{54}p^{21}$<br>$+2\varphi_4(p^{51}+p^{32})(p^{42}+p^{31})+\varphi_5(p^{54})^2$<br>Singular variety is a double cubic,<br>$(u^1)^2-u^4(u^2)^2=0$ | 4 |
| $e^2\wedge e^3+e^4\wedge e^5$<br>$e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^4$<br>$e^1\wedge e^3$<br>(no. 36 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2p^{41}p^{31}+\varphi_3(p^{41})^2-\varphi_4(p^{54}+p^{32})^2$<br>$+\varphi_4(p^{52}+p^{43})^2+2\varphi_5(p^{54}+p^{32})(p^{52}+p^{43})$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^2u^3-u^4=0$ | 4 |
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^2$<br>(no. 41 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2(p^{41}+p^{32})p^{21}-2\varphi_3(p^{52}+p^{43})p^{21}+\varphi_3(p^{41}+p^{32})^2$<br>$+2\varphi_4(p^{51}+p^{42})p^{21}+\varphi_5(p^{51}+p^{42})^2+2\varphi_6(p^{52}+p^{43})(p^{41}+p^{32})$<br>Singular variety is a double cubic,<br>$u^1u^2u^3+(u^1)^2u^4-(u^2)^3=0$ | 5 |
| $e^3\wedge e^5$<br>$e^1\wedge e^5+e^2\wedge e^3$<br>$+e^2\wedge e^4$<br>$e^1\wedge e^4$<br>$e^1\wedge e^2$<br>(no. 44 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{41}p^{21}+\varphi_3(p^{41})^2+2\varphi_4(p^{51}+p^{42}+p^{32})p^{21}+\varphi_5(p^{53})^2$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^1+u^2u^3=0$ | 5 |
| $e^1\wedge e^5$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^2+e^2\wedge e^4$<br>$e^3\wedge e^4$<br>(no. 45 in [13]) | $g=\varphi_1(p^{43})^2+2\varphi_2(p^{42}+p^{21})p^{43}-\varphi_2(p^{41}+p^{32})^2+\varphi_3(p^{42}+p^{21})^2$<br>$+2\varphi_4(p^{41}+p^{32})p^{43}+2\varphi_5(p^{42}+p^{21})(p^{41}+p^{32})+\varphi_6(p^{51})^2$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^1u^4+u^2u^3-(u^4)^2=0$ | 5 |
| $e^2\wedge e^4+e^3\wedge e^5$<br>$e^2\wedge e^3$<br>$e^1\wedge e^5$<br>$e^1\wedge e^4$<br>(no. 48 in [13]) | $g=\varphi_1(p^{41})^2+2\varphi_2p^{51}p^{41}+\varphi_3(p^{51})^2+\varphi_4(p^{32})^2+2\varphi_5(p^{53}+p^{42})p^{32}$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^2u^4+u^3=0$ | 5 |

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Table 1 – continued from previous page

| Subspace in $\Lambda^2(V^5)$ | Monge metric $g$ / singular variety | Stab |
|---|---|---|
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^3$<br>$e^1\wedge e^4$<br>$e^1\wedge e^3$<br>(no. 50 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2p^{41}p^{31}+\varphi_3(p^{41})^2+2\varphi_4(p^{51}+p^{32})p^{31}$<br>$+2\varphi_5(p^{52}+p^{43})p^{31}+\varphi_5(p^{51}+p^{32})^2$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^1-u^2u^3=0$ | 6 |
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$e^1\wedge e^2$<br>$e^2\wedge e^3$<br>(no. 52 in [13]) | $g=\varphi_1(p^{32})^2+2\varphi_2p^{32}p^{21}+\varphi_3(p^{21})^2+2\varphi_4(p^{51}+p^{42})p^{21}+2\varphi_5(p^{52}+p^{43})p^{32}$<br>Singular variety is a double plane and a double quadric,<br>$u^2=0,\quad u^1u^3-(u^2)^2=0$ | 6 |
| $e^1\wedge e^5+e^3\wedge e^4$<br>$e^2\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^3$<br>(no. 53 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2(p^{41}+p^{32})p^{31}+2\varphi_3p^{42}p^{31}+\varphi_3(p^{41}+p^{32})^2$<br>$+2\varphi_4(p^{41}+p^{32})p^{42}+\varphi_5(p^{42})^2+2\varphi_6(p^{51}+p^{43})p^{31}$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^1u^4-u^2u^3=0$ | 6 |
| $e^1\wedge e^3+e^2\wedge e^5$<br>$e^2\wedge e^4$<br>$e^1\wedge e^5$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>(no. 54 in [13]) | $g=2\varphi_1(p^{52}+p^{31})p^{42}+\varphi_1(p^{41}+p^{32})^2++2\varphi_2(p^{41}+p^{32})p^{51}+\varphi_2(p^{52}+p^{31})^2$<br>$+\varphi_3(p^{51})^2+2\varphi_4(p^{41}+p^{32})p^{42}+2\varphi_5p^{51}p^{42}+2\varphi_5(p^{52}+p^{31})(p^{41}+p^{32})$<br>$+\varphi_6(p^{42})^2+2\varphi_7(p^{52}+p^{31})p^{51}$<br>Singular variety is a double cubic,<br>$u^1u^2u^3-(u^1)^2u^4-(u^2)^2=0$ | 6 |
| $e^3\wedge e^5$<br>$e^1\wedge e^3+e^1\wedge e^4$<br>$e^2\wedge e^4$<br>$e^1\wedge e^2$<br>(no. 55 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{42}p^{21}+\varphi_3(p^{42})^2+2\varphi_4(p^{41}+p^{31})p^{21}+$<br>$\varphi_5(p^{41}+p^{31})^2+\varphi_6(p^{53})^2$<br>Singular variety consists of 3 double planes,<br>$u^1=0,\quad u^2=0,\quad u^3=0$ | 6 |
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^5+e^2\wedge e^4$<br>$e^1\wedge e^3$<br>$e^1\wedge e^2$<br>(no. 56 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{31}p^{21}+\varphi_3(p^{31})^2+2\varphi_4(p^{51}+p^{42})p^{21}$<br>$+2\varphi_5(p^{52}+p^{43})p^{21}+2\varphi_5(p^{51}+p^{42})p^{31}$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^1u^3-(u^2)^2=0$ | 7 |
| $e^1\wedge e^5$<br>$e^2\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^3$<br>(no. 58 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2(p^{41}+p^{32})p^{31}+2\varphi_3p^{42}p^{31}+\varphi_3(p^{41}+p^{32})^2$<br>$+2\varphi_4(p^{41}+p^{32})p^{42}+\varphi_5(p^{42})^2+2\varphi_6p^{51}p^{31}+\varphi_7(p^{51})^2$<br>Singular variety is a double plane and a double quadric,<br>$u^1=0,\quad u^1u^4-u^2u^3=0$ | 7 |
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^4$<br>$e^1\wedge e^3$<br>$e^2\wedge e^3$<br>(no. 59 in [13]) | $g=\varphi_1(p^{32})^2+2\varphi_2p^{32}p^{31}+\varphi_3(p^{31})^2+2\varphi_4p^{41}p^{31}$<br>$+\varphi_5(p^{41})^2+2\varphi_6(p^{52}+p^{43})p^{32}$<br>Singular variety consists of 3 double planes,<br>$u^1=0,\quad u^2=0,\quad u^3=0$ | 7 |

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Table 1 – continued from previous page

| Subspace in $\Lambda^2(V^5)$ | Monge metric $g$ / singular variety | Stab |
|---|---|---|
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^3+e^2\wedge e^3$<br>$e^1\wedge e^4$<br>$e^1\wedge e^2$<br>(no. 60 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{41}p^{21}+\varphi_3(p^{41})^2+2\varphi_4(p^{32}+p^{31})p^{21}$<br>$-2\varphi_5(p^{52}+p^{43})p^{21}+2\varphi_5(p^{32}+p^{31})p^{41}+\varphi_6(p^{32}+p^{31})^2$<br>Singular variety consists of 3 double planes,<br>$u^1=0,\quad u^2=0,\quad u^1+u^2=0$ | 8 |
| $e^4\wedge e^5$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^2$<br>$e^1\wedge e^3$<br>(no. 63 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2p^{31}p^{21}+\varphi_3(p^{21})^2+2\varphi_4(p^{41}+p^{32})p^{31}$<br>$+2\varphi_5(p^{41}+p^{32})p^{21}+\varphi_6(p^{54})^2$<br>Singular variety consists of 2 planes,<br>$u^1=0$ (quadruple),$\quad u^4=0$ (double) | 8 |
| $e^2\wedge e^4$<br>$e^1\wedge e^5$<br>$e^1\wedge e^3$<br>$e^2\wedge e^3$<br>(no. 64 in [13]) | $g=\varphi_1(p^{32})^2+2\varphi_2p^{32}p^{31}+\varphi_3(p^{31})^2+2\varphi_4p^{51}p^{31}+\varphi_5(p^{51})^2+2\varphi_6p^{42}p^{32}$<br>$+\varphi_7(p^{42})^2$<br>Singular variety consists of 3 double planes,<br>$u^1=0,\quad u^2=0,\quad u^3=0$ | 8 |
| $e^2\wedge e^5+e^3\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^3$<br>$e^1\wedge e^2$<br>(no. 65 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{31}p^{21}+\varphi_3(p^{31})^2+2\varphi_4(p^{41}+p^{32})p^{21}$<br>$+2\varphi_5(p^{41}+p^{32})p^{31}-2\varphi_6(p^{52}+p^{43})p^{21}+\varphi_6(p^{41}+p^{32})^2$<br>Singular variety consists of 2 planes,<br>$u^1=0$ (quadruple),$\quad u^2=0$ (double) | 9 |
| $e^1\wedge e^3+e^2\wedge e^3$<br>$e^2\wedge e^4$<br>$e^1\wedge e^5$<br>$e^1\wedge e^2$<br>(no. 66 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{51}p^{21}+\varphi_3(p^{51})^2+2\varphi_4p^{42}p^{21}+\varphi_5(p^{42})^2$<br>$+2\varphi_6(p^{32}+p^{31})p^{21}+\varphi_7(p^{32}+p^{31})^2$<br>Singular variety consists of 3 double planes,<br>$u^1=0,\quad u^2=0,\quad u^1+u^2=0$ | 9 |
| $e^1\wedge e^5+e^2\wedge e^4$<br>$e^2\wedge e^3$<br>$e^1\wedge e^4$<br>$e^1\wedge e^3$<br>(no. 67 in [13]) | $g=\varphi_1(p^{31})^2+2\varphi_2p^{41}p^{31}+\varphi_3(p^{41})^2+2\varphi_4p^{32}p^{31}+2\varphi_5(p^{51}+p^{42})p^{31}$<br>$+2\varphi_5p^{41}p^{32}+\varphi_6(p^{32})^2+2\varphi_7(p^{51}+p^{42})p^{41}$<br>Singular variety consists of 2 planes,<br>$u^1=0$ (quadruple),$\quad u^3=0$ (double) | 9 |
| $e^1\wedge e^5+e^2\wedge e^4$<br>$e^2\wedge e^3$<br>$e^1\wedge e^4$<br>$e^1\wedge e^2$<br>(no. 71 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{41}p^{21}+\varphi_3(p^{41})^2+2\varphi_4p^{32}p^{21}+\varphi_5(p^{32})^2$<br>$+2\varphi_6(p^{51}+p^{42})p^{21}+2\varphi_7(p^{51}+p^{42})p^{41}$<br>Singular variety consists of 2 planes,<br>$u^1=0$ (quadruple),$\quad u^2=0$ (double) | 10 |
| $e^1\wedge e^5+e^2\wedge e^4$<br>$e^1\wedge e^4+e^2\wedge e^3$<br>$e^1\wedge e^3$<br>$e^1\wedge e^2$<br>(no. 73 in [13]) | $g=\varphi_1(p^{21})^2+2\varphi_2p^{31}p^{21}+\varphi_3(p^{31})^2+2\varphi_4(p^{41}+p^{32})p^{21}+2\varphi_5(p^{41}+p^{32})p^{31}$<br>$+2\varphi_6(p^{51}+p^{42})p^{31}+\varphi_6(p^{41}+p^{32})^2+2\varphi_7(p^{51}+p^{42})p^{21}$<br>Singular variety is a single six-tuple plane,<br>$u^1=0$ | 11 |

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Table 1 – continued from previous page

| Subspace in $\Lambda^2(V^5)$ | Monge metric $g$ / singular variety | Stab |
|---|---|---|
| $e^2 \wedge e^3$<br>$e^1 \wedge e^4$<br>$e^1 \wedge e^5$<br>$e^1 \wedge e^2$<br>(no. 79 in [13]) | $g = \varphi_1(p^{21})^2 + 2\varphi_2 p^{51}p^{21} + \varphi_3(p^{51})^2 + 2\varphi_4 p^{41}p^{21} + 2\varphi_5 p^{51}p^{41} + \varphi_6(p^{41})^2$<br>$+2\varphi_7 p^{32}p^{21} + \varphi_8(p^{32})^2$<br>Singular variety consists of 2 planes,<br>$u^1 = 0$ (quadruple), $\quad u^2 = 0$ (double) | 12 |
| $e^1 \wedge e^5 + e^2 \wedge e^3$<br>$e^1 \wedge e^4$<br>$e^1 \wedge e^2$<br>$e^1 \wedge e^3$<br>(no. 86 in [13]) | $g = \varphi_1(p^{31})^2 + 2\varphi_2 p^{31}p^{21} + \varphi_3(p^{21})^2 + 2\varphi_4 p^{41}p^{31} + 2\varphi_5 p^{41}p^{21} + \varphi_6(p^{41})^2$<br>$+2\varphi_7(p^{51} + p^{32})p^{31} + 2\varphi_8(p^{51} + p^{32})p^{21}$<br>Singular variety is a single six-tuple plane,<br>$u^1 = 0$ | 14 |
| $e^1 \wedge e^2$<br>$e^1 \wedge e^3$<br>$e^1 \wedge e^4$<br>$e^1 \wedge e^5$<br>(no. 97 in [13]) | $g = \varphi_1(p^{51})^2 + 2\varphi_2 p^{51}p^{41} + \varphi_3(p^{41})^2 + 2\varphi_4 p^{51}p^{31} + 2\varphi_5 p^{41}p^{31} + \varphi_6(p^{31})^2$<br>$+2\varphi_7 p^{51}p^{21} + 2\varphi_8 p^{41}p^{21} + 2\varphi_9 p^{31}p^{21} + \varphi_{10}(p^{21})^2$<br>Singular variety is a single six-tuple plane,<br>$u^1 = 0$ | 20 |

**Remark 1.** Most of the cases in Table 1 can be uniquely distinguished by dimensions of stabilisers and the geometry of singular varieties.

**Remark 2.** Singular varieties appearing in Table 1 are nothing but normal forms of determinantal cubics defined as

$$rk(A^1u, A^2u, A^3u, A^4u) < 4,$$

where $A^i$ are $5 \times 5$ skew-symmetric matrices, and $u$ is a 5-component column vector (note that all $4 \times 4$ minors of this $5 \times 4$ matrix have one and the same cubic factor).

**Remark 3.** Monge metrics from Table 1 depend on auxiliary parameters which, in some cases, can be removed by using the stabiliser (the action of the stabiliser may have several non-equivalent orbits). Thus, one can show that every Monge metric with 14-dimensional stabiliser is projectively equivalent to one of the three canonical forms,

$$(du^3 + u^2du^1 - u^1du^2)du^2 + du^1du^4,$$
$$(du^3 + u^2du^1 - u^1du^2)du^2 + (du^1)^2 + (du^4)^2,$$
$$(du^3 + u^2du^1 - u^1du^2)du^2 + (du^1)^2 + du^2du^4;$$

here the first canonical form corresponds to Example 6 of Sect 2.

### 5.5 Remarks on the multi-component case

For $n = 5$ formula (11) involves a 5-dimensional subspace $A = \langle A^1, \dots, A^5 \rangle$ in the space of $6 \times 6$ skew-symmetric forms, equivalently, a point in $Gr_5(\Lambda^2 V^6)$. Condition (12) imposes 15 constraints for the 15 entries of $\phi$ (note that $\dim S^2 A = \dim \Lambda^4 V^6 = 15$). For a *generic* subspace, these conditions are linearly independent, and imply $\phi = 0$. Thus, to get a nontrivial Monge metric one has to select a subspace $A$ such that the forms $A^\beta \wedge A^\gamma$ are linearly dependent. This defines a hypersurface $M$ in $Gr_5(\Lambda^2 V^6)$, which is of degree 6 in the Plücker coordinates. Given a smooth generic point of $M$, there exists a unique non-degenerate $\phi$ satisfying these conditions. Thus, 5-component Hamiltonian operators are parametrised by points of an algebraic hypersurface (of degree 6) in $Gr_5(\Lambda^2 V^6)$. Unfortunately, it is highly unlikely that one can obtain an effective classification of orbits of the associated $SL(6)$-action, as well as to classify Monge metrics corresponding to singular points of $M$.

**Example.** Consider the subspace spanned by the following bivectors,

$$
\begin{aligned}
&2e^1\wedge e^2+e^3\wedge e^4+e^5\wedge e^6,\\
&e^1\wedge e^3+e^1\wedge e^4+e^4\wedge e^6+\alpha e^2\wedge e^4,\\
&e^2\wedge e^6-e^3\wedge e^5,\\
&e^1\wedge e^6-e^2\wedge e^3+e^4\wedge e^5,\\
&e^1\wedge e^5+2e^2\wedge e^5+e^3\wedge e^4.
\end{aligned}
$$

One can show that the associated system (12) has rank 15 for $\alpha\neq 0$, and therefore implies $\phi=0$; for $\alpha=0$ it has rank 14, and the corresponding (unique) nonzero solution $\phi$ is non-degenerate.

For general $n$ the situation is similar: given a point in $Gr_n(\Lambda^2V^{n+1})$, condition (12) imposes $C^4_{n+1}$ constraints for $C^2_n$ components of $\phi$. Thus, to get a nontrivial solution one has to require that the number of independent constraints is less than $C^2_n$. This defines an algebraic subvariety in $Gr_n(\Lambda^2V^{n+1})$ whose geometry/singularities are yet to be investigated.

## 6 Concluding remarks

We have obtained a classification of third-order Hamiltonian operators of differential-geometric type with the number of components $n\leq 4$. For $n=1,2$ any such operator can be transformed to constant coefficients, for $n=3$ we have 6 non-equivalent canonical forms, for $n=4$ there are 32 multi-parameter families.

- Our approach is based on the classification of $SL(n+1)$-orbits in $Gr_n(\Lambda^2V)$, which is only available for $n\leq 4$. Apparently, for $n=5$ the problem becomes 'wild', and no reasonable classification is possible.
- Examples suggest that every homogeneous third-order Hamiltonian operator arises as a Hamiltonian structure of some *local* conservative system of hydrodynamic type of the form $u^i_t=(V^i(u))_x$, with non-local Hamiltonian (here $u^i$ are the flat coordinates). In the 'generic' case, any system of this kind is linearly degenerate and non-diagonalisable. Furthermore, in the generic case the fluxes $V^i$ are rational functions of the form $V^i=\frac{S^i}{S}$ where $S$ is the polynomial defining the singular variety of the corresponding Monge metric, and $S^i$ are polynomials of degree one higher. It would be interesting to clarify the geometric meaning of such systems. It would also be interesting to classify higher-order conservative systems possessing homogeneous third-order Hamiltonian structures (the compatibility conditions between right-hand sides of such systems with the corresponding Hamiltonian operators are currently under investigation).

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