LINEAR RESPONSE IN THE INTERMITTENT FAMILY: DIFFERENTIATION IN A WEIGHTED C^0-NORM

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ABSTRACT. We provide a general framework to study differentiability of SRB measures for one dimensional non-uniformly expanding maps. Our technique is based on inducing the non-uniformly expanding system to a uniformly expanding one, and on showing how the linear response formula of the non-uniformly expanding system is inherited from the linear response formula of the induced one. We apply this general technique to interval maps with a neutral fixed point (Pomeau-Manneville maps) to prove differentiability of the corresponding SRB measure. Our work covers systems that admit a finite SRB measure and it also covers systems that admit an infinite SRB measure. In particular, we obtain a linear response formula for both finite and infinite SRB measures. To the best of our knowledge, this is the first work that contains a linear response result for infinite measure preserving systems.

CONTENTS

1. **Introduction.** In physical applications of dynamical systems, it is important to understand how statistical properties of a perturbed physical system are related to statistical properties of the original system; i.e., before the occurrence of the perturbation. In particular, it is always desirable to write a first order approximation of the Sinai-Ruelle-Bowen (SRB) measure of the perturbed system in terms of the SRB measure of the original system. In smooth ergodic theory, this direction of research, which was pioneered by David Ruelle, is called differentiation (with respect to noise) of SRB measures. In the physics literature the equivalent term is called 'linear response'.

Linear response has been proved for several classes of smooth dynamical systems that admit exponential, or at least summable, decay of correlations [4, 6, 8, 9, 10, 11, 16]. Negative results, where linear response does not hold, are also known [4, 5]. A recent survey on the progress in this area of research is [5]. More recently, results on the linear response of polynomially mixing systems that admit a probabilistic SRB measure were announced in [7, 12]. Such systems have attracted the attention of both mathematicians [14, 17] and physicists because of their importance in the study of intermittent transition to turbulence [15].

In this work we provide a general framework to study differentiability of SRB measures for one dimensional non-uniformly expanding maps. We use this general framework to study linear response of maps with neutral fixed points. In particular, we apply our results to study linear response of Pomeau-Manneville type maps [14, 15]. The difference between our result and those of [7, 12] is two-fold: in [7, 12] the authors obtain results only for probabilistic SRB measures. Moreover, they obtain a weak form of differentiability. While in our work, we cover both the finite and infinite SRB measure cases and we prove differentiability in norm[1]. Moreover, we provide a linear response formula that covers both the finite and infinite SRB measure cases.

In Section 2 we introduce a general setup for the systems we study and we state our assumptions on this general setup. Section 3 includes the statement of our main results (Theorems 3.1 and 3.2). Section 4 contains the proof of the theorems through several lemmas. In Section 5 we show that the assumptions of Section 2 are satisfied by the intermittent maps studied in [14].

2. **Setup and assumptions.**

2.1. **Interval maps with an inducing scheme.** We introduce now a class of (family of) interval maps which are non-uniformly expanding with two branches, for which one can construct an inducing scheme which allow to inherit the linear response formula from the one for the induced system.

- Let V be a neighbourhood of 0. For any $\varepsilon \in V$, $T_\varepsilon \colon [0,1] \to [0,1]$ is a non-singular map, with respect to Lebesgue measure, m, with two onto branches $T_{0,\varepsilon} \colon [0,1/2] \to [0,1]$ and $T_{1,\varepsilon} \colon [1/2,1] \to [0,1]$. The inverse branches of $T_{0,\varepsilon}$, $T_{1,\varepsilon}$ are respectively denoted by $g_{0,\varepsilon}$ and $g_{1,\varepsilon}$. We call $T_0 := T$ the unperturbed map, and T_ε, for $\varepsilon \neq 0$, the perturbed map.
- We assume that for each $i = 0, 1$ and $j = 0, 1, 2$ the following partial derivatives exist and satisfy the commutation relation

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[1]Theorem 1.2 of Korepanov [12] implies differentiability in norm for the LSV map but only for probabilistic SRB measures. See the discussion on page 2 of [12]. We would also like to stress here that Theorem 1.2 of [12] uses the explicit formula of LSV maps and it does not cover the infinite SRB measure case.

$$\partial_\varepsilon g_{i,\varepsilon}^{(j)} = (\partial_\varepsilon g_{i,\varepsilon})^{(j)}. \tag{1}$$

- We assume that T_ε has a unique absolutely continuous invariant measure[2] (up to multiplication) whose Radom-Nykodim derivative will be denoted by h_ε, and we denote for simplicity $h = h_0$.
- Let $\hat{T}_\varepsilon$, be the first return map of T_ε to Δ, where $\Delta := [1/2, 1]$; i.e., for $x \in \Delta$

$$\hat{T}_\varepsilon(x) = T_\varepsilon^{R_\varepsilon(x)}(x),$$

where

$$R_\varepsilon(x) = \inf\{n \geq 1 : T_\varepsilon^n(x) \in \Delta\}.$$

We assume that $\hat{T}_\varepsilon$ has a unique acim (up to multiplication) with a continuous density denoted $\hat{h}_\varepsilon \in C^0$.

- Let Ω be the set of finite sequences of the form $\omega = 10^n$, for $n \in \mathbb{N} \cup \{0\}$. We set $g_{\omega,\varepsilon} = g_{1,\varepsilon} \circ g_{0,\varepsilon}^n$. Then for $x \in [0,1]$ we have $T_\varepsilon^{n+1} \circ g_{\omega,\varepsilon}(x) = x$. The cylinder sets $[\omega]_\varepsilon = g_{\omega,\varepsilon}(\Delta)$, form a partition of Δ (mod 0). For $x \in [0,1]$, we assume

$$\sup_\omega \sup_{\varepsilon \in V} \sup_{x \in [0,1]} |g'_{\omega,\varepsilon}(x)| < \infty; \tag{2}$$

$$\sup_\omega \sup_{\varepsilon \in V} \sup_{x \in [0,1]} |\partial_\varepsilon g_{\omega,\varepsilon}(x)| < \infty; \tag{3}$$

$$\sum_\omega \sup_{\varepsilon \in V} ||g'_{\omega,\varepsilon}||_\mathcal{B} < \infty; \tag{4}$$

and

$$\sum_\omega \sup_{\varepsilon \in V} ||\partial_\varepsilon g'_{\omega,\varepsilon}||_\mathcal{B} < \infty, \tag{5}$$

where $\mathcal{B}$ denotes the set of continuous functions on $(0,1]$ with the norm

$$\| f \|_\mathcal{B} = \sup_{x \in (0,1]} |x^\gamma f(x)|,$$

for a fixed[3] $\gamma > 0$. When equipped with the norm $\| \cdot \|_\mathcal{B}$, $\mathcal{B}$ is a Banach space.

For $\Phi \in L^1$, let

$$F_\varepsilon(\Phi) := 1_\Delta \Phi + (1 - 1_\Delta) \sum_{\omega \in \Omega} \Phi \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon}. \tag{6}$$

Note that F_ε is a linear operator. In fact, for $x \in [0,1] \setminus \Delta$, the formula of F_ε can be re-written using the Perron-Frobenius operator of T_ε:

$$F_\varepsilon(\Phi) := 1_\Delta \Phi + (1 - 1_\Delta) \sum_{k \geq 1} L_\varepsilon^k (\Phi \cdot 1_{\{R_\varepsilon > k\}}),$$

where L_ε is the Perron-Frobenius operator associated with T_ε; i.e., for $\varphi \in L^\infty$ and $\psi \in L^1$

$$\int \varphi \circ T_\varepsilon \cdot \psi dm = \int \varphi \cdot L_\varepsilon \psi dm.$$

It is well known, see for instance [3], that the densities of the original system and the induced one are related (modulo normalization in the finite measure case) by

$$h_\varepsilon = F_\varepsilon(\hat{h}_\varepsilon). \tag{7}$$

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[2]The T_ε absolutely continuous invariant measure is not assumed to be probabilistic; we allow for T_ε to admit a σ-finite absolutely continuous invariant measure.

[3]In (4) and (5) we need the assumptions to hold only for a single γ.

We also define the following operator, which will represent $\partial_\varepsilon F_\varepsilon \Phi|_{\varepsilon=0}$

$$Q\Phi = (1 - 1_\Delta) \sum_\omega \Phi' \circ g_\omega \cdot a_\omega g'_\omega + \Phi \circ g_\omega \cdot b_\omega, \tag{8}$$

where $a_\omega = \partial_\varepsilon g_{\omega,\varepsilon}|_{\varepsilon=0}$ and $b_\omega = \partial_\varepsilon g'_{\omega,\varepsilon}|_{\varepsilon=0}$.

2.2. **Interval maps with countable number of branches.** We introduce here a class of (family of) interval maps which are uniformly expanding, with a finite or countable number of branches, for which we will be able to prove a linear response formula. The induced map in Subsection 2.1 is a particular case of such uniformly expanding maps.

Let Δ be an interval and V be a neighborhood of 0. Let Ω be a finite or countable set. We assume that the maps $\hat{T}_\varepsilon \colon \Delta \to \Delta$ satisfy

- For each $\varepsilon \in V$, there exists a partition (mod 0) of Δ into open intervals $\Delta_{\omega,\varepsilon}$, $\omega \in \Omega$ such that the restriction of $\hat{T}_\varepsilon$ to $\Delta_{\omega,\varepsilon}$ is piecewise C^3, onto and uniformly expanding in the sense that $\inf_\omega \inf_{\Delta_{\omega,\varepsilon}} |\hat{T}'_{\omega,\varepsilon}| > 1$. We denote by $g_{\omega,\varepsilon}$ the inverse branches of $\hat{T}_\varepsilon$ on $\Delta_{\omega,\varepsilon}$.
- We assume that for each $\omega \in \Omega$ and $j = 0, 1, 2$ the following partial derivatives exist and satisfy the commutation relation[4]

$$\partial_\varepsilon g_{\omega,\varepsilon}^{(j)} = (\partial_\varepsilon g_{\omega,\varepsilon})^{(j)}. \tag{9}$$

- We assume

$$\sum_\omega \sup_{\varepsilon \in V} \sup_{x \in \Delta} |g'_{\omega,\varepsilon}(x)| < \infty; \tag{10}$$

and

$$\sup_\omega \sup_{\varepsilon \in V} \sup_{x \in \Delta} \left| \frac{g''_{\omega,\varepsilon}(x)}{g'_{\omega,\varepsilon}(x)} \right| < \infty; \tag{11}$$

and for $i = 1, 2$

$$\sum_\omega \sup_{\varepsilon \in V} \sup_{x \in \Delta} |\partial_\varepsilon g_{\omega,\varepsilon}^{(i)}(x)| < \infty. \tag{12}$$

Let $\hat{L}_\varepsilon$ denote the Perron-Frobenius operator of the map $\hat{T}_\varepsilon$; i.e., for $\Phi \in L^1(\Delta)$

$$\hat{L}_\varepsilon \Phi(x) := \sum_{\omega \in \Omega} \Phi \circ g_{\omega,\varepsilon}(x) g'_{\omega,\varepsilon}(x)$$

for a.e. $x \in \Delta$. Under these conditions it is well known that $\hat{T}_\varepsilon$ admits a unique (up to multiplication) invariant absolutely continuous finite measure. We denote its density by $\hat{h}_\varepsilon$. Hence $\hat{L}_\varepsilon \hat{h}_\varepsilon = \hat{h}_\varepsilon$. Moreover, $\hat{L}_\varepsilon$ has a spectral gap when acting on C^k, $k = 1, 2$ (see for instance [13]). We denote the Perron-Frobenius operator of the unperturbed map $\hat{T}$ by $\hat{L}$; i.e., $\hat{L} := \hat{L}_0$ and let $\hat{h} := \hat{h}_0$.

3. **Statement of the main results.**

3.1. **Statement of the main results.** A first general statement is that the differentiability of the T_ε absolutely continuous measure is inherited from that of the induced system.

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[4]Note that (9) is satisfied when $\hat{T}_\varepsilon$ is an induced map as in Subsection 2.1. In particular, for each $i = 0, 1$ and $j = 0, 1, 2$ the following partial derivatives exist and satisfy the commutation relation $\partial_\varepsilon g_{i,\varepsilon}^{(j)} = (\partial_\varepsilon g_{i,\varepsilon})^{(j)}$.

Theorem 3.1. *Let T_ε be a family of maps of the interval as described in Subsection 2.1. If the density $\hat{h}_\varepsilon$ of the induced map $\hat{T}_\varepsilon$ is differentiable as a C^0 element, that is there exists $\hat{h}^* \in C^0$ such that*

$$\lim_{\varepsilon \to 0} ||\frac{\hat{h}_\varepsilon - \hat{h}}{\varepsilon} - \hat{h}^*||_{C^0} = 0, \tag{13}$$

for some $\hat{h} \in C^0$, then

  a) *there exists $h^* \in \mathcal{B}$ such that*

$$\lim_{\varepsilon \to 0} ||\frac{h_\varepsilon - h}{\varepsilon} - h^*||_{\mathcal{B}} = 0;$$

  i.e., h_ε is differentiable as an element of $\mathcal{B}$ with respect to ε;
  b) *in particular, if the conditions hold for some $\gamma < 1$*

$$\lim_{\varepsilon \to 0} ||\frac{h_\varepsilon - h}{\varepsilon} - h^*||_1 = 0.$$

  c) *The function h^* is given by* [5]

$$h^* = F_0(\hat{h}^*) + Q\hat{h}.$$

Next, we show that for the family of maps with countable number of branches introduced in Subsection 2.2 the invariant density is differentiable as an element of C^0.

Theorem 3.2. *Let $\hat{T}_\varepsilon \colon \Delta \to \Delta$ be a family of maps of the interval as described in Subsection 2.2. Then the density $\hat{h}_\varepsilon$ of the map $\hat{T}_\varepsilon$ is differentiable as a C^0 element, that is there exists $\hat{h}^* \in C^0$ such that (13) holds. Moreover, we have the linear response formula*

$$\hat{h}^* := (I - \hat{L})^{-1}\hat{L}[A_0\hat{h}' + B_0\hat{h}],$$

where $\hat{h}'$ is the spatial derivative of $\hat{h}$ and

$$A_0 = -\left(\frac{\partial_\varepsilon \hat{T}_\varepsilon}{\hat{T}'_\varepsilon}\right)\Big|_{\varepsilon=0}, \qquad B_0 = \left(\frac{\partial_\varepsilon \hat{T}_\varepsilon \cdot \hat{T}''_\varepsilon}{\hat{T}'^2_\varepsilon} - \frac{\partial_\varepsilon \hat{T}'_\varepsilon}{\hat{T}'_\varepsilon}\right)\Big|_{\varepsilon=0}.$$

Corollary 3.3. *If T_ε satisfies the assumptions of Subsections 2.1 and 2.2, then*

$$h^* = F_0(I - \hat{L})^{-1}\hat{L}(A_0\hat{h}' + B_0\hat{h}) + Q\hat{h}. \tag{14}$$

Proof. The proof follows from Theorems 3.1 and 3.2. $\square$

Remark 3.4 (Moving inducing sets)**.** We notice that Theorem 3.1 generalizes easily to the case where the inducing sets Δ_ε are allowed to depend on ε in a C^1 way. Indeed, any C^1 family of C^1 diffeomorphism $S_\varepsilon \colon [0,1] \to [0,1]$ such that $S_\varepsilon(\Delta_\varepsilon) = \Delta$, $S_0 = id$, will conjugate T_ε with a map $\bar{T}_\varepsilon$ whose inducing set is Δ. Applying Theorem 3.1 to the map $\bar{T}_\varepsilon$, with the obvious notation, we obtain:

$$\bar{h}_\varepsilon = \bar{h} + \varepsilon\bar{h}^* + o(\varepsilon) \tag{15}$$

Then using (15) and the fact that $h_\varepsilon = \bar{h}_\varepsilon \circ S_\varepsilon \cdot S'_\varepsilon$ we obtain

$$\partial_\varepsilon h_\varepsilon|_{\varepsilon=0} = \bar{h}' \cdot \partial_\varepsilon S_\varepsilon|_{\varepsilon=0} + \bar{h}^* + \bar{h} \cdot \partial_\varepsilon S'_\varepsilon|_{\varepsilon=0}. \tag{16}$$

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[5]Note that in the finite measure case, h^* is the derivative of the non-normalized density h_ε. The advantage in working with h_ε is reflected in keeping the operator F_ε linear and to accommodate the infinite measure preserving case. In the finite measure case, once the derivative of h_ε is obtained, the derivative of the normalized density can be easily computed. Indeed, $h_\varepsilon = h + \varepsilon h^* + o(\varepsilon)$. Consequently, $\int h_\varepsilon = \int h + \varepsilon \int h^* + o(\varepsilon)$. Hence, $\partial_\varepsilon(\frac{h_\varepsilon}{\int h_\varepsilon})|_{\varepsilon=0} = h^* - h \int h^*$.

3.2. **Rigorous numerical approximation of the derivative.** An important
feature of our approach is that it could be amenable to obtain rigorous numerical
approximation of h^*. In particular, since $\hat{L}$ has a spectral gap on C^k, $k = 1, 2$, using
ideas of [2] one can approximate $(I - \hat{L})^{-1}\hat{L}[A_0\hat{h}' + B_0\hat{h}]$ as a first step, and in the
second step one can follow the path of [1] and pull back the computed formula of
the first step to the full system and obtain a numerical approximation of h^* in $\mathcal{B}$.

4. **Proof of the results.** We use the letter C to denote positive constants whose
values may change when estimating various expressions but are independent of
both ε and ω (or n). In the following, we first present in Subsection 4.1 the proof
of Theorem 3.2, and then in Subsection 4.2 we present the proof of Theorem 3.1.

4.1. **Proof of Theorem 3.2.** We first prove a lemma that will be used in the
linear response formula in Theorem 3.2.

Lemma 4.1. *For any differentiable function* $\Phi\colon \Delta \to \mathbb{R}$*, the function* $\Phi \circ g_{\omega,\varepsilon}g'_{\omega,\varepsilon}$
is differentiable with respect to ε *and we have on* Δ

$$\partial_\varepsilon(\Phi \circ g_{\omega,\varepsilon}g'_{\omega,\varepsilon}) = [\Phi'A_\varepsilon + \Phi B_\varepsilon] \circ g_{\omega,\varepsilon}g'_{\omega,\varepsilon}, \tag{17}$$

where

$$A_\varepsilon = -\left(\frac{\partial_\varepsilon\hat{T}_\varepsilon}{\hat{T}'_\varepsilon}\right), \qquad B_\varepsilon = \left(\frac{\partial_\varepsilon\hat{T}_\varepsilon \cdot \hat{T}''_\varepsilon}{\hat{T}'^2_\varepsilon} - \frac{\partial_\varepsilon\hat{T}'_\varepsilon}{\hat{T}'_\varepsilon}\right).$$

Proof. We start from the relation $\hat{T}_\varepsilon \circ g_{\omega,\varepsilon}(x) = x$ and differentiate it with respect
to ε and get $\hat{T}'_\varepsilon \circ g_{\omega,\varepsilon}\partial_\varepsilon g_{\omega,\varepsilon} + \partial_\varepsilon\hat{T}_\varepsilon \circ g_{\omega,\varepsilon} = 0$. This gives $\partial_\varepsilon g_{\omega,\varepsilon} = A_\varepsilon \circ g_{\omega,\varepsilon}$. This
also implies that $\partial_\varepsilon g'_{\omega,\varepsilon} = A'_\varepsilon \circ g_{\omega,\varepsilon}g'_{\omega,\varepsilon} = B_\varepsilon \circ g_{\omega,\varepsilon}g'_{\omega,\varepsilon}$. The conclusion follows from
the following differentiation with respect to ε:

$$\begin{aligned}
\partial_\varepsilon(\Phi \circ g_{\omega,\varepsilon}g'_{\omega,\varepsilon}) &= \partial_\varepsilon(\Phi \circ g_{\omega,\varepsilon})g'_{\omega,\varepsilon} + \Phi \circ g_{\omega,\varepsilon}\partial_\varepsilon g'_{\omega,\varepsilon} \\
&= \Phi' \circ g_{\omega,\varepsilon}\partial_\varepsilon g_{\omega,\varepsilon}g'_{\omega,\varepsilon} + \Phi \circ g_{\omega,\varepsilon}\partial_\varepsilon g'_{\omega,\varepsilon}.
\end{aligned} \tag{18}$$

$\square$

Strategy of the proof of Theorem 3.2. The general strategy starts from the
identity

Lemma 4.2. *We have* $\hat{h}_\varepsilon = (I - \hat{L}_\varepsilon)^{-1}(\hat{L}_\varepsilon - \hat{L})\hat{h} + \hat{h}$.

Proof. One easily checks that

$$(I - \hat{L}_\varepsilon)(\hat{h}_\varepsilon - \hat{h}) = (\hat{L}_\varepsilon - \hat{L})\hat{h}.$$

Since $\hat{L}_\varepsilon$ has a spectral gap on C^1 it eventually contracts exponentially on the
subset of zero average functions C^1_0. Since the ranges of $(\hat{L}_\varepsilon - \hat{L}_0)$ and $(I - \hat{L}_\varepsilon)$ are
contained in C^1_0, the composition below is well defined

$$(I - \hat{L}_\varepsilon)^{-1}(I - \hat{L}_\varepsilon)(\hat{h}_\varepsilon - \hat{h}) = (I - \hat{L}_\varepsilon)^{-1}(\hat{L}_\varepsilon - \hat{L})\hat{h}.$$

This completes the proof of the lemma. $\square$

Setting $H_\varepsilon = \hat{L}_\varepsilon - \hat{L}$ and $G_\varepsilon = (I - \hat{L}_\varepsilon)^{-1}$, Lemma 4.2 reads

$$\hat{h}_\varepsilon = G_\varepsilon H_\varepsilon\hat{h} + \hat{h}. \tag{19}$$

We then obtain, using Lemma 4.3 below, the following first order expansion in C^1_0

$$H_\varepsilon\hat{h} = \varepsilon q + o(\varepsilon).$$

We then show, see second statement of Lemma 4.5 below, that G_ε is uniformly bounded in $\mathcal{L}(C_0^1, C^0)$ to obtain the following expansion in C^0

$$G_\varepsilon H_\varepsilon \hat{h} = \varepsilon G_\varepsilon q + o(\varepsilon).$$

Finally, using the two expansions above with (19) and showing that $G_\varepsilon(q) \to G_0(q)$ in C^0, see the first statement of Lemma 4.5 below, we obtain in C^0

$$\hat{h}_\varepsilon = \hat{h} + \varepsilon G_0(q) + o(\varepsilon),$$

which proves the theorem.

Lemma 4.3. *We have*

$$\frac{H_\varepsilon \hat{h}}{\varepsilon} \to q \ \text{in} \ C_0^1,$$

where $q = \hat{L}[A_0 \hat{h}' + B_0 \hat{h}]$.

Proof. Recall that $H_\varepsilon = \hat{L}_\varepsilon - \hat{L}$ hence we need to show that $\varepsilon \mapsto \hat{L}_\varepsilon \hat{h}$ is differentiable as a C^1 element, on some neighborhood V of 0. To this end, recall that $\hat{L}_\varepsilon \hat{h} = \sum_\omega \hat{h} \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon}$. It suffices to show that

(i) for each ω, the map $\varepsilon \in V \mapsto \hat{h} \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon} \in C^1$ is differentiable;

(ii) the series $\sum_\omega \sup_{\varepsilon \in V} \|\partial_\varepsilon(\hat{h} \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon})\|_{C^1} < \infty$.

We first prove (i). Drop for simplicity the subscript ω and write $g_\varepsilon = g_{\omega,\varepsilon}$ and let $f_\varepsilon = \hat{h} \circ g_\varepsilon g'_\varepsilon$. We have

$$f_\varepsilon = \hat{h} \circ g_\varepsilon g'_\varepsilon$$
$$f'_\varepsilon = \hat{h}' \circ g_\varepsilon (g'_\varepsilon)^2 + \hat{h} \circ g_\varepsilon g''_\varepsilon.$$

By the commutation relations given by assumption (9) we have

$$\partial_\varepsilon f_\varepsilon^{(i)} = (\partial_\varepsilon f_\varepsilon)^{(i)}, \quad i = 0, 1 \tag{20}$$

and these are continuous functions on $\Delta \times V$.

Let $\nu \in V$ and ε be small. We have

$$\|f_{\varepsilon+\nu} - f_\nu - \varepsilon(\partial_\delta f_\delta|_{\delta=\nu})\|_{C^1} = \sum_{i=0}^1 \|f_{\varepsilon+\nu}^{(i)} - f_\nu^{(i)} - \varepsilon(\partial_\delta f_\delta|_{\delta=\nu})^{(i)}\|_{C^0}. \tag{21}$$

For each x, by the mean value theorem, there exists $\eta_{x,\varepsilon}^i$ such that $f_{\varepsilon+\nu}^{(i)}(x) - f_\nu^{(i)}(x) = \varepsilon \partial_\delta f_\delta^{(i)}|_{\delta=\eta_{x,\varepsilon}^i}$, with $|\eta_{x,\varepsilon}^i - \nu| < \varepsilon$. Therefore

$$\sum_{i=0}^1 \|f_{\varepsilon+\nu}^{(i)} - f_\nu^{(i)} - \varepsilon(\partial_\delta f_\delta^{(i)}|_{\delta=\nu})\|_{C^0} \le |\varepsilon| \sum_{i=0}^1 \|\partial_\delta f_\delta^{(i)}|_{\delta=\eta_{\cdot,\varepsilon}^i} - \partial_\delta f_\delta^{(i)}|_{\delta=\nu}\|_{C^0} = o(\varepsilon).$$

We conclude by (21) and the commutation relation (20). We now prove (ii).

$$\sum_\omega \sup_{\varepsilon \in V} \|\partial_\varepsilon f_{\omega,\varepsilon}\|_{C^1} = \sum_\omega \sup_{\varepsilon \in V} \sum_{i=0}^1 \|\partial_\varepsilon f_{\omega,\varepsilon}^{(i)}\|_{C^0}. \tag{22}$$

We write for $i = 0, 1$

$$\partial_\varepsilon f_{\omega,\varepsilon}^{(i)} = \sum_{k=0}^{i+1} a_k^{(i)} \partial_\varepsilon g_{\omega,\varepsilon}^{(k)}, \tag{23}$$

where the coefficients $a_k^{(i)}$ are given respectively by

$$a_0^{(0)} = \hat{h}' \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon}, \quad a_1^{(0)} = \hat{h} \circ g_{\omega,\varepsilon}$$

then differentiating again in space we get

$$a_0^{(1)} = \hat{h}'' \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}'^2 + \hat{h}' \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}'', \quad a_1^{(1)} = 2\hat{h}' \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}', \quad a_2^{(1)} = \hat{h} \circ g_{\omega,\varepsilon}.$$

By assumptions (11) and (2), we have $a_0^{(i)} \leq C g_{\omega,\varepsilon}'$ and $a_k^{(i)} \leq C$ for any $i = 0, 1$ and $k \neq 0$. Moreover, by assumption (12), for $k = 1, 2$,

$$\sum_\omega \sup_{\varepsilon \in V} \sup_{x \in \Delta} |\partial_\varepsilon g_{\omega,\varepsilon}^{(k)}(x)| \leq C.$$

Putting these estimates together with (23) imply that (22) is finite, proving (ii). Moreover, we have

$$\partial_\varepsilon H_\varepsilon \hat{h}|_{\varepsilon=0} = \sum_\omega \partial_\varepsilon (\hat{h} \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}')|_{\varepsilon=0} = \hat{L}[A_0 \hat{h}' + B_0 \hat{h}],$$

where we have used (17).   □

Lemma 4.4. *For any $\Phi \in C^1$ we have $\hat{L}_\varepsilon \Phi \to \hat{L}\Phi$ in C^1 as $\varepsilon \to 0$.*

Proof. We have $\hat{L}_\varepsilon \Phi = \sum_\omega \Phi \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}'$. It suffices to show that for some neighborhood V of 0,
   (i) for each ω, the map $\varepsilon \in V \mapsto \Phi \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}' \in C^1$ is continuous in ε;
   (ii) the series $\sum_\omega \sup_{\varepsilon \in V} \|\Phi \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}'\|_{C^1} < \infty$. We skip the proof of (i) since it is similar to (i) in the proof of Lemma 4.3. (ii) follows from the identity

$$(\Phi \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}')' = \Phi' \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}'^2 + \Phi \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}''$$

and conditions (10) and (11).   □

Lemma 4.5. *We have $G_\varepsilon(q) \to G_0(q)$ in C^0 and G_ε is uniformly bounded in $\mathcal{L}(C_0^1, C^0)$*

Proof. We use the fact that the family of operators $\hat{L}_\varepsilon$ has a uniform spectral gap on C_0^1, for ε in a neighborhood of 0. Hence, these operators are invertible on this space and we have $\|(1 - \hat{L}_\varepsilon)^{-1}\|_{C_0^1 \to C_0^1} \leq C < \infty$. This proves in particular the second statement. Note that

$$(G_\varepsilon - G_0)(q) = (I - \hat{L}_\varepsilon)^{-1}(\hat{L}_\varepsilon - \hat{L})(1 - \hat{L})^{-1}(q).$$

By Lemma 4.4 with $\Phi = (I - \hat{L})^{-1}(q)$ and the previous observations this proves the first statement.   □

4.2. **Proof of Theorem 3.1.** We first prove a lemma that will be used in the linear response formula in Theorem 3.1.

Lemma 4.6. *For any differentiable function Φ, the function $\Phi \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}'$ is differentiable with respect to ε and we have on $[0, 1]$*

$$\partial_\varepsilon (\Phi \circ g_{\omega,\varepsilon} g_{\omega,\varepsilon}') = \Phi' \circ g_{\omega,\varepsilon} \partial_\varepsilon g_{\omega,\varepsilon} g_{\omega,\varepsilon}' + \Phi \circ g_{\omega,\varepsilon} \partial_\varepsilon g_{\omega,\varepsilon}'. \tag{24}$$

Proof. The proof follows by differentiating with respect to ε and is similar to (18).
                                                                            □

Strategy of the proof of Theorem 3.1. The argument starts from the first order expansion for $\hat{h}_\varepsilon$ in C^0

$$\hat{h}_\varepsilon = \hat{h} + \varepsilon \hat{h}^* + o(\varepsilon).$$

Using this, we then obtain, by the second statement of Lemma 4.8 below and relation (7) the following expansion in $\mathcal{B}$

$$h_\varepsilon = F_\varepsilon(\hat{h}_\varepsilon) = F_\varepsilon(\hat{h}) + \varepsilon F_\varepsilon(\hat{h}^*) + o(\varepsilon).$$

Finally, we obtain by Lemma 4.7 below and the first statement of Lemma 4.8 below the first order expansion of h_ε in $\mathcal{B}$

$$h_\varepsilon = h + \varepsilon(Q\hat{h} + F_0(\hat{h}^*)) + o(\varepsilon),$$

which finishes the proof of the theorem.

Lemma 4.7. *The map $\varepsilon \mapsto F_\varepsilon \hat{h}$ is differentiable as an element in $\mathcal{B}$ and $\partial_\varepsilon F_\varepsilon \hat{h}|_{\varepsilon=0} = Q\hat{h}$, where Q is defined in (8).*

Proof. It suffices to show that
  (i) for each ω, the map $\varepsilon \in V \mapsto \hat{h} \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon} \in \mathcal{B}$ is differentiable;
  (ii) the series $\sum_\omega \sup_{\varepsilon \in V} \|\partial_\varepsilon(\hat{h} \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon})\|_\mathcal{B} < \infty$.
  We skip the proof of (i) as, by using (1), it follows similar steps as in the proof of (i) in Lemma 4.3. For (ii), using (24) of Lemma 4.6 we have

$$\sum_\omega \sup_{\varepsilon \in V} \|\partial_\varepsilon(\hat{h} \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon})\|_\mathcal{B} \leq \sum_\omega \sup_{\varepsilon \in V} \|\hat{h}' \circ g_{\omega,\varepsilon} \cdot \partial_\varepsilon g_{\omega,\varepsilon} \cdot g'_{\omega,\varepsilon}\|_\mathcal{B}$$
$$+ \sum_\omega \sup_{\varepsilon \in V} \|\hat{h} \circ g_{\omega,\varepsilon} \cdot \partial_\varepsilon g'_{\omega,\varepsilon}\|_\mathcal{B} \tag{25}$$
$$\leq C \sum_\omega \sup_{\varepsilon \in V} \|g'_{\omega,\varepsilon}\|_\mathcal{B} + C \sum_\omega \sup_{\varepsilon \in V} \|\partial_\varepsilon g'_{\omega,\varepsilon}\|_\mathcal{B},$$

where we have used the fact that $\hat{h}$ is C^1 and assumptions (2) and (3). The rest of the proof follows from assumptions (4) and (5). $\qquad\square$

Lemma 4.8. *$F_\varepsilon(\hat{h}^*) \to F_0(\hat{h}^*)$ in $\mathcal{B}$ and F_ε is uniformly bounded in $\mathcal{L}(C^0, \mathcal{B})$.*

Proof. To prove uniform boundedness we use assumption (4) to get, for $\Phi \in C^0$,

$$\|F_\varepsilon(\Phi)\|_\mathcal{B} = \|1_\Delta \Phi + (1 - 1_\Delta) \sum_{\omega \in \Omega} \Phi \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon}\|_\mathcal{B}$$
$$\leq \|\Phi\|_{C^0} + \|\Phi\|_{C^0} \sum_{\omega \in \Omega} \sup_{\varepsilon \in V} \|g'_{\omega,\varepsilon}\|_\mathcal{B} \leq C\|\Phi\|_{C^0}.$$

Next, the map $g_{\omega,\varepsilon}$ converges to $g_{\omega,0}$ in the C^1 norm. Hence for the continuous function $\Phi = \hat{h}^* \in C^0$ we have $\Phi \circ g_{\omega,\varepsilon} g'_{\omega,\varepsilon}$ converges uniformly to $\Phi \circ g_{\omega,0} g'_{\omega,0}$. This together with the normal convergence above shows the continuity of $F_\varepsilon(\hat{h}^*) \in \mathcal{B}$ at $\varepsilon = 0$. $\qquad\square$

5. **Verifying the assumptions for Pomeau-Manneville type maps.** We verify the assumptions of Section 2 for the family of intermittent maps studied by Liverani-Saussol-Vaienti [14] which is a version of the Pomeau-Manneville family [15]. Let $0 < \alpha < \infty$, and define

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & x \in [0, \frac{1}{2}] \\ 2x - 1 & x \in (\frac{1}{2}, 1] \end{cases} . \tag{26}$$

Note that $x = 0$ is a neutral fixed point for the map T_α which is consequently a non-uniformly expanding map of the interval (on two pieces). Following Korepanov [12], we use the following notation

$$\mathrm{logg}(n) = \begin{cases} 1 & n \leq e \\ \log(n) & n > e \end{cases},$$

and we let $E_\alpha : [0, 1/2] \to [0, 1]$, $E_\alpha x = T_\alpha x$ be the left branch of T_α. Let $z \in [0, 1]$, and write $z_n := E_\alpha^{-n}(z)$; $z := z_0$. Let $\hat{T}_\omega := \hat{T}_\alpha|_{[\omega]}$ as defined in Section 2, then $\hat{T}_\omega(z) = E_\alpha^n(T_\alpha(z)) = E_\alpha^n(2z - 1)$ for $z \in [\omega]$, and for $z \in [1/2, 1]$ $T_\alpha(g_\omega(z)) = 2g_\omega(z) - 1 = z_n$. Note that $z_0 = z$, $z_0' = 1$, $z_0'' = z_0''' = 0$, for $n \geq 1$ $z_n \leq 1/2$, and

$$z_n = z_{n+1}(1 + 2^\alpha z_{n+1}^\alpha); \tag{27}$$

$$z_n' = (1 + (\alpha + 1)2^\alpha z_{n+1}^\alpha)z_{n+1}'. \tag{28}$$

It is well known, see for example [17], that $z_n \sim \frac{1}{2\alpha^{1/\alpha}} n^{-1/\alpha}$. In [12] Korepanov proved

Lemma 5.1. *We have*

a) $\frac{C}{n}z_0^\alpha \leq z_n^\alpha \leq \frac{C}{n}$, *and* $-\log(z_n) \leq C[\mathrm{logg}(n) - \log z_0]$;

b)

$$0 \leq z_n' \leq C(1 + nz_0^\alpha \alpha 2^\alpha)^{-1/\alpha - 1}; \tag{29}$$

c) $0 \leq \frac{z_n''}{z_n'} \leq C z_0^{-2}/\max\{n, 1\}$;

d) $\frac{\partial_\alpha z_n}{z_n} \leq C\mathrm{logg}(n)[\mathrm{logg}(n) - \log z_0]$ *and*

$$\partial_\alpha z_n \leq C\frac{\mathrm{logg}(n)}{n^{1/\alpha}}[\mathrm{logg}(n) - \log z_0]; \tag{30}$$

e) $|\frac{\partial_\alpha z_n'}{z_n'}| \leq C(\mathrm{logg}(n))^2[\mathrm{logg}(n) - \log z_0]$;

f) $|\frac{\partial_\alpha z_n''}{z_n'}| \leq C z_0^{-2}(\mathrm{logg}(n))^2[\mathrm{logg}(n) - \log z_0]$.

The above list shows that our assumptions (10), (11), (12) and (2) are satisfied for the LSV family. We still have to show that assumptions (3), (4) and (5) hold. This will be done in the following lemma.

Lemma 5.2. *Let[6] $\alpha_0 < \gamma$. Let U be a neighbourhood of α_0 such that $\gamma \notin U$. We have*

a) $\sum_n \sup_{\alpha \in U} \sup_{z \in (0,1]} |z^\gamma(g^n)'(z)| < C$;

b) $\sup_n \sup_{\alpha \in U} \sup_{z \in (0,1]} |\partial_\alpha g^n(z)| < \infty$;

c) $\sum_n \sup_{\alpha \in U} \sup_{z \in (0,1]} |z^\gamma \partial_\alpha (g^n)'(z)| < \infty$.

Proof. For (a), by (29), we have

$$\sum_n \sup_{\alpha \in U} \sup_{z \in [0,1]} |z^\gamma(g^n)'(z)| \leq C \sum_n \sup_{\alpha \in U} \sup_{z \in (0,1]} z^\gamma(1 + nz^\alpha \alpha 2^\alpha)^{-1/\alpha - 1} < C.$$

For (b), we only discuss the case for $z \in (0, 1/2)$. The other case is the same[7], with a small change in notation. Using (27) we have

$$\partial_\alpha z_{j+1} = \frac{\partial_\alpha z_j + 2^\alpha z_{j+1}^{\alpha+1}(-\log 2z_{j+1})}{1 + (\alpha + 1)2^\alpha z_{j+1}^\alpha} > 0. \tag{31}$$

---

[6]Here α_0 is understood as the parameter corresponding to the unperturbed map; i.e., equivalent to the case $\varepsilon = 0$ in Section 2.

[7]In fact Lemma 5.6 of [12] provides an estimate which works only for $z \in (1/2, 1]$.

Consequently
$$0 < \partial_\alpha z_{j+1} - \partial_\alpha z_j \leq 2^\alpha z_{j+1}^{\alpha+1}(-\log 2z_{j+1}).$$
Noticing that $\partial_\alpha z_0 = 0$, and summing up, we get
$$\partial_\alpha z_{n+1} \leq 2^\alpha \sum_{j=1}^{n+1} z_j^{\alpha+1}(-\log 2z_j). \tag{32}$$

Therefore, using (32), we have
$$\sup_n \sup_{\alpha \in U} \sup_{z \in [0,1/2]} |\partial_\alpha g^n|$$
$$\leq C \sup_n \sup_{\alpha \in U} \sum_{j=1}^n ((j)^{-1/\alpha})^{\alpha+1}(-\log(j)^{-1/\alpha}) \tag{33}$$
$$\leq C \sup_n \sup_{\alpha \in U} \sum_{j=1}^n j^{-1-1/\alpha} \log(j) < \infty.$$

For (c), using the commutation relation $\partial_\alpha z_n' = (\partial_\alpha z_n)'$, (31) and (28), we get
$$\frac{\partial_\alpha z_j'}{z_j'} - \frac{\partial_\alpha z_{j+1}'}{z_{j+1}'} = \frac{2^\alpha z_{j+1}^\alpha + (\alpha+1)2^\alpha z_{j+1}^\alpha \log(2z_{j+1}) + \alpha(\alpha+1)2^\alpha z_{j+1}^{\alpha-1}\partial_\alpha z_{j+1}}{1 + (\alpha+1)2^\alpha z_{j+1}^\alpha}.$$
Noticing that $\partial_\alpha z_0 = 0$, and summing up, we get
$$-\frac{\partial_\alpha z_{n+1}'}{z_{n+1}'} = \sum_{j=0}^n \frac{2^\alpha z_{j+1}^\alpha + (\alpha+1)2^\alpha z_{j+1}^\alpha \log(2z_{j+1}) + \alpha(\alpha+1)2^\alpha z_{j+1}^{\alpha-1}\partial_\alpha z_{j+1}}{1 + (\alpha+1)2^\alpha z_{j+1}^\alpha},$$
which is equivalent to
$$-\partial_\alpha z_{n+1}' = z_{n+1}' \sum_{j=1}^{n+1} \frac{2^\alpha z_j^\alpha + (\alpha+1)2^\alpha z_j^\alpha \log(2z_j) + \alpha(\alpha+1)2^\alpha z_j^{\alpha-1}\partial_\alpha z_j}{1 + (\alpha+1)2^\alpha z_j^\alpha}.$$

Therefore,
$$\sum_n \sup_{\alpha \in U} \sup_{z \in (0,1/2]} |z^\gamma \partial_\alpha(g^n)'| \leq C \sum_{n=1}^\infty \sup_{\alpha \in U} \sup_{z \in (0,1/2]} |z^\gamma \cdot z_n'| \sum_{j=1}^n z_j^\alpha$$
$$+ C \sum_{n=1}^\infty \sup_{\alpha \in U} \sup_{z \in (0,1/2]} |z^\gamma \cdot z_n'| \sum_{j=1}^n z_j^\alpha |\log(z_j)| \tag{34}$$
$$+ C \sum_{n=1}^\infty \sup_{\alpha \in U} \sup_{z \in (0,1/2]} |z^\gamma \cdot z_n'| \sum_{j=1}^n z_j^{\alpha-1}|\partial_\alpha z_j|$$
$$:= (I) + (II) + (III).$$

We use (29) to show that (I) and (II) are finite, and (29), (32) to show that (III) is finite. Indeed,
$$(I) \leq C \sum_{n=1}^\infty \sup_{\alpha \in U} \sup_{z \in (0,1/2]} z^\gamma \cdot (1 + nz^\alpha \alpha 2^\alpha)^{-1/\alpha-1} \sum_{j=1}^n j^{-1} < \infty;$$
$$(II) \leq C \sum_{n=1}^\infty \sup_{\alpha \in U} \sup_{z \in (0,1/2]} z^\gamma \cdot (1 + nz^\alpha \alpha 2^\alpha)^{-1/\alpha-1} \sum_{j=1}^n j^{-1} \log(j) < \infty;$$

$$(III) \leq C \sum_{n=1}^{\infty} \sup_{\alpha \in U} \sup_{z \in (0,1/2]} z^{\gamma} \cdot (1 + nz^{\alpha}\alpha 2^{\alpha})^{-1/\alpha - 1} \sum_{j=1}^{n} j^{-1} \sum_{k=1}^{j} k^{-1-1/\alpha} \log k < \infty.$$

$\square$

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