ON A NOTION OF SPECIALITY OF LINEAR SYSTEMS IN $\mathbb{P}^n$

MARIA CHIARA BRAMBILLA, OLIVIA DUMITRESCU, AND ELISA POSTINGHEL

Abstract. Given a linear system in $\mathbb{P}^n$ with assigned multiple general points we compute the cohomology groups of its strict transforms via the blow-up of its linear base locus. This leads us to give a new definition of expected dimension of a linear system, which takes into account the contribution of the linear base locus, and thus to introduce the notion of linear speciality. We investigate such a notion giving sufficient conditions for a linear system to be linearly non-special for arbitrary number of points, and necessary conditions for small numbers of points.

1. Introduction

The study of linear systems of hypersurfaces in complex projective spaces with finitely many assigned base points of given multiplicities is a fundamental problem in algebraic geometry, related to polynomial interpolation in several variables, to the Waring problem for polynomials and to the classification of defective higher secant varieties to projective varieties.

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \dots, m_s)$ be the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through a general union of s points with multiplicities respectively $m_1, \dots, m_s$. The *virtual dimension* of $\mathcal{L}$ is

$$\operatorname{vdim}(\mathcal{L}) = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n} - 1,$$

the *expected dimension* of $\mathcal{L}$ is $\operatorname{edim}(\mathcal{L}) = \max(\operatorname{vdim}(\mathcal{L}), -1)$. The dimension of $\mathcal{L}$ is upper-semicontinuous in the position of the points in $\mathbb{P}^n$; it achieves its minimum value when they are in general position. The inequality $\dim(\mathcal{L}) \geq \operatorname{edim}(\mathcal{L})$ is always satisfied. If the conditions imposed by the assigned points are not linearly independent, then the actual dimension of $\mathcal{L}$ is strictly greater that the expected one: in that case we say that $\mathcal{L}$ (or a divisor D in $\mathcal{L}$) is *special*. Otherwise, if the actual and the expected dimension coincide, we say that $\mathcal{L}$ is *non-special*.

Recall that if Z is a collection of general fat points in $\mathbb{P}^n$ of multiplicities $m_1, \dots, m_s$, then the sheaf associated to the linear system $\mathcal{L}$ is $\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z$. For this reason, by abuse of notation, we will use the same letter $\mathcal{L}$ to denote such a

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sheaf, when no confusion arises. From the restriction exact sequence of sheaves

$$0 \to \mathcal{L} \to \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_Z \to 0,$$

taking cohomology we get

$$0 \to H^0(\mathbb{P}^n, \mathcal{L}) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(Z, \mathcal{O}_Z) \to H^1(\mathbb{P}^n, \mathcal{L}) \to 0,$$

being $h^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$. Moreover we obtain $h^i(\mathbb{P}^n, \mathcal{L}) = 0$, for all $i \geq 2$. Thus $\mathcal{L}$ is non-special if and only if

$$h^0(\mathbb{P}^n, \mathcal{L}) \cdot h^1(\mathbb{P}^n, \mathcal{L}) = 0.$$

The problem of classifying special linear systems attracted the attention of many researchers in the last century. In the case of the plane, the well-known Segre-Harbourne-Gimigliano-Hirschowitz conjecture describes all special linear systems, but even in this case, in spite of many partial results (see e.g. [10], [11] and the reference therein), the conjecture is still open. In the case of $\mathbb{P}^3$, there is an analogous conjecture formulated by Laface and Ugaglia, see Section 6.2 for more details.

In the direction of extending such conjectures to $\mathbb{P}^n$, and possibly to other projective varieties, we start with a very natural and general question.

Question 1.1. *Consider any non-empty linear system $\mathcal{L}$ in $\mathbb{P}^n$ and a divisor $D \in \mathcal{L}$. Denote by $\widetilde{D}$ the strict transform of D in the blow-up X of $\mathbb{P}^n$ along the base locus of $\mathcal{L}$. Is $\widetilde{D}$ non-special, namely, does $h^i(X, \mathcal{O}_X(\widetilde{D}))$ vanish for all $i \geq 1$?*

In order to answer this question one has to tackle two problems: the first one is to describe the base locus of a linear system, the second one is to understand the contribution given by each component of the base locus to the speciality of the linear system.

In this paper we start considering the case when the base locus of the linear system is linear, that is given by the union of linear subspaces of $\mathbb{P}^n$ of dimensions $1 \leq r \leq n-1$. In this case, it is possible to give arithmetical criteria, see Section 2, which tell when a linear subspace is contained in the base locus and with which multiplicity. Moreover the contribution to the speciality of the linear system is easy to compute.

In order to show this, let us consider first an example. The well-known Alexander-Hirschowitz theorem states that a linear system in $\mathbb{P}^n$ with only double points is special either if the degree is 2 and the number of points is $2 \leq s \leq n$, or if the linear system is one of the four exceptional cases, see e.g. [4],[26] for more details. The first case is easily understood, indeed the linear system $\mathcal{L} = \mathcal{L}_{n,2}(2^s)$ satisfies $\operatorname{vdim}(\mathcal{L}) = \binom{n+2}{2} - s(n+1) - 1$. On the other hand it is easy to see that any hypersurface in $\mathcal{L}$ is a quadric cone with vertex the linear subspace $\mathbb{P}^{s-1}$, hence $\dim(\mathcal{L}) = \binom{n-s+2}{2} - 1$, which is the dimension of the complete linear system of quadric hypersurfaces in $\mathbb{P}^{n-s}$. If $s = 2$, then $\dim(\mathcal{L}) - \operatorname{vdim}(\mathcal{L}) = 1$; therefore we may conjecture that the double line which is contained in the base locus gives an obstruction of 1. Similarly if $s = 3$, then $\dim(\mathcal{L}) - \operatorname{vdim}(\mathcal{L}) = 3$ and the presence of three double lines in the base locus, each of them contribuiting by 1, would highlight the same phenomenon; it is interesting to note that the plane spanned by the three base points, which is doubly contained in the base locus as well, does not give any contribution.

The example of quadrics can be extended to any linear system with degree d and points of multiplicities d, see Section 3. More generally, any linear subspace $\mathbb{P}^r$

which is contained in the base locus with multiplicity $k \geq r+1$ gives a contribution to the speciality of $\mathcal{L}$, which depends on r and k.

This suggests a new extended definition of expected dimension which takes in account such *linear obstructions*. We call this new notion *linear expected dimension* of $\mathcal{L}$ and we denote it by $\mathrm{ldim}(\mathcal{L})$ (see Definition 3.2 for the precise formulation). Consequently, we say that $\mathcal{L}$ is *linearly special* if $\dim(\mathcal{L}) \neq \mathrm{ldim}(\mathcal{L})$. In this sense the quadrics with double points are no longer special.

The linear expected dimension is meant to be a refined version of the expected dimension. We surmise to have $\mathrm{edim}(\mathcal{L}) \leq \mathrm{ldim}(\mathcal{L}) \leq \dim(\mathcal{L})$. The second inequality is in fact also predicted by the so-called *weak Fröberg-Iarrobino conjecture*, see Section 6.1 and [9] for more details.

In this paper we investigate the notion of linear speciality and we study conditions for a linear system to be not linearly special.

The first step in this direction, which also allows to partially answer Question 1.1, is to give a detailed description of the cohomology of the strict transform of $\mathcal{L}$ with respect to the blow-up of $\mathbb{P}^n$ along the linear base locus. In particular we will show that any r-dimensional cycle in the base locus of $\mathcal{L}$ gives a contribution at the level of the r-th cohomology group of the strict transform of $\mathcal{L}$, after blowing up all linear base cycles of dimension at most $r-1$. See Theorem 4.6 for a more precise statement.

A consequence of this result is that a non-empty linear system $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$ in $\mathbb{P}^n$ is linearly non-special as soon as $s \leq n+2$, (see Corollary 4.8).

In our opinion the description of the cohomology groups of the strict transform of the linear system is our main original contribution to this subject. See Section 6.1 for a discussion on the connections between our approach and the Fröberg-Iarrobino conjecture, and with a comparison with previous results by Chandler.

When the points are more than $n+2$, we are able to give a sufficient condition for a linear systems to be linearly non-special, see Theorem 5.3 for the precise statement. In this case, it is easy to find linear systems which are linearly special. For instance the speciality might be given by rational normal curves or by quadric hypersurfaces contained in the base locus. In Section 6.2 we give a short account on future directions of our research.

We would like to point out, finally, that our results have also an interesting interpretation in the setting of moduli space of stable rational curves with marked points, see Section 6.3 for more details.

The paper is organized as follows. In Section 2 we describe the linear components of the base locus of any linear system. In Section 3 we give the definition of linear expected dimension and of linear speciality, and we discuss some basic examples. Sections 4 and 5 contain the main results of the paper; precisely Theorem 4.6 and Corollary 4.8 are devoted to the case $s \leq n+2$, while Theorem 5.3 concerns the case $s \geq n+3$. In Section 6 we discuss the links with other approaches, interesting connections and future directions.

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2. The linear components of the base locus

Let $\mathcal{L} := \mathcal{L}_{n,d}(m_1, ..., m_s)$ be a linear system with multiple base points supported at general points $p_1, \ldots, p_s \in \mathbb{P}^n$. Let moreover $I(r) \subseteq \{1, \ldots, s\}$ be any multi-index of length $|I(r)| = r + 1$, for $0 \leq r \leq \min(n, s) - 1$. We denote by $L_{I(r)}$ the unique r-linear cycle through the points p_i, for $i \in I(r)$. We introduce the following notation:

$$k_{I(r)} := \max\left(\left(\sum_{i \in I(r)} m_i\right) - rd, 0\right). \tag{2.1}$$

The following lemma is equivalent to [9, Cor. 5.2], however we include here the proof for the sake of completeness.

Lemma 2.1 (Linear Base Locus Lemma). *Let $\mathcal{L} := \mathcal{L}_{n,d}(m_1, ..., m_s)$ be a non-empty linear system. In the notation of above, assume that $0 \leq r \leq n - 1$ and $k_{I(r)} > 0$. Then $\mathcal{L}$ contains in its base locus the cycle $L_{I(r)}$ with multiplicity at least $k_{I(r)}$.*

Proof. We use induction on r. If $r = 1$ the statement follows from Bézout's theorem. If $r > 1$ consider the r-cycle $L_{I(r)}$ spanned by $p_1, \ldots, p_r, p_{r+1}$ as the cone over the $(r-1)$-cycle $L_{I(r-1)}$ spanned by $p_1, \ldots, p_r$ and with vertex p_{r+1}. Notice that $k_{I(r-1)} \geq k_{I(r)}$, being $d \geq m_{r+1}$. By the inductive hypothesis, any point $q \in L_{I(r-1)}$ is contained in the base locus of $\mathcal{L}$ with multiplicity at least $k_{I(r-1)} = m_1 + ... + m_r - (r-1)d > 0$. Therefore the line spanned by q and p_{r+1} is contained in the base locus with multiplicity at least $k_{I(r-1)} + m_{r+1} - d = k_{I(r)}$. □

Given the general points $p_1, \ldots, p_s$ in $\mathbb{P}^n$, we denote by

$$\pi^n_{(0)} : X^n_{(0)} \to \mathbb{P}^n$$

the blow-up of $\mathbb{P}^n$ at $p_1, \ldots, p_s$, with $E_1, \ldots, E_s$ exceptional divisors. The index (0) indicates that the space $\mathbb{P}^n$ is blown-up at 0-dimensional schemes; in the same way $X^n_{(r)}$ will denote the n-dimensional projective space blown-up along arrangements of linear cycles of dimension $\leq r$ spanned by the points p_i (see Section 4.1 for more details).

The Picard group of $X^n_{(0)}$ is spanned by the class H of a general hyperplane and the exceptional divisors E_i, $i = 1, \ldots, s$. As in [7] we introduce a symmetric bilinear form on the blown up $\mathbb{P}^n$ by

$$\langle E_i, E_j \rangle = -\delta_{i,j},\ \langle E_i, H \rangle = 0,\ \langle H, H \rangle = n - 1.$$

We recall that the standard Cremona transformation along the coordinate points of $\mathbb{P}^n$ is defined to be the birational map

$$[x_0, \ldots, x_n] \to [x_1 \ldots x_n, \ldots, x_0 \ldots x_{n-1}].$$

This map is given by the linear system of hypersurfaces of degree n with multiplicity $n-1$ at each of the $n+1$ coordinate points. Moreover it induces an automorphism of the Picard group of the blow-up $X^n_{(0)}$ at $s \geq n+1$ points by sending any divisor $dH - m_1E_1 - \cdots - m_sE_s$ to

$$(d-c)H - (m_1 - c)E_1 - \cdots - (m_{n+1} - c)E_{n+1} - m_{n+2}E_{n+2} - \cdots - m_sE_s,$$

where $c = m_1 + \cdots + m_{n+1} - (n-1)d$ and the first $n+1$ points are chosen to be the coordinate points of $\mathbb{P}^n$.

Let W denote the Weyl group of the blow-up $X^n_{(0)}$ at s points. Recall that every element of W corresponds to a birational map of $\mathbb{P}^n$ lying in the group generated by standard Cremona transformations and projective automorphisms of $\mathbb{P}^n$. For more information on the properties of the Cremona transformations and the Weyl group see [5, Section 2.3] or [13, 14].

The study of the effective cone of a linear system is extremely difficult in general, for instance in the case $n = 2$ and $s \geq 10$ Nagata's conjecture as well as the computation of the effective cone $\overline{M_{0,n}}$ are still open problems (see also Section 6.3). In the following lemma we describe the effective cone of a linear system with a small number of points and we briefly sketch the proof, see also [5, Lemma 4.8].

Lemma 2.2 (Effectivity Lemma). *A linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ is non-empty if*

$$m_i \leq d \quad \forall i \quad \text{and} \quad \sum_{i=1}^{s} m_i \leq nd. \tag{2.2}$$

Moreover if $s \leq n+2$, then $\mathcal{L}$ is non-empty if and only if conditions (2.2) are satisfied. In particular the faces of the effective cone of the blow-up of $\mathbb{P}^n$ at s points are given by $\{d = m_1\}, \ldots, \{d = m_s\}$ if $s \leq n$ and by $\{d = m_1\}, \ldots, \{d = m_s\}, \{nd = \sum_{i=1}^{s} m_i\}$ if $s = m+1, n+2$.

Proof. By [7, Lemma 4.24], we know that conditions (2.2) imply that $\mathcal{L}$ is not empty.

Conversely, consider an effective divisor $D \in \mathcal{L}$ and note that we must have $d \geq m_1, \ldots, d \geq m_s$. Assume now by contradiction that $nd < \sum_{i=1}^{s} m_i$.

If $s = n+1$, then since $d \geq m_i$, by Lemma 2.1 we get that each of the $n+1$ hyperplanes H_i, spanned by all but the i-th point, is in the base locus at least $k(i) = m_1 + \cdots + \widehat{m_i} + \cdots + m_{n+1} - (n-1)d > 0$ times. We obtain $D = \sum_{i=1}^{n+1} k(i) H_i + \mathrm{Res}$, where Res is the residual system. But the degree of Res is negative, indeed it is

$$d - \sum_{i=1}^{n+1} k(i) = d - \sum_{i=1}^{n+1} (m_1 + \ldots \widehat{m_i} + \cdots + m_{n+1} - (n-1)d) =$$

$$= d - (n \sum_{i=1}^{n+1} m_i - (n+1)(n-1)d) = n(nd - \sum_{i=1}^{n+1} m_i) < 0,$$

therefore D is not effective and this gives a contradiction.

If $s = n+2$, then since $nd < \sum_{i=1}^{n+2} m_i$ and $d \geq m_{n+2}$, we have $c = \sum_{i=1}^{n+1} m_i - (n-1)d > 0$. So performing a Cremona transformation based at the first $n+1$ points, since we have $d - c = nd - \sum_{i=1}^{n+1} m_i < 0$, we obtain a linear system with negative degree, so D is again not effective, and we have a contradiction.

Notice, finally, that if $s \leq n$ the inequality $\sum_{i=1}^{s} m_i \leq nd$ is redundant. $\square$

We now improve Lemma 2.1, giving results which predict the exact multiplicity with which a cycle is contained in the base locus of the linear system.

The following proposition is more general, concerning also non linear divisors, anyway it applies in particular to the case of hyperplanes $L_{I(n-1)}$.

Proposition 2.3. *Let $\mathcal{L}_{n,d}(m_1,...,m_s)$ be a linear system and $D \in \mathcal{L}$. Let W be the Weyl group of the blow-up, $X^n_{(0)}$, at the s base points of $\mathcal{L}$ and let F be a divisor in any Weyl orbit of E_i. If $\langle D, F\rangle \leq 0$, then F is contained in the base locus of $\mathcal{L}$ with multiplicity $-\langle D, F\rangle$.*

Proof. Let $F = w^{-1}(E_i)$ for $w \in W$. We note that $\langle E_i, w(D)\rangle = \langle w(F), w(D)\rangle = \langle F, D\rangle < 0$, hence [5, Lemma 4.3] implies that any exceptional divisor E_i is contained in the base locus of the divisor $w(D)$ with multiplicity equal to $-\langle E_i, w(D)\rangle$. Applying the birational map of $\mathbb{P}^n$ corresponding to the element w^{-1} we obtain that F is contained in the base locus of D with the same multiplicity, that equals $-\langle F, D\rangle$. □

In particular, the above proposition shows that, for $n = 2$, the multiplicity of containment of any (-1)-curve is given by the intersection product.

The following result is an easy consequence of Lemma 2.1 and concerns the case where the multiplicities of the first points equal the degree of the linear system.

Proposition 2.4. *Let $\mathcal{L} := \mathcal{L}_{n,d}(d^t, m_1,...,m_s)$ be a non-empty linear system. Given $r \leq t-1$, let $L_{I(r)}$ be the r-cycle spanned by $r+1$ among the first t points. Then $\mathcal{L}$ contains in its base locus the cycle $L_{I(r)}$ with multiplicity d.*

Proof. Since the first $r+1$ points have multiplicity d we obtain $k_{I(r)} = d$. Hence by Lemma 2.1 the cycle $L_{I(r)}$ is contained in the base locus with multiplicity at least d. Obviously the multiplicity can not be higher than the degree, being $\mathcal{L}$ not empty. □

Finally we give a result concerning the case of linear systems with at most $n+2$ points.

Proposition 2.5. *Assume that $s \leq n+2$ and let $\mathcal{L}_{n,d}(m_1,...,m_s)$ be a non-empty linear system. If $0 \leq r \leq \min(n,s)-1$, then $\mathcal{L}_{n,d}(m_1,...,m_s)$ contains in its base locus the cycle $L_{I(r)}$ with multiplicity $k_{I(r)}$, for any $I(r) \subseteq \{1,\ldots,s\}$.*

Proof. For every multi-index $I(l) \subseteq \{1,\ldots,s\}$ we write $K_{I(l)} := \sum_{i\in I(l)} m_i - ld$, so that $k_{I(l)} = \max(K_{I(l)}, 0)$.

Fixed a multi-index $I(r) \subseteq \{1,\ldots,s\}$, we consider separately the following cases:

(1) $k_{I(r)} = K_{I(r)} \geq 0$
(2) $k_{I(r)} \neq K_{I(r)} < 0$.

Case (1). We denote by

$$R := \max\{l | K_{I(l)} \geq 0, I(r) \subseteq I(l)\},$$

If $r = n-1 (= R)$ then our claim is true by Proposition 2.3. Next, we consider separately the following cases:

(i) $r < R$
(ii) $r = R$.

Case (i). *We assume that for any $I(R)$ such that $K_{I(R)} \geq 0$ the cycle $L_{I(R)}$ is contained in $\mathcal{L}$ with multiplicity $K_{I(R)}$, and we prove that any cycle $L_{I(r)}$ is contained in* $\mathrm{Bs}(\mathcal{L})$ *with multiplicity $K_{I(r)}$.*

Notice that $0 \leq K_{I(R)} \leq K_{I(r)}$, since $m_i \leq d$. Therefore all linear subcycles $L_{I(l)}$ of $L_{I(R)}$ are contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity at least $K_{I(l)} \geq 0$, by Lemma 2.1; in particular $L_{I(r)}$ is contained at least $K_{I(r)}$ times. Assume now by contradiction that $L_{I(r)}$ is contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity at least $1+K_{I(r)}$. We know that the linear cycle $L_{J(R-r-1)}$ is contained in the base locus with multiplicity at least $K_{J(R-r-1)} \geq 0$, where $J(R-r-1) := I(R) \setminus I(r)$. For any point p in the cycle $L_{I(r)}$ and p' in the cycle $L_{J(R-r-1)}$ we get that the line spanned by p and p' is contained in the base locus with multiplicity at least $1+K_{I(r)}+K_{J(R-r-1)}-d = 1+K_{I(R)}$ and this is a contradiction. Hence $L_{I(r)}$ is contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity $K_{I(r)}$.

Case (ii). *We prove that for any $I(R)$ such that $K_{I(R)} \geq 0$ the cycle $L_{I(R)}$ is contained in $\mathcal{L}$ with multiplicity $K_{I(R)}$.*

We prove this claim for any non-empty linear system in $\mathbb{P}^n$ by using backward induction on R. Given $R \leq n-2$, assume now that for every non-empty linear system $\mathcal{L}$ in $\mathbb{P}^n$ such that

$$\max\{l | K_{I(l)} \geq 0, I(r) \subseteq I(l)\} = R+1,$$

for any multi-index $I(R+1)$ such that $K_{I(R+1)} \geq 0$, we know that the cycle $L_{I(R+1)}$ is contained in the base locus with multiplicity $K_{I(R+1)}$. We prove the statement for a non-empty linear system with

$$\max\{l | K_{I(l)} \geq 0, I(r) \subseteq I(l)\} = R.$$

Let $I(R) = \{i_1, \ldots, i_{R+1}\}$ such that $K_{I(R)} \geq 0$.

We first consider the case $s \leq n$. By the effectivity of $\mathcal{L}$ we have $nd \geq m_1 + \cdots + m_s$. If $m_i = d$ for all $i \in I(r)$ we conclude by Proposition 2.4, hence we can assume that $m_i < d$ for some $i \in I(r)$. This implies that $0 < d - K_{I(R)} \leq d$. We consider now the subsystem $\mathcal{L}'$ of $\mathcal{L}$ obtained by adding another general point p_{s+1} of multiplicity $d - K_{I(R)}$. This is a linear system based at, at least, $R+2$ points, and which is clearly effective, by Lemma 2.2. Now, for the linear system $\mathcal{L}'$, we have $K_{I(R+1)} = 0$ by construction, for all $I(R+1) \ni s+1$, and therefore by induction $L_{I(R+1)}$ is contained in $\mathrm{Bs}(\mathcal{L}')$ with multiplicity $K_{I(R+1)}$. Hence by applying case (i) we deduce that the cycle $L_{I(R)}$ is contained in $\mathrm{Bs}(\mathcal{L}')$ with multiplicity $K_{I(R)}$. We finally note that $\mathrm{Bs}(\mathcal{L}') \supseteq \mathrm{Bs}(\mathcal{L})$, hence we deduce that $L_{I(R)}$ is also contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity exactly $K_{I(R)}$.

Now we consider the case $n+1 \leq s \leq n+2$. We define

$$\begin{aligned} q_{\mathcal{L}} :&= \min\{(R+1)d - (m_{i_1} + \cdots + m_{i_{R+1}} + m_j) | j \notin I(R)\} \\ &= \min\{d - m_j - K_{I(R)} | j \notin I(R)\} = \min\{-K_{I(R+1)} | I(R+1) = I(R) \cup \{j\}\}. \end{aligned}$$

Let $I(R+1) = \{i_1, \ldots, i_{R+1}, i_{R+2}\}$ for some $i_{R+2} \notin I(R)$ such that $-K_{I(R+1)} = q_{\mathcal{L}}$. Note that $q_{\mathcal{L}} > 0$, by definition of R, and recall that $K_{I(R)} \geq 0$, hence we have $d > m_{i_{R+2}}$. Now if $m_i = d$ for all $i \in I(R)$, then we apply Proposition 2.4 to conclude that $L_{I(R)}$ is contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity d. Otherwise, after possibly reordering the points, we can assume also that $m_{i_{R+1}} < d$. Now we restrict $\mathcal{L}$ to a hyperplane passing through n points, such that $p_{i_1}, \ldots, p_{i_R}$ are among them and $p_{i_{R+1}}, p_{i_{R+2}}$ are not. The residual linear system Res is non-empty by construction, in fact it satisfies conditions (2.2), because $\mathcal{L}$ does. It is easy to check that $q_{\mathrm{Res}} = q_{\mathcal{L}} - 1$. We repeat this procedure, after possibly reordering the points, until either all points on $L_{I(R)}$ have multiplicity equal to the degree, or we get a residual, that we denote again by Res abusing notation, for which $q_{\mathrm{Res}} = 0$. Since, the linear system $\mathcal{L}$ contains the subsystem given by the hypersurfaces reducible to the sum of hypersurfaces of Res and hyperplanes not containing the cycle $L_{I(R)}$, then clearly we have $\mathrm{Bs}(\mathcal{L}) \subseteq \mathrm{Bs}(\mathrm{Res})$. Now note that $q_{\mathrm{Res}} = 0$, in particular in

Res there is a cycle $L_{I(R+1)}$ with $K_{I(R+1)}=0$. Hence we apply step (i) to conclude that $L_{I(R)}$ is contained in Bs(Res) with multiplicity $K_{I(R)}$. Hence it follows that $L_{I(R)}$ is contained in Bs$(\mathcal{L})$ with multiplicity $K_{I(R)}$.

Case (2). An easy remark is that $L_{I(r)}$ contains at least two points, say $p_{i_r}, p_{i_{r+1}}$, with $d > m_{i_r}, m_{i_{r+1}}$, since $d \geq 1$. We consider the restriction of $\mathcal{L}$ to a hyperplane passing through $\min(n,s)$ points of $\mathcal{L}$, such that $p_{i_1},\ldots,p_{i_{r-1}} \in L_{I(r)}$ are among them and p_r, p_{r+1} are not. In this way, as above, the residual system Res satisfies conditions (2.2) hence it is non-empty. We can repeat this procedure until we get a residual for which the corresponding number $K_{I(r)}$ is null. Therefore, by the previous case, $L_{I(r)}$ is not contained in the Bs(Res), so it is not contained in Bs$(\mathcal{L})$. □

3. Linear expected dimension and linear speciality

In this section we will introduce the main notion of the paper, which is the notion of linear speciality.

We consider first, as an illustrative example, the case of cones which is easy and well-understood. Note that any divisor in the linear system $\mathcal{L}_{n,d}(d^s)$ is a cone with vertex the linear space spanned by the s points, for any $0 \leq s \leq n$ (this immediately follows from Proposition 2.4). On the other hand if $s \geq n+1$ the linear system $\mathcal{L}_{n,d}(d^s)$ is empty. We get then the following result.

Proposition 3.1. *Given $0 \leq s \leq n+1$ and the linear system $\mathcal{L} = \mathcal{L}_{n,d}(d^s)$, we have*

$$\mathrm{h}^1(\mathbb{P}^n,\mathcal{L}) = \sum_{i=2}^{s}(-1)^i\binom{n+d-i}{n}\binom{s}{i}. \tag{3.1}$$

Proof. Let Z be a collection of s multiplicity-d points in $\mathbb{P}^n$. We have:

$$\mathrm{h}^0(\mathbb{P}^n,\mathcal{I}_Z\otimes\mathcal{O}(d)) = \mathrm{h}^0(\mathbb{P}^{n-s},\mathcal{O}(d)) = \binom{n-s+d}{d}$$

if $s \leq n$, and $\mathrm{h}^0(\mathbb{P}^n,\mathcal{I}_Z\otimes\mathcal{O}(d)) = 0$ if $s = n+1$. Hence

$$\mathrm{h}^1(\mathbb{P}^n,\mathcal{I}_Z\otimes\mathcal{O}(d)) = \binom{n-s+d}{d} - \binom{n+d}{d} + s\binom{n+d-1}{n}.$$

It is easy to prove that

$$\binom{n-s+d}{d} - \binom{n+d}{d} + s\binom{n+d-1}{n} = \sum_{i=2}^{s}(-1)^i\binom{n+d-i}{n}\binom{s}{i} \tag{3.2}$$

by double induction on $n \geq 1$ and $d \geq 1$. This concludes the proof. □

We can interpret formula (3.1) in the following way: there are $\binom{s}{i}$ linear $(i-1)$-cycles contained in the base locus, each of them having multiplicity of containment d and giving an obstruction equal to of $(-1)^i\binom{n+d-i}{n}$ to the speciality.

This example suggests that, in general, if a linear subspace $\mathbb{P}^r$ is contained in the base locus of any linear system $\mathcal{L}$ with multiplicity k, then its contribution to the speciality of $\mathcal{L}$ is

$$(-1)^{r+1}\binom{n+k-r-1}{n}.$$

Therefore we give the following new definition of virtual (and expected) dimension which takes in account these linear obstructions.

Definition 3.2. Given a linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \dots, m_s)$, for any integer $-1 \le r \le s-1$ and for any multi-index $I(r) = \{i_1, \dots, i_{r+1}\} \subseteq \{1, \dots, s\}$, with the convention $I(-1) := \emptyset$, we adopt the following notation, which extends formula (2.1):

$$k_{I(r)} := \max(m_{i_1} + \ldots + m_{i_{r+1}} - rd, 0).$$

The *linear virtual dimension* of $\mathcal{L}$ is the number

$$\sum_{r=-1}^{s-1} \sum_{I(r) \subseteq \{1,\dots,s\}} (-1)^{r+1} \binom{n + k_{I(r)} - r - 1}{n} - 1. \tag{3.3}$$

The *linear expected dimension* of $\mathcal{L}$, denoted by $\operatorname{ldim}(\mathcal{L})$ is defined as follows: if the linear system $\mathcal{L}$ is contained in a linear system whose linear virtual dimension is negative, then we set $\operatorname{ldim}(\mathcal{L}) = -1$, otherwise we define $\operatorname{ldim}(\mathcal{L})$ to be the maximum between the linear virtual dimension of $\mathcal{L}$ and -1.

We say that $\mathcal{L}$ is *linearly special* if $\dim(\mathcal{L}) \neq \operatorname{ldim}(\mathcal{L})$. Otherwise we say that $\mathcal{L}$ is *linearly non-special.*

Notice that, according to this definition, we argue in Proposition 3.1 that any linear system of the form $\mathcal{L}_{n,d}(d^s)$ is linearly non-special.

Remark 3.3. Note that if $\mathcal{L}$ is non-empty, then the terms corresponding to $n \le r \le s-1$ in the sum of formula (3.3) vanish. In the next section we will prove that, in this case,

$$\operatorname{ldim}(\mathcal{L}) = \chi(\tilde{\mathcal{L}}) - 1,$$

where $\chi(\tilde{\mathcal{L}})$ is the Euler characteristic of the sheaf associated to the strict transform of $\mathcal{L}$ after blowing up the linear base locus.

Remark 3.4. It is easy to prove that $\operatorname{vdim}(\mathcal{L}) \le \operatorname{ldim}(\mathcal{L})$ for any linear system $\mathcal{L}$. On the other hand, we expect that $\operatorname{ldim}(\mathcal{L}) \le \dim(\mathcal{L})$. This inequality is not obvious and it is indeed equivalent to the *weak Fröberg-Iarrobino conjecture*, see [9, Conjecture 4.5]. Note also that our definition of linear virtual dimension is equivalent to [9, Definition 6].

Also the family of homogeneous linear systems with only triple points $\mathcal{L}_{n,d}(3^s)$, for arbitrary s, is well suited to test our approach. By means of computer-aided computations [18], we calculated the dimensions of these linear systems for $n = 3, 4, 5$ and low degree, getting the following (partial) classification:

In $\mathbb{P}^3$, if $d \le 38$, the only special linear systems are:

- $\mathcal{L}_{3,3}(3^2), \mathcal{L}_{3,3}(3^3)$ linearly non-special,
- $\mathcal{L}_{3,4}(3^2), \mathcal{L}_{3,4}(3^3), \mathcal{L}_{3,4}(3^4)$ linearly non-special,
- $\mathcal{L}_{3,6}(3^9)$ linearly special.

In $\mathbb{P}^4$, if $d \le 10$, the only special linear systems are:

- $\mathcal{L}_{4,3}(3^2), \mathcal{L}_{4,3}(3^3), \mathcal{L}_{4,3}(3^4)$ linearly non-special,
- $\mathcal{L}_{4,4}(3^2), \mathcal{L}_{4,4}(3^3), \mathcal{L}_{4,4}(3^4), \mathcal{L}_{4,4}(3^5)$ linearly non-special,
- $\mathcal{L}_{4,6}(3^{14})$ linearly special.

In $\mathbb{P}^5$, if $d \le 7$, the only special linear systems are:

- $\mathcal{L}_{5,3}(3^2), \dots, \mathcal{L}_{5,3}(3^5)$ linearly non-special,
- $\mathcal{L}_{5,4}(3^2), \dots, \mathcal{L}_{5,4}(3^6)$ linearly non-special.

In other words our experiments show that there are no linearly special linear systems with triple points in $\mathbb{P}^3$ with degree ≤ 38 except for the case $\mathcal{L}_{3,6}(3^9)$, there are no linearly special linear systems with triple points in $\mathbb{P}^4$ with degree ≤ 10 except for $\mathcal{L}_{4,6}(3^{14})$, and there are no linearly special linear systems with triple points in $\mathbb{P}^5$ with degree ≤ 7.

The first line in the three groups of examples corresponds precisely to the case of cones, see Proposition 3.1.

Notice that the two exceptional cases $\mathcal{L}_{3,6}(3^9)$ and $\mathcal{L}_{4,6}(3^{14})$ can be explained by the fact that there is a quadric in the base locus which gives speciality. They were predicted both the by Fröberg-Iarrobino conjecture and by the Laface-Ugaglia conjecture, see Section 6 for a more detailed explanation about these conjectures.

Besides this two special cases, all other cases have $s \leq n+2$. In Corollary 4.8, we will give a precise description and an explicit computation of their speciality.

4. Linear systems with at most $n+2$ points

4.1. **Blowing up: construction and notation.** Let $p_1, \dots, p_s$ be general points in $\mathbb{P}^n$. For every integer $0 \leq r \leq \min(n,s)-1$ we denote by $I(r) \subseteq \{1,\dots,s\}$ a multi-index of length $|I| = r+1$, and by $L_{I(r)}$ the unique r-cycle through the points $\{p_i,\ i \in I(r)\}$: $L_{I(r)} \cong \mathbb{P}^r \subseteq \mathbb{P}^n$. Notice that $L_{I(0)} = p_i$ is a point.

Let $\mathcal{I}$ be a set of subsets of $\{1,\dots,s\}$ such that

(1) $\{i\} \in \mathcal{I}$, for all $i \in \{1,\dots,s\}$;
(2) if $I \subset J$ and $J \in \mathcal{I}$, then $I \in \mathcal{I}$.

Let $\Lambda = \Lambda(\mathcal{I}) \subset \mathbb{P}^n$ be the subspace arrangement corresponding to $\mathcal{I}$, i.e. the (finite) union of the linear cycles L_I for $I \in \mathcal{I}$. Let $\bar{r}$ be the dimension of the biggest linear cycle in Λ, i.e. $\bar{r} = \max_{I \in \mathcal{I}}(|I|) - 1$. Write $\Lambda = \Lambda_{(1)} + \cdots + \Lambda_{(\bar{r})}$, where $\Lambda_{(r)} = \cup_{I(r) \in \mathcal{I}} L_{I(r)}$.

Assume moreover that $\mathcal{I}$ satisfies the following condition

(3) if $I, J \in \mathcal{I}$, then $L_I \cap L_J = L_{I \cap J}$.

Notice that this condition is obviously satisfied when $s \leq n+1$.

As in Section 2, we denote by $\pi^n_{(0)} : X^n_{(0)} \to \mathbb{P}^n$ the blow-up of $\mathbb{P}^n$ at $p_1, \dots, p_s$, with $E_1, \dots, E_s$ exceptional divisors. Let us also consider the following sequence of blow-ups:

$$X^n_{(\bar{r})} \xrightarrow{\pi^n_{(\bar{r})}} \cdots \xrightarrow{\pi^n_{(3)}} X^n_{(2)} \xrightarrow{\pi^n_{(2)}} X^n_{(1)} \xrightarrow{\pi^n_{(1)}} X^n_{(0)},$$

where $X^n_{(r)} \xrightarrow{\pi^n_{(r)}} X^n_{(r-1)}$ denotes the blow-up of $X^n_{(r-1)}$ along the strict transform of $\Lambda_{(r)} \subset \mathbb{P}^n$, via $\pi^n_{(r-1)} \circ \cdots \circ \pi^n_{(0)}$. Let us denote by $E_{I(r)}$ the exceptional divisors of the cycles $L_{I(r)}$, for any $I(r) \in \mathcal{I}$. We will denote, abusing notation, by H the pull-back in $X^n_{(r)}$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ and by $E_{I(\rho)}$, for $0 \leq \rho \leq r-1$, the pull-backs in $X^n_{(r)}$ of the exceptional divisors of $X^n_{(\rho)}$, respectively.

Remark 4.1. Notice that, in the case $\bar{r} = n-1$, the map $X^n_{(n-1)} \xrightarrow{\pi^n_{(n-1)}} X^n_{(n-2)}$ is an isomorphism and in particular $\mathrm{Pic}(X^n_{(n-1)}) \cong (\pi^n_{(n-1)})^* \mathrm{Pic}(X^n_{(n-2)})$. Thus, in our notation, for every $I(n-1) \subseteq \{1,\dots,s\}$ we have

$$E_{I(n-1)} = H - \sum_{\substack{I(\rho) \subseteq I(n-1),\\ 0 \leq \rho \leq n-2}} E_{I(\rho)}.$$

Intersection theory on the blow-up $X^n_{(r)}$. The Picard group of $X^n_{(r)}$ is

$$\mathrm{Pic}(X^n_{(r)}) = \langle H, E_{I(\rho)} : 0 \leq \rho \leq r-1 \rangle.$$

Remark 4.2. For $r = 1, \dots, n-1$, if F is any divisor on $X^n_{(r-1)}$, then for any $i \geq 0$, we have

$$h^i(X^n_{(r)}, (\pi^n_{(r)})^* F) = h^i(X^n_{(r-1)}, F).$$

It follows from Zariski connectedness Theorem and by the projection formula (see for instance [19] or [24, Lemma 1.3] for a more detailed proof.).

Let $\pi = \pi^n_{(r)} \circ \dots \circ \pi^n_{(0)}$. Given $0 \leq \rho \leq \min(n,s) - 1$ and any multi-index $I(\rho) = \{i_1, \dots, i_{\rho+1}\}$, we denote by $H_{I(\rho)}$ the strict transform via π of a hyperplane $\mathcal{H}$ of $\mathbb{P}^n$ containing the points $p_{i_1}, \dots, p_{i_{\rho+1}}$.

Note that the total transform of $\mathcal{H}$ is

$$\pi^*(\mathcal{H}) = H_{I(\rho)} + \sum_{\substack{J \subseteq I(\rho), \\ |\rho| \leq r+1}} E_J,$$

since the hyperplane $\mathcal{H}$ contains the cycle L_J, for any $J \subseteq I(\rho)$, and the cycle has been blown up if its length is at most $r+1$.

Notice that $H_{I(\rho)}$ is the blow-up of the hyperplane $\mathcal{H}$ at the points $p_{i_1}, \dots, p_{i_{\rho+1}}$ and at all the cycles L_J for any $J \subseteq I(\rho)$ of length at most $r+1$. Denoting by h the pull-back of $\mathcal{O}_{\mathcal{H}}(1)$ and by e_J the corresponding exceptional divisors, we have

$$\mathrm{Pic}(H_{I(\rho)}) = \langle h, e_J : J \subset I(\rho), |J| \leq \min(n-2, r+1) \rangle.$$

Then we have:

$$h = H_{|H_{I(\rho)}},$$

and, for any multi-index J,

$$\begin{aligned} E_{J|H_{I(\rho)}} &= 0, && \text{if } J \cap I(\rho) = \emptyset, |J| \leq n-2, \\ E_{J|H_{I(\rho)}} &= e_{J \cap I(\rho)}, && \text{if } J \cap I(\rho) \neq \emptyset, |J| \leq n-2. \end{aligned}$$

If $r = n-2$, we also have some exceptional divisor E_J with $|J| = n-1$, and in this case, if $J \subseteq I(\rho)$ we have

$$E_{J|H_{I(\rho)}} = h - \sum_{K \subsetneq J} e_K.$$

The geometry of the exceptional divisors. Fix $0 \leq r \leq \min(n,s) - 1$ and consider an exceptional divisor $E_{I(r)}$ in $X^n_{(r)}$, for $I(r) = \{i_1, \dots, i_{r+1}\} \in \mathcal{I}$. Notice that

$$E_{I(r)} \cong X^r_{(r-1)} \times \mathbb{P}^{n-r-1} \subset X^n_{(r)},$$

where $X^r_{(r-1)}$ denotes the blow-up of $L_{I(r)} \cong \mathbb{P}^r$ along all linear ρ-cycles, $0 \leq \rho \leq r-1$, spanned by the points $p_{i_1}, \dots, p_{i_{r+1}}$.

Let us denote

$$\mathrm{Pic}(X^r_{(r-1)}) = \langle h, e_{I(\rho)} : I(\rho) \subset I(r), \rho \leq r-2 \rangle.$$

Recall that the canonical sheaf of $X^r_{(r-1)}$ is

$$\mathcal{O}_{X^r_{(r-1)}}(-(r+1)h + (r-1)\sum e_i + (r-2)\sum e_{I(1)} + \dots + \sum e_{I(r-2)}) \tag{4.1}$$

For any multi-index $I(r)$ we will denote by $x_{I(r)}$ the following divisor on $X^r_{(r-1)}$

$$x_{I(r)} = rh - \Big((r-1)\sum e_i + (r-2)\sum e_{I(1)} + \dots + \sum e_{I(r-2)}\Big) \tag{4.2}$$

Note that $x_{I(r)}$ is the Cremona transform of the hyperplane of $L_{I(r)}$.

Lemma 4.3. *For any exceptional divisor $E_{I(r)}$ in $X^n_{(r)}$, we have*

$$E_{I(r)|E_{I(r)}} \cong \mathcal{O}_{X^r_{(r-1)}\times\mathbb{P}^{n-r-1}}(-x_{I(r)}, -1), \tag{4.3}$$

where $x_{I(r)}$ is defined in (4.2).

Proof. Let $I = I(r) \in \mathcal{I}$ be the set of indices parametrizing the $r+1$ fundamental points of the linear cycle $L_{I(r)}$ in $X^n_{(0)}$ whose corresponding exceptional divisor in $X^n_{(r)}$ is $E_{I(r)}$. For $1 \leq \rho \leq r-1$, set $\varepsilon_{(\rho)} := \sum_{I(\rho)\subset I} e_{I(\rho)}$ to be the exceptional divisor of all linear cycles of dimension ρ of $\mathbb{P}^n$ that are contained in $L_{I(r)}$, namely of $\Lambda_{(\rho)|L_{I(r)}} \subset L_{I(r)} \cong \mathbb{P}^r$. Notice that, being $\pi^r_{(r-1)} : X^r_{(r-1)} \to X^r_{(r-2)}$ an isomorphism, then $\varepsilon_{(r-1)}$ is the pull-back of the sum of fundamental hyperplanes $H_{I(r-1)}$ of $X^r_{(0)}$:

$$\begin{aligned}
\varepsilon_{(r-1)} &= \sum_{I(r-1)\subset I} (h - \sum_{i\in I(r-1)} e_i - \sum_{I(1)\subset I(r-1)} e_{I(1)} - \cdots - \sum_{I(r-2)\subset I(r-1)} e_{I(r-2)}) \\
&= (r+1)h - r\sum_{i\in I} e_i - (r-1)\sum_{I(1)\subset I} e_{I(1)} - \cdots - 2\sum_{I(r-2)\subset I} e_{I(r-2)}.
\end{aligned}$$

Set now $\psi^* := (\pi^r_{(r-1)})^* \circ (\pi^r_{(r-2)})^* \cdots \circ (\pi^r_{(1)})^*$. By using [17, B.6.10], we compute the normal bundle $N_{X^r_{(r-1)}|X^n_{(r)}}$ of $X^r_{(r-1)}$ in $X^n_{(r)}$ and we get:

$$\begin{aligned}
N_{X^r_{(r-1))}|X^n_{(r-1)}} &= \psi^*(N_{X^r_{(0)}|X^n_{(0)}}) \otimes -\left(\sum_{\rho=1}^{r-2} \varepsilon_{(\rho)} + \varepsilon_{(r-1)}\right) \\
&= \mathcal{O}_{X^r_{(r-1)}}\left(\left(h - \sum_{i\in I} e_i\right) - \sum_{\rho=1}^{r-2} \varepsilon_{(\rho)} - \varepsilon_{(r-1)})\right)^{\oplus n-r} \\
&= \mathcal{O}_{X^r_{(r-1)}}(-x_{I(r)})^{\oplus n-r}
\end{aligned}$$

hence we deduce (4.3). $\square$

Remark 4.4. Given a multi-index I and F any divisor in $X^r_{(r-1)}$, if F contains k times the cycle L_I in its base locus, then we have

$$F_{|E_I} \cong \mathcal{O}_{X^r_{(r-1)}\times\mathbb{P}^{n-r-1}}(-kx_{I(r)}, 0)$$

where $x_{I(r)}$ is as in (4.2).

4.2. **The main theorem.** In this section we give our main result, Theorem 4.6, concerning the cohomology of all the strict transforms of a linear system. Note that as a consequence of this technical result we obtain that a non-empty linear system $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$ with $s \leq n+2$ is always linearly non-special (see Corollary 4.8).

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ be a non-empty linear system on $\mathbb{P}^n$. Elements D of $\mathcal{L}$ are in bijection with divisors on $X^n_{(0)}$ of the following form:

$$D_{(0)} := dH - \sum_{i=1}^{s} m_i E_i.$$

In the blow-up $X^n_{(r)}$ of $X^n_{(r-1)}$ along the union of the strict transforms of all r-cycles $L_{I(r)}$, the total transform of $D_{(r-1)} \subset X^n_{(r-1)}$ is

$$(\pi^n_{(r)})^* D_{(r-1)} = dH - \sum_{\substack{I(\rho),\\ 0\leq\rho\leq r-1}} k_{I(\rho)} E_{I(\rho)}, \tag{4.4}$$

while the strict transform of $D_{(r-1)}$ is

$$D_{(r)} = dH - \sum_{\substack{I(\rho),\\ 0\leq\rho\leq r-1}} k_{I(\rho)} E_{I(\rho)} - \sum_{I(r)} k_{I(r)} E_{I(r)} = dH - \sum_{\substack{I(\rho),\\ 0\leq\rho\leq r}} k_{I(\rho)} E_{I(\rho)}, \tag{4.5}$$

For the sake of simplicity throughout this paper we will abbreviate by $H^i(X^n_{(r)}, D_{(r)})$, or by $H^i(D_{(r)})$, the cohomology group $H^i(X^n_{(r)}, \mathcal{O}_{X^n_{(r)}}(D_{(r)}))$.

Remark 4.5. If the linear base locus of $\mathcal{L}$ has maximal dimension $n-1$, then there exists an effective divisor Δ in $X^n_{(0)}$ such that

$$H^i(X^n_{(n-1)}, D_{(n-1)}) \cong H^i(X^n_{(n-2)}, \Delta_{(n-2)}),$$

where $\Delta_{(n-2)}$ is the strict transform of Δ in $X^n_{(n-2)}$.

Proof. Recall that a hyperplane $L_{I(n-1)}$ through n points of multiplicity respectively $m_{i_1}, \ldots, m_{i_n}$ is contained in the base locus of $D = dH - \sum_{i=1}^s m_i E_i$ if and only if $k_{I(n-1)} = \sum_{j=1}^n m_{i_j} - (n-1)d \geq 1$, by Proposition 2.3. All hyperplanes for which $k_{I(n-1)} \geq 1$ split off D $k_{I(n-1)}$-many times, and the residual part is a divisor $\Delta = \delta H - \sum_{i=1}^s \mu_i E_i$ on $X^n_{(0)}$, with $\delta = d - \sum_{I(n-1)} k_{I(n-1)} \geq 0$ and $\mu_i = m_i - \sum_{I(n-1)\ni i} k_{I(n-1)} \geq 0$, for $i = 1, \ldots, s$.

Clearly Δ is effective and its linear base locus has maximal dimension $\leq n-2$. Denote by $\Delta_{(n-2)}$ the strict transform of Δ in $X^n_{(n-2)}$. The conclusion easily follows as the strict transform $D_{(n-1)}$ equals the total transform in $X^n_{(n-1)}$ of $\Delta_{(n-2)}$ in $X^n_{(n-2)}$. $\square$

The following theorem is the main result of this section:

Theorem 4.6. *Given a non-empty linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ and $D \in \mathcal{L}$, then the following holds.*

(i) Assume that r is an integer such that $1 \leq r \leq \min(n,s) - 1$. Then $h^i(X^n_{(r)}, D_{(r)}) = h^i(X^n_{(r-1)}, D_{(r-1)})$, for $i \leq r-1$, and $h^i(X^n_{(r)}, D_{(r)}) = 0$, for $i \geq r+2$. Moreover

$$h^r(X^n_{(r)}, D_{(r)}) - h^{r+1}(X^n_{(r)}, D_{(r)}) = h^r(X^n_{(r-1)}, D_{(r-1)}) - \sum_{I(r)} \binom{n + k_{I(r)} - r - 1}{n}.$$

(ii) Assume that $s \leq n+2$ and that r is such that $0 \leq r \leq \min(n,s) - 1$. Then $h^i(X^n_{(r)}, D_{(r)}) = 0$, for $i \geq 1$, $i \neq r+1$. Moreover, if $\bar{r}$ is the dimension of the linear base locus in $\mathcal{L}$, then $h^{r+1}(X^n_{(r)}, D_{(r)}) = 0$, for $r \geq \bar{r}$.

Remark 4.7. Notice that Theorem 4.6, part (ii) states that for a linear system $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$, with $s \leq n+2$, and any divisor $D \in \mathcal{L}$ we have

$$h^i(X^n_{(\bar{r})}, D_{(\bar{r})}) = 0, \qquad \text{for all } i \geq 1.$$

This means that after blowing up the linear base locus we get non-speciality of the strict transform $D_{(\bar{r})}$. In other words if $s \leq n+2$ we are able to affirmatively answer to Question 1.1.

The following is an immediate consequence of Theorem 4.6.

Corollary 4.8. *If $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ is a non-empty linear system with $s \leq n+2$, then its speciality is given by*

$$h^1(\mathcal{L}) = \sum_{\substack{I(r),\\ 1 \leq r \leq \bar{r}}} (-1)^{r-1} \binom{n + k_{I(r)} - r - 1}{n}, \tag{4.6}$$

where $\bar{r}$ is the dimension of the base locus.

In particular $\dim(\mathcal{L}) = \operatorname{ldim}(\mathcal{L})$ and $\mathcal{L}$ is linearly non-special.

More precisely the following corollary describes the cohomology of the strict transform $D_{(r)}$ for any r.

Corollary 4.9. *Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ be a non-empty linear system with $s \leq n+2$ and with linear base locus of dimension $\bar{r}$. For any divisor $D \in \mathcal{L}$ and for $0 \leq r \leq \bar{r}$ we have*

$$h^i(D_{(r)}) = 0, \quad i \neq 0, r+1,$$

$$h^{r+1}(D_{(r)}) = \sum_{\substack{I(\rho),\\ r+1 \leq \rho \leq \bar{r}}} (-1)^{\rho - r - 1} \binom{n + k_{I(\rho)} - \rho - 1}{n}.$$

Note that, by Lemma 2.2, the previous corollaries hold for all linear systems satisfying conditions (2.2).

4.3. **Proof of the main theorem.** We will split the proof of Theorem 4.6 in various steps and for this purpose we need to introduce the following notation.

Given the integers $n \geq 1$, $s \geq 0$ and r with $1 \leq r \leq \min(n, s) - 1$, we abbreviate by $(A_{n,s,r})$, $(B_{n,s,r})$ the following statements.

$(A_{n,s,r})$: For any linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ and $D \in \mathcal{L}$ we have:

$$h^i(D_{(r)}) = h^i(D_{(r-1)}), \quad \text{for } i \leq r-1,$$

$$h^i(D_{(r)}) = 0, \quad \text{for } i \geq r+2,$$

$$h^r(D_{(r)}) - h^{r+1}(D_{(r)}) = h^r(D_{(r-1)}) - \sum_{I(r)} \binom{n + k_{I(r)} - r - 1}{n}.$$

$(B_{n,s,r})$: For any linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ and $D \in \mathcal{L}$, for any integer $l_{I(r)}$ with $0 \leq l_{I(r)} \leq \min(r, k_{I(r)})$ we have:

$$h^i(D_{(r)}) = h^i(D_{(r)} + \sum_{I(r)} l_{I(r)} E_{I(r)}), \quad \text{for } i \geq 0.$$

Given the integers n, s, R, with $n \geq 1$, $0 \leq s \leq n+2$ and $0 \leq R \leq \min(n, s) - 1$, we abbreviate by $(C_{n,s,R})$ the following statement.

$(C_{n,s,R})$: For any linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ with linear base locus of dimension R, any $D \in \mathcal{L}$ and any r with $0 \leq r \leq \min(n,s) - 1$ we have:

$$h^i(D_{(r)}) = 0, \quad \text{for } i \geq 1,\, i \neq r+1,$$
$$h^{r+1}(D_{(r)}) = 0, \quad \text{for } r \geq R.$$

The proof of Theorem 4.6 will be by induction and the two following propositions provide the inductive steps.

Proposition 4.10. *Let $n \geq 2$, $s \geq 0$ and $1 \leq r \leq \min(n,s) - 1$. If for any ρ such that $1 \leq \rho \leq r-1$ statements $(A_{r,r+1,\rho})$, $(B_{r,r+1,\rho})$ and statement $(C_{r,r+1,r-1})$ hold, then statements $(A_{n,s,r})$ and $(B_{n,s,r})$ hold.*

Remark 4.11. Notice that, by applying Proposition 4.10 in the case $r = 1$, we get that statement $(C_{1,2,0})$, which holds trivially true, implies $(A_{n,s,1})$ and $(B_{n,s,1})$, for any n and s. This result was already proved by Laface and Ugaglia, see [24, Theorem 2.1].

Proof of Proposition 4.10. We define $\mathcal{I} := \{I \subset \{1, \ldots, s\} : k_I > 0\}$, so that the set of linear cycles $\Lambda(\mathcal{I})$ is the support of the linear base locus of $\mathcal{L}$. Notice that, under the effectivity assumption on $\mathcal{L}$, the set $\Lambda(\mathcal{I})$ is a subspace arrangement satisfying conditions (1),(2) and (3) of Section 4.1. In particular (2) is satisfied since $m_i \leq d$, and (3) holds because it is easy to prove, by using (2.2), that two disjoint index sets I and J belong to $\mathcal{I}$ only if $|I| + |J| \leq n+1$.

Fix a multi-index $I = I(r) \in \mathcal{I}$. Let E_I be the exceptional divisor of the corresponding linear r-cycle $L_I \subset \mathbb{P}^n$ and denote by F_I the following divisor on $X^n_{(r)}$:

$$F_I = (\pi^n_{(r)})^* D_{(r-1)} - \sum_{J \prec I} k_J E_J,$$

where $\prec$ is the lexicographic order on the set of index sets in $\mathcal{I}$ of cardinality $r+1$. For $0 \leq l \leq k-1$, consider the exact sequences of sheaves

$$0 \to F_I - (l+1)E_I \to F_I - lE_I \to (F_I - lE_I)_{|E_I} \to 0.$$

By Lemma 4.3 and Remark 4.4, we have

$$(F_I - lE_I)_{|E_I} \cong \mathcal{O}_{X^r_{(r-1)} \times \mathbb{P}^{n-r-1}}((l-k)x, l) \tag{4.7}$$

where $x = x_I$ is defined by (4.2).

Now let us compute the dimension of the cohomology groups of (4.7). Clearly we have $h^i(\mathcal{O}_{\mathbb{P}^{n-r-1}}(l)) = 0$ for all $i \neq 1$ and

$$h^0(\mathcal{O}_{\mathbb{P}^{n-r-1}}(l)) = \binom{n-r-1+l}{l}.$$

In order to compute $h^i(\mathcal{O}_{X^r_{(r-1)}}((l-k)x))$, notice that, by Serre duality and (4.1), we have

$$h^i(\mathcal{O}_{X^r_{(r-1)}}((l-k)x)) = h^{r-i}(\mathcal{O}_{X^r_{(r-1)}}((k-l-1)x - h)).$$

Set $a := (k-l-1)$. Notice that $0 \le a \le k-1$ and consider on $X^r_{(0)}$ the following divisor:

$$y = (ar-1)h - \sum_{j=1}^{r+1} a(r-1)e_j.$$

By using Proposition 2.5 one easily computes the strict transform of y via $\pi = \pi^r_{(r-1)} \circ \dots \circ \pi^r_{(1)}$:

$$\widetilde{y} = (ar-1)h - \sum a(r-1)e_j - \sum (a(r-2)+1)e_{ij} - \ldots - \sum (r-1)e_{J(r-1)}.$$

Since

$$ax - h = \widetilde{y} + \left(\sum e_{ij} + \ldots + \sum (r-1)e_{J(r-1)} \right),$$

by $(B_{r,r+1,r-1})$ we conclude that $h^i(ax-h) = h^i(\widetilde{y})$, for $i \ge 0$. On the other hand, as $(C_{r,r+1,r-1})$ holds for y, then $h^i(\widetilde{y}) = 0$ for all $i \ge 1$. Furthermore one computes

$$h^0(\widetilde{y}) = \binom{(a+1)r-1}{r} + \sum_{\substack{I(\rho),\\ 0 \le \rho \le r-1}} (-1)^{\rho-1} \binom{(a+1)(r-\rho-1)+\rho}{r} = \binom{a}{r},$$

obtaining the first equality by a combined application of statements $(A_{r,r+1,\rho})$, $(B_{r,r+1,\rho})$, and $(C_{r,r+1,r-1})$, that hold true for y, and the second equality by induction on r. It follows that $h^i((l-k)x) = 0$ for all $i \ne r$, and that and $h^r((l-k)x) = 0$ for $k-r \le l \le k-1$, while $h^r((l-k)x) = \binom{k-l-1}{r}$ for $0 \le l \le k-r-1$.

By means of Künneth formula (see e.g. [27, Section 2.1.7]), one easily computes that the only non-vanishing cohomology group of the sheaf introduced in (4.7) is

$$h^r((F_I - lE_I)_{|E_I}) = \binom{k-l-1}{r}\binom{n-r-1+l}{l}$$

for $0 \le l \le k-r-1$. From this we derive immediately statement $(B_{n,s,r})$ and the fact that $h^i(D_{(r)}) = h^i(D_{(r-1)})$, for $i \ne r, r+1$ and for every $1 \le r \le \min(n,s)-1$. In particular we obtain, for $i \ge r+2$, that $h^i(D_{(r)}) = h^i(D_{(r-1)}) = \cdots h^i(D_{(0)}) = 0$, the last equality being trivially true. Finally from

$$\sum_{l=0}^{k-r-1} \binom{k-l-1}{r}\binom{n-r-1+l}{l} = \binom{n+k-r-1}{n},$$

which can be easily proved by induction, we get $h^r(D_{(r-1)}) = h^r(D_{(r)}) - h^{r+1}(D_{(r)}) + \sum \binom{n+k_{I(r)}-r-1}{n}$, where the summation ranges over all the multi-indices $I(r) \subseteq \{1,\ldots,s\}$. This completes the proof of $(A_{n,s,r})$. $\square$

Remark 4.12. The geometric meaning of $(B_{n,s,r})$ is that if a linear r-cycle is contained with multiplicity at most r in the base locus of a linear system $\mathcal{L}$, then it does not contribute to the speciality of $\mathcal{L}$ (see Example 4.15).

Proposition 4.13. *Let $n \ge 2$, $s \ge 0$ and $1 \le \bar{r} \le \min(n-1,s)-1$. If statements $(A_{n,s,\rho})$ and $(B_{n,s,\rho})$ hold for any $1 \le \rho \le \bar{r}$ and moreover statements $(C_{n,s,R-1})$ and $(C_{n-1,s-1,R})$ hold for any $1 \le R \le \bar{r}$, then statement $(C_{n,s,\bar{r}})$ holds.*

Proof. Recall that $\mathcal{L} = \mathcal{L}_{n,d}(m_1,\ldots,m_s)$ satisfies condition (2.2), by Lemma 2.2.

Set $\sigma := \min(n,s)$. Define the linear system $\mathcal{L}' := \mathcal{L}_{n-1,d}(m_1,\ldots,m_\sigma)$ in $\mathbb{P}^{n-1}$ and notice that it also satisfies (2.2). Moreover define $\hat{\mathcal{L}}$ to be the linear system in $\mathbb{P}^n$ given either by $\hat{\mathcal{L}} := \mathcal{L}_{n,d-1}(m_1-1,\ldots,m_\sigma-1)$ if $s \le n$, or by $\hat{\mathcal{L}} := \mathcal{L}_{n,d-1}(m_1-$

$1,\dots,m_n-1,m_{n+1})$ if $s=n+1$, or by $\hat{\mathcal{L}} := \mathcal{L}_{n,d-1}(m_1-1,\dots,m_n-1,m_{n+1},m_{n+2})$ if $s=n+2$. It also satisfies (2.2), for all $s\leq n+2$.

Let now

$$D_{(\bar{r})} = dH - \sum_{i=1}^{s} m_i E_i - \sum_{\substack{I(\rho),\\ 1\leq\rho\leq\bar{r}}} k_{I(\rho)} E_{I(\rho)},$$

$$H_{(\bar{r})} = H - \sum_{i=1}^{\sigma} E_i - \sum_{\substack{I(\rho),\\ I(\rho)\subseteq\{1,\dots,\sigma\},\\ 1\leq\rho\leq\bar{r}}} E_{I(\rho)}$$

be the strict transforms respectively of $\mathcal{L}$ and of a hyperplane containing the points $p_1,\dots,p_\sigma$ in $X^n_{(\bar{r})}$ and consider the following Castelnuovo exact sequence:

$$0 \to D_{(\bar{r})} - H_{(\bar{r})} \to D_{(\bar{r})} \to D_{(\bar{r})|H_{(\bar{r})}} \to 0. \tag{4.8}$$

We can identify the restricted divisor $D_{(\bar{r})|H_{(\bar{r})}}$ of the sequence (4.8) with the strict transform $D'_{(\bar{r})}$ of the general divisor in $\mathcal{L}'$. Moreover if $\hat{D}_{(\bar{r})}$ is the strict transform of a general divisor in $\hat{\mathcal{L}}$ in $X^n_{(\bar{r})}$, then if $s\leq n+1$ we can identify the divisors $D_{(\bar{r})}-H_{(\bar{r})}$ and $\hat{D}_{(\bar{r})}$, while if $s=n+2$ we have

$$D_{(\bar{r})} - H_{(\bar{r})} = \hat{D}_{(\bar{r})} + \sum_{\substack{I(\rho)\\ \{n+1,n+2\}\subseteq I(\rho),\\ 1\leq\rho\leq\bar{r}}} E_{I(\rho)}.$$

From $(B_{n,s,\rho})$, $1\leq\rho\leq\bar{r}$, we derive the equalities $h^i(D_{(\bar{r})}-H_{(\bar{r})}) = h^i(\hat{D}_{(\bar{r})})$, $i\geq 0$. Notice that the dimension of the linear base locus of of $\hat{\mathcal{L}}$ is $\leq\bar{r}$. We iterate the procedure, each time reordering the points with respect to their multiplicity from the highest to the lowest, until the kernel corresponds to a divisor whose linear base locus has dimension $\leq\bar{r}-1$. We conclude that $h^i(D_{(\bar{r})})=0$, for $i\geq 1$, by using $(C_{n,s,R-1})$ and $(C_{n-1,s-1,R})$, with $R\leq\bar{r}$,

Finally, if $r<\bar{r}$, then statements $(A_{n,s,\rho})$, with $1\leq\rho\leq\bar{r}$, imply $h^i(D_{(r)})=0$, for $i\geq 1$, $i\neq r+1$. Indeed if $1\leq i\leq r$, from $(A_{n,s,\rho})$, $1\leq\rho\leq r$, one deduces the equalities $h^i(D_{(r)}) = h^i(D_{(r+1)}) = \cdots = h^i(D_{(\bar{r})}) = 0$; while, on the other hand, if $i\geq r+2$ then statements $(A_{n,s,r})$ implies $h^i(D_{(r)})=0$. If $r>\bar{r}$ then $D_{(r)}$ is the total transform in $X_{(r)}$ of $D_{(\bar{r})}$, therefore $h^i(D_{(r)}) = h^i(D_{(\bar{r})}) = 0$, $i\geq 1$. □

Proof of Theorem 4.6. We will prove statements $(A_{n,s,r})$, $(B_{n,s,r})$, with $n\geq 1, s\geq 0$ and $1\leq r\leq \min(n,s)-1$, by induction on n, and statement $(C_{n,s,R})$, with $n\geq 1, s\geq 0$ and $0\leq R\leq\min(n,s)-1$, by induction on n and R.

If $n=1$, then statements $(A_{1,s,r})$ and $(B_{1,s,r})$ trivially hold; furthermore statement $(C_{1,s,R})$ holds as well because any linear system in $\mathbb{P}^1$ is non-special, namely $h^i(X^1_{(0)}, D_{(0)})=0$, $i\geq 1$.

In order to prove $(A_{n,s,r})$ for $n\geq 2$ and $1\leq r\leq\min(n,s)-1$, we may assume by induction on n that $(A_{r,r+1,\rho})$, $(B_{r,r+1,\rho})$, for any $1\leq\rho\leq r-1$ and $(C_{r,r+1,r-1})$ hold so that we can apply Proposition 4.10 to obtain $(A_{n,s,r})$ and $(B_{n,s,r})$. This proves part (i).

Now we prove $(C_{n,s,R})$ by induction on R. If $R=0$ then $\mathcal{L}$ contains only points in its base locus and it is well-known to be non-special for $n\geq 1$ and $s\leq n+2$, hence $(C_{n,s,0})$ holds; while if $R=n-1$, we reduce to the case $R\le n-2$ exploiting Remark 4.5 . In order to prove $(C_{n,s,R})$ for the pair (n,R), with $n\ge 2$ and $1\le R\le \min(n-1,s)-1$, we assume that $(A_{n,s,\rho})$ and $(B_{n,s,\rho})$, for any ρ with $1\le\rho\le R-1$, and $(C_{n-1,s-1,R})$, $(C_{n,s,R-1})$ hold. By applying Proposition 4.13 we get $(C_{n,s,R})$ and this complete the proof of part (ii). □

4.4. **Examples.**

Example 4.14. Consider the case $\mathcal{L}=\mathcal{L}_{n,3}(3^3)$, with $n\ge 3$. Notice that the dimension of the linear base locus of $\mathcal{L}$ is $\bar{r}=2$. Indeed, by Proposition 2.5, all the lines and the plane spanned by the three base points are triply contained in the base locus of $\mathcal{L}$, i.e. $k_{12}=k_{13}=k_{23}=k_{123}=3$. Let us compute the cohomologies of the strict transforms $D_{(r)}$, $r\ge 0$, by means of Theorem 4.6:

- $h^i(X^n_{(0)},D_{(0)})=0$, for $i\ge 2$ and $h^1(X^n_{(0)},D_{(0)})=-h^2(X^n_{(1)},D_{(1)})+3(n+1)$;
- $h^i(X^n_{(1)},D_{(1)})=0$, for $i\ge 1, i\ne 2$ and $h^2(X^n_{(1)},D_{(1)})=1$;
- $h^i(X^n_{(r)},D_{(r)})=0$, for $i\ge 1, r\ge 2$.

Therefore $h^1(\mathbb{P}^n,\mathcal{L})=h^1(X^n_{(0)},D_{(0)})=3n+2$ and one can hence easily compute $h^0(\mathbb{P}^n,\mathcal{L})=h^0(X^n_{(r)},D_{(r)})=\binom{n}{3}$, as it was already obtained in Proposition 3.1.

Example 4.15. Consider the linear system $\mathcal{L}_{4,6}(5^3,4,3,2)$. It has virtual dimension $\operatorname{vdim}(\mathcal{L})=-56$ and linear virtual (and expected) dimension $\operatorname{ldim}(\mathcal{L})=6$. The linear base locus of $\mathcal{L}$ is formed by multiple points, lines and planes and a 3-dimensional linear cycle, by Proposition 2.5, namely $\bar{r}=3$.

Computing the values of the integers $k_{I(r)}$, for $1\le r\le\bar{r}$, we see that the contributions of the multiple lines to the speciality of $\mathcal{L}$ is 63 and that only the plane through the three points of multiplicity 5, which is triply contained in the base locus, gives a correction equal to -1. Moreover the 3-dimensional cycle is simply contained in the base locus so, as we noticed in Remark 4.12, it does not create speciality. Exploiting Theorem 4.6, we get

- $h^1(X^n_{(0)},D_{(0)})=-h^2(X^n_{(1)},D_{(1)})+63$;
- $h^2(X^n_{(1)},D_{(1)})=1$;

and all the other cohomologies vanish. Therefore $\mathcal{L}$ is linearly non-special and $h^1(\mathbb{P}^4,\mathcal{L})=62$ and $h^0(\mathbb{P}^4,\mathcal{L})=6$.

5. Linear systems with more than $n+2$ points

In this section we will obtain a sufficient condition to be linearly non-special for a linear system $\mathcal{L}_{n,d}(m_1,\dots,m_s)$ when $s\ge n+3$.

Lemma 5.1. *Given integers n,d, $m_i\le d$, consider the two linear systems $\mathcal{L}=\mathcal{L}_{n,d}(d,m_1,\dots,m_s)$ and $\mathcal{L}'=\mathcal{L}_{n-1,d}(m_1,\dots,m_s)$. If $\mathcal{L}$ and $\mathcal{L}'$ are non-empty, then we have* $\dim(\mathcal{L})=\dim(\mathcal{L}')$, *and* $\operatorname{ldim}(\mathcal{L})=\operatorname{ldim}(\mathcal{L}')$.

Proof. The first equality is obvious since any divisor in $\mathcal{L}$ is a cone with vertex in the point of multiplicity d. The second equality is easily proved. □

Remark 5.2. Given $\mathcal{L}=\mathcal{L}_{n,d}(m_1,\dots,m_s)$ and $1<t<s$, assume that the set $I=\{(i,j): t+1\le i<j\le s, m_i+m_j-d>0\}$ contains at most two pairs $(i_1,j_1),(i_2,j_2)$. Consider $\mathcal{L}'=\mathcal{L}_{n-1,d}(m_1,\dots,m_t,m_{i_1}+m_{j_1}-d,m_{i_2}+m_{j_2}-d)$

and $\mathcal{L}'' = \mathcal{L}_{n,d-1}(m_1 - 1, \ldots, m_t - 1, m_{t+1}, \ldots, m_s)$. If $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ are non-empty, then it is easy to prove that

$$\mathrm{ldim}(\mathcal{L}) = \mathrm{ldim}(\mathcal{L}') + \mathrm{ldim}(\mathcal{L}'') + 1.$$

The following theorem is the main result of this section:

Theorem 5.3. *Given the integers $n \geq 1$, $d \geq 2$, $s \geq n+3$, $d \geq m_1 \geq \ldots \geq m_s \geq 1$, consider the linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$. Let $s(d) \geq 0$ be the number of points of multiplicity d, that is $m_{s(d)} = d$ and $m_{s(d)+1} \leq d-1$. Define*

$$b(\mathcal{L}) := \min\{n - s(d), s - n - 2\}.$$

Then we have $\dim(\mathcal{L}) = \mathrm{ldim}(\mathcal{L})$, if the following condition is satisfied:

$$\sum_{i=1}^{s} m_i \leq nd + b(\mathcal{L}) \tag{5.1}$$

We split the proof of the theorem in the two Lemmas 5.4 and 5.7.

Lemma 5.4. *Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ such that (5.1) is satisfied. Then*

$$\dim(\mathcal{L}) \leq \mathrm{ldim}(\mathcal{L}).$$

Proof. Notice that it is enough to prove the statement when equality holds in (5.1) for a collection of s general points. Indeed any collection Z of s fat points which strictly satisfies inequality (5.1) can be thought as subscheme of a collection Z' of fat points given by Z and a suitable number s' of simple (general) points, which satisfies equality in (5.1); moreover if $\mathcal{L}' \subset \mathcal{L}$ is the linear system formed by the divisors in $\mathcal{L}$ that pass through these extra simple points, then $\dim(\mathcal{L}') = \dim(\mathcal{L}) - s'$ and $\mathrm{ldim}(\mathcal{L}') = \mathrm{ldim}(\mathcal{L}) - s'$.

We prove the statement by induction on d and n based on the cases $n = 1$ and $d = 2$, which are easily checked.

If $s(d) \geq 1$, then applying Lemma 5.1 $s(d)$ times we obtain $\dim(\mathcal{L}) = \dim(\mathcal{L}')$ and $\mathrm{ldim}(\mathcal{L}) = \mathrm{ldim}(\mathcal{L}')$, where $\mathcal{L}' := \mathcal{L}_{n-s(d),d}(m_{s(d)+1}, \ldots, m_s)$. Notice moroever that $b(\mathcal{L}') = b(\mathcal{L})$, and that from (5.1) it immediately follows that

$$\sum_{i=s(d)+1}^{s} m_i \leq (n - s(d))d + b(\mathcal{L}'),$$

hence by induction we have $\mathrm{ldim}(\mathcal{L}') = \dim(\mathcal{L}')$, so $\mathrm{ldim}(\mathcal{L}) = \dim(\mathcal{L})$.

Assume $s(d) = 0$. For any pair (n, d), we will assume the statement true for $(n-1, d)$ as well as for $(n, d-1)$ and consider suitable specializations of the points on a hyperplane $H \subset \mathbb{P}^n$ in such a way that, in the so obtained Castelnuovo exact sequence

$$0 \to \hat{\mathcal{L}} \to \mathcal{L} \to \mathcal{L}_H \to 0,$$

both the restricted system $\mathcal{L}_H$ and the kernel system $\hat{\mathcal{L}}$ satisfy the hypotheses of the theorem. We split the proof in the following subcases:

(i) $m_1 + m_2 \leq d$;
(ii) $m_1 + m_2 \geq d+1$ and $b(\mathcal{L}) = s - n - 2 < n$;
(iii) $m_1 + m_2 \geq d+1$, $m_1 + m_s \geq d+1$ and $b(\mathcal{L}) = n$;
(iv) $m_1 + m_2 \geq d+1$, $m_1 + m_s \leq d$ and $b(\mathcal{L}) = n$.

Case (i). Consider the hyperplane H through the first n points and notice that the trace $\mathcal{L}_H$ satisfies $\dim(\mathcal{L}_H) \leq \dim(\mathcal{L}_{n-1,d}(m_1,\ldots,m_n))$. From the assumption it follows that

$$\sum_{i<j,i,j=1}^{n}(m_i+m_j-d) = (n-1)\sum_{i=1}^{n} m_i - \binom{n}{2}d \leq 0.$$

Therefore, since $n \geq 2$, we have $\sum_{i=1}^{n} m_i \leq \frac{n}{2}d \leq (n-1)d$, so the linear system is not empty by Lemma 2.2. Hence by Corollary 4.8 we have $\dim(\mathcal{L}_{n-1,d}(m_1,\ldots,m_n)) = \operatorname{ldim}(\mathcal{L}_{n-1,d}(m_1,\ldots,m_n))$. The residual is $\hat{\mathcal{L}} = \mathcal{L}_{n,d-1}(m_1-1,\cdots,m_n-1,m_{n+1}\ldots,m_s)$. Since $d \geq 3$ and $m_1+m_2-d \leq 0$ we have that there are no points of multiplicity $d-1$ in $\hat{\mathcal{L}}$. If $m_n \geq 2$, then it is based on s points, and we have $b(\hat{\mathcal{L}}) = b(\mathcal{L})$. Otherwise, let $l \in \{1,\ldots,n\}$ be the integer such that $m_{n-l+1} = \cdots = m_s = 1$. As $b(\mathcal{L}) = \sum_{i=1}^{s} m_i - nd \leq (n-l)(d-1)+(s-n+l)-nd = s-2n-(d-2)l \leq s-n-2-l$, then we have $b(\hat{\mathcal{L}}) = b(\mathcal{L}) = n$. In both cases from (5.1) it follows that

$$\sum_{i=1}^{n}(m_i-1) + \sum_{i=n+1}^{s} m_i = n(d-1)+b(\hat{\mathcal{L}}),$$

hence by induction we obtain $\dim(\hat{\mathcal{L}}) = \operatorname{ldim}(\hat{\mathcal{L}})$. We conclude $\dim(\mathcal{L}) \leq \dim(\mathcal{L}_H) + \dim(\hat{\mathcal{L}})+1 \leq \operatorname{ldim}(\mathcal{L}_d^{n-1}(m_1,\ldots,m_n)) + \operatorname{ldim}(\hat{\mathcal{L}})+1 = \operatorname{ldim}(\mathcal{L})$ where the last equality follows from Remark 5.2.

Case (ii). In this case we specialize the last $s-2$ points on a general hyperplane H. The trace is $\mathcal{L}_H = \mathcal{L}_{n-1,d}(m_1+m_2-d,m_3,\ldots,m_s)$. From the assumption it follows $b(\mathcal{L}_H) = \min(n-1,(s-1)-(n-1)-2) = b(\mathcal{L})$ and so we have

$$(m_1+m_2-d)+\sum_{i=3}^{s} m_i \leq (n-1)d + b(\mathcal{L}_H),$$

and we get $\dim(\mathcal{L}_H) = \operatorname{ldim}(\mathcal{L}_H)$ by induction. The residual is $\hat{\mathcal{L}} = \mathcal{L}_{n,d-1}(m_1,m_2,m_3-1,\ldots,m_s-1)$. From (5.1) we have

$$m_1+m_2+\sum_{i=3}^{s}(m_i-1) = nd+b(\mathcal{L})-(s-2) = n(d-1).$$

Let $\hat{s}$ the number of points where $\hat{\mathcal{L}}$ is supported. We conclude that $\dim(\hat{\mathcal{L}}) = \operatorname{ldim}(\hat{\mathcal{L}})$, using Corollary 4.8 if $\hat{s} \leq n+2$, and using induction if $\hat{s} \geq n+3$. As in the previous case, by Remark 5.2, we conclude $\dim(\mathcal{L}) \leq \dim(\mathcal{L}_H)+\dim(\hat{\mathcal{L}})+1 = \operatorname{ldim}(\mathcal{L}_H)+\operatorname{ldim}(\hat{\mathcal{L}})+1 = \operatorname{ldim}(\mathcal{L})$.

Case (iii). Notice that $m_{s-1}+m_s \leq d$. Indeed if $m_i+m_j \geq d+1$ for all $1 \leq i,j \leq s$, then $(s-1)\sum_{i=1}^{s} m_i \geq \binom{s}{2}(d+1)$, so that $\sum_{i=1}^{s} m_i \geq \frac{s(d+1)}{2} \geq (n+1)(d+1) > nd+n$, and this leads to a contradiction with (5.1).

Since $m_1+m_s-d \geq 1$, then there exists an integer $\iota \in \{1,\ldots,s-2\}$ such that $m_\iota+m_s \geq d+1$, and $m_{\iota+1}+m_s \leq d$. Let us specialize all points but $m_\iota,m_{\iota+1},m_s$ on a hyperplane H. Since $m_\iota+m_{\iota+1} \geq m_\iota+m_s \geq d+1$, it follows that the trace is $\mathcal{L}_H = \mathcal{L}_d^{n-1}(m_1,\ldots,m_{\iota-1},m_{\iota+2}\ldots,m_{s-1},m_\iota+m_{\iota+1}-d,m_\iota+m_s-d)$ and $b(\mathcal{L}_H) = \min(n-1,(s-1)-(n-1)-2) = n-1$. Since $m_\iota < d$ and $b(\mathcal{L}_H) = n-1$

then

$$\sum_{i=1}^{s} m_i + m_\iota - 2d \leq (n-1)d + n + (m_\iota - d) \leq (n-1)d + b(\mathcal{L}_H),$$

and thus we have $\dim(\mathcal{L}_H) = \mathrm{ldim}(\mathcal{L}_H)$ by induction. The residual linear system is $\hat{\mathcal{L}} = \mathcal{L}_{n,d-1}(m_1 - 1, \ldots, m_{\iota-1} - 1, m_\iota, m_{\iota+1}, m_{\iota+2} - 1, \ldots, m_{s-1} - 1, m_s)$. Note that from the assumptions $m_\iota + m_s \geq d+1$ and $s(d) = 0$ we have $m_s \geq 2$, hence $\hat{\mathcal{L}}$ is supported at s points and $b(\hat{\mathcal{L}}) \geq 1$. It easily follows, being $s \geq 2n+2$ by assumption that

$$\sum_{i=1}^{s} m_i - (s-3) = nd + n - s + 3 \leq n(d-1) + 1 \leq n(d-1) + b(\hat{\mathcal{L}}),$$

and we conclude that $\dim(\hat{\mathcal{L}}) = \mathrm{ldim}(\hat{\mathcal{L}})$ by induction. We conclude, like in the previous cases, that $\dim(\mathcal{L}) \leq \mathrm{ldim}(\mathcal{L})$.

Case (iv). We specialize all points but m_1, m_2, m_s on a hyperplane H. The trace is $\mathcal{L}_H = \mathcal{L}_d^{n-1}(m_1 + m_2 - d, m_3, \ldots, m_{s-1})$ and $b(\mathcal{L}_H) = \min(n-1, (s-2) - (n-1) - 2) = b(\mathcal{L}) - 1 = n - 1$. Since

$$m_1 + m_2 - d + \sum_{i=3}^{s-1} m_i \leq nd + b(\mathcal{L}) - m_s - d \leq (n-1)d + b(\mathcal{L}_H),$$

we have $\dim(\mathcal{L}_H) = \mathrm{ldim}(\mathcal{L}_H)$ by induction. The residual linear system is $\hat{\mathcal{L}} = \mathcal{L}_{n,d-1}(m_1, m_2, m_3 - 1, \ldots, m_{s-1} - 1, m_s)$, and, as in the previous case $\sum_{i=1}^{s} m_i - (s-3) \leq n(d-1) + 1$. Let $\hat{s}$ the number of points where $\hat{\mathcal{L}}$ is supported. If $\hat{s} \geq n+3$, then clearly $b(\hat{\mathcal{L}}) \geq 1$ and we conclude by induction. If $\hat{s} \leq n+2$ then it has to be $m_{s-1} = 1$. Therefore the sum of the multiplicities of the points in $\hat{\mathcal{L}}$ is $\sum_{i=1}^{s-2} m_i - (s-2) + m_s \leq \sum_{i=1}^{s} m_i - (s-2) \leq n(d-1)$, being $s \geq 2n+2$. Hence it is non-empty by Lemma 2.2 and we conclude by Corollary 4.8. Finally we conclude, as in the previous cases, that $\dim(\mathcal{L}) \leq \mathrm{ldim}(\mathcal{L})$. □

Lemma 5.5. [9, Lemma 6.3] *Let $k \geq 1$, $d-1 \geq m_1 \geq \ldots \geq m_s \geq 1$ and consider the linear systems $\mathcal{L} = \mathcal{L}_{n,d}(k, m_1, \ldots, m_s)$ and $\mathcal{L}' = \mathcal{L}_{n,d}(k-1, m_1, \ldots, m_s)$. Denote $c_i = \max\{k + m_i - d - 1, 0\}$ and $\overline{s} = \max\{1 \leq i \leq s : c_i > 0\}$ and $\mathcal{W} = \mathcal{L}_{n-1,k-1}(c_1, \ldots, c_{\overline{s}})$. Then we have*

$$\dim(\mathcal{L}) \geq \dim(\mathcal{L}') - \dim(\mathcal{W}) - 1$$

Remark 5.6. With the notation of the previous lemma it is easy to check that if $\mathcal{L}, \mathcal{L}', \mathcal{W}$ are non-empty, then

$$\mathrm{ldim}(\mathcal{L}) = \mathrm{ldim}(\mathcal{L}') - \mathrm{ldim}(\mathcal{W}) - 1.$$

Lemma 5.7. *Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ such that (5.1) is satisfied. Then*

$$\dim(\mathcal{L}) \geq \mathrm{ldim}(\mathcal{L}).$$

Proof. We may assume $s(d) = 0$, thanks to Lemma 5.1.

The proof is by induction on $\sum_{i=1}^{s} m_i$, based on the case $\sum_{i=1}^{s} m_i = s$ for which the statement trivially holds being $\mathrm{ldim}(\mathcal{L}) = \mathrm{vdim}(\mathcal{L})$.

If $\sum_{i=1}^{s} m_i > s$, by induction we assume that the statement holds for any subscheme strictly contained in $\mathcal{L}$. Consider the linear systems

$$\mathcal{L}' := \mathcal{L}_{n,d}(m_1 - 1, m_2, \ldots, m_s) \ \text{ and } \ \mathcal{W} := \mathcal{L}_{n-1,m_1-1}(c_1, \ldots, c_{\overline{s}}),$$

where $c_i := \max\{k_{1,i+1} - 1, 0\}$, for $2 \leq i \leq s$ and $\bar{s} := \max\{2 \leq i \leq s : c_i > 0\} - 1$. Clearly $\mathcal{L}'$ satisfies condition (5.1) and by induction we have $\dim(\mathcal{L}') \geq \operatorname{ldim}(\mathcal{L}')$. We claim that

$$\sum_{i=1}^{\bar{s}} c_i \leq (n-1)(m_1 - 1).$$

It is easily verified when $\bar{s} \leq n-1$, while if $\bar{s} \geq n$, since

$$\begin{aligned} \sum_{i=1}^{\bar{s}} c_i = \sum_{i=2}^{\bar{s}+1} (m_1 + m_i - d - 1) &= \sum_{i=2}^{\bar{s}+1} m_i + \bar{s}(m_1 - d - 1) \\ &\leq \sum_{i=1}^{s} m_i - m_1 - (s - \bar{s} - 1) + \bar{s}(m_1 - d - 1), \end{aligned}$$

then from condition (5.1) and the fact that $b(\mathcal{L}) \leq s - n$ we get

$$\begin{aligned} \sum_{i=1}^{\bar{s}} c_i &\leq nd + b(\mathcal{L}) - m_1 - (s-1) + \bar{s}(m_1 - d) \\ &\leq (n-1)(m_1 - 1). \end{aligned}$$

Now, if $\bar{s} \leq n+2$ we already proved in Corollary 4.8 that $\dim(\mathcal{W}) = \operatorname{ldim}(\mathcal{W})$; if $\bar{s} \geq n+3$ then by Lemma 5.4 we have that $\dim(\mathcal{W}) \leq \operatorname{ldim}(\mathcal{W})$. Finally, by Lemma 5.5 and Remark 5.6 we have

$$\dim(\mathcal{L}) \geq \dim(\mathcal{L}') - \dim(\mathcal{W}) - 1 \geq \operatorname{ldim}(\mathcal{L}') - \operatorname{ldim}(\mathcal{W}) - 1 = \operatorname{ldim}(\mathcal{L}).$$

□

6. Concluding remarks and future directions

6.1. **The Fröberg-Iarrobino conjecture and Chandler results.** The results of this paper are connected with the Fröberg-Iarrobino conjecture, which is the geometrical version of an important conjecture formulated by Fröberg in the commutative algebra setting.

More precisely, let $R = \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial ring in $n+1$ variables over $\mathbb{C}$ and $I = (f_1, \ldots, f_s)$ be an ideal generated by s general forms of degrees $m_1, \ldots, m_s$. In [16], Fröberg conjectures that the Hilbert series of the quotient ring R/I is

$$\operatorname{Hilb}_{R/I}(t) = \left[\frac{\Pi_{i=1}^{s}(1 - t^{m_i})}{(1-t)^{n+1}} \right] \tag{6.1}$$

where we use the notation: $[\sum a_i t^i] = \sum b_i t^i$ with $b_i = a_i$ if $a_j > 0$ for all $j \leq i$, and $b_i = 0$ otherwise.

The Fröberg conjecture has been proved to be true for $n = 1$ [16] and $n = 2$ [1]. Moreover, it is easily verified when the number of generator is $s \leq n+1$, and in the case $s = n+2$ has been proved by Stanley (see [16, Example 2]). The conjecture is known to be true in some other special cases but it is still open in general.

The Fröberg-Iarrobino conjecture concerns the case when every form $f_i = (l_i)^{m_i}$ is a power of a general linear form l_i (see [9, 21] for more details). The Fröberg-Iarrobino conjecture implies the Fröberg conjecture, but they are not equivalent, because a power of a general linear form is not a general form.

More precisely, the Fröberg-Iarrobino conjecture deals with the homogeneous case, i.e. $m_i = m$, $i = 1, \ldots, s$, and states that formula (6.1) holds except for a given list of exceptions (see [9, Conjecture 4.8]). It is natural to generalize such a conjecture to the non-homogeneous case, namely when the m_i's are different, as Chandler pointed out. Since in the case of powers of linear forms, the ideal I can be seen as the ideal of a collection of fat points, it is possible to give a geometric interpretation of such a conjecture in terms of our Definition 3.2, namely a linear system is always linearly non-special but in a list of exceptions.

In [9, Proposition 9.1] it is proved that the the generalized Fröberg-Iarrobino conjecture is true if either $s \leq n+1$ or $\sum_{i=1}^{s} m_i \leq dn+1$. Our Corollary 4.8 and Theorem 5.3 improve Chandler's result and show that the generalized Fröberg-Iarrobino conjecture holds true if either $s \leq n+2$ or condition (5.1) is satisfied.

As already mentioned in Remark 3.4 there exists a weak version of the Fröberg-Iarrobino conjecture, see [9, Conjecture 4.5], which states that for any linear system $\mathcal{L}$ the inequality $\operatorname{ldim}(\mathcal{L}) \leq \dim(\mathcal{L})$ is verified. The weak Fröberg-Iarrobino conjecture for the case of $\mathbb{P}^n$, $n \leq 3$, is established in [9, Theorem 1.2]. Moreover in Lemma 5.7 we prove that such a conjecture holds true for any n and arbitrary number of points if condition (5.1) is satisfied.

6.2. **The Laface-Ugaglia conjecture and future directions.** In view of extending the well-known Segre-Harbourne-Gimigliano-Hirschowitz conjecture to $\mathbb{P}^3$, Laface and Ugaglia, in [23, Conjecture 4.1] and [25, Conjecture 6.3], formulated the following conjecture.

Conjecture 6.1 (Laface-Ugaglia). *If $\mathcal{L} = \mathcal{L}_{3,d}(m_1, \ldots, m_s)$ is Cremona reduced, i.e. $2d \geq m_{i_1} + m_{i_2} + m_{i_3} + m_{i_4}$, for any $\{i_1, i_2, i_3, i_4\} \subseteq \{1, \ldots, s\}$, then $\mathcal{L}$ is special if and only if one of the following holds:*

1. *there exists a line $L = \langle p_i, p_j \rangle$, for some $i, j \in \{1, \ldots, s\}$ such that $\mathcal{L} \cdot L \leq -2$;*
2. *there exists a quadric $Q = \mathcal{L}_{3,2}(1^9)$ such that $Q \cdot (\mathcal{L} - Q) \cdot (\mathcal{L} - K_{\mathbb{P}^3}) < 0$.*

The Laface-Ugaglia conjecture is known to be true when the number of points is at most 8 [12], and when the points are at most quartuple [2, 15] or quintuple [3].

We remark that this conjecture can be reformulated, according to our definition, saying that a Cremona reduced linear system in $\mathbb{P}^3$ either is linearly non-special, or contains in its base locus a quadric surface which gives speciality.

Our Corollary 4.8 proves that if the points are $s \leq n+2$, then such a conjecture holds true also in $\mathbb{P}^n$ for any n, i.e. a non-empty linear system (not necessarily Cremona reduced) with at most $n+2$ points is special if and only if $\mathcal{L} \cdot L \leq -2$ for some line $L = \langle p_i, p_j \rangle$. Moreover we prove that, in this case, $\dim(\mathcal{L}) = \operatorname{ldim}(\mathcal{L}) > \operatorname{edim}(\mathcal{L})$.

When the points are $s \geq n+3$, from Theorem 5.3 it follows that the same is true under the assumption (5.1). Such an assumption is, in particular, a sufficient condition for the base locus to contain no multiple rational normal curves. In fact, we expect that when multiple rational normal curves appear in the base locus, they give a contribution to the speciality of the system.

We point out that a Cremona reduced linear system in $\mathbb{P}^3$ does not contain rational normal curves in its base locus, but this fact is no longer true in $\mathbb{P}^4$. Indeed consider the following example:

Example 6.2. Set $\mathcal{L} = \mathcal{L}_{4,10}(6^7)$. Then $\mathcal{L}$ is Cremona reduced, but its base locus contains the double rational curve through the seven base points. On can easily see that $\mathcal{L}$ is linearly special, since $\dim(\mathcal{L}) = 140$, while

$$\operatorname{ldim}(\mathcal{L}) = \binom{14}{4} - 7\binom{9}{4} + 21 - 1 = 139.$$

It seems natural then to think that the double rational normal curve gives exactly a contribution of 1, similarly to a double line.

The following example suggests that only the contribution of rational normal curves is important, even when there are other multiple curves in the base locus.

Example 6.3. Consider the linear system $\mathcal{L} = \mathcal{L}_{5,6}(4^9)$. It has dimension $\dim(\mathcal{L}) = 2$. Indeed there is a normal elliptic curve C of degree 6 through the 9 points. The secant variety $\sigma_2(C)$ is a threefold cut out by two cubics hypersurfaces Σ_1 and Σ_2, defined by the equations $F_1 = 0$ and $F_2 = 0$. Then the equations $F_1^2 = 0$, $F_1F_2 = 0$, $F_2^2 = 0$ define three independent hypersurfaces of degree 6 which have multiplicity 4 along C and which generate $\mathrm{H}^0(\mathcal{L})$.

On the other hand, taking into account the contributions of the 36 double lines in the base locus, one computes

$$\operatorname{ldim}(\mathcal{L}) = \binom{11}{5} - 9\binom{8}{5} + 36 - 1 = -7,$$

hence $\mathcal{L}$ is linearly special. Furthermore, it is possible to prove that the base locus of $\mathcal{L}$ contains the 9 double rational normal curves through each set of 8 points and, assuming that each of their contribution is the same as the contribution of a double line (that is 1), one gets exactly: $\operatorname{ldim}(\mathcal{L}) + 9 = 2 = \dim(\mathcal{L})$.

The above examples, besides their intrinsic interest, show that the existence in the base locus of multiple rational normal curves, that are not removable by Cremona transformations, plays an important role. Hence it seems natural to extend our definition of dimension of a linear system, taking into account also the contribution of such curves. We plan to develop furtherly these ideas.

On the other hand, as already noticed by Laface and Ugaglia in $\mathbb{P}^3$, also the existence of quadric hypersurfaces passing through 9 general points in the base locus can give contribution to the speciality of a linear system. In this case it does not seem very clear how to quantify such contribution. Consider for example the following list, which contains all the special linear systems in $\mathbb{P}^3$ of degree 10 and with at most quintuple points (see [3, Table 4]). In the columns we write, respectively, the expected dimension, the dimension, the expected base locus, the residual system (which is non-special), and the difference between the dimension and the virtual dimension.

	edim	dim	base locus	residual	dim − vdim
$\mathcal{L}_{3,10}(5^9)$	-1	0	$5Q$	$\emptyset$	30
$\mathcal{L}_{3,10}(5^8, 4)$	-1	1	$4Q$	$\mathcal{L}_{3,2}(1^8)$	16
$\mathcal{L}_{3,10}(5^8, 3, 2)$	-1	0	$3Q_1 \cup 2Q_2$	$\emptyset$	9
$\mathcal{L}_{3,10}(5^8, 3)$	-1	2	$3Q$	$\mathcal{L}_{3,4}(2^8)$	7
$\mathcal{L}_{3,10}(5^8, 2^2)$	-1	1	$2Q_1 \cup 2Q_2$	$\mathcal{L}_{3,2}(1^8)$	4
$\mathcal{L}_{3,10}(5^8, 2)$	1	3	$2Q$	$\mathcal{L}_{3,6}(3^8)$	2
$\mathcal{L}_{3,10}(5^7, 4^2, 2)$	-1	1	$2Q$	$\mathcal{L}_{3,6}(3^7, 2^3)$	5
$\mathcal{L}_{3,10}(5^7, 4, 3^2)$	0	1	$Q_1 \cup Q_2$	$\mathcal{L}_{3,6}(3^7, 2^3)$	1
$\mathcal{L}_{3,10}(5^7, 4^2)$	0	5	$2Q$	$\mathcal{L}_{3,6}(3^7, 2^2)$	5

Special linear systems which contain fixed quadrics through simple points appear also in $\mathbb{P}^4$ (for example $\mathcal{L}_{4,4}(2^{14})$ and $\mathcal{L}_{4,6}(3^{14})$), but quite surprisingly we are not able to find examples in $\mathbb{P}^n$ for $n \geq 5$. Understanding better this phenomenon is another goal of our future work.

6.3. **Divisors on $\overline{M_{0,n}}$.** Let $\overline{M_{0,n}}$ be the moduli space of stable rational curves with n marked points. Kapranov's construction identifies it with a projective variety isomorphic to the projective space $\mathbb{P}^{n-3}$ successively blown up along $L_{I(r)}$, r-dimensional cycles spanned by $(r+1)$-subsets of a set with $n-1$ general points, for r increasing from 0 to $n-4$. In our paper, see Section 4, $\overline{M_{0,n}}$ is denoted by $X^{n-3}_{(n-4)}$ for $s = n-1$. This space has been well studied, however basic questions are still open.

We recall here Fulton's conjectures, concerning the description of the Nef cone and the Effective cone of $\overline{M_{0,n}}$. Fulton's weak conjecture states that 1-dimensional boundary strata, whose components are called F-curves, generate the Mori cone of curves, $\overline{NE}_1(\overline{M_{0,n}})$. This is proven for $n \leq 7$ in [22]. Fulton's strong conjecture, saying that the boundary divisors generate the effective cone of $\overline{M_{0,n}}$, is known to be false. For $n = 6$ the effective cone is described in [20, 6] as being spanned by boundary divisors and by the Keel-Vermeire divisor. Even though pull-backs of the Keel-Vermeire divisor under the forgetful morphism are extremal rays that are not boundary divisors, few things are known regarding the effective cone of $\overline{M_{0,n}}$ for $n \geq 7$. We should also mention the conjecture of Castravet and Tevelev regarding the effective cone of $\overline{M_{0,n}}$ for $n \geq 7$, see [8].

In our paper, Theorem 4.6 for $s = n-1$ points, computes the dimension of all cohomology groups for special type of divisors in $\overline{M_{0,n}}$. We prove that it depends exclusively on the dimension of the linear base locus, $L_{I(r)}$, the multiplicity of containment, $k_{I(r)}$, and n, these three information being elegantly encoded in a binomial formula. As showed in the examples of Section 6.2, we expect a similar result to hold for divisors interpolating a higher number of points and having non-linear base locus. We point out that the divisors we consider in Section 4 live on $\overline{M_{0,n}}$ and not on the blow-up of the projective space in points, suggesting that $\overline{M_{0,n}}$ is the natural space where interpolation problems on linear cycles, $L_{I(r)}$, based on the set with $n-1$ fixed points should be formulated. For this we believe that our work extends and connects the algebraic approach of Fröberg and the geometric perspective of Chandler and Laface-Ugaglia to the geometry of $\overline{M_{0,n}}$.

We believe that interpolating higher dimensional linear cycles, $L_{I(r)}$ for positive r, is a possible direction for describing the effective cone of $\overline{M_{0,n}}$. On the other hand, showing that the F-divisors are globally generated by the techniques that we developed here, is a possible approach to the F-Nef conjecture.

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E-mail address: `brambilla@dipmat.univpm.it`

Università Politecnica delle Marche, via Brecce Bianche, I-60131 Ancona, Italy

E-mail address: `dolivia@math.ucdavis.edu`

University of California, Davis, Department of Mathematics, MSB 2107, Davis, CA 95616

E-mail address: `epostinghel@impan.pl`

Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, P.O. Box 21, 00-956 Warszawa, Poland