Finite-dimensional integrable systems: a collection of research problems

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Abstract

This article suggests a series of problems related to various algebraic and geometric aspects of integrability. They reflect some recent developments in the theory of finite-dimensional integrable systems such as bi-Poisson linear algebra, Jordan-Kronecker invariants of finite dimensional Lie algebras, the interplay between singularities of Lagrangian fibrations and compatible Poisson brackets, and new techniques in projective geometry.

1 Introduction

The idea to publish such a paper was on the one hand inspired by our discussions with both the organisers and the participants of the conference "Finite Dimensional Integrable Systems in Geometry and Mathematical Physics (FDIS 2015)" held in July 2015 at the Mathematical Research and Conference Center (MRCC) in Będlewo. On the other, for a number of years our research group has been working on a list of open problems [5] published in Russian and almost unavailable for the general audience. That list included about 70 problems with various level of difficulty — from the diploma and the PhD levels to serious conjectures which could rather be considered as possible directions for further research. The present paper is based on [5] and is to some extent part of it. The problems offered herein are selected and extended according to our preferences and adapted for publication as a journal paper. So, first of all, we would like to acknowledge the contribution of Andrey Konyaev and Andrey Oshemkov, our co-authors in [5] with whom we are currently working on the English version of the latter. It will contain a background section for beginners, comments and updates and, hopefully, will soon appear on the arXiv. For a similar paper of open problems the reader may wish to refer to a recently published work by S. Rosemann and K. Schöbel [39].

The immediate goal of this paper is to introduce the reader to a bunch of interesting open problems. There is no section with preliminaries but we have tried all our best to define the main concepts as well as to properly motivate the problems to follow. Our selection of problems concerns *bi-Poisson algebra and geometry*, *Jordan-Kronecker invariants*, *Applications to projective geometry* and *Singular Lagrangian fibrations*. It is worth mentioning that many of the problems under discussion are, directly or indirectly, related to the argument shift method [34] which originated from a two-page note by

S.Manakov [33] published in 1976. It is quite surprising that 40 years later this very elegant and simple idea still remains a source of non-trivial problems, new constructions and applications. Therefore, we would like to take this opportunity to emphasise its importance in a context much wider than one could have expected forty years ago. It is our intention to try to convince the reader that not only there are reasonable open problems in the area but also that some of the ideas/solutions may have far reaching applications.

This set of problems have been discussed with many of our colleagues, including the authors of [39], and we are very grateful to all of them. We are especially thankful to A. Borisov, D. Dowell, L. Guglielmi, I. Kozlov, V. Matveev, E. Miranda, A. Panasyuk, C. Wacheux, N.T. Zung.

2 Existence of integrable systems and chaotic Poisson brackets

Let M be a smooth manifold. A *Poisson bracket* on M is a bilinear (over $\mathbb{R}$) skew-symmetric operation $\{\,,\,\}$ on the space of smooth functions $C^\infty(M)$ which satisfies the Leibniz rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g$$

and the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

for any $f, g, h \in C^\infty(M)$. This operation turns $C^\infty(M)$ into an infinite-dimensional Lie algebra. A manifold M endowed with a Poisson bracket is called a *Poisson manifold*.

A Poisson bracket on M can be given by means of a smooth tensor field $\mathcal{A} = (\mathcal{A}^{ij})$, called a *Poisson tensor*, or a *Poisson structure*:

$$\{f, g\} = \mathcal{A}(\mathrm{d}f, \mathrm{d}g) = \sum_{i,j} \mathcal{A}^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \qquad f, g \in C^\infty(M).$$

The *rank* of a Poisson bracket at a point $x \in M$ is the rank of the corresponding tensor $\mathcal{A}$ at this point. The rank of the Poisson tensor (equivalently, of the Poisson bracket) on M is defined to be $\operatorname{rank}\mathcal{A} = \max_{x \in M} \operatorname{rank}\mathcal{A}(x)$. A bracket is called *non-degenerate* if $\det(\mathcal{A}^{ij}) \neq 0$. The non-degeneracy of the Poisson bracket everywhere on M is equivalent to the fact that the 2-form $\omega = \mathcal{A}^{-1}$ defines a symplectic structure on M.

Our first question is about the existence of integrable Hamiltonian systems on Poisson and symplectic manifolds. In the case of symplectic manifolds (equivalently, non-degenerate Poisson), by an integrable system we understand a collection of $n = \frac{1}{2}\dim M$ Poisson commuting functions $f_1, \ldots, f_n$ which are independent almost everywhere on M. If the Poisson bracket is degenerate then we need more commuting functions, namely,

$$k = \frac{1}{2}(\dim M + \operatorname{corank}\mathcal{A}). \tag{1}$$

If one ignores the structure of the singular set where the differentials of commuting functions become linearly dependent, then, by using partition of unity argument,

one can easily construct a C^∞-smooth integrable system on any Poisson manifold [8]. However, in applications we have to deal with polynomial or real-analytic Poisson structures, so it is natural to require that commuting functions (or at least one of them, considered as a Hamiltonian) belong to a natural functional space, e.g., polynomial, rational or real analytic. Alternatively, one may impose some additional conditions on common singularities of these functions (i.e., to require that all of them are non-degenerate). These additional assumptions lead us to a natural existence problem "Do any integrable systems exist on a given Poisson manifold?" or, conversely, to a question on obstructions to integrability. In the algebraic context, this problem can be stated as follows:

Problem 1. Let V be a vector space endowed with a polynomial Poisson structure $\mathcal{A}$, i.e., the bracket of two polynomials is again a polynomial. Do there exist any polynomial integrable systems on $(V, \mathcal{A})$? This question can be naturally generalised by replacing V with an arbitrary algebraic Poisson variety or polynomial Poisson algebra.

One of the important achievements in this direction was the proof of the Mischenko-Fomenko conjecture by S.Sadetov [41] (see also [3, 51]), which says that for linear Poisson structures on V the answer to the above question is always positive.

There are, however, many other examples of polynomial Poisson brackets for which the above problem remains open. In the theory of integrable geodesic flows on Riemannian manifolds, the following question naturally appears. Consider a homogeneous space G/H of a compact Lie group G assuming, for simplicity, that both G and H are connected (one may, of course, consider an arbitrary homogeneous space). Let $\mathcal{F}$ be the algebra of all G-invariant functions on $T^*(G/H)$ which are polynomial in momenta. This algebra can be naturally considered as a polynomial Lie algebra (see [7, 35]).

Problem 2. For which homogeneous spaces $M = G/H$, does $\mathcal{F}$ admit a complete commutative subalgebra? Notice that such a subalgebra can be equivalently understood as an integrable system on the orbit space T^*M/G which has a natural structure of an algebraic Poisson variety.

Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g} = \mathrm{Lie}(G)$ endowed with the standard (linear) Lie-Poisson bracket and $\mathcal{P}(\mathfrak{g})$ denote the space of all polynomials on $\mathfrak{g}^*$ considered as polynomial Poisson algebra. Consider the coadjoint action of H on $\mathfrak{g}^*$ and the subalgebra $\mathcal{P}(\mathfrak{g})^\mathfrak{h} \subset \mathcal{P}(\mathfrak{g})$ of all H-invariant functions. Equivalently, $\mathcal{P}(\mathfrak{g})^\mathfrak{h}$ can be characterised as the centraliser of $\mathfrak{h} = \mathrm{Lie}(H)$ in $\mathcal{P}(\mathfrak{g})$.

Problem 3. Does $\mathcal{P}(\mathfrak{g})^\mathfrak{h}$ admit a complete commutative subalgebra?

The algebras $\mathcal{P}(\mathfrak{g})^\mathfrak{h}$ and $\mathcal{F}$ from Problems 2 and 3 are closely related. Geometrically, $\mathcal{P}(\mathfrak{g})^\mathfrak{h}$ can be understood as the algebra of functions on the orbit space $\mathfrak{g}^*/H$ whereas $\mathcal{F}$ is the algebra of functions on the Poisson submanifold $\mathfrak{h}^\perp/H \subset \mathfrak{g}^*/H$ where $\mathfrak{h}^\perp \subset \mathfrak{g}^*$, the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$, is considered as an invariant subspace of the coadjoint action of H on $\mathfrak{g}^*$. Problems 2 and 3 have been discussed for various classes of homogeneous spaces in the context of integrability of geodesic flows (e.g., see the review paper [7] and the recent paper by I.Mykytyuk [35]) but, in the general setting, they remain open. Although it is hard to expect a complete answer, developing new methods for constructing integrable systems in polynomial Poisson algebras (as well

as finding algebraic obstructions to integrability) is, in our opinion, an interesting and promising direction of research.

Possible obstructions to existence of integrable systems could be related to the chaoticity of the Poisson structure itself. We illustrate this idea by a couple of simple models. In the theory of integrable Hamiltonian systems, formula (1) can be interpreted as follows: if the rank of the Poisson bracket is $2r$, then for complete integrability we need to have r nontrivial integrals in addition to Casimir functions (which are usually considered as trivial integrals). In particular, if $r = 1$, then the Hamiltonian itself can be taken as a nontrivial integral and therefore in this case one should expect that every Hamiltonian system is integrable.

In some concrete problems in mechanics, however, A.V. Borisov and I.S. Mamaev [16] found a series of examples of Hamiltonian systems on Poisson manifolds of rank 2 that demonstrate chaotic dynamics, i.e., are not integrable.

One such system is a particular case of the Suslov problem. Without going into the mechanical nature of the problem and following the original notation from [16] we shall explicitly define the corresponding Poisson bracket. Consider $\mathbb{R}^5$ with coordinates $\gamma_1, \gamma_2, \gamma_3, \omega_1, \omega_2$ and set:

$$\{\omega_1, \gamma_3\} = -\frac{1}{A}\frac{\partial V}{\partial \gamma_2}, \quad \{\omega_2, \gamma_3\} = \frac{1}{B}\frac{\partial V}{\partial \gamma_1},$$

$$\{\gamma_1, \gamma_3\} = -\omega_2, \quad \{\gamma_2, \gamma_3\} = \omega_1,$$

where A, B are constants and $V(\gamma_1, \gamma_2)$ is an arbitrary function of γ_1 and γ_2. It can be shown [16] that for $A = B$ and a suitable choice of V (potential of some non-integrable natural system on the plane), the Hamiltonian $H = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$ from the Suslov problem determines a non-integrable system.

This effect is due to the fact that the Poisson structure itself is chaotic in the sense that its symplectic leaves are embedded in the Poisson manifold in some chaotic manner. In particular, the Casimir functions are not globally defined.

Problem 4. Give a rigorous definition of chaotic Poisson brackets and construct examples of such brackets.

The minimal research programme would be to clarify the situation when a dynamical system, which is Hamiltonian with respect to a Poisson structure of rank 2, might be non-integrable and even chaotic. The following example helps to better understand this phenomenon.

Let v be an arbitrary vector field on $\mathbb{R}^n$. Consider the space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, where τ is taken as an additional coordinate on $\mathbb{R}$ and define the bi-vector $A = v \wedge \partial_\tau$ on it. Since the vector fields v and ∂_τ commute, A is a Poisson structure of rank 2. If we take the Hamiltonian $H = \tau$ then the corresponding Hamiltonian vector field precisely coincides with v. If v is chaotic on $\mathbb{R}^n$, then it will still remain chaotic on $\mathbb{R}^{n+1}$.

Problem 5. a) What will the dynamics be in the example just described if we take the Hamiltonian $H = \sum x_i^2 + \tau^2$? This Hamiltonian is interesting because its isoenergy surfaces $\{H = \varepsilon\}$ for small ε are spheres on which local Casimir functions are well-defined. It follows from this that for small ε almost all trajectories are closed and therefore there should be no chaos. Can chaos appear with the growth of ε?

b) A more general question is: can chaotic dynamics arise if the Poisson structure has rank 2 and the level sets of the Hamiltonian H are compact?

c) Does there exist an analytic (polynomial) Poisson structure of rank 2 for which every analytic (polynomial) Hamiltonian gives a chaotic system? Note that such a Poisson structure would give an example of a Poisson algebra which does not admit a complete commutative subalgebra.

d) Generalise the above example for Poisson structures of an arbitrary rank.

In classical mechanics there are quite simple examples of Poisson brackets whose symplectic leaves are not separated by global Casimir functions which lead to interesting dynamical phenomena related to Problem 5, see [1].

3 Bi-Poisson linear algebra

A vector space V is called *bi-Poisson* if it is endowed with a pair of skew-symmetric bilinear forms A and B. The reader may think of these forms as (constant) Poisson tensors defined on V^* or, alternatively, as two (compatible) Poisson tensors on $T_x^* M$, i.e., at a fixed point x. Both interpretations make sense and give us a natural motivation to study the properties of bi-Poisson vector spaces as an object of linear algebra.

The main tool of the bi-Poisson linear algebra is the *Jordan–Kronecker* decomposition theorem (see [9, 44]) that describes a canonical form of the pair A, B (over $\mathbb{C}$).

Theorem 1. *A pair of skew-symmetric bilinear forms A and B on $V = \mathbb{C}^m$ can be simultaneously reduced to a block-diagonal matrix form by means of an appropriate change of coordinates:*

$$A \mapsto \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}, \quad B \mapsto \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix} \tag{2}$$

where the corresponding pairs of blocks have one of the following three possible types:

1) *Jordan type μ_i-block:* $\quad A_i = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & J(\mu_i) \\ -J^\top(\mu_i) & 0 \end{pmatrix},$

where $J(\mu_i)$ is a Jordan block of size $k_i \times k_i$ with eigenvalue μ_i and Id is the identity matrix of the same size;

2) *Jordan type ∞-block:* $\quad A_i = \begin{pmatrix} 0 & J(0) \\ -J^\top(0) & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix},$

where $J(0)$ is a Jordan block of size $k_i \times k_i$ with eigenvalue 0 and Id is the identity matrix of the same size;

3) *Kronecker block:* $\quad A_i = \begin{pmatrix} 0 & D \\ -D^\top & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & D' \\ -D'^\top & 0 \end{pmatrix},$

where D and D' are both matrices of size $k_i \times (k_i + 1)$ of the form:

$$D = \begin{pmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \quad D' = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{pmatrix}$$

Bi-Poisson linear algebra seems to be a rather new topic. Albeit some basic results can be found in [9, 54], there are still a lot of natural questions to clarify. Below we list some of them.

Instead of A and B, it is more convenient to work with the *pencil* of skew-symmetric bilinear forms $J = \{A_\lambda = A + \lambda B\}_{\lambda \in \bar{\mathbb{C}}}$ generated by them (we may consider the forms from the pencil up to proportionality and formally set $A_\infty = B$). The automorphism group of the pencil is an algebraic group defined by

$$\mathrm{Aut}(V, J) = \{\phi \in \mathrm{End}(V) \mid A_\lambda(\phi(\xi), \phi(\eta)) = A_\lambda(\xi, \eta), \text{ for all } A_\lambda \in J\} \qquad (3)$$

Obviously, the structure of this group essentially depends on the algebraic type of J (see [54]), that is, the type of the canonical Jordan–Kronecker decomposition (2).

Problem 6. Consider the action of $\mathrm{Aut}(V, J)$ on the set of all subspaces $U \subset V$ of dimension $k < \dim V$ (or equivalently, on the Grassmannian $\mathrm{Gr}(k, V)$). It is not hard to see that this action admits infinitely many orbits. What are the orbits (equivalently, subspaces) of generic type?

Problem 7. What happens to the algebraic type of a pencil J under reduction? This question is very important in the differential-geometric context for various applications related to (bi)-Hamiltonian reduction. In bi-Poisson linear algebra, by *reduction* one can understand the restriction of a pencil to a certain subspace $U \subset V$. The simplest version of this question is as follows. Consider a pencil J of a certain algebraic type and take its restriction to a subspace $U \subset V$ of codimension 1. What may happen to its algebraic structure? Which scenarios are typical (i.e. what happens for a generic subspace U)? The converse is also interesting: what happens to the algebraic structure of a pencil if the dimension is increased by one and the pencil is prolonged to the extended space? What is the typical scenario?

Problem 8. The reductions occurring in applications are not "generic" in the sense of linear algebra; they correspond to subspaces with some special properties. One of them can be defined as follows. We will say that a subspace $U \subset V$ of a bi-Poisson vector space is *admissible* if its A_λ-orthogonal subspace $\{v \in V \mid A_\lambda(v, U) = 0\}$ does not depend on the choice of a (generic) skew-symmetric form A_λ from the pencil. We want to describe such subspaces and study their properties. In particular, we want to understand how the algebraic structure of the pencil J changes under reduction to an admissible subspace $U \subset V$.

In the theory of bi-Poisson manifolds, an analog of the subspace $U \subset V$ is a bi-Poisson subalgebra $\mathcal{F} \subset C^\infty(M)$, and the above property in terms of Poisson subalgebras means that the polar subalgebras[1] of $\mathcal{F}$ with respect to the compatible Poisson structures $\mathcal{A}$ and $\mathcal{B}$ coincide, which is a very useful property in the bi-Hamiltonian context.

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[1]Following S. Lie, by the *polar* subalgebra of a Poisson algebra $\mathcal{F}$ we understand the collection of functions f satisfying $\{f, \mathcal{F}\} = 0$.

Problem 9. What happens to the algebraic structure of a pencil $A + \lambda B$ under continuous deformation $A_\alpha + \lambda B_\alpha$? Which pencils are "more stable" than the others?

More precisely, consider a pencil $J_\alpha = \{A_\alpha + \lambda B_\alpha\}$ which depends (continuously, smoothly, analytically, algebraically) on a parameter α in such a way that its algebraic type remains the same for all values of α except for a certain bifurcation value α_0. What the algebraic type of J_{α_0} could be? In other words, the goal is to study bifurcations of pencils. Which bifurcations are generic? From a more formal point of view, the problem is to describe the so-called "closure of orbits" diagram for the natural action of $\mathrm{GL}(V)$ on the set of all pencils.

Without loss of generality we may assume that A and B are both regular, that is, they have a maximal rank within the pencil. We shall say that a subspace $L \subset V$ is *bi-Lagrangian* if it is simultaneously Lagrangian with respect to both forms A and B. The set $\mathrm{LG}(V, J)$ of all bi-Lagrangian subspaces is called the *bi-Lagrangian Grassmannian*. This is a projective algebraic variety with rather non-trivial properties (a detailed description of $\mathrm{LG}(V, J)$ is an important open problem, see the discussion in [39]), which essentially depends on the algebraic type of the pencil J. For example, if J is of Kronecker type, then there is a unique bi-Lagrangian subspace, i.e., $\mathrm{LG}(V, J)$ consists of one single point. If, on the contrary, J is of Jordan type and is diagonalisable with simple spectrum, i.e., all the blocks A_i, B_i in Theorem 1 are 2×2 and all μ_i are distinct, then $\mathrm{LG}(V, J)$ is diffeomorphic to the torus $T^k = (S^1)^k$, (in the real case) or $(\mathbb{C}P^1)^k$ (in the complex case), $k = \frac{1}{2} \dim V$. Moreover, the action of $\mathrm{Aut}(V, J)$ on $\mathrm{LG}(V, J)$ is transitive. However, if the structure of Jordan blocks is more complicated, then $\mathrm{LG}(V, J)$ may consist of several orbits.

Problem 10. Describe the partition of the bi-Lagrangian Grassmannian $\mathrm{LG}(V, J)$ into orbits of the automorphism group $\mathrm{Aut}(V, J)$.

In a symplectic space every isotropic space is contained in a Lagrangian one. In a bi-Poisson vector space, this is not true any more.

Problem 11. Describe bi-isotropic subspaces $W \subset V$ that can be extended up to bi-Lagrangian ones. In bi-Poisson geometry, a non-linear analog of this question is "which sets of functions in bi-involution can be (at least locally) extended up to a bi-integrable system (or, equivalently, up to a complete set of functions in bi-involution).

4 Bi-Poisson geometry and the argument shift method

4.1 Existence of bi-integrable systems

It is well known that compatible Poisson structures are one of the most effective tools for constructing integrable systems, see for example [31, 32] . In particular, in almost all non-trivial examples of Poisson algebras $\mathcal{P}(\mathfrak{g})^\natural$ and $\mathcal{F}$ from Problems 2 and 3 for which complete commutative subalgebras (i.e. integrable systems) have been found, this was done by using an appropriate pencil of compatible Poisson brackets. Recall that two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are called *compatible* if any linear combination of them is again a Poisson bracket (a necessary and sufficient condition for this is the Jacobi

identity for the sum $\{\ ,\ \}_1 + \{\ ,\ \}_2$). The set $\{\lambda\{\ ,\ \}_1 + \mu\{\ ,\ \}_2 \mid (\lambda, \mu) \in \mathbb{R}^2 \setminus (0,0)\}$ of all non-trivial linear combinations of two compatible Poisson brackets is called a *Poisson pencil*.

The rank of a Poisson pencil at a point $x \in M$ is defined as the maximum of the ranks of the brackets $\lambda\{\ ,\ \}_1 + \mu\{\ ,\ \}_2$ at that point. The rank of the pencil in a neighbourhood of a point $x \in M$ (or on the whole manifold M) is the maximum of the ranks of the pencil over all points of that neighbourhood (manifold).

It deserves to be noticed that by virtue of the Jordan–Kronecker theorem (Theorem 1) we can distinguish three essentially different *types* of Poisson pencils according to the Jordan–Kronecker decomposition of a given pencil at a generic point. Thus, we shall say that a Poisson pencil is of *Kronecker type* if its canonical form only consists of Kronecker blocks. Similarly, if the canonical form of a pencil consists only of Jordan blocks, we shall call it *Jordan* or *symplectic*. In the most general case, we shall just say that the pencil is of a *mixed* algebraic type.

As pointed out in Section 3, every bi-Poisson vector space $(V, J = \{A + \lambda B\})$ admits a bi-Lagrangian subspace $L \subset V$ (at least one), i.e., a subspace which is simultaneously Lagrangian (i.e. maximal isotropic) with respect to all regular forms $A + \lambda B$.

An analogous statement in the case of compatible Poisson brackets would be the existence of a complete subalgebra $\mathcal{F} \subset C^\infty(M)$ consisting of functions in involution with respect to all the brackets in a given pencil. By completeness in this case we mean that the subspace generated by the differentials of functions $f \in \mathcal{F}$ in the cotangent space $T_x^* M$ is maximal isotropic. It is natural to refer to such a subalgebra (or to a collection of its generators) as a *bi-integrable* system.

In the case of Kronecker pencils such a subalgebra $\mathcal{F}$ is generated by the (local) Casimir functions of all the brackets in the pencil. It is well known that $\mathcal{F}$ so obtained is complete and commutative with respect to all the brackets in the pencil ([2, 23]). Similar results in the symplectic case can be found in the works of P. Olver [36] and H. Turiel [48].

Of great interest, however, is the mixed case.

Problem 12. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two compatible Poisson structures of an arbitrary type on a manifold M and $x \in M$ be a generic point in the sense that the structure of the Jordan-Kronecker decomposition of the pencil does not change in a neighbourhood of x. Does there exist, in a neighbourhood of x, a complete set of functions in bi-involution?

In fact, the answer to this question is positive but to the best of our knowledge this result has never been published. We would like, however, to formulate a more interesting version of this local problem. As pointed out in Section 3, bi-Lagrangian subspaces $L \subset V$ may have different algebraic types (in other words, the bi-Lagrangian Grassmannian $\mathrm{LG}(V, J)$ may consist of several orbits of the automorphism group of the pencil J). In this view, bi-integrable systems may be of different algebraic types too. More precisely, let $f_1, \ldots, f_k$ be a complete set of functions in bi-involution, i.e., a bi-integrable system. We restrict our considerations to a small neighbourhood U of a generic point $x \in M$ such that for each $y \in U$ the Jordan-Kronecker type of the pencil defined on $T_y^* M$ by a pair of compatible Poisson brackets is the same. If we think of $T_y^* M$ as a bi-Poisson vector space, then the subspace $L_y = \mathrm{span}\{df_1(y), \ldots, df_k(y)\} \subset T_y^* M$ is bi-Lagrangian and therefore can be characterised by its algebraic type (i.e., by the type of the orbits of the automorphism group action in the bi-Lagrangian Grassmannian to

which L belongs). If this type remains the same for all points $y \in U$, we refer to it as the *algebraic type* of the bi-integrable system $f_1, \ldots, f_k$.

Problem 13. Do bi-integrable systems of a given algebraic type exist? In the local setting, this question can be reformulated as follows. Let us consider a (co)-distribution on U all of whose subspaces $L_y \subset T_y^* M$ are bi-Lagrangian and belong to the same algebraic type. Do there exist integrable (in the sense of Frobenius) co-distributions of this kind?

4.2 Argument shift method, the generalised argument shift conjecture, and Jordan-Kronecker invariants

The argument shift method was developed by A.S. Mischenko and A.T. Fomenko [34] as a generalisation of S.V. Manakov's construction suggested in [33]. In the context of bi-integrable systems, it is perhaps the first non-trivial example which illustrates almost all phenomena that one can observe in finite-dimensional bi-Poisson geometry.

Let $\mathfrak{g}^*$ be the dual space of a finite-dimensional Lie algebra $\mathfrak{g}$. It is well known that $\mathfrak{g}^*$ possesses two natural compatible Poisson brackets. The first one is the standard linear Lie-Poisson bracket

$$\{f, g\}(x) = \langle x, [df(x), dg(x)] \rangle, \tag{4}$$

and the second one is a constant bracket given by

$$\{f, g\}_a(x) = \langle a, [df(x), dg(x)] \rangle, \tag{5}$$

where $a \in \mathfrak{g}^*$ is a fixed element. Here we assume a to be regular although formula (5) makes sense for an arbitrary a (see the next section).

Now, let $f \in \mathcal{P}(\mathfrak{g})$ be a polynomial invariant of the coadjoint action. Then its shift $f(x + \lambda a)$ is Casimir for the linear combination $\{\,,\,\} + \lambda \{\,,\,\}_a$. Therefore, for any two coadjoint invariant polynomials $f, g \in C^\infty(\mathfrak{g}^*)$, their shifts $f(x + \lambda a)$ and $g(x + \mu a)$ commute with respect to both brackets (4) and (5). In this way, one can, as a rule, construct a large (bi-)commutative subalgebra of $\mathcal{P}(\mathfrak{g})$. However, in many examples, there are either no polynomial coadjoint invariants, or their number is too small. In this case, one needs to modify the construction. To that end, consider local analytic invariants $f_1, \ldots, f_s$, $s = \operatorname{ind} \mathfrak{g}$ defined in a neighbourhood of $a \in \mathfrak{g}^*$ such that their differentials $df_i(a)$ form a basis of $\operatorname{Ann} a$ (recall that a is regular so that such invariants do exist). Take the Taylor expansions of f_i at a:

$$f_i(a + \lambda x) = f_i^{(0)} + \lambda f_i^{(1)}(x) + \lambda^2 f_i^{(2)}(x) + \lambda^3 f_i^{(3)}(x) + \ldots, \tag{6}$$

and consider the commutative subalgebra $\mathcal{F}_a \subset \mathcal{P}(\mathfrak{g})$ generated by polynomials $f_i^{(k)}$ where $i = 1, \ldots, \operatorname{ind} \mathfrak{g}$, and $k > 0$. We call $\mathcal{F}_a$ the *algebra of (polynomial) shifts*. In terms of the algebra $\mathcal{F}_a$, the main result of [34] can be stated as follows

Theorem 2 (A.S. Mischenko, A.T. Fomenko [34])**.**
    1) *The functions from $\mathcal{F}_a$ pairwise commute with respect to both brackets $\{\,,\,\}$ and $\{\,,\,\}_a$.*
    2) *If $\mathfrak{g}$ is semisimple, then $\mathcal{F}_a$ is complete, i.e., contains $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$ algebraically independent polynomials.*

Although in general $\mathcal{F}_a$ is not necessarily complete, A.S. Mischenko and A.T. Fomenko stated the following well known conjecture.

Mischenko–Fomenko conjecture. *On the dual space $\mathfrak{g}^*$ of an arbitrary Lie algebra $\mathfrak{g}$ there exists a complete family $\mathcal{F}$ of commuting polynomials.*

In other words, for each $\mathfrak{g}$ one can construct a (polynomial) completely integrable system on $\mathfrak{g}^*$ or, speaking in algebraic terms, the Lie-Poisson algebra $\mathcal{P}(\mathfrak{g})$ always contains a complete commutative subalgebra.

This conjecture was proved in 2004 by S.T.Sadetov [41], see also [3, 51]. However, Sadetov's family $\mathcal{F} \subset \mathcal{P}(\mathfrak{g})$ is essentially different from the algebra $\mathcal{F}_a$ of shifts. Thus, it is still an open question whether or not one can modify the argument shift method to construct a complete family of polynomials in bi-involution, that is, commuting with respect to both brackets (4) and (5). In many examples we have already studied, the answer turned out to be positive which led us to the following bi-Hamiltonian version of the Mischenko–Fomenko conjecture.

Generalised argument shift conjecture ([9]). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then for every regular element $a \in \mathfrak{g}^*$, there exists a complete family $\mathcal{G}_a \subset \mathcal{P}(\mathfrak{g})$ of polynomials in bi-involution, i.e., in involution w.r.t. the two brackets $\{\,,\,\}$ and $\{\,,\,\}_a$.*

In fact, our conjecture can be reformulated in the following equivalent way: *the algebra $\mathcal{F}_a$ of polynomial shifts can always be extended up to a complete subalgebra $\mathcal{G}_a \subset \mathcal{P}(\mathfrak{g})$ of polynomials in bi-involution.*

Problem 14. Either prove this conjecture or construct a counterexample.

Some results in this direction can be found in [9, 25] where this conjecture has been verified for several classes of Lie algebras. However, there is one example for which we could not do it.

Problem 15. Let $\mathfrak{g} = \mathrm{gl}(4) + (\mathbb{R}^4 + \mathbb{R}^4)$ be the semidirect sum of $\mathrm{gl}(4)$ with two copies of $\mathbb{R}^4$ on each of which $\mathrm{gl}(4)$ acts in the standard way[2]. Construct a complete family of polynomials in bi-involution or prove that no such family exists.

Possible difficulties are related to the algebraic structure of the "argument shift" pencil on $\mathfrak{g} = \mathrm{gl}(4) + (\mathbb{R}^4 + \mathbb{R}^4)$ which can be naturally described by means of the so-called Jordan-Kronecker invariants introduced in [9].

Consider an arbitrary Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$. For an arbitrary pair of elements $x, a \in \mathfrak{g}^*$ we define two skew-symmetric forms on $\mathfrak{g}$ by

$$\mathcal{A}_x(\xi, \eta) = \langle x, [\xi, \eta] \rangle \quad \text{and} \quad \mathcal{A}_a(\xi, \eta) = \langle a, [\xi, \eta] \rangle.$$

These forms can be understood as the Poisson tensors corresponding to the brackets (4) and (5) (at the point $x \in \mathfrak{g}^*$). By the Jordan-Kronecker theorem (Theorem 1) the forms $\mathcal{A}_x$ and $\mathcal{A}_a$ can be reduced to some canonical form. We are interested in their discrete invariants:

1) the number j_m and the sizes $n_1, \ldots, n_{j_m}$ of the Jordan blocks for each eigenvalue μ_m;

2) the number i and the sizes $2p_1 + 1, \ldots, 2p_i + 1$ of the Kronecker blocks.

---

[2]The matrix representation of this Lie algebra is very simple: all 6×6 matrices with two zero rows.

Consider a generic pencil of this type, i. e., such that these invariants do not change under small perturbations of its parameters $x, a \in \mathfrak{g}^*$. It can be shown that the set of such generic pairs $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is Zariski open. Then, the numbers j_m, n_r and p_s are invariants of the Lie algebra $\mathfrak{g}$ and are called *Jordan–Kronecker invariants* (for more details see [9]). They are closely related to some important properties of $\mathfrak{g}$. For instance, the completeness of the algebra of shifts $\mathcal{F}_a$ is equivalent to the absence of Jordan blocks, i. e., $j = \sum j_m = 0$. Lie algebras of purely Jordan type (i.e., with no Kronecker blocks) are exactly Frobenius Lie algebras $\mathfrak{g}$, i.e. can be characterised by the condition ind $\mathfrak{g} = 0$ (see [20]). Furthermore, the numbers p_s in the semisimple case are related to the so-called Chevalley indices (exponents of $\mathfrak{g}$) and in the general case may be considered as lower estimates for the degrees of Ad*-invariant polynomials of $\mathfrak{g}$ (A.S. Vorontsov [52]).

Problem 16. Compute the Jordan–Kronecker invariants for the most interesting classes of Lie algebras, and particularly for the following
(a) semidirect sums $\mathfrak{g} +_\rho V$, where $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of a semisimple Lie algebra $\mathfrak{g}$ and V is assumed to be commutative;
(b) Borel subalgebras of simple Lie algebras;
(c) parabolic subalgebras of simple Lie algebras;
(d) the centralisers of singular elements $a \in \mathfrak{g}$, where $\mathfrak{g}$ is simple;
(e) Lie algebras of small dimensions.

For Lie algebras of dimension ≤ 5, the Jordan–Kronecker invariants were computed by P. Zhang. Her results, as well as some other examples of computing Jordan–Kronecker invariants, can be found in the arXiv version of [9].

Problem 17. Can an arbitrary set of Jordan–Kronecker invariants j_m, n_r, p_s be realised by means of an appropriately chosen Lie algebra? Or there are some non-trivial restrictions?

A partial, but quite substantial, solution to Problem 17 has been obtained by Ivan Kozlov, see [9]. Roughly speaking, the answer is obtained for Lie algebras of Jordan and Kronecker type. It turns out that in the Jordan case there are non-trivial restrictions on the sizes of blocks, while in the Kronecker case any combination of blocks is allowed. The mixed case remains open because of possible non-trivial interaction between Kronecker and Jordan blocks.

Coming back to Problem 15, the Lie algebra $\mathfrak{g} = \mathrm{gl}(4) + (\mathbb{R}^4 + \mathbb{R}^4)$ is Frobenius, i.e., of pure Jordan type. The Jordan-Kronecker decomposition of a generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ contains 6 Jordan blocks of size 4×4 (i.e., non-trivial) with distinct characteristic numbers $\lambda_1, \ldots, \lambda_6$. It can be shown that locally bi-integrable systems on $\mathfrak{g}^*$ exist but it is not clear if the corresponding bi-Lagrangian integrable (co)-distribution can be defined by means of polynomial functions.

4.3 The limit of the argument shift method

Let us recall, in more detail, how the argument shift method works in the case of a simple Lie algebra $\mathfrak{g}$. Identify $\mathfrak{g}$ with its dual space by means of the Killing form. As before, let $\mathcal{P}(\mathfrak{g})$ denote the symmetric algebra of $\mathfrak{g}$, i.e., the algebra of polynomials on $\mathfrak{g}^* = \mathfrak{g}$, endowed with the standard Lie-Poisson bracket. Further, let $\mathcal{I}(\mathfrak{g}) \subset \mathcal{P}(\mathfrak{g})$ be

the subalgebra of polynomial invariants of the adjoint, or, which is the same, coadjoint, representation (recall that $\mathcal{I}(\mathfrak{g})$ is exactly the centre of $\mathcal{P}(\mathfrak{g})$ with respect to the Lie-Poisson bracket). Then it is known that $\mathcal{I}(\mathfrak{g})$ is freely generated by certain homogeneous polynomials $I_1, \ldots, I_n$, where n is the rank of the Lie algebra $\mathfrak{g}$. Consider an arbitrary element $a \in \mathfrak{g}$ and the corresponding expansion

$$I_i(x + \lambda a) = \sum_j \lambda^j f_{ij}(x, a).$$

Each coefficient f_{ij} in this expansion is a certain homogeneous polynomial of degree $\deg I_i - j$. In this way we obtain exactly $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g})$ non-constant functions f_{ij}.

The subalgebra $\mathcal{F}_a \subset \mathcal{P}(\mathfrak{g})$ generated by the polynomials f_{ij} is called the *Mischenko-Fomenko subalgebra* corresponding to the element $a \in \mathfrak{g}$. Note that the symmetric algebra $\mathcal{P}(\mathfrak{g})$ has a natural grading by polynomial degrees

$$\mathcal{P} = \bigoplus_{i \geq 0} \mathcal{P}^i,$$

and since the subalgebra $\mathcal{F}_a \subset \mathcal{P}(\mathfrak{g})$ is generated by homogeneous functions, it inherits the grading:

$$\mathcal{F}_a = \bigoplus_{i \geq 0} \mathcal{F}_a^i,$$

where $\mathcal{F}_a^i = \mathcal{F}_a \cap \mathcal{P}^i$.

It follows from Theorem 2 that the subalgebra $\mathcal{F}_a \subset \mathcal{P}(\mathfrak{g})$ is commutative with respect to the Lie-Poisson bracket. Furthermore, if a is a regular element, then $\mathcal{F}_a$ has maximal (for a Poisson-commutative subalgebra) transcendence degree $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g})$. Such subalgebras are called *complete*.

In the case of a singular element $a \in \mathfrak{g}$, the corresponding subalgebra $\mathcal{F}_a$ is not complete. However, as was shown by E.B. Vinberg [50], it can always be included in some bigger complete commutative subalgebra. To see this, consider an analytic curve $a(t) \in \mathfrak{g}$ such that $a(t)$ is regular for $0 < |t| < \varepsilon$, and $a(0) = a$ is a prescribed singular element. Considering the corresponding Mischenko–Fomenko subalgebras and their homogeneous parts $\mathcal{F}_{a(t)}^i \subset \mathcal{P}^i$, we obtain a family of subspaces such that $k^i = \dim \mathcal{F}_{a(t)}^i$ is independent of t for $0 < |t| < \varepsilon$. Now, we claim that if $\mathcal{F}_{a(t)}^i$ is regarded as a point in the corresponding Grassmannian $\operatorname{Gr}(k^i, \mathcal{P}^i)$, then there exists the limit $\lim_{t \to 0} \mathcal{F}_{a(t)}^i$. Indeed, let $g_1^t, \ldots, g_{k^i}^t$ be a basis of $\mathcal{F}_{a(t)}^i$ chosen in such a way that the coefficients of every polynomial g_j^t are analytic functions of t. Since the Mischenko–Fomenko subalgebra is free, such a basis can be constructed by taking products of polynomials of the form $f_{ij}(x, a(t))$ of suitable degrees. Let also

$$\omega(t) = g_1^t \wedge \cdots \wedge g_{k^i}^t \in \Lambda^{k^i} \mathcal{P}^i.$$

Now, observe that since the form $\omega(t)$ depends analytically on t and does not vanish identically, it follows that there exists k such that $\omega(t) = t^k \tilde{\omega}(t)$, where $\tilde{\omega}(0) \neq 0$. Further, notice that the coefficients of the form $\omega(t)$ are, by definition, the Plücker coordinates of the subspace $\mathcal{F}_{a(t)}^i \subset \mathcal{P}^i$. Therefore, since the image of the Plücker embedding

$$\operatorname{Gr}(k^i, \mathcal{P}^i) \hookrightarrow \mathbb{P}(\Lambda^{k^i} \mathcal{P}^i)$$

is closed, the coefficients of the form $\tilde{\omega}(0)$ are also Plücker coordinates of a certain subspace of $\mathcal{P}^i$, which can be naturally viewed as the limit $\lim_{t\to 0} \mathcal{F}^i_{a(t)}$. Now, taking the direct sum

$$\bigoplus_{i\geq 0} \lim_{t\to 0} \mathcal{F}^i_{a(t)},$$

we obtain a Poisson-commutative graded algebra which we denote by $\lim_{t\to 0} \mathcal{F}_{a(t)}$. Note that, in contrast to the actual algebra of shifts $\mathcal{F}_a$, this subalgebra may depend on the curve $a(t)$, and not only on a itself. However, we always have $\lim_{t\to 0} \mathcal{F}_{a(t)} \supset \mathcal{F}_a$. So, it remains to show that the subalgebra $\lim_{t\to 0} \mathcal{F}_{a(t)}$ is complete. To that end, recall that for any graded algebra $\mathcal{F} = \bigoplus_i \mathcal{F}^i$, its Poincaré series is defined by $F(t) = \sum_i n_i t^i$, where $n_i = \dim \mathcal{F}^i$. In the case of subalgebras of finitely generated algebras, this series uniquely determines the transcendence degree [50]. Now, notice that, by construction the Poincaré series of a pre-limit algebra $\mathcal{F}_{a(t)}$ coincides with the Poincaré series of the limit algebra $\lim_{t\to 0} \mathcal{F}_{a(t)}$. Therefore, the number of algebraically independent polynomials in the algebra $\lim_{t\to 0} \mathcal{F}_{a(t)}$ is equal to $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g})$, i.e., the limit algebra is complete, as desired.

Problem 18. Generalize the limit argument shift method to the case of an arbitrary Lie algebra. Find a necessary condition for the completeness of the limit algebra of shifts.

Let $a(t)$ be a polynomial in t whose coefficients lie in the same Cartan subalgebra. In this case, an explicit description of the generators for the limit Mischenko–Fomenko subalgebra was obtained by V.V. Shuvalov [42]. Namely, assume that

$$a(t) = a_0 + a_1 t + \cdots + a_m t^m.$$

Define

$$C_i = C_{\mathfrak{g}}(a_0) \cap \cdots \cap C_{\mathfrak{g}}(a_i),$$

where $C_{\mathfrak{g}}(a_i)$ is the centraliser of a_i in $\mathfrak{g}$. Then the limit Mischenko–Fomenko subalgebra $\lim_{t\to 0} \mathcal{F}_{a(t)}$ is generated by the conventional Mischenko–Fomenko subalgebra $\mathcal{F}_{a_0}$, and by the union of the shifts of invariants for the Lie algebras C_i along the elements a_{i+1}.

Problem 19. Describe the generators of the limit Mischenko–Fomenko algebra in the case of an arbitrary analytic curve $a(t)$ in a simple Lie algebra. If Problem 18 is solved, then generalize the description of the generators to the case of an arbitrary analytic curve $a(t) \in \mathfrak{g}^*$ for an arbitrary Lie algebra $\mathfrak{g}$.

4.4 Quantization of the argument shift method

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathcal{P}(\mathfrak{g})$ be its Lie-Poisson algebra that consists of polynomial functions on $\mathfrak{g}^*$, and let $\mathcal{F}_a \subset \mathcal{P}(\mathfrak{g})$ be the commutative subalgebra of shifts for a regular element $a \in \mathfrak{g}^*$.

The problem of quantization consists in constructing the corresponding (quantum) commutative subalgebra $\widehat{\mathcal{F}}_a$ in the universal enveloping algebra $U(\mathfrak{g})$. The correspondence between the quantum algebra of shifts $\widehat{\mathcal{F}}_a$ and the classical algebra of shifts $\mathcal{F}_a$ is required to be as follows.

The universal enveloping algebra $U(\mathfrak{g})$ has a natural filtration by degree

$$U_0 \subset U_1 \subset \cdots \subset U_{k-1} \subset U_k \subset \ldots, \quad U(\mathfrak{g}) = \bigcup_{k=0}^{\infty} U_k.$$

The quotient space U_k/U_{k-1} is naturally identified with the subspace $P_k \subset \mathcal{P}(\mathfrak{g})$ of homogeneous polynomials of degree k. Let $\hat{p} \in \widehat{\mathcal{F}}_a \cap U_k$ be some element of degree k. It is required that its "principal symbol" $p \in U_k/U_{k-1} \simeq P_k$ lies in the algebra of shifts $\mathcal{F}_a$, and $\mathcal{F}_a$ must be generated by such elements.

Equivalently, this question can be reformulated as a problem of lifting of the homogeneous generators $p_1, p_2, \ldots, p_s$ of the classical algebra of shifts $\mathcal{F}_a$ to the universal enveloping algebra $U(\mathfrak{g})$ in such a way that the lifted generators still commute.

A method for constructing the quantum algebra of shifts $\widehat{\mathcal{F}}_a$ for a semisimple Lie algebra $\mathfrak{g}$ was suggested by L.G. Rybnikov [40]. However, it is not clear whether or not a similar construction works for an arbitrary Lie algebra $\mathfrak{g}$.

Problem 20. Quantize the algebra of shifts $\mathcal{F}_a$ in the case of an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ (or find an obstruction for quantization).

4.5 Flat Pencils

A Poisson pencil is called *flat* if there exists a local coordinate system in which the Poisson tensors of all brackets of the pencil are constant. The problem of flatness for Poisson pencils has been actively studied [22, 23, 36, 47, 48]. At present, some fundamental results are known.

Recall that a Poisson pencil on a manifold M is called *Kronecker* at $x \in M$ if the Jordan-Kronecker normal form for the corresponding Poisson tensors at the point x has no Jordan blocks. A Poisson pencil is called *Kronecker* on a manifold M if it is Kronecker at a generic point $x \in M$. Note that the problem of flatness is mainly of interest in the Kronecker case, because if a flat pencil has Jordan blocks, then the corresponding eigenvalues must be constant. At the same time, the latter case can be easily reduced to the Kronecker situation [49].

To formulate a criterion for flatness of Kronecker pencils, we need the notion of a *Lenard chain*. A sequence f_i of functions on M is called a *Lenard chain* for compatible Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ if

$$\{f_{i+1}, g\}_1 = \{f_i, g\}_2$$

for any two consecutive members f_i, f_{i+1} of the chain and any $g \in C^\infty(M)$.

Theorem 3. *Let $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ be two compatible Poisson brackets defining a Kronecker pencil with Kronecker blocks of sizes $2n_1 - 1, \ldots, 2n_k - 1$. Such a pencil is flat in a neighbourhood of a point x if and only if there exist k Lenard chains*

$$0 \to f_1^1 \to \cdots \to f_{n_1}^1 \to 0,$$
$$\cdots$$
$$0 \to f_1^k \to \cdots \to f_{n_k}^k \to 0,$$

where f_i^j's are local smooth functions with linearly independent differentials.

In the case when there is only one Kronecker block and all objects are analytic this theorem was proved by I.M. Gelfand and I.S. Zakharievich [22]. The general case follows from the results of A. Panasyuk [37] and F.J. Turiel [47].

The paper [26] gives a more explicit criterion for flatness of Kronecker pencils with a single block in terms of *bi-invariant volume forms*. A volume form ω on M is called *bi-invariant* with respect to compatible Poisson brackets $\{\,,\,\}_1$ and $\{\,,\,\}_2$ if ω is preserved by all vector fields Hamiltonian with respect to $\{\,,\,\}_1$, as well as by all vector fields Hamiltonian with respect to $\{\,,\,\}_2$. According to [26], a Kronecker pencil with a single block is locally flat if and only if it admits a local bi-invariant volume form.

For arbitrary Kronecker pencils, the existence of a bi-invariant volume form no longer implies flatness, see Remark 4.4 of [26].

Problem 21. Generalize the results of [26] to the case of arbitrary Kronecker pencils.

In the case of the argument shift pencil on the dual of a Lie algebra, that is a pencil generated by brackets (4) and (5), Theorem 3 implies the following.

Theorem 4. *Let $\mathfrak{g}$ be a Lie algebra whose ring of polynomial invariants of the coadjoint action has maximal transcendence degree* ind $\mathfrak{g}$ *and is generated by polynomials of total degree* $1/2(\dim\mathfrak{g} + \operatorname{ind}\mathfrak{g})$. *Assume also that the codimension of the singular set in $\mathfrak{g}^*$ is at least two. Then, for any regular $a \in \mathfrak{g}^*$, the corresponding argument shift pencil is Kronecker and flat in a neighbourhood of a generic point.*

In particular, we have the following

Theorem 5 (A. Panasyuk [37], I.S. Zakharievich [53])**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and $a \in \mathfrak{g}^* = \mathfrak{g}$ be regular. Then the argument shift pencil is flat in a neighbourhood of a generic point.*

Note that the condition of Theorem 4 is not necessary for flatness. For example, the three-dimensional Lie algebra with relations $[x, y] = y$, $[x, z] = z$ does not have polynomial invariants, but the corresponding argument shift pencil is Kronecker and flat [24]. Also, note that even if a Lie algebra $\mathfrak{g}$ admits a complete (i.e., of transcendence degree ind $\mathfrak{g}$) set of polynomial invariants but their degrees are two high, then the corresponding argument shift pencil does not have to be flat (consider, for instance, the algebra discussed in Remark 4.4 of [26]).

Problem 22. Find necessary and sufficient algebraic conditions under which the argument shift pencil on the dual space of a Lie algebra is flat.

In [24], Problem 22 is solved for all three-dimensional Lie algebras. More generally, the technique of [26] can be used to study argument shift pencils with a single Kronecker block. In particular, from the flatness criterion of [26] it follows that any such pencil associated with a unimodular Lie algebra is flat.

Analogous questions may be asked for other natural pencils of compatible Poisson brackets.

Example 1. Consider the Lie algebra $\mathrm{gl}(n)$. Let $A \in \mathrm{gl}(n)$ be fixed. Define a new operation on $\mathrm{gl}(n)$ by
$$[X, Y]_A = XAY - YAX.$$

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It is easy to check that this operation satisfies the Jacobi identity and defines a structure of a Lie algebra on gl(n) which is compatible with the standard one. Therefore, on the dual space gl(n)* we obtain a pair of compatible Poisson brackets.

Furthermore, it is easy to see that if the matrix A is taken to be symmetric, then so(n) $\subset$ gl(n) will be a subalgebra with respect to both commutators. Hence, on the dual space to so(n), we also get a pair of compatible Poisson brackets.

Problem 23. Are the pencils on gl(n) and so(n) defined in Example 1 flat?

This question might be of interest in connection with the unsolved problem of separation of variables for the Manakov top.

5 Applications to Differential Geometry

5.1 Sectional operators, projective equivalence and holonomy groups

An interesting relationship between a special class of integrable Hamiltonian systems on semisimple Lie algebras, projectively equivalent metrics, and pseudo-Riemannian metrics with special holonomy groups has been discovered recently [12, 13, 14].

Let $\mathfrak{g} = $ so(g) be the Lie algebra of the (pseudo)-orthogonal group related to a non-degenerate bilinear form g defined on a finite-dimensional vector space V. Consider a linear operator $R :$ so(n) $\to$ so(n), symmetric w.r.t. the Killing form and satisfying the relation

$$[R(X), A] = [X, B] \quad \text{for all } X \in \text{so}(g),$$

where A and B are some fixed g-symmetric matrices.

Such operators play an important role in the theory of finite-dimensional integrable systems since the corresponding Hamiltonian equations on so(n)

$$\dot{X} = [R(X), X]$$

can be viewed as natural generalisations of the Euler equations describing the dynamics of an n-dimensional rigid body [33, 34]. These equations admit a family of commuting integrals $\text{Tr}(X + \lambda A)^n$ which are sufficient for complete integrability if A is regular. Following [45, 46], we will call such operators *sectional* (see also [11, 12, 15, 29]).

Two (pseudo)-Riemannian metrics are called *projectively equivalent* if they have the same unparametrised geodesics and *affinely equivalent* if their geodesics coincide with parametrisation.

The following two facts illustrate the relationship between sectional operators and projectively equivalent metrics.

Theorem 6 (A.V. Bolsinov, V. Kiosak, V.S. Matveev [12])**.** *Let g and $\bar{g}$ be projectively equivalent metrics on M. Then the Riemann curvature operator[3] of g at each point $x \in M$:*

$$R : \Lambda^2(T_x M) \simeq \text{so}(g) \to \text{so}(g) \tag{7}$$

---

[3]Recall that the Riemann curvature tensor of a metric g can (pointwise) be considered as a linear map from the space of bivectors to the space of g-skew symmetric operators, and these two spaces can be naturally identified by means of g.

is a sectional operator in the above sense. Namely, it satisfies

$$[R(X), A] = [X, B]$$

where $A = \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{-1} g$ *and* $B = \frac{1}{2} \nabla (\operatorname{grad} \operatorname{tr} A)$.

If g and $\bar{g}$ are affinely equivalent, then the operator A from this theorem is covariantly constant, the above identity takes the form

$$[R(X), A] = 0$$

and the Lie algebra $\mathfrak{hol}(\nabla)$ of the (local) holonomy group of g is contained in $\mathfrak{g}_A = \{Y \in \operatorname{so}(g), \ YA = AY\}$.

Is it possible to construct examples of metrics g for which $\mathfrak{hol}(\nabla)$ coincides with $\mathfrak{g}_A$? In other words, is $\mathfrak{g}_A$ a holonomy algebra (for an appropriate (pseudo)-Riemannian metric)? Not only the answer is positive, but there is also an elegant construction, based on the idea of sectional operators, that gives a series of explicit examples of such metrics.

Theorem 7 (A.V. Bolsinov, D. M. Tsonev [14]). *For any g-symmetric operator A, the algebra $\mathfrak{g}_A$ can be realised as a holonomy Lie algebra $\mathfrak{hol}(\nabla)$ of some (pseudo)-Riemannian metric g. Moreover, such metrics can be constructed explicitly.*

The definition of sectional operators is naturally generalised to the case of $\mathbb{Z}_2$-graded Lie algebras of the form $\mathfrak{g} = \mathfrak{k} + \mathfrak{v}$, where $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{v}] \subset \mathfrak{v}$ and $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{k}$:

$$R : \mathfrak{k} \to \mathfrak{k} \quad \text{is such that} \quad [R(x), a] = [b, x] \quad \text{for all } x \in \mathfrak{k}$$

with $a, b \in \mathfrak{v}$ being fixed (they play the role of parameters for R). In the sense of this definition, the sectional operators (7) correspond to the standard decomposition of $\operatorname{gl}(n)$ into g-skew symmetric and g-symmetric matrices.

Problem 24. Is it possible to generalise the above theorems to the case of the other $\mathbb{Z}_2$-graded Lie algebras? Can we use such sectional operators to construct examples of metrics with non-standard holonomy groups in the (pseudo)-Kähler and hyper-Kähler setting by means of explicit formulas similar to the so-called magic formulas from [14]? Does the notion of projective equivalence has a natural meaning in the case of hyper-Kähler manifolds?

These questions have been motivated by two recent results. The first one is a description of all possible algebras of parallel endomorphisms obtained by C. Boubel in [17]. The centralisers of these algebras are exactly the non-standard holonomy groups mentioned in Problem 24. The second result is an analog of Theorem 6 for the so-called c-projectively equivalent Kähler metrics which, in algebraic terms, corresponds to the Z_2-grading $\operatorname{gl}(n, \mathbb{C}) = \operatorname{u}(n) + \mathfrak{v}$ where $\mathfrak{v}$ denotes the space of Hermitian matrices. The proof of this latter result can be found in [13] .

5.2 Singularities of bi-Hamiltonian systems and bi-Hamiltonian reduction

A recent series of papers [6, 10, 27] has been devoted to the study of singularities of bi-Hamiltonian systems. The underlying bi-Hamiltonian structure discussed in these papers is related to Poisson pencils $\mathcal{P} = \{\mathcal{A}_0 + \lambda\mathcal{A}_1\}$ of *Kronecker* type (see Section 4.1). Such pencils appear in many interesting integrable problems in geometry and physics and usually the verification of the property of being Kronecker is not very difficult. Recall that this property is in fact algebraic and means that at a generic point $x \in M$ the Jordan–Kronecker decomposition of the pair of skew-symmetric forms $\mathcal{A}_0(x)$ and $\mathcal{A}_1(x)$ consists only of Kronecker blocks (see Theorem 1).

However, there are important examples of Poisson pencils for which such a verification is not obvious at all. In particular, the situation becomes unclear for Poisson pencils obtained by a bi-Hamiltonian reduction. Here is one example of this kind. Consider the Lie-Poisson pencil on the space of skew-symmetric matrices from Example 1:

$$[X, Y]_{E+\lambda A} = X(E + \lambda A)Y - Y(E + \lambda A)X$$

where E is the identity matrix and $A = \mathrm{diag}(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots a_k)$ is a (singular) diagonal matrix. This pencil is not Kronecker as $\lambda_i = -a_i^{-1}$ is a characteristic number and as a result the bracket $[\ ,\ ]_{E-a_i^{-1}A}$ admits a non-trivial centre isomorphic to $\mathrm{so}(n_i)$ where n_i is the multiplicity of a_i. There is, however, a natural way to "kill" all these centres.

To that end, consider the natural decomposition $\mathrm{so}(n) = \mathfrak{k} + \mathfrak{v}$, where $\mathfrak{k} = \mathrm{so}(n_1) \oplus \mathrm{so}(n_2) \oplus \cdots \oplus \mathrm{so}(n_k) \subset \mathrm{so}(n)$ and $\mathfrak{v}$ is the natural complement to $\mathfrak{k}$ in $\mathrm{so}(n)$. Consider on $\mathrm{so}(n) \simeq \mathrm{so}(n)^*$ the subalgebra $\mathcal{P}\big(\mathrm{so}(n)\big)^{\mathfrak{k}} \subset \mathcal{P}\big(\mathrm{so}(n)\big)$ consisting of all the functions commuting with $\mathfrak{k}$ (w.r.t. the standard $\mathrm{so}(n)$-bracket). It is easy to see that $\mathcal{P}(\mathrm{so}(n))^{\mathfrak{k}}$ is closed w.r.t. $\{\ ,\ \}_A$ so that we still have a Poisson pencil on $\mathcal{P}\big(\mathrm{so}(n)\big)^{\mathfrak{k}}$. Moreover, this pencil is also naturally defined on the space $\mathcal{P}(\mathfrak{v})^{\mathfrak{k}}$ of all $\mathfrak{k}$-invariant functions on $\mathfrak{v} \simeq \mathfrak{v}^*$, cf. Problems 2 and 3.

Problem 25. What are the algebraic properties of the Poisson pencil on $\mathcal{P}(\mathfrak{v})^{\mathfrak{k}}$ obtained from $\{\ ,\ \}_{E+\lambda A}$ by the above reduction? Are they of Kronecker type? Is there any general mechanism describing the behaviour of (algebraic properties of) Poisson pencils under reduction?

This particular question is directly related to the integrability of geodesic flows on the homogeneous space $\mathrm{SO}(n)/\big(\mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \times \cdots \times \mathrm{SO}(n_k)\big)$. It follows from the paper by I. Mykytyuk [35] that the reduced pencil on $\mathcal{P}(\mathfrak{v})^{\mathfrak{k}}$ is of Kronecker type. The same conclusion follows from [19]. However, the approaches suggested in [35] and [19] do not use the reduced pencil directly (notice that the sizes of the corresponding Kronecker blocks are still unknown). In fact, the construction in [35] is applied to arbitrary semisimple Lie algebras, and in this more general setting, Problem 25 remains open.

Could we use the technology developed in [10] to study this problem? The idea of such an alternative approach is as follows. The reduced Poisson pencil, as a rule, is more complicated than the original one. For example, even if we start with quite simple brackets (e.g., linear as in our example), the new reduced brackets are not linear any more which leads to many technical problems (e.g., we do not have any natural coordinate system for explicit computations). The main idea of [10] is to "replace"

a (non-linear) Poisson pencil by its linearisation. For our purposes, this linearisation needs to be done at an equilibrium point of the corresponding reduced Hamiltonian system (more precisely, at a common equilibrium point for all reduced Hamiltonians). A "good" linearisation may guarantee that the given (non-linear) pencil is of Kronecker type.

Problem 26. What are the common equilibrium points for the reduced pencil on $\mathcal{P}(\mathfrak{v})^{\mathfrak{k}}$? Do they exist? Can we describe them explicitly? (In the case of a generic A with simple eigenvalues, the answer is well known; see for example [27].) What is the linearisation of the reduced pencil at such points? Can one say anything about the algebraic type of the reduced pencil (we want it to be Kronecker) if its linearisation is known? In the more general setting: under which conditions on its linearisation a (non-linear) Poisson pencil is of Kronecker type?

6 Action variables as symplectic invariants

This section is related to symplectic topology of integrable systems (see [18, 38, 43, 55]). Consider a singular Lagrangian fibration $\phi : M \to B$ associated with a certain finite-dimensional integrable system. The base B of this fibration can be understood as a stratified manifold whose strata correspond to different levels of degeneracy of singular fibers. Recall that the action variables are defined on the regular part $B_{\mathrm{reg}} \subset B$ and they, in turn, define an integer affine structure on it. In a neighbourhood of a singular stratum, this affine structure, as a rule, is not smooth but we still may study its asymptotic behaviour and think of it as a singular affine structure on B as a whole.

Since the action variables are preserved under symplectomorphisms, we may think of them as symplectic invariants of (singular) Lagrangian fibrations. How much information do they contain? Examples show that very often the action variables (equivalently, singular affine structure on B) are sufficient to reconstruct completely the structure of the Lagrangian fibration up to symplectomorphisms. The classical result illustrating this principle is the Delzant theorem [18] stating that in the case of Lagrangian fibrations related to Hamiltonian torus actions, the base B is an affine polytope (with some special properties) which determines the structure of the Lagrangian fibration (as well as the torus action) up to a symplectomorphism.

This situation, however, is almost trivial from the point of view of singularities of action variables as they are smooth functions both on M^{2n} and B (here we think of B as a manifold with corners). We believe that an analog of the Delzant theorem holds true in a much more general situation. Ideally, such an analog could be formulated as the following principle (it is not a theorem as a counterexample is easy to construct!):

Let $\phi : M \to B$ and $\phi' : M' \to B'$ be two singular Lagrangian fibrations. If B and B' are affinely equivalent (as stratified manifolds with singular affine structures), then these Lagrangian fibrations are fiberwise symplectomorphic.

Problem 27. Under which additional conditions, does this principle become a rigorous theorem?

In our opinion, this is an important general question which is apparently quite difficult to answer in full generality. Some more specific questions could be of interest too.

Let $\phi : M^4 \to B$ be an almost toric fibration (see [30, 43]), which means in particular that its singularities are all non-degenerate and can be of elliptic and focus type only. Consider the most typical situation when the base B of such a fibration is a two-dimensional region with boundary (having some corners) and some isolated singular points of focus type lying inside B. This domain is endowed with an integer affine structure having singularities at focus points.

Problem 28. Consider two Lagrangian fibrations $\phi : M^4 \to B$ and $\phi' : M'^4 \to B'$. Assume that B and B' are affinely equivalent in the sense that there exists an affine diffeomorphism $\psi : B \to B'$. Is it true that under these assumptions the corresponding Lagrangian fibrations are symplectomorphic?

In the semi-global setting, we may ask a similar question for a Lagrangian fibration in a neighbourhood of a singular fiber. Non-degenerate singularities satisfy one important property: locally, every action variable can be written as $I = I_{\mathrm{sing}} + I_{\mathrm{reg}}$, where the singular part I_{sing} is the same for all singularities of a given topological type and I_{reg} is a smooth function which can be arbitrary. This reflects the fact that the singular part I_{sing} has a local nature and is defined by the structure of a fibration in a neighbourhood of a singular point. According to the Eliasson theorem [21], in the non-degenerate case there are no local symplectic invariants and hence I_{sing} is uniquely defined by the topological type of a singular point.

Problem 29. What happens in the case of degenerate singularities? Do local symplectic invariants exist (for diffeomorphic singularities)? How many and of what kind are they? This question makes sense even in the simplest case of one-degree-of-freedom systems.

As we know from concrete examples of integrable systems, many degenerate singularities are stable in the sense that they cannot be avoided by a small perturbation of a system. The problem of description of stable degenerate singularities is very important on its own but for many degrees of freedom it is quite complicated. However, in the case of two (and two and a half) degrees of freedom there is a number of well-known examples of stable singularities (e.g., Hamiltonian Hopf bifurcation). It would be interesting to clarify the situation with asymptotic behaviour of angle variables at least for such singularities.

V. Kalashnikov [28] has described all stable singularities of rank 1 for Hamiltonian systems of two degrees of freedom.

Problem 30. Describe the symplectic invariants of stable singularities from [28]. For such singularities one of the action variables, say I_1, is smooth, and the other I_2 is singular. Is it true that the singular part I_{sing} of I_2 is always "standard", i.e., is it the same for all singularities of a given topological type?

One could also ask the converse question.

Problem 31. Assume that we know explicit formulas for the action variables $I_1, \ldots, I_n$ so that we are able to analyse the asymptotic behaviour of them in a neighbourhood of a singular fiber. Can we recover the topology of this singularity from the asymptotic behaviour (or at least to distinguish between different types of singularities)? For example, we know that in the case of non-degenerate hyperbolic singularities, the singular

part I_{sing} is of the form $h \ln h + \dots$. Is this property a characteristic for non-degenerate singularities? The answer is apparently positive so that we could use this property for verifying the non-degeneracy condition. For degenerate singularities, this question becomes, of course, more interesting and important.

Another particular question in the same spirit is related to geodesic flows on 3-dimensional manifolds. Among the eight 3-dimensional geometries, there is only one which does not admit integrable geodesic flows, namely, the hyperbolic one. The situation with $\mathrm{SL}(2, \mathbb{R})$ geometry is not quite clear. The geodesic flow on a compact $\mathrm{SL}(2, \mathbb{R})$-manifold M^3 seems to be "half"-integrable in the sense that the phase space $T^* M^3$ can be partitioned into two open domains one of which carries "integrable" dynamics and is fibered into invariant Liouville 3-tori while the other does not admit three commuting integrals.

Problem 32. Do the action variables of integrable geodesic flows on 3-dimensional closed manifolds "feel" the underlying geometry? In other words, can we recover the geometry (i.e., distinguish between 7 possibilities) from the action variables?

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