"Explanatory" Talk in Mathematics Research Papers

Juan Pablo Mejia-Ramos[1] and Matthew Inglis[2]
[1]Rutgers University [2]Loughborough University

In this paper we explore the ways in which mathematicians talk about explanation in their research papers. We analyze the use of the words explain/explanation (and various related words) in a large corpus of text containing research papers in both mathematics and physical sciences. We found that mathematicians do not frequently use this family of words and that their use is considerably more prevalent in physics papers than in mathematics papers. In particular, we found that physicists talk about explaining why disproportionately more often than mathematicians. We discuss some possible accounts for these differences.

Key words: corpus linguistics, mathematical language, mathematical explanation.

The notion of explanation in mathematics has received a lot of attention in both mathematics education and the philosophy of mathematics. In mathematics education, scholars have been particularly interested in proofs that explain mathematical theorems (i.e. proofs that provide an insight into why a mathematical claim is true) and their role in the mathematics classroom (e.g. Hanna, 1990). Philosophers of mathematics have discussed at length possible equivalents for mathematics of existing philosophical theories of scientific explanation (e.g. Steiner, 1978). Some of these discussions bring to bear the extent to which explanation is relevant to the actual practice of mathematicians and often cite individual mathematicians' views on mathematical explanation (more often than not that mathematician seems to be Henri Poincaré, Paul Halmos, or William Thurston). In this report we explore the extent to which mathematicians talk about explanation in their research papers, and the ways in which they do so.

Literature review

In an influential paper in mathematics education, de Villiers (1990) argued that proof serves several different roles in mathematics, that proof is not only used in mathematics as a way to verify results, to provide conviction of the truth of those results (see also Bell, 1976). One of those other functions of proof was to *explain* mathematical results, to provide an insight or understanding into why these results were true, as opposed to just evidence in support of that result. Hanna (1990) made a similar distinction in the context of the teaching and learning of mathematics, discussing the idea that certain proofs fulfilled this explanatory function better than others, to the point that among the set of all proofs one could identify proofs that explain why a theorem is true, while others simply demonstrate that a theorem is true. Mathematics educators have generally suggested that in the mathematics classroom, mathematical explanation should be an important, if not the primary role of proof (de Villiers, 1990; Hanna, 1990; Hersh, 1993).

This distinction between proofs that explain and proofs that demonstrate has a longer history in the philosophy of mathematics. Steiner (1978) put forward a model of mathematical explanation, arguing that a mathematical proof could be better defined in terms of what he called *a characterizing property* of a concept in the theorem, as opposed to other alternative defining characteristics such as the abstractness or the generality of the proof. Steiner's top-down approach to modeling mathematical explanation by providing a general definition of explanatory proof (and thus creating an absolute distinction between explanatory and non-explanatory proofs) has been criticized by other philosophers of mathematics. In particular, Hafner and Mancosu (2005) argued that ascribing explanatoriness to specific proofs should be done based on practicing mathematicians' evaluations, not philosophers' own intuitions (such as Steiner's). The extent to which practicing mathematicians not only agree with philosophers' characterization of mathematical explanation, but simply talk about explanation in their practice plays an important role in the general argument for the *existence* of explanation in mathematics (which not all philosophers believe). As such, it is not uncommon for a discussion of mathematical explanation to mention how much mathematicians talk about it. For example, Steiner claimed that "mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (p.135), and Hafner and Mancosu (2005) supported their claim that mathematicians seek and value explanation in mathematics by presenting several examples of what they called *"explanatory" talk* in mathematical practice: passages of research mathematics papers in which the authors explicitly discuss the role of explanation in their own work. However, we do not currently have empirical evidence, other than these small selections of introspective accounts, about the extent to which talk about mathematical explanation is part of mathematical discourse. We believe one of the reasons this has not been studied at a larger scale may be methodological: a researcher would have to be able to process and analyze a large number of mathematical research papers or conversations among mathematicians.

One method of studying mathematical discourse at such a scale is to use the techniques of corpus linguistics, a branch of linguistics that statistically investigates large collections of naturally occurring text, known as corpora. Methods developed by corpus linguists can be used to investigate many different types of linguistic questions. Here, we report a study that employs some of these techniques to address the following questions: to what extent do mathematicians discuss explanation in their research papers, how does it compare to the extent to which they discuss other important related notions (such as *showing* or *proving* given mathematical results), and how does it compare to discussions about explanation in other types of scientific discourse?

Theoretical perspective

Discussions about mathematical explanation tend to differentiate between mathematical explanations of other mathematics (i.e. mathematics X explains mathematics Y, or X is an explanatory proof of theorem Y), and mathematical explanations of physical phenomena (i.e. mathematics X explains physical phenomenon Y). Colyvan (2011) refers to these two types of explanation as *intra-mathematical* and *extra-mathematical*, respectively. Here we focus on intra-mathematical explanations.

Hafner and Mancosu (2005) further differentiated between two uses of intra-mathematical explanations: those that are "instructions" on how to master the tools of the trade (as in explaining how to employ a certain mathematical technique), and those that "call for an account of the mathematical facts themselves, the reason why" (p. 217). While Hafner and Mancosu considered the latter to be a "deeper" use of mathematical explanation, which is also the focus of the larger philosophical discussion around explanatory proofs, others have emphasized the importance of the former type of explanation in mathematical practice. For instance, Rav (1999) insisted that one of the main reasons mathematicians read proofs is because of all the mathematical know-how embedded in them, emphasizing the mathematical methodologies and problem solving strategies/techniques contained in proofs. According to Rav, "proofs are for the mathematician what experimental procedures are for the experimental scientist: in studying them one learns of new ideas, new concepts, new strategies—devices which can be assimilated for one's own research and be further developed.” (p. 20) Indeed, researchers have obtained empirical evidence (in both small scale interview studies and large scale surveys) that practicing mathematicians maintain that one of the main reasons they read proofs is to gain insights into how they can solve problems that they are working on (Weber & Mejía-Ramos, 2011, Mejía-Ramos & Weber, 2014).

An interesting question related to the specific ways in which mathematicians talk about explanation in their papers, relates to these two types of *“explanatory” talk*: to what extent do mathematicians discuss explanations of why a certain mathematical statement is true, compared to their talk about explanations of how to do something in mathematics?

Methods

One of the main ways in which mathematicians around the world communicate about mathematics is through research papers stored in the ArXiv. The ArXiv is an online repository of electronic preprints of scientific papers in the fields of mathematics, physics, astronomy, computer science, quantitative biology, quantitative finance, and statistics. These papers constitute a large corpus of scientific text that can be used to analyze mathematical discourse.

We downloaded the bulk source files (mostly TeX/LaTeX) and converted the source code to plain text, which we could then analyze using standard software packages for corpus analysis. We then sorted these articles based on their primary and secondary subject classification (Alcock et al., 2017, discussed the details about the processing of these source files). All analyses reported here are based on a proper subset of this corpus, containing all mathematics and physics articles (based on their primary subject classification) uploaded in the first four months of 2009. This left us with 6988 mathematics papers (30,892,695 words) and 14861 physics papers (58,859,660 words).

Results

Frequency of explicit *“explanatory” talk* in mathematics papers

Table 1 shows the frequencies of all words linguistically related to the word *explain* (henceforth *explain-words*) in our corpus of 6988 mathematics papers. Explain-words showed up 4871 times in this set of papers, approximately once every 0.7 papers. While this certainly provides an existence proof of explicit *“explanatory” talk* in this corpus, it is not very surprising (it would very rare if no word based on the word explain showed up in these many mathematics papers). In order to get a sense of the extent to which these frequencies were high or low in this type of mathematical discourse, we compared them against the frequencies of words related to other important mathematical activities.

Tables 2 presents the frequencies of words linguistically related to the notions of showing, solving, and proving, which were chosen based on their relevance in mathematical explanation. Measured against these other frequencies, mathematicians used explain-words rather infrequently. Indeed, mathematicians used explain-words in their papers approximately 11 times less frequently than show-words or solve-words and nearly 23 times less often than prove-words.

One possibility is that explain-words are simply not used much in this kind of scientific discourse in general. Thus, even though the importance of scientific explanation is so obvious that it does not need to be justified by looking at *“explanatory” talk* in science, it could be the case that this type of talk is not that common in scientific research papers either. In order to test this hypothesis we studied the use of explain-words in the 14861 physics papers in our corpus (Table 3). Explain-words showed up 21305 times in this set of papers, approximately once every

1.45 papers, or twice as often as they showed up in the mathematics papers. Thus, based on the comparison of the use of explain-words in mathematics and physics papers, it seems that mathematicians discuss explanations much more infrequently than physicists.

Explain-words	Frequency
explain	1827
explained	1690
explanation	498
explains	484
explaining	175
explanations	119
explanatory	51
unexplained	22
unexplainable	4
explainable	1
Total	4871

Table 1. Frequency of words related to *explanation* appearing in the mathematics papers.

Show-words	Frequency
show	31691
shows	12890
shown	10235
showed	2414
showing	2129
Total	59359

Solve-words	Frequency
solution	25845
solutions	15956
solve	2204
solving	1717
solvable	1618
solved	1342
solves	1071
solvability	429
solver	145
unsolved	95
solvers	56
nonsolvable	39
unsolvable	32
cosolvable	29
equisolvable	18
unsolvability	12
Total	50608

Prove-words	Frequency
proof	56452
prove	29481
proved	12842
proves	4160
proofs	3892
proving	2661
proven	1902
provable	159
reprove	58
disprove	43
provability	29
reproved	29
disproved	17
unprovable	13
unproven	12
reproving	11
disproving	10
reproves	10
prover	7
unproved	7
subproof	5
disproof	4
Total	111804

Table 2. Frequencies of words linguistically related to the notions of *showing, solving*, and *proving* appearing in the mathematics papers.

Explain-words	Frequency
explain	7768
explained	6513
explanation	3564
explains	1601
explaining	914
explanations	675
unexplained	177
explanatory	62
explainable	23
unexplainable	8
Total	21305

Table 3. Frequency of words linguistically related to the notion of *explaining* appearing in the physics papers.

Finally, the search for explain-words may be thought of as requiring an extremely explicit discussion of explanation, one that would leave unnoticed a significant amount of the *"explanatory" talk* in these papers. Hafner and Mancosu (2005) offered a list of eight expressions that they had found to be commonly used in the mathematics and philosophy of mathematics literature to describe the search for explanations. Table 4 presents these expressions along with the specific concordance search we made to investigate their prevalence in both the mathematics and physics papers, and the frequencies with which these alternative expressions appeared. We note that the total number of occurrences of these expressions is only about 10% of the total amount of explain-words in each set of papers (with disproportionately more occurrences of these expressions in the physics papers than the mathematics ones) and thus this analysis does not affect the finding made by only investigating the use of explain-words.

Alternative expression	Concordance search	Mathematics	Physics
"the deep reasons"	deep* reason*	5	16
"an understanding of the essence"	understand* the essence	0	5
"a better understanding"	better understand*	161	767
"a satisfying reason"	satisfy* reason	0	0
"the reason why"	reason* why	312	924
"the true reason"	true reason	3	1
"an account of the fact"	an account of the fact	0	0
"the causes of"	cause* of	16	609
	Total	497	2322

Table 4. Frequencies of alternative expressions of related to *"explanatory" talk*

Explaining why vs. explaining how

In order to investigate mathematicians' discussion of *explanations of why* a certain mathematical statement is true (Hafner and Mancosu's "deep" explanation), in comparison to their talk about *explanations of how* to do something in mathematics (related to Rav's notion of mathematical know-how), we created a concordance of the corpus of papers and identified every instance an explain-word had been immediately followed by the words why or how (e.g. unexplained why, explanation how). We did this by searching the concordance for *expla* why and *expla* how, and checking that all results were indeed uses of explain-words. We then repeated the process with the corpus of physics papers. As, shown in Table 5, there is a clear difference between the ways that explain-words show up in the mathematics and the physics research papers.

	Mathematics	Physics
expla why	247	952
expla how	458	353
Total	705	1305

Table 5. Frequencies of explain-words immediately followed by the words why or how in the mathematics and physics research papers

We note that when taken together the total of *expla*-why and *expla*-how expressions were roughly as common in math papers as they were in physics papers, with approximately one of these expressions showing up every 10-11 papers in the corresponding set, and also a relatively small subset of the wider use of explain-words (roughly 14% and 6% of explain-word usage in mathematics and physics, respectively). However, the distribution of these two different types of expressions in the two sets of papers was significantly different (Fisher's exact test, $p < .001$), with mathematicians using nearly twice as many *expla*-how expressions than *expla*-why expressions, and physicists on the other hand using a little under three times as many *expla*-why expressions than *expla*-how expressions.

Discussion

Our analysis of *"explanatory" talk* in a large sample of mathematics papers does not offer support for a claim often made in the philosophy of mathematics: that this type of talk is prevalent in mathematical discourse. When compared to explicit discussion of other related mathematical practices (showing results, solving problems, and proving theorems), mathematicians do not seem to discuss explanation nearly as much. Furthermore, when compared to another scientific discourse, we found that mathematical discourse contains only a fraction of *"explanatory" talk* as research papers in physics. Indeed, we believe these findings suggest that the prevalence of *"explanatory" talk* in mathematical discourse has been widely exaggerated.

Furthermore, by analyzing the frequency with which variations of the expressions *explain why* and *explain how* occur in mathematics and physics research papers, we found that, to the extent to which they engage in *"explanatory" talk*, mathematicians seem to be much more interested in discussing explanations of how to do something in mathematics, than in explanations of why things are the way they are in mathematics. In physics we found the situation to be the opposite. This is particularly interesting given mathematics educators' and philosophers' of mathematics preoccupation with the type of intra-mathematical explanations of the form X explains why Y (where X and Y are mathematical assertions), and particularly with the notion of explanatory proofs (in which proof X explains why theorem Y is true). This focus may have been inherited from the more traditional study of the notion of scientific explanation, which is not only naturally concerned with this type of explanations (the desire to explain the real world is full of why-questions), but according to our findings may also be more commonly discussed in scientific discourse in terms of answers to why-questions. However, our findings suggest that this focus may also be misguided for those interested in studying the notion of mathematical explanation as it more commonly occurs in the discourse of professional mathematicians. Indeed, as suggested by Rav (1999), it seems that when it comes to proofs and explanations, mathematicians are primarily interested in learning how to solve other problems, possibly over learning the reasons why some mathematical results hold true.

Now, one must be careful about several inferential jumps made in this kind of analysis. First, while the ArXiv may well be the largest, most widely used repository of this type of preprints and postprints in the world, we have analyzed a very specific type of mathematical discourse, leaving open the possibility that studies of mathematical discourse in others settings (conversational or other digital communications) could lead to contrasting findings. Second, we have analyzed these research papers for a limited type of *"explanatory" talk*, one required to contain explain-words or a limited number of alternative, related expressions. While this was an obvious place to start to investigate *"explanatory" talk* in mathematical discourse, it is certainly possible that the analysis of other expressions related to mathematical explanation may skew our results. These limitations of the present study indicate clear avenues for future empirical research on mathematical explanation.

Acknowledgements

This project was funded by the British Academy.

References

Alcock, L., Inglis, M., Lew, K., Mejia-Ramos, J.P., Rago, P. & Sangwin, C. (2017). Comparing expert and learner mathematical language: A corpus linguistics approach. *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education.*

Bell, A. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational studies in mathematics, 7*, 23-40.

Colyvan, M. (2011). *An Introduction to the Philosophy of Mathematics.* Sydney: University of Sydney.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Hafner, J., & Mancosu, P. (2005). The Varieties of Mathematical Explanation. In P. Mancosu et al. (Eds.), *Visualization, Explanation and Reasoning Styles in Mathematics* (pp. 215–250). Berlin: Springer.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6–13.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24*, 389–399.

Mejía-Ramos, J. P., & Weber, K. (2014). Why and how mathematicians read proofs: Further evidence from a survey study. *Educational Studies in Mathematics, 85*(2), 161-173.

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica, 7*(3), 5-41.

Steiner, M. (1978). Mathematical Explanation. *Philosophical Studies, 34*, 135–151.

Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics, 76*, 329-344.