Linear degeneracy in multidimensions

by

Jonathan Moss

A Doctoral Thesis

Submitted in partial fulfillment of the requirements for the award of Doctor of Philosophy of Loughborough University

September 2015

Abstract

Linear degeneracy of a PDE is a concept that is related to a number of interesting geometric constructions. We first take a quadratic line complex, which is a three-parameter family of lines in projective space $\mathbb{P}^3$ specified by a single quadratic relation in the Plücker coordinates. This complex supplies us with a conformal structure in $\mathbb{P}^3$. With this conformal structure, we associate a three-dimensional second order quasilinear wave equation. We show that any PDE arising in this way is linearly degenerate, furthermore, any linearly degenerate PDE can be obtained by this construction. We classify Segre types of quadratic complexes for which the structure is conformally flat, as well as Segre types for which the corresponding PDE is integrable. These results were published in [1]. We then introduce the notion of characteristic integrals, discuss characteristic integrals in 3D and show that, for certain classes of second-order linearly degenerate dispersionless integrable PDEs, the corresponding characteristic integrals are parameterised by points on the Veronese variety. These results were published in [2].

Keywords

Second order PDEs, hydrodynamic reductions, integrability, conformal structures, quadratic line complexes, linear degeneracy, characteristic integrals, principal symbol.

Acknowledgments

I would like to express many thanks to Prof E.V. Ferapontov for his guidance, understanding and support. I would also like to thank Dr. V. Novikov, Dr K. Khusnutdinova, Dr. G. Brown and Dr. A. Bolsinov for help and clarifying discussions throughout my time of study. Thanks to all the staff of the School of Mathematics at Loughborough University for the discussion-friendly and encouraging environment, constant seminar and conference activity, and useful and effective administrative support.

I also would like to thank the School of Mathematics of Loughborough University for the financial support during my PhD studies. Finally, thanks to BAST Inc Ltd and to Dr J. Grevel for allowing me to continue with my studies whilst in full-time employment.

Contents

Chapter 1

Introduction

The theory of nonlinear differential equations which can be, in some sense, solved exactly, is often referred to as the *theory of integrable systems*. This subject has developed rapidly over the last half a century, partly due to its applicability to a wide range of physical situations. The concept of a 'completely integrable system' first appeared in the 19th century in the context of finite-dimensional classical mechanics. It began when Hamilton reformulated Newton's equations by introducing the so-called canonical coordinates $x_1, ..., x_n$ to describe general positions, and $p_1, ..., p_n$ to describe their general momenta. The Hamiltonian formalism of the Kepler problem is one of the earliest examples of an integrable system.

Later, the soliton phenomenon was discovered by Zabusky and Kruskal in 1965. These results were a progression of work that was done over one hundred years previous. The solitary wave, so-called because it often occurs as a single entity and is localised, was first observed by J. Scott Russell on the Edinburgh-Glasgow canal in 1834. He called it the 'great wave of translation'. The KdV equation was later derived by Korteweg and de Vries in 1895. Its dimensionless form is

$$\phi_t + 6\phi\phi_x + \phi_{xxx} = 0, \tag{1.1}$$

where $\phi = \phi(x, t)$. Computer calculations by Zabusky and Kruskal showed that the solitary waves are very robust objects. Collide two together and they both emerge unchanged, with the same shapes and velocities as before the collision, which is unexpected as the KdV equation is nonlinear. These properties led to the term soliton. Kruskal and his coworkers had shown that the KdV equation has an infinite number of conservation laws, and continuing work by Zakharov and Faddeev (1971) showed that the KdV equation can be viewed as an infinite-dimensional classical integrable system. More recently the number of nonlinear partial differential equations in two space-time variables known to admit soliton solutions has increased, and there are now many other examples other than the KdV. In all of these equations, additional structural features have been found. There is in fact no universally accepted definition of 'integrability' for classical systems with infinitely many degrees of freedom. The term is instead used whenever certain structural properties are present. For example, the KdV equation has what is known as a 'Lax pair', named after Peter Lax who studied solitons in continuous media. A Lax pair is a pair of linear operators L and A associated with a partial differential equation which can be used to solve the equation. The KdV equation $u_t = 6uu_x - u_{xxx}$ can be reformulated as

$$L_t = [P, L].$$

Here $[P, L] = PL - LP$ is the operator commutator and

$$L = -\partial_x^2 + u, \qquad P = -4\partial_x^3 + 3(u\partial_x + \partial_x u).$$

As another example, take the 3-dimensional dKP equation

$$u_{xt} - u_x u_{xx} - u_{yy} = 0,$$

where $u = u(x, y, t)$. The dKP equation has what is known as a *dispersionless* Lax pair

$$v_y - \frac{1}{2}v_x^2 - u_x = 0, \qquad v_t - \frac{1}{3}v_x^3 - v_x u_x - u_y = 0,$$

with $v = v(x, y, t)$ and $u = u(x, y, t)$. The dKP equation results from the above on elimination of v, that is, via the compatibility condition $v_{yt} = v_{ty}$. Similarly, the elimination of u leads to the modified dKP (mdKP) equation $v_{xt} - (v_y - \frac{1}{2}v_x^2)v_{xx} - v_{yy} = 0$. This is an example of a Bäcklund transformation between the dKP and mdKP equations, another remarkable structural property.

Chapter 2 of this thesis focuses on a number of general concepts. Systems of the type

$$u_t^i + \sum_{j=1}^{n} v_j^i(\mathbf{u}) u_x^j = 0,$$

are known as $1+1$ dimensional systems of hydrodynamic type. Here $\mathbf{u} = (u^1(t, x), u^2(t, x), ..., u^n(t, x))$ is an n-component vector of dependent variables.

The functions $v^i_j(\mathbf{u})$, which could also be considered as matrix elements of an $n \times n$ matrix $\mathbf{V}$, are assumed to be smooth and, in general, non-constant. Systems of this type arise in applications in differential geometry, general relativity and fluid dynamics. We say that this system possesses Riemann invariants if we can find suitable variables $R^1(\mathbf{u}), ..., R^n(\mathbf{u})$ such that the system becomes diagonal,

$$R^i_t = \lambda^i(\mathbf{R})R^i_x.$$

These new variables $\mathbf{R} = (R^1, ..., R^n)$ are called Riemann invariants. Note that for 2×2 systems Riemann invariants always exist, whereas for higher dimensional systems they do not necessarily exist. This diagonal system is said to be *semi-Hamiltonian* if,

$$\partial_k \left(\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \right) = \partial_j \left(\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i} \right), \quad i \neq j \neq k.$$

The generalised hodograph method [12] can be used to find a general solution to semi-Hamiltonian systems of the form shown above. As an example, consider a simple scalar equation, the so-called Hopf equation,

$$R_t = RR_x.$$

The general solution of this equation is given by the implicit formula $f(R) = x + Rt$, where f is an arbitrary function of one variable. The generalised hodograph method extends this formula to multi-component hydrodynamic type systems.

The method of hydrodynamic reductions was developed in [7] (see also references therein). Consider the case of $(2 + 1)-$dimensional quasilinear systems of the form

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0,$$

where $u = (u^1, ..., u^l)^T$ is an l-component column vector of dependent variables, and A, B, C are $m \times l$ matrices, where m is the number of equations. The key construction in the method of hydrodynamic reductions is to seek multi-phase solutions in the form $u(R^1, ..., R^n)$ where the 'phases' $R^i(x, y, t)$ are the Riemann invariants satisfying a pair of commuting diagonal $(1 + 1)$-dimensional systems of hydrodynamic type,

$$R^i_y = \mu^i(\mathbf{R})R^i_x, \quad R^i_t = \lambda^i(\mathbf{R})R^i_x.$$

We then say that a $(2+1)$-dimensional quasilinear system is said to be *integrable* if, for any number of phases n, it possesses infinitely many n-phase solutions parameterised by $2n$ arbitrary functions of a single variable.

Chapter 2 focuses on quasilinear wave equations of the form

$$f_{11}u_{x_1x_1} + f_{22}u_{x_2x_2} + f_{33}u_{x_3x_3} + 2f_{12}u_{x_1x_2} + 2f_{13}u_{x_1x_3} + 2f_{23}u_{x_2x_3} = 0$$

where $u(x_1, x_2, x_3)$ is a function of three independent variables, and the coefficients f_{ij} depend on the first order derivatives $u_{x_1}, u_{x_2}, u_{x_3}$ only, we also assume the non-degeneracy condition $\det f_{ij} \neq 0$. Equations of this type were studied in [4], it was shown that this class is invariant under the equivalence group $\mathbf{SL}(4)$. Furthermore, it was demonstrated that it is natural to associate with this PDE the conformal structure $f_{ij}(P)dp^i dp^j$, where f_{ij} are taken as the coefficients of the equation shown above and $p^i = x_i$, $p^j = x_j$. To define the concept of linear degeneracy for these types of equations, we first consider quasilinear equations of the form

$$\mathbf{u}_t - A(\mathbf{u})\mathbf{u}_x = 0,$$

where $\mathbf{u} = (u_1, ..., u_n)^T$ is an $n-$component column vector of dependent variables, $u_i = u_i(x, t)$ are functions of two independent variables and A is an $n \times n$ matrix. We call this PDE linearly degenerate if the directional derivative of the eigenvalues of A along their corresponding right eigenvectors is zero. In general, this can be verified by introducing the characteristic polynomial of A and imposing a constraint which, in the 2 component case, simplifies to

$$\nabla(trA)A = \nabla(detA),$$

where ∇ is the operator of the gradient, $\nabla f = (\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2})$. Linear degeneracy is known to prevent breakdown of smooth initial data, leading to global solvability of the Cauchy problem [10]. In the 1-component case, this can be seen by considering the 2 systems

$$u_t + uu_x = 0, \qquad u_t + cu_x = 0,$$

where $u = u(x, t)$ and c is a constant.

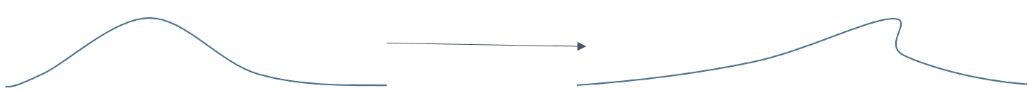

Figure 1.1: Time evolution of $u_t = uu_x$ with a Gaussian initial profile. The profile becomes steeper, and breaks down in finite time.

Figure 1.2: Time evolution of $u_t = cu_x$ with a Gaussian initial profile. The initial profile translates with finite speed c without changing its shape.

The two figures above illustrate what is meant by global solvability of the Cauchy problem. For the equation $u_t = uu_x$ which is clearly not linearly degenerate, we see that the solution to the Cauchy problem breaks down after sufficient time. The equation $u_t = cu_x$ clearly *is* linearly degenerate and the image above illustrates that the wave form does not change it's shape.

Similarly, a 2D second order quasilinear wave equation of the form

$$a(u_x, u_t)u_{xx} + 2b(u_x, u_t)u_{xt} + c(u_x, u_t)u_{tt} = 0$$

is linearly degenerate if this is the case for the corresponding first order system (obtained by setting $u^1 = u_x, u^2 = u_t$).

Finally, in order to define linear degeneracy for 3D quasilinear wave equations, one must first take traveling wave reductions by setting $u(x_1, x_2, x_3) = u(\xi_1, \xi_2)$, where $\xi_1 = x_1 + \alpha x_3, \xi_2 = x_2 + \beta x_3$, we then get a $2D$ equation for $u(\xi_1, \xi_2)$

$$(f_{11}+2\alpha f_{13}+\alpha^2 f_{33})u_{\xi_1\xi_1}+2(f_{12}+\alpha f_{23}+\beta f_{13}+\alpha\beta f_{33})u_{\xi_1\xi_2}+(f_{22}+2\beta f_{23}+\beta^2 f_{33})u_{\xi_2\xi_2} = 0.$$

We require that the above is linearly degenerate for all α, β in the sense of the 2D case, this second order quasilinear equation takes first order quasilinear form in the variables $u^1 = u_{\xi_1}, u^2 = u_{\xi_2}$. The requirement of linear degeneracy for *any* α, β imposes strong constraints on the coefficients f_{ij}. This is in fact our first main result:

Theorem 1 *A quasilinear wave equation is linearly degenerate if and only if the corresponding conformal structure* $f_{ij}dp^i dp^j$ *satisfies the constraint*

$$\partial_{(k} f_{ij)} = \phi_{(k} f_{ij)},$$

here $\partial_k = \partial_{p^k}$, ϕ_k *is a covector, and brackets denote a complete symmetrisation in the indices* i, j, k *which take values* $1, 2, 3$. *Here* $f_{ij}(p^1, p^2, p^3)$ *coincide with the coefficients* $f_{ij}(u_{x_1}, u_{x_2}, u_{x_3})$ *upon setting* $p^1 = u_{x_1}, p^2 = u_{x_2}, p^3 = u_{x_3}$.

Remarkably, the above constraint for f_{ij} arises in the theory of quadratic complexes of lines in projective space. Let us begin with the simplest non-trivial case of $\mathbb{P}^3$. Consider a line r in $\mathbb{P}^3$ passing through the points $\mathbf{p} = (p^1 : p^2 : p^3 : p^4)$ and $\mathbf{q} = (q^1 : q^2 : q^3 : q^4)$. The so-called Plücker coordinates p^{ij} are given by the six 2×2 minors of the matrix

$$\begin{pmatrix} p^1 & p^2 & p^3 & p^4 \\ q^1 & q^2 & q^3 & q^4 \end{pmatrix}.$$

Explicitly, $p^{ij} = p^i q^j - p^j q^i$. The p^{ij} are not arbitrary, rather they satisfy a quadratic relation, $p^{12}p^{34} + p^{13}p^{42} + p^{14}p^{23} = 0$. It can be shown that there exists a bijection between the Plücker coordinates and lines in $\mathbb{P}^3$. In the space $\mathbb{P}^5$ we can now define a point as $(p^{12} : p^{13} : p^{14} : p^{23} : p^{24} : p^{34})$, where the quadratic relation represents a four dimensional subset of $\mathbb{P}^5$, called the Plücker quadric. The lines in $\mathbb{P}^3$ whose coordinates p^{ij} satisfy an extra equation $Q(p^{ij}) = 0$, where Q is a homogeneous polynomial of degree n, give an algebraic complex (that is, a 3-parameter family of lines) of degree n. For a quadratic complex, fixing a point $\mathbf{p}$ in $\mathbb{P}^3$ and taking the lines of the complex which pass through $\mathbf{p}$, one obtains a quadratic cone with vertex at $\mathbf{p}$. The family of these cones supplies $\mathbb{P}^3$ with a conformal structure. Its equation can be obtained by setting $q^i = p^i + dp^i$ and passing to a system of affine coordinates, say, $p^4 = 1$, $dp^4 = 0$. The expressions for the Plücker coordinates take the form $p^{4i} = dp^i$, $p^{ij} = p^i dp^j - p^j dp^i$, $i, j = 1, 2, 3$ and the equation of the complex takes the so-called Monge form,

$$Q(dp^i, p^i dp^j - p^j dp^i) = f_{ij} dp^i dp^j = 0.$$

We can now associate a PDE to the given Monge form of quadratic complex in the following way; make the substitution $u_{x_i} = p^i$ and $u_{x_i x_j} = dp^i dp^j$, so that we obtain a PDE of our required form [4],

$$f_{11}u_{11} + f_{22}u_{22} + f_{33}u_{33} + 2f_{12}u_{12} + 2f_{13}u_{13} + 2f_{23}u_{23} = 0, \quad u_{ij} = u_{x_i x_j}.$$

Thus, there is a correspondence between linearly degenerate wave equations and quadratic line complexes. In [9], it was shown that the general equation for the quadratic complex can be reduced to eleven different canonical forms. This was done in the following way: First take the Plücker quadric in matrix form, call this matrix Ω. Next take the matrix of the equation of the complex, call this Q. Now calculate $Q\Omega^{-1}$ and bring to Jordan normal form. From the Jordan normal form we can extract the so-called Segre symbol, given by the number of Jordan blocks of the matrix. For example, one 2×2 block and one 4×4 block gives Segre symbol [24]. If it so happens that the eigenvalues of the different blocks coincide we then use round brackets as well [(24)]. This leads us to our second main result.

Theorem 2 *Any linearly degenerate 3D quasilinear wave equation can be brought by an equivalence transformation to one of the eleven canonical forms, labeled by Segre symbols of the associated quadratic complexes.*
The first three are presented here, the rest can be found in the text.

Case 1: Segre symbol [111111]

$$(a_1 + a_2u_3^2 + a_3u_2^2)u_{11} + (a_2 + a_1u_3^2 + a_3u_1^2)u_{22} + (a_3 + a_1u_2^2 + a_2u_1^2)u_{33}+$$

$$2(\alpha u_3 - a_3u_1u_2)u_{12} + 2(\beta u_2 - a_2u_1u_3)u_{13} + 2(\gamma u_1 - a_1u_2u_3)u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

Case 2: Segre symbol [11112]

$$(\lambda u_2^2 + \mu u_3^2 + 1)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}+$$

$$2(\alpha u_3 - \lambda u_1u_2)u_{12} + 2(\beta u_2 - \mu u_1u_3)u_{13} + 2\gamma u_1u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

Case 3: Segre symbol [1113]

$$(\lambda u_2^2 + \mu u_3^2 + 2u_3)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}+$$

$$2(\mu u_3 - \lambda u_1u_2 - 1)u_{12} + 2(\beta u_2 - \mu u_1u_3 - u_1)u_{13} + 2\gamma u_1u_{23} = 0,$$

$\mu + \beta + \gamma = 0.$

An additional property of interest is the conformal flatness of the metric $f_{ij}dp^idp^j$. We say that $f_{ij}dp^idp^j$ is 'conformally flat' if, after multiplication by some function, an appropriate change of variables can be made such that the coefficients are made constant. There is a classical result from differential geometry which states that, for any metric on a 3-manifold, the vanishing of the Cotton tensor is equivalent to the metric being conformally flat. This leads to our third main result:

Theorem 3 *A quadratic complex defines a flat conformal structure if and only if its Segre symbol is one of the following:*

$$[111(111)]^*,\ [(111)(111)],\ [(11)(11)(11)],$$

$$[(11)(112)],\ [(11)(22)],\ [(114)],\ [(123)],\ [(222)],\ [(24)],\ [(33)].$$

Here the asterisk denotes a particular sub-case of $[111(111)]$ *where the matrix* $Q\Omega^{-1}$ *has eigenvalues* $(1, \epsilon, \epsilon^2, 0, 0, 0)$*,* $\epsilon^3 = 1$*.*
Modulo equivalence transformations this gives a complete list of normal forms of the associated PDEs, which are shown in the text.

Finally, we present our fourth and last main result of this section:

Theorem 4 *A linearly degenerate* $3D$ *quasilinear wave equation is integrable if and only if the corresponding complex has one of the following Segre types:*

$$[(11)(11)(11)],\ [(11)(112)],\ [(11)(22)],\ [(123)],\ [(222)],\ [(33)].$$

Modulo equivalence transformations, this leads to the five canonical forms of linearly degenerate integrable PDEs (we exclude the linearisable case with Segre symbol $[(222)]$*). Each integrable equation is presented with its Lax pair in the form* $[X, Y] = 0$ *where* X *and* Y *are parameter-dependent vector fields which commute modulo the corresponding equation:*

Segre symbol $[(11)(11)(11)]$

$$\alpha u_3 u_{12} + \beta u_2 u_{13} + \gamma u_1 u_{23} = 0,$$

$\alpha + \beta + \gamma = 0$*. Setting* $\alpha = a - b,\ \beta = b - c,\ \gamma = c - a$ *we obtain the Lax pair:* $X = \partial_{x^3} - \frac{\lambda - b}{\lambda - c}\frac{u_3}{u_1}\partial_{x^1},\ Y = \partial_{x^2} - \frac{\lambda - b}{\lambda - a}\frac{u_2}{u_1}\partial_{x^1}.$

Segre symbol $[(11)(112)]$

$$u_{11} + u_1 u_{23} - u_2 u_{13} = 0,$$

Lax pair: $X = \partial_{x^1} - \lambda u_1 \partial_{x^3},\ Y = \partial_{x^2} + (\lambda^2 u_1 - \lambda u_2)\partial_{x^3}.$

Segre symbol $[(11)(22)]$

$$u_{12} + u_2 u_{13} - u_1 u_{23} = 0,$$

Lax pair: $X = \lambda \partial_{x^1} - u_1 \partial_{x^3},\ Y = (\lambda - 1)\partial_{x^2} - u_2 \partial_{x^3}.$

Segre symbol [(123)]

$$u_{22} + u_{13} + u_2 u_{33} - u_3 u_{23} = 0,$$

Lax pair: $X = \partial_{x^2} + (\lambda - u_3)\partial_{x^3},\ Y = \partial_{x^1} + (\lambda^2 - \lambda u_3 + u_2)\partial_{x^3}$.

Segre symbol [(33)]

$$u_{13} + u_1 u_{22} - u_2 u_{12} = 0,$$

Lax pair: $X = \lambda\partial_{x^1} - u_1\partial_{x^2},\ Y = \partial_{x^3} + (\lambda - u_2)\partial_{x^2}$.

In equivalent forms, these PDEs have appeared before in literature, although the Lax pairs presented are new. The results of this chapter were published in [1].

Chapter 4 of this thesis focuses on linear degeneracy and characteristic integrals. Let Σ be a partial differential equation (PDE) in n independent variables $x_1, \ldots, x_n$. A conservation law is an $(n-1)$-form Ω which is closed on the solutions of Σ: $d\Omega = 0 \bmod \Sigma$. Since any $(n-1)$-form in n variables possesses a unique annihilating direction, there exists a vector field F such that $\Omega(F) = 0$. We say that Ω is a characteristic integral (conservation law) if F is a characteristic direction of Σ. For a conservation law represented as $\partial_{x_1}F_1 + ... + \partial_{x_n}F_n = 0$, we have $F = (F_1, ..., F_n)$.

For systems of hydrodynamic type,

$$u^i_t = v^i_j(\mathbf{u})u^i_x,$$

this goes as follows. Let λ^i be the eigenvalues (characteristic speeds) of V, and let ξ^i be the corresponding eigenvectors, so that $V\xi^i = \lambda^i\xi^i$. Characteristic directions are defined as $dx + \lambda^i dt = 0$, and the characteristic integral in i-th direction is a 1-form $h(\mathbf{u})(dx + \lambda^i dt)$ which is closed on solutions. The i-th characteristic direction is called linearly degenerate if the Lie derivative of λ^i in the direction of the corresponding eigenvector ξ^i vanishes, $L_{\xi^i}\lambda^i = 0$. It is well known that if there exists a characteristic integral in the i-th direction, then the corresponding characteristic speed λ^i must be linearly degenerate.

In chapter 4 we primarily concentrate on characteristic integrals of quasilinear wave equations discussed earlier. It has been shown in [4] that any integrable quasilinear wave equation must admit exactly four non-trivial conservation laws. Taking the linear combination of these conservation laws with constant coefficients, adding the trivial conservation laws and imposing the characteristic condition $Fg^{-1}F^t = 0$, where $g = f_{ij}$

is the 3×3 symmetric matrix of the corresponding principal symbol, we obtain our 5th main result,

Theorem 5

(i) If a 3D quasilinear PDE of the form discussed above possesses 'sufficiently many' characteristic integrals, then it must be linearly degenerate. Here 'sufficiently many' means that the corresponding vector F satisfies no extra algebraic constraints other than the characteristic condition itself, $Fg^{-1}F^t = 0$.
(ii) Any linearly degenerate integrable 3D quasilinear wave equation possesses a V^3-worth of characteristic integrals.

The V^3 above refers to a Veronese variety. A Veronese variety is an algebraic manifold that is realised by the Veronese embedding of projective space given by the complete linear system of quadrics. For a mapping $\mathbb{P}^2 \to \mathbb{P}^5$ we have a Veronese surface V^2, the embedding is given by,

$$\mathbb{P}^2 \to \mathbb{P}^5 : \qquad [x : y : z] \to [x^2 : y^2 : z^2 : yz : xz : xy].$$

Let us give an example.

Equation 1

$$\mu u_t u_{xy} + \nu u_y u_{xt} + \eta u_x u_{yt} = 0.$$

The general conservation law is

$$F^1_x + F^2_y + F^3_t = 0,$$

where

$$F^1 = J_1(\eta u_y u_t) + J_2\nu\left(\frac{u_y}{u_t}\right) + J_3\mu\left(\frac{u_t}{u_y}\right) - J_5 u_y + J_6 u_t + J_8,$$

$$F^2 = J_1\nu(u_x u_t) + J_2\eta\left(\frac{u_x}{u_t}\right) + J_4\mu\left(\frac{u_t}{u_x}\right) + J_5 u_x - J_7 u_t + J_9,$$

$$F^3 = J_1\mu(u_x u_y) + J_3\eta\left(\frac{u_x}{u_y}\right) + J_4\nu\left(\frac{u_y}{u_x}\right) - J_6 u_x + J_7 u_y + J_{10}.$$

Here, $J_1, ..., J_{10}$ are arbitrary constants. Imposing the characteristic condition, we get:

$$J_1 = \alpha^2, \quad J_2 = \frac{1}{4\nu\eta}\beta^2, \quad J_3 = \frac{1}{4\eta\mu}\delta^2, \quad J_4 = \frac{1}{4\nu\mu}\gamma^2, \quad J_5 = \alpha\beta,$$

$$J_6 = \alpha\delta, \quad J_7 = \alpha\gamma, \quad J_8 = -\frac{1}{2\eta}\beta\delta, \quad J_9 = -\frac{1}{2/nu}\beta\gamma, \quad J_{10} = -\frac{1}{2\mu}\delta\gamma.$$

These equations define the Veronese embedding of $\mathbb{P}^3(\alpha:\beta:\gamma:\delta)$ in $\mathbb{P}^9(J_1:J_2:...:J_{10})$. The results of this characterisation were published in [2].

We also consider characteristic integrals of linearly degenerate systems of hydrodynamic type

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0,$$

where $u(x, y, t)$ is a function of three independent variables. We conjecture that characteristic integrals of linearly degenerate, integrable 2-component systems of hydrodynamic type can be parameterised by the Veronese surface V^2.
Finally, we give an example of a first order system of the form

$$\mathbf{F}(u_x, u_y, u_t, v_x, v_y, v_t) = 0, \quad \mathbf{G}(u_x, u_y, u_t, v_x, v_y, v_t) = 0,$$

and see that the same principle holds, this time characteristic integrals are parameterised by a Veronese variety V^4.

Characteristic integrals are considered an important concept in 2D, as they are used in defining Darboux integrability. However, Darboux integrability currently has no meaning in 3D, and it is thought that this work could form a basis for further investigations into how Darboux integrability could be defined in higher dimensions.

Chapter 2

Hydrodynamic type systems and other general concepts

2.1 What are integrable systems?

The concept of a "completely integrable system" arose in the 19th century in the context of finite-dimensional classical mechanics. It began when Hamilton reformulated Newton's equations. He introduced the so-called canonical coordinates $x_1, ..., x_n$ to describe general positions, and $p_1, ..., p_n$ to describe their general momenta. Together, these coordinates describe a mechanical system with n degrees of freedom. The time-evolution of an initial state $(x_0, p_0) \in \mathbb{R}^{2n}$ is then governed by Hamilton's equations of motion $\dot{x} = \nabla_p H$, $\dot{p} = -\nabla_x H$ (where the dot denotes the time derivative). The Hamiltonian $H(x, p)$ then describes the total energy of the system.

The Poisson bracket is defined as $\{A, B\} = \nabla_x A \cdot \nabla_p B - \nabla_p A \cdot \nabla_x B$ for functions A, B on the phase space Ω (the domain of the canonical coordinates (x, p)). Hamilton's equations can now be re-written as $\dot{x}_j = \{x_j, H\}$, $\dot{p}_j = \{p_j, H\}$, $j = 1, ..., n$. A conserved quantity I, called a "first integral", can then be characterised by it's zero Poisson bracket with H, $\{I, H\} = 0$. Clearly, $\{H, H\} = 0$, which is the law of conservation of energy. Transformations of the phase space $\Omega \to \Omega'$, $(x, p) \mapsto (x', p')$ that preserve Hamilton's equations are called *canonical*. The formalism of Hamilton is invariant under such canonical maps. In particular, we have $\{x_j, x_k\} = 0$, $\{p_j, p_k\} = 0$ and $\{x_j, p_k\} = \delta_{jk}$ for $j, k = 1, ..., n$. A dynamical system defined by a given Hamiltonian H on a $2n$-dimensional phase space Ω is called *integrable* if there exists additional functions

$H_1, ..., H_n$ on Ω such that $H_1, ..., H_n$ are independent and in involution, i.e. all Poisson brackets $\{H_j, H_k\}$ vanish.

Example 1 Consider the system of n independent harmonic oscillators. In the phase space $\mathbb{R}^{2n}(q,p) = \mathbb{R}^{2n}(q_1, ..., q_n, p_1, ..., p_n)$, the equations of motion are

$$\dot{q}_i = \frac{p_i}{m_i} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -a_i q_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, ..., n,$$

$(m_i, a_i > 0)$, where the Hamiltonian of the system $H(q,p) = \sum_{i=1}^{n}(p_i^2/m_i + a_i q_i^2)/2$ represents the total mechanical energy of the system. This system has n independent integrals:

$$f_i(q,p) = f_i(q_i, p_i) = \frac{1}{2m_i}p_i^2 + \frac{a_i}{2}q_i^2, \quad i = 1, ..., n.$$

These integrals are all in involution, therefore this system is integrable. Here, $H = \sum_i^n = f_i$.

Example 2 (Kepler problem) In classical mechanics, the Kepler problem is a special case of the two-body problem, in which the two bodies interact by a central force F that varies in strength as the inverse square of the distance r between them. The force may be either attractive or repulsive. The "problem" to be solved is to find the position or speed of the two bodies over time given their masses and initial positions and velocities. In the 6 dimensional phase space $\mathbb{R}^6(q,p) = \mathbb{R}^6(q_1, q_2, q_3, p_1, p_2, p_3)$, the equations of motion are

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{kq_i}{|q|^3}, \quad i = 1, ..., 3.$$

The Hamiltonian of this system is then given as $H = \frac{1}{2}|p|^2 - \frac{k}{|q|}$. From the law of the conservation of angular momentum [27], we know that the components of angular momentum $M = p \times q$ are integrals of the Kepler system. A remarkable fact of the Kepler system is that the components of the Laplace' vector $l = p \times M + k\frac{q}{|q|}$ are integrals. One can verify that H, M_i and l_j are in involution for any $i, j = 1, ..., 3$. Therefore, the Kepler system is integrable. This result gives rise to Keplers famous laws of planetary motion:

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time (constant "sectorial" velocity).

3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Although various integrable systems were discovered in the 19th century, the subject lay dormant during the first seventy years of the 20th century. Results by Poincare, to the effect that integrability is a highly exceptional property for the systems usually considered in classical mechanics, were an important factor contributing to this lack of interest.

The situation changed dramatically after the discovery of the soliton phenomenon by Zabusky and Kruskal (1965). These results were a progression of work that was done over one hundred years previous. The solitary wave, so-called because it often occurs as a single entity and is localised, was first observed by J. Scott Russell on the Edinburgh-Glasgow canal in 1834. He called it the 'great wave of translation'. Russell reported his observations to the British Association in his 1844 'Report on Waves' in the following words: "I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly being drawn along a narrow channel by a pair of horses, when the boats suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel."

Russell also performed laboratory experiments, generating solitary waves by dropping a weight at one end of a water channel. He was able to deduce empirically that the volume of water in the wave is equal to the volume of water displaced and, further, that the speed, c, of the solitary wave is obtained from $c^2 = g(h + a)$, where a is the amplitude of the wave, h the undisturbed depth of water and g the acceleration of gravity. The solitary wave is therefore a *gravity wave*. There are two obvious facts from this; higher waves travel faster, and this equation applies only to waves of elevation. Any attempt to generate a wave of depression results in a train of oscillatory waves, as Russell found in his own experiments. Developing Russell's formula, both Boussinesq (1871) and Lord Rayleigh (1876) assumed that a solitary wave has a length scale much greater than the depth of water. They managed to show that the wave profile $z = \xi(x,t)$ for Russell's solitary wave is given by

$$\xi(x,t) = a\,\mathrm{sech}^2\{\beta(x-ct)\}, \tag{2.1}$$

where $\beta^{-2} = 4h^2(h+a)/3a$ for any $a > 0$. These authors did not, however, write down a simple equation for $\xi(x,t)$ which admits (2.1) as a solution. This final step was completed by Korteweg and de Vries in 1895. They derived what is now know as the Korteweg de Vries (KdV) equation. Its dimensionless form is

$$\phi_t + 6\phi\phi_x + \phi_{xxx} = 0, \tag{2.2}$$

where $\phi = \phi(x,t)$.

The extraordinary stability properties of these solitary waves were discovered much later in computer calculations by Zabusky and Kruskal. They studied collisions of n solitary waves, and found that these waves emerge unscathed, with the same velocities and shapes as before the collision. This was unexpected as the KdV equation is nonlinear, so solutions cannot be linearly superposed. Once the pertinent solutions were found in explicit form, the presence of a nonlinear interaction became clear. The positions of the solitary waves became shifted, compared to the positions arising from a linear superposition. In fact, the shifts can be written as sums of pairwise shifts, leading to a physical picture of individual entities scattering independently in pairs. These particle-like properties led to the coining of the term soliton.

The connection with the concept of completely integrable system was first made by Zakharov and Faddeev (1971). Kruskal and coworkers had shown that the KdV equation has an infinite number of conservation laws, and that there exists a linearizing transformation, which maps the initial value $u(0,x)$ for the KdV Cauchy problem to spectral and scattering data of the Schrödinger operator $-\frac{d^2}{dx^2} - u(0,x)$. The nonlinear evolution yielding $u(t,x)$ then transforms into an essentially linear time evolution of these data, so that $u(t,x)$ can be constructed via the inverse map, the so-called Inverse Scattering Transform (IST). Inspired by these findings, Zakharov and Faddeev showed that the KdV equation may be viewed as an infinite-dimensional classical integrable system. Ever since these pioneering works, the number of nonlinear partial differential equations in two space-time variables admitting n-soliton solutions has steadily increased, the most well-known examples being the KdV, modified KdV, sine-Gordon and nonlinear Schrödinger equation. For all of these equations additional structural features have been shown to be present. Examples of these additional features include Lax pairs and Bäcklund transformations, both of which will be discussed later.

There is in fact no universally accepted definition of 'integrability' for classical systems with infinitely many degrees of freedom. Rather, the term is used whenever certain structural features are present. As well as Lax pairs and Bäcklund transformations, these include exact soliton-like solutions, infinitely many conservation laws or infinitely many symmetries. We will see that the definition of integrability depends on the class of equations under study. In particular, for multidimensional quasilinear systems the integrability is understood as the existence of infinitely many hydrodynamic reductions. This will be discussed in detail later.

2.2 Hydrodynamic type systems in $1+1$ dimensions

Systems of the type

$$u_t^i + \sum_{j=1}^{n} v_j^i(\mathbf{u}) u_x^j = 0, \tag{2.3}$$

are known as $1+1$ dimensional systems of hydrodynamic type. Here $\mathbf{u} = (u^1(t,x), u^2(t,x), ..., u^n(t,x))$ is an n-component vector of dependent variables. The functions $v_j^i(\mathbf{u})$, which could also be considered as matrix elements of an $n \times n$ matrix $\mathbf{V}$, are assumed to be smooth and, in general, non-constant. Systems of this type arise in applications in differential geometry, general relativity and fluid dynamics.

Definition The system (2.3) is called *strictly hyperbolic* if and only if all eigenvalues of $\mathbf{V}$ are real and distinct.

Example The equations of motion for an ideal barotropic gas are given by

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + u u_x + \gamma \rho^{\gamma-2} \rho_x &= 0, \end{aligned} \tag{2.4}$$

where ρ is the density, u the velocity and γ is a polytropic constant. The equations can be expressed as (2.3) in the following way

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ \gamma\rho^{\gamma-2} & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0,$$

so that we have

$$\mathbf{u} = \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} u & \rho \\ \gamma\rho^{\gamma-2} & u \end{pmatrix}.$$

The eigenvalues $\lambda_{1,2}$ of $\mathbf{V}$ are

$$\lambda_1 = u + \sqrt{\gamma\rho^{\gamma-1}}, \qquad \lambda_2 = u - \sqrt{\gamma\rho^{\gamma-1}},$$

It can be seen that the system (2.4) is strictly hyperbolic if and only if we have $\gamma\rho^{\gamma-1} > 0$.

Remark All systems that we consider in this work are assumed to be strictly hyperbolic.

2.3 Riemann invariants

We say that system (2.3) possesses Riemann invariants if we can find suitable variables

$$R^1(\mathbf{u}), ..., R^n(\mathbf{u})$$

such that system (2.3) becomes diagonal,

$$R^i_t = \lambda^i(\mathbf{R})R^i_x. \tag{2.5}$$

These new variables $\mathbf{R} = (R^1, ..., R^n)$ are called Riemann invariants. Note that for 2×2 systems Riemann invariants always exist, whereas for higher component systems they do not necessarily exist. The reason for this can be seen below

There exists a standard procedure for transforming a 2×2 system to Riemann invariants:

1. Bring the system into the form (2.3) and solve the characteristic equation

$$\det(\mathbf{V} - \lambda I) = 0,$$

to find the roots $\lambda^1(u)$ and $\lambda^2(u)$.

2. Fix $\lambda^1(u)$ and $\lambda^2(u)$ and calculate their corresponding left eigenvectors such that

$$(\xi_1, \xi_2)(\mathbf{V} - \lambda^1 I) = 0,$$

$$(\xi_3, \xi_4)(\mathbf{V} - \lambda^2 I) = 0.$$

3. Choose (ξ_1, ξ_2) and (ξ_3, ξ_4), which are defined up to a scaling factor, to be the gradients of $R^1(u)$ and $R^2(u)$ respectively. Note that whereas this is always possible for $n = 2$, it is not always possible for $n > 2$. Next we solve the system:

$$\left(\frac{\partial R^1}{\partial u^1}, \frac{\partial R^1}{\partial u^2}\right) = (\xi_1, \xi_2),$$

$$\left(\frac{\partial R^2}{\partial u^1}, \frac{\partial R^2}{\partial u^2}\right) = (\xi_3, \xi_4).$$

Example Let us show how to reduce the equations of motion for an ideal barotropic gas (2.4) to a system in Riemann invariants. We have

$$\rho_t + u_x\rho + u\rho_x = 0, \quad u_t + uu_x + \gamma\rho^{\gamma-2}\rho_x = 0. \tag{2.6}$$

First we have to solve the characteristic equation $\gamma\rho^{\gamma-1} - (u - \lambda)^2 = 0$, which has solutions

$$\lambda_{1,2} = u \pm (\gamma\rho^{\gamma-1})^{\frac{1}{2}}.$$

Next we require

$$\begin{pmatrix} R^i_\rho & R^i_u \end{pmatrix} \begin{pmatrix} u - \lambda_i & \rho \\ \gamma\rho^{\gamma-2} & u - \lambda_i \end{pmatrix} = 0,$$

where $i = 1, 2$. We can now find R^1 and R^2 in terms of u and ρ first by finding the corresponding left eigenvectors of λ_1 and λ_2, then setting these to be the gradients of $R^1(u)$ and $R^2(u)$ respectively. Solving this system gives the solutions for R^1 and R^2:

$$R^1 = u + \frac{2\gamma^{\frac{1}{2}}\rho^{\frac{\gamma-1}{2}}}{\gamma - 1}, \quad R^2 = u - \frac{2\gamma^{\frac{1}{2}}\rho^{\frac{\gamma-1}{2}}}{\gamma - 1}.$$

Finally, we can substituTe back and express the eigenvalues λ_1 and λ_2 as functions of R^1 and R^2,

$$\lambda_1 = \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^1 - R^2)}{4},$$
$$\lambda_2 = \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^2 - R^1)}{4}.$$

Thus we can re-write the system in the diagonal form

$$\begin{aligned} R^1_t + \left(\frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^1 - R^2)}{4}\right) R^1_x = 0, \\ R^2_t + \left(\frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^2 - R^1)}{4}\right) R^2_x = 0. \end{aligned} \tag{2.7}$$

One can verify directly that the change of variables

$$\frac{R^1+R^2}{2}=u, \quad R^1-R^2=\frac{4\gamma^{\frac{1}{2}}\rho^{\frac{\gamma-1}{2}}}{\gamma-1}$$

brings the system (2.7) back to the form (2.6). Note that for this method to work, we needed the hydrodynamic type system (2.6) to be strictly hyperbolic.

2.4 Commuting flows

Consider 2 hydrodynamic type systems of the form (2.3),

$$\mathbf{u}_t=V(\mathbf{u})\mathbf{u}_x, \quad \mathbf{u}_y=W(\mathbf{u})\mathbf{u}_x. \tag{2.8}$$

We say that these systems *commute* if $\mathbf{u}_{ty}=\mathbf{u}_{yt}$.

Theorem *Consider two systems of the form (2.5),*

$$R^i_t=\lambda^i(R)R^i_x, \quad R^i_y=\mu^i(R)R^i_x, \quad i=1,...,n. \tag{2.9}$$

Here t and y are the corresponding "times". For these equations to be consistent, the following condition must be true,

$$\frac{\partial_j\lambda^i}{(\lambda^j-\lambda^i)}=\frac{\partial_j\mu^i}{(\mu^j-\mu^i)}, \quad i\neq j. \tag{2.10}$$

where $\partial_j=\frac{\partial}{\partial R^j}$.

Proof For the equations to be consistent, we require

$$R^i_{ty}=R^i_{yt}. \tag{2.11}$$

Explicitly, we have

$$\begin{aligned} R^i_{ty}&=(\lambda^iR^i_x)_y=\partial_j\lambda^iR^j_yR^i_x+\lambda^iR^i_{xy}=\partial_j\lambda^i\mu^jR^j_xR^i_x+\lambda^i(\mu^iR^i_x)_x \\ &=\partial_j\lambda^i\mu^jR^j_xR^i_x+\lambda^i\partial_j\mu^iR^j_xR^i_x+\lambda^i\mu^iR^i_{xx}, \end{aligned} \tag{2.12}$$

similarly,

$$\begin{aligned} R^i_{yt}&=(\mu^iR^i_x)_t=\partial_j\mu^iR^j_tR^i_x+\mu^iR^i_{xt}=\partial_j\mu^i\lambda^jR^j_xR^i_x+\mu^i(\lambda^iR^i_x)_x \\ &=\partial_j\mu^i\lambda^jR^j_xR^i_x+\mu^i\partial_j\lambda^iR^j_xR^i_x+\mu^i\lambda^iR^i_{xx}. \end{aligned} \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.11), we get

$$\partial_j\lambda^i\mu^jR^j_xR^i_x+\lambda^i\partial_j\mu^iR^j_xR^i_x+\lambda^i\mu^iR^i_{xx}=\partial_j\mu^i\lambda^jR^j_xR^i_x+\mu^i\partial_j\lambda^iR^j_xR^i_x+\mu^i\lambda^iR^i_{xx},$$

which simplifies to

$$\partial_j \lambda^i (\mu^j - \mu^i) = \partial_j \mu^i (\lambda^j - \lambda^i).$$

(2.10) is referred to as the *commutativity condition.* If (2.10) is satisfied then the systems (2.9) are said to be *commuting flows.*

2.5 Conservation laws

Consider a PDE of the form

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...). \tag{2.14}$$

A relation of the form

$$[f(\mathbf{u})]_t = [g(\mathbf{u})]_x, \tag{2.15}$$

which holds identically modulo (2.14) is called a *conservation law* of (2.14). The functions $f(\mathbf{u})$ and $g(\mathbf{u})$ are called the conserved density and flux respectively. Note that neither of these functions involve derivatives with respect to t. Usually, $f(\mathbf{u})$ and $g(\mathbf{u})$ are polynomials in u and its higher order x-derivatives. Let us assume that $\mathbf{u}(x)$ tends to zero sufficiently fast as x tends to infinity, and that both $f(\mathbf{u})$ and $g(\mathbf{u})$ also tend to zero, so that the integrals are convergent. Integrating (2.15) over x we have

$$\partial_t \int_{-\infty}^{+\infty} f\, dx = \int_{-\infty}^{+\infty} f_t\, dx = \int_{-\infty}^{+\infty} g_x\, dx = g(\infty) - g(-\infty) = 0,$$

which shows that the quantity

$$H = \int_{-\infty}^{+\infty} f\, dx$$

is conserved. We call H an *integral of motion.*

Now consider a diagonal system in Riemann invariants of the form (2.5). A conservation law is therefore a relation

$$[f(\mathbf{R})]_t = [g(\mathbf{R})]_x \tag{2.16}$$

which holds modulo (2.5). A conservation law is said to be of *hydrodynamic type* if f and g depend on R only, and not on any higher derivatives of R. By substituting (2.5) into (2.16) we get

$$\partial_i f \lambda^i R^i_x = \partial_i g R^i_x$$

so that $\partial_i f \lambda^i = \partial_i g$ for any i. For consistency we require that $\partial_j \partial_i g = \partial_i \partial_j g$. From this condition we obtain

$$\partial_i \partial_j f = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \partial_i f + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \partial_j f, \quad i \neq j.$$

Example Consider the system in Riemann invariants

$$R^1_t = R^2 R^1_x, \qquad R^2_t = R^1 R^2_x, \tag{2.17}$$

here $\lambda^1 = R^2$, $\lambda^2 = R^1$. Substituting into (2.16), the equation for conserved quantities takes the form

$$\partial_2 \partial_1 f = \frac{1}{R^1 - R^2} \partial_1 f + \frac{1}{R^2 - R^1} \partial_2 f = \frac{\partial_1 f - \partial_2 f}{R^1 - R^2}.$$

It's general solution is

$$f = \frac{p(R^1) - q(R^2)}{R^1 - R^2},$$

which can be verified by differentiation. Here $p(R^1)$ and $q(R^2)$ are arbitrary functions of one variable.

Remark The condition (2.16) is equivalent to the 1-form $fdx + gdt$ being closed. Indeed, if we apply the differential to $dP = fdx + gdt$,

$$f_x dx \wedge dx + f_t dt \wedge dx + g_x dx \wedge dt + g_t dt \wedge dt = 0,$$

$$= (f_t - g_x) dt \wedge dx.$$

Thus, $f_t = g_x$.

2.6 The semi-Hamiltonian property

The diagonal system (2.5) is said to be *semi-Hamiltonian* if,

$$\partial_k \left(\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \right) = \partial_j \left(\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i} \right), \quad i \neq j \neq k. \tag{2.18}$$

Let $a_{ij} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$, then (2.18) becomes

$$\partial_k a_{ij} = \partial_j a_{ik}. \tag{2.19}$$

Tsarev [12] has shown that for semi-Hamiltonian systems, commuting flows and conserved densities depend on n arbitrary functions of one argument. Before continuing with the proofs it is worth noting that calculating $\partial_k\partial_j\lambda^i$ by expanding equation (2.19) (note that $\partial_j\partial_k\lambda^i = \partial_k\partial_j\lambda^i$) and substituting this into (2.19) we obtain

$$\partial_k a_{ij} = a_{ij}a_{jk} - a_{ij}a_{ik} + a_{ik}a_{kj}. \tag{2.20}$$

Theorem (Tsarev) *If (2.18) is satisfied then commuting flows depend on n arbitrary functions of one argument.*

Proof:

The proof is taken from [12]. A diagonal system (2.5) that satisfies (2.18) has commuting flows governed by the linear system

$$\partial_j\mu^i = a_{ij}(\mu^j - \mu^i), \quad i \neq j. \tag{2.21}$$

Changing j to k where $k \neq j$, we have

$$\partial_k\mu^i = a_{ik}(\mu^k - \mu^i), \quad i \neq k. \tag{2.22}$$

We need to take partial derivatives of equtions (2.21), (2.22) and show that $\partial_k\partial_j\mu^i = \partial_j\partial_k\mu^i$. Using (2.20), (2.21) and (2.22) we obtain

$$(a_{ij}a_{jk} - a_{ij}a_{ik} + a_{ik}a_{kj})(\mu^j - \mu^i) + a_{ij}(a_{jk}(\mu^k - \mu^j) - a_{ik}(\mu^k - \mu^i)) =$$

$$(a_{ik}a_{kj} - a_{ik}a_{ij} + a_{ij}a_{jk})(\mu^k - \mu^i) + a_{ik}(a_{kj}(\mu^j - \mu^k) - a_{ij}(\mu^j - \mu^i)).$$

Equating coefficients of μ^i, μ^j, μ^k we observe that everything vanishes. We have shown that all partial derivatives of (2.21) are consistent identically in μ^i, μ^j, μ^k. Note that, in equation (2.21), all derivatives $\partial_j\mu^i$ are known for $i \neq j$. We are left with the unknown $\partial_i\mu^i$ as an unknown, so we can specify $\mu^i(R^i)$ on the R^i axis. We can do this for all $i \in 1...n$, so commuting flows depend on n arbitrary functions of a single argument.

Theorem (Tsarev) *If (2.18) is satisfied then conserved densities depend on n arbitrary functions of one argument.*

Proof:

Recall that for a system (2.5) to possess a conservation law $[f(\mathbf{R})]_t = [g(\mathbf{R})]_x$, we must have

$$\partial_i\partial_j f = \frac{\partial_j\lambda^i}{\lambda^j - \lambda^i}\partial_i f + \frac{\partial_i\lambda^j}{\lambda^i - \lambda^j}\partial_j f, \quad i \neq j.$$

Now let

$$\partial_j\partial_i f = a_{ij}\partial_i f + a_{ji}\partial_j f,$$

$$\partial_k \partial_i f = a_{ik} \partial_i f + a_{ki} \partial_k f,$$

and compute $\partial_k \partial_j \partial_i f = \partial_j \partial_k \partial_i f$. Doing this we get

$$\partial_k a_{ij} \partial_i f + a_{ij}(a_{ik} \partial_i f + a_{ki} \partial_k f) + \partial_k a_{ji} \partial_j f + a_{ji}(a_{jk} \partial_j f + a_{kj} \partial_k f) =$$

$$\partial_j a_{ik} \partial_i f + a_{ik}(a_{ij} \partial_i f + a_{ji} \partial_j f) + \partial_j a_{ki} \partial_k f + a_{ki}(a_{kj} \partial_k f + a_{jk} \partial_j f).$$

Equating coefficients of $\partial_i f, \partial_j f, \partial_k f$, we observe that everything cancels. So we know that all mixed partial derivatives of f are consistent. In order to see n functions of 1 variable, note that we can arbitrarily present the value of f on each R^i axis. Thus, conserved densities depend on n arbitrary functions of one variable.

2.7 The generalised hodograph method

The generalised hodograph method [12] can be used to find a general solution to semi-Hamiltonian systems of the form (2.5). As an example, consider a simple scalar equation, the so-called Hopf equation,

$$R_t = R R_x. \tag{2.23}$$

The general solution of this equation is a well known result and is given by

$$f(R) = x + Rt, \tag{2.24}$$

where f is an arbitrary function of one variable. Calculating partial derivatives of (2.24) with respect to x and t, we obtain

$$f_R R_x = 1 + R_x t,$$

$$f_R R_t = R + R_t t.$$

Solving for R_x and R_t we have,

$$R_x = \frac{1}{f_R - t},$$

$$R_t = \frac{R}{f_R - t}.$$

It follows that $R_t = R R_x$, so we see that the above is indeed the general solution for (2.23).

Theorem (Generalised hodograph method) *If $\lambda^i(R)$ satisfies (2.18) then the general solution of the diagonal system*

$$R^i_t = \lambda^i(R) R^i_x, \tag{2.25}$$

is given by

$$\mu^i(R) = \lambda^i(R)t + x, \tag{2.26}$$

where the characteristic speeds of commuting flows $\mu^i(R)$ satisfy the equations:

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}, \quad i \neq j. \tag{2.27}$$

Proof:

The proof is taken from [12]. First, use substitution of (2.26) into (2.27) to obtain,

$$\partial_j \mu^i = \partial_j \lambda^i t. \tag{2.28}$$

Next, differentiate (2.26) by x and t so that

$$\begin{aligned} \partial_j \mu^i R^j_x + \partial_i \mu^i R^i_x &= \partial_j \lambda^i R^j_x t + \partial_i \lambda^i R^i_x t + 1, \\ \partial_j \mu^i R^j_t + \partial_i \mu^i R^i_t &= \partial_j \lambda^i R^j_t t + \partial_i \lambda^i R^i_t t + \lambda^i. \end{aligned} \tag{2.29}$$

Substituting (2.28) into (2.29) we get

$$R^i_x = \frac{1}{\partial_i \mu^i - \partial_i \lambda^i t}, \quad R^i_t = \frac{\lambda^i}{\partial_i \mu^i - \partial_i \lambda^i t}. \tag{2.30}$$

Finally, we prove that (2.26) is a general solution of (2.25) by substituting (2.30) into (2.25). Note here that by general solution we mean that (2.26) has the same amount of freedom as the system (2.25). Taking the data for the initial value problem of (2.25) we have for $t = 0$,

$$R^i(x, 0) = f^i(x),$$

thus we have the freedom of n arbitrary functions of one variable in the system being solved. In (2.26), μ^i also depend on n arbitrary functions of one variable.

2.8 The method of hydrodynamic reductions. Example of dKP

The theory of integrability of one-dimensional hydrodynamic type systems provides the framework for studying the integrability of higher dimensional hydrodynamic type systems. In this section, we present the method of hydrodynamic reductions for the case of $(2+1)-$dimensional quasilinear systems of the form

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0, \tag{2.31}$$

where $u = (u^1, ..., u^l)^T$ is an m-component column vector of dependent variables, and A, B, C are $m \times l$ matrices, where m is the number of equations.

The key construction in the method of hydrodynamic reductions is as follows; we seek multi-phase solutions in the form $u(R^1, ..., R^n)$ where the "phases" $R^i(x, y, t)$ are the Riemann invariants satisfying a pair of commuting diagonal (1+1)-dimensional systems of hydrodynamic type,

$$R^i_y = \mu^i(\mathbf{R})R^i_x, \quad R^i_t = \lambda^i(\mathbf{R})R^i_x. \tag{2.32}$$

The consistency condition $R^i_{yt} = R^i_{ty}$ for these systems is equivalent to the following linear system for the characteristic speeds λ^i and μ^i,

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j, \quad \partial_i = \partial/\partial_{R^i}. \tag{2.33}$$

Specifically, we decouple a (2+1)-dimensional system of hydrodynamic type into a pair of commuting (1 + 1)-dimensional systems (2.32). Solutions of this type are known as nonlinear interactions of n planar waves.

Substituting (2.32) and (2.33) into (2.31), we get

$$(A + \mu^i B + \lambda^i C)\partial_i u = 0, \quad i = 1, ..., n. \tag{2.34}$$

In the case of square matrices A, B and C, equation (2.34) implies that both λ^i and μ^i satisfy the dispersion relation

$$\det(A + \mu^i B + \lambda^i C) = 0.$$

Combining equations (2.33) and (2.34), we end up with a system of equations for u, $\lambda^i(R)$ and $\mu^i(R)$ (so called *Gibbons-Tsarev system*).

Definition [7] A (2+1)-dimensional quasilinear system (2.31) is said to be *integrable* if, for any number of phases n, it possesses infinitely many n-phase solutions parameterised by $2n$ arbitrary functions of a single variable.

Example (dKP equation) Let us derive the Gibbons-Tsarev system for the dKP equation

$$u_{xt} - u_x u_{xx} - u_{yy} = 0,$$

where $u = u(x, y, t)$. All through this example, $\partial_i = \partial/\partial_{R^i}$. Introducing the variables $a = u_x$, $b = u_y$, $c = u_t$, we have the following system of four equations in three unknowns,

$$a_y = b_x, \quad a_t = c_x, \quad b_t = c_y, \quad a_t - aa_x - b_y = 0. \tag{2.35}$$

We look for solutions in the form $a = a(R^1, ..., R^n)$, $b = b(R^1, ..., R^n)$, $c = c(R^1, ..., R^n)$, where the Riemann invariants R^i satisfy (2.32). Substituting this ansatz into $a_y = b_x$ we get:

$$a_y = \partial_i a R^i_y = \partial_i a \mu^i R^i_x, \quad b_x = \partial_i b R^i_x.$$

This simplifies to $\partial_i b = \mu^i \partial_i a$. Similarly, $a_t = c_x$ yields $\partial_i c = \lambda^i \partial_i a$. From $a_t - aa_x - b_y = 0$ we get the dispersion relation $\lambda^i - a - (\mu^i)^2 = 0$. Next, computing the compatibility conditions $\partial_i \partial_j b = \partial_j \partial_i b$ and $\partial_i \partial_j c = \partial_j \partial_i c$ we get the expression,

$$\partial_i \partial_j a = \frac{\partial_j \mu^i}{\mu^j - \mu^i} \partial_i a - \frac{\partial_i \mu^j}{\mu^j - \mu^i} \partial_j a. \tag{2.36}$$

Differentiating the dispersion relation $\lambda^i - a - (\mu^i)^2 = 0$ with respect to R^i, we reduce the system (2.33) to the form

$$\partial_j \mu^i = \frac{\partial_j a}{\mu^j - \mu^i}, \quad i \neq j.$$

Substitution of the last equation into (2.36) yields a consistent system for $a(R)$ and $\mu^i(R)$ (Gibbons-Tsarev system) [7],

$$\partial_j \mu^i = \frac{\partial_j a}{\mu^j - \mu^i}, \quad \partial_i \partial_j a = 2\frac{\partial_i a \partial_j a}{(\mu^j - \mu^i)^2}, \quad i \neq j. \tag{2.37}$$

It is clear that the consistency of this system is equivalent to existence of infinity of hydrodynamic reductions (2.32) of the dKP. To get the general solution of this system, we prescribe $2n$ functions of a single variable as the Goursat data along the R^i-axes, precisely $\mu^i(R^i)$ and $a(R^i)$. As the Gibbons-Tsarev system is invariant under re-parameterisation $R^i \leftarrow f^i(R^i)$, where f^i are arbitrary functions of their arguments, the parametric freedom reduces to n functions of a single variable. A general solution of the system (2.32) is then given by the generalised hodograph method. This brings n arbitrary functions to the parametric freedom of an n-phase solution $u(R^1, ..., R^n)$ (n-component hydrodynamic reduction) of the dKP equation.

2.9 Lax pairs

In $1+1$D, a Lax pair is a pair of linear operators L and A associated with a partial differential equation which can be used to solve the equation. Lax pairs were introduced by Peter Lax to discuss solitons in continuous media. The inverse scattering transform makes use of Lax pairs to solve certain systems. The best way to look at Lax pairs is through an example.

Example 1 The KdV equation

$$u_t = 6uu_x - u_{xxx}$$

can be reformulated as

$$L_t = [P, L].$$

Here $[P, L] = PL - LP$ is the operator commutator and,

$$L = -\partial_x^2 + u,$$

$$P = -4\partial_x^3 + 3(u\partial_x + \partial_x u).$$

In $2+1$D, dispersionless Lax pairs are quite different objects. For instance, the dKP equation

$$(u_t - uu_x)_x = u_{yy},$$

has the Lax pair

$$\begin{aligned} S_y &= \frac{1}{2}S_x^2 + u, \\ S_t &= \frac{1}{3}S_x^3 + uS_x + w. \end{aligned} \tag{2.38}$$

Here, the dKP equation can be recovered by first calculating the compatibility conditions $S_{yt} = S_{ty}$ to get

$$u_t - uu_x = w_y, \quad u_y = w_x. \tag{2.39}$$

Eliminating w, we get $(u_t - uu_x)_x = u_{yy}$ as required. The equations (2.38) are known as a pair of nonlinear Hamilton-Jacobi type equations [24].

Later, it will be explained what is meant by linearly degenerate systems in $2+1$D. For these systems, Lax pairs are given by commuting λ-dependent vector fields. For example, the linearly degenerate (integrable) quasilinear wave equation

$$u_{xt} + u_x u_{yy} - u_y u_{xy}, \tag{2.40}$$

where $u = u(x, y, t)$ is representable as $[X, Y] = 0$. Here

$$X = \lambda\partial_x - u_x\partial_y,$$

$$Y = \partial_t + (\lambda - u_y)\partial_y.$$

Indeed calculating $XY - YX$, where all derivatives act on all objects to the right, yields equation (2.40). Equivalently, the Lax pair can be written as a system,

$$X\phi = \lambda\phi_x - u_x\phi_y = 0,$$

$$Y\phi = \phi_t + (\lambda - u_y)\phi_y = 0.$$

The existence of dispersionless Lax pairs is closely related to integrability by the method of hydrodynamic reductions [7].

2.10 Bäcklund transformations

Bäcklund transformations (named after the Swedish mathematician Albert Victor Bäcklund) first appeared in 1880 when they were used in differential geometry and the theory of differential equations. A Bäcklund transformation is a system of equations (generally nonlinear) that relates a solution of a given differential equation either with another solution of the same equation (auto-Bäcklund) or with a solution of a different differential equation. In general, it is not always known when a PDE possesses a Bäcklund transformation. There are however a limited number of cases where it is known. Every evolution equation that can be solved through the inverse scattering transform possesses a corresponding Bäcklund transformation.

Example: Auto-Bäcklund transformation. Consider the pair of equations

$$\frac{1}{2}(u+v)_x = a\sin\left(\frac{u-v}{2}\right), \quad \frac{1}{2}(u-v)_t = \frac{1}{a}\sin\left(\frac{u+v}{2}\right), \tag{2.41}$$

where $a \neq 0$ and $u = u(x, t)$, $v = v(x, t)$. Differentiating the first equation with respect to t and the second with respect to x, we get

$$\begin{aligned} \frac{1}{2}(u+v)_{xt} &= \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \frac{1}{2}(u-v)_{tx} &= \sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right). \end{aligned} \tag{2.42}$$

From (2.42) and using the fact that $u_{xt} = u_{tx}$ and $v_{xt} = v_{tx}$, we have

$$\begin{aligned} u_{xt} &= \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) + \sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right) \\ &= \sin\left(\frac{u+v}{2} + \frac{u-v}{2}\right) \\ &= \sin u, \end{aligned} \tag{2.43}$$

$$\begin{aligned} v_{xt} &= \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) - \sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right) \\ &= \sin\left(\frac{u+v}{2} - \frac{u-v}{2}\right) \\ &= \sin v, \end{aligned} \tag{2.44}$$

which means that u and v independently satisfy the sine-Gordon equation

$$u_{xt} = \sin u, \quad v_{xt} = \sin v.$$

Hence, equations (2.41) are an auto-Bäcklund transformation for the sine-Gordon equation. Putting $\xi = x - t$ and $\eta = x + t$ gives the more familiar form of the sine-Gordon equation $u_{\eta\eta} - u_{\xi\xi} = \sin u$.

We will now use a Bäcklund transformation to find a solution of the sine-Gordon equation $u_{xt} = \sin u$, starting from a known solution. The sine-Gordon equation has the trivial solution $u(x,t) = 0$. From (2.41) in the last example with $v = 0$, we have

$$u_x = 2a\sin\frac{u}{2}, \quad u_t = \frac{2}{a}\sin\frac{u}{2}. \tag{2.45}$$

Integrating the first equation in (2.45) we get

$$\frac{du}{\sin\frac{u}{2}} = 2a dx \;\Rightarrow\; 2ax = 2\ln|\tan\frac{u}{4}| + f(t). \tag{2.46}$$

Integrating the second equation in (2.45) we get

$$\frac{du}{\sin\frac{u}{2}} = \frac{2}{a}dt \;\Rightarrow\; \frac{2}{a}t = 2\ln|\tan\frac{u}{4}| + g(x) \tag{2.47}$$

Differentiating (2.46) and (2.47) with respect to t and x respectively, we get

$$f'(t) = -\frac{2}{a} \;\Rightarrow\; f(t) = -\frac{2}{a}t + c_1$$

$$g'(x) = -2a \;\Rightarrow\; g(x) = -2ax + c_2.$$

Substituting $f(t)$ and $g(x)$ in equations (2.46), (2.47) and adding them we obtain

$$2ax + \frac{2}{a}t = 4\ln|\tan\frac{u}{4}| - 2ax - \frac{2}{a}t + c_1 + c_2,$$

so that

$$\tan\frac{u}{4} = \pm e^{-\frac{c_1+c_2}{4}} e^{ax+\frac{t}{a}} = Ce^{ax+\frac{t}{a}}, \quad C \in \mathbb{R}.$$

Thus $u(x,t) = 4\arctan\left(Ce^{ax+\frac{t}{a}}\right)$, and we have found a solution of the sine-Gordon equation starting from the trivial solution $u = 0$.

Example 1 Here we take an example from the classification of linearly degenerate quasilinear PDEs in the next section. Let α, β, γ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ be two triplets of numbers such that $\alpha + \beta + \gamma = 0$ and $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}=0$. Consider the system of two first order relations for the functions $u(x,y,t)$ and $v(x,y,t)$,

$$\alpha\tilde{\gamma}v_x u_t - \gamma\tilde{\alpha}v_t u_x = 0, \quad \alpha\tilde{\beta}v_y u_t - \beta\tilde{\alpha}v_t u_y = 0.$$

Eliminating v (that is, solving the above relations for v_x and v_y and imposing the compatibility condition $v_{xy} = v_{yx}$), we obtain the second order equation $\alpha u_t u_{xy} + \beta u_y u_{xt} + \gamma u_x u_{yt} = 0$. Similarly, eliminating u we obtain the analogous equation for v, $\tilde{\alpha}v_t v_{xy} + \tilde{\beta}v_y v_{xt} + \tilde{\gamma}v_x v_{yt} = 0$. We will see later that this example illustrates that any two integrable equations of the Segre type [(11)(11)(11)] are related by a Bäcklund transformation.

Example 2 (dKP equation) The dKP equation has a dispersionless Lax pair consisting of two first-order relations,

$$v_y - \frac{1}{2}v_x^2 - u_x = 0, \quad v_t - \frac{1}{3}v_x^3 - v_x u_x - u_y = 0,$$

with $v = v(x,y,t)$ and $u = u(x,y,t)$. The dKP equation results from the above on elimination of v, that is, via the compatibility condition $v_{yt} = v_{ty}$. Similarly, the elimination of u leads to the modified dKP (mdKP) equation $v_{xt} - (v_y - \frac{1}{2}v_x^2)v_{xx} - v_{yy} = 0$. Thus, the relations above provide a Bäcklund-type transformation connecting dKP and mdKP equations.

2.11 Linearly degenerate systems

Let us first consider the two dimensional case, that is, first order quasilinear equations of the form

$$\mathbf{u}_t - A(\mathbf{u})\mathbf{u}_x = 0, \tag{2.48}$$

where $\mathbf{u} = (u_1, ..., u_n)^T$ is an $n-$component column vector of dependent variables, $u_i = u_i(x,t)$ are functions of two independent variables and A is an $n \times n$ matrix. We call the PDE (2.48) linearly degenerate if the directional derivative of the eigenvalues of A along their corresponding right eigenvectors is zero. In general, this can be verified by introducing the characteristic polynomial of A [6],

$$det(\lambda I - A(\mathbf{u})) = \lambda^n + f_1(\mathbf{u})\lambda^{n-1} + f_2(\mathbf{u})\lambda^{n-2} + \ldots + f_n(\mathbf{u}),$$

and imposing the constraint

$$\nabla f_1 A^{n-1} + \nabla f_2 A^{n-2} + \ldots + \nabla f_n = 0,$$

where ∇ is the operator of the gradient, $\nabla f = (\frac{\partial f}{\partial u^1}, \ldots, \frac{\partial f}{\partial u^n})$, and A^k denotes k-th power of the matrix A. In the $2-$component case this condition simplifies to

$$\nabla(trA)A = \nabla(detA). \tag{2.49}$$

Example Consider the simple 2-component system

$$v_t = wv_x, \quad w_t = vw_x.$$

We can see straight away that the directional derivative of the eigenvalues of A along their corresponding right eigenvectors is zero, and equivalently the constraint $\nabla(trA)A = \nabla(detA)$ is satisfied.

Linear degeneracy is known to prevent breakdown of smooth initial data, leading to global solvability of the Cauchy problem. Later, this notion of linear degeneracy will be extended to quasilinear wave equations in $2 + 1$D.

As an illustrative example, consider the two systems

$$u_t = uu_x, \quad u_t = cu_x,$$

where $u = u(x,t)$ and c is a constant.

Figure 2.1: Time evolution of $u_t = uu_x$ with a Gaussian initial profile.

Figure 2.2: Time evolution of $u_t = cu_x$ with a Gaussian initial profile.

The two figures above illustrate what is meant by global solvability of the Cauchy problem. For the equation $u_t = uu_x$ which is clearly not linearly degenerate, we see that the solution to the Cauchy problem breaks down after sufficient time. This can be seen by the waveform 'breaking' and the function becoming multi-valued. The equation $u_t = cu_x$ clearly *is* linearly degenerate and the image above illustrates that the wave form does not 'break' for all time.

2.12 Characteristics and the symbol of a PDE

In the study of linear partial differential equations a measure of the "strength" of a differential operator in a certain direction is given by the notion of characteristics. If $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ is a linear differential operator of order k on Ω in $\mathbb{R}^n$, then its *characteristic form* (or *principal symbol*) at $x \in \Omega$ is the homogeneous polynomial of degree k on $\mathbb{R}^n$ defined by

$$\chi_L(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi_\alpha,$$

A vector ξ is *characteristic* for L at x if

$$\chi_L(x,\xi) = 0.$$

The *characteristic variety* is the set of all characteristic covectors x_i, i.e.

$$Char_x(L) = \{\xi \neq 0 : \chi_L(x,\xi) = 0\}.$$

A hypersurface S is called characteristic for L at x if the normal vector $\nu(x)$ is in $Char_x(L)$ and S is called non-characteristic otherwise. An important property of the characteristic variety is contained in the following:

Let F be a smooth one-to-one mapping of Ω onto $\Omega' \subset \mathbb{R}^n$ and set $y = F(x)$. Assume that the Jacobian matrix

$$J_x = \left[\frac{\partial y_i}{\partial x_j}\right](x)$$

is nonsingular for $x \in \Omega$, so that $\{y_1, y_2, ..., y_n\}$ is a coordinate system on Ω'. We have

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial y_i}$$

which we can write symbolically as $\partial_x = J_x^T \partial_y$, where J_x^T is the transpose of J_x. The operator L is then transformed into

$$L' = \sum_{|\alpha| \le k} a_\alpha(F^{-1}(y)) \left(J^T_{F^{-1}(y)}\partial_y\right)_\alpha \quad on \quad \Omega'.$$

When this expression is expanded out, there will be some differentiations of $J^T_{F^{-1}(y)}$, but such derivatives are only formed by 'using up' some of the ∂_y on $J^T_{F^{-1}(y)}$, so they do not enter in the computation of the principal symbol in the y coordinates, i.e. they do not enter the highest order terms. We find that

$$\chi_L(x,\xi) = \sum_{|\alpha|=k} a_\alpha(F^{-1}(y)) \left(J^T_{F^{-1}(y)}\xi\right)_\alpha.$$

Now since $F^{-1}(y) = x$, on comparing with the expression

$$\chi_L(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi_\alpha$$

we see that $Char_x(L)$ is the image of $Char_y(L')$ under the linear map $J^T_{F^{-1}(y)}$.

Note that if $\xi \neq 0$ is a vector in the x_j-direction (i.e. $\xi_i = 0$ for $i \neq j$), then $\xi \in Char_x(L)$ if and only if the coefficient of ∂_j^k in L vanishes at x. Now, given any $\xi \neq 0$, by a rotation of coordinates we can arrange for ξ to lie in a coordinate direction. Thus the condition $\xi \in Char_x(L)$ means that, in some sense, L fails to be "genuinely kth order" in the ξ direction at x. L is said to be *elliptic* at x if $Char_x(L) = \emptyset$ and *elliptic on* Ω if it is elliptic at each $x \in \Omega$. Elliptic operators exert control on all derivatives of all order.

Examples.

1. $L = \partial_1 : Char_x(L) = \{\xi \neq 0 : \xi_1 = 0\}$.
2. $L = \partial_1\partial_2 : Char_x(L) = \{\xi \neq 0 : \xi_1 = 0 \quad or \quad \xi_2 = 0\}$.
3. $L = \frac{1}{2}(\partial_1 + i\partial_2)$: *$L$ is elliptic on $\mathbb{R}^2$.*

4. $L = \partial_1 - \sum_{j=2}^{n} \partial_j^2$ *(Heat Operator)*: $Char_x(L) = \{\xi \neq 0 : \xi_j = 0, \text{ for } j \geq 2\}$.

5. $L = \partial_1^2 - \sum_{j=2}^{n} \partial_j^2$ *(Wave Operator)*: $Char_x(L) = \{\xi \neq 0 : \xi_1^2 = \sum_{j=2}^{n} \xi_j^2\}$.

Note that the principal symbol only exists for a linear PDE. Consider the following example which utilizes the formal linearisation of PDEs to find the principle symbol of the dKP equation.

Example Recall that the dKP equation is defined as

$$u_{xt} - u_x u_{xx} - u_{yy} = 0,$$

where $u = u(x, y, t)$. First we set $u = u + \epsilon v$ and substitute into the above. Keeping only linear terms in ϵ, we get

$$v_{xt} - u_x v_{xx} - v_{yy} - u_{xx} v_x = 0.$$

Only the higher order derivatives of v contribute to the symbol. Following the definition outlined earlier in this section, the principal symbol of the dKP equation is

$$\xi^1 \xi^3 - u_x (\xi^1)^2 - (\xi^2)^2.$$

Later, we will see how the notion of a characteristic integral can be defined by using the principal symbol of a PDE. We will draw a parallel between linear degeneracy and characteristic integrals, and see how this can lead to a notion of *Darboux integrability* in higher dimensions.

Remark Throughout this section, the dKP equation has been given in 2 different forms which are equivalent. Taking the following form of the dKP equation

$$(u_t - u u_x)_x = u_{yy},$$

and setting $u = U_x$, we get the equivalent form,

$$U_{xt} - U_x U_{xx} = U_{yy}.$$

So both forms are in fact equivalent.

Chapter 3

Quasilinear wave equations

3.1 Summary of main results

In this section we study second order quasilinear equations of the form

$$f_{11}u_{x_1x_1} + f_{22}u_{x_2x_2} + f_{33}u_{x_3x_3} + 2f_{12}u_{x_1x_2} + 2f_{13}u_{x_1x_3} + 2f_{23}u_{x_2x_3} = 0, \tag{3.1}$$

where $u(x_1, x_2, x_3)$ is a function of three independent variables, and the coefficients f_{ij} depend on the first order derivatives $u_{x_1}, u_{x_2}, u_{x_3}$ only. Throughout this section we assume the non-degeneracy condition $\det f_{ij} \neq 0$. PDEs of this type, which can be called quasilinear wave equations, arise in a wide range of applications in mechanics, general relativity, differential geometry and the theory of integrable systems. One of the most familiar examples is the dispersionless Kadomtsev-Petviashvilli equation,

$$u_{xt} - u_x u_{xx} - u_{yy} = 0,$$

which arises in non-linear acoustics. Another example is the Boyer-Finley equation,

$$u_{xx} + u_{yy} - e^{u_t}u_{tt} = 0,$$

which has been discussed in the field of general relativity. The integrability of equations of the form (3.1) was extensively investigated in [4] where the method of hydrodynamic reductions [7] was used to create a classification of all integrable types of equation (3.1). Details of this classification are given later.

The class of equations (3.1) is invariant under the group $\mathbf{SL}(4)$ of linear transformations of the dependent and independent variables x_i, u, which constitute the natural equivalence group of the problem. Transformations from the equivalence group act projectively on the space $\mathbb{P}^3$ of first order derivatives $p^i = u_{x_i}$, and preserve conformal class of the quadratic form

$$f_{ij}(\mathbf{p})dp^i dp^j. \tag{3.2}$$

Here we concentrate on the particular class of equations (3.1) which are associated with quadratic complexes of lines in projective space $\mathbb{P}^3$. Recall that the Plücker coordinates of a line through the points $\mathbf{p} = (p^1 : p^2 : p^3 : p^4)$ and $\mathbf{q} = (q^1 : q^2 : q^3 : q^4)$ are defined as $p^{ij} = p^i q^j - p^j q^i$. They satisfy the quadratic Plücker relation, $\Omega = p^{23}p^{14} + p^{31}p^{24} + p^{12}p^{34} = 0$. A quadratic line complex is a three-parameter family of lines in $\mathbb{P}^3$ specified by an additional homogeneous quadratic relation among the Plücker coordinates,

$$Q(p^{ij}) = 0.$$

Quadratic line complexes can be classified according to their associated Segre symbol. The details of how this is done are given later, where we show how Jessop classified quadratic complexes into eleven canonical forms [9]. Fixing a point $\mathbf{p}$ in $\mathbb{P}^3$ and taking the lines of the complex which pass through $\mathbf{p}$ one obtains a quadratic cone with vertex at $\mathbf{p}$. This family of cones supplies $\mathbb{P}^3$ with a conformal structure. Its equation can be obtained by setting $q^i = p^i + dp^i$ and passing to a system of affine coordinates, say, $p^4 = 1,\ dp^4 = 0$. Expressions for the Plücker coordinates take the form $p^{4i} = dp^i,\ p^{ij} = p^i dp^j - p^j dp^i,\ i, j = 1, 2, 3$, and the equation of the complex takes the so-called Monge form,

$$Q(dp^i,\ p^i dp^j - p^j dp^i) = f_{ij}(\mathbf{p})dp^i dp^j = 0.$$

This provides the required conformal structure (3.2), and the associated equation (3.1) by putting $u_{x_i} = p^i,\ u_{x_i x_j} = dp^i dp^j$. The singular surface of the complex is defined as the locus of points in $\mathbb{P}^3$ where the conformal structure (3.2) degenerates, $\det f_{ij} = 0$. This is also known as Kummer's quartic with 16 double points. It can be viewed as the locus where equation (3.1) changes its type. Notice that taking two different affine projections of the same complex will lead to two seemingly different PDEs, but they will be related by a change of variables or *equivalence transformation*, and are in fact the same equation. Later we use [4] to check the integrability of these PDEs and thus come up with a complete classification of PDEs associated with quadratic line complexes.

Quadratic line complexes have been extensively investigated in the classical works by Plücker, Kummer, Klein and many other prominent geometers of 19-20th centuries.

Lie studied certain classes of PDEs associated with line complexes. These included first order PDEs governing surfaces which are tangential to the cones of the associated conformal structure, and second order PDEs for surfaces whose asymptotic tangents belong to a given line complex (as well as surfaces conjugate to a given complex). Large part of this theory has nowadays become textbook material. We point out that the correspondence between quadratic complexes and three-dimensional nonlinear wave equations described above has not been discussed in the literature. Our first result gives a characterisation of PDEs (3.1) associated with quadratic complexes.

The following conditions are equivalent:
(1) Equation (3.1)/conformal structure (3.2) is associated with a quadratic line complex.
(2) Equation (3.1) is linearly degenerate.
(3) Conformal structure (3.2) satisfies the condition

$$\partial_{(k}f_{ij)} = \varphi_{(k}f_{ij)}, \tag{3.3}$$

here $\partial_k = \partial_{p^k}$, φ_k *is a covector, and brackets denote complete symmetrization in* $i, j, k \in \{1, 2, 3\}$. *The above equivalence holds in any dimension* ≥ 3.

The equivalence of (1) and (3) is a well-known result [3]. Indeed, (3.3) means that the conformal structure possesses a quadratic complex of null lines. The equivalence of (2) and (3) is the statement of Theorem 1.
Based on the projective classification of quadratic complexes by their Segre types [9], we obtain a complete list of eleven normal forms of linearly degenerate PDEs of the form (3.1) (Theorem 2). For example, the most general linearly degenerate PDE corresponds to the Segre symbol [111111]:

$$(a_1 + a_2u_{x_3}^2 + a_3u_{x_2}^2)u_{x_1x_1} + (a_2 + a_1u_{x_3}^2 + a_3u_{x_1}^2)u_{x_2x_2} + (a_3 + a_1u_{x_2}^2 + a_2u_{x_1}^2)u_{x_3x_3}+$$

$$2(\alpha u_{x_3} - a_3u_{x_1}u_{x_2})u_{x_1x_2} + 2(\beta u_{x_2} - a_2u_{x_1}u_{x_3})u_{x_1x_3} + 2(\gamma u_{x_1} - a_1u_{x_2}u_{x_3})u_{x_2x_3} = 0,$$

here $a_i, \alpha, \beta, \gamma$ are constants such that $\alpha+\beta+\gamma = 0$. The particular choice $\alpha = \beta = \gamma = 0$, $a_1 = a_2 = a_3 = 1$, leads to the equation for minimal hypersurfaces in the Euclidean space E^4,

$$(1 + u_{x_3}^2 + u_{x_2}^2)u_{x_1x_1} + (1 + u_{x_3}^2 + u_{x_1}^2)u_{x_2x_2} + (1 + u_{x_2}^2 + u_{x_1}^2)u_{x_3x_3}+$$

$$-2u_{x_1}u_{x_2}u_{x_1x_2} - 2u_{x_1}u_{x_3}u_{x_1x_3} - 2u_{x_2}u_{x_3}u_{x_2x_3} = 0,$$

while the choice $a_1 = a_2 = a_3 = 0$ results in the nonlinear wave equation,

$$\alpha u_{x_3} u_{x_1 x_2} + \beta u_{x_2} u_{x_1 x_3} + \gamma u_{x_1} u_{x_2 x_3} = 0,$$

$\alpha + \beta + \gamma = 0$.

From this list of eleven normal forms of linearly degenerate PDEs, we can look at the geometry of the associated conformal structures. In our case, a conformal structure is an example of a metric in $\mathbb{P}^3$, considered up to a scalar factor. We call this metric conformally flat if it can be brought to constant coefficients by a change of variables, modulo multiplication by a function. Details of how to calculate whether a metric is conformally flat are given later, where we investigate the flatness of the conformal structures and establish the following result. Although the subject is fairly classical, to the best of our knowledge this is new information.

A quadratic complex defines a flat conformal structure if and only if its Segre symbol is one of the following:

$$[111(111)]^*,\ [(111)(111)],\ [(11)(11)(11)],$$

$$[(11)(112)],\ [(11)(22)],\ [(114)],\ [(123)],\ [(222)],\ [(24)],\ [(33)].$$

Here the asterisk denotes a particular sub-case of $[111(111)]$ *where the matrix* $Q\Omega^{-1}$ *has eigenvalues* $(1, \epsilon, \epsilon^2, 0, 0, 0)$*,* $\epsilon^3 = 1$.

This is our second main result (Theorem 3). Later we give a complete list of normal forms of linearly degenerate *integrable* equations of the form (3.1). In general, the integrability aspects of quasilinear wave equations (3.1) (not necessarily linearly degenerate) were investigated in [4], based on the method of hydrodynamic reductions [7]. It was shown that the moduli space of integrable equations is 20-dimensional. For linearly degenerate PDEs, the integrability is equivalent to the existence of a linear Lax pair of the form

$$\psi_{x_2} = f(u_{x_1}, u_{x_2}, u_{x_3}, \lambda)\psi_{x_1}, \quad \psi_{x_3} = g(u_{x_1}, u_{x_2}, u_{x_3}, \lambda)\psi_{x_1},$$

where λ is an auxiliary spectral parameter, so that (3.1) follows from the compatibility condition $\psi_{x_2 x_3} = \psi_{x_3 x_2}$. It was pointed out in [4] that the flatness of the conformal structure (3.2) is a necessary condition for integrability.

A quadratic complex corresponds to an integrable PDE if and only if its Segre symbol is one of the following:

$$[(11)(11)(11)],\ [(11)(112)],\ [(11)(22)],\ [(123)],\ [(222)],\ [(33)].$$

Modulo equivalence transformations (which are allowed to be complex-valued) this leads to a complete list of normal forms of linearly degenerate integrable PDEs:

Segre symbol [(11)(11)(11)]

$$\alpha u_{x_3}u_{x_1x_2} + \beta u_{x_2}u_{x_1x_3} + \gamma u_{x_1}u_{x_2x_3} = 0, \quad \alpha + \beta + \gamma = 0,$$

Segre symbol [(11)(112)]

$$u_{x_1x_1} + u_{x_1}u_{x_2x_3} - u_{x_2}u_{x_1x_3} = 0,$$

Segre symbol [(11)(22)]

$$u_{x_1x_2} + u_{x_2}u_{x_1x_3} - u_{x_1}u_{x_2x_3} = 0,$$

Segre symbol [(123)]

$$u_{x_2x_2} + u_{x_1x_3} + u_{x_2}u_{x_3x_3} - u_{x_3}u_{x_2x_3} = 0,$$

Segre symbol [(222)]

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = 0,$$

Segre symbol [(33)]

$$u_{x_1x_3} + u_{x_1}u_{x_2x_2} - u_{x_2}u_{x_1x_2} = 0.$$

The canonical forms listed here are a statement of our third main result; Theorem 4. They are not new: in different contexts, they have appeared before in literature. In particular, the same normal forms appeared in [11] in the alternative approach to linear degeneracy based on the requirement of 'non-singular' structure of generalised Gibbons-Tsarev systems which govern hydrodynamic reductions of PDEs in question. The final section contains remarks about the Cauchy problem for linearly degenerate PDEs. We observe that for some linearly degenerate PDEs (3.1), the coefficients f_{ij} can be represented in the form $f_{ij} = \eta_{ij} + \varphi_{ij}$ where η is a constant-coefficient matrix with diagonal entries $1, -1, -1$, while φ_{ij} vanish at the 'origin' $u_{x_1} = u_{x_2} = u_{x_3} = 0$. PDEs of this type can be viewed as nonlinear perturbations of the linear wave equation.

Under the so-called 'null conditions' of Klainerman, the paper [10] establishes global existence of smooth solutions with small initial data for multi-dimensional nonlinear wave equations. It remains to point out that both null conditions are automatically satisfied for linearly degenerate PDEs: they follow from the condition (3.3) satisfied in the vicinity of the origin. Our numerical simulations clearly demonstrate that solutions with small initial data do not break down, and behave essentially like solutions to the linear wave equation. For larger initial conditions we point out that there is no breakdown of solutions, although this is not fully understood and would be an area for further work. The results of this section were published in [1].

3.2 Integrability of quasilinear wave equations

Recall that the method of hydrodynamic reductions is a technique used to check the integrability of differential equations and was first derived in [7], where it was applied to the problem of integrability of (2+1)-dimensional quasilinear systems

$$\mathbf{u}_{x_3} + A(\mathbf{u})\mathbf{u}_{x_1} + B(\mathbf{u})\mathbf{u}_{x_2} = 0, \tag{3.4}$$

where x_1, x_2, x_3 are independant variables, $\mathbf{u}$ is an m-component column vector and $A(\mathbf{u})$, $B(\mathbf{u})$ are $m \times m$ matrices. We now reproduce a result from [4], where integrability conditions for the general equations of interest were first derived. Recall our equations of interest; second order quasilinear PDEs in $(2+1)$ dimensions of the form (3.1),

$$f_{11}u_{x_1x_1} + f_{22}u_{x_2x_2} + f_{33}u_{x_3x_3} + 2f_{12}u_{x_1x_2} + 2f_{13}u_{x_1x_3} + 2f_{23}u_{x_2x_3} = 0,$$

where $u = u(x_1, x_2, x_3)$ and $f_{ij} = f_{ij}(u_{x_1}, u_{x_2}, u_{x_3})$. We now apply the method of hydrodynamic reductions to this general case. Putting $a = u_{x_1}$, $b = u_{x_2}$, $c = u_{x_3}$ transforms equation (3.4) into the required quasilinear form;

$$a_{x_2} = b_{x_1}, \quad a_{x_3} = c_{x_1}, \quad b_{x_3} = c_{x_2},$$

$$f_{11}a_{x_1} + f_{22}b_{x_2} + f_{33}c_{x_3} + 2f_{12}a_{x_2} + 2f_{13}a_{x_3} + 2f_{23}b_{x_3} = 0.$$

Now look for solutions of the form $a = a(R^1, .., R^n)$, $b = b(R^1, .., R^n)$, $c = c(R^1, .., R^n)$, where $R^i(x_1, x_2, x_3)$ are arbitrary solutions of a pair of commuting equations of the form (2.5). Substituting this ansatz into (3.4) and using (2.5) we get

$$b_i = \mu^i a_i, \quad c_i = \lambda^i a_i,$$

together with the dispersion relation

$$D(\lambda^i, \mu^i) = f_{11} + f_{22}(\mu^i)^2 + f_{33}(\lambda^i)^2 + 2f_{12}\mu^i + 2f_{13}\lambda^i + 2f_{23}\mu^i\lambda^i = 0,$$

where the lower indices denote derivative with respect to R^i (i.e. a_i is equivalent to a_{R^i}). Applying consistency conditions, that is, $b_{ij} = b_{ji}$ and $c_{ij} = c_{ji}$ we get

$$a_{ij} = \frac{\lambda^i_j}{\lambda^j - \lambda^i} a_i + \frac{\lambda^j_i}{\lambda^i - \lambda^j} a_j.$$

Differentiating the dispersion relation with respect to R^j and using the above we obtain

$$\lambda^i_j = (\lambda^i - \lambda^j) B_{ij} a_j, \quad \mu^i_j = (\mu^i - \mu^j) B_{ij} a_j,$$

where B_{ij} are rational functions of $\lambda^i, \lambda^j, \mu^i, \mu^j$, the coefficients depend on f_{ij} and their first order derivatives. We then have

$$a_{ij} = -(B_{ij} + B_{ji}) a_i a_j.$$

It now remains to calculate the consistency conditions $\lambda^i_{jk} = \lambda^i_{kj}$, $\mu^i_{jk} = \mu^i_{kj}$ and $a_{ij,k} = a_{ik,j}$. Notice that these conditions involve triples of indices only, thus proving that integrability is equivalent to the existence of 3−component reductions. These relations are manifestly conformally invariant, and without any loss of generality one can set, say, $f_{22} = 1$. The calculation of these conditions for the remaining coefficients $f_{11}, f_{12}, f_{13}, f_{23}, f_{33}$ is a very lengthy calculation and the result is 30 differential relations for the coefficients f_{ij} which are linear in the second order derivatives. They can be represented as

$$d^2 f_{ij} = \frac{1}{F} G(f_{kl}, df_{kl}),$$

with $F = det[f_{ij}]$, and G is a quadratic polynomial in both f_{kl} and their first order derivatives. Thus, we can see that these 30 equations depend only on f_{ij} and their first order derivatives. One can show that these equations are in involution; all consistency conditions are satisfied identically. Since the values of the five functions $f_{11}, f_{12}, f_{13}, f_{23}, f_{33}$ and their first order derivatives are not restricted by any additional constraints, we can conclude that the moduli space of integrable equations is $5 + 3 \times 5 = 20$-dimensional. Programming these equations into Maple, we are now able to verify the integrability of any non-degenerate equation of the form (3.1).

3.3 Linearly degenerate quasilinear wave equations

Here we explain what is meant by linear degeneracy for quasilinear wave equations. First, consider the 2D case of our equations of interest,

$$f_{11}(u_1, u_2)u_{11} + 2f_{12}(u_1, u_2)u_{12} + f_{22}(u_1, u_2)u_{22} = 0, \quad u_i = u_{x_i}. \tag{3.5}$$

Setting $u_1 = p^1$, $u_2 = p^2$ we obtain the equivalent first order quasilinear representation

$$p^1_2 = p^2_1, \quad f_{11}(p^1, p^2)p^1_1 + 2f_{12}(p^1, p^2)p^1_2 + f_{22}(p^1, p^2)p^2_2 = 0. \tag{3.6}$$

We call the second order PDE (3.5) linearly degenerate if this is the case for the corresponding quasilinear system (3.6). Recall the condition of linear degeneracy for a general quasilinear system,

$$\mathbf{v}_2 + A(\mathbf{v})\mathbf{v}_1 = 0,$$

where $\mathbf{v} = (v_1, ..., v_n)$ is the vector of dependent variables, and A is an $n \times n$ matrix. We call the system linearly degenerate if the directional derivative of the eigenvalues of A along their corresponding eigenvectors is zero. In our 2-component case this can be expressed as the condition

$$\nabla(trA)A = \nabla(detA). \tag{3.7}$$

Lemma. *Applying the condition (3.7) to the system (3.6) with* $\mathbf{v} = (p^1, p^2)$ *we obtain the conditions for linear degeneracy in the form*

$$2\partial_1\left(\frac{f_{12}}{f_{11}}\right) + \partial_2\left(\ln\frac{f_{11}}{f_{22}}\right) = 0, \quad 2\partial_2\left(\frac{f_{12}}{f_{22}}\right) + \partial_1\left(\ln\frac{f_{22}}{f_{11}}\right) = 0, \quad \partial_k = \partial_{p^k}. \tag{3.8}$$

Proof:

From (3.6) we obtain our matrix A

$$A = \begin{pmatrix} 0 & -1 \\ \frac{a}{c} & \frac{2b}{c} \end{pmatrix}.$$

Where $a = f_{11}$, $b = f_{12}$, $c = f_{22}$. Now imposing (3.7) we get two equations for a, b, c;

$$\frac{a}{c}\partial_2\left(\frac{-2b}{c}\right) + \partial_1\left(\frac{a}{c}\right) = 0,$$

$$\partial_1\left(\frac{2b}{c}\right) - \left(\frac{2b}{c}\right)\partial_2\left(\frac{2b}{c}\right) + \partial_2\left(\frac{a}{c}\right) = 0.$$

The result $2\partial_2\left(\frac{b}{c}\right)+\partial_1\left(\ln\frac{c}{a}\right)=0$ follows directly from multiplying the first equation by $\frac{c}{a}$. Multiplication of the second equation by $\frac{c}{a}$ gives

$$2\partial_1\left(\frac{b}{c}\right)\frac{c}{a}-4\left(\frac{b}{a}\right)\partial_2\left(\frac{b}{c}\right)+\partial_2\ln\left(\frac{a}{c}\right)=0.$$

Substitution of the first equation for $\partial_2\left(\frac{b}{c}\right)$ gives the required result; $2\partial_1\left(\frac{b}{a}\right)+\partial_2\left(\ln\frac{a}{c}\right)=0$. These equations can be integrated implicitly leading to the following form of linearly degenerate second order PDEs:

Proposition. *The general linearly degenerate PDE of the form (3.5) can be represented in the form*

$$r^1u_{11}-(1+r^1r^2)u_{12}+r^2u_{22}=0.$$

Here the functions $r^1(u_1,u_2)$ and $r^2(u_1,u_2)$ are defined by implicit relations

$$f(r^1)=u_2-r^1u_1,\quad g(r^2)=u_1-r^2u_2,$$

where f,g are two arbitrary functions.

Proof:

Setting

$$\frac{f_{11}}{f_{12}}=-2\frac{r^1}{1+r^1r^2},\quad \frac{f_{22}}{f_{12}}=-2\frac{r^2}{1+r^1r^2},$$

and substituting into (3.8) we get, after simplification

$$-\frac{r^1_1}{(r^1)^2}-r^2_1+\frac{r^1_2}{r^1}-\frac{r^2_2}{r^2}=0,\quad -\frac{r^2_2}{(r^2)^2}-r^1_2+\frac{r^2_1}{r_2}-\frac{r^1_1}{r^1}=0,\quad r^i_j=\partial_jr^i.$$

This leads to a pair of uncoupled Hopf equations for r^1 and r^2, $\partial_1r^1+r^1\partial_2r^1=0, \partial_2r^2+r^2\partial_1r^2=0$. Their implicit solutions lead to the required result.

Remark 1. The choice $f=\sqrt{r^1}, g=\sqrt{r^2}$ leads to the so-called Born-Infeld equation,

$$u_2^2u_{11}-(1+2u_1u_2)u_{12}+u_1^2u_{22}=0,$$

while the complex choice $f=i\sqrt{1+(r^1)^2}, g=i\sqrt{1+(r^2)^2}$ leads to the (elliptic) equation for minimal surfaces,

$$(1+u_2^2)u_{11}-2u_1u_2u_{12}+(1+u_1^2)u_{22}=0.$$

Remark 2. The conditions (3.8) can be represented in tensorial form,

$$\partial_{(k}f_{ij)}=\phi_{(k}f_{ij)}, \tag{3.9}$$

here $\partial_k = \partial_{p^k}$, ϕ_k is a covector, and brackets denote a complete symmetrization in the indices i, j, k which take values $1, 2$. Explicitly, this gives

$$\partial_1 f_{11} = \phi_1 f_{11}, \quad \partial_2 f_{22} = \phi_2 f_{22},$$

$$\partial_2 f_{11} + 2\partial_1 f_{12} = \phi_2 f_{11} + 2\phi_1 f_{12}, \quad \partial_1 f_{22} + 2\partial_2 f_{12} = \phi_1 f_{22} + 2\phi_2 f_{12},$$

and the elimination of ϕ_1, ϕ_2 from the first two relations lead to the conditions of linear degeneracy (3.8).

Now consider our 3D case, we say that a PDE of the form (3.1),

$$f_{11}u_{x_1x_1} + f_{22}u_{x_2x_2} + f_{33}u_{x_3x_3} + 2f_{12}u_{x_1x_2} + 2f_{13}u_{x_1x_3} + 2f_{23}u_{x_2x_3} = 0,$$

is linearly degenerate if all its traveling wave reductions to two dimensions are linearly degenerate in the 2D sense as described above. More precisely, setting $u(x_1, x_2, x_3) = u(\xi_1, \xi_2)$, where $\xi_1 = x_1 + \alpha x_3, \xi_2 = x_2 + \beta x_3$, and substituting into (3.1) we get

$$(f_{11}+2\alpha f_{13}+\alpha^2 f_{33})u_{\xi_1\xi_1}+2(f_{12}+\alpha f_{23}+\beta f_{13}+\alpha\beta f_{33})u_{\xi_1\xi_2}+(f_{22}+2\beta f_{23}+\beta^2 f_{33})u_{\xi_2\xi_2} = 0. \tag{3.10}$$

We require that (3.10) is linearly degenerate for all α, β in the sense of the 2D case. The requirement of linear degeneracy for *any* α, β imposes strong constraints on the coefficients f_{ij}. They are, in tensorial form;

$$\partial_1 f_{11} = \phi_1 f_{11}, \quad \partial_2 f_{22} = \phi_2 f_{22}, \quad \partial_3 f_{33} = \phi_3 f_{33},$$

$$\partial_2 f_{11} + 2\partial_1 f_{12} = \phi_2 f_{11} + 2\phi_1 f_{12}, \quad \partial_1 f_{22} + 2\partial_2 f_{12} = \phi_1 f_{22} + 2\phi_2 f_{12},$$

$$\partial_3 f_{11} + 2\partial_1 f_{13} = \phi_3 f_{11} + 2\phi_1 f_{13}, \quad \partial_1 f_{33} + 2\partial_3 f_{13} = \phi_1 f_{33} + 2\phi_3 f_{13},$$

$$\partial_2 f_{33} + 2\partial_3 f_{23} = \phi_2 f_{33} + 2\phi_3 f_{23}, \quad \partial_3 f_{22} + 2\partial_2 f_{23} = \phi_3 f_{22} + 2\phi_2 f_{23}$$

$$\partial_1 f_{23} + \partial_2 f_{13} + \partial_3 f_{12} = \phi_1 f_{23} + \phi_2 f_{13} + \phi_3 f_{12}.$$

This is in fact the first main result of this study, and shows how linearly degenerate PDEs in 3D are related to quadratic line complexes.

3.4 Quadratic line complexes

In this section we introduce a notion from projective geometry, allowing us to look at the integrability of linearly degenerate PDEs. This section is based on [18].

To begin with, recall the concept of projective space;

Definition. *In projective geometry, real projective space* $\mathbb{P}^n(\mathbb{R})$ *is defined as*

$$\mathbb{P}^n(\mathbb{R}) := (\mathbb{R}^{n+1}\backslash\{0\}),\sim$$

with the equivalence relation $(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n)$ where $x_i \in \mathbb{R}$ and λ is any non-zero real number. In other words, projective space $\mathbb{P}^n$ is the set of all real $n+1$ tuples defined up to a common multiple, excluding the zero tuple. Equivalently, it can be thought of as the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin $(0, ..., 0)$.

Consider a line r in $\mathbb{P}^3$ passing through the points $\mathbf{p} = (p^1 : p^2 : p^3 : p^4)$ and $\mathbf{q} = (q^1 : q^2 : q^3 : q^4)$. The so called Plücker coordinates p^{ij} are given by the six 2×2 minors of the matrix

$$\begin{pmatrix} p^1 & p^2 & p^3 & p^4 \\ q^1 & q^2 & q^3 & q^4 \end{pmatrix}.$$

Explicitly, $p^{ij} = p^i q^j - p^j q^i$. From linear algebra we know that the determinant of a matrix with at least two common rows or columns is zero. We use this fact to see that the p^{ij} are not arbitrary, rather they satisfy a quadratic relation

$$\begin{vmatrix} p^1 & p^2 & p^3 & p^4 \\ q^1 & q^2 & q^3 & q^4 \\ p^1 & p^2 & p^3 & p^4 \\ q^1 & q^2 & q^3 & q^4 \end{vmatrix} = 2(p^{12}p^{34} + p^{13}p^{42} + p^{14}p^{23}) = 0.$$

From the previous result, we see that the Plücker coordinates satisfy the relation

$$F(p^{ij}) = p^{12}p^{34} + p^{13}p^{42} + p^{14}p^{23} = 0. \tag{3.11}$$

Proposition. *There exists a bijection between the Plücker coordinates and lines in* $\mathbb{P}^3$.

Proof:

We begin by showing that the ratios of the p^{ij} depend only on the line r, not the points $\mathbf{p}$**,** $\mathbf{q}$ taken on r. Indeed, if we take two other distinct points $\mathbf{p'}$**,** $\mathbf{q'}$ on r where

$$(p')^i = \lambda p^i + \mu q^i, \quad (q')^i = \lambda' p^i + \mu' q^i, \qquad i = 1, 2, 3, 4, \quad \lambda\mu' - \lambda'\mu \neq 0,$$

then we find that

$$(p')^{ij} = \begin{vmatrix} \lambda p^i + \mu q^i & \lambda p^j + \mu q^j \\ \lambda' p^i + \mu' q^i & \lambda' p^j + \mu' q^j \end{vmatrix} = (\lambda\mu' - \lambda'\mu)p^{ij}.$$

So we have shown that to each line in $\mathbb{P}^3$ there are associated six numbers p^{ij} defined up to a common factor that are not all zero. We can then associate each line in $\mathbb{P}^3$ to a point in $\mathbb{P}^5$.

Now, let $p^{12}, p^{13}, p^{14}, p^{23}, p^{24}, p^{34}$ be arbitrary Plücker coordinates, satisfying (3.11) and consider the points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{P}^3$

$$\mathbf{p} = (0 : p^{12} : p^{13} : p^{14}), \quad \mathbf{q} = (p^{12} : 0 : p^{23} : p^{24}).$$

Suppose that $p^{12} \neq 0$ so that $\mathbf{p} \neq \mathbf{q}$ and consider the line $r_{\mathbf{pq}}$ formed by $\mathbf{p}$ and $\mathbf{q}$. The Plücker coordinates $(p^{ij})'$ of this line are given by the second order minors of the matrix

$$\begin{pmatrix} 0 & p^{12} & p^{13} & p^{14} \\ p^{21} & 0 & p^{23} & p^{24} \end{pmatrix}.$$

Using the fact that $p^{ij} = -p^{ji}$ and bearing in mind equation (3.11), we find that

$$(p^{12})' = -p^{12}p^{21} = (p^{12})^2,$$

$$(p^{13})' = -p^{21}p^{13} = p^{12}p^{13},$$

$$(p^{14})' = -p^{21}p^{14} = p^{12}p^{14},$$

$$(p^{23})' = p^{12}p^{23},$$

$$(p^{24})' = p^{12}p^{24},$$

$$(p^{34})' = p^{13}p^{24} - p^{14}p^{23} = p^{13}p^{24} + p^{14}p^{32} = p^{12}p^{34}.$$

Notice that p^{12} is a common factor and so the $(p^{ij})'$ are proportional to p^{ij}. This shows that the line $r_{\mathbf{pq}}$ is determined by the numbers p^{ij}. Thus we have shown that there exists a bijection between lines in $\mathbb{P}^3$ and the Plücker coordinates p^{ij}.

In $\mathbb{P}^5$ we can now define a point as $(p^{12} : p^{13} : p^{14} : p^{23} : p^{24} : p^{34})$, where the equation (3.11) represents a four dimensional subset of $\mathbb{P}^5$, called the Plücker quadric. All the points of this quadric are in bijective correspondence with lines in $\mathbb{P}^3$. The lines in $\mathbb{P}^3$ whose coordinates p^{ij} satisfy an extra equation

$$G(p^{ij}) = 0, \tag{3.12}$$

where G is a homogeneous polynomial of degree n, gives an algebraic complex of degree n. We say that (3.12) is the equation of the complex. In $\mathbb{P}^5$ it can be thought of as the intersection of the Plücker quadric by the surface given by equation (3.12). For $n = 1$ we have a linear complex, for $n = 2$ we have a quadratic complex.

Remark. *The lines of an algebraic complex of order n that pass through a point $p \in \mathbb{P}^3$ form an algebraic cone of order n having vertex at p.*

For a quadratic complex, fixing a point $\mathbf{p}$ in $\mathbb{P}^3$ and taking the lines of the complex which pass through $\mathbf{p}$, one obtains a quadratic cone with vertex at $\mathbf{p}$. The family of these cones supplies $\mathbb{P}^3$ with a conformal structure. Its equation can be obtained by setting $q^i = p^i + dp^i$ and passing to a system of affine coordinates, say, $p^4 = 1$, $dp^4 = 0$. The expressions for the Plücker coordinates take the form $p^{4i} = dp^i$, $p^{ij} = p^i dp^j - p^j dp^i$, $i, j = 1, 2, 3$ and the equation of the complex takes the so-called Monge form,

$$Q(dp^i, p^i dp^j - p^j dp^i) = f_{ij} dp^i dp^j = 0.$$

It is important to notice that, when passing to a system of affine coordinates, any projection can be used, i.e. setting $p^1 = 1$, $dp^1 = 0$ is equivalent. We now make the most of a useful result from projective geometry.

Lemma. *Let $Q = f_{ij} dp^i dp^j$ be the Monge form of a quadratic complex defined as before. The following are equivalent.*
(a) $\partial_{(k} f_{ij)} = \phi_{(k} f_{ij)}$.
(b) A manifold of cones is generated by a quadratic complex in $\mathbb{P}^3$.

This is a known result and the proof is omitted here. Those interested in the proof can find it in [3], page 282. What this Lemma is basically saying is that if you take a line in $\mathbb{P}^3$, then all the quadratic cones with a vertex on this line in fact lie tangential to this line.

We now associate a PDE to the given Monge form of quadratic complex in the following way; make the substitution $u_{x_i} = p^i$ and $u_{x_i x_j} = dp^i dp^j$, so that we obtain a PDE of our required form,

$$f_{11}u_{11} + f_{22}u_{22} + f_{33}u_{33} + 2f_{12}u_{12} + 2f_{13}u_{13} + 2f_{23}u_{23} = 0, \quad u_{ij} = u_{x_i x_j}.$$

Notice here that two different affine projections of the complex will give two different PDEs, but they will be equal via a change of variables or *equivalence transformation.* In other words, in the world of the quadratic complexes we have different projections, in the world of PDEs we have the equivalence relation. From the Monge form of the complex we can also find the associated Kummer surface. In algebraic geometry, a Kummer quartic surface, first studied by Kummer (1864), is an irreducible algebraic surface of degree 4 in $\mathbb{P}^3$. In the context of quadratic complexes, it is defined as $det(f_{ij}) = 0$. It can be described as the boundary when the associated PDE passes from hyperbolic to being elliptic, or vice versa. It turns out that looking at the Kummer surface can give interesting insights into the integrability of the associated PDE, in fact, in all integrable cases the associated Kummer surface degenerates into a collection of planes in $\mathbb{P}^3$. However, we see that the converse is not true. Looking at the Kummer surface is also a good way of finding out if two PDEs are equivalent or not; if the conformal structures of the respective PDEs admit different Kummer surfaces, then they are not related via an equivalence relation. The process of deriving a PDE from a quadratic complex is best illustrated through looking at some examples.

Example 1. The so-called tetrahedral complex, see [9], chapter 7, is defined by the equation $Q = b_1p^{41}p^{23} + b_2p^{42}p^{31} + b_3p^{43}p^{12} = 0$. Its Monge form is

$$b_1dp^1(p^2dp^3 - p^3dp^2) + b_2dp^2(p^3dp^1 - p^1dp^3) + b_3dp^3(p^1dp^2 - p^2dp^1) = 0,$$

or, equivalently,

$$(b_3 - b_2)p^1dp^2dp^3 + (b_1 - b_3)p^2dp^1dp^3 + (b_2 - b_1)p^3dp^1dp^2 = 0,$$

which corresponds to the integrable dispersionless Hirota equation,

$$(b_3 - b_2)u_1u_{23} + (b_1 - b_3)u_2u_{13} + (b_2 - b_1)u_3u_{12} = 0.$$

The associated Kummer surface is

$$det\begin{pmatrix} 0 & (b_1 - b_2)u_3 & (b_3 - b_1)u_2 \\ (b_1 - b_2)u_3 & 0 & (b_2 - b_3)u_1 \\ (b_3 - b_1)u_2 & (b_2 - b_3)u_1 & 0 \end{pmatrix} = -2(b_1-b_2)u_3(b_2-b_3)u_1(-b_3+b_1)u_2 = 0.$$

We can see that this is a product of four planes: $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, plus the plane at infinity.

Example 2. The so-called special complex, see [9], chapter 7, is defined by the equation $Q = (p^{12})^2 + (p^{13})^2 + (p^{23})^2 - (p^{14})^2 - (p^{24})^2 - (p^{34})^2 = 0$. Its Monge form is

$$(p^1dp^2 - p^2dp^1)^2 + (p^1dp^3 - p^3dp^1)^2 + (p^2dp^3 - p^3dp^2)^2 - (dp^1)^2 - (dp^2)^2 - (dp^3)^2 = 0,$$

or, equivalently,

$$((p^2)^2 + (p^3)^2 - 1)(dp^1)^2 + ((p^1)^2 + (p^3)^2 - 1)(dp^2)^2 + ((p^1)^2 + (p^2)^2 - 1)(dp^3)^2$$

$$-2p^1p^2dp^1dp^2 - 2p^1p^3dp^1dp^3 - 2p^2p^3dp^2dp^3 = 0.$$

It corresponds to the equation

$$(u_2^2+u_3^2-1)u_{11}+(u_1^2+u_3^2-1)u_{22}+(u_1^2+u_2^2-1)u_{33}-2u_1u_2u_{12}-2u_1u_3u_{13}-2u_2u_3u_{23} = 0.$$

The associated Kummer surface is the sphere $(p^1)^2 + (p^2)^2 + (p^3)^2 = 1$ (taken with multiplicity two). The external part of the sphere is the domain of hyperbolicity of our equation: quadratic cones of the complex are tangential to the sphere. We point out that the equation for minimal surfaces is not integrable in dimensions higher than two.

We now present our first main result of this study.

Theorem 1 *A PDE (3.1) is linearly degenerate if and only if the corresponding conformal structure $f_{ij}dp^idp^j$ satisfies the constraint*

$$\partial_{(k}f_{ij)} = \phi_{(k}f_{ij)}, \tag{3.13}$$

here $\partial_k = \partial_{p^k}$, ϕ_k is a covector, and brackets denote a complete symmetrisation in the indices i, j, k which take values $1, 2, 3$.

Proof:

Let us seek traveling wave reductions in the form $u(x_1, x_2, x_3) = u(\xi, \eta)+\alpha x_1+\beta x_2+\gamma x_3$ where $\xi = x_1 + \lambda x_3,\ \eta = x_2 + \mu x_3$, and $\alpha, \beta, \gamma, \lambda, \mu$ are arbitrary constants. We have

$$u_{x_1} = u_\xi + \alpha, \quad u_{x_2} = u_\eta + \beta, \quad u_{x_3} = \lambda u_\xi + \mu u_\eta + \gamma,$$

as well as

$$u_{x_1x_1} = u_{\xi\xi}, \quad u_{x_1x_2} = u_{\xi\eta}, \quad u_{x_2x_2} = u_{\eta\eta},$$
$$u_{x_1x_3} = \lambda u_{\xi\xi} + \mu u_{\xi\eta}, \quad u_{x_2x_3} = \lambda u_{\xi\eta} + \mu u_{\eta\eta}, \quad u_{x_3x_3} = \lambda^2 u_{\xi\xi} + 2\lambda\mu u_{\xi\eta} + \mu^2 u_{\eta\eta}.$$

The reduced equation (3.1) takes the form

$$au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} = 0,$$

where

$$a = f_{11} + 2\lambda f_{13} + \lambda^2 f_{33}, \quad b = f_{12} + \lambda f_{23} + \mu f_{13} + \lambda\mu f_{33}, \quad c = f_{22} + 2\mu f_{23} + \mu^2 f_{33},$$

we point out that the coefficients a, b, c are now viewed as functions of u_ξ and u_η. For the reduced equation, the conditions of linear degeneracy take the form

$$\partial_{u_\xi} a = \varphi_1 a, \quad \partial_{u_\eta} c = \varphi_2 c, \quad \partial_{u_\eta} a + 2\partial_{u_\xi} b = \varphi_2 a + 2\varphi_1 b, \quad \partial_{u_\xi} c + 2\partial_{u_\eta} b = \varphi_1 c + 2\varphi_2 b.$$

Let us take the first condition, $\partial_{u_\xi} a = \varphi_1 a$. The calculation of $\partial_{u_\xi} a$ gives

$$\partial_{u_\xi} a = \partial_1 f_{11} + \lambda\partial_3 f_{11} + 2\lambda(\partial_1 f_{13} + \lambda\partial_3 f_{13}) + \lambda^2(\partial_1 f_{33} + \lambda\partial_3 f_{33}),$$

which is polynomial in λ of degree three. We point out that, due to the presence of arbitrary constants α, β, γ in the expressions for $u_{x_1}, u_{x_2}, u_{x_3}$, the coefficients of this polynomial can be viewed as independent of λ, μ. Thus, φ_1 must be linear in λ, so that we can set $\varphi_1 \to \varphi_1 + \lambda\varphi_3$ (keeping the same notation φ_1 for the first term). Ultimately, the relation $\partial_{u_\xi} a = \varphi_1 a$ takes the form

$$\partial_1 f_{11} + \lambda\partial_3 f_{11} + 2\lambda(\partial_1 f_{13} + \lambda\partial_3 f_{13}) + \lambda^2(\partial_1 f_{33} + \lambda\partial_3 f_{33}) =$$

$$(\varphi_1 + \lambda\varphi_3)(f_{11} + 2\lambda f_{13} + \lambda^2 f_{33}).$$

Equating terms at different powers of λ we obtain four relations,

$$\partial_1 f_{11} = \varphi_1 f_{11}, \quad \partial_3 f_{33} = \varphi_3 f_{33},$$

$$\partial_3 f_{11} + 2\partial_1 f_{13} = \varphi_3 f_{11} + 2\varphi_1 f_{13}, \quad \partial_1 f_{33} + 2\partial_3 f_{13} = \varphi_1 f_{33} + 2\varphi_3 f_{13}.$$

Similar analysis of the three remaining conditions of linear degeneracy of the reduced equation (where one should set $\varphi_2 \to \varphi_2 + \mu\varphi_3$) leads to the full set (3.3) of conditions of linear degeneracy in $3D$:

$$\partial_1 f_{11} = \varphi_1 f_{11}, \quad \partial_2 f_{22} = \varphi_2 f_{22}, \quad \partial_3 f_{33} = \varphi_3 f_{33},$$

$$\partial_2 f_{11} + 2\partial_1 f_{12} = \varphi_2 f_{11} + 2\varphi_1 f_{12}, \quad \partial_1 f_{22} + 2\partial_2 f_{12} = \varphi_1 f_{22} + 2\varphi_2 f_{12},$$

$$\partial_3 f_{11} + 2\partial_1 f_{13} = \varphi_3 f_{11} + 2\varphi_1 f_{13}, \quad \partial_1 f_{33} + 2\partial_3 f_{13} = \varphi_1 f_{33} + 2\varphi_3 f_{13},$$

$$\partial_2 f_{33} + 2\partial_3 f_{23} = \varphi_2 f_{33} + 2\varphi_3 f_{23}, \quad \partial_3 f_{22} + 2\partial_2 f_{23} = \varphi_3 f_{22} + 2\varphi_2 f_{23},$$

$$\partial_1 f_{23} + \partial_2 f_{13} + \partial_3 f_{12} = \varphi_1 f_{23} + \varphi_2 f_{13} + \varphi_3 f_{12}.$$

On elimination of φ's, these conditions give rise to seven first order differential constraints for f_{ij}. This finishes the proof of Theorem 1.

This theorem provides a link between the world of the quadratic complex and our linearly degenerate PDEs. It directly follows from this theorem that there is a bijection between quadratic line complexes and linearly degenerate PDEs of our type. Now, [9] provides a systematic classification of these complexes, so using this we are able to come up with a classification of integrable linearly degenerate PDEs of our type.

Side note 1: Classification of quadratic line complexes, Jessop 1903

In this section we go into some detail of how Jessop categorised quadratic complexes and see that the general equation for the complex can be reduced to eleven different canonical forms. Recall that a quadratic line complex is defined as the intersection of the Plücker quadric by the hyperquadric Q, where Q is a quadratic relation between the Plücker coordinates p^{ij}. The general form of a quadratic complex is then given as

$$\alpha(p^{12})^2 + \beta(p^{13})^2 + ... = 0$$

where, including all mixed terms, there are 21 terms in total. Now, we associate symmetric 6×6 matrices to the equations for the Plücker quadric and the quadratic complex. The equation for the Plücker quadric, $2(p^{12}p^{34} + p^{13}p^{42} + p^{14}p^{23}) = 0$, is represented as

	p^{12}	p^{13}	p^{14}	p^{23}	p^{42}	p^{34}
p^{12}						1
p^{13}					1	
p^{14}				1		
p^{23}			1			
p^{42}		1				
p^{34}	1					

We now take the 6×6 matrix of entries and put 0 for a blank space. For the Plücker quadric we call this matrix Ω. Now take the equation of the complex and proceed in the same way, call the associated matrix Q. Now calculate $Q\Omega^{-1}$ and bring it to Jordan normal form. From the Jordan normal form we extract the so-called Segre symbol of the complex, this is given by the number of Jordan blocks of the matrix. For example, two 3×3 Jordan blocks gives Segre symbol [33], one 2×2 block and one 4×4 block gives Segre symbol [24]. A single 2×2 block and four single 1×1 blocks gives Segre symbol [21111]. In addition, if the eigenvalues of different blocks are the same, then we use the rounded brackets notation (refined Segre symbol). So, again if we have a lone 2×2 block and four single blocks, with two of the single blocks equal, then this has Segre symbol [211(11)]. The derivation of the Segre symbol from a given quadratic complex is best seen through an example.

Example. Recall from earlier, the so-called special complex, see [9], chapter 7, is defined by the equation $Q = (p^{12})^2 + (p^{13})^2 + (p^{23})^2 - (p^{14})^2 - (p^{24})^2 - (p^{34})^2 = 0$. The matrix Q is therefore

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Calculating $Q\Omega^{-1}$ and bringing to Jordan normal form we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can see that the Segre symbol in this case is [(111)(111)]. Jessop used a similar system and classified the general equation for the quadratic complex. He found that there were eleven canonical forms given by eleven different Segre symbols. Some of the forms have sub-cases that occur when certain Jordan blocks have the same eigenvalues, for more details see [9]. We can now take these eleven canonical forms, derive the corresponding linearly degenerate PDEs and thus come up with a complete list of linearly degenerate equations of our type.

Side note 2: Conformal flatness of a metric

Here we go into some detail of how to check whether a metric is conformally flat. Recall that the family of quadratic cones generated by a given quadratic complex endows $\mathbb{P}^3$ with a conformal structure $f_{ij}dp^idp^j$.

Definition *We call the metric $f_{ij}dp^idp^j$ conformally flat if, after multiplication by some* *function, an appropriate change of variables can be made such that the coefficients are made constant.*

Example. Take the metric

$$\alpha p^1 dp^2 dp^3 + \beta p^2 dp^1 dp^3 + \gamma p^3 dp^1 dp^2.$$

If we multiply through by $\frac{1}{p^1p^2p^3}$ we get

$$\alpha \frac{dp^2 dp^3}{p^2p^3} + \beta \frac{dp^1 dp^3}{p^1p^3} + \gamma \frac{dp^1 dp^2}{p^1p^2}.$$

We can now use the fact that $\frac{dp^i}{p^i} = d\ln p^i$, and make the change of variables $\widetilde{p^i} = \ln p^i$ so that we get

$$\alpha d\widetilde{p}^2 d\widetilde{p}^3 + \beta d\widetilde{p}^1 d\widetilde{p}^3 + \gamma d\widetilde{p}^1 d\widetilde{p}^2.$$

So we have been able to bring this metric to constant coefficients. Therefore, it is conformally flat.

Lemma. *If a PDE of the form (3.1) is integrable, then the corresponding conformal structure is conformally flat.*

The proof to this Lemma is not presented here, but can be found in [4]. This result highlights the links between conformal geometry and integrable systems. Before this study, we conjectured that there is a one-to-one correspondence between integrable, linearly degenerate PDEs and conformally flat metrics. In other words, if the conformal structure of a given quadratic complex is conformally flat, then the associated PDE is integrable. However, as we see later, this is not universally the case.

We need a systematic way of working out whether a metric is conformally flat or not, we can then apply this method to each of the eleven cases and come up with a classification. We do this by calculating the Cotton tensor of the metric. There is a classical result from differential geometry that states that, for any metric in $\mathbb{P}^3$, the vanishing of the Cotton tensor is equivalent to the metric being conformally flat. The Cotton tensor is calculated in the following way [25].

Take the symmetric matrix of coefficients f_{ij} from the conformal structure. We denote f^{kl} the entries of the matrix inverse $(f_{ij})^{-1}$ and calculate the Levi-Civita connection,

$$\Gamma^i_{jk} = \frac{1}{2}\sum_r f^{ir}[\partial_j f_{rk} + \partial_k f_{jr} - \partial_r f_{jk}].$$

Next calculate the Riemann curvature tensor

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}.$$

Next we sum over i to get the Ricci tensor

$$R_{jl} = R^i_{jil}.$$

Finally, we calculate the scalar curvature

$$R = f^{ik} R_{ki},$$

and form the result $R_{ij} - \frac{1}{4}Rf_{ij}$. We can now calculate the Cotton tensor, and the requirement that is equals zero is equivalent to

$$\nabla_k(R_{ij} - \frac{1}{4}Rf_{ij}) = \nabla_j(R_{ik} - \frac{1}{4}Rf_{ik}), \quad \forall i, j, k.$$

In reality, it is impossible to calculate this by hand, we used the computer program Maple to calculate the Cotton tensor for all eleven cases of the quadratic complex.

3.5 Normal forms of quadratic complexes and linearly degenerate PDEs

In this section we utilise the projective classification of quadratic line complexes following [9]. Recall that quadratic complexes are characterised by the so-called Segre symbols which can be derived from their Monge form. The corresponding conformal structures result from the equation of the complex upon setting $p^{ij} = p^i dp^j - p^j dp^i$, $p^4 = 1$, $dp^4 = 0$, as explained in the previous section (in some cases it will be more convenient to use different affine projections, say, $p^1 = 1$, $dp^1 = 0$: this will be indicated explicitly where appropriate). Here is the summary of our results. Theorem 2 gives a complete list of normal forms of linearly degenerate PDEs based on the classification of quadratic complexes (for simplicity, we use the notation $u_{x_i} = u_i$, $u_{x_i x_j} = u_{ij}$, etc). Theorem 3 provides a classification of complexes with the flat conformal structure $f_{ij} dp^i dp^j$, and Theorem 4 characterises complexes corresponding to integrable PDEs. Theorems 2-4 will be proved simultaneously by going through the list of normal forms of quadratic complexes.

Theorem 2 *Any linearly degenerate PDE of the form (3.1) can be brought by an equivalence transformation to one of the eleven canonical forms, labeled by Segre symbols of the associated quadratic complexes.*

Case 1: Segre symbol [111111]

$$(a_1 + a_2u_3^2 + a_3u_2^2)u_{11} + (a_2 + a_1u_3^2 + a_3u_1^2)u_{22} + (a_3 + a_1u_2^2 + a_2u_1^2)u_{33}+$$

$$2(\alpha u_3 - a_3u_1u_2)u_{12} + 2(\beta u_2 - a_2u_1u_3)u_{13} + 2(\gamma u_1 - a_1u_2u_3)u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

Case 2: Segre symbol [11112]

$$(\lambda u_2^2 + \mu u_3^2 + 1)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}+$$

$$2(\alpha u_3 - \lambda u_1u_2)u_{12} + 2(\beta u_2 - \mu u_1u_3)u_{13} + 2\gamma u_1u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

Case 3: Segre symbol [1113]

$$(\lambda u_2^2 + \mu u_3^2 + 2u_3)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33}+$$

$$2(\mu u_3 - \lambda u_1u_2 - 1)u_{12} + 2(\beta u_2 - \mu u_1u_3 - u_1)u_{13} + 2\gamma u_1u_{23} = 0,$$

$\mu + \beta + \gamma = 0.$

Case 4: Segre symbol [1122]

$$(\lambda u_2^2 + 1)u_{11} + (\lambda u_1^2 + 4)u_{22} + \lambda u_{33} + 2(\alpha u_3 - \lambda u_1u_2)u_{12} + 2\beta u_2u_{13} + 2\gamma u_1u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

Case 5: Segre symbol [114]

$$\lambda u_{11} + (\lambda u_3^2 + 4)u_{22} + (\lambda u_2^2 - 2u_1)u_{33} + 2\alpha u_3u_{12} + 2(u_3 - \alpha u_2)u_{13} - 2\lambda u_2u_3u_{23} = 0.$$

Case 6: Segre symbol [123]

$$\lambda u_{11}+(\lambda u_3^2+4)u_{22}+[\lambda u_2^2+2u_2]u_{33}+2\alpha u_3u_{12}+2(1-\lambda u_2)u_{13}+2(\gamma u_1-\lambda u_2u_3-u_3)u_{23} = 0,$$

$\alpha - \lambda + \gamma = 0.$

Case 7: Segre symbol [222]

Subcase 1:

$$u_{11} + u_{22} + u_{33} + 2\alpha u_3 u_{12} + 2\beta u_2 u_{13} + 2\gamma u_1 u_{23} = 0,$$

Subcase 2:

$$(u_2^2 + u_3^2)u_{11} + (u_1^2 + u_3^2)u_{22} + (u_1^2 + u_2^2)u_{33}$$

$$+2(\alpha u_3 - u_1 u_2)u_{12} + 2(\beta u_2 - u_1 u_3)u_{13} + 2(\gamma u_1 - u_2 u_3)u_{23} = 0,$$

$\alpha + \beta + \gamma = 0.$

Case 8: Segre symbol [15]

$$\lambda u_{11} + (\lambda u_3^2 - 2u_3)u_{22} + (\lambda u_2^2 - 4u_1)u_{33} + 2(\lambda u_3 + 1)u_{12} + 2(2u_3 - \lambda u_2)u_{13}$$

$$+2(u_2 - \lambda u_2 u_3)u_{23} = 0.$$

Case 9: Segre symbol [24]

Subcase 1:

$$u_{11} + u_{22} - 2u_1 u_{33} + 2\lambda u_3 u_{12} + 2(u_3 - \lambda u_2)u_{13} = 0.$$

Subcase 2:

$$u_3^2 u_{22} + (1 + u_2^2)u_{33} + 2u_{12} + 2\lambda u_2 u_{13} - 2(\lambda u_1 + u_2 u_3)u_{23} = 0.$$

Case 10: Segre symbol [33]

$$\lambda u_{11} + (\lambda u_3^2 - 2u_3)u_{22} + (\lambda u_2^2 - 2u_2)u_{33} + 2(\lambda u_3 + 1)u_{12}$$

$$+2(\lambda u_2 + 1)u_{13} - 2(2\lambda u_1 + \lambda u_2 u_3 - u_2 - u_3)u_{23} = 0.$$

Case 11: Segre symbol [6]

Subcase 1:

$$2u_3 u_{11} + u_{22} + 2u_2 u_{33} - 2u_1 u_{13} - 2u_3 u_{23} = 0.$$

Subcase 2:

$$(u_3^2 - 2u_2)u_{11} - 2u_3 u_{22} + u_1^2 u_{33} + 2u_1 u_{12} - 2u_1 u_3 u_{13} + 2u_2 u_{23} = 0.$$

Calculating the Cotton tensor (whose vanishing is responsible for conformal flatness in three dimensions) we obtain a complete list of quadratic complexes with the flat conformal structure. Recall that the flatness of $f_{ij}dp^i dp^j$ is a necessary condition for integrability of the corresponding PDE [4]. We observe that the requirement of conformal flatness imposes further constraints on the parameters appearing in cases 1-11 of Theorem 2, which are characterised by certain coincidences among eigenvalues of the corresponding Jordan normal forms of $Q\Omega^{-1}$ (some Segre types do not possess conformally flat specialisations at all). In what follows we label conformally flat sub-cases by their 'refined' Segre symbols, e.g., the symbol [(11)(11)(11)] denotes the sub-case of [111111] with three pairs of coinciding eigenvalues, the symbol [(111)(111)] denotes the sub-case with two triples of coinciding eigenvalues, etc, see [9]. Although the subject sounds very classical, we were not able to find a reference to the following result.

Theorem 3 *A quadratic complex defines a flat conformal structure if and only if its Segre symbol is one of the following:*

$$[111(111)]^*, \ [(111)(111)], \ [(11)(11)(11)],$$

$$[(11)(112)], \ [(11)(22)], \ [(114)], \ [(123)], \ [(222)], \ [(24)], \ [(33)].$$

Here the asterisk denotes a particular sub-case of [111(111)] *where the matrix* $Q\Omega^{-1}$ *has eigenvalues* $(1, \epsilon, \epsilon^2, 0, 0, 0)$, $\epsilon^3 = 1$. *Modulo equivalence transformations this gives the following list of normal forms of the associated PDEs:*

Segre symbol $[111(111)]^*$

$$(1 - 2u_2u_3)u_{11} + (1 - 2u_1u_3)u_{22} + 2(u_1 - u_2)u_{33} +$$

$$2(1 + u_1u_3 + u_2u_3)u_{12} + 2(u_1u_2 - u_3 - u_2^2)u_{13} + 2(u_1u_2 + u_3 - u_1^2)u_{23} = 0,$$

Segre symbol $[(111)(111)]$

$$(u_2^2+u_3^2-1)u_{11}+(u_1^2+u_3^2-1)u_{22}+(u_1^2+u_2^2-1)u_{33}-2u_1u_2u_{12}-2u_1u_3u_{13}-2u_2u_3u_{23} = 0,$$

Segre symbol $[(11)(11)(11)]$

$$\alpha u_3u_{12} + \beta u_2u_{13} + \gamma u_1u_{23} = 0, \quad \alpha + \beta + \gamma = 0,$$

Segre symbol $[(11)(112)]$

$$u_{11} + u_1u_{23} - u_2u_{13} = 0,$$

Segre symbol [(11)(22)]

$$u_{12} + u_2u_{13} - u_1u_{23} = 0,$$

Segre symbol [(114)]

$$u_{22} + u_1u_{33} - u_3u_{13} = 0,$$

Segre symbol [(123)]

$$u_{22} + u_{13} + u_2u_{33} - u_3u_{23} = 0,$$

Segre symbol [(222)]

$$u_{11} + u_{22} + u_{33} = 0,$$

Segre symbol [(24)]

$$u_3^2u_{22} + (1 + u_2^2)u_{33} + 2u_{12} - 2u_2u_3u_{23} = 0,$$

Segre symbol [(33)]

$$u_{13} + u_1u_{22} - u_2u_{12} = 0.$$

Since conformal flatness is a necessary condition for integrability, a complete list of linearly degenerate integrable PDEs can be obtained by going through the list of Theorem 3 and either calculating the integrability conditions as derived in [4], or verifying the existence of a Lax pair. A direct computation shows that the requirement of integrability eliminates Segre types $[111(111)]^*$, $[(111)(111)]$, $[(114)]$, $[(24)]$, leading to the following result:

Theorem 4 *A linearly degenerate PDE of the form (3.1) is integrable if and only if the corresponding complex has one of the following Segre types:*

$$[(11)(11)(11)],\ [(11)(112)],\ [(11)(22)],\ [(123)],\ [(222)],\ [(33)].$$

Modulo equivalence transformations, this leads to the five canonical forms of linearly degenerate integrable PDEs (we exclude the linearisable case with Segre symbol [(222)]*). For each integrable equation we calculated its Lax pair in the form* $[X, Y] = 0$ *where* X *and* Y *are parameter-dependent vector fields which commute modulo the corresponding equation:*

Segre symbol [(11)(11)(11)]

$$\alpha u_3 u_{12} + \beta u_2 u_{13} + \gamma u_1 u_{23} = 0,$$

$\alpha + \beta + \gamma = 0$. *Setting* $\alpha = a - b,\ \beta = b - c,\ \gamma = c - a$ *we obtain the Lax pair:* $X = \partial_{x^3} - \frac{\lambda - b}{\lambda - c}\frac{u_3}{u_1}\partial_{x^1},\ Y = \partial_{x^2} - \frac{\lambda - b}{\lambda - a}\frac{u_2}{u_1}\partial_{x^1}.$

Segre symbol [(11)(112)]

$$u_{11} + u_1 u_{23} - u_2 u_{13} = 0,$$

Lax pair: $X = \partial_{x^1} - \lambda u_1 \partial_{x^3},\ Y = \partial_{x^2} + (\lambda^2 u_1 - \lambda u_2)\partial_{x^3}.$

Segre symbol [(11)(22)]

$$u_{12} + u_2 u_{13} - u_1 u_{23} = 0,$$

Lax pair: $X = \lambda \partial_{x^1} - u_1 \partial_{x^3},\ Y = (\lambda - 1)\partial_{x^2} - u_2 \partial_{x^3}.$

Segre symbol [(123)]

$$u_{22} + u_{13} + u_2 u_{33} - u_3 u_{23} = 0,$$

Lax pair: $X = \partial_{x^2} + (\lambda - u_3)\partial_{x^3},\ Y = \partial_{x^1} + (\lambda^2 - \lambda u_3 + u_2)\partial_{x^3}.$

Segre symbol [(33)]

$$u_{13} + u_1 u_{22} - u_2 u_{12} = 0,$$

Lax pair: $X = \lambda \partial_{x^1} - u_1 \partial_{x^2},\ Y = \partial_{x^3} + (\lambda - u_2)\partial_{x^2}.$

Remark 1 The five canonical forms from Theorem 4 are not new: in different contexts, they have appeared before [11]. The non-equivalence of the above PDEs can also be seen by calculating the Kummer surfaces of the corresponding line complexes. In all cases the Rummer surfaces degenerate into a collection of planes:

– case 1: four planes in general position, one of them at infinity.

– case 2: two double planes, one of them at infinity.

– case 3: three planes, one of them double, with the double plane at infinity.

– case 4: one quadruple plane at infinity.

– case 5: two planes, one of them triple, with the triple plane at infinity.

Remark 2 Although all equations from Theorem 3 are not related via the equivalence group **SL**(4), there may exist more complicated Bäcklund-type links between them.

Thus, let α, β, γ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ be two triplets of numbers such that $\alpha + \beta + \gamma = 0$ and $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}$=0. Consider the system of two first order relations for the functions u and v,

$$\alpha\tilde{\gamma}v_1u_3 - \gamma\tilde{\alpha}v_3u_1 = 0, \quad \alpha\tilde{\beta}v_2u_3 - \beta\tilde{\alpha}v_3u_2 = 0.$$

Eliminating v (that is, solving the above relations for v_1 and v_2 and imposing the compatibility condition $v_{12} = v_{21}$), we obtain the second order equation $\alpha u_3u_{12} + \beta u_2u_{13} + \gamma u_1u_{23} = 0$. Similarly, eliminating u we obtain the analogous equation for v, $\tilde{\alpha}v_3v_{12} + \tilde{\beta}v_2v_{13} + \tilde{\gamma}v_1v_{23} = 0$. This construction first appeared in [19] in the context of Veronese webs in 3D. It shows that any two integrable equations of the Segre type [(11)(11)(11)] are related by a Bäcklund transformation. Similarly, the relations

$$(\lambda - 1)v_2 - u_2v_3 = 0, \quad \lambda v_1 - u_1v_3 = 0$$

provide a Bäcklund transformation between the equation for u, $u_{12}+u_2u_{13}-u_1u_{23} = 0$, and the equation for v, $v_3v_{12}+(\lambda-1)v_2v_{13}-\lambda v_1v_{23} = 0$, thus establishing the equivalence of integrable equations of the types [(11)(22)] and [(11)(11)(11)].

Proof of Theorems 2–4:

We follow the classification of quadratic complexes as presented in [9], p. 206-232. This constitutes eleven canonical forms which are analysed case-by-case below. In each case we calculate the conditions of vanishing of the Cotton tensor (responsible for conformal flatness in three dimensions), as well as the integrability conditions as derived in [4]. Recall that conformal flatness is a necessary condition for integrability: this requirement already leads to a compact list of conformally flat sub-cases which can be checked for integrability by calculating the Lax pair. Our results are summarised as follows.

Case 1 (generic): Segre symbol [111111]**.** The equation of the complex is

$$\lambda_1(p^{12}+p^{34})^2-\lambda_2(p^{12}-p^{34})^2+\lambda_3(p^{13}+p^{42})^2-\lambda_4(p^{13}-p^{42})^2+\lambda_5(p^{14}+p^{23})^2-\lambda_6(p^{14}-p^{23})^2 = 0,$$

here λ_i are the eigenvalues of $Q\Omega^{-1}$. Its Monge form is

$$[a_1+a_2(p^3)^2+a_3(p^2)^2](dp^1)^2+[a_2+a_1(p^3)^2+a_3(p^1)^2](dp^2)^2+[a_3+a_1(p^2)^2+a_2(p^1)^2](dp^3)^2+$$

$$2[\alpha p^3 - a_3p^1p^2]dp^1dp^2 + 2[\beta p^2 - a_2p^1p^3]dp^1dp^3 + 2[\gamma p^1 - a_1p^2p^3]dp^2dp^3 = 0,$$

where $a_1 = \lambda_5 - \lambda_6,\ a_2 = \lambda_3 - \lambda_4,\ a_3 = \lambda_1 - \lambda_2,\ \alpha = \lambda_5 + \lambda_6 - \lambda_3 - \lambda_4,\ \beta = \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6,\ \gamma = \lambda_3 + \lambda_4 - \lambda_1 - \lambda_2$, notice that $\alpha + \beta + \gamma = 0$. The corresponding

PDE takes the form

$$(a_1 + a_2u_3^2 + a_3u_2^2)u_{11} + (a_2 + a_1u_3^2 + a_3u_1^2)u_{22} + (a_3 + a_1u_2^2 + a_2u_1^2)u_{33} +$$

$$2(\alpha u_3 - a_3u_1u_2)u_{12} + 2(\beta u_2 - a_2u_1u_3)u_{13} + 2(\gamma u_1 - a_1u_2u_3)u_{23} = 0,$$

which is the 1st case of Theorem 2. The analysis of integrability/conformal flatness leads to the four subcases, depending on how many a's equal zero.

Sub-case 1: $a_1 = a_2 = a_3 = 0$. This sub-case, which corresponds to the so-called tetrahedral complex, is integrable and conformally flat, leading to the nonlinear wave equation [19],

$$\alpha u_3u_{12} + \beta u_2u_{13} + \gamma u_1u_{23} = 0.$$

The Kummer surface of this complex consists of four planes in $\mathbb{P}^3$ in general position. The lines of the complex intersect these planes at four points with constant cross-ratio (depending on α, β, γ). The corresponding affinor $Q\Omega^{-1}$ has three pairs of coinciding eigenvalues. The notation for such complexes is [(11)(11)(11)].

Sub-case 2: $a_1 = a_2 = 0$. This sub-case possesses no non-degenerate integrable specialisations. The conditions of conformal flatness imply $\alpha = -2\beta,\ a_3 = \pm\beta$. For any choice of the sign the corresponding affinor $Q\Omega^{-1}$ has two triples of coinciding eigenvalues. Complexes of this type are denoted [(111)(111)], and are known as 'special': they consist of tangent lines to a non-degenerate quadric surface in P^3. A particular example of this type is the PDE for minimal surfaces in Minkowski space.

Sub-case 3: $a_1 = 0$. The further analysis splits into two essentially different branches. The first branch corresponds to $\gamma = 0,\ a_3 = \pm a_2$, in this case we have both conformal flatness and integrability. The corresponding complexes are the same as in sub-case 1, with Segre symbols [(11)(11)(11)]. The second branch corresponds to $\beta = \alpha,\ a_2 = \pm\alpha,\ a_3^2 + 3\alpha^2 = 0$ or $\beta = \alpha,\ a_3 = \pm\alpha,\ a_2^2 + 3\alpha^2 = 0$. All these sub-cases are conformally flat, but not integrable. They are projectively equivalent to each other, with the same Segre symbol $[111(111)]^*$ where the asterisk indicates that the eigenvalues of the (traceless) operator $Q\Omega^{-1}$ are proportional to $(1, \epsilon, \epsilon^2, 0, 0, 0)$, here ϵ is a cubic root of unity, $\epsilon^3 = 1$. There exists an equivalent real normal form of complexes of this type, the simplest one we found is

$$(p^{24} + p^{14})^2 + 2(p^{12} + p^{34})(p^{23} + p^{31}) = 0.$$

The corresponding Monge form is

$$[1-2p^2p^3](dp^1)^2+[1-2p^1p^3](dp^2)^2+2(p^1-p^2)(dp^3)^2+$$

$$2[1+p^1p^3+p^2p^3]dp^1dp^2+2[p^1p^2-p^3-(p^2)^2]dp^1dp^3+2[p^1p^2+p^3-(p^1)^2]dp^2dp^3=0,$$

with the associated PDE

$$(1-2u_2u_3)u_{11}+(1-2u_1u_3)u_{22}+2(u_1-u_2)u_{33}+$$

$$2(1+u_1u_3+u_2u_3)u_{12}+2(u_1u_2-u_3-u_2^2)u_{13}+2(u_1u_2+u_3-u_1^2)u_{23}=0,$$

which is not integrable, although the corresponding conformal structure is flat. The associated Kummer surface is a double quadric, $2p^3+(p^1)^2-(p^2)^2$. This is the first case of Theorem 3.

Sub-case 4: All $a's$ are nonzero. Here we have three essentially different branches which, however, give no new examples. Thus, the first branch corresponds to $a_1=\epsilon_1\gamma,\ a_2=\epsilon_2\beta,\ a_3=\epsilon_3\alpha,\ \epsilon_i=\pm1$, in all these cases we have both conformal flatness and integrability. The corresponding complexes are the same as in sub-case 1, with Segre symbols [(11)(11)(11)]. The second branch is $\alpha=\beta=\gamma=0,\ a_2=\epsilon_2a_1,\ a_3=\epsilon_3a_1,\ \epsilon_i=\pm1$. This coincides with sub-case 2, with Segre symbol [(111)(111)]. The third branch is $a_1=\epsilon_1\frac{\gamma^2}{\alpha-\beta},\ a_2=\epsilon_2\frac{\beta^2}{\gamma-\alpha},\ a_3=\epsilon_3\frac{\alpha^2}{\beta-\gamma},\ \epsilon_i=\pm1$, where $\alpha,\beta,\gamma\in\{1,\epsilon,\epsilon^2\}$ are three distinct cubic roots of unity. This is the same as sub-case 3, with Segre symbol $[111(111)]^*$.

Case 2: Segre symbol [11112]**.** The equation of the complex is

$$\lambda_1(p^{12}+p^{34})^2-\lambda_2(p^{12}-p^{34})^2+\lambda_3(p^{13}+p^{42})^2-\lambda_4(p^{13}-p^{42})^2+4\lambda_5p^{14}p^{23}+(p^{14})^2=0.$$

Its Monge form is

$$[\lambda(p^2)^2+\mu(p^3)^2+1](dp^1)^2+[\lambda(p^1)^2+\mu](dp^2)^2+[\mu(p^1)^2+\lambda](dp^3)^2+$$

$$2[\alpha p^3-\lambda p^1p^2]dp^1dp^2+2[\beta p^2-\mu p^1p^3]dp^1dp^3+2\gamma p^1dp^2dp^3=0,$$

where $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$, $\alpha=-\lambda_3-\lambda_4+2\lambda_5$, $\beta=\lambda_1+\lambda_2-2\lambda_5$, $\gamma=-\alpha-\beta$, so that the corresponding PDE is

$$(\lambda u_2^2+\mu u_3^2+1)u_{11}+(\lambda u_1^2+\mu)u_{22}+(\mu u_1^2+\lambda)u_{33}+$$

$$2(\alpha u_3 - \lambda u_1 u_2)u_{12} + 2(\beta u_2 - \mu u_1 u_3)u_{13} + 2\gamma u_1 u_{23} = 0.$$

This is the 2nd case of Theorem 2. We verified that in this case conditions of integrability are equivalent to conformal flatness, leading to the following subcases.

Subcase 1: $\lambda = \mu = 0,\ \alpha = 0$ (the possibility $\lambda = \mu = 0,\ \beta = 0$ is equivalent to $\alpha = 0$ via the interchange of indices 2 and 3), which simplifies to

$$u_{11} + 2\beta(u_2 u_{13} - u_1 u_{23}) = 0.$$

Modulo a rescaling this gives the corresponding sub-cases of Theorems 3-4.

Sub-case 2: $\beta = -\alpha,\ \lambda = \epsilon_1\alpha,\ \mu = \epsilon_2\alpha,\ \epsilon_i = \pm 1$. One can show that sub-case 2 is equivalent to sub-case 1: all such complexes have the same Segre type [(11)(112)].

Case 3: Segre symbol [1113]**.** The equation of the complex is

$$\lambda_1(p^{12}+p^{34})^2-\lambda_2(p^{12}-p^{34})^2-\lambda_3(p^{13}-p^{42})^2+\lambda_4(p^{13}+p^{42})^2+4\lambda_4 p^{14}p^{23}+2p^{14}(p^{13}+p^{42}) = 0.$$

Its Monge form is

$$[\lambda(p^2)^2 + \mu(p^3)^2 + 2p^3](dp^1)^2 + [\lambda(p^1)^2 + \mu](dp^2)^2 + [\mu(p^1)^2 + \lambda](dp^3)^2 +$$

$$2[\mu p^3 - \lambda p^1 p^2 - 1]dp^1 dp^2 + 2[\beta p^2 - \mu p^1 p^3 - p^1]dp^1 dp^3 + 2\gamma p^1 dp^2 dp^3 = 0,$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_4 - \lambda_3$, $\beta = \lambda_1 + \lambda_2 - 2\lambda_4$, $\gamma = -\mu - \beta$, so that the corresponding PDE is

$$(\lambda u_2^2 + \mu u_3^2 + 2u_3)u_{11} + (\lambda u_1^2 + \mu)u_{22} + (\mu u_1^2 + \lambda)u_{33} +$$

$$2(\mu u_3 - \lambda u_1 u_2 - 1)u_{12} + 2(\beta u_2 - \mu u_1 u_3 - u_1)u_{13} + 2\gamma u_1 u_{23} = 0.$$

This is the 3rd case of Theorem 2. One can show that it possesses no non-degenerate integrable/conformally flat sub-cases.

Case 4: Segre symbol [1122]**.** The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + 4\lambda_3 p^{13}p^{42} + 4\lambda_4 p^{14}p^{23} + (p^{13})^2 + 4(p^{23})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^3 = 1,\ dp^3 = 0$ we obtain the associated Monge equation,

$$[\lambda(p^2)^2 + 1](dp^1)^2 + [\lambda(p^1)^2 + 4](dp^2)^2 + \lambda(dp^4)^2$$

$$+2[\alpha p^4 - \lambda p^1 p^2]dp^1 dp^2 + 2\beta p^2 dp^1 dp^4 + 2\gamma p^1 dp^2 dp^4,$$

where $\lambda = \lambda_1 - \lambda_2,\ \alpha = 2\lambda_4 - 2\lambda_3,\ \beta = 2\lambda_3 - \lambda_1 - \lambda_2,\ \gamma = -\alpha - \beta$, so that the corresponding PDE is

$$(\lambda u_2^2 + 1)u_{11} + (\lambda u_1^2 + 4)u_{22} + \lambda u_{44} + 2(\alpha u_4 - \lambda u_1 u_2)u_{12} + 2\beta u_2 u_{14} + 2\gamma u_1 u_{24} = 0.$$

Relabelling independent variables gives the 4th case of Theorem 2. In this case conditions of conformal flatness are equivalent to the integrability, leading to $\lambda = \alpha = 0$,

$$u_{11} + 4u_{22} + 2\beta(u_2 u_{14} - u_1 u_{24}) = 0.$$

Modulo elementary changes of variables this gives the corresponding sub-cases of Theorems 3-4, with Segre symbol [(11)(22)].

Case 5: Segre symbol [114]**.** The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + 4\lambda_3(p^{14}p^{23} + p^{42}p^{13}) + 2p^{14}p^{42} + 4(p^{13})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation,

$$\lambda(dp^2)^2 + [\lambda(p^4)^2 + 4](dp^3)^2 + [\lambda(p^3)^2 - 2p^2](dp^4)^2 +$$

$$2\alpha p^4 dp^2 dp^3 + 2[p^4 - \alpha p^3]dp^2 dp^4 - 2\lambda p^3 p^4 dp^3 dp^4 = 0,$$

where $\lambda = \lambda_1 - \lambda_2$, $\alpha = 2\lambda_3 - \lambda_1 - \lambda_2$, so that the corresponding PDE is

$$\lambda u_{22} + (\lambda u_4^2 + 4)u_{33} + (\lambda u_3^2 - 2u_2)u_{44} + 2\alpha u_4 u_{23} + 2(u_4 - \alpha u_3)u_{24} - 2\lambda u_3 u_4 u_{34} = 0.$$

Relabeling independent variables gives the 5th case of Theorem 2. One can show that this equation is not integrable. The condition of conformal flatness gives $\lambda = \alpha = 0$,

$$4u_{33} - 2u_2 u_{44} + 2u_4 u_{24} = 0.$$

Such complexes are denoted [(114)]. Modulo elementary changes of variables this gives the corresponding sub-case of Theorem 3.

Case 6: Segre symbol [123]**.** The equation of the complex is

$$-\lambda_1(p^{12} - p^{34})^2 + 4\lambda_2 p^{13}p^{42} + 4(p^{13})^2 + \lambda_3(4p^{14}p^{23} + (p^{12} + p^{34})^2) + 2p^{14}(p^{12} + p^{34}) = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation,

$$\lambda(dp^2)^2 + [\lambda(p^4)^2 + 4](dp^3)^2 + [\lambda(p^3)^2 + 2p^3](dp^4)^2 +$$

$$2\alpha p^4dp^2dp^3+2[1-\lambda p^3]dp^2dp^4+2[\gamma p^2-\lambda p^3p^4-p^4]dp^3dp^4=0,$$

where $\lambda=\lambda_3-\lambda_1$, $\alpha=2\lambda_2-\lambda_1-\lambda_3$, $\gamma=\lambda-\alpha$, so that the corresponding PDE is

$$\lambda u_{22}+(\lambda u_4^2+4)u_{33}+(\lambda u_3^2+2u_3)u_{44}+2\alpha u_4u_{23}+2(1-\lambda u_3)u_{24}+2(\gamma u_2-\lambda u_3u_4-u_4)u_{34}=0.$$

Relabelling independent variables gives the 6th case of Theorem 2. In this case conditions of conformal flatness are equivalent to the integrability. One can show that both require $\lambda=\alpha=\gamma=0$, which gives

$$2u_{33}+u_{24}+u_3u_{44}-u_4u_{34}=0.$$

Appropriate relabeling and rescaling give the corresponding sub-cases of Theorems 3-4, denoted [(123)].

Case 7: Segre symbol [222]**.** Here we have two (projectively dual) sub-cases. In sub-case 1 the equation of the complex is

$$2\lambda_1p^{12}p^{34}+2\lambda_2p^{13}p^{42}+2\lambda_3p^{14}p^{23}+(p^{12})^2+(p^{13})^2+(p^{14})^2=0.$$

Setting $p^{ij}=p^idp^j-p^jdp^i$ and using the affine projection $p^1=1,\ dp^1=0$ we obtain the associated Monge equation,

$$(dp^2)^2+(dp^3)^2+(dp^4)^2+2\alpha p^4dp^2dp^3+2\beta p^3dp^2dp^4+2\gamma p^2dp^3dp^4=0,$$

where $\alpha=\lambda_2-\lambda_1,\ \beta=\lambda_1-\lambda_3,\ \gamma=\lambda_3-\lambda_2$, so that the corresponding PDE is

$$u_{22}+u_{33}+u_{44}+2\alpha u_4u_{23}+2\beta u_3u_{24}+2\gamma u_2u_{34}=0.$$

Setting $\alpha=\beta=\gamma=0$ we obtain the linear equation. The corresponding Segre symbol is [(222)]. One can show that the above PDE is not integrable/conformally flat for nonzero values of constants. This is the linearisable sub-case of Theorems 3-4.

In sub-case 2 the equation of the complex is

$$2\lambda_1p^{12}p^{34}+2\lambda_2p^{13}p^{42}+2\lambda_3p^{14}p^{23}+(p^{23})^2+(p^{24})^2+(p^{34})^2=0.$$

Setting $p^{ij}=p^idp^j-p^jdp^i$ and using the affine projection $p^1=1,\ dp^1=0$ we obtain the associated Monge equation,

$$((p^3)^2+(p^4)^2)(dp^2)^2+((p^2)^2+(p^4)^2)(dp^3)^2+((p^2)^2+(p^3)^2)(dp^4)^2+$$

$$2(\alpha p^4 - p^2p^3)dp^2dp^3 + 2(\beta p^3 - p^2p^4)dp^2dp^4 + 2(\gamma p^2 - p^3p^4)dp^3dp^4 = 0,$$

so that the corresponding PDE is

$$(u_3^2 + u_4^2)u_{22} + (u_2^2 + u_4^2)u_{33} + (u_2^2 + u_3^2)u_{44} + 2(\alpha u_4 - u_2u_3)u_{23}$$

$$+2(\beta u_3 - u_2u_4)u_{24} + 2(\gamma u_2 - u_3u_4)u_{34} = 0.$$

One can show that this sub-case possesses no non-degenerate integrable/conformally flat specialisations (notice that for $\alpha = \beta = \gamma = 0$ this PDE becomes degenerate). Relabeling independent variables gives the 7th case of Theorem 2.

Case 8: Segre symbol [15]**.** The equation of the complex is

$$-\lambda_1(p^{12} - p^{34})^2 + \lambda_2(4p^{14}p^{23} + 4p^{13}p^{42} + (p^{12} + p^{34})^2) + 4p^{14}p^{42} + 2p^{13}(p^{12} + p^{34}) = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation,

$$\lambda(dp^2)^2 + [\lambda(p^4)^2 - 2p^4](dp^3)^2 + [\lambda(p^3)^2 - 4p^2](dp^4)^2 +$$

$$2[\lambda p^4 + 1]dp^2dp^3 + 2[2p^4 - \lambda p^3]dp^2dp^4 + 2[p^3 - \lambda p^3p^4]dp^3dp^4 = 0,$$

where $\lambda = \lambda_2 - \lambda_1$, so that the corresponding PDE is

$$\lambda u_{22} + (\lambda u_4^2 - 2u_4)u_{33} + (\lambda u_3^2 - 4u_2)u_{44} + 2(\lambda u_4 + 1)u_{23} + 2(2u_4 - \lambda u_3)u_{24} + 2(u_3 - \lambda u_3u_4)u_{34} = 0.$$

One can show that this PDE possesses no integrable/conformally flat specialisations. Relabeling independent variables gives the 8th case of Theorem 2.

Case 9: Segre symbol [24]**.** Here we have two (projectively dual) sub-cases. In sub-case 1 the equation of the complex is

$$2\lambda_1 p^{12}p^{34} + (p^{12})^2 + 2\lambda_2(p^{14}p^{23} + p^{13}p^{42}) + 2p^{14}p^{42} + (p^{13})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation,

$$(dp^2)^2 + (dp^3)^2 - 2p^2(dp^4)^2 + 2\lambda p^4dp^2dp^3 + 2[p^4 - \lambda p^3]dp^2dp^4 = 0,$$

where $\lambda = \lambda_2 - \lambda_1$, so that the corresponding PDE is

$$u_{22} + u_{33} - 2u_2u_{44} + 2\lambda u_4u_{23} + 2(u_4 - \lambda u_3)u_{24} = 0.$$

One can show that this sub-case possesses no integrable/conformally flat specialisations. In sub-case 2 the equation of the complex is

$$2\lambda_1 p^{12}p^{34} + (p^{34})^2 + 2\lambda_2(p^{14}p^{23} + p^{13}p^{42}) + 2p^{13}p^{23} + (p^{42})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^3 = 1,\ dp^3 = 0$ we obtain the associated Monge equation,

$$(p^4)^2(dp^2)^2 + (1 + (p^2)^2)(dp^4)^2 + 2dp^1dp^2 + 2\lambda p^2 dp^1 dp^4 - 2[\lambda p^1 + p^2p^4]dp^2dp^4 = 0,$$

where $\lambda = \lambda_2 - \lambda_1$, so that the corresponding PDE is

$$u_4^2 u_{22} + (1 + u_2^2)u_{44} + 2u_{12} + 2\lambda u_2 u_{14} - 2(\lambda u_1 + u_2u_4)u_{24} = 0.$$

One can show that this PDE is not integrable, however, the corresponding conformal structure is flat for $\lambda = 0$. This Segre type is known as [(24)], giving the corresponding sub-case of Theorem 3. The associated Kummer surface consists of three planes, with a double plane at infinity. We point out that in sub-case 1 the Kummer surface of the complex [(24)] consists of a quadratic cone and a double plane at infinity. This gives an invariant characterisation of sub-case 2 of the complex [(24)].
Relabeling independent variables gives the 9th case of Theorem 2.

Case 10: Segre symbol [33]**.** The equation of the complex is

$$\lambda_1(4p^{31}p^{24} + (p^{12} + p^{34})^2) + 2p^{13}(p^{12} + p^{34})$$

$$+\lambda_2(4p^{23}p^{14} - (p^{12} - p^{34})^2) + 2p^{14}(p^{12} - p^{34}) = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation,

$$\lambda(dp^2)^2 + [\lambda(p^4)^2 - 2p^4](dp^3)^2 + [\lambda(p^3)^2 - 2p^3](dp^4)^2 +$$

$$2[\lambda p^4 + 1]dp^2dp^3 + 2[\lambda p^3 + 1]dp^2dp^4 - 2[2\lambda p^2 + \lambda p^3p^4 - p^3 - p^4]dp^3dp^4 = 0,$$

where $\lambda = \lambda_1 - \lambda_2$, so that the corresponding PDE is

$$\lambda u_{22} + (\lambda u_4^2 - 2u_4)u_{33} + (\lambda u_3^2 - 2u_3)u_{44}$$

$$+2(\lambda u_4 + 1)u_{23} + 2(\lambda u_3 + 1)u_{24} - 2(2\lambda u_2 + \lambda u_3u_4 - u_3 - u_4)u_{34} = 0.$$

Relabeling independent variables gives the 10th case of Theorem 2. One can show that the conditions of integrability are equivalent to conformal flatness, leading to $\lambda = 0$,

$$u_4 u_{33} + u_3 u_{44} - u_{23} - u_{24} - (u_3 + u_4) u_{34} = 0.$$

The corresponding complex is denoted [(33)]. Introducing the new independent variables x, y, t such that $\partial_3 = \partial_x + \partial_y,\ \partial_4 = \partial_x - \partial_y,\ \partial_2 = -2\partial_t$ one can reduce the above PDE to the canonical form

$$u_{xt} + u_x u_{yy} - u_y u_{xy} = 0.$$

This is the last case of Theorems 3-4.

Case 11: Segre symbol [6]**.** Here we have two (projectively dual) sub-cases. In sub-case 1 the equation of the complex is

$$2\lambda(p^{23}p^{14} + p^{31}p^{24} + p^{12}p^{34}) + 2p^{14}p^{34} + 2p^{12}p^{42} + (p^{13})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation,

$$2p_4(dp^2)^2 + (dp^3)^2 + 2p^3(dp^4)^2 - 2p^2 dp^2 dp^4 - 2p^4 dp^3 dp^4 = 0,$$

so that the corresponding PDE is

$$2u_4 u_{22} + u_{33} + 2u_3 u_{44} - 2u_2 u_{24} - 2u_4 u_{34} = 0.$$

In the second sub-case the equation of the complex is

$$2\lambda(p^{23}p^{14} + p^{31}p^{24} + p^{12}p^{34}) + 2p^{23}p^{12} + 2p^{34}p^{13} + (p^{42})^2 = 0.$$

Setting $p^{ij} = p^i dp^j - p^j dp^i$ and using the affine projection $p^1 = 1,\ dp^1 = 0$ we obtain the associated Monge equation,

$$((p^4)^2 - 2p_3)(dp^2)^2 - 2p^4(dp^3)^2 + (p^2)^2(dp^4)^2 + 2p^2 dp^2 dp^3 - 2p^2 p^4 dp^2 dp^4 + 2p^3 dp^3 dp^4 = 0,$$

so that the corresponding PDE is

$$(u_4^2 - 2u_3)u_{22} - 2u_4 u_{33} + u_2^2 u_{44} + 2u_2 u_{23} - 2u_2 u_4 u_{24} + 2u_3 u_{34} = 0.$$

One can show that both sub-cases are not integrable/conformally flat. Relabeling independent variables gives the last case of Theorem 2. This finishes the proof of Theorems 2-4.

3.6 Conservation laws of linearly degenerate integrable equations

According to [4], any integrable equation of the form (3.1) possesses exactly four first order conservation laws. In the previous section we identified that there are exactly five integrable, linearly degenerate quasilinear wave equations. Here we present their conservation laws (they will be used in section 4 in the discussion of characteristic integrals):

Segre symbol [(11)(11)(11)]

$$\alpha u_3 u_{12} + \beta u_2 u_{13} + \gamma u_1 u_{23} = 0.$$

Conservation laws are:

$$\gamma(u_2u_3)_1 + \beta(u_1u_3)_2 + \alpha(u_1u_2)_3 = 0,$$
$$\beta\left(\frac{u_2}{u_3}\right)_1 + \gamma\left(\frac{u_1}{u_3}\right)_2 = 0,$$
$$\alpha\left(\frac{u_3}{u_2}\right)_1 + \gamma\left(\frac{u_1}{u_2}\right)_3 = 0,$$
$$\alpha\left(\frac{u_3}{u_1}\right)_2 + \beta\left(\frac{u_2}{u_1}\right)_3 = 0.$$

Segre symbol [(11)(112)]

$$u_{11} + u_1u_{23} - u_2u_{13} = 0.$$

Conservation laws are:

$$\left(\frac{u_2}{2u_1^2}\right)_1 + \left(\frac{1}{2u_1}\right)_2 - \left(\frac{u_2^2}{2u_1^2}\right)_3 = 0,$$
$$(u_1 - u_2u_3)_1 + (u_1u_3)_2 = 0,$$
$$(2u_1u_3 - u_2u_3^2)_1 + (u_3^2u_1)_2 - (u_1^2)_3 = 0,$$
$$-\left(\frac{1}{u_1}\right)_1 + \left(\frac{u_2}{u_1}\right)_3 = 0.$$

Segre symbol [(11)(22)]

$$u_{12} + u_2u_{13} - u_1u_{23} = 0.$$

Conservation laws are:

$$(u_2u_3)_1 + (u_1 - u_1u_3)_2 = 0,$$

$$(u_2u_3^2)_1 + (-u_1u_3^2 + 2u_1u_3 - u_1)_2 - (u_1u_2)_3 = 0,$$
$$\left(\frac{1}{u_2}\right)_1 - \left(\frac{u_1}{u_2}\right)_3 = 0,$$
$$-\left(\frac{1}{u_1}\right)_2 - \left(\frac{u_2}{u_1}\right)_3 = 0.$$

Segre symbol [(123)]

$$u_{22} + u_{13} + u_2u_{33} - u_3u_{23} = 0.$$

Conservation laws are:

$$(u_2 - u_3^2)_2 + (u_1 + u_2u_3)_3 = 0,$$
$$(\frac{1}{2}u_3^2)_1 + (u_2u_3 - \frac{1}{2}u_3^3)_2 + (-u_2u_3^2 - \frac{1}{2}u_2^2 + \frac{3}{2}u_3^2u_2)_3 = 0,$$
$$(u_2u_3 - u_3^3)_1 + (-u_1u_3 + u_2^2 - 3u_2u_3^2 + u_3^4)_2 + (u_1u_2 + 2u_3u_2^2 - u_2u_3^3)_3 = 0,$$
$$(u_2^2 - 2u_2u_3^2 + u_3^4)_1 + (-2u_1u_2 + 2u_1u_3^2 - 3u_2^2u_3 + 4u_2u_3^3 - u_3^5)_2$$
$$+(-2u_1u_2u_3 - u_1^2 - 3u_2^2u_3^2 + u_2u_3^4 + u_2^3)_3 = 0.$$

Segre symbol [(33)]

$$u_{13} + u_1u_{22} - u_2u_{12} = 0.$$

Conservation laws are:

$$\left(\frac{u_2}{u_1}\right)_2 - \left(\frac{1}{u_1}\right)_3 = 0,$$
$$(-u_2^2)_1 + (u_1u_2)_2 + (u_1)_3 = 0,$$
$$(-u_3^2 + 2u_3u_2^2 - u_2^4)_1 + (-2u_1u_2u_3 + u_1u_2^3)_2 + (u_1u_2^2)_3 = 0,$$
$$(u_3u_2 - u_2^3)_1 + (-u_1u_3 + u_1u_2^2)_2 + (u_1u_2)_3 = 0.$$

3.7 Remarks on the Cauchy problem for quasilinear wave equations

In 1+1 dimensions, linearly degenerate systems are known to be quite exceptional from the point of view of solvability of the Cauchy problem: generic smooth initial data do not develop shocks in finite time [15]. The conjecture of Majda [16], p. 89, suggests that the same statement should be true in higher dimensions, namely, for linearly degenerate systems the shock formation never happens for smooth initial data. To the best of our knowledge this conjecture is largely open, and has only been established for particular classes of multi-dimensional linearly degenerate PDEs, see [10, 5] and references therein. Klainerman established global existence results for $3+1$ dimensional nonlinear wave equations with small initial data. In the more subtle case of $2+1$ dimensions, the results of Klainerman imply long time existence for sufficiently small initial conditions. The approach of [10, 5] applies to second order quasilinear PDEs which can be viewed as nonlinear deformations of the wave equation,

$$\Box u = g_{ij}(u_k)u_{ij}, \tag{3.14}$$

here $\Box = \partial_1^2 - \partial_2^2 - ... - \partial_n^2$ is the wave operator, and the coefficients g_{ij}, which depend on the first order derivatives of u, are required to vanish at the origin $u_k = 0$. Under the conditions of Klainerman imposed on g_{ij} (which are automatically satisfied for linearly degenerate PDEs of the form (3.14), in fact, these conditions follow from the requirement of linear degeneracy (3.3) in the vicinity of the origin), one has global existence of classical solutions with small initial data. Since some of the linearly degenerate examples from Theorem 2 can be put into the form (3.14), one can automatically guarantee global existence. For instance, the PDE for minimal hypersurfaces is

$$u_{11}-u_{22}-u_{33} = -(u_2^2+u_3^2)u_{11}+(u_3^2-u_1^2)u_{22}+(u_2^2-u_1^2)u_{33}+2u_1u_2u_{12}+2u_1u_3u_{13}-2u_2u_3u_{23}, \tag{3.15}$$

take case [111111] of Theorem 2 and set $a_1 = -1,\ a_2 = a_3 = 1,\ \alpha = \beta = \gamma = 0,\ u \to iu$. It can be obtained as the Euler-Lagrange equation for the area functional, $\int \sqrt{1+u_2^2+u_3^2-u_1^2}\, dx$. In this particular case global existence was established in [17]. Further examples of this type include the equation

$$u_{11} - u_{22} - u_{33} = 2\alpha u_3u_{12} + 2\beta u_2u_{13} + 2\gamma u_1u_{23}, \tag{3.16}$$

take case [222] of Theorem 2 and set $x_2 \to ix_2,\ x_3 \to ix_3$. For PDEs of this type, solutions with small initial data essentially behave like solutions of the linear wave equation. As an illustration we present Mathematica snapshots of numerical solutions for equations (3.15) and (3.16) with hump-like initial data at $x_1 = 0$: $u = 0.8e^{-x_2^2-x_3^2},\ u_{x_1} = 0$.

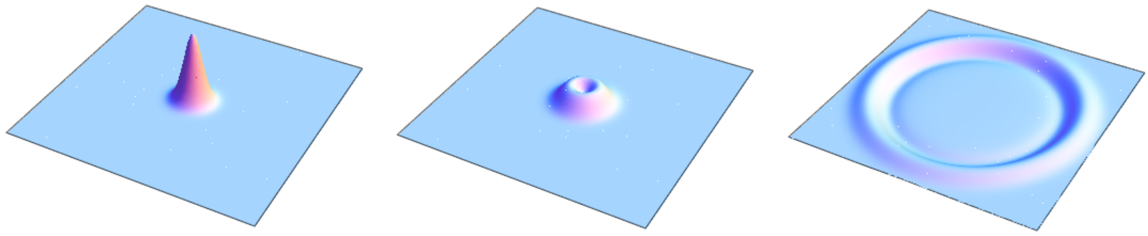

Figure 3.1: Numerical solution of equation (3.15) for $x_1 = 0, 1, 8$.

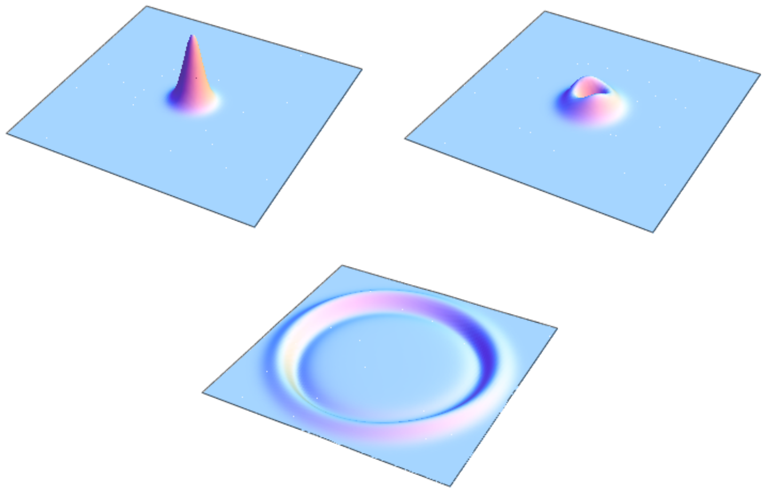

Figure 3.2: Numerical solution of equation (3.16) for $x_1 = 0, 1, 8$. $\alpha = \beta = 1, \gamma = -2$.

Figure 3.1 shows how equation (3.15) develops. We see that it is very similar to the wave equation, and there is no breakdown of solutions. Figure 3.2 shows how equation (3.16) develops. We see that for $x_1 = 1$ it shows behaviour quite different to that of the wave equation, an interesting saddle like shape. However, for $x_1 = 8$ the behaviour looks more like that of the wave equation. These results fit with those of [17], which imply that in the limit as $x_1 \to \infty$, behaviour of linearly degenerate equations tend to that of the wave equation. Experimenting with the initial condition shows that there is a point where the initial condition is too large and the solution breaks down. This is not currently understood and is an area for possible development.

Chapter 4

Linear degeneracy and characteristic integrals

Conservation laws that vanish along characteristic directions of a given system of PDEs are known as characteristic conservation laws, or characteristic integrals. In 2D, they are important in the theory of Darboux-integrable systems. In this section we introduce the notion of a characteristic integral in 2D, before extending this to 3D. We go on to demonstrate that for a class of second-order linearly degenerate dispersionless integrable PDEs, the corresponding characteristic integrals are parameterised by points on a sub-manifold in projective space, called a Veronese variety. The results of this chapter were published in [2].

4.1 Preliminaries

We begin this topic by recapping some basic PDE theory. An n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers is called a multi-index. We define

$$|\alpha| = \sum_{k=1}^{n} \alpha_k, \quad x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n}, \quad \forall x \in \mathbb{R}^n.$$

For partial derivitives of a function u of $x \in \mathbb{R}^n$ we use the notation $u_j = \partial_j u = \frac{\partial u}{\partial x_j}$. For higher order partial derivatives we have

$$u_\alpha = \partial^\alpha u = (\partial_1)^{\alpha_1} ... (\partial_n)^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

A *partial differential equation of order k* is an equation of the form

$$F(x_1, x_2, ..., x_n, u, \partial_1 u, ..., \partial_n u, ..., \partial_n^\alpha u) = 0, \qquad |\alpha| = k, \tag{4.1}$$

relating a function u of $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and its partial derivatives of order $\leq k$.

The equation (4.1) is called *linear* if F is an affine-linear function of the vector of variables,

$$\sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha u = f(x),$$

so that the coefficients a_α depend on x only. Here we can define the *differential operator* $L = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$ and write $Lu = f$. More generally, we have *quasi-linear* equations which have the form

$$\sum_{|\alpha| \leq k} a_\alpha(x, \partial^\beta u) \partial^\alpha u = b(x, \partial^\beta u), \quad |\beta| \leq k - 1.$$

The general form of a linear 2^{nd} order partial differential equation in 2D is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

with $u = u(x, y)$ and all coefficients are functions of x and y. For example,

$$xu_{xx} + yu_{xy} + u_x = 0$$

is linear and

$$uu_{xx} + u_y u_{xy} + u_x = 0$$

is nonlinear but quasi-linear. If $G = 0$ the equation is *homogeneous.*

Linear equations are often classified as being elliptic, hyperbolic, or parabolic. An example of an elliptic partial differential equation is the Laplace equation,

$$\nabla^2 u = u_{x_1 x_1} + ... + u_{x_n x_n} = 0, \quad u = u(x_1, ..., u_n).$$

One of the simplest examples of a hyperbolic equation is the wave equation,

$$u_{tt} = c^2 \Delta u_{xx}.$$

Here $u(x, t)$ represents the displacement of a point x on an infinite string at time t. The heat equation,

$$u_t(x, t) = \Delta u(x, t),$$

is an example of a parabolic equation. Some general concerns in the study of partial differential equations include the existence, uniqueness, representations and behaviour of the solutions.

Characteristics and Symbol of a PDE (recap)

In the study of linear partial differential equations a measure of the "strength" of a differential operator in a certain direction is given by the notion of characteristics. If $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ is a linear differential operator of order k on Ω in $\mathbb{R}^n$, then its *characteristic form* (or *principal symbol*) at $x \in \Omega$ is the homogeneous polynomial of degree k on $\mathbb{R}^n$ defined by

$$\chi_L(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi_\alpha,$$

A covector ξ is *characteristic* for L at x if

$$\chi_L(x,\xi) = 0.$$

The *characteristic variety* is the set of all characteristic covectors ξ, i.e.

$$Char_x(L) = \{\xi \neq 0 : \chi_L(x,\xi) = 0\}.$$

A hypersurface S is called characteristic for L at x if the normal vector $\nu(x)$ is in $Char_x(L)$, and S is called non-characteristic if it is not characteristic at any point. An important property of the characteristic variety is its transformation rule:

Let F be a smooth one-to-one mapping of Ω onto $\Omega' \subset \mathbb{R}^n$ and set $y = F(x)$. Assume that the Jacobian matrix

$$J_x = \left[\frac{\partial y_i}{\partial x_j}\right](x)$$

is nonsingular for $x \in \Omega$, so that $\{y_1, y_2, ..., y_n\}$ is a coordinate system on Ω'. We have

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial y_i}$$

which we can write symbolically as $\partial_x = J_x^T\partial_y$, where J_x^T is the transpose of J_x. The operator L is then transformed into

$$L' = \sum_{|\alpha|\leq k} a_\alpha(F^{-1}(y))\left(J^T_{F^{-1}(y)}\partial_y\right)_\alpha \quad on \quad \Omega'.$$

When this expression is expanded out, there will be some differentiations of $J^T_{F^{-1}(y)}$, but such derivatives are only formed by "using up" some of the ∂_y on $J^T_{F^{-1}(y)}$, so they do not enter in the computation of the principal symbol in the y coordinates, i.e. they do not enter the highest order terms. We find that

$$\chi_L(x,\xi) = \sum_{|\alpha|=k} a_\alpha(F^{-1}(y))\left(J^T_{F^{-1}(y)}\xi\right)_\alpha.$$

Now since $F^{-1}(y) = x$, on comparing with the expression

$$\chi(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi_\alpha$$

we see that $Char_x(L)$ is the image of $Char_y(L')$ under the linear map $J^T_{F^{-1}(y)}$.

Note that if $\xi \neq 0$ is a vector in the x_j-direction (i.e. $\xi_i = 0$ for $i \neq j$), then $\xi \in Char_x(L)$ if and only if the coefficient of ∂_j^k in L vanishes at x. Now, given any $\xi \neq 0$, by a rotation of coordinates we can arrange for ξ to lie in a coordinate direction. Thus the condition $\xi \in Char_x(L)$ means that, in some sense, L fails to be "genuinely kth order" in the ξ direction at x. L is said to be *elliptic* at x if $Char_x(L) = \emptyset$ and *elliptic on* Ω if it is elliptic at each $x \in \Omega$. Elliptic operators exert control on all derivatives of all order.

Veronese variety

Throughout this section, we will encounter the Veronese variety. A Veronese variety is an algebraic manifold that is realised by the Veronese embedding; the embedding of projective space given by the complete linear system of quadrics. For a mapping $\mathbb{P}^2 \to \mathbb{P}^5$ we have a Veronese surface, the embedding is given by,

$$[x : y : z] \to [x^2 : y^2 : z^2 : yz : xz : xy],$$

where $[x : ...]$ denotes homogeneous coordinates. In this section we will also consider Veronese varieties given by the embeddings $\mathbb{P}^3 \to \mathbb{P}^9$ and $\mathbb{P}^4 \to \mathbb{P}^{14}$.

4.2 Characteristic integrals

Let Σ be a partial differential equation (PDE) in n independent variables $x_1, \ldots, x_n$. A conservation law is an $(n-1)$-form Ω which is closed on the solutions of Σ: $d\Omega = 0 \bmod \Sigma$. Since any $(n-1)$-form in n variables possesses a unique annihilating direction, there exists a vector field F such that $\Omega(F) = 0$. We say that Ω is a characteristic integral (conservation law) if F is a characteristic direction of Σ[1]. If a conservation law is represented in conventional form,

$$(F_1)_{x_1} + \cdots + (F_n)_{x_n} = 0 \bmod \Sigma,$$

[1]The set of characteristic directions is projectively dual to the more conventional variety of characteristic covectors determined by the principal symbol of the equation.

the corresponding vector field is $F = (F_1, \dots, F_n)$. The characteristic condition becomes particularly simple for scalar second order PDEs, in which case F can be interpreted as a null vector of the conformal structure defined by the principal symbol of the equation. Let us begin with illustrating examples.

Example 1. Consider the $2+1$ dimensional wave equation,

$$u_{tt} = u_{xx} + u_{yy}. \tag{4.2}$$

It possesses four first order conservation laws,

$$(u_x)_x + (u_y)_y - (u_t)_t = 0,$$

$$(u_x^2 + u_t^2 - u_y^2)_x + (2u_xu_y)_y - (2u_xu_t)_t = 0,$$

$$(2u_yu_x)_x - (u_x^2 - u_y^2 - u_t^2)_y - (2u_yu_t)_t = 0,$$

$$(2u_tu_x)_x + (2u_tu_y)_y - (u_x^2 + u_y^2 + u_t^2)_t = 0.$$

Let us denote them

$$(f_i)_x + (g_i)_y + (h_i)_t = 0,$$

$i = 1, ..., 4$. Taking their linear combination with constant coefficients $J_1, \dots, J_4$, and adding trivial conservation laws, we obtain the expression $(F_1)_x + (F_2)_y + (F_3)_t = 0$ where

$$F_1 = J_if_i - J_5u_y + J_6u_t + J_8, \quad F_2 = J_ig_i + J_5u_x - J_7u_t + J_9, \quad F_3 = J_ih_i - J_6u_x + J_7u_y + J_{10},$$

(summation over $i = 1, \dots, 4$ is assumed). Here the constants J_5, J_6, J_7 correspond to trivial conservation laws of the form $(u_x)_y - (u_y)_x = 0$, etc., and J_8, J_9, J_{10} are three extra arbitrary constants. Although the constants $J_5 - J_{10}$ correspond to trivial conservation laws, they effect non-trivially the characteristic condition, $Fg^{-1}F^t = 0$, where g is the 3×3 symmetric matrix of the corresponding principal symbol,

$$g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(in this particular example g coincides with g^{-1}). The characteristic condition imposes a system of quadratic constraints for $J_1, \dots, J_{10}$, which specify a Veronese threefold $V^3 \subset \mathbb{P}^9$ with parametric equations

$$J_1 = \frac{\sqrt{2}}{2}(\gamma\alpha + \gamma\beta + \delta\alpha - \delta\beta), \quad J_2 = -\alpha\beta, \quad J_3 = \frac{\alpha^2 - \beta^2}{2}, \quad J_4 = -\frac{\alpha^2 + \beta^2}{2},$$

$$J_5=\frac{\sqrt{2}}{2}(\gamma\alpha-\gamma\beta-\delta\alpha-\delta\beta),\quad J_6=-\frac{\sqrt{2}}{2}(\gamma\alpha+\gamma\beta-\delta\alpha+\delta\beta),$$
$$J_7=\frac{\sqrt{2}}{2}(\gamma\alpha-\gamma\beta+\delta\alpha+\delta\beta),\quad J_8=\frac{\delta^2-\gamma^2}{2},\quad J_9=\delta\gamma,\quad J_{10}=\frac{\gamma^2+\delta^2}{2}.$$

We use $\alpha,\beta,\gamma,\delta$ as homogeneous coordinates in $\mathbb{P}^3$, and $J_1,\ldots,J_{10}$ as homogeneous coordinates in $\mathbb{P}^9$. Recall that the Veronese threefold V^3 is the image of the projective embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$ defined by a complete system of quadrics. Thus, we have a whole V^3-worth of characteristic integrals. It turns out that this example is not isolated, and similar phenomena take place for other classes of 3D linearly degenerate dispersionless integrable PDEs.

Example 2. Let us consider the equation

$$\mu u_t u_{xy}+\nu u_y u_{xt}+\eta u_x u_{yt}=0, \tag{4.3}$$

$\mu+\nu+\eta=0$, which appeared in the context of Veronese webs in 3D [19]. This equation possesses four first order conservation laws,

$$\eta(u_yu_t)_x+\nu(u_xu_t)_y+\mu(u_xu_y)_t=0,$$
$$\nu\left(\frac{u_y}{u_t}\right)_x+\eta\left(\frac{u_x}{u_t}\right)_y=0,$$
$$\mu\left(\frac{u_t}{u_y}\right)_x+\eta\left(\frac{u_x}{u_y}\right)_t=0,$$
$$\mu\left(\frac{u_t}{u_x}\right)_y+\nu\left(\frac{u_y}{u_x}\right)_t=0.$$

Let us denote them

$$(f_i)_x+(g_i)_y+(h_i)_t=0,$$

$i=1,...,4$. Taking their linear combination with coefficients $J_1,\ldots,J_4$, and adding trivial conservation laws, we obtain the expression $(F_1)_x+(F_2)_y+(F_3)_t=0$ where

$$F_1=J_if_i-J_5u_y+J_6u_t+J_8,\quad F_2=J_ig_i+J_5u_x-J_7u_t+J_9,\quad F_3=J_ih_i-J_6u_x+J_7u_y+J_{10},$$

As in Example 1, the constants J_5-J_{10} correspond to trivial conservation laws. The characteristic condition takes the form $Fg^{-1}F^t=0$ where g is the 3×3 symmetric matrix of the corresponding principal symbol:

$$g=\begin{pmatrix}0 & \mu u_t & \nu u_y\\ \mu u_t & 0 & \eta u_x\\ \nu u_y & \eta u_x & 0\end{pmatrix}.$$

The characteristic condition imposes a system of quadratic constraints for $J_1, \ldots, J_{10}$, which specify a Veronese threefold $V^3 \subset \mathbb{P}^9$ with parametric equations

$$J_1 = \alpha^2, \quad J_2 = \frac{1}{4\nu\eta}\beta^2, \quad J_3 = \frac{1}{4\eta\mu}\delta^2, \quad J_4 = \frac{1}{4\nu\mu}\gamma^2,$$
$$J_5 = \alpha\beta, \quad J_6 = \alpha\delta, \quad J_7 = \alpha\gamma, \quad J_8 = -\frac{1}{2\eta}\beta\delta, \quad J_9 = -\frac{1}{2\nu}\beta\gamma, \quad J_{10} = -\frac{1}{2\mu}\delta\gamma.$$

Both of these examples are quasilinear wave-type equations, a complete classification of all linearly degenerate, integrable equations was given in section 3.

4.3 Characteristic integrals in 2D and linear degeneracy

For definiteness we restrict the discussion to systems of hydrodynamic type,

$$u^i_t = v^i_j(\mathbf{u})u^i_x, \tag{4.4}$$

where $\mathbf{u} = (u^1, \ldots, u^n)$ denotes dependent variables, and $V = v^i_j$ is an $n \times n$ matrix. Let λ^i be the eigenvalues (characteristic speeds) of V, and let ξ^i be the corresponding eigenvectors, so that $V\xi^i = \lambda^i\xi^i$. Characteristic directions are defined as $dx + \lambda^i dt = 0$, and the characteristic integral in i-th direction is a 1-form $h(\mathbf{u})(dx + \lambda^i dt)$ which is closed on solutions of (4.4). We will assume that the density h depends on $\mathbf{u}$ only, although, in principle, nontrivial dependence on higher order x-derivatives of $\mathbf{u}$ may also be allowed. Recall that the i-th characteristic direction is called linearly degenerate if the Lie derivative of λ^i in the direction of the corresponding eigenvector ξ^i vanishes, $L_{\xi^i}\lambda^i = 0$. The following result is well-known:

Proposition. *If there exists a characteristic integral in the i-th direction, then the corresponding characteristic speed λ^i must be linearly degenerate.*

Proof:

The closedness of $h(\mathbf{u})(dx + \lambda^i dt)$ is equivalent to $h_t = (\lambda^i h)_x$. This implies

$$(\nabla h)v = h\nabla\lambda^i + \lambda^i\nabla h,$$

where $\nabla = (\partial_{u^1}, \ldots, \partial_{u^n})$ denotes the gradient. Evaluating both sides of this identity (which are 1-forms) on the vector ξ^i, and using $V\xi^i = \lambda^i\xi^i$, one can see that the left hand side cancels with the second term on the right hand side, leading to $\xi^i \nabla \lambda^i = L_{\xi^i} \lambda^i = 0$. This finishes the proof.

The requirement of the existence of characteristic integrals for all characteristic directions implies that all characteristic speeds must be linearly degenerate. Such systems are known as (totally) linearly degenerate, they have been thoroughly investigated in the literature, see e.g. [15]. For linearly degenerate systems the gradient catastrophe, which is typical for genuinely nonlinear systems, does not occur, and one has global existence results for an open set of initial data.
There exist systems which possess infinitely many characteristic integrals.
Example. The 2-component linearly degenerate system,

$$v_t = wv_x, \quad w_t = vw_x,$$

possesses functionally many characteristic integrals in both characteristic directions:

$$\frac{\phi(v)}{w-v}(dx + wdt), \qquad \frac{\psi(w)}{v-w}(dx + vdt),$$

here ϕ and ψ are arbitrary functions of w and v respectively.

4.4 Characteristic integrals of second order PDEs in 3D

In this section we consider quasilinear wave-type equations of the form (3.1),

$$f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0,$$

where $u(x, y, t)$ is a function of three independent variables, and the coefficients f_{ij} depend on the first order derivatives u_x, u_y, u_t only. Equations of this form generalise examples 1 and 2 from section 4.2. It was shown in [4] that any integrable equation of the form (3.1) possesses exactly four conservation laws

$$(f_i)_x + (g_i)_y + (h_i)_t = 0,$$

$i = 1, \ldots, 4$, where f_i, g_i, h_i are functions of u_x, u_y, u_t only. Taking their linear combination with constant coefficients $J_1, \ldots, J_4$, and adding trivial conservation laws, we obtain the expression $(F_1)_x + (F_2)_y + (F_3)_t = 0$ where

$$F_1 = J_i f_i - J_5 u_y + J_6 u_t + J_8, \quad F_2 = J_i g_i + J_5 u_x - J_7 u_t + J_9, \quad F_3 = J_i h_i - J_6 u_x + J_7 u_y + J_{10}.$$

Although the constants $J_5 - J_{10}$ give trivial contribution to conservation laws, they do effect non-trivially the characteristic condition, $Fg^{-1}F^t = 0$, where $g = f_{ij}$ is the 3×3 symmetric matrix of the corresponding principal symbol. The characteristic condition imposes a system of quadratic constraints for the coefficients $J_1, \dots, J_{10}$ which, in linearly degenerate integrable cases, specify a Veronese threefold $V^3 \subset \mathbb{P}^9$. Recall that for 3D equations of the form (3.1), the concept of linear degeneracy can be defined as follows.

Looking for travelling wave solutions in the form $u(x, y, t) = u(\xi, \eta) + \zeta$ where ξ, η, ζ are arbitrary linear forms in the variables x, y, t, we obtain a second order PDE for $u(\xi, \eta)$,

$$au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} = 0,$$

where the coefficients a, b, c are certain functions of u_ξ and u_η. Setting $v = u_\xi,\ w = u_\eta$, one can rewrite this PDE as a two-component system of hydrodynamic type. We say that Equation (3.1) is linearly degenerate if all its travelling wave reductions are linearly degenerate in the sense of Sect. 2. The condition of linear degeneracy is equivalent to the identity (set $u_x, u_y, u_t = p_1, p_2, p_3$ and consider f_{ij} as functions of p_1, p_2, p_3):

$$f_{(ij,k)} = c_{(k}f_{ij)}, \tag{4.5}$$

here $f_{ij,k} = \partial_{p_k} f_{ij}$, c_k is a covector, and brackets denote complete symmetrisation in i, j, k. Linearly degenerate integrable PDEs of the form (3.1) were classified in the previous section. The result of theorem 4 is the following:

The following examples constitute a complete list of linearly degenerate integrable PDEs of the type (3.1):

$$\mu u_t u_{xy} + \nu u_y u_{xt} + \eta u_x u_{yt} = 0, \quad \mu + \nu + \eta = 0,$$

$$u_{xx} + u_x u_{yt} - u_y u_{xt} = 0,$$

$$u_{xy} + u_y u_{xt} - u_x u_{yt} = 0,$$

$$u_{yy} + u_{xt} + u_y u_{tt} - u_t u_{yt} = 0,$$

$$u_{xt} + u_x u_{yy} - u_y u_{xy} = 0,$$

$$u_{tt} - u_{xx} - u_{yy} = 0.$$

In different contexts, the canonical forms of Theorem 1 have appeared in [19, 11].

The main result of this section is the following.

Theorem 5

(i) If a 3D quasilinear PDE of the form (3.1) possesses 'sufficiently many' characteristic integrals, then it must be linearly degenerate. Here 'sufficiently many' means that the corresponding vector F satisfies no extra algebraic constraints other than the characteristic condition itself, $Fg^{-1}F^t = 0$.
(ii) Any linearly degenerate integrable PDE of the form (3.1) possesses a V^3-worth of characteristic integrals.

Proof:

To demonstrate (i) we recall the result of [4] according to which the functions F_i defining a conservation law must satisfy the identity $F_{(i,j)} = sf_{ij}$, where $F_{i,j} = \partial_{p_j} F_i$, brackets denote symmetrisation in i, j, and s is a coefficient of proportionality (all entries are viewed as functions of p's). The characteristic constraint takes the form

$$(f^{-1})^{ij} F_i F_j = 0,$$

which can be rewritten as $f_{ij}F^iF^j = 0$ where we use the notation $F_i = f_{ij}F^j$. Differentiating the characteristic condition by p_k we obtain

$$-(f^{-1})^{ip} f_{pq,k} (f^{-1})^{qj} F_i F_j + 2(f^{-1})^{ij} F_{i,k} F_j = 0,$$

which can be rewritten as

$$f_{pq,k} F^p F^q = 2F_{i,k} F^i.$$

Contracting this identity with F^k, using the condition $F_{(i,j)} = sf_{ij}$ and the characteristic constraint $f_{ij}F^iF^j = 0$ we obtain the additional algebraic condition

$$f_{ij,k} F^i F^j F^k = 0. \tag{4.6}$$

The requirement that this condition is satisfied identically modulo the characteristic constraint, $f_{ij}F^iF^j = 0$, is equivalent to saying that the cubic (4.6) is divisible by the quadric $f_{ij}F^iF^j = 0$,

$$f_{ij,k} F^i F^j F^k = (c_i F^i)(f_{ij} F^i F^j),$$

for some linear form c_iF^i. Symmetrisation of this identity implies the condition of linear degeneracy (4.5).

The proof of (ii) is a case-by-case calculation. Details are included after the following examples.

Note that linearly non-degenerate or non-integrable equations may also possess characteristic integrals (although not 'as many' as linearly degenerate integrable ones).

Example 1. The integrable, linearly non-degenerate equation

$$u_t u_{xy} + u_y u_{xt} + u_x u_{yt} = 0,$$

admits the following conservation laws;

$$(u_y u_t)_x + (u_x u_t)_y + (u_x u_y)_t = 0,$$

$$(u_x^2 u_t)_y + (u_x^2 u_y)_t = 0,$$
$$(u_y^2 u_t)_x + (u_y^2 u_x)_t = 0,$$
$$(u_t^2 u_y)_x + (u_t^2 u_x)_y = 0.$$

It turns out here that the second, third and fourth conservation laws shown above are independent characteristic integrals. There are no others.

Example 2. The integrable, linearly non-degenerate dKP equation

$$u_{xt} - u_x u_{xx} - u_{yy} = 0,$$

admits the following conservation laws;

$$(u_x^2 - u_t)_x + (2u_y)_y - (u_x)_t = 0,$$
$$\left(\frac{2}{3}u_x^3 - u_y^2\right)_x + (2u_x u_y)_y - (u_x^2)_t = 0,$$
$$(u_x^2 u_y - u_t u_y)_x + (u_y^2 - \frac{1}{3}u_x^2 + u_x u_t)_y - (u_x u_y)_t = 0,$$
$$(u_x^2 u_t - u_t^2)_x + (2u_y u_t)_y - \left(\frac{1}{3}u_x^3 + u_y^2\right)_t = 0.$$

In this case, it turns out that the only characteristic integral is a trivial one, $(J_8)_x = 0$.

Example 3. The linearly degenerate, non-integrable quasilinear wave equation of Segre type [114],

$$2u_{yy} - u_x u_{tt} + u_t u_{xt} = 0,$$

admits only trivial conservation laws, these are

$$(u_t)_x - (u_x)_t = 0, \qquad (J_{10})_t = 0.$$

Proof of Theorem 5-(ii):

Quasilinear wave equations are equations of the form (3.1),

$$f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0,$$

where $u(x, y, t)$ is a function of three independent variables, and the coefficients f_{ij} depend on the first order derivatives u_x, u_y, u_t only. The non-degeneracy condition $\det f_{ij} \neq 0$ is also taken as an assumption. In the previous chapter it was shown that there are exactly five canonical forms (plus one linearisable example) of (3.1) that satisfy both the integrability and linear degeneracy condition. We know that each equation must have exactly four conservation laws of the form $f(u_x, u_y, u_t)_x + g(u_x, u_y, u_t)_y + h(u_x, u_y, u_t)_t = 0$. The five canonical forms, together with their respective conservation laws and Veronese variety parameterisation are listed below (the linearisable example is omitted). Here $\alpha, \beta, \gamma, \delta$ are homogeneous coordinates in $\mathbb{P}^3$.

Equation 1 (discussed earlier)

$$\mu u_t u_{xy} + \nu u_y u_{xt} + \eta u_x u_{yt} = 0.$$

Four conservation laws:

$$\eta(u_y u_t)_x + \nu(u_x u_t)_y + \mu(u_x u_y)_t = 0,$$
$$\nu\left(\frac{u_y}{u_t}\right)_x + \eta\left(\frac{u_x}{u_t}\right)_y = 0,$$
$$\mu\left(\frac{u_t}{u_y}\right)_x + \eta\left(\frac{u_x}{u_y}\right)_t = 0,$$
$$\mu\left(\frac{u_t}{u_x}\right)_y + \nu\left(\frac{u_y}{u_x}\right)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = \alpha^2, \quad J_2 = \frac{1}{4\nu\eta}\beta^2, \quad J_3 = \frac{1}{4\eta\mu}\delta^2, \quad J_4 = \frac{1}{4\nu\mu}\gamma^2, \quad J_5 = \alpha\beta,$$

$$J_6 = \alpha\delta, \quad J_7 = \alpha\gamma, \quad J_8 = -\frac{1}{2\eta}\beta\delta, \quad J_9 = -\frac{1}{2/nu}\beta\gamma, \quad J_{10} = -\frac{1}{2\mu}\delta\gamma.$$

Equation 2

$$u_{xx} + u_x u_{yt} - u_y u_{xt} = 0.$$

Four conservation laws:

$$\left(\frac{u_y}{2u_x^2}\right)_x + \left(\frac{1}{2u_x}\right)_y - \left(\frac{u_y^2}{2u_x^2}\right)_t = 0,$$

$$(u_x - u_y u_t)_x + (u_x u_t)_y = 0,$$

$$(2u_x u_t - u_y u_t^2)_x + (u_t^2 u_x)_y - (u_x^2)_t = 0,$$

$$-\left(\frac{1}{u_x}\right)_x + \left(\frac{u_y}{u_x}\right)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = \alpha^2, \quad J_2 = -\delta\beta, \quad J_3 = \frac{1}{2}\beta^2, \quad J_4 = \alpha\gamma, \quad J_5 = \frac{1}{2}\delta^2,$$

$$J_6 = \beta\gamma, \quad J_7 = \alpha\beta, \quad J_8 = -\alpha\beta - \gamma\delta, \quad J_9 = \alpha\delta, \quad J_{10} = -\frac{1}{2}\gamma^2.$$

Equation 3

$$u_{xy} + u_y u_{xt} - u_x u_{yt} = 0.$$

Four conservation laws:

$$(u_y u_t)_x + (u_x - u_x u_t)_y = 0,$$

$$(u_y u_t^2)_x + (2u_x u_t - u_x u_t^2 - u_x)_y - (u_x u_y)_t = 0,$$

$$\left(\frac{1}{u_y}\right)_x - \left(\frac{u_x}{u_y}\right)_t = 0,$$

$$-\left(\frac{1}{u_x}\right)_y - \left(\frac{u_y}{u_x}\right)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = \alpha\beta, \quad J_2 = \alpha^2, \quad J_3 = \frac{1}{4}\delta^2, \quad J_4 = \frac{1}{4}\gamma^2, \quad J_5 = -\frac{1}{4}\beta^2,$$

$$J_6 = -\alpha\delta, \quad J_7 = \alpha\gamma, \quad J_8 = -\frac{1}{2}\beta\delta, \quad J_9 = \alpha\gamma - \frac{1}{2}\beta\gamma, \quad J_{10} = -\frac{1}{2}\delta\gamma.$$

Equation 4

$$u_{yy} + u_{xt} + u_y u_{tt} - u_t u_{yt} = 0.$$

Four conservation laws:

$$(u_y - u_t^2)_y + (u_x + u_y u_t)_t = 0,$$

$$(\frac{1}{2}u_t^2)_x + (u_y u_t - \frac{1}{2}u_t^3)_y + (-u_y u_t^2 - \frac{1}{2}u_y^2 + \frac{3}{2}u_t^2 u_y)_t = 0,$$

$$(u_y u_t - u_t^3)_x + (-u_x u_t + u_y^2 - 3u_y u_t^2 + u_t^4)_y + (u_x u_y + 2u_t u_y^2 - u_y u_t^3)_t = 0,$$

$$(u_y^2 - 2u_y u_t^2 + u_t^4)_x + (-2u_x u_y + 2u_x u_t^2 - 3u_y^2 u_t + 4u_y u_t^3 - u_t^5)_y$$

$$+(-2u_x u_y u_t - u_x^2 - 3u_y^2 u_t^2 + u_y u_t^4 + u_y^3)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = -\alpha\delta - \frac{1}{2}\beta\gamma, \quad J_2 = 2\alpha\gamma + \frac{1}{2}\delta^2, \quad J_3 = \frac{1}{2}\delta\gamma, \quad J_4 = \frac{1}{4}\gamma^2, \quad J_5 = \alpha\gamma,$$

$$J_6 = -\alpha\delta, \quad J_7 = \alpha^2 + \frac{1}{2}\beta\delta, \quad J_8 = \alpha^2, \quad J_9 = \alpha\beta, \quad J_{10} = -\frac{1}{4}\beta^2.$$

Equation 5

$$u_{xt} + u_x u_{yy} - u_y u_{xy} = 0.$$

Four conservation laws:

$$\left(\frac{u_y}{u_x}\right)_y - \left(\frac{1}{u_x}\right)_t = 0,$$

$$(-u_y^2)_x + (u_x u_y)_y + (u_x)_t = 0,$$

$$(-u_t^2 + 2u_t u_y^2 - u_y^4)_x + (-2u_x u_y u_t + u_x u_y^3)_y + (u_x u_y^2)_t = 0,$$

$$(u_t u_y - u_y^3)_x + (-u_x u_t + u_x u_y^2)_y + (u_x u_y)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = \alpha^2, \quad J_2 = -\frac{1}{2}\beta\gamma - \frac{1}{4}\delta^2, \quad J_3 = -\frac{1}{4}\beta^2, \quad J_4 = -\frac{1}{2}\beta\delta, \quad J_5 = -\frac{1}{2}\gamma\delta,$$

$$J_6 = -\frac{1}{2}\beta\gamma, \quad J_7 = \beta\alpha, \quad J_8 = \frac{1}{4}\gamma^2, \quad J_9 = \gamma\alpha, \quad J_{10} = \delta\alpha.$$

4.5 Systems of hydrodynamic type

Recall that systems of hydrodynamic type are systems of the form

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0, \tag{4.7}$$

where $u(x, y, t)$ is a function of three independent variables, and the coefficients are matrices that depend on u only. We shall consider only 2-component systems. We know that an integrable equation of this type must have exactly three conservation laws of the form $f(u_1, u_2)_x + g(u_1, u_2)_y + h(u_1, u_2)_t = 0$ [26].

For equations of hydrodynamic type, the notion of a characteristic conservation law takes the following form. We take linear combinations of the three conservation laws,

$$J_i : \quad (f_i)_x + (g_i)_y + (h_i)_t = 0, \quad i = 1, ..., 3.$$

We then add three constants J_4, J_5 and J_6 and express the general conservation law:

$$\frac{\partial}{\partial x} \underbrace{\left(\sum J_i f_i + J_4\right)}_{F_1} + \frac{\partial}{\partial y} \underbrace{\left(\sum J_i g_i + J_5\right)}_{F_2} + \frac{\partial}{\partial y} \underbrace{\left(\sum J_i h_i + J_6\right)}_{F_3} = 0.$$

As before, the condition of characteristic integral means we must find $J_i's$ such that

$$F(g^{-1})F^t = 0,$$

where $F = (F_1, F_2, F_3)$ and g is a symmetric matrix found in the following way. From (4.7) we construct,

$$det(A\lambda + B\mu + C\nu) = g^{11}\lambda^2 + g^{22}\mu^2 + g^{33}\nu^2 + 2g^{12}\lambda\mu + 2g^{13}\lambda\nu + 2g^{23}\mu\nu.$$

This is called the dispersion relation for (4.7), and the coefficients g^{ij} form the entries of g.

Conjecture

(i) If a PDE of the form (4.7) possesses 'sufficiently many' characteristic integrals, then it must be linearly degenerate. Here 'sufficiently many' means that the corresponding vector F satisfies no extra algebraic constraints other than the characteristic condition itself, $Fg^{-1}F^t = 0$.
(ii) Any linearly degenerate integrable PDE (4.7) possesses a V^2-worth of characteristic integrals.

Due to time constraints, a proof is not presented here. The idea of a proof would be based on that of theorem 5.

Example. The linearly degenerate, integrable hydrodynamic system

$$v_t + w_x = 0, \quad w_t + w_y + vw_x - wv_x = 0,$$

possesses the following three conservation laws;

$$v_t + w_x = 0$$

$$(v^2 - w)_t + (vw)_x - w_y = 0,$$
$$\left(\frac{1}{w}\right)_t + \left(\frac{v}{w}\right)_x + \left(\frac{1}{w}\right)_y = 0.$$

As this system is integrable and linearly degenerate, characteristic integrals lie on a Veronese surface $V^2 \subset \mathbb{P}^5$. The parameterisation is given by

$$J_1 = \alpha\beta, \quad J_2 = \alpha^2 \quad J_3 = -\frac{1}{4}\gamma^2, \quad J_4 = -\frac{1}{2}\beta\gamma, \quad J_5 = \alpha\gamma, \quad J_6 = \frac{1}{4}\beta^2 + \alpha\gamma.$$

Where α, β, γ are homogeneous coordinates in $\mathbb{P}^2$.

4.6 Another class of first order systems

Here we consider systems of the form

$$\mathbf{F}(u_x, u_y, u_t, v_x, v_y, v_t) = 0, \quad \mathbf{G}(u_x, u_y, u_t, v_x, v_y, v_t) = 0, \tag{4.8}$$

where $u(x, y, t)$ and $v(x, y, t)$ are both functions of three independent variables. There currently exists no general classification of linear degeneracy, we shall instead consider an example. It is proposed, but not proven, that an integrable system of this form must have exactly six non-trivial conservation laws of the form

$$f(u_x, u_y, u_t, v_x, v_y, v_t)_x + g(u_x, u_y, u_t, v_x, v_y, v_t)_y + h(u_x, u_y, u_t, v_x, v_y, v_t)_t = 0.$$

The notion of a characteristic integral then takes the following form. As before, we take linear combinations of the six conservation laws,

$$J_i : \quad (f_i)_x + (g_i)_y + (h_i)_t = 0, \quad i = 1, ..., 6.$$

We then add the six trivial conservation laws and three constants to express the general conservation law:

$$\frac{\partial}{\partial x}\underbrace{\left(\sum J_i f_i + J_7 u_y + J_8 u_t + J_{10} v_y + J_{11} v_t + J_{13}\right)}_{F_1}$$
$$+\frac{\partial}{\partial y}\underbrace{\left(\sum J_i g_i - J_7 u_x + J_9 u_t - J_{10} v_x + J_{12} v_t + J_{14}\right)}_{F_2}$$
$$+\frac{\partial}{\partial y}\underbrace{\left(\sum J_i h_i - J_8 u_x - J_9 u_y - J_{11} v_x - J_{12} v_y + J_{15}\right)}_{F_3} = 0.$$

Again, the condition of characteristic integrals means we must find $J_i's$ such that

$$F(g^{-1})F^t = 0,$$

where g is a symmetric matrix found from the dispersion relation in the usual way.

Consider the following example of a linearly degenerate, integrable system,

$$(\lambda - \nu)v_y = u_y v_x, \quad (\lambda - \mu)v_t = u_t v_x. \tag{4.9}$$

Using the equations (4.9), we can eliminate v_y and v_t. The six conservation laws are then found to be

$$(\nu - \mu)(v_x^2 u_y u_t)_x + (\nu - \lambda)^2 (v_x^2 u_t)_y - (\mu - \lambda)^2 (v_x^2 u_y)_t = 0,$$

$$(\nu - \mu)(v_x u_t u_y)_x - (\nu - \lambda)(u_t v_x u_x)_y - (\lambda - \mu)(u_x u_y v_x + (\nu - \mu)u_y v_x)_t = 0,$$
$$\left(\frac{u_y}{u_t}\right)_x + (\nu - \mu)\left(\frac{1}{u_t}\right)_y = 0,$$
$$(\nu - \mu)(u_y u_t)_x + (u_x^2 u_t)_y - ((\nu - \mu)^2 u_y + 2(\nu - \mu)u_x u_y + u_x^2 u_y)_t = 0,$$
$$\left(\frac{u_t}{u_y}\right)_x + (\mu - \nu)\left(\frac{1}{u_y}\right)_t = 0.$$
$$(u_x u_t)_y + ((\mu - \nu)u_y - u_y u_x)_t = 0.$$

The characteristic integrals of the system (4.9) are found to lie on a Veronese surface $V^4 \subset \mathbb{P}^{14}$, with coordinates $(J_1 : ... : J_{15})$. The parameterisation of this surface is given by

$$J_1 = \frac{1}{4}\eta^2, \quad J_2 = \alpha\eta \quad J_3 = \frac{1}{4(\nu - \mu)}\beta^2, \quad J_4 = \alpha^2, \quad J_5 = \frac{1}{4(\nu - \mu)}\gamma^2,$$
$$J_6 = \alpha\delta, \quad J_7 = \alpha\beta, \quad J_8 = \alpha\gamma, \quad J_9 = \frac{1}{4}\delta^2, \quad J_{10} = \frac{1}{2}\beta\eta,$$
$$J_{11} = \frac{1}{2}\gamma\eta, \quad J_{12} = -\frac{1}{2}\delta\eta, \quad J_{13} = \frac{1}{2(\nu - \mu)}\beta\gamma,$$
$$J_{14} = -\frac{1}{2}\beta\delta, \quad J_{15} = -(\nu - \mu)\alpha\gamma - \frac{1}{2}\gamma\delta.$$

Where $\alpha, \beta, \gamma, \delta, \eta$ are homogeneous coordinates in $\mathbb{P}^4$. To the best of our knowledge, there currently exists no complete classification of all integrable, linearly degenerate PDE systems of the form (4.8). However, we conjecture that the same theorem holds for systems of this type. That is, the characteristic integrals for an integrable, linearly degenerate PDE systems of the form (4.8) correspond to a Veronese surface $V^4 \subset \mathbb{P}^{14}$.

Open conjecture.

Any linearly degenerate integrable PDE (4.8) possesses a V^4-worth of characteristic integrals.

The first stage in proving this conjecture would be to show that integrable equations of the form (4.8) admit exactly six conservation laws.

4.7 Darboux integrability

In this section we introduce the notion of Darboux integrability in 2D. We start with a simple example.

Consider the Liouville equation

$$u_{xy} = e^u, \tag{4.10}$$

where $u = u(x, y)$. Characteristic directions of (4.10) are ∂_x and ∂_y. A characteristic integral is a conserved quantity that vanishes along a characteristic direction. In this example characteristic directions are simple, and a x-characteristic integral is a quantity $F = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ such that

$$\partial_x F = 0 \mod (4.10).$$

One can immediately see the $F = y$ is an integral. Another integral is $F = u_{yy} - \frac{1}{2}(u_y)^2$ which can be found after a short calculation. Similarly for characteristic integrals in the y-direction we have $F = x$ and $F = u_{xx} - \frac{1}{2}(u_x)^2$. We can then write

$$\begin{aligned} u_{yy} - \frac{1}{2}(u_y)^2 &= f(y) \\ u_{xx} - \frac{1}{2}(u_x)^2 &= g(x) \end{aligned} \tag{4.11}$$

Equations (4.10) and (4.11) now define a compatible system for any functions f and g.

In general, an equation of the form

$$u_{xy} = f(x, y, u, u_x, u_y), \tag{4.12}$$

is said to be *Darboux integrable* if it possesses x- and y- characteristic integrals,

$$\partial_y(F) = 0 \mod (4.12),$$

$$\partial_x(G) = 0 \mod (4.12).$$

Then we can express

$$F = f(x), \quad G = g(y).$$

where $f(x)$ and $g(y)$ are arbitrary.

For the most general case, a 2nd order PDE in 2D,

$$\phi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{4.13}$$

is said to be *Darboux integrable* if each characteristic direction possesses *two* characteristic integrals. If we denote the integrals from the first characteristic direction as I_1, I_2 and the ones from the second characteristic direction as J_1, J_2 then we can write

$$I_1 = f(I_2), \quad J_1 = g(J_2).$$

where f, g are arbitrary.

Example. Consider the equation

$$v_x v_{tt} - (v_t + v v_x) v_{xt} + v v_t v_{xx} = 0, \tag{4.14}$$

where $v = v(x,t)$. This can be re-written in the form

$$v_t = w v_x, \quad w_t = v w_x, \tag{4.15}$$

where $w = \frac{v_t}{v_x}$. Equivalently, using the 'wedge' notation we can write $dv \wedge (dx + wdt) = 0$, $dw \wedge (dx + vdt) = 0$. The characteristic directions of this system are $dx + wdt$ and $dx + vdt$. In section 4.3 we showed that this system has functionally many characteristic integrals in both directions:

$$\frac{\phi(v)}{w - v}(dx + wdt), \qquad \frac{\psi(w)}{v - w}(dx + vdt),$$

where ϕ and ψ are arbitrary functions of w and v respectively. This is clearly Darboux integrable. We can now use the characteristic integrals to find the solution. First introduce new independant variables

$$d\tilde{x} = \frac{\phi(v)}{w - v}(dx + wdt), \quad d\tilde{t} = \frac{\psi(w)}{v - w}(dx + vdt). \tag{4.16}$$

From (4.15), we can now write

$$v = f(\tilde{x}), \quad w = g(\tilde{t}),$$

where f and g are arbitrary functions. From (4.16) we can find dx and dt:

$$dt = \frac{d\tilde{x}}{\phi(v)} + \frac{d\tilde{t}}{\psi(w)}, \qquad dx = -\frac{v d\tilde{x}}{\phi(v)} - \frac{w d\tilde{t}}{\psi(w)}.$$

So the general solution in parametric form is

$$v = f(\tilde{x}), \qquad w = g(\tilde{t}),$$

$$x = -\int^{\tilde{x}} \frac{v d\tilde{x}}{\phi(v)} - \int^{\tilde{t}} \frac{w d\tilde{t}}{\psi(w)},$$

$$t = \int^{\tilde{x}} \frac{d\tilde{x}}{\phi(v)} + \int^{\tilde{t}} \frac{d\tilde{t}}{\psi(w)},$$

where f, g, ϕ and ψ are arbitrary functions.

Remark. Darboux integrability is as yet undefined in 3D. However, we have shown that there exists interesting structures when looking for characteristic integrals in 3D equations. This may be a step towards defining Darboux integrability in 3D.

Chapter 5

Concluding remarks

In this work we first studied the integrability and corresponding geometric structure of second order quasilinear equations of the form (3.1),

$$f_{11}u_{x_1x_1} + f_{22}u_{x_2x_2} + f_{33}u_{x_3x_3} + 2f_{12}u_{x_1x_2} + 2f_{13}u_{x_1x_3} + 2f_{23}u_{x_2x_3} = 0,$$

where $u = u(x_1, x_2, x_3)$ and the coefficients f_{ij} are functions of the first order derivatives $u_{x_1}, u_{x_2}, u_{x_3}$ only, with det $f_{ij} \neq 0$. A *conformal structure* can be associated to this equation (3.2),

$$f_{ij}(\mathbf{p})dp^i dp^j.$$

We focused on the particular class of equations (3.1) which are associated with quadratic complexes of lines in projective space $\mathbb{P}^3$. Using Jessop's classification of quadratic line complexes, we showed that the following conditions are equivalent:

(1) Equation (3.1)/conformal structure (3.2) is associated with a quadratic line complex.

(2) Equation (3.1) is linearly degenerate.

(3) Conformal structure (3.2) satisfies the condition

$$\partial_{(k}f_{ij)} = \varphi_{(k}f_{ij)},$$

here $\partial_k = \partial_{p^k}$, φ_k is a covector, and brackets denote complete symmetrization in $i, j, k \in \{1, 2, 3\}$. Our second main result is that a quadratic complex defines a flat conformal structure if and only if its Segre symbol is one of the following:

$$[111(111)]^*,\ [(111)(111)],\ [(11)(11)(11)],$$

$$[(11)(112)],\ [(11)(22)],\ [(114)],\ [(123)],\ [(222)],\ [(24)],\ [(33)].$$

Here the asterisk denotes a particular sub-case of [111(111)] where the matrix $Q\Omega^{-1}$ has eigenvalues $(1, \epsilon, \epsilon^2, 0, 0, 0)$, $\epsilon^3 = 1$. Our third main result is that a quadratic complex corresponds to an integrable PDE if and only if its Segre symbol is one of the following:

$$[(11)(11)(11)],\ [(11)(112)],\ [(11)(22)],\ [(123)],\ [(222)],\ [(33)].$$

Modulo equivalence transformations (which are allowed to be complex-valued) this leads to a complete list of normal forms of linearly degenerate integrable PDEs, which were given in the text. We highlighted that equations of the form (3.1) can be viewed as nonlinear perturbations of the linear wave equation and pointed out that the 'null conditions' of Klainerman, which establishes global existence of smooth solutions to the Cauchy problem, are automatically satisfied for linearly degenerate PDEs.

We next studied linear degeneracy in the context of characteristic integrals. First we showed that if a 3D quasilinear wave equation possesses 'sufficiently many' characteristic integrals, then it must be linearly degenerate. We then showed that a linearly degenerate, integrable quasilinear wave equation admits characteristic integrals which can be parameterised by a Veronese variety: $\mathbb{P}^3 \to \mathbb{P}^9$. Next, we showed that an equivalent property exists for systems of hydrodynamic type (4.7),

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0,$$

where $u(x, y, t)$ is a function of three independent variables, and the coefficients are matrices that depend on u only. For the 2-component case, we showed that if a hydrodynamic type system possesses 'sufficiently many' characteristic integrals, then it must be linearly degenerate. We then showed that a linearly degenerate, integrable system of hydrodynamic type admits characteristic integrals which can this time be parameterised by a lower dimensional Veronese variety: $\mathbb{P}^2 \to \mathbb{P}^5$. Finally, we gave an example for another class of first-order systems (4.8),

$$\mathbf{F}(u_x, u_y, u_t, v_x, v_y, v_t) = 0, \quad \mathbf{G}(u_x, u_y, u_t, v_x, v_y, v_t) = 0,$$

where $u(x, y, t)$ and $v(x, y, t)$ are both functions of three independent variables. We conjectured that any linearly degenerate, integrable system of this type will possess characteristic integrals that can be parameterised by a higher dimensional Veronese variety: $\mathbb{P}^4 \to \mathbb{P}^{15}$, although much work is needed in order to first classify and identify linearly degeneracy and integrability in these equations. We finished by introducing the notion of Darboux integrability, and mentioning that this work may be a first step in defining what is meant by Darboux integrability in 3D.

The next stage in this work would definitely be to explore ideas about Darboux integrability in 3D, as the entire notion would have to be re-defined. It would also be of interest to extend our results to linearly degenerate PDEs in 4D.

Bibliography

[1] S.I. Agafonov, E.V. Ferapontov, Systems of conservation laws in the setting of the projective theory of congruences: reducible and linearly degenerate systems, Differential Geometry and its Applications 7 (2002) 153-173.

[2] M.A. Akivis and V.V. Goldberg, Projective differential geometry of submanifolds, Elsevier Science Publishers (1993) 362pp.

[3] P.A. Burovskii, E.V. Ferapontov and S.P. Tsarev, Second order quasilinear PDEs and conformal structures in projective space, International J. Math. **21**, no. 6 (2010) 799-841.

[4] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, Comm. Pure Appl. Math. **39** (1986) 267–282.

[5] Eisenhart, Luther P. (1977). Riemannian Geometry. Princeton, NJ: Princeton University Press. ISBN 0-691-08026-7.

[6] Erwin Kreyszig, Differential geometry, Toronto University Press (1964)

[7] E.V. Ferapontov, Integration of weakly nonlinear hydrodynamic systems in Riemann invariants, Phys. Lett. A **158** (1991) 112-118.

[8] E.V. Ferapontov and J. Moss. Linearly degenerate partial differential equations and quadratic line complexes, Communications in Analysis and Geometry, Volume 23, Number 1 , p.91.

[9] E.V. Ferapontov and J. Moss. Characteristic integrals in 3D and linear degeneracy. Journal of Nonlinear Mathematical Physics, Volume 21, Issue 2, 2014.

[10] E.V. Ferapontov and K.R. Khusnutdinova, On integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys. **248** (2004) 187–206.

[11] E.V. Ferapontov, K.R. Khusnutdinova and C. Klein, On linear degeneracy of integrable quasilinear systems in higher dimensions, Lett. Math. Phys. (2011) 96:5-35.

[12] E.V. Ferapontov and K.R Khusnutdinova. The characterisation of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen **37**, no. 8 (2004).

[13] Forsyth A.R., Theory of differential equations, vol. 6, Dover Press, New York, 1959.

[14] R.W.H.T. Hudson, Kummer's quartic surface, Cambridge University Press, 1905.

[15] C.M. Jessop, A treatise on the line complex, Chelsea Publishing Co., New York (1969) 364 pp.

[16] S. Klainerman, The null condition and global existence to nonlinear wave equations. Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293–326, Lectures in Appl. Math. **23**, Amer. Math. Soc., Providence, RI, 1986.

[17] H. Lindblad, A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time, Proc. Amer. Math. Soc. **132**, no.4 (2004) 1095–1102.

[18] A. Majda, Compressible fluid flows and systems of conservation laws in several space variables, Appl. Math. Sci., Springer-Verlag, NY, **53** (1984) 159 pp.

[19] Mauro C. Beltrmetti, Ettore Carletti, Dionisio Gallarati and Giacomo Monti Bragadin, Lectures on Curves, Surfaces and Projective Varieties - A classical view of algebraic geometry, European Mathematical Society, 2003.

[20] J. Moss. Lecture notes on the undergraduate course 'Regular and chaotic dynamics", Loughborough university, 2012.

[21] A. Odesskii and V. Sokolov, Integrable (2+1)-dimensional systems of hydrodynamic type, Theoretical and Mathematical Physics **163**, no. 2 (2010) 549–586.

[22] J. Plücker, Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement, Bd. I, Teubner, Leipzig, 1868; ibid. Bd. II, 1869.

[23] B.L. Rozdestvenskii and A.D. Sidorenko, On the impossibility of 'gradient catastrophe' for weakly nonlinear systems, Z. Vycisl. Mat. i Mat. Fiz. **7** (1967) 1176-1179.

[24] S.P. Tsarev, Geometry of Hamiltonian systems of hydrodynamic type. Generalized hodograph method, Izvestija AN USSR Math. **54** (1990) 1048–1068.

[25] E. Vessiot, Sur les équations aux dérivées partielles du second order $F(x, y, u, p, q, r, s, t)$=0 intégrable par la méthose de Darboux, J. Math. Pure Appl. 18(1939), 1-61.

[26] I. Zakharevich, Nonlinear wave equation, nonlinear Riemann problem, and the twistor transform of Veronese webs, arXiv:math-ph/0006001.

[27] V.E. Zakharov. Dispersionless limit of integrable systems in $2 + 1$ dimensions.