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Ultraviolet Renormalization of the Nelson Hamiltonian through Functional Integration

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Abstract

Starting from the N-particle Nelson Hamiltonian defined by imposing an ultraviolet cutoff, we perform ultraviolet renormalization by showing that in the ultraviolet cutoff limit a self-adjoint operator exists after a logarithmically divergent term is subtracted from the original Hamiltonian. We obtain this term as the diagonal part of a pair interaction appearing in the density of a Gibbs measure derived from the Feynman-Kac representation of the Hamiltonian. Also, we show existence of a weak coupling limit of the renormalized Hamiltonian and derive an effective Yukawa interaction potential between the particles.

Keywords: Nelson model, ultraviolet cutoff, energy renormalization, Yukawa potential, Feynman-Kac representation, stochastic integrals, Gibbs measure

1 Introduction

In this paper we consider the N-particle Nelson model, which describes the interaction of N electrically charged spinless particles with a scalar boson field. In Fock space representation the model can be given by the Hamiltonian

$$H = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + H_{\mathrm{I}} \tag{1.1}$$

acting on

$$\mathscr{H} = L^2(\mathbb{R}^{3N}) \otimes \mathscr{F}_{\mathrm{b}},$$

where $L^2(\mathbb{R}^{3N})$ is the particle space and $\mathscr{F}_{\mathrm{b}}$ denotes the symmetric Fock space over $L^2(\mathbb{R}^3)$ describing the bosons. Recall that $\mathscr{F}_{\mathrm{b}} = \bigoplus_{n=0}^{\infty} \mathscr{F}_{\mathrm{b}}{}^{(n)}$, where $\mathscr{F}_{\mathrm{b}}{}^{(n)} = \otimes_{\mathrm{sym}}^n L^2(\mathbb{R}^3)$ is the n-boson subspace and $\mathscr{F}_{\mathrm{b}}{}^{(0)} = \mathbb{C}$ (where the subscript indicates symmetrized tensor product), for which the infinite direct sum norm $\|F\|_{\mathscr{F}_{\mathrm{b}}}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathscr{F}_{\mathrm{b}}{}^{(n)}}^2$ is finite. We denote the Fock vacuum by $\mathbb{1}_{\mathscr{F}_{\mathrm{b}}} = 1 \oplus 0 \oplus 0 \oplus \ldots \in \mathscr{F}_{\mathrm{b}}$, and write simply $\mathbb{1}$ when no confusion arises.

The components of the N-particle Nelson Hamiltonian are as follows. The N-particle Schrödinger operator

$$H_{\mathrm{p}} = -\frac{1}{2}\sum_{j=1}^{N} \Delta_j + V$$

is the Hamiltonian of the free particles with an external potential $V : \mathbb{R}^{3N} \to \mathbb{R}$, which acts as a multiplication operator, and where $\Delta_j = \Delta_{x_j}$ denotes the 3-dimensional Laplacian.

Let $a^*(f)$ and $a(f)$, $f \in L^2(\mathbb{R}^3)$, be the boson creation and annihilation operators, respectively, satisfying the canonical commutation relations

$$[a(f), a^*(g)] = (\bar{f}, g), \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)].$$

We formally denote $a^\sharp(f) = \int_{\mathbb{R}^3} a^\sharp(k) f(k) dk$, where $a^\sharp$ stands for a or a^*. Denote by $\omega(k)$ the dispersion relation, which we will choose in the main part of the paper to be $\omega(k) = |k|$, describing massless bosons. The free field Hamiltonian H_{f} of $\mathscr{F}_{\mathrm{b}}$ is then defined by the second quantization of ω, i.e.,

$$H_{\mathrm{f}} \prod_{j=1}^{n} a^*(f_j)\mathbb{1} = \sum_{j=1}^{n} a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n)\mathbb{1} \quad \text{and} \quad H_{\mathrm{f}}\mathbb{1} = \mathbb{1},$$

formally expressed as

$$H_{\mathrm{f}} = \int_{\mathbb{R}^3} \omega(k) a^*(k) a(k) dk.$$

The interaction Hamiltonian is formally defined by

$$H_{\mathrm{I}}(x) = g\sum_{j=1}^{N}\int_{\mathbb{R}^3}\frac{1}{\sqrt{2\omega(k)}}\left(\widehat{\varrho}(k)\mathrm{e}^{ik\cdot x_j}a(k)+\widehat{\varrho}(-k)\mathrm{e}^{-ik\cdot x_j}a^*(k)\right)dk \tag{1.2}$$

for every $x\in\mathbb{R}^{3N}$. We make the identification $\mathscr{H}\cong L^2(\mathbb{R}^{3N};\mathscr{F}_{\mathrm{b}})$, i.e., $F\in\mathscr{H}$ will be regarded as a function $\mathbb{R}^{3N}\ni x\mapsto F(x)\in\mathscr{F}_{\mathrm{b}}$ such that $\int_{\mathbb{R}^{3N}}\|F(x)\|^2_{\mathscr{F}_{\mathrm{b}}}dx<\infty$. Under this identification the interaction Hamiltonian becomes $(H_{\mathrm{I}}F)(x) = H_{\mathrm{I}}(x)F(x)$ on $\mathscr{H}$. The function φ (featured in its Fourier transform $\widehat{\varrho}$) is a function describing a charge distribution so that the total charge is $\int_{\mathbb{R}^3}\varphi(x)dx=1$. In the Hamiltonian it has a regularising role, making the operator well defined, and physically it has the meaning of an ultraviolet (UV) cutoff. The prefactor g is a coupling constant between the particles and the field.

Under the assumptions

$$\widehat{\varrho}/\omega^{1/2},\widehat{\varrho}/\omega\in L^2(\mathbb{R}^3),\quad \overline{\widehat{\varrho}(k)}=\widehat{\varrho}(-k) \tag{1.3}$$

the interaction H_{I} is well defined, symmetric and infinitesimally $\mathbb{1}\otimes H_{\mathrm{f}}$-bounded. Thus by the Kato-Rellich theorem H is self-adjoint on $D(H_{\mathrm{p}}\otimes\mathbb{1})\cap D(\mathbb{1}\otimes H_{\mathrm{f}})$. If, moreover, an infrared (IR) cutoff is imposed by the condition

$$\widehat{\varrho}/\omega^{3/2}\in L^2(\mathbb{R}^3), \tag{1.4}$$

then H has a unique ground state [Spo98, BFS98, Ger00, Ara01, Sas05], i.e., an eigenfunction $\Psi\in\mathscr{H}$ at the bottom of its spectrum. As it was shown in [LMS02, Hir06], condition (1.4) is also necessary for a ground state to exist in this space.

In this paper we are interested in an appropriate definition of H in the point charge limit, i.e., when $\varphi(x)\to(2\pi)^{3/2}\delta(x)$ or, equivalently, $\widehat{\varrho}(k)\to 1$. This is a physically interesting but mathematically singular case, when condition (1.3) fails to hold. In order to analyse this limit, we regularise the Hamiltonian by choosing the UV cutoff function $\widehat{\varrho}_\varepsilon(k)=\mathrm{e}^{-\varepsilon|k|^2/2}$. With this choice in (1.2) we define the Hamiltonian H_ε, and by regarding $\varepsilon>0$ as a UV cutoff parameter we will analyse the limit of $H_\varepsilon - E_\varepsilon$ as $\varepsilon\downarrow 0$, where E_ε is an energy renormalization term, which will be determined below.

The main results of this paper are as follows.

(1) By using functional integration we derive the energy renormalization term E_ε from the diagonal part of a pair interaction, and show the existence of the renormalized Hamiltonian $H_{\mathrm{ren}}=\lim_{\varepsilon\downarrow 0}(H_\varepsilon-E_\varepsilon)$ in the sense of strong convergence of the related semigroups.

(2) We derive the pair interaction potential in the path measure associated with H_{ren}.

(3) We show existence of the weak coupling limit of $e^{-tH_{\rm ren}}$ and compare it with that of the Nelson model with UV cutoff.

Here are some comments to these points.

(1) The first part of this problem was already investigated in [Nel64a] by using Gross transform. This is a unitary transform implemented by $e^{i\pi_\varepsilon(x)}$ with the generator

$$\pi_\varepsilon(x) = i\sum_{j=1}^N \int_{|k|>\sigma} \frac{1}{\sqrt{2\omega(k)}} \frac{1}{\omega(k)+|k|^2/2}(-a(k)\widehat{\varrho}_\varepsilon(k)e^{ikx_j} + a^*(k)\widehat{\varrho}_\varepsilon(-k)e^{-ikx_j})dk. \tag{1.5}$$

Here $\sigma > 0$ is introduced for π_ε to be well defined. Nelson has shown that the Gross transformed operator $e^{i\pi_\varepsilon(x)}(H_\varepsilon - E_\varepsilon)e^{-i\pi_\varepsilon(x)}$ converges to a self-adjoint operator in norm resolvent sense as $\varepsilon \downarrow 0$, and $e^{i\pi_\varepsilon(x)} \to e^{i\pi_0(x)}$ in strong sense.

In contrast to this approach, in the following we present a UV renormalization by using path measure methods. Nelson did consider a path integral approach [Nel64b], however, this remained incomplete since the approach based on Gross transform may have appeared simpler and satisfactory for the purposes of his investigation. Taking the Gross transformation of $H_\varepsilon - E_\varepsilon$, a cancellation of diverging terms occurs and the limit $\varepsilon \downarrow 0$ can be analysed to define a UV renormalised Hamiltonian. However, in this paper we do not take Gross transform and derive E_ε from the diagonal part of the pair interaction potential associated with a Gibbs measure instead.

Following our previous work on the Nelson model [LMS02, BHLMS02] we find it worthwhile to analyse this problem by using functional integration not just for the extra insights it gives (applicable also to other cases, e.g., UV renormalization of the Nelson model with variable coefficients [GHPS12]), but also because this approach allows to prove existence of a ground state of the UV renormalized Hamiltonian. As far as we are aware, the existence of a ground state of $H_{\rm ren}$ was shown for sufficiently weak couplings only [HHS05]. This problem is another illustration of the fact that direct analytic and path integral methods complement each other, and both have specific advantages.

A key point in this paper is to show that

$$\lim_{\varepsilon\downarrow 0}(f\otimes \mathbb{1}, e^{-T(H_\varepsilon - E_\varepsilon)}g\otimes \mathbb{1}) \tag{1.6}$$

can be expressed in terms of a path measure representation (Lemma 2.14 below), and

$$H_\varepsilon - E_\varepsilon > C \tag{1.7}$$

holds with a constant C, uniformly in $\varepsilon > 0$ (Corollary 2.18). Although these were established by operator analysis methods in [Nel64a] by using the Gross transform, we derive them directly by using Feynman-Kac type formulae for $H_\varepsilon - E_\varepsilon$, so our strategy is completely different from Nelson's.

(2) By constructing a functional integral representation of the Nelson Hamiltonian with UV cutoff and integrating out the field part, an expression is obtained in terms of an expectation with respect to Wiener measure on Brownian paths under V and a pair interaction potential W (see Chapter 6 in [LHB11]). Using this as a density, the path measure can be seen as a Gibbs measure on paths. The pair interaction has the form

$$\int_{-T}^{T} ds \int_{-T}^{T} dt W(B_t - B_s, t - s), \tag{1.8}$$

where W depends on the UV cutoff [BHLMS02]. On the other hand, a similar construction for the Pauli-Fierz model in non-relativistic quantum electrodynamics yields a Gibbs measure with pair interaction formally given by [Spo87, BH09]

$$\int_{-T}^{T} dB_s^{\mu} \int_{-T}^{T} dB_t^{\nu} W_{\mu\nu}(B_t - B_s, t - s). \tag{1.9}$$

It is remarkable that the double Riemann integral in (1.8) is replaced by a double stochastic integral. In this paper we will consider the finite volume Gibbs measure associated with the Nelson model without UV cutoff and obtain that the exponent in the Boltzmann-Gibbs density has the form (Corollary 2.20)

$$\int_{-T}^{T} ds \int_{s}^{T} dB_t W(B_t - B_s, t - s) + \int_{-T}^{T} ds W(B_T - B_s, T - s). \tag{1.10}$$

It is interesting to see that the Gibbs measure without a UV cutoff has the intermediate form of (1.8) and (1.9). The representation of the renormalized pair potential in terms of a mixed integral is obtained via an Itô formula. This technique has been used widely to study the intersection local time of Brownian motion [Yo85a, Yo85b, Yo86] and can be used to study the related polymer measure in two dimensions [LeG85, LeG94]. A further application of the Itô formula would transform the pair potential given by a double Lebesgue integral (1.8) into a pair potential given by a double Itô integral similar to (1.9), see [GL09]. However, as analyzed in some depth in [FG02] in the context of a stochastic model for 3d vortex filaments, pair potentials given by double stochastic integrals are difficult to handle analytically and do not allow for uniform exponential estimates. The mixed representation we have chosen is better suited for bounds which are valid for any strength of the coupling constant g.

(3) Finally we consider the weak coupling limit, which is a scaling limit such that the creation operators a^* and the annihilation operators a are scaled to κa and κa^*, respectively, with a scaling parameter $\kappa > 0$. When $\varepsilon > 0$, the scaled Nelson Hamiltonian $H_\varepsilon(\kappa)$ converges in the limit $\kappa \to \infty$ to a Schrödinger operator with an effective potential, and furthermore it converges to a Schrödinger operator with Yukawa potential when in a subsequent limit $\varepsilon \downarrow 0$. We are interested in obtaining

the weak coupling limit of $H_{\text{ren}}(\kappa)$ and comparing it with that of the UV-regularized Hamiltonian (Corollary 3.3).

We note that in this paper the space dimension is fixed to the physical $d = 3$, however, our arguments can be carried out for any other dimension in a similar way. We will see that the energy renormalization term behaves like $E_\varepsilon \sim -\int_1^\infty e^{-\varepsilon r^2} r^{d-4} dr$. When $d = 1$ or $d = 2$, there is no need for an energy renormalization, and when $d \geq 3$, energy renormalization becomes important to deal with the UV divergences in the $\varepsilon \downarrow 0$ limit. Also, it will be seen that our arguments do not need the assumption of a pinning external potential, in fact, we do not need any. However, we keep a V in our formulae with the understanding that the results are valid also for $V \equiv 0$.

The plan of this paper is as follows. In the main Section 2 we perform renormalization on the level of the density of the Gibbs measure, determine the pair interaction potential, and prove existence of a UV renormalized Hamilton operator. In Section 3 we study a weak coupling limit of the renormalized Hamiltonian and derive an effective interaction potential between the particles.

2 Energy renormalization by path measures

2.1 Functional integral representation of regularized Hamiltonians

First we define the version of the Nelson model which will be the main object studied in this paper. Throughout this paper we choose

$$\omega(k) = |k|. \tag{2.1}$$

Notice that the dispersion relation $\omega(k)$ we can choose can be more general form. For example $\omega(k) = \sqrt{|k|^2 + \nu^2}$ and $\omega(k) = 1$, but we take (2.1) for simplicity. Let

$$\mathbb{1}_\sigma(k) = \begin{cases} 1, & \omega(k) < \sigma \\ 0, & \omega(k) \geq \sigma \end{cases}$$

and $\mathbb{1}_\sigma^\perp(k) = \mathbb{1} - \mathbb{1}_\sigma(k)$. We assume that $\sigma > 0$, which is needed in (2.31), Lemma 2.9 and Corollary 2.18 below. For simplicity we will use the following standing assumption throughout below.

Assumption 2.1 *The external potential V is a bounded continuous function. In particular, it is of Kato-class, i.e., it satisfies*

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^3} \mathbb{E}^x \left[\int_0^t |V(B_s)| ds \right] = 0. \tag{2.2}$$

Note that in what follows $V \equiv 0$ is a possible choice without changing the statements. We summarize some properties of Kato-class potentials in Appendix A for the reader's convenience; they will be used below on some objects which take the role of a potential.

Consider the cutoff function

$$\widehat{\varrho}_\varepsilon(k) = \mathrm{e}^{-\varepsilon|k|^2/2} \mathbb{1}_\sigma^\perp(k), \quad \varepsilon \geq 0 \tag{2.3}$$

and define the regularized Hamiltonian

$$H_\varepsilon = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + H_{\mathrm{I}}^\varepsilon, \quad \varepsilon > 0, \tag{2.4}$$

by

$$H_{\mathrm{I}}^\varepsilon(x) = g \sum_{j=1}^N \int_{\mathbb{R}^3} \frac{1}{\sqrt{2\omega(k)}} \left(\widehat{\varrho}_\varepsilon(k) \mathrm{e}^{ik\cdot x_j} a(k) + \widehat{\varrho}_\varepsilon(-k) \mathrm{e}^{-ik\cdot x_j} a^*(k) \right) dk. \tag{2.5}$$

Here $\varepsilon > 0$ is regarded as the UV cutoff parameter. The main purpose of this paper is to consider the limit $\varepsilon \downarrow 0$ of H_ε. We show that this limit can be sensibly defined by an energy renormalization. Define

$$E_\varepsilon = -\frac{g^2}{2} N \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\varepsilon|k|^2}}{\omega(k)} \beta(k) \mathbb{1}_\sigma^\perp(k) dk, \tag{2.6}$$

where

$$\beta(k) = \frac{1}{\omega(k) + |k|^2/2}. \tag{2.7}$$

Notice that $E_\varepsilon \to -\infty$ as $\varepsilon \downarrow 0$.

Our main theorem states that $H_\varepsilon - E_\varepsilon$ converges in the $\varepsilon \downarrow 0$ limit to a non-trivial self-adjoint operator H_{ren} in a specific sense, which we call the UV renormalized Nelson Hamiltonian.

Theorem 2.2 *There exists a self-adjoint operator H_{ren} such that*

$$\mathrm{s-}\lim_{\varepsilon\downarrow 0} \mathrm{e}^{-t(H_\varepsilon - E_\varepsilon)} = \mathrm{e}^{-tH_{\mathrm{ren}}}, \quad t \geq 0. \tag{2.8}$$

We carry out the proof by functional integration and will obtain E_ε as the diagonal term of a pair interaction potential on the paths of a Brownian motion.

In the following we will fix a time interval $[-T, T]$ once and for all, and track the dependence on T of the various estimates. A Feynman-Kac formula holds for $(F, \mathrm{e}^{-2TH_\varepsilon} G)$ (see [LHB11, Theorem 6.3]). In particular, for $F = f \otimes \mathbb{1}$ and $G = h \otimes \mathbb{1}$ it can be described in terms of a family $(B_t)_{t\in\mathbb{R}} = (B_t^1, ..., B_t^N)_{t\in\mathbb{R}}$ of N indepedent, two-sided $\mathbb{R}^3$-valued Brownian motions $(B_t^j)_{t\in\mathbb{R}}$, $j = 1, ..., N$. It is convenient to take $(B_t)_{t\in\mathbb{R}}$ to be the canonical process on the space of $\mathbb{R}^{3N}$-valued continuous paths indexed by the whole real line, endowed with Wiener measure P^x starting from $x \in (\mathbb{R}^3)^N$ at $t = -T$. We will denote by $\mathbb{E}^x$ the associated expectation. Note that with respect to P^x the process $(B_t)_{t \geq -T}$ is a martingale with respect to the forward filtration

$$\mathcal{F}^T = (\mathcal{F}_t^T)_{t \geq -T} \tag{2.9}$$

with $\mathcal{F}_t^T = \sigma(B_s : -T \leq s \leq t)$.

Proposition 2.3 *Let $f, h \in L^2(\mathbb{R}^{3N})$. Then*

$$(f \otimes \mathbb{1}, \mathrm{e}^{-2TH_\varepsilon} h \otimes \mathbb{1}) = \int_{\mathbb{R}^{3N}} dx \, \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) \mathrm{e}^{-\int_{-T}^T V(B_s) ds} \mathrm{e}^{\frac{g^2}{2} S_\varepsilon^T} \right], \tag{2.10}$$

where

$$S_\varepsilon^T = \sum_{i,j=1}^N \int_{-T}^T ds \int_{-T}^T dt \, W_\varepsilon(B_t^i - B_s^j, t - s) \tag{2.11}$$

is the pair interaction given by the pair interaction potential $W_\varepsilon : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ given by

$$W_\varepsilon(x,t) = \int_{\mathbb{R}^3} \frac{1}{2\omega(k)} \mathrm{e}^{-\varepsilon |k|^2} \mathrm{e}^{-ik \cdot x} \mathrm{e}^{-\omega(k)|t|} \mathbb{1}_\sigma^\perp(k) dk. \tag{2.12}$$

2.2 Renormalized action

In this section we perform UV renormalization on the level of the density of the path measure (2.10). Consider the function

$$\varphi_\varepsilon(x,t) = \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\varepsilon |k|^2} \mathrm{e}^{-ik \cdot x - \omega(k)|t|}}{2\omega(k)} \beta(k) \mathbb{1}_\sigma^\perp(k) dk, \quad \varepsilon \geq 0, \tag{2.13}$$

where $\beta(k)$ is given by (2.7).

Proposition 2.4 *There exists a functional S_0^{ren} such that*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^T V(B_s) ds} \mathrm{e}^{\frac{g^2}{2}(S_\varepsilon^T - 4NT\varphi_\varepsilon(0,0))} \right] = \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^T V(B_s) ds} \mathrm{e}^{\frac{g^2}{2} S_0^{\mathrm{ren}}} \right]. \tag{2.14}$$

Notice that $W_\varepsilon(x,t)$ is smooth, and $W_\varepsilon(x,t) \to W_0(x,t)$ as $\varepsilon \downarrow 0$ for every $(x,t) \neq (0,0)$, where

$$W_0(x,t) = \int_{\mathbb{R}^3} \frac{1}{2\omega(k)} \mathrm{e}^{-ik \cdot x} \mathrm{e}^{-\omega(k)|t|} \mathbb{1}_\sigma^\perp(k) dk. \tag{2.15}$$

It is seen, however, that $W_\varepsilon(0,0) \to \infty$ as $\varepsilon \downarrow 0$, i.e., $W_0(x,t)$ has a power-like singularity at $(0,0)$, thus (2.14) is non-trivial to obtain. We will prove the above proposition through a sequence of lemmas below.

As $\varepsilon \downarrow 0$ only the diagonal part of the interaction has a singular term. Fix $0 < \tau \le T$ and denote $[t]_T = -T \vee t \wedge T$. We decompose the regularized interaction into diagonal and off-diagonal parts as

$$S_\varepsilon^T = S_\varepsilon^{\mathrm{D},T} + S_\varepsilon^{\mathrm{OD},T}, \tag{2.16}$$

where

$$S_\varepsilon^{\mathrm{D},T} = 2 \sum_{i,j=1}^N \int_{-T}^T ds \int_s^{[s+\tau]_T} dt\, W_\varepsilon(B_t^i - B_s^j, t-s) \tag{2.17}$$

and

$$S_\varepsilon^{\mathrm{OD},T} = 2 \sum_{i,j=1}^N \int_{-T}^T ds \int_{[s+\tau]_T}^T dt W_\varepsilon(B_t^i - B_s^j, t-s). \tag{2.18}$$

$S_\varepsilon^{\mathrm{D}}$ denotes the integral of S_ε in a neighborhood of the diagonal $\{(t,t) \in \mathbb{R}^2 | |t| \le T\}$, and $S_\varepsilon^{\mathrm{OD}}$ on its complement. Notice that for $\tau = T$ we have $S_\varepsilon^{\mathrm{OD}} = 0$. The next lemma is easy to prove and we omit the details.

Lemma 2.5 *We have that* $\lim_{\varepsilon\downarrow 0} S_\varepsilon^{\mathrm{OD},T} = S_0^{\mathrm{OD},T}$ *pathwise.*

A representation in terms of a stochastic integral will help us deal with the more difficult term $S_\varepsilon^{\mathrm{D},T}$. In the following lemma first we derive some bounds on $\varphi_\varepsilon(x,t)$ and its gradient.

Lemma 2.6 *There exists a constant $c > 0$ such that the bounds*

$$|\nabla_x \varphi_\varepsilon(x,t)| \le c|t|^{-1}, \quad t \ne 0$$
$$|\nabla_x \varphi_\varepsilon(x,t)| \le c|x|^{-1}, \quad |x| \ne 0$$

hold uniformly in ε. Furthermore, similar bounds hold for the function $\varphi_0 - \varphi_\varepsilon$ with a constant $\mathrm{c}_\varepsilon > 0$ such that $\lim_{\varepsilon\downarrow 0} \mathrm{c}_\varepsilon = 0$, i.e.,

$$|\nabla_x \varphi_\varepsilon(x,t) - \nabla_x \varphi_0(x,t)| \le \mathrm{c}_\varepsilon |t|^{-1}, \quad t \ne 0,$$
$$|\nabla_x \varphi_\varepsilon(x,t) - \nabla_x \varphi_0(x,t)| \le \mathrm{c}_\varepsilon |x|^{-1}, \quad |x| \ne 0.$$

Proof. The first bound on the gradient follows directly by

$$|\nabla_x \varphi_\varepsilon(x,t)| \le \int_{\mathbb{R}^3} \frac{1}{2(\omega(k) + |k|^2/2)} \mathrm{e}^{-\varepsilon|k|^2} \mathrm{e}^{-\omega(k)|t|} \mathbb{1}_\sigma^\perp(k) dk \le c \int_\sigma^\infty \mathrm{e}^{-rt} dr.$$

Next consider the second. After integration over the angular variables we obtain

$$\varphi_\varepsilon(x,t) = 2\pi \int_\sigma^\infty \frac{\mathrm{e}^{-\varepsilon r^2 - \omega(r)|t|}}{(2\omega(r) + r^2)} \frac{\sin(r|x|)}{|x|} dr. \tag{2.19}$$

Differentiation in (2.19) gives

$$\nabla_x \varphi_\varepsilon(x,t) = \frac{2\pi x}{|x|^2} \int_{\sigma|x|}^{\infty} \frac{\mathrm{e}^{-\varepsilon r^2/|x|^2 - |t|\omega(r)/|x|}}{(2|x|\omega(r) + r^2)} (r\cos r - \sin r)\, dr, \tag{2.20}$$

and estimating the right-hand side we have

$$|\nabla_x \varphi_\varepsilon(x,t)| \leq \frac{1}{|x|} \left(\int_0^1 \frac{Cr^3}{r^2} dr + \left| \int_1^\infty \frac{\mathrm{e}^{-\varepsilon r^2/|x|^2 - |t|\omega(r)/|x|}}{(2|x| + r)} \cos r dr \right| + \int_1^\infty \frac{1}{r^2} dr \right)$$

using that $\omega(r)/r = 1$, $|r\cos r - \sin r| \leq Cr^3$, for a constant $C > 0$ and all $r \in [0,1]$. Since the integral in the middle term above is bounded, the lemma follows. □

Define

$$X_\varepsilon^T = 2 \sum_{i \neq j}^{N} \int_{-T}^{T} \varphi_\varepsilon(B_s^i - B_s^j, 0) ds, \tag{2.21}$$

$$Y_\varepsilon^T = 2 \sum_{i,j=1}^{N} \int_{-T}^{T} ds \int_s^{[s+\tau]_T} \nabla_x \varphi_\varepsilon(B_t^i - B_s^j, t-s) \cdot dB_t^i, \tag{2.22}$$

$$Z_\varepsilon^T = -2 \sum_{i,j=1}^{N} \int_{-T}^{T} \varphi_\varepsilon(B_{[s+\tau]_T}^i - B_s^j, [s+\tau]_T - s) ds. \tag{2.23}$$

Lemma 2.7 *The representation formula*

$$S_\varepsilon^{\mathrm{D},T} = 4TN\varphi_\varepsilon(0,0) + X_\varepsilon^T + Y_\varepsilon^T + Z_\varepsilon^T \tag{2.24}$$

holds for all $\varepsilon > 0$.

Proof. Note that $\varphi_\varepsilon(x,t)$ solves the equation

$$\left(\partial_t + \frac{1}{2}\Delta\right) \varphi_\varepsilon(x,t) = -W_\varepsilon(x,t), \qquad x \in \mathbb{R}^3, \quad t \geq 0. \tag{2.25}$$

Fix i and j. The Itô formula yields that

$$\begin{aligned} &\varphi_\varepsilon(B_{[s+\tau]_T}^i - B_s^j, [s+\tau]_T - s) - \varphi_\varepsilon(B_s^i - B_s^j, 0) \\ &= \int_s^{[s+\tau]_T} \nabla_x \varphi_\varepsilon(B_t^i - B_s^j, t-s) \cdot dB_t^i + \int_s^{[s+\tau]_T} \left(\partial_t + \frac{1}{2}\Delta\right) \varphi_\varepsilon(B_t^i - B_s^j, t-s) dt. \end{aligned} \tag{2.26}$$

Hence by (2.25)

$$\int_s^{[s+\tau]_T} W_\varepsilon(B_t^i - B_s^j, t-s)dt$$

$$= \varphi_\varepsilon(B_s^i - B_s^j, 0) - \varphi_\varepsilon(B_{[s+\tau]_T}^i - B_s^j, [s+\tau]_T - s) + \int_s^{[s+\tau]_T} \nabla_x \varphi_\varepsilon(B_t^i - B_s^j, t-s) \cdot dB_t^i \tag{2.27}$$

follows. Inserting this expression into $S_\varepsilon^{\mathrm{D},T}$ proves the claim. $\square$

Lemma 2.7 suggests the definition

$$S_\varepsilon^{\mathrm{ren}} = S_\varepsilon^T - 4NT\varphi_\varepsilon(0,0), \tag{2.28}$$

as a renormalized action. This can be expressed as

$$S_\varepsilon^{\mathrm{ren}} = S_\varepsilon^{\mathrm{OD},T} + X_\varepsilon^T + Y_\varepsilon^T + Z_\varepsilon^T. \tag{2.29}$$

Lemma 2.8 *Let $\varepsilon \geq 0$. There exist constants c_Z, c_S such that $|S_\varepsilon^{\mathrm{OD},T}| \leq c_S(T+1)$ and $|Z_\varepsilon^T| \leq c_Z T$, uniformly in the paths and in $\varepsilon \geq 0$.*

PROOF. We see that

$$|Z_\varepsilon^T| \leq 4\pi N^2 \left(\int_{-T}^{T-\tau} ds \int_\sigma^\infty \frac{\mathrm{e}^{-\omega(r)\tau}}{\omega(r)/r + r/2} dr + \int_{T-\tau}^T ds \int_\sigma^\infty \frac{\mathrm{e}^{-\omega(r)(T-s)}}{\omega(r)/r + r/2} dr \right) \leq c_z T$$

with some $c_z > 0$. It is also easy to see that $|S_\varepsilon^{\mathrm{OD},T}| \leq \mathrm{const}\ \left((\frac{2T}{\tau} - 1) + \log(\frac{\tau}{2T}) \right)$. Then the bound $|S_\varepsilon^{\mathrm{OD},T}| \leq c_s(T+1)$ follows. $\square$

Lemma 2.9 *Let $\varepsilon \geq 0$. There exists a constant c_X independent of ε such that for all $\alpha > 0$ and $T > 0$ we have*

$$\sup_{x \in \mathbb{R}^{3N}} \mathbb{E}^x[e^{\alpha |X_\varepsilon^T|}] \leq e^{\alpha c_X T}.$$

PROOF. We notice that

$$X_\varepsilon^T = \sum_{i \neq j}^N \int_{-T}^T ds \frac{2\pi}{|B_s^i - B_s^j|} \int_\sigma^\infty \frac{\sin \sqrt{r |B_s^i - B_s^j|}}{\omega(r) + r^2/2} \mathrm{e}^{-\varepsilon r^2} dr, \quad \varepsilon \geq 0. \tag{2.30}$$

The assumption $\sigma > 0$ implies

$$a = 2\pi \int_\sigma^\infty \frac{1}{\omega(r) + r^2/2} dr < \infty. \tag{2.31}$$

Hence

$$|X_\varepsilon^T| \leq a \sum_{i\neq j}^{N} \int_{-T}^{T} \frac{ds}{|B_s^i - B_s^j|} \tag{2.32}$$

Since $\sum_{i\neq j}^N |x^i - x^j|^{-1}$ is a Kato-class potential on $L^2(\mathbb{R}^{3N})$ (see Appendix A), we have the claim. □

By the stochastic Fubini theorem we can interchange the stochastic and Lebesgue integrals when $\varepsilon > 0$. Thus Y_ε^T has the representation

$$Y_\varepsilon^T = \sum_{i=1}^{N} \int_{-T}^{T} \Phi_{\varepsilon,t}^i dB_t^i, \quad \varepsilon > 0, \tag{2.33}$$

where $\Phi_{\varepsilon,t} = (\Phi_{\varepsilon,t}^1, \ldots, \Phi_{\varepsilon,t}^N)$ is the process with values in $\mathbb{R}^{3N}$ given by

$$\Phi_{\varepsilon,t}^i = 2 \sum_{j=1}^{N} \int_{[t-\tau]_T}^{t} \nabla_x \varphi_\varepsilon(B_t^i - B_s^j, t-s) ds.$$

Define

$$Y_0^T = \sum_{i=1}^{N} \int_{-T}^{T} \Phi_{0,t}^i dB_t^i. \tag{2.34}$$

Lemma 2.10 *Let $\varepsilon \geq 0$. Then there exists a constant c_Y independent of ε such that for all $\alpha > 0$ it follows that $\sup_{x\in\mathbb{R}^{3N}} \mathbb{E}^x[e^{\alpha Y_\varepsilon^T}] \leq e^{c_Y \alpha^2 T}$, and $\lim_{\varepsilon\downarrow 0} \mathbb{E}^x[|Y_\varepsilon^T - Y_0^T|^2] = 0$ for all $x \in \mathbb{R}^{3N}$.*

Proof. The process $(\Phi_{\varepsilon,t}^i)_{t\in\mathbb{R}}$ is adapted to the forward filtration $\mathcal{F}^T$ given in (2.9) so Y_ε^T is the terminal value of a martingale with quadratic variation given by the $L^2([-T,T];\mathbb{R}^{3N})$ norm of $\Phi_{\varepsilon,\cdot}$. We have

$$\begin{aligned}\int_{-T}^{T} |\Phi_{\varepsilon,t}|^2 dt &\leq 4 \sum_{i=1}^{N} \int_{-T}^{T} \left[\sum_{j=1}^{N} \int_{[t-\tau]_T}^{t} |\nabla_x \varphi_\varepsilon(B_t^i - B_s^j, t-s)| ds \right]^2 dt \\ &\leq 4c^2 N \sum_{i,j=1}^{N} \int_{-T}^{T} \left[\int_{[t-\tau]_T}^{t} |B_t^i - B_s^j|^{-\theta} |t-s|^{-(1-\theta)} ds \right]^2 dt,\end{aligned}$$

where we used Jensen's inequality, Lemma 2.6 and an interpolation to obtain the bound

$$|\nabla_x \varphi_\varepsilon(x,t)| \leq c|x|^{-\theta}|t|^{-(1-\theta)}, \quad \theta \in [0,1], \tag{2.35}$$

which is uniform in $\varepsilon \in [0,1]$. With some $\frac{1}{2} < \theta < 1$, the Cauchy-Schwarz inequality applied to the latter integral gives

$$\begin{aligned}\int_{-T}^{T} |\Phi_{\varepsilon,t}|^2 dt &\le 4c^2 N \sum_{i,j=1}^{N} \int_{-T}^{T} \left[\int_{[t-\tau]_T}^{t} |B_t^i - B_s^j|^{-2\theta} ds\right] \left(\int_{[t-\tau]_T}^{t} |t-s|^{-2(1-\theta)} ds\right) dt \\ &\le 4c^2 \tau^{2\theta-1} N \sum_{i,j=1}^{N} \int_{-T}^{T} \left[\int_{[t-\tau]_T}^{t} |B_t^i - B_s^j|^{-2\theta} ds\right] dt \\ &\le 4c^2 \tau^{2\theta-1} N Q,\end{aligned}$$

where c is the constant in Lemma 2.6, which is independent of ε, and where

$$Q = \sum_{i,j=1}^{N} \int_{-T}^{T} ds \int_{s}^{[s+\tau]_T} |B_t^i - B_s^j|^{-2\theta} dt.$$

Then by Girsanov's theorem,

$$\begin{aligned}\left(\mathbb{E}^x\left[\mathrm{e}^{\alpha Y_\varepsilon^T}\right]\right)^2 &\le \mathbb{E}^x\left[\mathrm{e}^{2\alpha \int_{-T}^{T} \Phi_{\varepsilon,t}\cdot dB_t - \frac{1}{2}(2\alpha)^2 \int_{-T}^{T} |\Phi_{\varepsilon,t}|^2 dt}\right] \mathbb{E}^x\left[\mathrm{e}^{2\alpha^2 \int_{-T}^{T} |\Phi_{\varepsilon,t}|^2 dt}\right] \\ &= \mathbb{E}^x\left[\mathrm{e}^{2\alpha^2 \int_{-T}^{T} |\Phi_{\varepsilon,t}|^2 dt}\right] \le \mathbb{E}^x\left[\mathrm{e}^{\gamma Q}\right]. \end{aligned} \tag{2.36}$$

where $\gamma = 8c\sqrt{N}\alpha^2\tau^{2\theta-1}$ and where we recall that we have chosen $\frac{1}{2} < \theta < 1$. Writing $Q = \int_{-T}^{T} \frac{ds}{2T} 2TK_s$, where $K_s = \sum_{i,j=1}^{N} \int_s^{[s+\tau]_T} |B_t^i - B_s^j|^{-2\theta} dt$, we take the probability measure $ds/2T$ on $[-T,T]$ and apply Jensen's inequality to the convex function $X \mapsto \mathrm{e}^{2T\gamma X}$ to get

$$\mathrm{e}^{\gamma Q} = \mathrm{e}^{2T\gamma \int_{-T}^{T} K_s \frac{ds}{2T}} \le \int_{-T}^{T} \mathrm{e}^{2T\gamma K_s} \frac{ds}{2T} \tag{2.37}$$

and hence further obtain

$$\mathbb{E}^x\left[\mathrm{e}^{\gamma Q}\right] \le \int_{-T}^{T} \frac{ds}{2T} \mathbb{E}^x\left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^{N} \int_s^{[s+\tau]_T} |B_t^i - B_s^j|^{-2\theta} dt}\right]. \tag{2.38}$$

Note that $[s+\tau]_T \le s+\tau$, thus the right hand side is bounded by

$$\mathbb{E}^x\left[\mathrm{e}^{\gamma Q}\right] \le \int_{-T}^{T} \frac{ds}{2T} \mathbb{E}^x\left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^{N} \int_s^{s+\tau} |B_t^i - B_s^j|^{-2\theta} dt}\right]. \tag{2.39}$$

Taking conditional expectation with respect to $(\mathcal{F}_s)_{s\ge 0}$ with $\mathcal{F}_t = \sigma(B_r, 0 \le t)$, and using the Markov property, we see that

$$\begin{aligned}\mathbb{E}^x\left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^{N} \int_0^{\tau} |B_{s+t}^i - B_s^j|^{-2\theta} dt}\right] &= \mathbb{E}^x\left[\mathbb{E}^x\left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^{N} \int_0^{\tau} |B_{s+t}^i - B_s^j|^{-2\theta} dt} \middle| \mathcal{F}_s\right]\right] \\ &= \mathbb{E}^x\left[\mathbb{E}^{B_s}\left[\mathrm{e}^{2T\gamma \sum_{i,j=1}^{N} \int_0^{\tau} |B_t^i - B_0^j|^{-2\theta} dt}\right]\right].\end{aligned}$$

Since $|x|^{-2\theta}$ is a Kato-class potential, we see that

$$\sup_{x,z\in\mathbb{R}^3} \mathbb{E}^x[\mathrm{e}^{\beta\int_0^\tau |B_s^i+z|^{-2\theta}ds}] = \sup_{x\in\mathbb{R}^3} \mathbb{E}^x[\mathrm{e}^{\beta\int_0^\tau |B_s^i|^{-2\theta}ds}] \le e^{c\tau\beta}, \tag{2.40}$$

for some $c > 0$ and all $\beta > 0$ (see Appendix A). From this it follows that

$$\sup_{x\in\mathbb{R}^{3N}} \mathbb{E}^x\left[\mathrm{e}^{\gamma Q}\right] \le \sup_{x\in\mathbb{R}^{3N}} \int_{-T}^{T} \frac{ds}{2T}\mathbb{E}^x\left[\mathrm{e}^{2T\gamma\sum_{i,j=1}^N\int_0^\tau |B_{s+t}^i-B_s^j|^{-2\theta}dt}\right] \le e^{\alpha^2 cT} \tag{2.41}$$

for a possibly different constant c (here and in the following formula). Hence we obtain

$$\sup_{\varepsilon\in(0,1]}\sup_{x\in\mathbb{R}^3} \mathbb{E}^x[\mathrm{e}^{2\alpha Y_\varepsilon^T}] \le e^{\alpha^2 cT} \tag{2.42}$$

for all $\alpha \in \mathbb{R}$. By similar computations we can establish that the process Φ_ε converges to Φ_0 in $L^2([-T,T],\mathbb{R}^{3N})$ almost surely under Wiener measure P^x, for all $x \in \mathbb{R}^{3N}$. For every $\varepsilon > 0$ indeed we have

$$\begin{aligned}
&\int_{-T}^{T} |\Phi_{\varepsilon,t} - \Phi_{0,t}|^2 dt \\
&\le 4\mathrm{c}_\varepsilon^2 N \sum_{i,j=1}^N \int_{-T}^{T} \left[\int_{[t-\tau]_T}^{t} |B_t^i - B_s^j|^{-2\theta} ds\right] \left(\int_{[t-\tau]_T}^{t} |t-s|^{-2(1-\theta)} ds\right) dt \\
&\le 4\mathrm{c}_\varepsilon^2 \tau^{2\theta-1} N \sum_{i,j=1}^N \int_{-T}^{T} \left[\int_{[t-\tau]_T}^{t} |B_t^i - B_s^j|^{-2\theta} ds\right] dt \\
&\le 4\mathrm{c}_\varepsilon^2 \tau^{2\theta-1} NQ,
\end{aligned}$$

where as above we used Lemma 2.6 and interpolation to obtain the bound

$$|\nabla_x\varphi_\varepsilon(x,t) - \nabla_x\varphi_0(x,t)| \le \mathrm{c}_\varepsilon|x|^{-\theta}|t|^{-(1-\theta)}, \quad \theta \in [0,1] \tag{2.43}$$

for all $\varepsilon > 0$, and where the constant $\mathrm{c}_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. The almost sure convergence is then a consequence of the fact that $Q < +\infty$ almost surely under P^x, for all $x \in \mathbb{R}^{3N}$, which we have already shown above. The convergence of Φ_ε implies also the convergence of the martingale Y_ε^T to Y_0^T, at least in $L^2(\Omega, \mathrm{P}^x)$, for all $x \in \mathbb{R}^{3N}$. □

Lemma 2.11 *There exists a constant $c_{\rm ren}$ such that for all $\alpha \in \mathbb{R}$ and every $f, h \in L^2(\mathbb{R}^{3N})$ we have*

$$\int_{\mathbb{R}^{3N}} \mathbb{E}^x[f(B_{-T})h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\alpha S_\varepsilon^{\rm ren}}]dx \le \|f\|\|h\|e^{c_{\rm ren}(\alpha^2T+\alpha T+\alpha)}$$

for all $\varepsilon \ge 0$.

PROOF. Recall the decomposition $S_\varepsilon^{\rm ren} = S_\varepsilon^{\rm OD} + X_\varepsilon + Y_\varepsilon + Z_\varepsilon$. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^{3N}} \mathbb{E}^x[|f(B_{-T})h(B_T)|e^{\alpha S_\varepsilon^{\rm ren}}]dx \\
&= \int_{\mathbb{R}^{3N}} \mathbb{E}^0\left[|f(x)h(B_T)|e^{-\int_{-T}^T V(B_s)ds} e^{\alpha(S_\varepsilon^{{\rm OD},T} + X_\varepsilon^T + Y_\varepsilon^T + Z_\varepsilon^T)}\right] dx \\
&\le \|f\|\|h\| \sup_{x\in\mathbb{R}^{3N}} \left(\mathbb{E}^x\left[e^{-2\int_{-T}^T V(B_s)ds} e^{2\alpha(S_\varepsilon^{{\rm OD},T} + X_\varepsilon^T + Y_\varepsilon^T + Z_\varepsilon^T)}\right]\right)^{1/2}. \qquad (2.44)
\end{aligned}$$

By Lemmas 2.8, 2.9 and 2.10 and the fact that V is Kato-class, we see that there exists a constant $c_{\rm ren}$ such that

$$\sup_{x\in\mathbb{R}^{3N}} \mathbb{E}^x\left[e^{-2\int_{-T}^T V(B_s)ds} e^{2\alpha(S_\varepsilon^{{\rm OD},T} + X_\varepsilon^T + Y_\varepsilon^T + Z_\varepsilon^T)}\right] \le e^{2c_{\rm ren}(\alpha^2 T + \alpha T + \alpha)},$$

and the lemma follows. □

2.3 Renormalized Hamiltonian

In this section we show that $H_\varepsilon + g^2 N \varphi_\varepsilon(0,0)$ converges to a self-adjoint operator $H_{\rm ren}$ as $\varepsilon \downarrow 0$.

2.3.1 Convergence of the renormalized action

Lemma 2.12 *If $\alpha \in \mathbb{R}$, then for every $x \in \mathbb{R}^{3N}$*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}^x[|e^{\alpha U_\varepsilon^T} - e^{\alpha U_0^T}|] = 0, \quad U = S^{\rm OD}, X, Y, Z. \qquad (2.45)$$

PROOF. Let $U = X$. We obtain that $|X_\varepsilon^T| \le \int_{-T}^T V_{\rm Coul}(B_s)ds$, where $V_{\rm Coul}(x) = C\sum_{i\neq j}^N |x^i - x^j|^{-1}$ with some constant C, and the fact that

$$\mathbb{E}^x\left[|e^{\alpha X_\varepsilon^T} - e^{\alpha X_0^T}|\right] \le 2\mathbb{E}^x\left[e^{\alpha\int_{-T}^T V_{\rm Coul}(B_s)ds}\right] < \infty.$$

Since $X_\varepsilon^T \to X_0^T$ a.s. with respect to P^x for every $x \in \mathbb{R}^{3N}$, the Lebesgue dominated convergence theorem implies (2.45).

Let $U = Y$. It suffices to show that $\sup_{x\in\mathbb{R}^{3N}} \mathbb{E}^x\left[\left|e^{\alpha(Y_\varepsilon^T - Y_0^T)} - 1\right|^2\right] \to 0$. We have

$$\mathbb{E}^x\left[\left(e^{\alpha(Y_\varepsilon^T - Y_0^T)} - 1\right)^2\right] = \mathbb{E}^x\left[e^{2\alpha(Y_\varepsilon^T - Y_0^T)}\right] + 1 - 2\mathbb{E}^x\left[e^{\alpha(Y_\varepsilon^T - Y_0^T)}\right]. \qquad (2.46)$$

We will show below that $\lim_{\varepsilon\downarrow 0}\mathbb{E}^x[\mathrm{e}^{\alpha(Y_\varepsilon^T-Y_0^T)}]=1$. Define the random process $\delta\Phi_t=\Phi_{\varepsilon,t}-\Phi_{0,t}$ so that

$$Y_\varepsilon^T-Y_0^T=\int_{-T}^T\delta\Phi_t\cdot dB_t.$$

By the Girsanov theorem

$$1=\mathbb{E}^x[\mathrm{e}^{\alpha\int_0^{2T}\delta\Phi_t\cdot dB_t-\frac{\alpha^2}{2}\int_0^{2T}|\delta\Phi_t|^2dt}] \tag{2.47}$$

for every $\alpha\in\mathbb{R}$, hence it follows that

$$\left(\mathbb{E}^x\left[\mathrm{e}^{\alpha(Y_\varepsilon^T-Y_0^T)}\right]-1\right)^2\le\mathbb{E}^x\left[\mathrm{e}^{2\alpha\int_{-T}^T\delta\Phi_t\cdot dB_t}\right]\mathbb{E}^x\left[\left(1-\mathrm{e}^{-\frac{\alpha^2}{2}\int_{-T}^T|\delta\Phi_t|^2dt}\right)^2\right]. \tag{2.48}$$

We see that by (2.47) again

$$\sup_{x\in\mathbb{R}^{3N}}\mathbb{E}^x\left[\mathrm{e}^{2\alpha\int_{-T}^T\delta\Phi_t\cdot dB_t}\right]\le\sup_{x\in\mathbb{R}^{3N}}\left(\mathbb{E}^x\left[\mathrm{e}^{4\alpha^2\int_{-T}^T|\delta\Phi_t|^2dt}\right]\right)^{1/2} \tag{2.49}$$

and furthermore

$$\mathbb{E}^x\left[\left(1-\mathrm{e}^{-\frac{\alpha^2}{2}\int_{-T}^T|\delta\Phi_t|^2dt}\right)^2\right]\le\mathbb{E}^x\left[\left|\frac{\alpha^2}{2}\int_{-T}^T|\delta\Phi_t|^2dt\right|^2\right]\to 0 \tag{2.50}$$

as $\varepsilon\downarrow 0$. Here (2.50) can be shown by Lemma 2.10. The right-hand side of (2.49) is uniformly bounded in ε, which can be proven in the same way as in the proof of Lemma 2.11. Hence (2.48) converges to zero as $\varepsilon\downarrow 0$ and (2.45) for $U=Y$ follows.

Let $U=Z$. It suffices to show that $\sup_{x\in\mathbb{R}^{3N}}\mathbb{E}^x\left[|\mathrm{e}^{\alpha(Z_\varepsilon-Z_0)}-1|\right]\to 0$. We have

$$\begin{aligned}Z_\varepsilon-Z_0=2\sum_{i,j=1}^N\int_{-T}^T ds\int_{\mathbb{R}^3}&\mathrm{e}^{-ik\cdot(B^i_{[s+\tau]_T-s}+x^i-B^j_{[s+\tau]_T-s}-x^j)}\mathrm{e}^{-([s+\tau]_T-s)\omega(k)}\\&\times\frac{1}{\omega(k)}\beta(k)\mathbb{1}_\sigma^\perp(k)(1-\mathrm{e}^{-\varepsilon|k|^2})dk.\end{aligned}$$

Let $\eta_\varepsilon=\alpha(Z_\varepsilon-Z_0)$. It can be directly seen that $|\eta_\varepsilon|^n\le c^n\alpha^nT^n\varepsilon^n$ for a constant c. Then we have $\mathbb{E}^x[\mathrm{e}^{\eta_\varepsilon}]=1+\sum_{n\ge1}\frac{1}{n!}\mathbb{E}^0[\eta_\varepsilon^n]$ and

$$\sum_{n\ge1}\frac{1}{n!}\mathbb{E}^x[|\eta_\varepsilon|^n]\le\sum_{n\ge1}\frac{1}{n!}c^nT^n\varepsilon^n\to 0$$

as $\varepsilon\downarrow 0$, uniformly in $x\in\mathbb{R}^{3N}$. Thus (2.45) for $U=Z$ follows. For $U=S^{\mathrm{OD}}$ we obtain (2.45) in a similar way. □

Lemma 2.13 *Let $\alpha \in \mathbb{R}$ and $f, h \in L^2(\mathbb{R}^{3N})$. Then*

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^{3N}} dx \mathbb{E}^x [f(B_{-T}) h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s) ds} \mathrm{e}^{\alpha S_\varepsilon^{\mathrm{ren}}}] = \int_{\mathbb{R}^{3N}} dx \mathbb{E}^x [f(B_{-T}) h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s) ds} \mathrm{e}^{\alpha S_0^{\mathrm{ren}}}]. \tag{2.51}$$

Proof. Write $S_\varepsilon = S_\varepsilon^{\mathrm{OD},T} + X_\varepsilon^T + Y_\varepsilon^T + Z_\varepsilon^T$. Then by telescoping we have that

$$\begin{aligned} &\left| \int_{\mathbb{R}^{3N}} dx \mathbb{E}^x \left[f(B_{-T}) h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s) ds} \left(\mathrm{e}^{\alpha S_\varepsilon^{\mathrm{ren}}} - \mathrm{e}^{\alpha S_0^{\mathrm{ren}}} \right) \right] \right| \\ &\leq \mathrm{e}^{2T\|V\|_\infty} \int_{\mathbb{R}^{3N}} dx |f(x)| \mathbb{E}^x \left[|h(B_T)| \left(\mathrm{e}^{\alpha S_\varepsilon} - \mathrm{e}^{\alpha S_0} \right) \right] \\ &\leq \mathrm{e}^{2T\|V\|_\infty} \int_{\mathbb{R}^{3N}} dx |f(x)| \left(\mathbb{E}^x \left[|h(B_T)|^2 \right] \right)^{1/2} E_\varepsilon(x), \end{aligned}$$

where $E_\varepsilon(x) = \left(\mathbb{E}^x \left[\left(\mathrm{e}^{\alpha S_\varepsilon} - \mathrm{e}^{\alpha S_0} \right)^2 \right] \right)^{1/2}$. Note that by reasoning like in Lemma 2.11 we can show that $\sup_{x \in \mathbb{R}^{3N}} E_\varepsilon(x) < \infty$, and by Lemma 2.12 we have that $\lim_{\varepsilon \downarrow 0} E_\varepsilon(x) = 0$ for every $x \in \mathbb{R}^3$. Hence the Lebesgue dominated convergence theorem implies that the second term converges to zero, and the lemma follows. □

Lemma 2.14 *It follows that*

$$\lim_{\varepsilon \downarrow 0} (f \otimes \mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon + g^2 N \varphi_\varepsilon(0,0))} h \otimes \mathbb{1}) = \int_{\mathbb{R}^{3N}} \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s) ds} \mathrm{e}^{\frac{g^2}{2} S_0^{\mathrm{ren}}} \right] dx, \tag{2.52}$$

where

$$\begin{aligned} S_0^{\mathrm{ren}} &= 2 \sum_{i \neq j}^{N} \int_{-T}^{T} \varphi_0(B_s^i - B_s^j, 0) ds + 2 \sum_{i,j=1}^{N} \int_{-T}^{T} ds \left(\int_{-T}^{t} \nabla_x \varphi_0(B_t^i - B_s^j, t - s) ds \right) \cdot dB_t \\ &\quad - 2 \sum_{i,j=1}^{N} \int_{-T}^{T} \varphi_0(B_T^i - B_s^j, T - s) ds, \end{aligned} \tag{2.53}$$

and the integrands are given by $\varphi_0(X,t) = \int_{\mathbb{R}^3} \frac{e^{-ikX} e^{-|t|\omega(k)}}{2\omega(k)} \beta(k) \mathbb{1}_\sigma^\perp(k) dk$ and

$$\nabla_x \varphi_0(X,t) = \int_{\mathbb{R}^3} \frac{-ike^{-ikX} e^{-|t|\omega(k)}}{2\omega(k)} \beta(k) \mathbb{1}_\sigma^\perp(k) dk.$$

Proof. We have

$$(f \otimes \mathbb{1}, \mathrm{e}^{-2T(H_\varepsilon + g^2 N \varphi_\varepsilon(0,0))} h \otimes \mathbb{1}) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) \mathrm{e}^{-\int_{-T}^{T} V(B_s) ds} \mathrm{e}^{\frac{g^2}{2} S_\varepsilon^{\mathrm{ren}}} \right] dx. \tag{2.54}$$

The right-hand side above converges to $\int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^{T} V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\mathrm{ren}}}]dx$ as $\varepsilon \downarrow 0$. Thus (2.52) follows. We also see that

$$S_0^{\mathrm{ren}} = 2\sum_{i\neq j}^{N} \int_{-T}^{T} \varphi_0(B_s^i - B_s^j, 0)ds + 2\sum_{i,j=1}^{N} \int_{-T}^{T} \left(\int_{[t-\tau]_T}^{t} \nabla_x \varphi_0(B_t^i - B_s^j, t-s)ds \right) \cdot dB_t$$
$$- 2\sum_{i,j=1}^{N} \int_{-T}^{T} \varphi_0(B_{[s+\tau]_T}^i - B_s^j, [s+\tau]_T - s)ds. \tag{2.55}$$

Taking $\tau = T$, we obtain (2.53). □

2.3.2 Extension beyond the vacuum vector

Now we extend the result in Section 2.3.1 from vectors of the form $f \otimes \mathbb{1}$ to more general vectors of the form $f \otimes F(\phi(f_1), \ldots, \phi(f_n))\mathbb{1}$, with $F \in \mathscr{S}(\mathbb{R}^n)$, where $\phi(f)$ stands for the scalar field given by $\frac{1}{\sqrt{2}}(a^*(\hat{f}) + a(\tilde{\hat{f}}))$, where $\tilde{\hat{f}}(k) = \hat{f}(-k)$. To do this we need a Feynman-Kac-type formula giving a representation of $\mathrm{e}^{-2TH_\varepsilon}$.

Denote

$$H_{-k}(\mathbb{R}^n) = \{f \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^n) \mid \widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n), |\cdot|^{-k/2}\widehat{f} \in L^2(\mathbb{R}^n)\} \tag{2.56}$$

endowed with the norm $\|f\|^2_{H_{-k}(\mathbb{R}^n)} = \displaystyle\int_{\mathbb{R}^n} |\widehat{f}(x)|^2|x|^{-k}dx$. Recall that a Euclidean field is a family of Gaussian random variables $\{\phi_{\mathrm{E}}(F),\, F \in H_{-1}(\mathbb{R}^4)\}$ on a probability space $(Q_{\mathrm{E}}, \Sigma_{\mathrm{E}}, \mu_{\mathrm{E}})$, such that the map $F \mapsto \phi_{\mathrm{E}}(F)$ is linear, and their mean and covariance are given by

$$\mathbb{E}_{\mu_{\mathrm{E}}}[\phi_{\mathrm{E}}(F)] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_{\mathrm{E}}}[\phi_{\mathrm{E}}(F)\phi_{\mathrm{E}}(G)] = \frac{1}{2}(F,G)_{H_{-1}(\mathbb{R}^4)}.$$

For the reader's convenience we summarize in Appendix B some basic facts on Euclidean fields and their representation in L^2 space (including the operators $\mathrm{J}_t : \mathscr{F}_{\mathrm{b}} \to L^2(Q_{\mathrm{E}}; \mu_{\mathrm{E}})$), which will be used here. In what follows, we identify $\mathscr{H}$ with the set of $\mathscr{F}_{\mathrm{b}}$-valued L^2 functions $L^2(\mathbb{R}^{3N}; \mathscr{F}_{\mathrm{b}})$, i.e., $F \in \mathscr{H}$ can be regarded as a function $\mathbb{R}^{3N} \ni x \mapsto F(x) \in \mathscr{F}_{\mathrm{b}}$ such that $\int_{\mathbb{R}^{3N}} \|F(x)\|^2_{\mathscr{F}_{\mathrm{b}}} dx < \infty$.

Proposition 2.15 *Let $F, G \in \mathscr{H}$. Then*

$$(F, \mathrm{e}^{-2TH_\varepsilon}G)$$
$$= \int_{\mathbb{R}^{3N}} dx \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^{T} V(B_s)ds} \mathbb{E}_{\mu_{\mathrm{E}}} \left[\mathrm{J}_{-T}F(B_{-T}) \cdot \mathrm{e}^{-\phi_{\mathrm{E}}(\int_{-T}^{T} \sum_{j=1}^{N} \delta_s \otimes \tilde{\varphi}(\cdot - B_s^j)ds)} \mathrm{J}_T G(B_T) \right] \right], \tag{2.57}$$

where $\tilde{\varphi}_\varepsilon(x) = \left(\mathrm{e}^{-\varepsilon|\cdot|^2/2}\mathbb{1}_\sigma^\perp/\sqrt{\omega}\right)^\vee(x)$, and $\delta_s(x) = \delta(x-s)$ is Dirac delta distribution with mass on s.

PROOF. See [LHB11, Theorem 6.3]. □

Lemma 2.16 *Let $\rho_j \in H_{-1/2}(\mathbb{R}^3)$ for $j=1,2$, $f,h \in L^2(\mathbb{R}^{3N})$ and $\alpha,\beta\in\mathbb{C}$. Then*

$$\lim_{\varepsilon\downarrow 0}(f\otimes \mathrm{e}^{\alpha\phi(\rho_1)}1\!\!1, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes \mathrm{e}^{\beta\phi(\rho_2)}1\!\!1)$$
$$= \int_{\mathbb{R}^{3N}}\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\mathrm{ren}}+\frac14\xi}\right]dx, \tag{2.58}$$

where

$$\xi=\xi(g)=\bar\alpha^2\|\rho_1/\sqrt\omega\|^2+\beta^2\|\rho_2/\sqrt\omega\|^2+2\bar\alpha\beta(\rho_1/\sqrt\omega,\mathrm{e}^{-2T\omega}\rho_2/\sqrt\omega)$$
$$+2\bar\alpha g\sum_{j=1}^N\int_{-T}^T ds\int_{\mathbb{R}^3}dk\frac{\widehat\rho_1(k)}{\sqrt{\omega(k)}}1\!\!1_\sigma^\perp(k)\mathrm{e}^{-|s-T|\omega(k)}\mathrm{e}^{-ikB_s^j}$$
$$+2\beta g\sum_{j=1}^N\int_{-T}^T ds\int_{\mathbb{R}^3}dk\frac{\widehat\rho_2(k)}{\sqrt{\omega(k)}}1\!\!1_\sigma^\perp(k)\mathrm{e}^{-|s+T|\omega(k)}\mathrm{e}^{-ikB_s^j}.$$

PROOF. By the functional integral representation (2.57) we have

$$(f\otimes \mathrm{e}^{\alpha\phi(\rho_1)}1\!\!1, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes \mathrm{e}^{\beta\phi(\rho_2)}1\!\!1)=\int_{\mathbb{R}^{3N}}dx\mathbb{E}^x\Big[\bar f(B_{-T})h(B_T)\mathrm{e}^{-\int_{-T}^TV(B_s)ds}$$
$$\times\ \mathbb{E}_{\mu_E}[\mathrm{e}^{\bar\alpha\phi_E(\delta_{-T}\otimes\rho_1)}\mathrm{e}^{\beta\phi_E(\delta_T\otimes\rho_2)}\mathrm{e}^{g\phi_E(-\sum_{j=1}^N\int_{-T}^T\delta_s\otimes\tilde\varphi_\varepsilon(\cdot-B_s^j)ds)}]\Big]\ \mathrm{e}^{-2Tg^2N\varphi_\varepsilon(0,0)}.$$

It can be directly seen that

$$\mathbb{E}_{\mu_E}\Big[\mathrm{e}^{\bar\alpha\phi_E(\delta_{-T}\otimes\rho_1)}\mathrm{e}^{\beta\phi_E(\delta_T\otimes\rho_2)}\mathrm{e}^{g\phi_E(-\sum_{j=1}^N\int_{-T}^T\delta_s\otimes\tilde\varphi_\varepsilon(\cdot-B_s^j)ds)}\Big]\ \mathrm{e}^{-2Tg^2N\varphi_\varepsilon(0,0)}=\mathrm{e}^{\frac{g^2}{2}S_\varepsilon^{\mathrm{ren}}+\frac14\xi_\varepsilon},$$

where ξ_ε is defined by ξ with $1\!\!1_\sigma^\perp(k)$ replaced by $1\!\!1_\sigma^\perp(k)\mathrm{e}^{-\varepsilon|k|^2/2}$. Thus

$$(f\otimes \mathrm{e}^{\alpha\phi(\rho_1)}1\!\!1, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes \mathrm{e}^{\beta\phi(\rho_2)}1\!\!1)$$
$$=\int_{\mathbb{R}^{3N}}dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^TV(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_\varepsilon^{\mathrm{ren}}+\frac14\xi_\varepsilon}\right].$$

Notice that with a constant C we have $\xi_\varepsilon\le C$ uniformly in the paths and $\varepsilon\ge 0$. Hence we can complete the proof of the lemma in a similar way to Lemma 2.14. □

Consider the dense subspace $\mathscr{D}$ of $\mathscr{H}$ given by

$$\mathscr{D}=\mathrm{L.H.}\,\{f\otimes 1\!\!1\,|\,f\in L^2(\mathbb{R}^{3N})\}\cup$$
$$\left\{f\otimes F(\phi(f_1),\ldots,\phi(f_n))\,|\,F\in\mathscr{S}(\mathbb{R}^n),f_j\in C_0^\infty(\mathbb{R}^3),1\le j\le n,f\in L^2(\mathbb{R}^{3N})\right\}.$$

By Lemma 2.16 the next result is immediate.

Lemma 2.17 *Let* $\Phi = f \otimes F(\phi(u_1), \ldots, \phi(u_n))$, $\Psi = h \otimes G(\phi(v_1), \ldots, \phi(v_m)) \in \mathscr{D}$. *Then*

$$\lim_{\varepsilon\downarrow 0}(\Phi, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}\Psi) = (2\pi)^{-(n+m)/2}\int_{\mathbb{R}^{n+m}} dK_1 dK_2 \overline{\widehat{F}(K_1)}\widehat{G}(K_2) \times \int_{\mathbb{R}^{3N}} dx \mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\mathrm{ren}}+\frac{1}{4}\xi(K_1,K_2)}\right], \tag{2.59}$$

where

$$\xi(K_1,K_2) = -\|K_1\cdot u/\sqrt{\omega}\|^2 - \|K_2\cdot v/\sqrt{\omega}\|^2 - 2(K_1\cdot u/\sqrt{\omega}, \mathrm{e}^{-2T\omega}K_2\cdot v/\sqrt{\omega}) - 2ig\sum_{j=1}^N\int_{-T}^T ds\int_{\mathbb{R}^3} dk\frac{K_1\cdot\widehat{u}(k)}{\sqrt{\omega(k)}}\mathbb{1}_\sigma^\perp(k)\mathrm{e}^{-|s-T|\omega(k)}\mathrm{e}^{-ikB_s^j} + 2ig\sum_{j=1}^N\int_{-T}^T ds\int_{\mathbb{R}^3} dk\frac{K_2\cdot\widehat{v}(k)}{\sqrt{\omega(k)}}\mathbb{1}_\sigma^\perp(k)\mathrm{e}^{-|s+T|\omega(k)}\mathrm{e}^{-ikB_s^j}$$

and $u = (u_1, ..., u_n)$, $v = (v_1, ..., v_m)$.

Proof. Notice that $F(\phi(f_1), \ldots, \phi(f_n)) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\widehat{F}(K)\mathrm{e}^{i\phi(K\cdot f)}dK$. Hence

$$(\Phi, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}\Psi) = \frac{1}{(2\pi)^{(n+m)/2}}\int_{\mathbb{R}^{m+n}} dK_1 dK_2\overline{\widehat{F}(K_1)}\widehat{G}(K_2) \times (f\otimes\mathrm{e}^{-i\phi(K_1\cdot f)}, \mathrm{e}^{-2T(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}h\otimes\mathrm{e}^{-i\phi(K_2\cdot h)}).$$

Thus the statement follows from Lemma 2.16. □

2.3.3 Uniform lower bound

In this section we show a crucial lower bound on $H_\varepsilon + g^2N\varphi_\varepsilon(0,0)$ uniform in $\varepsilon > 0$, and give the proof of Theorem 2.2.

Corollary 2.18 *There exists* $C \in \mathbb{R}$ *such that* $H_\varepsilon + g^2N\varphi_\varepsilon(0,0) > C$, *uniformly in* $\varepsilon > 0$.

Proof. Consider the function

$$W(x^1, ..., x^N) = \sum_{j=1}^N |x^j|^2.$$

We denote H_ε with V replaced by δW by $H_\varepsilon(\delta)$, for $\delta \geq 0$. Then $-\frac{1}{2}\sum_{j=1}^N\Delta_j + \delta W$, $\delta > 0$, has a compact resolvent, which implies that $H_\varepsilon(\delta)$ for $\delta > 0$ has a unique ground state $\Psi_{\rm g}(\delta)$ by [Spo98, Ger00], see Remark 2.19 below. By the Feynman-Kac formula (2.57) we see that $\mathrm{e}^{-TH_\varepsilon(\delta)}$ is positivity improving for $\varepsilon > 0$, i.e., $(F, \mathrm{e}^{-TH_\varepsilon(\delta)}G) > 0$ for $F, G \in \mathscr{H}$ such that $F, G \geq 0$. Hence it follows that $\Psi_{\rm g}(\delta) > 0$. In particular, $(f \otimes 1\!\!1, \Psi_{\rm g}(\delta)) \neq 0$, for every $0 \leq f \in L^2(\mathbb{R}^{3N})$, where $f \not\equiv 0$. Thus

$$\inf \sigma\left(H_\varepsilon(\delta) + g^2 N\varphi_\varepsilon(0,0)\right) = -\lim_{T\to\infty} \frac{1}{T} \log(f \otimes 1\!\!1, \mathrm{e}^{-T(H_\varepsilon(\delta)+g^2N\varphi_\varepsilon(0,0))} f \otimes 1\!\!1), \quad (2.60)$$

for every $0 \leq f \in L^2(\mathbb{R}^{3N})$. By Lemma 2.11 there exists a constant b such that

$$\begin{aligned}
&(f \otimes 1\!\!1, \mathrm{e}^{-2T(H_\varepsilon(\delta)+g^2N\varphi_\varepsilon(0,0))} f \otimes 1\!\!1) = \int_{\mathbb{R}^{3N}} dx \mathbb{E}^x[f(B_{-T})f(B_T)\mathrm{e}^{-\int_{-T}^T \delta W(B_s)ds} \mathrm{e}^{S_\varepsilon^{\rm ren}}] \\
&\leq \int_{\mathbb{R}^{3N}} dx \mathbb{E}^x[|f(B_{-T})||f(B_T)|\mathrm{e}^{S_\varepsilon^{\rm ren}}] \\
&\leq \|f\|^2 \mathrm{e}^{b(1+2T)},
\end{aligned}$$

which implies, together with (2.60), that

$$\inf \sigma\left(H_\varepsilon(\delta) + g^2N\varphi_\varepsilon(0,0)\right) + b \geq 0, \quad \delta > 0. \quad (2.61)$$

Note that b is independent of δ. Thus

$$|(F, \mathrm{e}^{-2T(H_\varepsilon(\delta)+g^2N\varphi_\varepsilon(0,0))}G)| \leq \|F\|\|G\|\mathrm{e}^{2bT} \quad (2.62)$$

follows for every $\delta > 0$. Let $F, G \in \mathscr{H}$. By the Feynman-Kac formula (2.57) we have

$$\begin{aligned}
&(F, \mathrm{e}^{-2TH_\varepsilon(\delta)}G) \\
&= \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\mathrm{e}^{-\int_{-T}^T \delta W(B_s)ds} \mathbb{E}_{\mu_{\rm E}} \left[\mathrm{J}_{-T}F(B_{-T}) \cdot \mathrm{e}^{-\phi_{\rm E}(\int_{-T}^T \sum_{j=1}^N \delta_s \otimes \tilde{\varphi}(\cdot - B_s^j)ds)} \mathrm{J}_T G(B_T)\right]\right].
\end{aligned}$$

The Lebesgue dominated convergence theorem furthermore implies

$$\lim_{\delta\downarrow 0}(F, \mathrm{e}^{-2T(H_\varepsilon(\delta)+g^2N\varphi_\varepsilon(0,0))}G) = (F, \mathrm{e}^{-2T(H_\varepsilon(0)+g^2N\varphi_\varepsilon(0,0))}G).$$

Taking the limit $\delta \downarrow 0$ on both sides of (2.62), we have

$$|(F, \mathrm{e}^{-2T(H_\varepsilon(0)+g^2N\varphi_\varepsilon(0,0))}G)| \leq \|F\|\|G\|\mathrm{e}^{2bT}. \quad (2.63)$$

This implies that (2.61) also holds for $\delta = 0$. Since $H_\varepsilon = H_\varepsilon(0) + V$ and V is bounded, we obtain

$$\inf \sigma(H_\varepsilon + g^2N\varphi_\varepsilon(0,0)) + b + \|V\|_\infty \geq 0.$$

Setting $C = -b - \|V\|_\infty$ yields the corollary. □

Remark 2.19 Let Σ be the infimum of the essential spectrum of the self-ajoint operator $h=-\frac{1}{2}\sum_{j=1}^N \Delta_j + V$ in $L^2(\mathbb{R}^{3N})$ and $E=\inf\sigma(-\frac{1}{2}\sum_{j=1}^N \Delta_j+V)$. Let $\Psi_{\rm p}$ be the ground state of h. Then it is known that H_ε has a ground state if and only if

$$\lim_{T\to\infty}\frac{(\Psi_{\rm p}\otimes 1\!\!1, e^{-TH_\varepsilon}\Psi_{\rm p}\otimes 1\!\!1)^2}{(\Psi_{\rm p}\otimes 1\!\!1, e^{-2TH_\varepsilon}\Psi_{\rm p}\otimes 1\!\!1)}>0.$$

This is shown in [Spo98] by using functional integrations, see also [LHB11, Section 6]. Then H_ε has a unique ground state if

$$\Sigma - E > \frac{N^2}{4}\int_{\mathbb{R}^3} \mathrm{e}^{-\varepsilon|k|^2}\beta(k)1\!\!1_\sigma^\perp(k)dk$$

see also [LHB11, Theorem 6.6]. In particular, in the case of $V(x^1,...,x^N)=\delta\sum_{j=1}^N|x^j|^2$, the operator H_ε has a unique ground state for every $\varepsilon>0$ and $\delta>0$, since $\Sigma-E=\infty$.

Now we can complete the proof of the main theorem.

Proof of Theorem 2.2. Let $F,G\in\mathscr{H}$ and $C_\varepsilon(F,G)=(F,\mathrm{e}^{-t(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}G)$. By Lemma 2.16 we obtain that $C_\varepsilon(F,G)$ is convergent as $\varepsilon\downarrow 0$, for every $F,G\in\mathscr{D}$. By the uniform bound

$$\|\mathrm{e}^{-t(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}\|<\mathrm{e}^{-tC}$$

obtained from Corollary 2.18 and since $\mathscr{D}$ is dense in $\mathscr{H}$, it follows that $\{C_\varepsilon(F,G)\}_\varepsilon$ is a Cauchy sequence for $F,G\in\mathscr{H}$. Let $C_0(F,G)=\lim_{\varepsilon\downarrow 0}C_\varepsilon(F,G)$. Hence we get $|C_0(F,G)|\le \mathrm{e}^{-tC}\|F\|\|G\|$. The Riesz theorem implies that there exists a bounded operator T_t such that

$$C_0(F,G)=(F,T_tG),\quad F,G\in\mathscr{H}.$$

Thus $\mathrm{s-}\lim_{\varepsilon\downarrow 0}\mathrm{e}^{-t(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}=T_t$ follows. Furthermore, we also have that

$$\mathrm{s-}\lim_{\varepsilon\downarrow 0}\mathrm{e}^{-t(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}\mathrm{e}^{-s(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}=\mathrm{s-}\lim_{\varepsilon\downarrow 0}\mathrm{e}^{-(t+s)(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}=T_{t+s}.$$

Since the left-hand side above is T_tT_s, the semigroup property of T_t follows. Since $\mathrm{e}^{-t(H_\varepsilon+g^2N\varphi_\varepsilon(0,0))}$ is a symmetric semigroup, T_t is also symmetric. By the functional integral representation (2.59) the functional (F,T_tG) is continuous at $t=0$ for every $F,G\in\mathscr{D}$. Given that $\mathscr{D}$ is in $\mathscr{H}$ and $\|T_t\|$ is uniformly bounded in a neighborhood of $t=0$, it also follows that T_t is strongly continuous at $t=0$. Then the semigroup version of Stone's theorem [LHB11, Proposition 3.26] implies that there exists a self-adjoint operator $H_{\rm ren}$, bounded from below, such that $T_t=\mathrm{e}^{-tH_{\rm ren}}$, $t\ge 0$. The proof is completed by setting $E_\varepsilon=-g^2N\varphi_\varepsilon(0,0)$. □

We established the existence of the renormalized Hamiltonian $H_{\rm ren}$. We can obtain explicitly the pair interaction potential associated with $H_{\rm ren}$.

Corollary 2.20 *The pair interaction potential associated with* $H_{\rm ren}$ *is given by* $\frac{g^2}{2}S_0^{\rm ren}$.

PROOF. By Lemma 2.14 we see that

$$(f\otimes 1\!\!1, \mathrm{e}^{-2TH_{\rm ren}}h\otimes 1\!\!1)=\int_{\mathbb{R}^{3N}}dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\rm ren}}\right]. \tag{2.64}$$

□

3 Effective potential in the weak coupling limit

In this section we consider the weak coupling limit of the renormalized Hamiltonian. In order to have a physically reasonable effective potential, we take the dispersion relation

$$\omega_\nu(k)=\sqrt{|k|^2+\nu^2}$$

with positive mass $\nu>0$ instead of $\omega(k)=|k|$, set the IR cutoff to zero, and take the cutoff function to be

$$\widehat{\varrho}_\varepsilon(k)=(2\pi)^{-3/2}\mathrm{e}^{-\varepsilon|k|^2/2}.$$

The Hamiltonian is defined on $L^2(\mathbb{R}^{3N})\otimes\mathscr{F}_{\rm b}$ and given by

$$H_\varepsilon=H_{\rm p}\otimes 1\!\!1+1\!\!1\otimes H_{\rm f}+H_{\rm I},$$

where $H_{\rm p}=\sum_{j=1}^N(-\frac{1}{2}\Delta_j)+V(x_1,...,x_N)$ denotes the N-body Schrödinger operator, and

$$H_{\rm f}=\int_{\mathbb{R}^3}\omega_\nu(k)a^*(k)a(k)dk$$

is the free massive boson field. We scale the Hamiltonian by replacing the annihilation operator a and the creation operator a^* by κa and κa^*, respectively, where $\kappa>0$ is the scaling parameter. Then H_ε changes to

$$H_\varepsilon(\kappa)=H_{\rm p}\otimes 1\!\!1+\kappa^2 1\!\!1\otimes H_{\rm f}+\kappa H_{\rm I}. \tag{3.1}$$

This scaling also implies the transformations $\omega\mapsto\kappa^2\omega$ and $\widehat{\varrho}\mapsto\kappa^2\widehat{\varrho}$, while the energy renormalization term scales as

$$E_\varepsilon(\kappa)=-g^2N\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\varepsilon|k|^2}}{2(2\pi)^3\omega_\nu(k)}\frac{\kappa^2}{\kappa^2\omega_\nu(k)+|k|^2/2}dk. \tag{3.2}$$

By Theorem 2.2 there exists a self-adjoint operator $H_{\rm ren}(\kappa)$ such that

$$\lim_{\varepsilon\downarrow 0}(f\otimes 1\!\!1, \mathrm{e}^{-t(H_\varepsilon(\kappa)-E_\varepsilon(\kappa))}h\otimes 1\!\!1)=(f\otimes 1\!\!1, \mathrm{e}^{-tH_{\rm ren}(\kappa)}h\otimes 1\!\!1). \tag{3.3}$$

The next proposition is established in [Dav79, Hir99].

Proposition 3.1 *We have*

$$\mathrm{s}-\lim_{\varepsilon\downarrow 0}\lim_{\kappa\to\infty} \mathrm{e}^{-t(H_\varepsilon(\kappa)-E_\varepsilon(\kappa))} = \mathrm{e}^{-th_{\rm eff}}\otimes P_\Omega,$$

where P_Ω *denotes the projection to* $\{z\mathbb{1}\,|\,z\in\mathbb{C}\}\subset\mathscr{F}_{\rm b}$ *and*

$$h_{\rm eff} = -\frac{1}{2}\sum_{j=1}^N \Delta_j + V(x^1,...,x^N) - \frac{g^2}{4\pi}\sum_{i<j}\frac{\mathrm{e}^{-\nu|x_i-x_j|}}{|x_i-x_j|}.$$

We are now interested in the scaling limit of $H_{\rm ren}(\kappa)$ when $\kappa\to\infty$. By Theorem 2.2 we see that

Lemma 3.2 *If* $f,h\in L^2(\mathbb{R}^{3N})$, *then*

$$\lim_{\kappa\to\infty}(f\otimes\mathbb{1}, \mathrm{e}^{-tH_{\rm ren}(\kappa)}h\otimes\mathbb{1}) = (f,\mathrm{e}^{-th_{\rm eff}}h). \tag{3.4}$$

Proof. By Lemma 2.16 we have

$$(f\otimes\mathbb{1},\mathrm{e}^{-2TH_{\rm ren}(\kappa)}h\otimes\mathbb{1}) = \int_{\mathbb{R}^{3N}} dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\rm ren}(\kappa)}\right], \tag{3.5}$$

where

$$\begin{aligned} S_0^{\rm ren}(\kappa) = {} & 2\sum_{i\neq j}^N\int_{-T}^T \varphi_0(B_s^i-B_s^j,0,\kappa)ds + 2\sum_{i,j=1}^N\int_{-T}^T\left(\int_{-T}^t\nabla_x\varphi_0(B_t-B_s,t-s,\kappa)ds\right)\cdot dB_t \\ & -2\sum_{i,j=1}^N\int_{-T}^T\varphi_0(B_T-B_s,T-s,\kappa)ds, \end{aligned} \tag{3.6}$$

and

$$\varphi_0(x,t,\kappa) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-ik\cdot x}\mathrm{e}^{-\kappa^2\omega(k)|t|}}{2\omega(k)}\frac{\kappa^2}{\kappa^2\omega(k)+|k|^2/2}\mathbb{1}_\sigma^\perp(k)dk. \tag{3.7}$$

In particular, for $t=0$ we have

$$g^2\sum_{i\neq j}^N\varphi_0(x^i-x^j,0,\kappa)ds \;\to\; \frac{g^2}{4\pi}\sum_{i<j}\frac{\mathrm{e}^{-\nu|x^i-x^j|}}{|x^i-x^j|},$$

and for $t\neq 0$,

$$|\nabla_x\varphi_0(X,t,\kappa)|\to 0,\quad |\varphi_0(X,t,\kappa)|\to 0$$

pointwise as $\kappa\to\infty$. It can be shown in the same way as in Lemma 2.14 that

$$\begin{aligned} & \lim_{\kappa\to\infty}\int_{\mathbb{R}^{3N}} dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{2}S_0^{\rm ren}(\kappa)}\right] \\ & = \int_{\mathbb{R}^{3N}} dx\mathbb{E}^x\left[\overline{f(B_{-T})}h(B_T)\mathrm{e}^{-\int_{-T}^T V(B_s)ds}\mathrm{e}^{\frac{g^2}{4\pi}\sum_{i<j}\int_{-T}^T\frac{\mathrm{e}^{-\nu|B_s^i-B_s^j|}}{|B_s^i-B_s^j|}ds}\right]. \end{aligned}$$

This completes the proof of the corollary. □

Corollary 3.3 *If $F, G \in \mathscr{D}$, then*

$$\lim_{\kappa\to\infty} (F, \mathrm{e}^{-tH_{\mathrm{ren}}(\kappa)} G) = (F, (\mathrm{e}^{-th_{\mathrm{eff}}} \otimes P_\Omega) G). \tag{3.8}$$

Proof. This follows from Lemmas 2.16 and 3.2. □

A Kato-class potentials

A potential $V : \mathbb{R}^d \to \mathbb{R}$ is said to belong to Kato-class relative to the Laplacian whenever

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\int_0^t |V(W_s^x)| ds\right] = 0 \tag{A.1}$$

where $(W_t^x)_{t\geq 0}$ is a standard d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$. We will denote by $\mathscr{K}_d$ the set of all such potentials. For details on Kato-class potentials we refer to [LHB11, Chapter 3.3] and [AS82, CFKS08].

Proposition A.1 *If $V \in \mathscr{K}_d$, then $\sum_{i\neq j}^N V(x^i - x^j) \in \mathscr{K}_{dN}$, with the notation $x = (x^1, ..., x^N) \in \mathbb{R}^{dN}$.*

For a proof we refer e.g. to [CFKS08, p.7]. An equivalent characterization of Kato-class potentials is as follows. A potential $V \in \mathscr{K}_d$ if and only if

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} |g(x-y)V(y)| dy = 0 \quad \text{with} \quad g(x) = \begin{cases} |x| & d = 1 \\ -\log|x| & d = 2 \\ |x|^{2-d} & d \geq 3. \end{cases} \tag{A.2}$$

Examples of Kato-class potentials include (1) $|x|^{-(2-\varepsilon)}$ with $d = 3$ for any $\varepsilon > 0$, (2) $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $p = 1$ for $d = 1$, and $p > d/2$ for $d \geq 2$. It is also known that the function $\mathrm{e}^{\int_0^t V(W_s^x) ds}$ of the d-dimensional Brownian motion $(W_t^x)_{t\geq 0}$ is integrable if V is Kato-class.

Proposition A.2 *Let $0 \leq V \in \mathscr{K}_d$. Then there exist $\beta, \gamma > 0$ such that*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}[\mathrm{e}^{\int_0^t V(W_s^x) ds}] \leq \gamma \mathrm{e}^{t\beta}. \tag{A.3}$$

Proof. See [LHB11, Lemma 3.38]. □

B Schrödinger representation and Euclidean field

In this section we consider the Hilbert spaces $H_{-1/2}(\mathbb{R}^3)$ and $H_{-1}(\mathbb{R}^4)$ as given by (2.56). It is well known that the boson Fock space $\mathscr{F}_{\mathrm{b}}$ is unitary equivalent to $L^2(Q,\mu)$, where this space consists of square integrable functions on a probability space (Q,Σ,μ). Consider the family of Gaussian random variables $\{\phi_0(f),\ f \in H_{-1/2}(\mathbb{R}^3)\}$ on (Q,Σ,μ) such that $\phi_0(f)$ is linear in $f \in H_{-1/2}(\mathbb{R}^3)$, and their mean and covariance are given by

$$\mathbb{E}_\mu[\phi_0(f)] = 0 \quad \text{and} \quad \mathbb{E}_\mu[\phi_0(f)\phi_0(g)] = \frac{1}{2}(f,g)_{H_{-1/2}(\mathbb{R}^3)}.$$

Given this space, the Fock vacuum $\mathbb{1}_{\mathscr{F}_{\mathrm{b}}}$ is unitary equivalent to $\mathbb{1}_{L^2(Q)} \in L^2(Q)$, and the scalar field $\phi(f)$ is unitary equivalent to $\phi_0(f)$ as operators, i.e., $\phi_0(f)$ is regarded as multiplication by $\phi_0(f)$. Then the linear hull of the vectors given by the Wick products $:\prod_{j=1}^n \phi_0(f_j):$ is dense in $L^2(Q)$, where recall that Wick product is recursively defined by

$$:\phi_0(f): \; = \phi_0(f)$$

$$:\phi_0(f)\prod_{j=1}^n \phi_0(f_j): \; = \phi_0(f) :\prod_{j=1}^n \phi_0(f_j): - \frac{1}{2}\sum_{i=1}^n (f,f_i)_{H_{-1/2}(\mathbb{R}^3)} :\prod_{j\neq i}^n \phi_0(f_j):$$

This allows to identify $\mathscr{F}_{\mathrm{b}}$ and $L^2(Q)$, which we have done in (2.57), i.e., $F \in \mathscr{H}$ can be regarded as a function $\mathbb{R}^{3N} \ni x \mapsto F(x) \in L^2(Q)$ such that $\int_{\mathbb{R}^{3N}} \|F(x)\|^2_{L^2(Q)} dx < \infty$.

To construct a Feynman-Kac-type representation we use a Euclidean field. Consider the family of Gaussian random variables $\{\phi_{\mathrm{E}}(F),\ F \in H_{-1}(\mathbb{R}^4)\}$ with mean and covariance

$$\mathbb{E}_{\mu_E}[\phi_E(F)] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_E}[\phi_E(F)\phi_E(G)] = \frac{1}{2}(F,G)_{H_{-1}(\mathbb{R}^4)}$$

on a chosen probability space (Q_E,Σ_E,μ_E). Note that for $f \in H_{-1/2}(\mathbb{R}^3)$ the relations

$$\delta_t \otimes f \in H_{-1}(\mathbb{R}^4) \quad \text{and} \quad \|\delta_t \otimes f\|_{H_{-1}(\mathbb{R}^4)} = \|f\|_{H_{-1/2}(\mathbb{R}^3)}$$

hold, where $\delta_t(x) = \delta(x-t)$ is Dirac delta distribution with mass on t. The family of identities used in (2.57) is then given by $\mathrm{J}_t : L^2(Q) \to L^2(Q_{\mathrm{E}})$, $t \in \mathbb{R}$, defined by the relations

$$\mathrm{J}_t \mathbb{1}_{L^2(Q)} = \mathbb{1}_{L^2(Q_{\mathrm{E}})} \quad \text{and} \quad \mathrm{J}_t :\prod_{j=1}^m \phi(f_j): \; = \; :\prod_{j=1}^m \phi_{\mathrm{E}}(\delta_t \otimes f_j):$$

Under the identification $\mathscr{F}_{\mathrm{b}} \cong L^2(Q)$ it follows that

$$(\mathrm{J}_t F, \mathrm{J}_s G)_{L^2(Q_{\mathrm{E}})} = (F, \mathrm{e}^{-|t-s|H_{\mathrm{f}}} G)_{\mathscr{F}_{\mathrm{b}}}$$

for $F, G \in \mathscr{F}_{\mathrm{b}}$. For an extensive discussion of the details we refer to [LHB11, Chapter 5].

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