

The Annals of Probability
2015, Vol. 43, No. 3, 1350–1398
DOI: 10.1214/13-AOP897
© Institute of Mathematical Statistics, 2015

POINTWISE EIGENFUNCTION ESTIMATES AND INTRINSIC ULTRAContractIVITY-TYPE PROPERTIES OF FEYNMAN–KAC SEMIGROUPS FOR A CLASS OF LÉVY PROCESSES

BY KAMIL KALETA¹ AND JÓZSEF LŐRINCZI

Wrocław University of Technology and Loughborough University

We introduce a class of Lévy processes subject to specific regularity conditions, and consider their Feynman–Kac semigroups given under a Kato-class potential. Using new techniques, first we analyze the rate of decay of eigenfunctions at infinity. We prove bounds on λ -subaveraging functions, from which we derive two-sided sharp pointwise estimates on the ground state, and obtain upper bounds on all other eigenfunctions. Next, by using these results, we analyze intrinsic ultracontractivity and related properties of the semigroup refining them by the concept of ground state domination and asymptotic versions. We establish the relationships of these properties, derive sharp necessary and sufficient conditions for their validity in terms of the behavior of the Lévy density and the potential at infinity, define the concept of borderline potential for the asymptotic properties and give probabilistic and variational characterizations. These results are amply illustrated by key examples.

1. Introduction. Jump Lévy processes differ in a number of essential ways from Brownian motion. In this paper, we focus on two aspects of this qualitatively different behavior under the effect of a potential (or penalty function) on the paths. One is a strong smoothing property of the semigroup of such a process called intrinsic ultracontractivity. The other is the rate of decay of its eigenfunctions. These two properties are related and in this paper we will discuss the extent of this relationship.

We consider a class of symmetric jump Lévy processes satisfying specific regularity conditions. One condition is given in terms of the convolution of their jump intensities by a restriction to a subset of the full state space in relation to large jumps. Another is existence of transition densities and their uniform boundedness in space after at least a sufficiently long time. A final condition requires sufficient regularity of the Green function for specially chosen balls. These conditions are formulated in Assumptions 2.1–2.3 below. As it will be seen, they are satisfied by

Received February 2013; revised September 2013.

¹Supported by the National Science Center (Poland) Grant on the basis of the decision no. DEC-2011/03/D/ST1/00311.

MSC2010 subject classifications. Primary 47D08, 60G51; secondary 47D03, 47G20.

Key words and phrases. Symmetric Lévy process, subordinate Brownian motion, Feynman–Kac semigroup, nonlocal operator, intrinsic ultracontractivity, entropy, ground state domination, decay of eigenfunctions, λ -subaveraging function.

important classes of Lévy processes, including many cases of interest of subordinate Brownian motion and also others.

Next, we introduce a potential function V and study the Lévy processes perturbed by it in terms of their Feynman–Kac semigroups $\{T_t : t \geq 0\}$. Under a suitable choice of V , which we call X -Kato class associated with Lévy process $(X_t)_{t \geq 0}$ (see Definition 2.1 below), the semigroup $\{T_t : t \geq 0\}$ is well defined and consists of symmetric operators. We additionally assume that $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, which implies that all T_t are compact and have a discrete spectrum. The corresponding eigenfunctions φ_n form an orthonormal basis in $L^2(\mathbf{R}^d, dx)$ and satisfy $T_t \varphi_n = e^{-\lambda_n t} \varphi_n$, where $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$. All φ_n are bounded continuous functions, and each λ_n has finite multiplicity. We call φ_0 ground state, which can be shown to be unique and strictly positive.

Since a ground state φ_0 exists, we can define the intrinsic Feynman–Kac semigroup

$$(1.1) \quad \tilde{T}_t f(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} T_t(\varphi_0 f)(x),$$

which is a Markov semigroup on $L^2(\mathbf{R}^d, \varphi_0^2 dx)$. Whenever T_t has an integral kernel $u(t, x, y)$, the operators \tilde{T}_t are given by the kernels

$$(1.2) \quad \tilde{u}(t, x, y) = \frac{e^{\lambda_0 t} u(t, x, y)}{\varphi_0(x)\varphi_0(y)}.$$

The intrinsic Feynman–Kac semigroup $\{\tilde{T}_t : t \geq 0\}$ describes a stationary Markov process which is called $P(\phi)_1$ -process associated with potential V [9, 47, 60, 75].

Intrinsic ultracontractivity (IUC) means that \tilde{T}_t is a bounded operator from $L^2(\mathbf{R}^d, \varphi_0^2 dx)$ to $L^\infty(\mathbf{R}^d)$ for every $t > 0$, or equivalently, $\tilde{u}(t, x, y) \leq C$ for all $x, y \in \mathbf{R}^d$, with an appropriate constant C dependent on V and t . IUC has been introduced in [32] for general semigroups of compact operators and it proved to be a strong regularity property [31]. Important examples include semigroups of elliptic operators and Schrödinger semigroups on function spaces over \mathbf{R}^d or bounded domains of \mathbf{R}^d with Dirichlet boundary conditions [1, 3, 4, 31, 48, 62, 77]. More recently, IUC has been investigated also in the case of semigroups generated by fractional Laplacians, and fractional and relativistic Schrödinger operators [26, 27, 46, 47, 55, 57, 58], as well as for more general symmetric [37] and non-symmetric [49] Lévy processes in bounded domains. In the context of parabolic partial differential equations [64, 65], obtained integral representations of the non-negative solutions of the Cauchy problem when intrinsic ultracontractivity holds.

As it follows from our previous analysis, it is of interest to consider also the property that $\tilde{u}(t, x, y) \leq C$, for all $x, y \in \mathbf{R}^d$ and sufficiently large t only, which we call asymptotic intrinsic ultracontractivity (AIUC). This is weaker than IUC, and we have seen in the case of fractional $P(\phi)_1$ -processes [47] that it has an

immediate impact on the support properties of their (Gibbs) path measure. Another important consequence is that AIUC is equivalent to

$$\left| \frac{e^{\lambda_0 t} u(t, x, y)}{\varphi_0(x)\varphi_0(y)} - 1 \right| \leq C e^{-(\lambda_1 - \lambda_0)t}, \quad t > t_0,$$

which means that the distribution of the corresponding $P(\phi)_1$ -process rapidly tends to equilibrium as $t \rightarrow \infty$ with decay rate given by the spectral gap $\lambda_1 - \lambda_0$. This, in turn, has an offshoot on the efficiency of practical sampling of conditioned processes; see, for example, [39, 40]. Also, it implies that the kernel $u(t, x, y)$ takes the shape of the ground state exponentially quickly, in particular, the decay of the eigenfunctions $\varphi_1, \varphi_2, \dots$ will be dominated by the decay of the ground state.

A basic question we address in this paper is that given the class of jump Lévy processes considered, what are conditions on V making the Feynman–Kac semigroup $\{T_t : t \geq 0\}$ IUC or AIUC. The answer is, roughly, that this is decided by how the asymptotic behaviors of V and $|\log \nu|$ at infinity compare, where ν is the Lévy density. We further refine IUC-type properties by considering a ground state domination (GSD) property and its asymptotic version for sufficiently long times (AGSD). We clarify the relationships of these properties (Theorem 2.5), and give precise necessary and sufficient conditions (Theorems 2.6 and 2.7) for them to hold. Our results recover the facts on IUC previously known for stable processes [46, 47] and relativistic stable processes [57] only, and establish these properties for many other processes, also shedding new light on existing results for diffusions [31, 32]. Corollary 2.3 gives a sharp description of the borderline potential $V(x) \asymp |\log \nu(x)|$ distinguishing (A)IUC from non(A)IUC. In comparison with the classic result which says roughly that IUC holds for a diffusion when V grows in leading order super-quadratically, it is seen that for a jump Lévy process it is “easier” to be (A)IUC than for a diffusion. We give an explanation of this in terms of a balance mechanism between the competition of killing and survival of paths (Proposition 2.2), and give a probabilistic characterization (Propositions 2.3 and 2.4). Furthermore (Theorem 2.8, Corollary 2.4), we obtain a second characterization of AIUC in terms of minimizing a free energy functional appearing as the difference of an energy and an entropy associated with the Lévy measure [see (2.9)–(2.11) below], and obtain the borderline potential as the solution of this variational problem. Due to the role played by the entropy this also explains why $\log \nu$ appears in this expression.

A second basic problem we address is to derive pointwise bounds on the eigenfunctions for a given Lévy process and V . We obtain sharp lower and upper bounds (Theorem 2.4) showing that the fall-off of the ground state follows the tail behavior of the Lévy density with corrections resulting from the contribution of the potential. Furthermore, we obtain upper bounds on all other eigenfunctions (Theorem 2.3, Corollary 2.1). Under a reasonable condition, we derive a more explicit expression of the dependence of the decay on V (Corollary 2.2). We note that,

importantly, the ground state estimates follow without any need to use results on the (A)IUC properties, unlike in the previous work [57]. Our considerations lead naturally to studying λ -subaveraging functions, which can be thought of as counterparts to λ -superaveraging functions known in potential theory, and we prove two results on them (Theorems 2.1 and 2.2). Although it makes the paper longer, we find it useful to discuss important (classes of) examples in relation to both the ground state bounds and the IUC-type properties. We also discuss in some detail which types of specific cases are covered by the Lévy processes we tackle in this paper, as well as interesting cases which fail the assumptions or the IUC-properties.

We note that our results can also be considered from the perspective of the correspondence between jump Lévy processes and nonlocal operators, which are their generators. Via a Feynman–Kac representation our results characterize the decay of eigenfunctions and IUC-type properties of semigroups related to generalized Schrödinger operators whose kinetic terms are given by the generators. A specific class of nonlocal operators covered to a large extent by our results are of the form $\Psi(-\Delta) + V$, studied in [41, 42], where Ψ is a Bernstein function with vanishing right limit at zero, giving the Lévy exponent of a subordinator [6, 73]. Some important specific cases are fractional Schrödinger operators $(-\Delta)^{\alpha/2} + V$, relativistic Schrödinger operators $(-\Delta + m^{2/\alpha})^{\alpha/2} - m + V$, jump-diffusion operators $a(-\Delta)^{\alpha/2} - b\Delta + V$, and many others. There is an increasing literature studying these operators from both a probabilistic and an analytic point of view [5, 11, 12, 18, 20, 28, 34, 45–47, 56, 57, 59, 61, 76].

The basic input object in this paper is a Lévy process, therefore, we mainly use probabilistic methods. Our argument builds on a completely new approach which combines sharp uniform estimates on the local extrema of functions harmonic with respect to the subprocess obtained under the Feynman–Kac functional of the Lévy process (Lemma 3.1) developed only recently in [14] (see also [51]), and new powerful self-improving estimates under the path measure of the process (see the proofs of Theorems 2.1 and 3.1). In the proofs, it will become evident that the pivotal Assumption 2.1(3) is very natural, and its generality will be seen by many examples of interest satisfying it. In particular, this will allow to study also processes with exponentially localized Lévy measures and derive sharp estimates, which are of special interest for various further investigations and have been little understood before (see [18]). Since IUC has been much studied in operator semigroup and PDE context, we find it important to develop a probabilistic understanding of it.

The remainder of the paper is organized as follows. In Section 2, we state the main results. First, we introduce the class of Lévy processes and potentials which will be considered. The next two subsections are devoted to presenting the estimates on λ -subaveraging functions, ground states and the other eigenfunctions. The last two subsections present the results on intrinsic ultracontractivity and ground state domination. In Section 3, we provide the proofs in a similar division of the material. We devote Section 4 to a detailed discussion of ground state decay and IUC-type behaviors of specific processes of interest.

2. Assumptions and main results.

2.1. *A class of Lévy processes.* Let $(X_t)_{t \geq 0}$ be a Lévy process in \mathbf{R}^d , $d \geq 1$. We use the notations \mathbf{P}^x and \mathbf{E}^x , respectively, for the probability measure and expected value of the process starting in $x \in \mathbf{R}^d$. Recall that $(X_t)_{t \geq 0}$ is a Markov process with respect to its own filtration satisfying the strong Markov property, and has right continuous paths with left limits (càdlàg paths). It is a basic fact that $(X_t)_{t \geq 0}$ is completely determined by its characteristic exponent ψ given by the Lévy–Khintchine formula, that is, for $\xi \in \mathbf{R}^d$

$$\mathbf{E}^0[e^{i\xi \cdot X_t}] = e^{-t\psi(\xi)}$$

holds with

$$\psi(\xi) = -i\gamma \cdot \xi + A\xi \cdot \xi + \int_{\mathbf{R}^d} (1 - e^{i\xi \cdot z} + i\xi \cdot z \mathbf{1}_{\{|z| < 1\}}(z)) \nu(dz).$$

Here, $\gamma \in \mathbf{R}^d$ (*drift coefficient*), A is a symmetric nonnegative definite $d \times d$ matrix (*diffusion or Gaussian coefficient*), and ν is a Radon measure on $\mathbf{R}^d \setminus \{0\}$ such that $\int_{\mathbf{R}^d} (1 \wedge |z|^2) \nu(dz) < \infty$ (*Lévy measure*). The defining parameters (γ, A, ν) are called the *Lévy triplet* of the process. $(X_t)_{t \geq 0}$ is said to be *symmetric* whenever X_t has the same distribution as $-X_t$ for all $t > 0$. In this case, $\psi(\xi) = \psi(-\xi)$, $\xi \in \mathbf{R}^d$, that is, $\gamma \equiv 0$ and $\nu(B) = \nu(-B)$ for all $B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\})$, and then the characteristic exponent reduces to

$$\psi(\xi) = A\xi \cdot \xi + \int_{\mathbf{R}^d} (1 - \cos(\xi \cdot z)) \nu(dz).$$

Whenever $\nu(dz)$ is absolutely continuous with respect to Lebesgue measure, we denote its density by $\nu(z)$ and call it the *Lévy (jump) intensity* of $(X_t)_{t \geq 0}$. When $A \equiv 0$ and $\nu \neq 0$, the process $(X_t)_{t \geq 0}$ is said to be a *purely jump Lévy process*. For more details on Lévy processes, we refer to [2, 7, 35, 44, 72].

We will use throughout the notation $C(a, b, c, \dots)$ for a positive constant dependent on parameters a, b, c, \dots , while dependence on the process $(X_t)_{t \geq 0}$ is indicated by $C(X)$, and dependence on the dimension d is assumed without being indicated. Since constants appearing in definitions, lemmas and theorems play a role later on, we use the numbering C_1, C_2, \dots to be able to track them. We will also use the notation $f \asymp Cg$ meaning that $C^{-1}g \leq f \leq Cg$ with a constant C , while $f \asymp g$ means that there is a constant C such that the latter holds.

For the remainder of the paper, we make three standing assumptions on the Lévy processes we consider.

ASSUMPTION 2.1. $(X_t)_{t \geq 0}$ is a symmetric Lévy process with Lévy measure absolutely continuous with respect to Lebesgue measure. The corresponding jump intensity $\nu(x)$ is strictly positive and satisfies the following three conditions:

(1) For every $0 < r \leq 1/2$ there is a constant $C_1 = C_1(X, r) \geq 1$ such that

$$v(x) \asymp C_1 v(y), \quad r \leq |y| \leq |x| \leq |y| + 1.$$

(2) There is a constant $C_2 = C_2(X) \geq 1$ such that

$$v(x) \leq C_2 v(y), \quad 1/2 \leq |y| \leq |x|.$$

(3) There is a constant $C_3 = C_3(X) \geq 1$ such that

$$\int_{|z-x|>1/2, |z-y|>1/2} v(x-z)v(z-y) dz \leq C_3 v(x-y), \quad |y-x| \geq 1.$$

Conditions (1) and (2) are clearly satisfied when, for example, $v(x) \asymp \kappa(|x|)$, $x \in \mathbf{R}^d$, where $\kappa : (0, \infty) \rightarrow (0, \infty)$ is a nonincreasing function such that $\kappa(s) \leq C\kappa(s+1)$, $s \geq 1/2$. Condition (3) provides a regularity of the convolutions of v with respect to large jumps. While (1)–(2) can be seen as technical conditions, (3) has a structural importance. Examples and counterexamples to conditions (1)–(3) above are discussed in Section 4.1.

Denote by $P_t f(x) = \mathbf{E}^x[f(X_t)]$ the transition operators of the process $(X_t)_{t \geq 0}$. Recall that $(X_t)_{t \geq 0}$ has the strong Feller property if $P_t(L^\infty(\mathbf{R}^d)) \subset C_b(\mathbf{R}^d)$, for all $t > 0$.

ASSUMPTION 2.2. The process $(X_t)_{t \geq 0}$ has the strong Feller property, or equivalently, the one-dimensional \mathbf{P}^x -distributions of $(X_t)_{t \geq 0}$ are absolutely continuous with respect to Lebesgue measure, that is, there exist transition probability densities $p(t, x, y) = p(t, y-x, 0) =: p(t, y-x)$. Furthermore, there exist $t_b > 0$ and $C_4 = C_4(X, t_b)$ such that $0 < p(t_b, x) \leq C_4$, for all $x \in \mathbf{R}^d$.

Note that the first part of the above assumption is satisfied by a large class of Lévy processes including subordinate Brownian motion [50] provided that they are not compound Poisson processes. In fact, our assumption is equivalent to $e^{-t_b \psi(\cdot)} \in L^1(\mathbf{R}^d)$, for some $t_b > 0$. In this case, $p(t_b, x)$ can be obtained by the Fourier inversion formula. Clearly, this property extends to all $t \geq t_b$ by the Markov property of $(X_t)_{t \geq 0}$. For more details on the existence and properties of transition probability densities for Lévy processes, we refer to [54] and references therein.

We note for later use that under Assumption 2.2 transition densities $p_D(t, x, y)$ of the process $(X_t)_{t \geq 0}$ killed upon exiting an open bounded set $D \subset \mathbf{R}^d$ also exist. In this case, the Hunt formula

$$(2.1) \quad p_D(t, x, y) = p(t, y-x) - \mathbf{E}^x[\tau_D < t; p(t - \tau_D, y - X_{\tau_D})], \quad x, y \in D,$$

holds, where $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ is the first exit time from D . The Green function of the process $(X_t)_{t \geq 0}$ on D is thus given by $G_D(x, y) = \int_0^\infty p_D(t, x, y) dy$, for all $x, y \in D$, and $G_D(x, y) = 0$ if $x \notin D$ or $y \notin D$.

Since our results rely on a use of potential theory, we need some more regularity of $(X_t)_{t \geq 0}$.

ASSUMPTION 2.3. For all $0 < p < q < R \leq 1$, we have

$$\sup_{x \in B(0,p)} \sup_{y \in B(0,q)^c} G_{B(0,R)}(x, y) < \infty.$$

In many cases, Assumption 2.3 is a direct consequence of time–space estimates of the function $p(t, x)$. Indeed, it is clearly satisfied when the boundedness condition holds with $G_{B(0,R)}(x, y)$ replaced by the potential kernel $G_{\mathbf{R}^d}(x, y) = \int_0^\infty p(t, x, y) dt$ or, as proved in [14], Proposition 2.3, the λ -potential kernel $G_{\mathbf{R}^d}^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y) dt$, $\lambda > 0$, whenever the process $(X_t)_{t \geq 0}$ is recurrent.

One of our key arguments following below uses some estimates (see Lemma 3.1) on the local extrema of functions harmonic with respect to the subprocess of $(X_t)_{t \geq 0}$ obtained under its Feynman–Kac functional. These bounds are a direct consequence of more general results of Bogdan, Kumagai and Kwaśnicki obtained recently in [14]. To borrow these results, we need to match some assumptions made in this paper, however, since we consider symmetric Lévy processes, Assumptions 2.1(1), 2.2 [without requiring boundedness of $p(t, x)$] and 2.3 provide sufficient regularity of $(X_t)_{t \geq 0}$ to allow a use of [14]. The remaining conditions in Assumption 2.1 are independent from this context and together with condition (1) for $r = 1/2$ only they will allow to draw more regularity of the Lévy intensity ν needed in controlling jumps in Section 3.2 below. Similarly, boundedness of $p(t, x)$ in Assumption 2.2 guarantees sufficient regularity of the process needed below.

Note that all of our assumptions above are satisfied by a wide class of symmetric Lévy processes including a large subclass of subordinate Brownian motions, Lévy processes with nondegenerate Brownian components, symmetric stable-like ones or processes with subexponentially localized Lévy measures. Some important examples with a verification of assumptions are discussed in Section 4.1.

Next we give the class of potentials which will be used in this paper.

DEFINITION 2.1 (*X*-Kato class). We say that the Borel function $V : \mathbf{R}^d \rightarrow \mathbf{R}$ belongs to the Kato-class \mathcal{K}^X associated with the Lévy process $(X_t)_{t \geq 0}$ if it satisfies

$$(2.2) \quad \lim_{t \rightarrow 0} \sup_{x \in \mathbf{R}^d} \mathbf{E}^x \left[\int_0^t |V(X_s)| ds \right] = 0.$$

We write $V \in \mathcal{K}_{\text{loc}}^X$ if $V \mathbf{1}_B \in \mathcal{K}^X$ for every ball $B \subset \mathbf{R}^d$. Moreover, we say that V is an *X-Kato decomposable potential*, whenever

$$V = V_+ - V_- \quad \text{with } V_- \in \mathcal{K}^X, V_+ \in \mathcal{K}_{\text{loc}}^X,$$

where V_+ and V_- denote the positive and negative parts of V , respectively.

For simplicity, in what follows we refer to X -Kato decomposable potentials as X -Kato class potentials. It is easy to see that $L^\infty \subset \mathcal{K}_{\text{loc}}^X$. Moreover, by stochastic continuity of $(X_t)_{t \geq 0}$ also $\mathcal{K}_{\text{loc}}^X \subset L^1_{\text{loc}}(\mathbf{R}^d)$, and thus an X -Kato class potential is always locally absolutely integrable. Note that condition (2.2) allows local singularities of V . For specific processes $(X_t)_{t \geq 0}$ condition (2.2) can be equivalently reformulated in terms of the potential kernel of the process in the transient case, and the so-called compensated potential kernel when the process is recurrent (for more details see, e.g., [12, 18, 78]).

We single out a restricted set of potentials which will be often used below.

ASSUMPTION 2.4. For a given Lévy process $(X_t)_{t \geq 0}$ let V be such that:

- (1) V is an X -Kato class potential
- (2) $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

Next, for a given X -Kato class potential V , we define

$$T_t f(x) = \mathbf{E}^x \left[e^{-\int_0^t V(X_s) ds} f(X_t) \right], \quad f \in L^2(\mathbf{R}^d), t > 0.$$

Using the Markov property and stochastic continuity of the process, it can be shown that $\{T_t : t \geq 0\}$ is a strongly continuous semigroup of symmetric operators on $L^2(\mathbf{R}^d)$, which we call the *Feynman–Kac semigroup* associated with the process $(X_t)_{t \geq 0}$ and potential V . In particular, by the Hille–Yoshida theorem, there exists a self-adjoint operator H bounded from below such that $e^{-tH} = T_t$. The operator H is often seen as a *generalized Schrödinger operator* based on the infinitesimal generator L of the process $(X_t)_{t \geq 0}$. Whenever V is relatively bounded with respect to L with relative bound less than 1 we can write $H = -L + V$ as an operator sum.

We now summarize the basic properties of the operators T_t , some of which will be explicitly used below.

LEMMA 2.1. *Let Assumptions 2.1–2.4 be satisfied. Then the following properties hold:*

- (1) For all $t > 0$, T_t are bounded operators on each $L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$. The operators $T_t : L^p(\mathbf{R}^d) \rightarrow L^p(\mathbf{R}^d)$ for $1 \leq p \leq \infty$, $t > 0$, $T_t : L^p(\mathbf{R}^d) \rightarrow L^\infty(\mathbf{R}^d)$ for $1 < p \leq \infty$, $t \geq t_b$, and $T_t : L^1(\mathbf{R}^d) \rightarrow L^\infty(\mathbf{R}^d)$ for $t \geq 2t_b$ are bounded.
- (2) For all $t \geq 2t_b$, T_t has a bounded, measurable, and symmetric kernel $u(t, \cdot, \cdot)$, that is, $T_t f(x) = \int_{\mathbf{R}^d} u(t, x, y) f(y) dy$, $f \in L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$.
- (3) For all $t > 0$ and $f \in L^\infty(\mathbf{R}^d)$, $T_t f(x)$ is a bounded continuous function.
- (4) For all $t \geq 2t_b$ the operators T_t are positivity improving, that is, $T_t f(x) > 0$ for all $x \in \mathbf{R}^d$ and $f \in L^2(\mathbf{R}^d)$ such that $f \geq 0$ and $f \neq 0$ a.e.
- (5) All operators $T_t : L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)$, $t > 0$, are compact.

Properties (1)–(4) can be established by standard arguments based on [30], Section 3.2. Property (5) is a consequence of $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and standard arguments based on approximation of T_t , $t \geq 2t_b$, by compact operators, see [47], Lemma 3.2. Clearly, compactness extends to all $t > 0$ by the fact that $T_t = e^{-tH}$ for a self-adjoint operator H and a use of the spectral theorem. Note that we do not assume that $p(t, x)$ is bounded for all $t > 0$, and thus in general the operators $T_t : L^p(\mathbf{R}^d) \rightarrow L^\infty(\mathbf{R}^d)$ need not be bounded for $t < t_b$.

The theory of operator semigroups implies that there exists an orthonormal basis in $L^2(\mathbf{R}^d)$ consisting of the eigenfunctions φ_n given by $T_t \varphi_n = e^{-\lambda_n t} \varphi_n$, $t > 0$, $n \geq 0$, and $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$. All φ_n are bounded continuous functions. Moreover, the first eigenfunction (or ground state) φ_0 has a strictly positive version ([69], Theorem XIII.43), which will be our choice throughout.

2.2. Estimates of λ -subaveraging functions. Recall that one of the fundamental objects in potential theory are λ -superaveraging (and related λ -excessive) functions, see [29], Section 2.1. Below it is useful to consider λ -subaveraging functions, which in some sense are counterparts of λ -superaveraging functions in the opposite direction of domination. We say that a nonnegative Borel function φ is λ -subaveraging for the semigroup $\{T_t : t \geq 0\}$ with $\lambda \geq 0$ if for every $t > 0$ and $x \in \mathbf{R}^d$ we have $e^{\lambda t} T_t \varphi(x) \geq \varphi(x)$.

For an open set $D \subset \mathbf{R}^d$, a Kato-class potential V and a nonnegative or bounded Borel function φ we define the V -Green operator for the semigroup $\{T_t : t \geq 0\}$ and set D (see Section 3.1),

$$G_D^V \varphi(x) = \mathbf{E}^x \left[\int_0^{\tau_D} e^{-\int_0^t V(X_s) ds} \varphi(X_t) dt \right], \quad x \in D,$$

where τ_D is the first exit time from D .

The following estimates for λ -subaveraging functions will be used in proving bounds on eigenfunctions and intrinsic ultracontractivity.

THEOREM 2.1. *Let Assumptions 2.1–2.4 hold. If φ is a bounded λ -subaveraging function for the semigroup $\{T_t : t \geq 0\}$ with $\lambda \geq 0$, then there is a constant $C_4 = C_4(X, V, \lambda)$ and $R = R(X, V, \lambda) > 0$ such that*

$$\varphi(x) \leq C_4 \|\varphi\|_\infty \nu(x), \quad |x| \geq R.$$

The proof of this theorem is probably the most involved and crucial part of the paper. The required bound is obtained inductively, stemming from a new idea based on a self-improving estimate iterated infinitely many times. The main difficulty is that we need to have a statement on strictly $\nu(x)$ rather than $\nu(cx)$ for some $c \in (0, 1)$. This is particularly critical in the case of exponentially localized Lévy measures, which are of special interest in our further investigations.

For simplicity, we write $\mathbf{1}(x)$ instead of $\mathbf{1}_{\mathbf{R}^d}(x)$ throughout below.

THEOREM 2.2. *Let Assumptions 2.1–2.4 hold.*

(1) *If φ is a bounded function (possibly negative) for which there exists $\lambda > 0$ such that for every $t > 0$ and $x \in \mathbf{R}^d$ we have $e^{\lambda t} T_t \varphi(x) = \varphi(x)$ (clearly, in this case $|\varphi|$ is λ -subaveraging), then there is a constant $C_5 = C_5(X, V, \lambda)$ and $R = R(X, V, \lambda) > 0$ such that*

$$|\varphi(x)| \leq C_5 \|\varphi\|_\infty G_{B(x,1)}^V \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

(2) *If φ is a strictly positive function for which there is $\lambda > 0$ such that for every $t > 0$ and $x \in \mathbf{R}^d$ we have $e^{\lambda t} T_t \varphi(x) = \varphi(x)$, then there is a constant $C_6 = C_6(X, \varphi)$ and $R = R(X, V, \lambda) > 0$ such that*

$$\varphi(x) \geq C_6 G_{B(x,1)}^V \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

2.3. Eigenfunction estimates. The following pointwise upper bounds for eigenfunctions and sharp two-sided bounds for the ground state of the operators T_t are the next main results of this paper.

THEOREM 2.3 (Upper bounds on eigenfunctions). *If Assumptions 2.1–2.4 hold, then for every $n \in \{0, 1, 2, \dots\}$ and $\eta \geq 0$ such that $\lambda_0 + \eta > 0$, there exists a constant $C_7 = C_7(X, V, n, \eta)$ and a radius $R = R(X, V, n, \eta) > 0$ such that*

$$|\varphi_n(x)| \leq C_7 G_{B(x,1)}^{V+\eta} \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

THEOREM 2.4 (Ground state estimates). *If Assumptions 2.1–2.4 hold, then for every $\eta \geq 0$ such that $\lambda_0 + \eta > 0$ there exist constants $C_8 = C_8(X, V, \eta)$, $C_9 = C_9(X, V, \eta)$ and a radius $R = R(X, V, \eta) > 0$ such that*

$$C_8 G_{B(x,1)}^{V+\eta} \mathbf{1}(x) \nu(x) \leq \varphi_0(x) \leq C_9 G_{B(x,1)}^{V+\eta} \mathbf{1}(x) \nu(x), \quad |x| \geq R.$$

We emphasize that the above bounds on the eigenfunctions are obtained by using a completely new idea in this context, without using any (intrinsic) ultracontractivity properties of $\{T_t : t \geq 0\}$, unlike in [57], which will be further discussed below.

The following domination property is an immediate consequence of the above theorems. We note that this is in contrast with Brownian motion, for which it does not occur if the growth of the potential V at infinity is not fast enough [see further discussion in Example 4.8(5) and compare with (2.3) below].

COROLLARY 2.1. *If Assumptions 2.1–2.4 hold, then for every $n \in \{1, 2, \dots\}$ there is a constant $C_{10} = C_{10}(X, V, n)$ such that*

$$|\varphi_n(x)| \leq C_{10} \varphi_0(x), \quad x \in \mathbf{R}^d.$$

By the estimates in (3.2), we also have the following corollary.

COROLLARY 2.2. *Let Assumptions 2.1–2.4 hold. Then for every $n \in \{0, 1, 2, \dots\}$ there exists a radius $R = R(X, V, n) > 0$ such that*

$$|\varphi_n(x)| \leq C_7 \frac{\nu(x)}{\inf_{y \in B(x,1)} V(y)}, \quad |x| \geq R$$

and

$$C_{11} \frac{\nu(x)}{\sup_{y \in B(x,1)} V(y)} \leq \varphi_0(x) \leq C_9 \frac{\nu(x)}{\inf_{y \in B(x,1)} V(y)}, \quad |x| \geq R$$

with some constant $C_{11} = C_{11}(X, V)$. In particular, if for some $n \in \{0, 1, 2, \dots\}$ there is a constant $C > 1$ such that for all unit balls $B \subset B(0, R)^c$ it holds that $\sup_{y \in B} V(y) \leq C \inf_{y \in B} V(y)$ (cf. Assumption 2.5 below), then

$$|\varphi_n(x)| \leq C_7 C \frac{\nu(x)}{V(x)}, \quad |x| \geq R + 1$$

and

$$C_{11} C^{-1} \frac{\nu(x)}{V(x)} \leq \varphi_0(x) \leq C_9 C \frac{\nu(x)}{V(x)}, \quad |x| \geq R + 1.$$

Ground state decays for specific examples are discussed in Section 4.2 below.

2.4. *Intrinsic ultracontractivity and ground state domination.* Under the given choice of potential the Feynman–Kac semigroup has strong smoothing properties which we define next. In particular, they imply degrees of regularity and the rate of decay of eigenfunctions. Recall that the intrinsic Feynman–Kac semigroup is given by (1.1).

DEFINITION 2.2 (IUC/AIUC). Consider the following ultracontractivity properties:

- (1) The semigroup $\{T_t : t \geq 0\}$ is *ultracontractive* if T_t is a bounded operator from $L^2(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$, for every $t > 0$.
- (2) The semigroup $\{T_t : t \geq 0\}$ is *intrinsically ultracontractive* (abbreviated as IUC) if \tilde{T}_t is a bounded operator from $L^2(\mathbf{R}^d, \varphi_0^2 dx)$ to $L^\infty(\mathbf{R}^d)$, for every $t > 0$.
- (3) The semigroup $\{T_t : t \geq 0\}$ is *t_0 -intrinsically ultracontractive* (abbreviated as t_0 -IUC) if the above boundedness property of \tilde{T}_t holds for some $t_0 > 0$. In this case, the semigroup property extends it to all $t \geq t_0$.
- (4) When $\{T_t : t \geq 0\}$ is t_0 -IUC, but the specific value of t_0 is not essential, we simply say that $\{T_t : t \geq 0\}$ is *asymptotically intrinsically ultracontractive* (abbreviated as AIUC).

A remarkable consequence of IUC-type properties is the following domination property for eigenfunctions. If for some $t > 0$, the semigroup $\{T_t : t \geq 0\}$ is t -IUC, then there is a constant $C = C(X, V, t)$ such that (see, e.g., [3], (1.7))

$$(2.3) \quad |\varphi_n(x)| \leq C e^{(\lambda_n - \lambda_0)t} \varphi_0(x), \quad x \in \mathbf{R}^d, n \geq 1.$$

Clearly, if $\{T_t : t \geq 0\}$ is IUC, then (2.3) holds for all $t > 0$. Unlike in Corollary 2.1, here the dependence on n of the expression on the right-hand side of the inequality is more explicit.

Since below we mainly use probabilistic arguments, it will be useful to consider the following property of the semigroup $\{T_t : t \geq 0\}$ which we call *ground state domination*. As it will be seen later, in general ground state domination is a weaker property than IUC.

DEFINITION 2.3 (GSD/AGSD). Consider the following boundedness properties:

(1) The semigroup $\{T_t : t \geq 0\}$ is *ground state dominated* (abbreviated as GSD) if for every $t > 0$ there is a constant $C_{12} = C_{12}(X, V, t)$ such that

$$(2.4) \quad T_t \mathbf{1}(x) \leq C_{12} \varphi_0(x), \quad x \in \mathbf{R}^d.$$

(2) The semigroup $\{T_t : t \geq 0\}$ is t_0 -*ground state dominated* (abbreviated as t_0 -GSD) if (2.4) holds for some $t_0 > 0$. In this case, the semigroup property extends this bound to all $t \geq t_0$ with constant $C_{12} e^{-\lambda_0(t-t_0)}$, where $C_{12} = C_{12}(X, V, t_0)$.

(3) When $\{T_t : t \geq 0\}$ is t_0 -GSD but the specific value of t_0 is not essential, we simply say that $\{T_t : t \geq 0\}$ is *asymptotically ground state dominated* (abbreviated as AGSD).

Before stating the main results of this subsection, it is worthwhile to discuss the relationship between these properties. Parts (1)–(2) in Definition 2.2 are standard, (4) has been introduced in [47]. It is immediate from the definitions that IUC and GSD imply t_0 -IUC and t_0 -GSD (for all $t_0 > 0$), respectively. However, as it will be seen below, IUC and GSD are essentially stronger properties than their asymptotic versions. We will show that under our assumptions AIUC and AGSD are equivalent, while IUC implies GSD and in general conversely this is not the case.

THEOREM 2.5 [(A)IUC and (A)GSD]. *Let Assumptions 2.1–2.4 be satisfied, specifically, let Assumption 2.2 hold with $t_b > 0$.*

(1) *Then*

$$\text{AIUC} \iff \text{AGSD} \quad \text{and} \quad \text{IUC} \implies \text{GSD}$$

in the sense that t_0 -IUC $\implies 2t_0$ -GSD, $t_0 > 0$, and t_0 -GSD $\implies 2t_0$ -IUC, whenever $t_0 \geq t_b$.

(2) If, moreover, $p(t, x) \leq C_4$ for all $t > 0$ and $x \in \mathbb{R}^d$ with $C_4 = C_4(X, t)$, then also

$$\text{GSD} \implies \text{IUC}.$$

In Proposition 2.1 below, we show that the assumption in part (2) of Theorem 2.5 is essential. This means that in general when $p(t, \cdot)$ may be unbounded for small t , IUC is a strictly stronger property than GSD. Intuitively, it is clear that IUC requires more smoothness of the semigroup $\{T_t : t \geq 0\}$ than GSD as it also depends on the local singularities of the semigroup, while GSD is in fact, roughly speaking, a decay property of the semigroup at infinity.

We now present characterization results on GSD/AGSD and IUC/AIUC.

THEOREM 2.6 (Sufficient and necessary conditions for AGSD). *Let Assumptions 2.1–2.4 hold.*

(1) If there exist a constant $C_{13} = C_{13}(X, V)$ and a radius $R > 0$ such that

$$\frac{V(x)}{|\log v(x)|} \geq C_{13}, \quad |x| \geq R,$$

then the semigroup $\{T_t : t \geq 0\}$ is t_0 -GSD with $t_0 = 4/C_{13}$.

(2) If the semigroup $\{T_t : t \geq 0\}$ is t_0 -GSD, then for every $\varepsilon \in (0, 1]$ there is $R_\varepsilon > 0$ such that

$$\frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log v(x)|} \geq \frac{1}{2t_0}, \quad |x| \geq R_\varepsilon.$$

THEOREM 2.7 (Sufficient and necessary conditions for GSD). *Let Assumptions 2.1–2.4 hold.*

(1) If

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log v(x)|} = \infty,$$

then the semigroup $\{T_t : t \geq 0\}$ is GSD.

(2) If the semigroup $\{T_t : t \geq 0\}$ is GSD, then for every $\varepsilon \in (0, 1]$ we have

$$\lim_{|x| \rightarrow \infty} \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log v(x)|} = \infty.$$

REMARK 2.1. By Theorem 2.5, the limit condition in Theorem 2.7(2) is necessary for IUC, and the condition in Theorem 2.7(1) is sufficient for IUC, whenever $p(t, \cdot)$ is bounded for all $t > 0$. Similarly, the growth condition in Theorem 2.6(2) is necessary for $t_0/2$ -IUC, and the condition in Theorem 2.6(1) is sufficient for t_0 -IUC with $t_0 = 2(t_b \vee 4/C_{13})$.

The following result is intuitively clear, however, for the reader’s convenience we include a short proof at the end of Section 3.1.

PROPOSITION 2.1 (Ultracontractivity). *Let Assumptions 2.1–2.4 hold, in particular, let Assumption 2.2 hold with $t_b > 0$. Furthermore, suppose that:*

- (1) *there exists $t < t_b$ such that $\lim_{|x| \rightarrow 0^+} p(t, x) = \infty$, and for all $s \in (0, t]$ we have that $p(s, x) \geq p(s, y)$ whenever $|x| \leq |y|$;*
- (2) *there exist $x_0 \in \mathbf{R}^d$ and $\varepsilon > 0$ such that V is bounded from above in $B(x_0, \varepsilon)$.*

Then for every $0 < t < t_b$ for which condition (1) is satisfied, the operator $T_{t/2}$ is not bounded from $L^2(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$. In particular, the semigroup $\{T_t : t \geq 0\}$ is not ultracontractive.

Since φ_0 is bounded, IUC implies ultracontractivity. Hence, the above result shows that the assumption in assertion (2) of Theorem 2.5 is essential. This means that there exists a class of random processes whose Feynman–Kac semigroups are GSD but not IUC (even if the potential grows to infinity at infinity very quickly). Typical examples of Lévy processes fitting the above proposition include subordinate Brownian motion with suitably slowly varying characteristic exponents such as geometric stable processes. This example will be discussed in more detail in Section 4.2.

For the remainder of this subsection, we restrict attention to a somewhat smaller class of potentials by imposing more regularity. This will also be used in the next subsection.

ASSUMPTION 2.5. There exist $R > 1$ and a constant $C_{14} = C_{14}(V)$ such that for every $|x| > R$

$$(2.5) \quad V(y) \leq C_{14}V(x), \quad y \in B(x, 1)$$

holds.

A straightforward consequence of the above theorems is the following result.

COROLLARY 2.3 (Borderline case). *Let Assumptions 2.1–2.5 hold, in particular, let Assumption 2.2 hold with $t_b > 0$. Then we have the following:*

- (1) *The semigroup $\{T_t : t \geq 0\}$ is GSD if and only if*

$$(2.6) \quad \lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log v(x)|} = \infty.$$

Moreover, condition (2.6) is necessary for IUC, and sufficient whenever $p(t, \cdot)$ is bounded for every fixed $t > 0$.

- (2) *The semigroup $\{T_t : t \geq 0\}$ is AGSD (or, equivalently, AIUC) if and only if there exist a constant C_{15} and $R > 0$ such that*

$$(2.7) \quad \frac{V(x)}{|\log v(x)|} \geq C_{15}, \quad |x| \geq R.$$

Specifically, if (2.7) is satisfied, then $\{T_t : t \geq 0\}$ always is t_0 -GSD with $t_0 = 4/C_{15}$, and it is t_0 -IUC with $t_0 = 2(t_b \vee 4/C_{15})$. If $\{T_t : t \geq 0\}$ is t_0 -GSD, then (2.7) holds with constant $C_{15} = 1/(2C_{14}t_0)$. Similarly, t_0 -IUC implies (2.7) with $C_{15} = 1/(4C_{14}t_0)$.

By the above results, we are now able to formally define borderline potentials.

DEFINITION 2.4 (Borderline potential). Let Assumptions 2.1–2.4 hold. We call V *borderline potential* for (A)GSD/(A)IUC of the semigroup $\{T_t : t \geq 0\}$ if there exist $t_0 > 0$ and $R > 0$ such that $t_0 V(x) = |\log \nu(x)|$, for every $x \in B(0, R)^c$.

Note that by Assumption 2.1 the borderline potentials always satisfy Assumption 2.5. Also, note that we speak of borderline potentials in the sense of equivalence classes given by the definition above. The examples of possible borderline potentials for different classes of Lévy processes are discussed in Section 4.2.

2.5. Probabilistic and variational interpretation of AGSD/AIUC. It was seen in the previous subsection that under Assumptions 2.1–2.4 AGSD/AIUC of $\{T_t : t \geq 0\}$ depends only on the intensity of large jumps of the process $(X_t)_{t \geq 0}$. This means that whenever $\nu \neq 0$, the Gaussian and small jump parts of the process have no impact on AGSD/AIUC. Indeed, the borderline growth of V is decided by the ratio $e^{-t_0 V(x)}/\nu(x)$ for x sufficiently far away from the origin and some time point $t_0 > 0$. More precisely, AGSD/AIUC occurs if and only if $e^{-t_0 V(x)}$ is uniformly dominated by the jump intensity $\nu(x)$ outside a bounded set in \mathbf{R}^d . We note that although this description gives a full picture of what AGSD/AIUC is in the case when $\nu \neq 0$, it does not help to understand what is behind this property when the process is strictly diffusive, that is, whenever $\nu = 0$. (In a sense, this situation confirms that Brownian motion is an exceptional Lévy process and processes with jumps are the more generic.) In this section, we discuss probabilistic and variational descriptions of these properties.

It is straightforward that the condition on V for $\{T_t : t \geq 0\}$ being AGSD/AIUC is much weaker than in the case of the Feynman–Kac semigroup for diffusions. This can be explained by the following heuristic interpretation. For our purposes here, it suffices to observe that the effect of the potential on the distribution of paths is a concurrence of killing at a rate of $e^{-\int_0^t V_+(X_s) ds}$ and mass generation at a rate of $e^{\int_0^t V_-(X_s) ds}$. When, however, $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, then outside a compact set only the killing effect occurs and $\mathbf{E}^x[e^{-\int_0^t V(X_s) ds}]$ gives the probability of survival of the process up to time t . The following characterization of AGSD/AIUC may be used as a probabilistic definition of these properties, valid for both our jump Lévy processes and Brownian motion. In fact, this property has a strong ergodic flavor; compare also with [33, 53].

PROPOSITION 2.2. *The semigroup $\{T_t : t \geq 0\}$ is AGSD/AIUC if and only if there exist $t > 0$, a bounded nonempty Borel set $D \subset \mathbf{R}^d$, and a constant $C_{16} = C_{16}(X, V, t)$ such that for every Borel set $A \subseteq \mathbf{R}^d$ we have*

$$(2.8) \quad \mathbf{E}^x \left[e^{-\int_0^t V(X_s) ds}; X_t \in A \right] \leq C_{16} \mathbf{E}^x \left[e^{-\int_0^t V(X_s) ds}; X_t \in D \right], \quad x \in \mathbf{R}^d.$$

(For a proof, see [47], Corollary 4.1, Proposition 4.1, and [57], equations (1.2)–(1.3).) Asymptotically, the probability of survival of the process staying around the starting point x (far from the region D) is approximately $e^{-tV(x)}$, while the probability of surviving by escaping to a region D with a lower killing rate is $\mathbf{P}^x(X_t \in D)$. By using (2.8), it is immediately seen that when $\{T_t : t \geq 0\}$ is AGSD/AIUC, then the probability that the process under V survives up to time t far from the location of $\inf V$ is bounded by the probability that the process survives up to time t and is in some bounded region D , no matter its starting point. It can be expected that the balance of the competing effects in fact will be decided roughly by the ratio $V(x)/|\log \mathbf{P}^x(X_t \in D)|$. Below we prove this intuition and show that for a large class of nondiffusive Lévy processes the expression $|\log \mathbf{P}^x(X_t \in D)|$ precisely determines the borderline potential. Note that this expression does not give the borderline potential for diffusions, however, it allows to identify the leading order of the borderline growth which is known to be quadratic [32]. Some further examples will be discussed below.

The following comparability condition will be used in Propositions 2.3–2.4 only. It is partly satisfied under our previous assumptions and it appears to be strongly related to Assumption 2.1. However, we are not aware of a general argument showing a possible implication, and thus we formulate it as an independent assumption.

ASSUMPTION 2.6. For every $t > 0$, there is $R = R(t) > 0$ such that

$$|\log v(x)| \asymp C_{17} |\log p(t, x)| \asymp C_{18} |\log \mathbf{P}^x(X_t \in B(0, 1))|, \quad |x| > R$$

with constants $C_{17} = C_{17}(X)$ and $C_{18} = C_{18}(X)$ (independent of t).

The next two propositions are direct consequences of Assumption 2.6 and Theorems 2.6–2.7.

PROPOSITION 2.3 (AGSD/AIUC probabilistically). *Let Assumptions 2.1–2.4 and 2.6 be satisfied. Then the following hold:*

(1) *If the semigroup $\{T_t : t \geq 0\}$ is t_0 -GSD (or $t_0/2$ -IUC), then for every $0 < \varepsilon \leq 1$ there is $R \geq 2$ such that*

$$\frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log \mathbf{P}^x(X_{t_0} \in B(0, 1))|} \geq \frac{1}{2C_{17}^2 C_{18} t_0} \quad \text{and} \quad \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log p(t_0, x)|} \geq \frac{1}{2C_{17} t_0},$$

$|x| \geq R.$

Moreover, if also Assumption 2.5 holds, then $\sup_{y \in B(x, \varepsilon)} V(y)$ can be replaced by $C_V V(x)$.

(2) If there exist $t > 0$, $R > 0$ and a constant $C_{19} = C_{19}(X, V)$ such that

$$\frac{V(x)}{|\log \mathbf{P}^x(X_t \in B(0, 1))|} \geq \frac{1}{C_{19}t} \quad \text{or} \quad \frac{V(x)}{|\log p(t, x)|} \geq \frac{1}{C_{19}t}, \quad |x| > R,$$

then $\{T_t : t \geq 0\}$ is t_0 -GSD with $t_0 = 4C_{18}C_{19}t$ and t_0 -IUC with $t_0 = 2(t_b \vee 4C_{18}C_{19}t)$ or t_0 -GSD with $t_0 = 4C_{17}C_{19}t$ and t_0 -IUC with $t_0 = 2(t_b \vee 4C_{17}C_{19}t)$, respectively.

PROPOSITION 2.4 (GSD/IUC probabilistically). *Let Assumptions 2.1–2.4 and 2.6 be satisfied. Then the following hold:*

(1) *If the semigroup $\{T_t : t \geq 0\}$ is GSD (or IUC), then for every $t > 0$ we have*

$$\lim_{|x| \rightarrow \infty} \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log \mathbf{P}^x(X_t \in B(0, 1))|} = \lim_{|x| \rightarrow \infty} \frac{\sup_{y \in B(x, \varepsilon)} V(y)}{|\log p(t, x)|} = \infty.$$

When in addition also Assumption 2.5 holds, then $\sup_{y \in B(x, \varepsilon)} V(y)$ may be replaced by $V(x)$.

(2) *If there is $t > 0$ such that*

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log \mathbf{P}^x(X_t \in B(0, 1))|} = \infty \quad \text{or} \quad \lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log p(t, x)|} = \infty,$$

then $\{T_t : t \geq 0\}$ is GSD. If, moreover, $p(t, \cdot)$ is bounded for all $t > 0$, then any of these two conditions also implies IUC.

Finally, we give another description of AGSD/AIUC. In order to do that, we need to put one more condition on the Lévy measure.

ASSUMPTION 2.7. For every $R > 0$, we have $\log \nu \in L^1(B(0, R)^c, \nu(x) dx)$.

Under Assumptions 2.1–2.5 and 2.7, and for all $A \in \mathcal{B}(\mathbf{R}^d)$ such that $\text{dist}(A, 0) > 0$ we define the functionals

$$(2.9) \quad E_A^V(\nu) = \int_A V(x)\nu(x) dx,$$

$$(2.10) \quad H_A(\nu) = - \int_A \nu(x) \log \nu(x) dx,$$

$$(2.11) \quad F_A^V(\nu) = E^V(\nu) - H_A(\nu).$$

Note that under Assumption 2.7 $F_A^V(\nu)$ is well defined. We call the functional $E_A^V(\nu)$ energy, $H_A(\nu)$ entropy and $F_A^V(\nu)$ free energy in set A for the given potential V and Lévy measure $\nu(dx)$. Note that since $\nu(x)$ is the Radon–Nikodým derivative of the Lévy measure with respect to Lebesgue measure, $H_A(\nu)$ is in

fact the relative entropy (or Kullback–Leibler functional) of the Lévy measure with respect to Lebesgue measure. Then we have the following characterization of AGSD/AIUC.

THEOREM 2.8 (Characterization of AGSD/AIUC). *Let Assumptions 2.1–2.5 and 2.7 hold. The potential V is such that the semigroup $\{T_t : t \geq 0\}$ is AGSD (or, equivalently, AIUC) if and only if there exists $t_0 > 0$ and $R > 0$ such that for every Borel set $A \subset B(0, R)^c$ we have that $F_A^{t_0 V}(\nu) \geq 0$. Specifically, if $\{T_t : t \geq 0\}$ is t_0 -GSD, then $F_A^{2C_{14}t_0 V}(\nu) \geq 0$. If $F_A^{t_0 V}(\nu) \geq 0$, then $\{T_t : t \geq 0\}$ is $8C_{14}t_0$ -GSD.*

Note that due to monotonicity of the free energy functional with respect to potential V , the inequality $F_A^{t_0 V}(\nu) \geq 0$ implies $F_A^t V(\nu) \geq 0$ for all $t \geq t_0$. Furthermore, we have the following variational result.

COROLLARY 2.4 (Variational principle for borderline potential). *Let Assumptions 2.1–2.4 and 2.7 hold, and the jump intensity ν and the potential V be continuous functions. Then V is the borderline potential for AGSD/AIUC of the semigroup $\{T_t : t \geq 0\}$ if and only if there exist $t_0 > 0$ and $R > 0$ such that $F_A^{t_0 V}(\nu) = 0$ for all Borel sets $A \subset B(0, R)^c$.*

We note that similar energy and entropy functionals have been used in [36] to determine heavy tailed probability distributions with prescribed asymptotics, satisfying the Fokker–Planck equation. Such optimization methods are widely used, however, in our context it is derived and rigorously justified by Theorem 2.8. Furthermore, the above variational problem can also be considered in the reverse direction. Roughly speaking, for a given sufficiently regular potential V we may be interested in finding the appropriate Lévy measures ν [i.e., Lévy processes $(X_t)_{t \geq 0}$] such that the corresponding free energy functional $F_A^{t_0 V}(\nu)$ is minimized for some $t_0 > 0$, $R > 0$ and every Borel set $A \subset B(0, R)^c$.

3. Proofs.

3.1. Preliminary results. Here, we recall some basic facts of potential theory for the Feynman–Kac semigroup related to process $(X_t)_{t \geq 0}$ needed for our purposes, and show some technical facts used in proving our results concerning intrinsic ultracontractivity and the eigenfunction estimates below. For background, we refer to [10–12, 26, 28, 30].

We adopt the convention that auxiliary constants appearing in proofs may change their values from one use to another (possibly from line to line). However, if necessary, we write $C, C^{(1)}, C^{(2)}, \dots$ to distinguish them. Recall that, in contrast, constants appearing in the statements of theorems, propositions and lemmas are fixed throughout the paper and can be tracked in the proofs.

Denote by

$$e_V(t) := e_V(t)(\omega) = e^{-\int_0^t V(X_s(\omega)) ds}, \quad t > 0.$$

The Feynman–Kac-functional for the Lévy process $(X_t)_{t \geq 0}$ for potential V . By standard arguments based on Khasminskii’s lemma (see [30], Proposition 3.8, and [60]), there are constants $C_{21} = C_{21}(X, V)$ and $C_{22} = C_{22}(X, V)$ such that

$$(3.1) \quad \sup_{x \in \mathbf{R}^d} \mathbf{E}^x [e_V(t)] \leq \sup_{x \in \mathbf{R}^d} \mathbf{E}^x [e_{-V_-}(t)] \leq C_{21} e^{C_{22}t}, \quad t > 0.$$

Recall that $\tau_D = \inf\{t > 0 : X_t \notin D\}$ denotes the first exit time of the process from the set D . The potential operator for the semigroup $\{T_t : t \geq 0\}$ is defined by

$$G^V f(x) = \int_0^\infty T_t f(x) dt = \mathbf{E}^x \left[\int_0^\infty e_V(t) f(X_t) dt \right]$$

for nonnegative or bounded Borel functions f on \mathbf{R}^d , while the V -Green operator for an open set D is given by

$$G_D^V f(x) = \int_0^\infty \mathbf{E}^x [t < \tau_D; e_V(t) f(X_t)] dt = \mathbf{E}^x \left[\int_0^{\tau_D} e_V(t) f(X_t) dt \right]$$

for nonnegative or bounded Borel functions f on D .

It can be seen directly that if $D \subset \mathbf{R}^d$ is a nonempty bounded open set and V is a nonnegative and not identically zero potential on D , then for all $x \in D$ we have

$$(3.2) \quad \left(1 - \exp\left(-\sup_{y \in D} V(y)\right) \right) \frac{\mathbf{P}^x(\tau_D > 1)}{\sup_{y \in D} V(y)} \leq G_D^V \mathbf{1}(x) \leq \frac{1}{\inf_{y \in D} V(y)}.$$

Here, we use the convention that $1/\infty = 0$ and $1/0^+ = \infty$.

Below we often use the fact that for all bounded Borel sets $D \subset \mathbf{R}^d$ and $x \in D$ we have $\mathbf{E}^x[\tau_D] \leq \mathbf{E}^x[\tau_{B(x, \text{diam } D)}] = \mathbf{E}^0[\tau_{B(0, \text{diam } D)}] < \infty$. Furthermore, when $D' \subset \mathbf{R}^d$ is an open set, $D \subset D'$ is open and bounded and f is a nonnegative or bounded Borel function on D' , then by the strong Markov property, it follows for every $x \in D$ that

$$(3.3) \quad G_{D'}^V f(x) = G_D^V f(x) + \mathbf{E}^x [X_{\tau_D} \in D' \setminus D; e_V(\tau_D) G_{D'}^V f(X_{\tau_D})].$$

A Borel function f on \mathbf{R}^d is called (X, V) -harmonic in an open set $D \subset \mathbf{R}^d$ if

$$(3.4) \quad f(x) = \mathbf{E}^x [\tau_U < \infty; e_V(\tau_U) f(X_{\tau_U})], \quad x \in U,$$

for every open set U with \bar{U} contained in D , and it is called regular (X, V) -harmonic in D if (3.4) holds for $U = D$. By the strong Markov property, every regular (X, V) -harmonic function in D is (X, V) -harmonic in D . We always assume that the expectation in (3.4) is absolutely convergent.

The following uniform estimates for local suprema of (X, V) -harmonic functions are an important ingredient in proving AGSD/GSD and eigenfunction bounds. Under Assumptions 2.1–2.3, they directly follow from the more general results in [14].

LEMMA 3.1. *Let Assumptions 2.1(1), 2.2 and 2.3 be satisfied. Then for every $0 < r < p < q < R \leq 1$ there exists a constant $C_{23} = C_{23}(X, r, p, q, R)$ such that for any $V \in \mathcal{K}_{\text{loc}}^X$, $V \geq 0$ on $B(x_0, R)$, and every nonnegative function f on \mathbf{R}^d that is (X, V) -regular harmonic on $B(x_0, R)$, we have*

$$f(y) \asymp C_{23} G_{B(x_0, p)}^V \mathbf{1}(y) \int_{B(x_0, q)^c} f(z) v(z - x_0) dz, \quad |y - x_0| < r.$$

PROOF. Under the assumptions of the lemma, Assumptions A–D in [14] are satisfied. Specifically, since $(X_t)_{t \geq 0}$ is a symmetric Lévy process satisfying the strong Feller property, Assumptions A and B hold directly, while Assumption C is a consequence of our Assumption 2.1(1), and our Assumption 2.3 is just Assumption D (for details of their verification, see [14], Example 5.5). Thus, the above estimates hold for $V \equiv 0$ as a consequence of [14], Lemma 3.2 and Theorem 3.4, for the set $D = B(x_0, R)$. By space homogeneity of $(X_t)_{t \geq 0}$, the constant C_{23} does not depend on the specific choice of x_0 . Similar estimates for an arbitrary $V \in \mathcal{K}_{\text{loc}}^X$, $V \geq 0$ on $B(x_0, R)$, that is, for the subprocess of $(X_t)_{t \geq 0}$ given by the multiplicative functional $M_t = e_V(t)$, follow from the latter with the same constant C_{23} (independent of V) by the argument in [14], Example 5.9. \square

Note that it is crucial below that C_{23} in the above bounds does not depend on the (local behavior of) the potential V . It is also essential for our further applications that due to space-homogeneity of the process $(X_t)_{t \geq 0}$ the constant is independent of the location of the ball $B(x_0, R)$ in space. This is also the reason why we cannot consider in this paper more general Markov processes that are not space-homogeneous. In fact, Lemma 3.1 will be used below in the following form which is sufficiently general and suitable for our purposes.

COROLLARY 3.1. *Let Assumptions 2.1(1), 2.2 and 2.3 be satisfied. Then there exists a constant $C_{24} = C_{24}(X)$ such that for any $V \in \mathcal{K}_{\text{loc}}^X$, $V \geq 0$ on $B(x_0, 1)$, and every nonnegative function f on \mathbf{R}^d that is (X, V) -regular harmonic on $B(x_0, 1)$, we have*

$$f(y) \asymp C_{24} G_{B(x_0, 1)}^V \mathbf{1}(y) \int_{B(x_0, 3/4)^c} f(z) v(z - x_0) dz, \quad |y - x_0| < \frac{1}{2}.$$

PROOF. By taking $r = 1/2$, $p = 5/8$, $q = 3/4$ and $R = 1$ in Lemma 3.1, we clearly have

$$f(y) \asymp C_{23} G_{B(x_0, 5/8)}^V \mathbf{1}(y) \int_{B(x_0, 3/4)^c} f(z) v(z - x_0) dz, \quad |y - x_0| < \frac{1}{2}.$$

Thus, it suffices to see that $G_{B(x_0, 1)}^V \mathbf{1}(y) \leq C G_{B(x_0, 5/8)}^V \mathbf{1}(y)$, $y \in B(x_0, 1/2)$, with a constant $C = C(X)$ (independent of V and x_0). By formula (3.3) for

$D' = B(x_0, 1)$, $D = B(x_0, 5/8)$ and $f = \mathbf{1}$, and by the fact that

$$\sup_{y \in B(x_0, 1)} G_{B(x_0, 1)}^V \mathbf{1}(y) = C^{(1)} < \infty$$

with $C^{(1)}$ independent of V and x_0 , we have

$$G_{B(x_0, 1)}^V \mathbf{1}(y) \leq G_{B(x_0, 5/8)}^V \mathbf{1}(y) + C^{(1)} \mathbf{E}^y[e_V(\tau_{B(x_0, 5/8)})], \quad |y - x_0| < \frac{1}{2}.$$

Let now

$$g(y) = \begin{cases} \mathbf{E}^y[e_V(\tau_{B(x_0, 5/8)})], & \text{if } y \in B(x_0, 5/8), \\ \mathbf{1}, & \text{if } y \notin B(x_0, 5/8). \end{cases}$$

By applying Lemma 3.1 for g with $r = 1/2$, $p = 17/32$, $q = 9/16$ and $R = 5/8$, we conclude that

$$\mathbf{E}^y[e_V(\tau_{B(x_0, 5/8)})] \leq C_{23} G_{B(x_0, 17/32)}^V \mathbf{1}(y) \int_{B(0, 9/16)^c} v(z) dz \leq C G_{B(x_0, 5/8)}^V \mathbf{1}(y), \quad |y - x_0| < \frac{1}{2}$$

with constant $C = C(X)$, independent of V and x_0 . \square

In fact, in order to obtain the above corollary it suffices to prove Lemma 3.1 only for two fixed sets of parameters r, p, q, R . Therefore, it would be enough to have Assumptions 2.1(1) and 2.3 in place only for some specially chosen, sufficiently small $r > 0$ and $p, q > 0$, respectively. However, this approach requires a detailed analysis of constants appearing in [14] and causes further technical difficulties (note that the parameters r, p, q in Lemma 3.1 do not correspond directly to r and p, q in the assumptions). Since the general Assumptions 2.1(1) and 2.3 are not restrictive for our further results, we included a general version of Lemma 3.1.

The following auxiliary results will also be used later.

LEMMA 3.2. *Let $D \subset \mathbf{R}^d$ be an arbitrary open set and V be an X -Kato class potential such that $V \geq 0$ on D . Then there are constants $C_{25} = C_{25}(X, V, t)$ and $C_{26} = C_{26}(X, V, t)$ such that for every $t > 0$ we have*

- (1) $\mathbf{E}^x[\frac{t}{2} \geq \tau_D; e_V(t)] \leq C_{25} \mathbf{E}^x[e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D})]$;
- (2) $\mathbf{E}^x[\frac{t}{2} < \tau_D; e_V(t)] \leq C_{26} G_D^V \mathbf{1}(x) \sup_{y \in D} T_{t/2} \mathbf{1}(y)$, $x \in D$.

PROOF. The proof of (1) and (2) with the expression on the right-hand side $G_D^V \mathbf{1}(x) \sup_{y \in D} T_{t/2} \mathbf{1}(y)$ replaced by $\mathbf{E}^x[\frac{t}{4} < \tau_D; e_V(\frac{t}{4})] \sup_{y \in D} T_{3t/4} \mathbf{1}(y)$ runs in the same way as in [47], Lemma 4.3. We complete the proof of (2) by the simple observation that

$$\mathbf{E}^x \left[\frac{t}{4} < \tau_D; e_V \left(\frac{t}{4} \right) \right] \leq \frac{4}{t} \mathbf{E}^x \left[\frac{t}{4} < \tau_D; \int_0^{t/4} e^{-\int_0^v V(X_s) ds} dv \right] \leq \frac{4}{t} G_D^V \mathbf{1}(x), \quad x \in D$$

and

$$\begin{aligned}
 T_{3t/4}\mathbf{1}(y) &= \mathbf{E}^y \left[e_V \left(\frac{t}{2} \right) \mathbf{E}^{X_{t/2}} \left[e_V \left(\frac{t}{4} \right) \right] \right] \\
 &\leq T_{t/2}(y) \sup_{z \in \mathbf{R}^d} \mathbf{E}^z \left[e_V \left(\frac{t}{4} \right) \right] \leq C_{V,t} T_{t/2}(y), \quad y \in D. \quad \square
 \end{aligned}$$

A short proof of the following fact was communicated to us by M. Kwaśnicki.

LEMMA 3.3. *Let $(X_t)_{t \geq 0}$ be a Lévy process with transition densities $p(t, x, y) = p(t, y - x)$ such that for some $t > 0$ and all $s \in (0, t]$ we have $p(s, x) \leq p(s, y)$ whenever $|x| \geq |y|$. Then for every bounded open set $D \subset \mathbf{R}^d$ and $r > 0$ such that $\{y \in D : \text{dist}(y, \partial D) \geq r\} \neq \emptyset$ there is a constant $C_{27} = C_{27}(r)$ such that*

$$p_D(t, x, y) \geq p(t, y - x) - C_{27}, \quad x, y \in D, \text{dist}(y, \partial D) \geq r.$$

PROOF. Observe that for every $|z| \geq r$ and $s \in (0, t]$ we have

$$|B(0, r)|p(s, z) \leq \int_{B(0,r)} p(s, w) dw \leq 1.$$

Thus, by Hunt’s formula (2.1) we get

$$\begin{aligned}
 p_D(t, x, y) &= p(t, y - x) - \mathbf{E}^x [\tau_D < t; p(t - \tau_D, y - X_{\tau_D})] \\
 &\geq p(t, y - x) - C_{27}
 \end{aligned}$$

for all $x, y \in D$ such that $\text{dist}(y, \partial D) \geq r$, with $C_{27} = |B(0, r)|^{-1}$. \square

3.2. *Jump estimates.* For our purposes below, we will need to control jumps between some carefully chosen regions. Let $n, k \in \mathbf{N}$, $n, k \geq n_0 \geq 2$ (with n_0 to be chosen below), and define

$$\begin{aligned}
 D_n &:= \{x \in \mathbf{R}^d : n - 2 < |x|\}, \quad n \geq n_0 + 2, \\
 D_{n_0} &= D_{n_0+1} := \mathbf{R}^d, \\
 R_k &:= \{x \in \mathbf{R}^d : k - 1 < |x| \leq k\}, \quad k \geq n_0 + 2, \\
 R_{n_0} &:= \{x \in \mathbf{R}^d : |x| \leq n_0\}, \\
 R_{n_0+1} &:= \{x \in \mathbf{R}^d : |x| \leq n_0 + 1\}.
 \end{aligned}$$

We will use the two stopping times

$$\begin{aligned}
 \tau_n = \tau_{D_n} &:= \inf\{t \geq 0 : X_t \notin D_n\}, \\
 \sigma_k = \sigma_{R_k} &:= \inf\{t \geq 0 : X_t \in R_k\}.
 \end{aligned}$$

Note that $\tau_{n_0} = \tau_{n_0+1} = \infty$. In the events in which we are interested, the process jumps from the complement of a ball D_n to a smaller shell R_k , which we will refer to as admissible jumps. We define for $k \geq n_0, n \geq k + 2$ and $t > 0$ the events

$$S(n, k, 1, t) = \{X_{\tau_n} \in R_k, \sigma_k, < t\},$$

$$S(n, k, l, t) = \bigcup_{p=k+2}^{n-2} S(n, p, l-1, t) \cap S(p, k, 1, t), \quad l > 1.$$

The first corresponds to the event that the process arrives in shell R_k before time t in just one jump after exiting D_n . The second event is defined inductively. Let $k + 2 \leq p \leq n - 2$. The event $S(n, p, 1, t) \cap S(p, k, 1, t)$ means that the process jumps to shell R_p on leaving D_n and then again jumps to shell R_k on leaving D_p , and all this occurs before time t . Note that the process may go elsewhere after arriving in R_p but the events which we are constructing only take account of admissible jumps, that is, those that are oriented to the origin through jumps into the shells R_k . Thus, the event $S(n, k, l, t)$ corresponds to the process arriving in shell R_k from D_n through l admissible jumps before time t . This scheme of keeping track of the so defined jumps has been first devised in [16] and used in [57]. Here, we also partially adopt the notation of [57].

The following technical lemma will be needed below.

LEMMA 3.4. *Let Assumption 2.1(1)–(2) be satisfied, and take $n, k \in \mathbf{N}$ such that $n - 2 > k \geq n_0$. Then the following hold:*

(1) *There is a constant $C_{28} = C_{28}(X) \geq 1$ (independent of n and k) such that*

$$\int_{R_k} v(z - y) dz \leq C_{28} \int_{R_k} v(z - x) dz, \quad x \in R_n, y \in D_n.$$

(2) *For any $m \in \mathbf{N}$, there is a constant $C_{29} = C_{29}(X, m) \geq 1$ (independent of k) such that*

$$\int_{R_{k+m} \cap \{z: |y-z| > 1/2\}} v(z - y) dz \leq C_{29} \int_{R_k} v(z - y) dz, \quad |y| \geq k + 1.$$

PROOF. First, we prove (1). By rotation symmetry of R_k and conditions (1)–(2) of Assumption 2.1, we deduce directly that

$$\int_{R_k} v(z - y) dz \leq C_1^2 \int_{R_k} v(z - x) dz, \quad x \in R_n, n - 2 < |y| \leq |x|$$

and

$$\int_{R_k} v(z - y) dz \leq C_2 \int_{R_k} v(z - x) dz, \quad x \in R_n, |x| \leq |y|,$$

respectively. Thus, (1) follows. Consider assertion (2) of the lemma. Define the dilations $S_{k,m}(w) = ((k + m - 1)/(k - 1))w, m \in \mathbf{N}$. Since $R_{k+m} \subset S_{k,m}(R_k)$, it

suffices to prove (2) for R_{k+m} replaced by $S_{k,m}(R_k)$. By changing variables, we obtain

$$\begin{aligned} & \int_{S_{k,m}(R_k) \cap \{z : |y-z| > 1/2\}} v(z-y) dz \\ &= \left(\frac{k+m-1}{k-1}\right)^d \int_{R_k \cap S_{k,m}^{-1}(\{z : |y-z| > 1/2\})} v\left(\frac{k+m-1}{k-1}w-y\right) dw \end{aligned}$$

for all $|y| \geq k+1$. Since $|\frac{k+m-1}{k-1}w-w| \leq 2m$, by Assumption 2.1(1) we have

$$v\left(\frac{k+m-1}{k-1}w-y\right) \leq C^{(1)}v(w-y)$$

with $w \in R_k \cap S_{k,m}^{-1}(\{z : |y-z| > 1/2\})$ and constant $C^{(1)} = C^{(1)}(X, m)$. Hence,

$$\begin{aligned} \int_{S_{k,m}(R_k) \cap \{z : |y-z| > 1/2\}} v(z-y) dz &\leq C^{(1)}(m+1)^d \int_{R_k} v(w-y) dw, \\ &|y| \geq k+1, \end{aligned}$$

which completes the proof of the lemma. \square

The next two lemmas are key tools to our further considerations. Lemma 3.5 builds on [57], Lemma 5.7, however, our argument is based on a completely new approach which combines sharp uniform upper estimates for local maxima of (X, V) -harmonic functions (Corollary 3.1) with an inductive procedure which substantially uses Assumption 2.1(3). We recall that a sufficiently general version of this uniform estimate necessary for our purposes in this paper was proved only recently in [14]. The second Lemma 3.6 is a corollary of Lemma 3.5 and Assumption 2.1. Notice that it is crucial for the applications below that the constants C_{30} and C_{31} in these lemmas are independent of t , unlike in [57]. This allows us to use them in proving estimates of λ -subaveraging functions and, in consequence, the bounds on the eigenfunctions. Both proofs below clearly show the significance of condition (3) in Assumption 2.1.

We note for later use that under condition (1) in Assumption 2.1 we have $\mathbf{P}^x(\tau_n < t) > 0$ and $\mathbf{P}^x(\sigma_k < t) > 0$, for all $n-2 \geq k \geq n_0$, $n-1 < |x| \leq n$ and $t > 0$.

LEMMA 3.5. *Let Assumptions 2.1–2.3 hold, and $n, k \in \mathbf{N}$ be such that $n-2 \geq k \geq n_0$. Then there is a constant $C_{30} = C_{30}(X)$ and $\theta_0 = \theta_0(X) \geq 1$ such that for every $t > 0$, for all $n-1 < |x| \leq n$ and $\theta > \theta_0$ we have*

$$\mathbf{E}^x[\tau_n < t, X_{\tau_n} \in R_k; e^{-\theta\tau_n}] \leq \frac{C_{30}}{\theta} \int_{R_k} v(y-x) dy.$$

PROOF. First, we assume that $n > k + 2$ and $\theta > 0$ is arbitrary. Note that $\text{dist}(D_n, R_k) \geq 1$. For $r > n - 2$ denote $\tau_{n,r} := \tau_{D_n \cap B(0,r)}$. Using the Ikeda–Watanabe formula [43], Theorem 1, we have

$$\begin{aligned} & \mathbf{E}^x[\tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta\tau_{n,r}}] \\ & \leq \int_{D_n \cap B(0,r)} \int_0^\infty e^{-\theta s} p_{D_n \cap B(0,r)}(s, x, y) \int_{R_k} \nu(z - y) dz ds dy \end{aligned}$$

and, consequently, by Lemma 3.4(1) and Fubini’s theorem we get

$$\begin{aligned} & \mathbf{E}^x[\tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta\tau_{n,r}}] \\ & \leq C \int_0^\infty e^{-\theta s} ds \int_{R_k} \nu(z - x) dz = \frac{C}{\theta} \int_{R_k} \nu(z - x) dz \end{aligned}$$

with constant $C = C(X)$. To complete the proof in this case, it suffices to show that

$$\mathbf{E}^x[\tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta\tau_{n,r}}] \rightarrow \mathbf{E}^x[\tau_n < t, X_{\tau_n} \in R_k; e^{-\theta\tau_n}]$$

as $r \rightarrow \infty$. Since $\tau_{n,r} = \tau_n$ when $X_{\tau_{n,r}} \in R_k$, we have

$$\begin{aligned} 0 & \leq \mathbf{E}^x[\tau_n < t, X_{\tau_n} \in R_k; e^{-\theta\tau_n}] - \mathbf{E}^x[\tau_{n,r} < t, X_{\tau_{n,r}} \in R_k; e^{-\theta\tau_{n,r}}] \\ & = \mathbf{E}^x[\tau_{n,r} < \tau_n < t, X_{\tau_n} \in R_k, X_{\tau_{n,r}} \in B(0, r)^c; e^{-\theta\tau_n}] \\ & \leq \mathbf{P}^x(\tau_{n,r} < t, X_{\tau_{n,r}} \in B(0, r)^c) \\ & \leq \mathbf{P}^x(\tau_{B(0,r)} < t) \leq \mathbf{P}^0(\tau_{B(0,r/2)} < t) \end{aligned}$$

for sufficiently large r . Clearly, $\mathbf{P}^0(\tau_{B(0,r/2)} < t) \rightarrow 0$ as $r \rightarrow \infty, t > 0$, and the claimed convergence follows.

Now consider the case $n = k + 2$. Denote $B_y = B(y, 1)$ and

$$f(y) = \begin{cases} \mathbf{E}^y[\tau_n < \infty, X_{\tau_n} \in R_k; e^{-\theta\tau_n}], & \text{if } y \in D_n, \\ \mathbf{1}_{R_k}(y), & \text{if } y \notin D_n. \end{cases}$$

Since f is an (X, θ) -regular harmonic function in D_n , we have

$$f(z) = \mathbf{E}^z[e^{-\theta\tau_{B_y}} f(X_{\tau_{B_y}})], \quad z \in B_y, |y| > n - 1.$$

We will show that there is a constant $C = C(X)$ and $\theta_0 \geq 1$ such that for every $\theta \geq \theta_0$ and $l \in \mathbf{N}$

$$(3.5) \quad f(y) \leq \frac{C}{\theta} \left(\sum_{i=1}^l \frac{1}{2^i} \int_{R_k} \nu(z - y) dz + \frac{1}{2^l} \int_{|z|>n+1} \nu(z - y) f(z) dz \right),$$

$n - 1 < |y| \leq n.$

If this holds, then by taking the limit $l \rightarrow \infty$ the required bound follows.

By Corollary 3.1, we have

$$f(z) \leq C_{24} G_{B_y}^\theta \mathbf{1}(z) \int_{B(y, 1/2)^c} f(w) v(w - y) dw, \quad |z - y| < 1/2, n - 1 < |y|$$

and, since $G_{B_y}^\theta \mathbf{1}(z) \leq 1/\theta$, we obtain

$$f(y) \leq \frac{2C_{24}}{2\theta} \left(\int_{R_k} v(z - y) dz + \int_{|z-y|>1/2, |z|>n-2} v(z - y) f(z) dz \right), \quad n - 1 < |y|.$$

(3.6)

Moreover, $0 \leq f \leq 1$ and a direct application of Lemma 3.4(2) to each of the three integrals (recall that $k = n - 2$)

$$\int_{R_{k+m} \cap \{z: |z-y|>1/2\}} v(z - y) dz, \quad m = 1, 2, 3,$$

separately gives that

$$f(y) \leq \frac{C}{2\theta} \left(\int_{R_k} v(z - y) dz + \int_{|z-y|>1/2, |z|>n+1} v(z - y) f(z) dz \right), \quad n - 1 < |y|$$

(3.7)

with constant $C = C(X)$. In particular, (3.5) holds for $l = 1$ and arbitrary $\theta > 0$.

Next, we use induction. Let $\theta_0 := CC_3 \vee 1$ and suppose that (3.5) is true for $l - 1 \in \mathbf{N}$ with constant C and $\theta \geq \theta_0$, where C is the constant in (3.7). By the induction hypothesis and (3.7), we have for $n - 1 < |y| \leq n$ and $\theta \geq \theta_0$

$$\begin{aligned} f(y) &\leq \frac{C}{\theta} \left(\sum_{i=1}^{l-1} \frac{1}{2^i} \int_{R_k} v(z - y) dz + \frac{1}{2^{l-1}} \int_{|z|>n+1} v(z - y) f(z) dz \right) \\ &\leq \frac{C}{\theta} \sum_{i=1}^{l-1} \frac{1}{2^i} \int_{R_k} v(z - y) dz \\ &\quad + \left(\frac{C}{\theta} \right)^2 \frac{1}{2^l} \int_{|z|>n+1} v(z - y) \int_{R_k} v(w - z) dw dz \\ &\quad + \left(\frac{C}{\theta} \right)^2 \frac{1}{2^l} \int_{|z|>n+1} v(z - y) \int_{|w-z|>1/2, |w|>n+1} v(w - z) f(w) dw dz. \end{aligned}$$

An application of Fubini’s theorem and Assumption 2.1(3) to the last two terms in the sum on the right-hand side above gives

$$f(y) \leq \frac{C}{\theta} \sum_{i=1}^l \frac{1}{2^i} \int_{R_k} v(w - y) dw + \frac{C}{\theta} \frac{1}{2^l} \int_{|w|>n+1} v(w - y) f(w) dw, \quad n - 1 < |y| \leq n.$$

□

LEMMA 3.6. *Let Assumptions 2.1–2.4 hold. Moreover, suppose that there is a nondecreasing sequence $g_n \rightarrow \infty$ as $n \rightarrow \infty$ and $n_0 \in \mathbf{N}$ large enough such that for $n \geq n_0$ we have*

$$1 < 2\theta_0 \leq g_n \leq \inf_{|y| \geq n} V(y) \quad \text{and} \quad 4C_3C_{28}C_{30} \leq g_{n_0},$$

where C_{28}, C_{30} are constants, and θ_0 is the parameter from Lemmas 3.4(1) and 3.5, respectively. Then for $n - 1 < |x| \leq n, n_0 \leq k \leq n - 2, n, k, l \in \mathbf{N}$, it follows that

$$(3.8) \quad \mathbf{E}^x[S(n, k, l, t); e^{-(1/2)\int_0^{\sigma_k} V(X_s) ds}] \leq \frac{C_{31}}{2^l g_{n-2}} \int_{R_k} v(y - x) dy$$

with $C_{31} = 4C_{30}$.

PROOF. We use induction in $l \in \mathbf{N}$. Let $l = 1$. Since we have $S(n, k, 1, t) = \{X_{\tau_n} \in R_k, \sigma_k < t\}$ and $\tau_n = \sigma_k$ for $X_{\tau_n} \in R_k$, we obtain by Lemma 3.5

$$\begin{aligned} \mathbf{E}^x[S(n, k, 1, t); e^{-(1/2)\int_0^{\sigma_k} V(X_s) ds}] &\leq \mathbf{E}^x[X_{\tau_n} \in R_k, \tau_n < t; e^{-\tau_n g_{n-2}/2}] \\ &\leq \frac{C_{31}}{2g_{n-2}} \int_{R_k} v(y - x) dy. \end{aligned}$$

Let now $l \geq 2$ and suppose that the bound (3.8) holds for $1, 2, \dots, l - 1$ and all n, k as in the statement of the lemma. The strong Markov property gives

$$\begin{aligned} &\mathbf{E}^x[S(n, k, l, t); e^{-(1/2)\int_0^{\sigma_k} V(X_s) ds}] \\ &= \sum_{p=k+2}^{n-2} \mathbf{E}^x[S(n, p, l - 1, t), S(p, k, 1, t); \\ &\quad e^{-(1/2)\int_0^{\sigma_p} V(X_s) ds} e^{-(1/2)\int_{\sigma_p}^{\sigma_k} V(X_s) ds}] \\ &\leq \sum_{p=k+2}^{n-2} \mathbf{E}^x[S(n, p, l - 1, t), S(p, k, 1, t + \sigma_p); \\ &\quad e^{-(1/2)\int_0^{\sigma_p} V(X_s) ds} e^{-(1/2)\int_{\sigma_p}^{\sigma_k} V(X_s) ds}] \\ &= \sum_{p=k+2}^{n-2} \mathbf{E}^x[S(n, p, l - 1, t); \\ &\quad e^{-(1/2)\int_0^{\sigma_p} V(X_s) ds} \mathbf{E}^{X_{\sigma_p}}[S(p, k, 1, t); e^{-(1/2)\int_0^{\sigma_k} V(X_s) ds}]]. \end{aligned}$$

By the induction hypothesis and Lemma 3.4(1), the last sum is bounded above by

$$\sum_{p=k+2}^{n-2} \frac{C_{31}}{2^{l-1} g_{n-2}} \int_{R_p} v(y - x) \frac{C_{31} C_{28}}{2g_{p-2}} \int_{R_k} v(z - y) dz dy.$$

Hence, Fubini’s theorem and Assumption 2.1(3) yield that

$$\begin{aligned}
 \mathbf{E}^x[S(n, k, l, t); e^{-(1/2) \int_0^k V(X_s) ds}] &\leq \frac{C_{31}}{2^l g_{n-2}} \frac{C_{31} C_{28}}{g_{n_0}} \int_{R_k} \sum_{p=k+2}^{n-2} \int_{R_p} v(y-x)v(z-y) dy dz \\
 &\leq \frac{C_{31}}{2^l g_{n-2}} \frac{4C_{30} C_{28} C_3}{g_{n_0}} \int_{R_k} v(y-x) dy \\
 &\leq \frac{C_{31}}{2^l g_{n-2}} \int_{R_k} v(y-x) dy. \quad \square
 \end{aligned}$$

3.3. Estimates of λ -subaveraging functions.

PROOF OF THEOREM 2.1. Recall that $C_{21} = C_{21}(X, V)$, $C_{22} = C_{22}(X, V)$ and $C_{29} = C_{29}(X, m)$ are the constants in (3.1) and Lemma 3.4(2), respectively. We write

$$\begin{aligned}
 C = C(X) &:= 2 \left(1 \vee \frac{C_2}{|B(0, 1)|v((6, 0, \dots, 0))} \right) \geq 2, \\
 C^{(1)} = C^{(1)}(X) &:= \max_{1 \leq m \leq 2} C_{29} \geq 1, \\
 C^{(2)} = C^{(2)}(X) &:= \frac{1}{4} \left(1 \wedge \left(\int_{B(0,1)^c} v(y) dy \right)^{-1} \right) \leq \frac{1}{4}.
 \end{aligned}$$

Notice that $C^{(2)} \int_{B(0,1)^c} v(y) dy \leq 1/4$. For $n \in \mathbf{N}$ we denote $g_n := \inf_{|y| \geq n} V_0(y)$ with $V_0 = V - \lambda$. Let n_0 be a natural number satisfying the assumptions of Lemma 3.6 for the potential V_0 and the sequence $(g_n)_{n \in \mathbf{N}}$, and such that

$$\begin{aligned}
 \max \left(C_{31} \left(\left(\int_{B(0,1)^c} v(y) dy \vee C_3 \right) + \frac{3C_{21} C C_{28} C^{(1)}}{C^{(2)}} \right), 2(\lambda + C_{22}) \right) \\
 \leq g_{n_0}, \tag{3.9}
 \end{aligned}$$

where C_{31} is the constant from Lemma 3.6. It is worth to note for later use that since $V_0(y) \leq 2V(y)$ for $|y| \geq n_0$, the number n_0 also satisfies the assumptions of Lemma 3.6 for the potential $2V$ and the same sequence $(g_n)_{n \in \mathbf{N}}$.

We will show that

$$\varphi(x) \leq C \|\varphi\|_\infty \left(\int_{R_{n_0}} v(y-x) dy \right)^{\sum_{i=1}^p 2^{-i}} \quad \text{for } |x| > n_0 + 3 \tag{3.10}$$

for all $p \in \mathbf{N}$. If this holds, then by taking the limit $p \rightarrow \infty$ and using Assumption 2.1(1) the theorem follows.

We use again induction. For more clarity, we divide the proof of (3.10) in two steps.

Step 1. First, we show that the bound (3.10) holds for $p = 1$. We have

$$\varphi(x) \leq e^{\lambda t} \mathbf{E}^x \left[e^{-\int_0^t V(X_s) ds} \varphi(X_t) \right] \leq \|\varphi\|_\infty \mathbf{E}^x \left[e^{-\int_0^t V_0(X_s) ds} \right],$$

$$x \in \mathbf{R}^d, t > 0.$$

To get an upper bound on the latter expectation, let $n - 1 < |x| \leq n, n \geq n_0 + 4$. For all $t > 0$, we have

$$(3.11) \quad \mathbf{E}^x \left[e^{-\int_0^t V_0(X_s) ds} \right] \leq \mathbf{E}^x \left[\tau_n > t; e^{-\int_0^t V_0(X_s) ds} \right] + \sum_{k=n_0}^{n-2} \sum_{l=1}^\infty \mathbf{E}^x \left[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \right].$$

Clearly, by (3.9) the first term on the right-hand side is estimated directly by

$$\mathbf{P}^x(\tau_n > t) e^{-g_{n-2}t} \leq \mathbf{P}^x(\tau_n > t) e^{-2(\lambda+C_{22})t} \leq e^{-(\lambda+C_{22})t}.$$

Lemma 3.6 and (3.9) yield for $k \geq n_0 + 2$

$$\begin{aligned} & \mathbf{E}^x \left[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \right] \\ & \leq \mathbf{E}^x \left[S(n, k, l, t), \tau_k > t; e^{-(1/2)\int_0^{\sigma_k} V_0(X_s) ds} e^{-(1/2)\int_0^t V_0(X_s) ds} \right] \\ & \leq e^{-(\lambda+C_{22})t} \mathbf{E}^x \left[S(n, k, l, t); e^{-(1/2)\int_0^{\sigma_k} V_0(X_s) ds} \right] \\ & \leq \frac{C_{31}}{g_{n_0} 2^l} e^{-(\lambda+C_{22})t} \int_{R_k} v(y-x) dy. \end{aligned}$$

Similarly, by the strong Markov property and (3.1), we have for $k = n_0$ and $k = n_0 + 1$ (recall $\tau_{n_0} = \tau_{n_0+1} = \infty$)

$$\begin{aligned} & \mathbf{E}^x \left[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \right] \\ & \leq e^{\lambda t} \mathbf{E}^x \left[S(n, k, l, t); e^{-\int_0^{\sigma_k} V_+(X_s) ds} e^{\int_0^{t+\sigma_k} V_-(X_s) ds} \right] \\ & = e^{\lambda t} \mathbf{E}^x \left[S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} e^{\int_{\sigma_k}^{t+\sigma_k} V_-(X_s) ds} \right] \\ & = e^{\lambda t} \mathbf{E}^x \left[S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} \mathbf{E}^{X_{\sigma_k}} \left[e^{\int_0^t V_-(X_s) ds} \right] \right] \\ & = e^{\lambda t} \sup_{y \in \mathbf{R}^d} \mathbf{E}^y \left[e^{\int_0^t V_-(X_s) ds} \right] \mathbf{E}^x \left[S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} \right] \\ & \leq C_{21} e^{(\lambda+C_{22})t} \mathbf{E}^x \left[S(n, k, l, t); e^{-(1/2)\int_0^{\sigma_k} 2V(X_s) ds} \right], \end{aligned}$$

which, in turn, by Lemmas 3.6 and 3.4(2), is smaller or equal to

$$\frac{C_{21} C_{31}}{g_{n_0} 2^l} e^{(\lambda+C_{22})t} \int_{R_k} v(y-x) dy \leq \frac{C_{21} C_{31} C^{(1)}}{g_{n_0} 2^l} e^{(\lambda+C_{22})t} \int_{R_{n_0}} v(y-x) dy.$$

By putting together the above estimates and choosing

$$t = -\frac{1}{2(\lambda + C_{22})} \log\left(C^{(2)} \int_{R_{n_0}} v(y - x) dy\right) > 0$$

in (3.11), and using (3.9) we conclude that

$$\begin{aligned} \varphi(x) &\leq \|\varphi\|_\infty \left(\left(C^{(2)} \int_{R_{n_0}} v(y - x) dy \right)^{1/2} \right. \\ &\quad + \frac{C_{31}}{g_{n_0}} \left(C^{(2)} \int_{R_{n_0}} v(y - x) dy \right)^{1/2} \sum_{k=n_0+2}^{n-2} \int_{R_k} v(y - x) dy \\ &\quad \left. + \frac{2C_{21}C_{31}C^{(1)}}{\sqrt{C^{(2)}}g_{n_0}} \left(\int_{R_{n_0}} v(y - x) dy \right)^{1/2} \right) \\ &\leq \|\varphi\|_\infty \left(1 + \frac{C_{31}}{g_{n_0}} \left(\int_{B(0,1)^c} v(y) dy + \frac{2C_{21}C^{(1)}}{\sqrt{C^{(2)}}} \right) \right) \left(\int_{R_{n_0}} v(y - x) dy \right)^{1/2} \\ &\leq 2\|\varphi\|_\infty \left(\int_{R_{n_0}} v(y - x) dy \right)^{1/2}, \end{aligned}$$

which completes the first step.

Step 2. Suppose now that (3.10) holds for p . We prove that it is also satisfied for $p + 1$. We consider two cases. First let $n_0 + 3 < |x| \leq n_0 + 4$. Denote $x_0 = ((n_0 - 1)/|x|)x$. Then by the fact that $1 \leq |y - x| \leq 6$ for $y \in B(x_0, 1)$ and by Assumption 2.1(2), we have

$$\begin{aligned} \frac{v((6, 0, \dots, 0))|B(0, 1)|}{C_2} &\leq \int_{B(x_0, 1)} v(y - x) dy \\ &\leq \int_{R_{n_0}} v(y - x) dy. \end{aligned}$$

By using this estimate and the definition of C , it is immediate to obtain the bound

$$\varphi(x) \leq \|\varphi\|_\infty \leq C \|\varphi\|_\infty \left(\int_{R_{n_0}} v(y - x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}}.$$

Let now $n - 1 < |x| \leq n, n \geq n_0 + 5$. Similarly as before, for all $t > 0$ we have

$$\begin{aligned} (3.12) \quad \varphi(x) &\leq \mathbf{E}^x[\tau_n > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t)] \\ &\quad + \sum_{k=n_0}^{n-2} \sum_{l=1}^\infty \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t)]. \end{aligned}$$

By (3.10), (3.9) and Lemma 3.4(1), we find the following bound for the first expectation in (3.12)

$$\begin{aligned}
 & \mathbf{E}^x[\tau_n > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t)] \\
 & \leq e^{-2(\lambda+C_{22})t} \sup_{|z|>n-2} \varphi(z) \\
 (3.13) \quad & \leq e^{-(\lambda+C_{22})t} C \|\varphi\|_\infty \sup_{|z|>n-2} \left(\int_{R_{n_0}} v(y-z) dy \right)^{\sum_{i=1}^p 2^{-i}} \\
 & \leq e^{-(\lambda+C_{22})t} C C_{28} \|\varphi\|_\infty \left(\int_{R_{n_0}} v(y-x) dy \right)^{\sum_{i=1}^p 2^{-i}}.
 \end{aligned}$$

We now estimate the expectations under the double sum on the right-hand side of (3.12). By using (3.10), (3.9) and the equality $\sum_{i=1}^p 2^{-i} = 1 - 2^{-p}$, we get for $k \geq n_0 + 3$

$$\begin{aligned}
 & \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t)] \\
 & \leq \sup_{|z|>k-2} \varphi(z) \mathbf{E}^x[S(n, k, l, t), \tau_k > t; \\
 (3.14) \quad & e^{-(1/2) \int_0^{\sigma_k} V_0(X_s) ds} e^{-(1/2) \int_0^t V_0(X_s) ds}] \\
 & \leq C e^{-(\lambda+C_{22})t} \|\varphi\|_\infty \sup_{|z|>k-2} \left(\int_{R_{n_0}} v(y-z) dy \right)^{1-2^{-p}} \\
 & \quad \times \mathbf{E}^x[S(n, k, l, t); e^{-(1/2) \int_0^{\sigma_k} V_0(X_s) ds}].
 \end{aligned}$$

By Lemmas 3.4(1) and 3.6, the right-hand side of (3.14) is less or equal to

$$\frac{C C_{31}}{g_{n_0} 2^t} e^{-(\lambda+C_{22})t} \|\varphi\|_\infty \left(C_{28} \inf_{z \in R_k} \int_{R_{n_0}} v(y-z) dy \right)^{1-2^{-p}} \int_{R_k} v(z-x) dz.$$

Furthermore, by Fubini's theorem,

$$\begin{aligned}
 & \inf_{z \in R_k} \int_{R_{n_0}} v(y-z) dy \int_{R_k} v(z-x) dz \\
 & \leq \int_{R_{n_0}} \int_{R_k} v(y-z) v(z-x) dz dy
 \end{aligned}$$

and again by Lemma 3.4(1),

$$\left(C_{28} \inf_{z \in R_k} \int_{R_{n_0}} v(y-z) dy \right)^{-2^{-p}} \leq \left(\int_{R_{n_0}} v(y-x) dy \right)^{-2^{-p}}.$$

Thus, the expectations on the left-hand side of (3.14) for $k \geq n_0 + 3$ are bounded above by

$$(3.15) \quad \frac{CC_{31}C_{28}}{g_{n_0}2^l} \|\varphi\|_\infty e^{-(\lambda+C_{22})t} \left(\int_{R_{n_0}} v(y-x) dy \right)^{-2^{-p}} \\ \times \int_{R_{n_0}} \int_{R_k} v(y-z)v(z-x) dz dy.$$

Similarly, by the strong Markov property, Lemmas 3.6 and 3.4(2), we estimate the expectations for $n_0 \leq k \leq n_0 + 2$ to obtain

$$(3.16) \quad \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V_0(X_s) ds} \varphi(X_t)] \\ \leq \|\varphi\|_\infty e^{\lambda t} \mathbf{E}^x[S(n, k, l, t); e^{-\int_0^{\sigma_k} V_+(X_s) ds} e^{\int_0^{t+\sigma_k} V_-(X_s) ds}] \\ = \|\varphi\|_\infty e^{\lambda t} \mathbf{E}^x[S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} e^{\int_{\sigma_k}^{t+\sigma_k} V_-(X_s) ds}] \\ = \|\varphi\|_\infty e^{\lambda t} \mathbf{E}^x[S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} \mathbf{E}^{X_{\sigma_k}}[e^{\int_0^t V_-(X_s) ds}]] \\ \leq C_{21} \|\varphi\|_\infty e^{(\lambda+C_{22})t} \mathbf{E}^x[S(n, k, l, t); e^{-(1/2)\int_0^{\sigma_k} 2V(X_s) ds}] \\ \leq \frac{C_{21}C_{31}}{g_{n_0}2^l} \|\varphi\|_\infty e^{(\lambda+C_{22})t} \int_{R_k} v(y-x) dy \\ \leq \frac{C_{21}C_{31}C^{(1)}}{g_{n_0}2^l} \|\varphi\|_\infty e^{(\lambda+C_{22})t} \int_{R_{n_0}} v(y-x) dy.$$

Combining the estimates (3.13)–(3.16) and choosing

$$t = -\frac{1}{\lambda + C_{22}} \left(\log(C_{28}C)^{-1} + 2^{-(p+1)} \log\left(C^{(2)} \int_{R_{n_0}} v(y-x) dy\right) \right) > 0$$

in (3.12), we obtain

$$\varphi(x) \leq \|\varphi\|_\infty \left(\left(\int_{R_{n_0}} v(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}} \right. \\ \left. + \frac{C_{31}}{g_{n_0}} \left(\int_{R_{n_0}} v(z-x) dz \right)^{2^{-(p+1)} - 2^{-p}} \right. \\ \left. \times \int_{R_{n_0}} \left(\sum_{k=n_0+3}^{n-2} \int_{R_k} v(y-z)v(y-x) dy \right) dz \right. \\ \left. + \frac{3C_{21}C_{31}CC_{28}C^{(1)}}{C^{(2)}g_{n_0}} \left(\int_{R_{n_0}} v(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}} \right).$$

Finally, by using Assumption 2.1(3) and (3.9), we get

$$\begin{aligned} \varphi(x) &\leq \|\varphi\|_\infty \left(1 + \frac{C_{31}C_3}{g_{n_0}} + \frac{3C_{21}C_{31}CC_{28}C^{(1)}}{C^{(2)}g_{n_0}} \right) \left(\int_{R_{n_0}} v(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}} \\ &\leq 2\|\varphi\|_\infty \left(\int_{R_{n_0}} v(y-x) dy \right)^{\sum_{i=1}^{p+1} 2^{-i}}, \end{aligned}$$

which completes the proof of the theorem. \square

PROOF OF THEOREM 2.2. By integrating in the equality $e^{-\lambda t} \varphi(x) = \mathbf{E}^x[e^{-\int_0^t V(X_s) ds} \varphi(X_t)]$ over $t \in (0, \infty)$, it follows that

$$\varphi(x) = \lambda G^V \varphi(x), \quad x \in \mathbf{R}^d,$$

and by (3.3) applied to $f = \varphi$, $D' = \mathbf{R}^d$, $D = B(x, 1)$, we obtain

$$(3.17) \quad \varphi(x) = \lambda G_D^V \varphi(x) + \mathbf{E}^x[e^{-\int_0^{\tau_D} V(X_s) ds} \varphi(X_{\tau_D})], \quad x \in \mathbf{R}^d.$$

We now prove part (1) of the statement. Let $R > 2$ be large enough so that $V(x) \geq 0$ for $|x| \geq R - 1$ and the assertion of Theorem 2.1 for the λ -subaveraging function $|\varphi|$ holds. Let $|x| \geq R + 2$. By (3.17), we have

$$|\varphi(x)| \leq \lambda G_D^V |\varphi(x)| + \mathbf{E}^x[e_V(\tau_D) |\varphi(X_{\tau_D})|] = \text{I} + \text{II}.$$

By Theorem 2.1 applied to $|\varphi|$, we have

$$\text{I} \leq C G_D^V \mathbf{1}(x) \sup_{y \in D} |\varphi(y)| \leq C \|\varphi\|_\infty G_D^V \mathbf{1}(x) v(x)$$

with $C = C(X, V, \lambda)$. To estimate II first note that by Theorem 2.1, Assumption 2.1(1), (3), the Ikeda–Watanabe formula and the fact that $\sup_{z \in D} \mathbf{E}^z[\tau_D] \leq \mathbf{E}^0[\tau_{B(0,2)}] < \infty$, we have for $z \in D \setminus B(x, 3/4)$ the estimates

$$\begin{aligned} &\mathbf{E}^z[e_V(\tau_D) |\varphi(X_{\tau_D})|] \\ &\leq \mathbf{E}^z[|\varphi(X_{\tau_D})|; X_{\tau_D} \in B(x, 2) \setminus D] + \mathbf{E}^z[|\varphi(X_{\tau_D})|; X_{\tau_D} \in B(x, 2)^c] \\ &\leq C \left(\|\varphi\|_\infty v(x) + \int_D G_D(z, y) \int_{B(x, 2)^c} |\varphi(w)| v(w - y) dw dy \right) \\ (3.18) \quad &\leq C \left(\|\varphi\|_\infty v(x) + \mathbf{E}^z[\tau_D] \int_{B(x, 2)^c} |\varphi(w)| v(w - x) dz \right) \\ &\leq C \|\varphi\|_\infty \left(v(x) + \int_{B(0, R)^c \cap B(x, 2)^c} v(w) v(w - x) dz \right. \\ &\quad \left. + \int_{B(0, R)} v(w - x) dw \right) \\ &\leq C \|\varphi\|_\infty v(x) \end{aligned}$$

with $C = C(X, V, \lambda)$. Thus, by using Corollary 3.1, the above estimate, Theorem 2.1 and Assumption 2.1(1), (3), we finally have

$$\begin{aligned} \Pi &\leq C G_D^V \mathbf{1}(x) \left(\int_{D \cap B(x, 3/4)^c} \mathbf{E}^z [e^V(\tau_D) |\varphi(X_{\tau_D})|] v(z-x) dz \right. \\ &\quad \left. + \int_{D^c} |\varphi(z)| v(z-x) dz \right) \\ &\leq C \|\varphi\|_\infty G_D^V \mathbf{1}(x) \left(v(x) \int_{D \cap B(x, 3/4)^c} v(z-x) dz \right. \\ &\quad \left. + \int_{D^c \cap B(0, R)^c} v(z)v(z-x) dz + \int_{B(0, R)} v(z-x) dz \right) \\ &\leq C \|\varphi\|_\infty G_D^V \mathbf{1}(x) v(x), \end{aligned}$$

where $C = C(X, V, \lambda)$. We conclude that $|\varphi(x)| \leq C_5 \|\varphi\|_\infty G_D^V \mathbf{1}(x) v(x)$ for all $|x| \geq R + 2$, with constant $C_5 = C_5(X, V, \lambda)$.

Now consider part (2) of the statement. Again, by (3.17), strict positivity of φ and Corollary 3.1 we have

$$\varphi(x) \geq \mathbf{E}^x \left[e^{-\int_0^{\tau_D} v(X_s) ds} \varphi(X_{\tau_D}) \right] \geq C_{24}^{-1} G_{B(x, 1)}^V \mathbf{1}(x) \int_{B(0, 1)} \varphi(z) v(x-z) dz, \quad |x| \geq R.$$

By Assumption 2.1(1), the last integral is greater than $C v(x) \int_{B(0, 1)} \varphi(z) dz$ and the required inequality follows again from the positivity of φ with constant $C_6 = C_6(X, \varphi)$. \square

3.4. Eigenfunction estimates.

PROOF OF THEOREM 2.3. Let $\eta \geq 0$ be such that $\lambda_0 + \eta > 0$ and let $n \geq 0$ be fixed. We thus clearly have $\varphi_n(x) = e^{\lambda t} \mathbf{E}^x [e^{V+\eta}(t) \varphi_n(X_t)]$, $x \in \mathbf{R}^d$, with $\lambda = \lambda_n + \eta > \lambda_0 + \eta > 0$, and the result immediately follows from Theorem 2.2(1) for $\varphi = \varphi_n$. \square

PROOF OF THEOREM 2.4. Let $\eta \geq 0$ be such that $\lambda_0 + \eta > 0$. The result directly follows from Theorems 2.3 and 2.2(2) for $\varphi = \varphi_0 > 0$ and $\lambda = \lambda_0 + \eta > 0$. \square

3.5. Intrinsic ultracontractivity.

PROOF OF THEOREM 2.5. First, we prove (1). By a standard argument based on the duality and symmetry of \tilde{T}_t , we have $\|\tilde{T}_t\|_{1 \rightarrow 2} = \|\tilde{T}_t\|_{2 \rightarrow \infty}$.

Since $\|\tilde{T}_{t_0}\|_{2 \rightarrow \infty} < \infty$, by the semigroup property it is straightforward that also $\|\tilde{T}_{2t_0}\|_{1 \rightarrow \infty} \leq \|\tilde{T}_{t_0}\|_{2 \rightarrow \infty}^2$. Hence,

$$\frac{e^{2\lambda_0 t_0}}{\varphi_0(x)} T_{2t_0} \mathbf{1}(x) = \tilde{T}_{2t_0} \left(\frac{1}{\varphi_0} \right) (x) \leq C_{t_0},$$

since by Theorem 2.3 we have $1/\varphi_0 \in L^1(\mathbf{R}^d, \varphi_0^2 dx)$. Hence, the implications t_0 -IUC \Rightarrow $2t_0$ -GSD and IUC \Rightarrow GSD follow. To show that t_0 -GSD \Rightarrow $2t_0$ -IUC ($t_0 \geq t_b$), it suffices to observe that for all $f \in L^2(\mathbf{R}^d, \varphi_0^2 dx)$ we have

$$\begin{aligned} T_{2t_0}(f\varphi_0)(x) &= T_{t_0}T_{t_0}(f\varphi_0)(x) \\ &\leq T_{t_0}\mathbf{1}(x)\|T_{t_0}\|_{2 \rightarrow \infty}\|f\varphi_0\|_2 \\ &\leq C_{t_0}\|f\varphi_0\|_2\varphi_0(x). \end{aligned}$$

The last inequality follows from $\|T_{t_0}\|_{2 \rightarrow \infty} < \infty$, coming from the boundedness of $p(t_0, x)$ in $x \in \mathbf{R}^d$. Moreover, since GSD means t -GSD for all $t > 0$, assertion (2) of the theorem follows again by the latter estimate. \square

LEMMA 3.7. *Let V be a X -Kato class potential, nonnegative outside a bounded subset of \mathbf{R}^d . Let Assumptions 2.1–2.3 be satisfied and consider the following two conditions:*

(1) *There exist $C_{32} = C_{32}(X, V, t)$ and $R > 0$ such that*

$$(3.19) \quad T_t \mathbf{1}(x) \leq C_{32} v(x), \quad |x| \geq R.$$

(2) *There exist $C_{33} = C_{33}(X, V, t)$ and $R > 0$ such that*

$$(3.20) \quad T_t \mathbf{1}(x) \leq C_{33} G_{B(x,1)}^V \mathbf{1}(x) v(x), \quad |x| \geq R.$$

Statements (1) and (2) are equivalent in the following sense. If (2) is true for some $t = s > 0$, then (1) also follows for $t = s$. If (1) holds true for some $t = s > 0$, then (2) follows for $t = 2s$.

PROOF. For the proof of the implication (2) \Rightarrow (1), it suffices to note that there is $R > 0$ large enough such that $G_{B(x,1)}^V \mathbf{1}(x) \leq G_{B(x,1)}^0 \mathbf{1}(x) = \mathbf{E}^x[\tau_{B(x,1)}] = \mathbf{E}^0[\tau_{B(0,1)}] < \infty$ for $|x| \geq R$.

For the converse implication, we suppose that (1) holds for some $t/2 > 0$ and find $R_1 \geq R \vee 2$ such that $V(x) \geq 0$ for $|x| \geq R_1 - 1$. Denote $D = B(x, 1)$ and let $|x| \geq R_1 + 2$. By Lemma 3.2 and Corollary 3.1, we have

$$\begin{aligned} T_t \mathbf{1}(x) &= \mathbf{E}^x \left[\frac{t}{2} < \tau_D; e_V(t) \right] + \mathbf{E}^x \left[\frac{t}{2} \geq \tau_D; e_V(t) \right] \\ &\leq C \left(G_D^V \mathbf{1}(x) \sup_{y \in D} T_{t/2} \mathbf{1}(y) \right) + \mathbf{E}^x [e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D})] \end{aligned}$$

$$\begin{aligned} &\leq CG_D^V \mathbf{1}(x) \left(\sup_{y \in B(x,1)} T_{t/2} \mathbf{1}(y) \right. \\ &\quad + \int_{B(x,1) \cap B(x,3/4)^c} \mathbf{E}^z [e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D})] v(z-x) dz \\ &\quad + \int_{B(x,1)^c \cap B(0,R_1)^c} T_{t/2} \mathbf{1}(z) v(z-x) dz \\ &\quad \left. + \sup_{y \in B(0,R_1)} T_{t/2} \mathbf{1}(y) \int_{B(0,R_1)} v(z-x) dz \right). \end{aligned}$$

Notice that by using (1) and exactly the same arguments as in (3.18) applied to $|\varphi(\cdot)|$ replaced by $T_{t/2} \mathbf{1}(\cdot)$, we get

$$(3.21) \quad \mathbf{E}^z [e_V(\tau_D) T_{t/2} \mathbf{1}(X_{\tau_D})] \leq C v(x), \quad z \in D \cap B(x, 3/4)^c$$

with constant $C = C(X, V, t)$. Thus, by estimate (1), Assumption 2.1(1), (3) and (3.21), we conclude similarly as in the proof of Theorem 2.2 that

$$T_t \mathbf{1}(x) \leq CG_{B(x,1)}^V \mathbf{1}(x) v(x), \quad |x| \geq R_1 + 2$$

with $C = C(X, V, t)$, which completes the proof. \square

THEOREM 3.1. *Let Assumptions 2.1–2.4 hold. If there exist a constant C_{13} and $R > 0$ such that*

$$(3.22) \quad \frac{V(x)}{|\log v(x)|} \geq C_{13}, \quad |x| \geq R,$$

then the bound (3.19) holds for all $t \geq t_0 = 2/C_{13}$. If, moreover,

$$(3.23) \quad \lim_{|x| \rightarrow \infty} \frac{V(x)}{|\log v(x)|} = \infty,$$

then this bound holds for every $t > 0$.

PROOF. First assume that the inequality (3.22) is satisfied for $R > 0$, and denote $t_0 = 2/C_{13}$. Choose $n_0 \geq R$ large enough such that

$$C_2 v((n, 0, \dots, 0)) < 1 \quad \text{and} \quad 2\theta_0 \leq g_n := \inf_{|y| \geq n} V(y) \quad \text{for } n \geq n_0$$

and

$$C_3 C_{28} C_{31} \leq g_{n_0},$$

where C_{31} is the constant and θ_0 is the parameter from Lemma 3.6. Thus, the assumptions of Lemma 3.6 are satisfied. Moreover, by (3.22) and Assumption 2.1(2),

$$(3.24) \quad g_n \geq -C_{13} \log(C_2 v((n, 0, \dots, 0))) \quad \text{for } n \geq n_0.$$

We show that for every $t \geq t_0$ condition (3.19) holds. Let $n - 1 < |x| \leq n$, $n \geq n_0 + 4$ and $t \geq t_0$. We have

$$(3.25) \quad T_t \mathbf{1}(x) \leq \mathbf{E}^x[\tau_n > t; e^{-\int_0^t V(X_s) ds}] + \sum_{k=n_0}^{n-2} \sum_{l=1}^{\infty} \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds}].$$

By (3.24) and Assumption 2.1(1), the first term at the right-hand side can be easily estimated by

$$\mathbf{P}^x(\tau_n > t) e^{-tg_{n-2}} \leq e^{C_{13}t \log(C_2 v((n-2, 0, \dots, 0)))} \leq C_1^2 C_2 v(x).$$

Similar arguments and Lemma 3.6 also yield

$$\begin{aligned} & \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds}] \\ & \leq \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-(1/2)\int_0^{\sigma_k} V(X_s) ds} e^{-(1/2)\int_0^t V(X_s) ds}] \\ & \leq e^{(1/2)C_{13}t \log(C_2 v((k-2, 0, \dots, 0)))} \mathbf{E}^x[S(n, k, l, t); e^{-(1/2)\int_0^{\sigma_k} V(X_s) ds}] \\ & \leq \frac{C_1^2 C_2 C_{31}}{2^l g_{n_0}} \int_{R_k} v(y) v(y - x) dy \end{aligned}$$

for $k \geq n_0 + 2$. For $k \in \{n_0, n_0 + 1\}$ we have

$$\begin{aligned} & \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds}] \\ & \leq \mathbf{E}^x[S(n, k, l, t); e^{-\int_0^{\sigma_k} V_+(X_s) ds} e^{\int_0^{t+\sigma_k} V(X_s) ds}] \\ & = \mathbf{E}^x[S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} e^{\int_{\sigma_k}^{t+\sigma_k} V_-(X_s) ds}] \\ & = \mathbf{E}^x[S(n, k, l, t); e^{-\int_0^{\sigma_k} V(X_s) ds} \mathbf{E}^{X_{\sigma_k}}[e^{\int_0^t V_-(X_s) ds}]] \\ & \leq C_{21} e^{C_{22}t} \mathbf{E}^x[S(n, k, l, t); e^{-(1/2)\int_0^{\sigma_k} 2V(X_s) ds}] \\ & \leq \frac{C_{21} C_{31} e^{C_{22}t}}{2^l g_{n_0}} \int_{R_k} v(y - x) dy \end{aligned}$$

by the strong Markov property, (3.1) and Lemma 3.6. Thus, by Assumption 2.1(1) and (3), the second term at the right-hand side of (3.25) is bounded above by

$$\begin{aligned} & \sum_{k=n_0}^{n-2} \sum_{l=1}^{\infty} \mathbf{E}^x[S(n, k, l, t), \tau_k > t; e^{-\int_0^t V(X_s) ds}] \\ & \leq C \sum_{l=1}^{\infty} 2^{-l} \left(\sum_{k=n_0}^{n_0+1} \int_{R_k} v(y - x) dy + \sum_{k=n_0+2}^{n-2} \int_{R_k} v(y) v(y - x) dy \right) \\ & \leq C v(x), \end{aligned}$$

where $C = C(X, V, t)$, and the first part of the theorem is proved. The second assertion follows from the first part by observing that (3.23) implies (3.22) with arbitrarily large constant C_{13} . \square

PROOF OF THEOREM 2.6. We first prove (1). By Theorems 3.1 and 2.4, Lemma 3.7, there is $R > 0$ such that for all $|x| > R$ condition (2.4) holds with $t_0 = 4/C_{13}$. The same is true for $|x| \leq R$ by boundedness of $T_t \mathbf{1}$, and by continuity and strict positivity of φ_0 .

To prove (2) fix $\varepsilon \in (0, 1]$ and first note that for every $t > 0$, we have that $\mathbf{P}^0(t < \tau_{B(0,\varepsilon)}) > 0$. This positivity property follows from [68] for small $t > 0$ and extends to all $t > 0$. By definition of t_0 -GSD and Theorem 2.4, there is $R > 0$ such that

$$e^{-t_0 \sup_{|y-x|<\varepsilon} V(y)} \mathbf{P}^x(t_0 < \tau_{B(x,\varepsilon)}) \leq T_{t_0} \mathbf{1}(x) \leq C \varphi_0(x) \leq C \nu(x)$$

with $C = C(X, V, t_0)$, for all $|x| > R$. Thus, by the fact that $|\log \nu(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$, we can choose $R_\varepsilon \geq R$ large enough such that

$$\frac{\sup_{|y-x|<\varepsilon} V(y)}{|\log \nu(x)|} \geq \frac{1}{t_0} \left(1 - \frac{\log(C/(\mathbf{P}^0(t_0 < \tau_{B(0,\varepsilon)}))}{|\log \nu(x)|} \right) \geq \frac{1}{2t_0}$$

for $|x| > R_\varepsilon$, which gives the required bound. \square

PROOF OF THEOREM 2.7. To prove (1), observe that for any $t > 0$ we can find $R > 0$ such that $V(x) \geq (4/t)|\log \nu(x)|$ for $|x| > R$, and we can proceed in the same way as in the proof of (1) of Theorem 2.6.

When the semigroup $\{T_t : t \geq 0\}$ is IUC, then by the same arguments as in the proof of Theorem 2.6(2) for every $t > 0$ there is $C = C(X, V, t)$ such that we have

$$\frac{\sup_{|y-x|<\varepsilon} V(y)}{|\log \nu(x)|} \geq \frac{1}{t} \left(1 - \frac{\log(C/(\mathbf{P}^0(t < \tau_{B(0,\varepsilon)}))}{|\log \nu(x)|} \right), \quad |x| > R, t > 0$$

for some $R > 0$. Thus, $\liminf_{|x| \rightarrow \infty} \frac{\sup_{|y-x|<\varepsilon} V(y)}{|\log \nu(x)|} = \frac{1}{t}, t > 0$, which completes the proof. \square

PROOF OF PROPOSITION 2.1. Denote $D = B(x_0, \varepsilon)$ and let $M = \sup_{y \in D} V(y)$. By assumption (1) and Lemma 3.3, $\lim_{y \rightarrow x} p_D(t, x, y) = \infty$ for every $x \in D$. Using this, we can derive that the transition operator of the process $(X_t)_{t \geq 0}$ killed in D , that is, $P_t^D f(x) = \int_D p_D(t, x, y) f(y) dy, f \in L^1(D)$, is not bounded from $L^1(D)$ to $L^\infty(D)$. Since for $f \geq 0$, we have

$$T_t f(x) \geq \mathbf{E}^x \left[e^{-\int_0^t V(X_s) ds} f(X_t); t < \tau_D \right] \geq e^{-Mt} \int_D p_D(t, x, y) f(y) dy, \quad x \in D,$$

this clearly means that T_t is not a bounded operator from $L^1(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$ as well. Thus, also $T_{t/2}$ cannot be a bounded operator from $L^2(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$. \square

PROOF OF THEOREM 2.8. First, assume that the potential V is such that the semigroup $\{T_t : t \geq 0\}$ is t_0 -GSD. From Corollary 2.3(2), we directly derive that there is $R > 0$ such that $2C_{14}t_0V(x)v(x) \geq -v(x)\log v(x)$, for all $x \in B(0, R)^c$. By integrating in this inequality with respect to Lebesgue measure over an arbitrary Borel set $A \subset B(0, R)^c$, we obtain $F_A^{2C_{14}t_0V}(v) \geq 0$.

Consider the converse implication. Since for some $t_0 > 0$, $R > 0$ and any Borel set $A \subset B(0, R)^c$, we have $\int_A v(x)(|\log v(x)| - t_0V(x)) dx \leq 0$, the bound $|\log v(x)| \leq t_0V(x)$ holds for almost every $x \in B(0, R)^c$. By Assumptions 2.1(1) and 2.5, it is immediate to deduce that there is $R_1 > 1$ such that $|\log v(x)| \leq 2C_{14}t_0V(x)$ for all $|x| > R_1$. Again, by Corollary 2.3(2), this implies $8C_{14}t_0$ -GSD of $\{T_t : t \geq 0\}$. \square

4. Discussion of examples.

4.1. *Verification of assumptions for the class of Lévy processes considered.* In the first example below, we show various choices of structure of the Lévy measure ν that satisfy conditions (1)–(3) in Assumption 2.1.

EXAMPLE 4.1. (1) Choosing $\nu(x) \asymp |x|^{-d-\alpha}(1 + |x|)^{\alpha-\beta}$, $x \in \mathbf{R}^d$, for $\alpha \in [0, 2)$ and $\beta > 0$, it can be directly seen that the conditions are verified.

(2) Also, if $\nu(x) \asymp \kappa(|x|)|x|^{-d-\alpha}$, $\alpha \in (0, 2)$, where $\kappa : [0, \infty) \rightarrow (0, 1]$ is a nonincreasing function such that $\kappa(0) = 1$ and $\kappa(a)\kappa(b) \leq C\kappa(a+b)$, $a, b, C > 0$, then all conditions on ν are verified directly. Examples include $\kappa(s) = 1/\log(e+s)$ and $\kappa(s) = 1/(\log(e + \log(1+s)))$.

(3) A case of special interest is $\nu(x) \asymp e^{-a|x|^\beta}|x|^{-d-\delta}(1 + |x|)^{d+\delta-\gamma}$ with $a > 0$, $\beta > 0$, $\delta \in [0, 2)$ and $\gamma > 0$. In this case condition (2) always holds, condition (1) is satisfied when $\beta \in (0, 1]$ without further restriction, and condition (3) is satisfied when moreover $\gamma > (d + 1)/2$.

We also give counterexamples to condition (3) in Assumption 2.1.

EXAMPLE 4.2. For $\beta > 1$ in case (3) of Example 4.1 the condition (3) of Assumption 2.1 is not satisfied. Similarly, at least in one dimension, it fails when $\beta = 1$ and $\gamma = (d + 1)/2$.

In the group of Examples 4.3–4.6 next we discuss specific classes of Lévy processes satisfying all of Assumptions 2.1–2.3.

EXAMPLE 4.3. *Subordinate Brownian motions* with characteristic exponents ψ such that $e^{-t_b\psi(\cdot)} \in L^1(\mathbf{R}^d)$ for some $t_b > 0$, whenever their Lévy measures satisfy Assumption 2.1. Since in this case $\nu(x)$ is radially decreasing, condition (2) of Assumption 2.1 is automatically satisfied, however, not necessarily the remaining

conditions (1) and (3). Condition (1) is always satisfied as long as $\nu(x) \leq C\nu(y)$ for all $|x| \geq 1, |y| = |x| + 1$, while, as seen in Example 4.1, condition (3) strongly depends on the specific form of the Lévy measure (in fact, the Lévy measure of the subordinator). The transition densities $p(t, y - x)$ are given by the subordination formula, that is, by the integral over time of the Brownian transition kernel with respect to the distribution of the given subordinator. Since also $e^{-t_b\psi(\cdot)}$ is integrable, Assumption 2.2 is satisfied [in particular, $(X_t)_{t \geq 0}$ is not a compound Poisson process]. Lastly, Assumption 2.3 follows from a similar bound for the potential or λ -potential kernel, which is again a consequence of the subordination formula and an easy estimate. Below we give specific examples of subordinate Brownian motion of special interest satisfying all of our assumptions. For properties of subordinate Brownian motion and further examples, see [8, 13, 50, 73].

(1) *Rotationally symmetric α -stable process.* Let $\psi(\xi) = |\xi|^\alpha, \alpha \in (0, 2)$. In this case, $\nu(x) = C(\alpha)|x|^{-d-\alpha}$.

(2) *Mixture of independent rotationally symmetric stable processes with indices α and β .* This is obtained for $\psi(\xi) = a|\xi|^\alpha + b|\xi|^\beta, 0 < \beta < \alpha < 2, a, b > 0$. We have $\nu(x) = aC(\alpha)|x|^{-d-\alpha} + bC(\beta)|x|^{-d-\beta}$.

(3) *Jump-diffusion process* [21, 25]. Let $\psi(\xi) = a|\xi|^\alpha + b|\xi|^2, 0 < \alpha < 2, a, b > 0$, that is, the process is a mixture of a rotationally symmetric α -stable process and an independent Brownian motion. In this case $\nu(x) \asymp aC(\alpha)|x|^{-d-\alpha}$.

(4) *Rotationally symmetric geometric α -stable process* [38, 74]. Let $\psi(\xi) = \log(1 + |\xi|^\alpha), 0 < \alpha < 2$. In this case, $\nu(x) \asymp |x|^{-d}(1 + |x|)^{-\alpha}$. Notice that ψ is now a slowly varying function at infinity. In contrast to the previous examples, in this case there is $t(\alpha) > 0$ for which the transition probability densities are unbounded for $0 < t < t(\alpha)$ ([13], page 117), though they are bounded for large t .

(5) *Relativistic rotationally symmetric α -stable process* [22, 71]. Let $\psi(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m, \alpha \in (0, 2), m > 0$. It is known that $\nu(x) \asymp e^{-m^{1/\alpha}|x|}|x|^{-d-\alpha}(1 + |x|^{(d+\alpha-1)/2})$ [57] [we take $a = m^{1/\alpha}, \beta = 1, \delta = \alpha, \gamma = (d + \alpha + 1)/2$ in (3) of Example 4.1].

EXAMPLE 4.4. *Symmetric Lévy processes with nondegenerate Brownian part* [52]. Let $(X_t)_{t \geq 0}$ be a Lévy process with characteristic exponent $\psi(\xi) = c|\xi|^2 + \int_{\mathbf{R}^d} (1 - \cos(z \cdot \xi))\nu(dz), c > 0$, that is, a sum of Brownian motion with rescaled time $(B_{2ct})_{t \geq 0}$ and an independent symmetric Lévy process $(Y_t)_{t \geq 0}$ with Lévy measure ν satisfying Assumption 2.1. In this case, the transition densities are given by the convolution of the Gaussian kernel and the distribution of the process $(Y_t)_{t \geq 0}$ [note that we do not need to assume that $(Y_t)_{t \geq 0}$ has transition densities] or by the Fourier inversion formula. They are clearly bounded for all $t > 0$, thus Assumption 2.2 also is satisfied. In one dimension, Assumption 2.3 easily follows from a similar bound for the corresponding λ -potential kernel. In higher dimensions, the required upper bound for the λ -potential kernel can be proved by showing that, for instance, for every $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that

$$(4.1) \quad \mathbf{P}^0(Y_t \in A) \leq C|A| \quad \text{whenever } A \subset \mathcal{B}(\mathbf{R}^d), \text{dist}(0, A) > \varepsilon, t \in (0, t_\varepsilon)$$

(here $|A|$ denotes Lebesgue measure of A) with a constant $C = C(Y, \varepsilon)$ independent of t and the specific A . Specific cases are jump-diffusions as above, and many similar processes in which the rotationally symmetric stable process is replaced by other symmetric Lévy processes $(Y_t)_{t \geq 0}$ satisfying Assumption 2.1 and condition (4.1).

EXAMPLE 4.5. *Symmetric stable-like Lévy processes* [23]. Let $\alpha \in (0, 2)$ and $(X_t)_{t \geq 0}$ be a purely jump (i.e., with no diffusion part) symmetric Lévy process with intensity $\nu(x) \asymp C|x|^{-d-\alpha}$, $x \in \mathbf{R}^d$. It is known [23] that $(X_t)_{t \geq 0}$ has bounded continuous transition probability densities $p(t, y - x) \asymp t^{-d/\alpha} \wedge t|y - x|^{-d-\alpha}$, $t > 0$, $x, y \in \mathbf{R}^d$. In this case Assumptions 2.1 and 2.2 are clearly satisfied, while Assumption 2.3 is an easy consequence of a similar bound for the potential kernel ($\alpha < d$) or the λ -potential kernel ($1 = d \leq \alpha < 2$) of $(X_t)_{t \geq 0}$. This class includes a subclass of strictly stable Lévy processes with intensities of the form $|x|^{-d-\alpha} f(x/|x|)$ with functions $f(x) = f(-x)$ that are bounded from above and below by positive constants [15].

EXAMPLE 4.6. *Symmetric Lévy processes with subexponentially localized Lévy measures*. Let $(X_t)_{t \geq 0}$ be a symmetric Lévy process with intensity $\nu(x) \asymp e^{-a|x|^\beta} |x|^{-d-\delta} (1 + |x|)^{d+\delta-\gamma}$, where $a > 0$, $\beta \in (0, 1]$, $\delta \in [0, 2)$ and $\gamma > (d + 1)/2$. Such processes were considered in more general settings in [19] (see also [24] and references therein). As discussed in Example 4.1(3), Assumption 2.1 is verified. Moreover, as proved in a greater generality in [19], Theorem 1.2(1), $(X_t)_{t \geq 0}$ is a strong Feller process with bounded continuous transition densities satisfying appropriate sharp two-sided bounds with respect to large and small times separately (see [19], (1.13) and (1.14)). Thus, also Assumption 2.2 holds. As before, the required bound on the Green function in Assumption 2.3 may be obtained by showing the same estimate for the λ -potential kernel of the process $(X_t)_{t \geq 0}$, which can be easily done by using the sharp transition density estimates referred to above. This class includes a large family of (exponentially) tempered symmetric stable processes [70] [$a > 0$, $\beta = 1$, $\gamma = \delta + d$, $\delta \in (0, 2)$] and the rotationally symmetric relativistic stable processes above.

We also give two examples of processes, which do not satisfy some of our assumptions.

EXAMPLE 4.7. (1) *Rotationally symmetric geometric 2-stable (gamma variance) process*. Let $\psi(\xi) = \log(1 + |\xi|^2)$. In this case the Lévy intensity is $\nu(x) \asymp |x|^{-d} e^{-|x|} (1 + |x|)^{(d-1)/2}$. As in Example 4.2, at least in dimension one, the Lévy measure does not satisfy condition (3) of Assumption 2.1.

(2) *Iterated rotationally symmetric geometric α -stable process*. Let $\psi(\xi) = \log(1 + \log^\alpha(1 + |\xi|^\alpha))$, $0 < \alpha < 2$. It can be checked directly that the transition densities are unbounded for any $t > 0$ (see [13], page 117) and the second part of Assumption 2.2 fails.

4.2. *Decay of ground state and intrinsic ultracontractivity-type properties.* It is useful to see how Theorem 2.4 translates to particular cases of processes. In the following, we give explicit examples of ground state decays and compare our results with others.

EXAMPLE 4.8. (1) *Rotationally symmetric non-Gaussian stable and related processes* discussed in Examples 4.3(1)–(4) and 4.5 above. In particular, this includes mixtures of two stable processes with different stability indices, jump-diffusion, rotationally symmetric geometric α -stable [with $\alpha \in (0, 2)$], and symmetric stable-like Lévy processes. In this case, we have $\nu(x) \asymp |x|^{-d-\alpha}$, $|x| > 1/2$, $\alpha \in (0, 2)$, and hence

$$\varphi_0(x) \asymp G_{B(x,1)}^V \mathbf{1}(x) |x|^{-d-\alpha}, \quad |x| > R.$$

Furthermore, when also the condition in Corollary 2.2 is satisfied, then

$$\varphi_0(x) \asymp \frac{1}{(1 + |x|)^{d+\alpha}(1 + V_+(x))}, \quad x \in \mathbf{R}^d.$$

This clearly recovers the results for non-Gaussian symmetric stable processes in [46, 47].

(2) *Symmetric Lévy process with nondegenerate Brownian part.* For this, see Example 4.4 above. In this case, Theorems 2.3 and 2.4 (also Corollary 2.2) allow to identify the leading order of decay of the ground state at infinity (as well as provide upper bounds for higher order eigenfunctions) as the contribution of the Lévy intensity ν and a correction from the potential V . However, since our constants are not optimal, it can be expected that the correct asymptotics should contain a further term of smaller order (similar to Carmona’s bound in [17]) coming from the Brownian component of the process. However, showing this requires a more subtle argument and cannot be seen from our present results.

(3) *Symmetric jump Lévy processes with exponentially localized Lévy measure.* See Example 4.6 above. It is a well-known result in [18], Proposition IV.4, that if $e^{-t\psi(\cdot)} \in L^1(\mathbf{R}^d)$, $t > 0$, and there is $b > 0$ such that $\int_{|x|>1} e^{b|x|} \nu(dx) < \infty$, then $|\varphi_n(x)| \leq C e^{-C'|x|}$, $x \in \mathbf{R}^d$, with $C = C(X, V, n)$, $C' = C'(X, V, n)$, that is, if the Lévy measure is exponentially localized, then the fall-off of the corresponding eigenfunctions is also exponential. Note that Theorems 2.3–2.4 (also Corollary 2.2) essentially improve this result under assumptions which are not more significantly restrictive than those of [18].

(4) *Rotationally symmetric relativistic stable process.* Compare Example 4.3(5). This is a special case of the class discussed in (3) above. It was proven in [57], Theorem 1.6, that if V is a nonnegative, locally bounded potential comparable to a rotationally symmetric function, radially nondecreasing and comparable on unit balls with $\lim_{|x| \rightarrow \infty} V(x)/|x| = \infty$ (i.e., the corresponding Feynman–Kac semigroup is IUC), then

$$\varphi_0(x) \asymp \frac{e^{-m^{1/\alpha}|x|}}{(1 + |x|)^{(d+\alpha+1)/2}(1 + V(x))}, \quad x \in \mathbf{R}^d.$$

Theorem 2.4 above generalizes this result to the substantially larger space of X -Kato class potentials with no restrictions on the order of growth of the potential at infinity and with no use being made of intrinsic ultracontractivity properties. We obtain the result of [57] as the second part of Corollary 2.2. Note also that Theorem 2.3 is completely new in this context.

(5) *Diffusions.* It is useful to compare our results to the classic facts known for the eigenfunctions of Feynman–Kac semigroups involving Brownian motion (i.e., Schrödinger semigroups generated by $-\Delta + V$). In the general case, t_0 -GSD and t_0 -IUC (see the definitions in Section 2.4) imply directly that for all $x \in \mathbf{R}^d$ and $n \geq 1$

$$(4.2) \quad \begin{aligned} |\varphi_n(x)| &\leq C(t_0)\|\varphi_n\|_\infty e^{(\lambda_n - \lambda_0)t_0} \varphi_0(x) \quad \text{and} \\ |\varphi_n(x)| &\leq C(t_0)e^{(\lambda_n - \lambda_0)t_0} \varphi_0(x), \end{aligned}$$

respectively. In Corollary 2.1, we get without any use of AGSD/AIUC-type properties that for a large class of jump Lévy processes

$$(4.3) \quad |\varphi_n(x)| \leq C(X, V, n)\varphi_0(x), \quad x \in \mathbf{R}^d, n \geq 1$$

with a constant $C(X, V, n)$. In comparison with (4.2), the dependence on n of the constant $C(X, V, n)$ is rather implicit, but (4.3) still says that for each fixed n the decay of φ_n at infinity is dominated by that of φ_0 . This markedly differs from the diffusive case, where the estimates as in (4.3) cannot be taken for granted in lack of AGSD/AIUC-type properties. This can be seen, for instance, in the example of the harmonic oscillator, for which $V(x) = |x|^2$ and the eigenfunctions are given by the Hermite functions. It is known that in this case the semigroup is not IUC (not even AIUC). A direct analysis shows that (4.3) does not occur either.

We now illustrate our results on intrinsic ultracontractivity-type properties by specific examples.

EXAMPLE 4.9. Theorems 2.6–2.7 and Propositions 2.3–2.4 apply directly to the following three classes of examples with different growth rate of borderline potentials. For any of these specific examples, Assumption 2.6 can be verified by using the time–space estimates of the related transition densities, as in Examples 4.3–4.6.

(1) *Borderline potentials of logarithmic order.* The borderline behavior

$$-\log v(x) \asymp -\log \mathbf{P}^x(X_t \in B(0, 1)) \asymp -\log p(t, x) \asymp \log |x|$$

occurs in the following cases:

(a) *Jump stable-type processes with bounded transition densities* [see Examples 4.3(1)–(3) and 4.5]: this includes non-Gaussian rotationally symmetric stable processes (our results recover and substantially improve the methods of

[46, 47]), mixtures of rotationally symmetric stable processes, jump-diffusions [in this case the Brownian component has no effect on (A)GSD and (A)IUC], and symmetric stable-like Lévy processes. In this case, Theorem 2.5 implies equivalence of GSD and IUC.

- (b) *Rotationally symmetric geometric α -stable processes*, $\alpha \in (0, 2)$ [see Example 4.3(4)]: by Proposition 2.1 the semigroup $\{T_t : t \geq 0\}$ is not IUC (i.e., it is not t -IUC for small $t > 0$, not even ultracontractive), while it is GSD and t -IUC for $t > 2t_b$ provided the potential V is pinning enough.

(2) *Borderline potentials of linear order*. The borderline behavior

$$-\log \nu(x) \asymp -\log \mathbf{P}^x(X_t \in B(0, 1)) \asymp -\log p(t, x) \asymp |x|$$

occurs for rotationally symmetric Lévy processes satisfying our assumptions provided that their Lévy intensities are exponentially decaying at infinity (the case $\beta = 1$ in Example 4.6). Important examples to this class are rotationally symmetric relativistic stable processes and tempered rotationally symmetric stable processes. For relativistic stable processes, it was proven in [57] that when V is a nonnegative, locally bounded potential comparable to a function which is rotationally symmetric, radially nondecreasing and comparable on unit balls, then the corresponding Feynman–Kac semigroup is IUC if and only if $\lim_{|x| \rightarrow \infty} V(x)/|x| = \infty$. The combination of Theorems 2.5 and 2.7 generalizes this result to the substantially larger class of X -Kato class potentials. Also, we obtain the result of [57] in Corollary 2.3(1) under Assumption 2.5.

(3) *Borderline potentials of sublinear but faster than logarithmic order*. The borderline behavior

$$-\log \nu(x) \asymp -\log \mathbf{P}^x(X_t \in B(0, 1)) \asymp -\log p(t, x) \asymp |x|^\beta, \quad 0 < \beta < 1,$$

appears in the case of processes with Lévy measures decaying subexponentially at infinity (the case $\beta < 1$ in Example 4.6).

Note that, roughly speaking, this is the complete range of possible borderline growths for the processes we consider. An asymptotic growth of the order $|x|^\beta$ with $\beta > 1$ is ruled out by Assumption 2.1(3), see the discussion in Example 4.2. Also, the borderline potential cannot be slower than logarithmic due to the integrability of the Lévy intensity ν outside a neighborhood of the origin. We also note that linear growth is the quickest possible as well for the class of subordinate Brownian motions obtained under subordinators whose Lévy exponents are complete Bernstein functions, see [50], Lemma 2.1.

A second type of example is about Feynman–Kac (in fact, Schrödinger) semigroups involving standard Brownian motion under a potential. Although the strictly diffusive case when $\nu \equiv 0$ is not covered by our paper, it is interesting to compare the results to better understand what mechanism lies behind IUC in the general case.

EXAMPLE 4.10. *Diffusions.* In the classic papers on IUC of Schrödinger semigroups generated by $-\Delta + V$, it was considered whether the property holds for some special ways of choosing the potential. In the one-dimensional case [32], Theorem 6.1, shows that when $V(x) = |x|^a$, $a > 0$, or $V(x) = |x|^2 \log(|x| + 2)^b$, $b > 0$, then the related semigroup is IUC if and only if $a > 2$ and $b > 2$ (i.e., fails for $a, b \leq 2$). When $d \geq 1$, it is shown in [32], Theorem 6.3, that IUC occurs whenever $C_1 + C_2|x|^b \leq V(x) \leq C_3 + C_4|x|^a$, with $a/2 + 1 < b$. To the best of our knowledge, AIUC was not considered before the paper [47]. However, by a use of the Mehler formula it follows that in the case of the harmonic oscillator AIUC does not occur, see [31], Theorem 4.3.2. Recently, in [1] a general sufficient condition for IUC was found for Schrödinger semigroups. For radial potentials V , this condition is also necessary and it is formulated as

$$(4.4) \quad \int_{r_0}^{\infty} \frac{1}{\sqrt{V(r)}} dr < \infty \quad \text{for some } r_0 > 0.$$

For instance, for the potential

$$V(x) = |x|^2 (\log |x|)^2 (\log \log |x|)^2 \cdots \underbrace{(\log \cdots \log |x|)^2}_{(m-1)\text{-times}} \underbrace{(\log \cdots \log |x|)^{2+\delta}}_{m\text{-times}},$$

$m \in \mathbf{N}, \delta \geq 0,$

this condition is satisfied if and only if $\delta > 0$. This means that IUC holds for an arbitrary choice of $m \in \mathbf{N}$ whenever $\delta > 0$, and suggests that in the diffusion case it is not possible to identify the borderline potential directly as in the case of jump processes. Note that all of the classic results discussed above were obtained by purely analytic arguments. We believe that it is possible to derive the analytic condition (4.4) by probabilistic methods based on sufficiently efficient estimates of the expression at the right-hand side of (2.8). For instance, when $V \geq 0$ satisfies Assumption 2.5 and the semigroup is IUC, then a rough estimate gives $\lim_{|x| \rightarrow \infty} V(x)/|\log \mathbf{P}^x(X_t \in D)| = \infty$, allowing correctly to identify $|x|^2$ as the leading order of borderline growth, as in Proposition 2.4(1).

In the context of diffusions, we also mention that a condition similar to part (3) of Assumption 2.1 has been used for Green functions of elliptic differential operators on domains in [63, 65, 77] and related papers, and it goes back to [66, 67], where it was introduced as a small-perturbation condition of an elliptic operator by another operator. In particular, it is shown that intrinsic ultracontractivity implies the small-perturbation condition.

Acknowledgments. The authors thank IHES, Bures-sur-Yvette, for a visiting fellowship providing an ideal environment to joint work. We are pleased to thank T. Kulczycki for discussions on intrinsic ultracontractivity and eigenfunction estimates. Our special appreciation goes to M. Kwaśnicki for reading a preliminary

version of the manuscript, and for many valuable comments and discussions on intrinsic ultracontractivity and estimates of local extrema of harmonic functions. We also thank the anonymous referee for a careful reading of the manuscript and observations which improved our paper.

REFERENCES

- [1] ALZIARY, B. and TAKÁČ, P. (2009). Intrinsic ultracontractivity of a Schrödinger semigroup in \mathbb{R}^N . *J. Funct. Anal.* **256** 4095–4127. [MR2521921](#)
- [2] APPLEBAUM, D. (2009). *Lévy Processes and Stochastic Calculus*, 2nd ed. Cambridge Univ. Press, Cambridge. [MR2512800](#)
- [3] BAÑUELOS, R. (1991). Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators. *J. Funct. Anal.* **100** 181–206. [MR1124298](#)
- [4] BAÑUELOS, R. and DAVIS, B. (1992). A geometrical characterization of intrinsic ultracontractivity for planar domains with boundaries given by the graphs of functions. *Indiana Univ. Math. J.* **41** 885–913. [MR1206335](#)
- [5] BAÑUELOS, R. and YOLCU, S. Y. (2013). Heat trace of nonlocal operators. *J. Lond. Math. Soc.* (2) **87** 304–318. [MR3022718](#)
- [6] BERG, C. and FORST, G. (1975). *Potential Theory on Locally Compact Abelian Groups*. Springer, New York. [MR0481057](#)
- [7] BERTOIN, J. (1996). *Lévy Processes*. Cambridge Univ. Press, Cambridge. [MR1406564](#)
- [8] BERTOIN, J. (1999). Subordinators: Examples and applications. In *École d'Été de Probabilités de St. Flour XXVII* (P. Bernard, ed.). *LNM* **1717** 4–79. Springer, Berlin.
- [9] BETZ, V. and LÓRINCZI, J. (2003). Uniqueness of Gibbs measures relative to Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **39** 877–889. [MR1997216](#)
- [10] BLUMENTHAL, R. M. and GETTOOR, R. K. (1968). *Markov Processes and Potential Theory*. Academic Press, New York. [MR0264757](#)
- [11] BOGDAN, K. and BYCZKOWSKI, T. (1999). Potential theory for the α -stable Schrödinger operator on bounded Lipschitz domains. *Studia Math.* **133** 53–92. [MR1671973](#)
- [12] BOGDAN, K. and BYCZKOWSKI, T. (2000). Potential theory of Schrödinger operator based on fractional Laplacian. *Probab. Math. Statist.* **20** 293–335. [MR1825645](#)
- [13] BOGDAN, K., BYCZKOWSKI, T., KULCZYCKI, T., RYZNAR, M., SONG, R. and VONDRAČEK, Z. (2009). *Potential Analysis of Stable Processes and Its Extensions* (P. GRACZYK, and A. STÓS, eds.). *Lecture Notes in Math.* **1980**. Springer, Berlin. [MR2569321](#)
- [14] BOGDAN, K., KUMAGAI, T. and KWAŚNICKI, M. (2015). Boundary Harnack inequality for Markov processes with jumps. *Trans. Amer. Math. Soc.* **367** 477–517. [MR3271268](#)
- [15] BOGDAN, K. and SZTONYK, P. (2007). Estimates of the potential kernel and Harnack's inequality for the anisotropic fractional Laplacian. *Studia Math.* **181** 101–123. [MR2320691](#)
- [16] BURDZY, K. and KULCZYCKI, T. (2003). Stable processes have thorns. *Ann. Probab.* **31** 170–194. [MR1959790](#)
- [17] CARMONA, R. (1978). Pointwise bounds for Schrödinger eigenstates. *Comm. Math. Phys.* **62** 97–106. [MR0505706](#)
- [18] CARMONA, R., MASTERS, W. C. and SIMON, B. (1990). Relativistic Schrödinger operators: Asymptotic behavior of the eigenfunctions. *J. Funct. Anal.* **91** 117–142. [MR1054115](#)
- [19] CHEN, Z.-Q., KIM, P. and KUMAGAI, T. (2011). Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.* **363** 5021–5055. [MR2806700](#)
- [20] CHEN, Z.-Q., KIM, P. and SONG, R. (2011). Stability of Dirichlet heat kernel estimates for non-local operators under Feynman–Kac perturbation. *Trans. Amer. Math. Soc.* **367** 5237–5270. [MR3335416](#)

- [21] CHEN, Z.-Q., KIM, P. and SONG, R. (2011). Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets. *J. Lond. Math. Soc. (2)* **84** 58–80. [MR2819690](#)
- [22] CHEN, Z.-Q., KIM, P. and SONG, R. (2012). Sharp heat kernel estimates for relativistic stable processes in open sets. *Ann. Probab.* **40** 213–244. [MR2917772](#)
- [23] CHEN, Z.-Q. and KUMAGAI, T. (2003). Heat kernel estimates for stable-like processes on d -sets. *Stochastic Process. Appl.* **108** 27–62. [MR2008600](#)
- [24] CHEN, Z.-Q. and KUMAGAI, T. (2008). Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Related Fields* **140** 277–317. [MR2357678](#)
- [25] CHEN, Z.-Q. and KUMAGAI, T. (2010). A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps. *Rev. Mat. Iberoam.* **26** 551–589. [MR2677007](#)
- [26] CHEN, Z.-Q. and SONG, R. (1997). Intrinsic ultracontractivity and conditional gauge for symmetric stable processes. *J. Funct. Anal.* **150** 204–239. [MR1473631](#)
- [27] CHEN, Z.-Q. and SONG, R. (2000). Intrinsic ultracontractivity, conditional lifetimes and conditional gauge for symmetric stable processes on rough domains. *Illinois J. Math.* **44** 138–160. [MR1731385](#)
- [28] CHEN, Z.-Q. and SONG, R. (2002). General gauge and conditional gauge theorems. *Ann. Probab.* **30** 1313–1339. [MR1920109](#)
- [29] CHUNG, K. L. and WALSH, J. B. (2005). *Markov Processes, Brownian Motion, and Time Symmetry*, 2nd ed. Springer, New York. [MR2152573](#)
- [30] CHUNG, K. L. and ZHAO, Z. X. (1995). *From Brownian Motion to Schrödinger's Equation*. Springer, Berlin. [MR1329992](#)
- [31] DAVIES, E. B. (1989). *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge. [MR0990239](#)
- [32] DAVIES, E. B. and SIMON, B. (1984). Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59** 335–395. [MR0766493](#)
- [33] DAVIS, B. (1991). Intrinsic ultracontractivity and the Dirichlet Laplacian. *J. Funct. Anal.* **100** 162–180. [MR1124297](#)
- [34] FRANK, R. L., LIEB, E. H. and SEIRINGER, R. (2008). Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators. *J. Amer. Math. Soc.* **21** 925–950. [MR2425175](#)
- [35] FRISTEDT, B. (1974). Sample functions of stochastic processes with stationary, independent increments. In *Advances in Probability and Related Topics* **3** 241–396. Dekker, New York. [MR0400406](#)
- [36] GARBACZEWSKI, P., STEPANOVICH, V. and KĘDZIERSKI, D. (2011). Heavy-tailed targets and (ab)normal asymptotics in diffusive motion. *Phys. A* **390** 990–1008.
- [37] GRZYWNY, T. (2008). Intrinsic ultracontractivity for Lévy processes. *Probab. Math. Statist.* **28** 91–106. [MR2445505](#)
- [38] GRZYWNY, T. and RYZNAR, M. (2012). Potential theory of one-dimensional geometric stable processes. *Colloq. Math.* **129** 7–40. [MR3007664](#)
- [39] HAIRER, M., STUART, A. M. and VOSS, J. (2007). Analysis of SPDEs arising in path sampling. II. The nonlinear case. *Ann. Appl. Probab.* **17** 1657–1706. [MR2358638](#)
- [40] HAIRER, M., STUART, A. M., VOSS, J. and WIBERG, P. (2005). Analysis of SPDEs arising in path sampling. I. The Gaussian case. *Commun. Math. Sci.* **3** 587–603. [MR2188686](#)
- [41] HIROSHIMA, F., ICHINOSE, T. and LÓRINCZI, J. (2012). Path integral representation for Schrödinger operators with Bernstein functions of the Laplacian. *Rev. Math. Phys.* **24** 1250013, 40. [MR2944475](#)
- [42] HIROSHIMA, F. and LÓRINCZI, J. (2012). Lieb–Thirring bound for Schrödinger operators with Bernstein functions of the Laplacian. *Commun. Stoch. Anal.* **6** 589–602. [MR3034006](#)

- [43] IKEDA, N. and WATANABE, S. (1962). On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. *J. Math. Kyoto Univ.* **2** 79–95. [MR0142153](#)
- [44] JACOB, N. (2003–2005). *Pseudo Differential Operators and Markov Processes: Markov Processes and Applications, Vols. 1–3*. Imperial College Press, London.
- [45] KALETA, K. (2012). Spectral gap lower bound for the one-dimensional fractional Schrödinger operator in the interval. *Studia Math.* **209** 267–287. [MR2944472](#)
- [46] KALETA, K. and KULCZYCKI, T. (2010). Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians. *Potential Anal.* **33** 313–339. [MR2726901](#)
- [47] KALETA, K. and LŐRINCZI, J. (2012). Fractional $P(\phi)_1$ -processes and Gibbs measures. *Stochastic Process. Appl.* **122** 3580–3617. [MR2956118](#)
- [48] KIM, P. and SONG, R. (2008). Intrinsic ultracontractivity of nonsymmetric diffusions with measure-valued drifts and potentials. *Ann. Probab.* **36** 1904–1945. [MR2440927](#)
- [49] KIM, P. and SONG, R. (2009). Intrinsic ultracontractivity for nonsymmetric Lévy processes. *Forum Math.* **21** 43–66. [MR2494884](#)
- [50] KIM, P., SONG, R. and VONDRAČEK, Z. (2012). Potential theory of subordinate Brownian motions revisited. In *Stochastic Analysis and Applications to Finance. Interdiscip. Math. Sci.* **13** 243–290. World Sci. Publ., Hackensack, NJ. [MR2986850](#)
- [51] KIM, P., SONG, R. and VONDRAČEK, Z. (2012). Uniform boundary Harnack principle for rotationally symmetric Lévy processes in general open sets. *Sci. China Math.* **55** 2317–2333. [MR2994122](#)
- [52] KIM, P., SONG, R. and VONDRAČEK, Z. (2013). Potential theory of subordinate Brownian motions with Gaussian components. *Stochastic Process. Appl.* **123** 764–795. [MR3005005](#)
- [53] KNOBLOCH, R. and PARTZSCH, L. (2010). Uniform conditional ergodicity and intrinsic ultracontractivity. *Potential Anal.* **33** 107–136. [MR2658978](#)
- [54] KNOPOVA, V. and SCHILLING, R. L. (2013). A note on the existence of transition probability densities of Lévy processes. *Forum Math.* **25** 125–149. [MR3010850](#)
- [55] KULCZYCKI, T. (1998). Intrinsic ultracontractivity for symmetric stable processes. *Bull. Pol. Acad. Sci. Math.* **46** 325–334. [MR1643611](#)
- [56] KULCZYCKI, T. (2013). Gradient estimates of q -harmonic functions of fractional Schrödinger operator. *Potential Anal.* **39** 69–98. [MR3065315](#)
- [57] KULCZYCKI, T. and SIUDEJA, B. (2006). Intrinsic ultracontractivity of the Feynman–Kac semigroup for relativistic stable processes. *Trans. Amer. Math. Soc.* **358** 5025–5057. [MR2231884](#)
- [58] KWAŚNICKI, M. (2009). Intrinsic ultracontractivity for stable semigroups on unbounded open sets. *Potential Anal.* **31** 57–77. [MR2507446](#)
- [59] LIEB, E. H. and SEIRINGER, R. (2010). *The Stability of Matter in Quantum Mechanics*. Cambridge Univ. Press, Cambridge. [MR2583992](#)
- [60] LŐRINCZI, J., HIROSHIMA, F. and BETZ, V. (2011). *Feynman–Kac-Type Theorems and Gibbs Measures on Path Space. With Applications to Rigorous Quantum Field Theory. de Gruyter Studies in Mathematics* **34**. de Gruyter, Berlin. [MR2848339](#)
- [61] LŐRINCZI, J. and MAŁECKI, J. (2012). Spectral properties of the massless relativistic harmonic oscillator. *J. Differential Equations* **253** 2846–2871. [MR2964646](#)
- [62] MÉNDEZ-HERNÁNDEZ, P. J. (2000). Toward a geometric characterization of intrinsic ultracontractivity for Dirichlet Laplacians. *Michigan Math. J.* **47** 79–99. [MR1755257](#)
- [63] MENDEZ-HERNANDEZ, P. J. and MURATA, M. (2009). Semismall perturbations, semi-intrinsic ultracontractivity, and integral representations of nonnegative solutions for parabolic equations. *J. Funct. Anal.* **257** 1799–1827. [MR2540992](#)
- [64] MURATA, M. (2002). Martin boundaries of elliptic skew products, semismall perturbations, and fundamental solutions of parabolic equations. *J. Funct. Anal.* **194** 53–141. [MR1929139](#)

- [65] MURATA, M. (2007). Integral representations of nonnegative solutions for parabolic equations and elliptic Martin boundaries. *J. Funct. Anal.* **245** 177–212. [MR2310806](#)
- [66] PINCHOVER, Y. (1989). Criticality and ground states for second-order elliptic equations. *J. Differential Equations* **80** 237–250. [MR1011149](#)
- [67] PINCHOVER, Y. (1999). Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations. *Math. Ann.* **314** 555–590. [MR1704549](#)
- [68] PRUITT, W. E. (1981). The growth of random walks and Lévy processes. *Ann. Probab.* **9** 948–956. [MR0632968](#)
- [69] REED, M. and SIMON, B. (1980). *Methods of Modern Mathematical Physics. Analysis of Operators* **4**. Academic Press, New York.
- [70] ROSIŃSKI, J. (2007). Tempering stable processes. *Stochastic Process. Appl.* **117** 677–707. [MR2327834](#)
- [71] RYZNAR, M. (2002). Estimates of Green function for relativistic α -stable process. *Potential Anal.* **17** 1–23. [MR1906405](#)
- [72] SATO, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press, Cambridge. [MR1739520](#)
- [73] SCHILLING, R. L., SONG, R. and VONDRAČEK, Z. (2010). *Bernstein Functions*. de Gruyter, Berlin. [MR2598208](#)
- [74] ŠIKIĆ, H., SONG, R. and VONDRAČEK, Z. (2006). Potential theory of geometric stable processes. *Probab. Theory Related Fields* **135** 547–575. [MR2240700](#)
- [75] SIMON, B. (2005). *Functional Integration and Quantum Physics*, 2nd ed. AMS Chelsea Publishing, Providence, RI. [MR2105995](#)
- [76] SONG, R. (2006). Two-sided estimates on the density of the Feynman–Kac semigroups of stable-like processes. *Electron. J. Probab.* **11** 146–161. [MR2217813](#)
- [77] TOMISAKI, M. (2007). Intrinsic ultracontractivity and small perturbation for one-dimensional generalized diffusion operators. *J. Funct. Anal.* **251** 289–324. [MR2353708](#)
- [78] ZHAO, Z. (1991). A probabilistic principle and generalized Schrödinger perturbation. *J. Funct. Anal.* **101** 162–176. [MR1132313](#)

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
WROCLAW UNIVERSITY OF TECHNOLOGY
WYB. WYSPIAŃSKIEGO 27
50-370 WROCLAW
POLAND
E-MAIL: kamil.kaleta@pwr.wroc.pl

DEPARTMENT OF MATHEMATICAL SCIENCES
LOUGHBOROUGH UNIVERSITY
LOUGHBOROUGH LE11 3TU
UNITED KINGDOM
E-MAIL: J.Lorinczi@lboro.ac.uk