On Restricting the Ambiguity in Morphic Images of Words

by

Joel D. Day

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Abstract

For alphabets Δ_1, Δ_2, a morphism $g : \Delta_1^* \to \Delta_2^*$ is *ambiguous* with respect to a word $u \in \Delta_1^*$ if there exists a second morphism $h : \Delta_1^* \to \Delta_2^*$ such that $g(u) = h(u)$ and $g \neq h$. Otherwise g is *unambiguous*. Hence unambiguous morphisms are those whose structure is fully preserved in their morphic images.

A concept so far considered in the free monoid, the first part of this thesis considers natural extensions of ambiguity of morphisms to free groups. It is shown that, while the most straightforward generalization of ambiguity to a free monoid results in a trivial situation, that all morphisms are (always) ambiguous, there exist meaningful extensions of (un)ambiguity which are non-trivial – most notably the concepts of (un)ambiguity up to inner automorphism and up to automorphism.

A characterization is given of words in a free group for which there exists an injective morphism which is unambiguous up to inner automorphism in terms of fixed points of morphisms, replicating an existing result for words in the free monoid. A conjecture is presented, which if correct, is sufficient to show an equivalent characterization for unambiguity up to automorphism. A rather counterintuitive statement is also established, that for some words, the only unambiguous (up to automorphism) morphisms are non-injective (or even periodic).

The second part of the thesis addresses words for which all non-periodic morphisms are unambiguous. In the free monoid, these take the form of *periodicity forcing words*. It is shown using morphisms that there exist ratio-primitive periodicity forcing words over arbitrary alphabets, and furthermore that it is possible to establish large and varied classes in this way. It is observed that the set of periodicity forcing words is spanned by chains of words, where each word is a morphic image of its predecessor. It is shown that the chains terminate in exactly one direction, meaning not all periodicity forcing words may be reached as the (non-trivial) morphic image of another. Such words are called *prime periodicity forcing words*, and some alternative methods for finding them are given.

The free-group equivalent to periodicity forcing words – a special class of *C-test words* – is also considered, as well as the ambiguity of *terminal-preserving* morphisms with respect to words containing terminal symbols, or constants. Moreover, some applications to pattern languages and group pattern languages are discussed.

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Contents

Chapter 1

Introduction

Words – strings of symbols or *letters* from a given set or *alphabet* – are, as far as mathematics is concerned, a reasonably new object of direct study. Nevertheless, they are arguably one of the most intuitive. One of the first things we learn as children is two write our own name: a simple act of joining certain symbols together (an operation we shall refer to as *concatenation*). Numbers are represented as words – for example the integers are words over the alphabet $\{0, 1, 2, ..., 9, -\}$ – and many basic numerical algorithms such as addition and multiplication can be done purely by combinatorial manipulation of the representative words. In fact, it is almost impossible to avoid encountering words, and everywhere we look we find information communicated using words in one form or another.

It is therefore no surprise that we are inclined to represent a wide range of things using words beyond the standard components of natural language, not least of which are numbers themselves as we have just mentioned. Other examples include DNA, which is often represented by words over the alphabet G, C, A, T, morse code (represented by 'dots' $\cdot$ and 'dashes' $-$), and binary data (words over the alphabet $\{0, 1\}$).

$$TGACATGGGTACACATGACGGG$$

$$0110110011011001110011101 0001$$

The origins of studying words as combinatorial mathematical objects are generally attributed to work from early 20th Century by Axel Thue [83] on the subject of avoiding certain patterns, such as repetitions in (infinite) words. Further publications on the subject (for example, those by Morse [57]) remained reasonably sparse for the following decades however, and the emergence of combinatorics on words as an established field of study can be traced alongside advances in computation in the latter half of the 20th Century. In 1983, the first book directly addressing the topic was published by a group of authors under the name Lothaire [47].

Nevertheless, as has already been stated, words represent a common and pervasive structure, and accordingly there are many results in various areas of mathematics and theoretical computer science which may be interpreted as concerning words, for example paths on graphs and elements of groups.

We have already mentioned the most fundamental operation on words, namely concatenation. From an algebraic point of view, the set of all words over a given alphabet Σ, along with the operation of concatenation, forms a free monoid Σ^*. The identity element of such a monoid is the empty-word ε, and is the word of length 0 (i.e., consisting of no letters). Similarly, for an alphabet $\Sigma = \{\mathtt{a}_1, \mathtt{a}_2, \dots, \mathtt{a}_n\}$, a free group $\mathcal{F}_\Sigma$ may be obtained as the monoid $(\Sigma \cup \Sigma^{-1})^*$ where Σ^{-1} is an alphabet $\{\mathtt{a}_1^{-1}, \mathtt{a}_2^{-1}, \dots \mathtt{a}_n^{-1}\}$, along with the relations $\mathtt{a}_i\mathtt{a}_i^{-1} = \mathtt{a}_i^{-1}\mathtt{a}_i = \varepsilon$ for $1 \leq i \leq n$.[1] For example, if $\Sigma := \{\mathtt{a}, \mathtt{b}\}$, the words $\mathtt{aaba}^{-1}\mathtt{b}$ and $\mathtt{aaaa}^{-1}\mathtt{ba}^{-1}\mathtt{babb}^{-1}\mathtt{a}$ both belong to the monoid $(\Sigma \cup \Sigma^{-1})^*$ and the free group $\mathcal{F}_\Sigma$. However, while they are considered distinct in the free monoid, due to the relations $\mathtt{a}_i\mathtt{a}_i^{-1} = \mathtt{a}_i^{-1}\mathtt{a}_i = \varepsilon$, they are considered equivalent in the free group:

$$\mathtt{a\ a\ b\ a}^{-1}\mathtt{b} = \mathtt{a\ a}\ \underbrace{\mathtt{a\ a}^{-1}}_{\varepsilon}\ \mathtt{b\ a}^{-1}\mathtt{b\ a}\ \underbrace{\mathtt{b\ b}^{-1}}_{\varepsilon}\ \mathtt{a}$$

In addition to concatenation, another fundamental operation on words, and the focus of the present thesis, is the morphism. A morphism is a mapping from one set of words to another which is compatible with the operation of concatenation. More formally, if $\mathcal{A}, \mathcal{B}$ are alphabets, then a morphism $h : \mathcal{A}^* \to \mathcal{B}^*$ (resp. $h : \mathcal{F}_\mathcal{A} \to \mathcal{F}_\mathcal{B}$) is a mapping such that for every $u, v \in \mathcal{A}^*$ (resp. $\mathcal{F}_\mathcal{A}$), $h(uv) = h(u)h(v)$. Consequently, we may define a morphism simply for each individual letter in $\mathcal{A}$, and treat its application to a word as a letter-by-letter substitution. For example, if Δ is the alphabet $\{1, 2, 3\}$ and Σ is the alphabet $\{\mathtt{a}, \mathtt{b}\}$, then we can define a morphism $h : \Delta^* \to \Sigma^*$ such that e.g., $h(1) := \mathtt{aa}$, $h(2) := \mathtt{bb}$ and $h(3) := \mathtt{bab}$, and for the word $1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2$, we have

$$h(1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2) = \overbrace{\mathtt{a\ a}}^{h(1)}\overbrace{\mathtt{b\ b}}^{h(2)}\overbrace{\mathtt{b\ a\ b}}^{h(3)}\overbrace{\mathtt{a\ a}}^{h(1)}\overbrace{\mathtt{b\ a\ b}}^{h(3)}\overbrace{\mathtt{b\ b}}^{h(2)}.$$

Similarly, we may define the morphism $h' : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ such that $h'(1) := \mathtt{aa}$,

[1] Thus, from a more technical perspective, the free group is a set of equivlance classes over the free monoid $(\Sigma \cup \Sigma^{-1})^*$.

$h'(2) := \mathtt{a}^{-1}\mathtt{b}$ and $h'(3) := \mathtt{ba}$, so that

$$h'(1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2) = \overbrace{\mathtt{a}\ \mathtt{a}}^{h(1)}\overbrace{\mathtt{a}^{-1}\mathtt{b}}^{h(2)}\overbrace{\mathtt{b}\ \mathtt{a}}^{h(3)}\overbrace{\mathtt{a}\ \mathtt{a}}^{h(1)}\overbrace{\mathtt{b}\ \mathtt{a}}^{h(3)}\overbrace{\mathtt{a}^{-1}\mathtt{b}}^{h(2)}\ .$$

$$= \mathtt{a}\ \mathtt{b}\ \mathtt{b}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{b}\ \mathtt{b}$$

In addition to their well-known algebraic importance, morphisms play a significant role in many areas and applications, and despite their conceptual simplicity, are connected to many deep and complex concepts. For example, the famous, undecidable Post Correspondence Problem (cf. Post [64]) is easily expressed using morphisms in the following way:

(Post Correspondence Problem). Let Δ, Σ be alphabets. Given two morphisms $g, h : \Delta^* \to \Sigma^*$, does there exist a word $u \in \Delta^+$ such that $g(u) = h(u)$?

The set of all words u such that $g(u) = h(u)$ is generally referred to as the *equality set* $E(g, h)$, and hence the Post Correspondence Problem is often presented as the emptiness problem for equality sets. Moreover, important complexity classes such as P, NP and even the recursively enumerable sets may also be expressed using equality sets of morphisms (cf. Mateescu et al. [54] and Culik II [4]).

A fundamental combinatorial property of morphisms which is closely related to equality sets is (un)ambiguity. For alphabets Δ, Σ, a morphism $g : \Delta^* \to \Sigma^*$ is *ambiguous* with respect to a word $u \in \Delta^*$ if there exists a morphism $h : \Delta^* \to \Sigma^*$ such that $g(u) = h(u)$ and $h \neq g$. Otherwise g is *unambiguous*. Hence for a morphism g and word u, g is ambiguous if there exists a morphism h such that $u \in E(g, h)$. For example, the morphism $h_1 : \{1, 2\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ given by $h_1(1) := \varepsilon$, $h_1(2) := \mathtt{a}$ is unambiguous with respect to the word $u := 1 \cdot 2 \cdot 1$, since any morphism from $\{1, 2\}^*$ to $\{\mathtt{a}, \mathtt{b}\}^*$ mapping u to $\mathtt{a}$ must necessarily erase 1 and map 2 onto $\mathtt{a}$. In other words, for any morphism $g_1 : \{1, 2\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ such that $g_1(u) = h_1(u)$ we have $g_1 = h_1$. On the other hand, the morphism $h_2 : \{1, 2\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ given by $h_2(1) := \mathtt{aba}$ and $h_2(2) := \mathtt{b}$ is ambiguous with respect to u, since for the morphism $g_2 : \{1, 2\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ given by $g_2(1) := \mathtt{a}$ and $g_2(2) := \mathtt{babab}$, we have:

$$g_2(1 \cdot 2 \cdot 1) = \overbrace{\mathtt{a}\ \mathtt{b}\ \mathtt{a}}^{h_2(1)}\overbrace{\mathtt{b}}^{h_2(2)}\overbrace{\mathtt{a}\ \mathtt{b}\ \mathtt{a}}^{h_2(2)} = h_2(1 \cdot 2 \cdot 1)$$

$$\underbrace{\mathtt{a}}_{g_2(1)}\underbrace{\mathtt{b}\ \mathtt{a}\ \mathtt{b}\ \mathtt{a}\ \mathtt{b}}_{g_2(2)}\underbrace{\mathtt{a}}_{g_2(1)}$$

and clearly $g \neq h$.

The (un)ambiguity of morphisms can be seen as both a property of the pair of words $(u,\, g(u))$, and of the morphism itself, and is essentially a measure of (non-)

determinism in the process of mapping u to $g(u)$. In this respect, unambiguity is similar to injectivity and determines to an extent the structural information lost when the morphism is applied. Indeed, unambiguity can even be seen as dual to injectivity. Consider two words u, v and a morphism g such that $g(u) = v$. It is possible to determine u from v and g when g is injective, while on the other hand it is possible to determine g from u and v when g is unambiguous. Of course the final configuration, that v may be determined by u and g is given when g is a function.

It is therefore surprising that despite the fundamental nature of (un)ambiguity as a property of morphisms, research explicitly addressing the topic is rather recent, and limited to words and morphisms of the free monoid. The origins come from pattern languages – the set of all morphic images of a given word – and has played an important role in proving their properties. More broadly, indirect references to, and implications of the (un)ambiguity of morphisms can be found in a wide range of topics including test words for automorphisms of free groups (cf. e.g., Turner [84]) and C-test words (cf. Ivanov [38]), word equations (cf. Lothaire [47], [48]), and as we have already mentioned, pattern languages (cf. Matescu, Salomaa [53]), equality sets (cf. Salomaa [76] and Engelfriet, Rozenberg [17]) and the Post Correspondence Problem (cf. Post [64]).

In the current thesis, we shall explore the topic of ambiguity of morphisms in free monoids and groups, with particular emphasis on *unambiguous* morphisms – addressing the question of when a word has (1) at least one unambiguous morphism, and (2) as many unambiguous morphisms as possible. Regarding the latter, all words posses some periodic[2] ambiguous morphisms, so we consider specifically words for which all non-periodic morphisms are unambiguous.

In Chapter 4, we address the problem of generalising the notion of ambiguity to a free group. Our first observation is that, due to the existence of non-trivial inner automorphisms, all morphisms are ambiguous with respect to all words in a free group. Of course this appears to be bad news for the concept of ambiguity in a free group and even contradictory to our claim that ambiguity is related to existing research in combinatorial group theory. However, we show that on closer inspection, the idea of unambiguity – and the question of how well the structure of a morphism is preserved in the morphic image – is still not only relevant, but leads to an intriguing and rich theory.

In particular, we are able to show that, provided a particular construction based on composition with inner automorphisms is disregarded, we once again have unambiguous morphisms. Since inner automorphisms are particularly closely

[2]A morphism is *periodic* if it maps each letter to repetitions of a single word. The images of periodic morphisms lose any structural artifacts from the pre-image pattern.

related – both combinatorially, and algebraically – to the identity morphism, a given morphism and its composition with an inner automorphism are also closely related. In terms of the structure preserved, we see that the "unambiguous" morphic images in this context preserve the structure of the morphism up to composition with inner automorphisms, which is demonstrably as much "preservation" as possible. We say such morphisms are *unambiguous up to inner automorphism*. Since the only inner automorphism in a free monoid is the identity morphism, our definition can be considered a direct generalization. Similarly, since automorphisms are a super-set of the inner automorphisms which are also closely related to the identity morphism, we define the slightly weaker notion of unambiguity up to automorphism in the same way.

In Chapter 5, we consider those words which possess an unambiguous morphism – with particular attention paid to injective morphisms. Since this question has been well addressed for words in a free monoid, we remain in the context of free groups, and consequently consider the notions of (un)ambiguity up to inner automorphism and (un)ambiguity up to automorphism.

We begin by producing some classes of words which do not possess an unambiguous morphism. Then, a characterization is given of words in a free group for which there exists an injective morphism which is unambiguous up to inner automorphism in terms of fixed points of morphisms, replicating an existing result for words in the free monoid. Furthermore, in the case that such a morphism exists, an explicit construction is given. A conjecture also is presented, which if correct, is sufficient to show an equivalent characterization for unambiguity up to automorphism.

We show that this (potential) characterization for unambiguity up to automorphism is also equivalent to a natural generalization of the notion of morphic primitivity in a free monoid, and hence that, subject to the correctness of our conjecture, two existing characterizations in the free monoid also exist for unambiguity up to automorphism (cf. Sections 3.1, 5.5). Interestingly, our second (potential) characterization does not hold when considering unambiguity up to inner automorphism, and so in this sense, we see that unambiguity up to automorphism is a better "fit" to unambiguity in a free monoid.

A rather counterintuitive statement is also established: that for some words, the only unambiguous (up to automorphism) morphisms are non-injective (or even periodic), and some classes of words which possess a non-injective morphism which is unambiguous up to automorphism are given in Section 5.6.

Finally, as a by-product of our reasoning earlier in the chapter, we are able to provide simple proofs of some properties of (terminal-free) pattern languages over a group alphabet: giving a characterization for the inclusion problem – and therefore also the equivalence problem, and moreover, characterizing when the union of two (terminal-free) group pattern languages is again a terminal-free group pattern language.

In Chapter 6, we consider words for which all non-periodic morphisms are unambiguous. In the free monoid, these take the form of *periodicity forcing words.* Periodicity forcing words are words which do not satisfy the so-called Dual Post Correspondence Problem – a decidable version of the PCP introduced by Culik II, Karhumäki [5]. While several examples and necessary/sufficient conditions are known for periodicity forcing words over a binary alphabet, very little is known regarding alphabets of larger sizes, as well as for so-called *ratio-primitive* examples. This is largely due to the fact that classifying periodicity forcing words essentially involves finding *all* solutions to certain word equations. Hence our focus is on better understanding the set of periodicity forcing words for alphabets with at least three letters.

Using an approach developed in [10] which avoids some of the difficulties associated with solving word equations by producing new examples as the morphic images of existing ones, it is shown that there exist ratio-primitive periodicity forcing words over arbitrary alphabets, and furthermore that it is possible to establish large and varied classes in this way – including words with any given prefix/suffix/factor.

We then consider the overall structure of the set of periodicity forcing words (denoted $\mathrm{DPCP}^{\neg}$), by dividing it into those words which may be reached by a non-trivial morphism from other elements of the set, and those which cannot. The latter form a "prime" subset of $\mathrm{DPCP}^{\neg}$ from which all periodicity forcing words may be generated using morphisms (which are characterized in [10]).

In order to find examples of prime periodicity forcing words – and therefore demonstrate that this subset is non-empty – it makes sense to consider the shortest periodicity forcing words, and as a result, we are able to give bounds on the length of the shortest periodicity forcing words over any given alphabet. Moreover, since we cannot produce prime periodicity forcing words as morphic images in a non-trivial way, we discuss some alternative methods for finding examples.

We are also able to exploit the connection between periodicity forcing words and word equations to establish some results concerning the closure properties of pattern languages before, finally, we consider the free-group equivalent of periodicity forcing words: a special class of C-test words. We show that our technique of producing new periodicity forcing words as the morphic images of existing ones can easily be adapted to the free group to produce new examples of C-test words, and then apply existing knowledge of C-test words to provide some elementary insights into the (un)ambiguity of terminal-preserving morphisms (morphisms which preserve a subset of letters, which are called *terminal symbols*, cf. Section 3.2) in a free group. Specifically, we show that, if terminal symbols are permitted, then unlike the terminal-free case, there exist completely (i.e., in the original, free monoid sense) unambiguous morphisms, and moreover, that for some words with terminal symbols, *all* morphisms are unambiguous. We also show that, rather counterintuitively, for a given terminal-preserving morphism h, there exists a word u (with terminal symbols) which is erased uniquely by that morphism (i.e., $h(u) = \varepsilon$ and h is unambiguous with respect to u).

The rest of the thesis is organised as follows. In the next chapter, we present the necessary notation and concepts which we shall use throughout this thesis. Chapter 3 provides a brief overview of the related literature. Our results are presented as described already in Chapters 4, 5 and 6, and in Chapter 7 we present our conclusions and discuss some open problems. We also remark that some of the results in this thesis have been published in [12], [13], [8] and [7].[3]

[3] Conference versions of the first three papers appear in [10], [11], and [9] respectively.

Chapter 2

Preliminaries

Before we progress further with the exposition of our thesis, the present chapter provides a detailed introduction to the concepts and notation we will use. The thesis is largely self-contained, although it is assumed that the reader is familiar with standard mathematical concepts, as well as elementary knowledge of abstract algebra and language theory/combinatorics on words. In particular, for an introduction to the latter, we recommend Rozenberg, Salomaa [74] and Lothaire [47].

2.1 Basics

For the majority of this thesis, we shall borrow our terminology from existing research into the ambiguity of morphisms, and hence also partly from the theory of pattern languages. In particular, we shall mostly use notation described in Section 2.3. Since we deal with many similar concepts separately in both the context of the free group and free monoid, we shall keep our notation as consistent as possible in both cases. However in the case that we do not distinguish between groups and monoids (such as Section 2.2), then we shall highlight this using a different choice of notation.

2.1.1 Sets, Semigroups, Monoids and Groups

We begin with the standard set operators $\subset, \subseteq, \supset, \supseteq$ which correspond to subset, proper subset, superset and proper superset respectively, and $\cap, \cup, \setminus$ denoting intersection, union and difference[1]. The complement of a set S is written $S^{\neg}$ and the empty set as $\emptyset$. By $\mathbb{N}$, we denote the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, while $\mathbb{Z}$ denotes the integers. For a set of integers $S \subset \mathbb{Z}$ we denote the greatest common divisor of elements of S by $\gcd(S)$, and lowest common multiple of elements of S by $\operatorname{lcm}(S)$. The cardinality of a set S is denoted by $|S|$. To

[1] Set-difference is also sometimes referred to as set-subtraction or relative complement.

indicate that an element s belongs to a set S, we write $s \in S$. A set S is *closed* under a binary operation $\circ$ if for every $s_1, s_2 \in S$, $s_1 \circ s_2 \in S$. For a set S the closure of S under an operation $\circ$ is the smallest set S' which contains S and is closed under $\circ$.

A set is a *semigroup* if it is closed under an associative binary operation. Consequently, for a set X and operation $\circ$, the closure of X under $\circ$ is a semigroup $\mathcal{S}_X$, and we say that $\mathcal{S}_X$ is *generated* by X. Similarly, we can refer to a *set of generators* (or *generating set*) X of $\mathcal{S}_X$ (note that this is not necessarily unique). If, for every $s \in \mathcal{S}_X$, s may be uniquely decomposed into elements of X (i.e., there are no non-trivial relations between elements of $\mathcal{S}_X$), then X is said to generate $\mathcal{S}$ *freely*. A semigroup is *free* if it has a free generating set. If there exists an *identity element* I in $\mathcal{S}_X$ (i.e., such that for every $s \in \mathcal{S}_X$, $s \circ I = I \circ s = s$), then $\mathcal{S}_X$ is a *monoid* $\mathcal{M}_X$. If there are no further non-trivial relations between the generators, then the monoid is freely generated by X. A monoid is free if it has a free generating set.

For example, if $X_1 = \{x_1, x_2\}$, and $\circ$ is the operation such that $y_1 \circ y_2 = y_1 y_2$ for all y_1, y_2, then the monoid $\mathcal{M}_{X_1}$ is the set

$$\{I,\ x_1,\ x_2,\ x_1x_1,\ x_1x_2,\ x_2x_1, x_2x_2,\ x_1x_1x_1,\ x_1x_1x_2,\ \ldots\}$$

where I is the identity element. Moreover, $\mathcal{M}_{X_1}$ is generated freely by X_1. Similarly, if $X_2 := \{x_1, x_1x_2, x_2\}$ then the monoid $\mathcal{M}_{X_2}$ is the set

$$\{I,\ x_1,\ x_1x_2,\ x_2,\ x_1x_1x_2,\ \ldots\}.$$

It is not freely generated by X_2, because the element $x_1\ x_2$ may be obtained in two ways from the generators X_2, and thus there is a non-trivial relation between them. Nevertheless, the monoid $\mathcal{M}_{X_2}$ *is* free, because it is freely generated by X_1.

For a set X, the *group* $\mathcal{G}_X$ generated by X is the monoid $\mathcal{M}_{X \cup X^{-1}}$, where X^{-1} is the set $\{x^{-1} \mid x \in X\}$, along with the additional relations $x \circ x^{-1} = x^{-1} \circ x = I$ for all $x \in X$, where I is the identity element. A group is *freely generated* by X if there are no further non-trivial relations between the generators. A group is *free* if it has a free generating set. We shall normally use $\mathcal{F}$ or $\mathcal{F}_X$ to denote a free group. For convenience, and with reference to our choice of notation in Section 2.2, we shall abreviate $\mathcal{F}_{\{1,2,\ldots,n\}}$ to $\mathcal{F}_n$.

For a group $\mathcal{G}$, a (proper) subset of $\mathcal{G}$ is a (proper) *subgroup* if it is also a group. Likewise, a (proper) subset of a monoid which is also a monoid is a (proper) submonoid. We note the following famous theorem (often called the Nielson-Schreier Theorem):

Theorem 1 (Schreier [78])**.** *Every subgroup of a free group is free.*

A subgroup is a *retract* if there exists an endomorphism (cf. Section 2.2) of the group which maps surjectively onto the subgroup, and is the identity on the subgroup. More precisely, a subgroup $\mathcal{G}'$ of a group $\mathcal{G}$ is a retract if there exists a morphism $h : \mathcal{G} \to \mathcal{G}$ such that $h(x) = x$ for all $x \in \mathcal{G}'$ and $h(y) \in \mathcal{G}'$ for all $y \in \mathcal{G}$.

The *rank* of a group $\mathcal{G}$ is the cardinality of the smallest generating set X and is denoted $\text{rank}(\mathcal{G})$. We define the rank of a monoid in the same way. If a group/monoid is free, then the cardinality is the same for any free generating set, and thus the rank is the cardinality of any generating set. It follows from Grushko's Theorem [26], below, that if X generates the group $\mathcal{G}_X$, and $|X| = \text{rank}(\mathcal{G})$, then X generates $\mathcal{G}_X$ freely.

Theorem 2 (Grushko's Theorem)**.** *Let $\mathcal{G}, \mathcal{G}'$ be finitely generated groups. Then the rank of the free product $\mathcal{G} * \mathcal{G}'$ is equal to the sum of the ranks of $\mathcal{G}$ and $\mathcal{G}'$.*

2.1.2 Combinatorics on Words

An *alphabet* is an enumerable set of symbols, sometimes called *letters*. A *word* over an alphabet Σ is a string/sequence of symbols from Σ, so that, for example $\mathtt{abaaba}$ is a word over the alphabet $\Sigma := \{\mathtt{a}, \mathtt{b}\}$. The set of letters occurring in a word u is $\text{symb}(u)$. For the remainder of the thesis, we shall use Σ to refer to the specific alphabet $\Sigma := \{\mathtt{a}, \mathtt{b}\}$ (unless it is explicitly stated otherwise). For two words u, v we define the operation *concatenation* $(\cdot)$ such that $u \cdot v = uv$. Hence a word is simply a concatenation of letters from a given alphabet. We shall generally omit the $\cdot$ symbol, and use it only when needed to avoid confusion (so for example when considering words over the alphabet $\mathbb{N}$, so we can distinguish between, e.g., $1 \cdot 1 \cdot 2$ and $11 \cdot 2$). The length of a word u is the number of letters, and denoted by $|u|$ so that e.g., $|\mathtt{abaaba}| = 6$. The word of length 0 is called the *empty word* and is denoted by ε. Hence the set of all words over a given alphabet X (including ε) forms a free monoid under the operation of concatenation, which we denote by X^*. Likewise, $X^* \backslash \{\varepsilon\}$ is a free semi-group, which we denote by X^+.

If, for words u, v, w, x, $u = vwx$, then w is a *factor* of u. It is a *proper* factor if $u \neq w$. If $v = \varepsilon$ then w is a *prefix* and if $x = \varepsilon$ then w is a *suffix* of u. If u has a non-empty word v as both a suffix and a prefix, it is *bordered*. Moreover, we say that the factor w *occurs* in u, and we may refer to specific (e.g., leftmost, rightmost, all, etc) *occurrences* of w in u. For example there are 3 occurrences of the factor $\mathtt{ab}$ in the word $\underline{\mathtt{ab}}\mathtt{ababa}$ and the leftmost occurrence is underlined. The number of occurrences of u in v is denoted by $|v|_u$. For a word $u := \mathtt{a}_1\mathtt{a}_2 \ldots \mathtt{a}_n$, two factors $v := \mathtt{a}_i \ldots \mathtt{a}_j$, $w := \mathtt{a}_k \ldots \mathtt{a}_\ell$, $1 \leq i, j, k, \ell \leq n$ *partially overlap* if $k \leq i$ and $i < \ell < j$ or $i \leq k$ and $k < j < \ell$.

The result of concatenating n occurrences of a single word u is denoted by u^n, and a word v is *primitive* if $v = u^n$ implies that $n = 1$. The word u is a *primitive root* of v if $v = u^n$ for some $n \in \mathbb{N}_0$ and u is primitive. The primitive root is unique for all words except ε. A word u is *ratio-imprimitive* if there exists $k \in \mathbb{N}$ and proper prefix v of u such that $|u|_x = k|v|_x$ for all $x \in \operatorname{symb}(u)$. Otherwise u is *ratio-primitive*. Let u be a word over an alphabet $\{\mathtt{a}_1, \mathtt{a}_2, \ldots, \mathtt{a}_n\}$. Then the *parikh* vector $\mathrm{P}(u)$ is the vector $(|u|_{\mathtt{a}_1}, |u|_{\mathtt{a}_2}, \ldots, |u|_{\mathtt{a}_n})$. The *basic Parikh vector* of u is the vector $\frac{1}{k}\,\mathrm{P}(u)$ where $k := \gcd\{|u|_{\mathtt{a}_1}, |u|_{\mathtt{a}_2}, \ldots, |u|_{\mathtt{a}_n}\}$. Two words u, v *commute* if $uv = vu$ and more generally, a set of words $\{u_1, u_2, \ldots, u_n\}$ commutes if $u_i u_j = u_j u_i$ for all i, j.

2.1.3 Words in a Free Group, Contractions

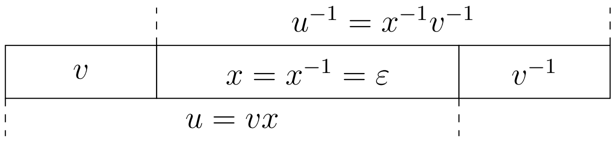
For an alphabet $X = \{\mathtt{a}_1, \mathtt{a}_2, \ldots, \mathtt{a}_n\}$, we define its *inverse* X^{-1} to be an alphabet $\{\mathtt{a}_1^{-1}, \mathtt{a}_2^{-1}, \ldots, \mathtt{a}_n^{-1}\}$ such that $X \cap X^{-1} = \emptyset$. The free monoid $(X \cup X^{-1})^*$, along with additional relations $\mathtt{a}_i\mathtt{a}_i^{-1} = \mathtt{a}_i^{-1}\mathtt{a}_i = \varepsilon$ for each $\mathtt{a}_i \in X$, forms the free group $\mathcal{F}_X$. For example, for the alphabet $\Sigma := \{\mathtt{a}, \mathtt{b}\}$, we have $\Sigma^{-1} = \{\mathtt{a}^{-1}, \mathtt{b}^{-1}\}$, and the words $u_1 := \mathtt{b}\mathtt{a}^{-1}\mathtt{b}\mathtt{b}^{-1}$, $u_2 := \mathtt{b}\mathtt{a}^{-1}$, and $u_3 := \mathtt{a}\mathtt{b}\mathtt{a}^{-1}\mathtt{b}^{-1}$ are all words belonging to both the free monoid $\{\mathtt{a}, \mathtt{b}, \mathtt{a}^{-1}, \mathtt{b}^{-1}\}^*$ and the free group $\mathcal{F}_\Sigma$, but while u_1 and u_2 are not graphically equal (and therefore not equal in the free monoid), they *are* equivalent in the free group since $\mathtt{b}\mathtt{b}^{-1} = \varepsilon$ in $\mathcal{F}_\Sigma$, so $u_1 = \mathtt{b}\mathtt{a}^{-1} \cdot \varepsilon = \mathtt{b}\mathtt{a}^{-1} = u_2$. In order to avoid confusion between the two, we will generally not refer to free monoids containing 'negative' letters $\mathtt{a}_i^{-1}$. Instead, if we refer to a free monoid, we will generally mean, e.g., Σ^*. Furthermore, when dealing with words in a free group, we will always assume that we consider the more general equality (i.e., equality in context of the free group) unless specifically stated.[2]

For an alphabet X, let $u \in \mathcal{F}_X$ be the word $u = \mathtt{a}_1^{p_1}\mathtt{a}_2^{p_2}\cdots\mathtt{a}_n^{p_n}$ where $\mathtt{a}_i \in X$ and $p_i \in \{1, -1\}$ for $1 \leq i \leq n$. The *inverse* of u in $\mathcal{F}_X$ is the word $u^{-1} = \mathtt{a}_n^{-p_n}\mathtt{a}_{n-1}^{-p_{n-1}}\cdots\mathtt{a}_1^{-p_1}$, so that $uu^{-1} = u^{-1}u = \varepsilon$, and it is unique (up to equivalence in the associated free group). Moreover u is *reduced* if $\mathtt{a}_i^{p_i}\mathtt{a}_{i+1}^{p_{i+1}} \neq \varepsilon$ for all $i, 1 \leq i < n$. Otherwise u is *unreduced*. For two reduced words $u, v \in \mathcal{F}_X$, $u = v$ if and only if u and v are graphically equal (i.e., equal in the free monoid). For every word $u \in \mathcal{F}_X$, there exists a unique reduced word v such that $u = v$. For two (potentially unreduced) words $u, v \in F_X$, $u = v$ if and only if their respective reduced words are equal.

$$\underline{\mathtt{a\,a\,b}}\,\mathtt{a\,a}^{-1}\,\underline{\mathtt{b}}\,\mathtt{b\,a}^{-1}\,\mathtt{a\,b}^{-1}\,\underline{\mathtt{b}} = \underline{\mathtt{a}}\,\mathtt{b\,b}^{-1}\,\underline{\mathtt{a\,b\,b\,b}} = \underline{\mathtt{a\,a\,b\,b\,b}}$$

[2] If we wish instead to express graphical/monoid equality, then usually we will state this explicitly, or by specifying whether a word should be taken as reduced or unreduced as defined in the next paragraph.

We can extend all the notions we have defined for words in a free monoid to words in a free group in the natural way, with a the following few notable differences. Firstly, it is important to observe that not all the properties we have defined for words so far are invariant between equivalent words in the free group. For example, the word u_1 as defined above has the suffix $\mathtt{bb}^{-1}$ while u_2 does not. It is also clear that the lengths of u_1 and u_2 are not the same, despite the fact that they represent the same element of the free group. We must therefore be careful when considering such properties (i.e., suffixes, prefixes, factors, length, symb, ratio-primitivity) of elements of the free group, that we consider the correct 'representation' of the word.[3] As such, if the representation of a word is not clear from context or otherwise specified, we shall generally assume the word is reduced. Moreover, when we refer to occurrences of a factor u in a word in the free group, we mean occurrences of both u and u^{-1}. If we wish to distinguish between the two, we call the former *positive* occurrences and the latter *negative* occurrences. It is straightforward to see that for any non-empty factor u, no positive occurrence overlaps or partially overlaps with a negative occurrence of u. In particular, it follows from the fact that the only word in the free group satisfying $x = x^{-1}$ is ε.



By $|u|_v$, we shall mean the number of positive occurrences of v in u minus the number of negative occurrences, and we shall refer to $|u|_v$ as the *balance* of v in u. Note that this also affects the way we interpret the Parikh vector and basic Parikh vector of words in the free group. We extend the notion of repetitions to cover negative exponents in the natural way, so that for a word u and $n \in \mathbb{N}$, $u^{-n} := (u^{-1})^n = \underbrace{u^{-1}u^{-1}\cdots u^{-1}}_{\text{n times}}$.

A *contraction* is a non-empty factor which is equal to ε. For example, if $u := \mathtt{aabb}^{-1}\mathtt{a}^{-1}\mathtt{abb}^{-1}\mathtt{a}^{-1}$, then *all* the contractions occurring in u are as follows: $\mathtt{bb}^{-1}$ (twice), $\mathtt{a}^{-1}\mathtt{a}$, $\mathtt{abb}^{-1}\mathtt{a}^{-1}$ (twice), $\mathtt{bb}^{-1}\mathtt{a}^{-1}\mathtt{a}$, $\mathtt{b}^{-1}\mathtt{a}^{-1}\mathtt{ab}$, $\mathtt{a}^{-1}\mathtt{abb}^{-1}$, $\mathtt{bb}^{-1}\mathtt{a}^{-1}\mathtt{abb}^{-1}$ and $\mathtt{abb}^{-1}\mathtt{a}^{-1}\mathtt{abb}^{-1}\mathtt{a}^{-1}$. We highlight a some examples below.

$$\mathtt{a}\ \mathtt{a}\ \overbrace{\mathtt{b}\ \mathtt{b}^{-1}}^{\varepsilon}\ \underbrace{\mathtt{a}^{-1}\ \mathtt{a}\ \mathtt{b}\ \mathtt{b}^{-1}}_{\varepsilon}\ \mathtt{a}^{-1}$$

Note that a word is reduced if and only if it contains no contractions. Let X be an alphabet and let $u := \mathtt{a}_1^{p_1}\mathtt{a}_2^{p_2}\cdots\mathtt{a}_n^{p_n}$ where $\mathtt{a}_i \in X$ and $p_i \in \{1, -1\}$

[3] This is generally only an issue occurring during some of our combinatorial proofs, and the correct representation will be clear from context and/or emphasized.

for $1 \leq i \leq n$. If the factor $v = \mathtt{a}_i^{p_i} \cdots \mathtt{a}_j^{p_j}$ is a contraction, and either $i = 1$, $j = n$, or $\mathtt{a}_{i-1}^{p_{i-1}}\mathtt{a}_{j+1}^{p_{j+1}} \neq \varepsilon$, then v is a *maximal* contraction. The maximal contractions of $u := \mathtt{aabb}^{-1}\mathtt{a}^{-1}\mathtt{abb}^{-1}\mathtt{a}^{-1}$ are: $\mathtt{abb}^{-1}\mathtt{a}^{-1}$ (twice), $\mathtt{bb}^{-1}\mathtt{a}^{-1}\mathtt{a}$, $\mathtt{a}^{-1}\mathtt{abb}^{-1}$, and $\mathtt{abb}^{-1}\mathtt{a}^{-1}\mathtt{abb}^{-1}\mathtt{a}^{-1}$. We show two examples below.

$$\mathtt{a}\;\overbrace{\mathtt{a}\;\mathtt{b}\;\mathtt{b}^{-1}\;\underbrace{\mathtt{a}^{-1}\;\mathtt{a}\;\mathtt{b}\;\mathtt{b}^{-1}}_{\varepsilon}\;\mathtt{a}^{-1}}^{\varepsilon}$$

A *primary* contraction is one which does not have a maximal contraction as a proper factor. For example, the contraction $\mathtt{aa}^{-1}\mathtt{bb}^{-1}$ is not a primary contraction, since e.g., it has the maximal contraction $\mathtt{bb}^{-1}$ as a proper factor, while $\mathtt{abb}^{-1}\mathtt{a}^{-1}$ *is* primary, as the only proper factor which is also a contraction is $\mathtt{bb}^{-1}$, which is not maximal. It is straightforward to see that the reduced version of a word may be obtained by removing a sequence of primary maximal contractions, although it is perhaps slightly less obvious that the choice of these contractions is not necessarily fixed. For example the word $\mathtt{aa}^{-1}\mathtt{a}$ has two maximal contractions: $\mathtt{aa}^{-1}$ and $\mathtt{a}^{-1}\mathtt{a}$, and removing either gives the (same) reduced word $\mathtt{a}$. Note that it is a straightforward observation that the reduced version of a word is unique (cf. e.g., Ang et al. [1]).

2.2 Morphisms

Let X, Y be alphabets and let $\mathcal{A}_X, \mathcal{B}_Y$ be free monoids or free groups generated by X and Y respectively. A *(homo)morphism* is a mapping $h : \mathcal{A}_X \to \mathcal{B}_Y$ such that, for all $u, v \in \mathcal{A}_X$, $h(uv) = h(u)h(v)$. Hence a morphism preserves the structure of the monoid/group, and is compatible with the associated operation (in our case, concatenation). It follows firstly from this definition that $h(\varepsilon) = \varepsilon$ (and, additionally if $\mathcal{A}_X, \mathcal{B}_Y$ are free groups, that $h(u)^{-1} = h(u^{-1})$), and secondly that a morphism is fully defined as soon as it is specified for each $x \in X$. Thus we shall usually define morphisms in this manner. For example, if $h : \{x_1, x_2\}^* \to \Sigma^*$ is the morphism such that $h(x_1) := \mathtt{aba}$ and $h(x_2) := \mathtt{ba}$, then for $u := x_1x_2x_2x_1$, we have:

$$h(u) = h(x_1x_2x_2x_1) = h(x_1)h(x_2)h(x_2)h(x_1) = \mathtt{a\,b\,a\;\;b\,a\;\;b\,a\;\;a\,b\,a}$$

$\mathcal{A}_X$ is the *domain* while $\mathcal{B}_Y$ is the *target* and similarly, X and Y are the *domain alphabet* and *target alphabet*. For $x \in \mathcal{A}_X$, we refer to $h(x)$ as the *image* of x under h (or simply image if the context is understood), and x is the *pre-image* of $h(x)$. The set $\{y \in \mathcal{B}_y \mid y = h(x), x \in \mathcal{A}_X\}$ is denoted by $h(\mathcal{A}_X)$, and is referred to as the *image* of h. The *composition* of two morphisms $g : \mathcal{A}_X \to \mathcal{B}_Y$, $h : \mathcal{B}_Y \to \mathcal{C}_Z$ is the morphism $h \circ g : \mathcal{A}_X \to \mathcal{C}_Z$ such that $h \circ g(x) = h(g(x))$ for all $x \in \mathcal{X}$. For a morphism $h : \mathcal{A}_X \to \mathcal{A}_X$ and $n \in \mathbb{N}$, we define $h^n := \underbrace{h \circ h \circ \ldots \circ h}_{\text{n times}}$.

A morphism $h : \mathcal{A}_X \to \mathcal{B}_Y$ is injective if, for any $u, v \in \mathcal{A}_X$ with $u \neq v$, $h(u) \neq h(v)$. In particular, if there exists $u \in \mathcal{A}_X$ such that $u \neq \varepsilon$ and $h(u) = \varepsilon$ then h is not injective. If, for every $y \in \mathcal{B}_Y$, there exists $x \in \mathcal{B}$ such that $h(x) = y$, then h is *surjective*. A morphism which is both injective and surjective is *bijective*. For a subset X' of X, h is *periodic* over X' if there exists $y \in \mathcal{B}_Y$ such that for every $x \in X'$, $h(x) = y^n$ for some $n \in \mathbb{N}_0$. If $X' = X$ then h is simply *periodic*. For a periodic morphism $h : \mathcal{A}_X \to \mathcal{B}_Y$, the primitive root of h is the unique[4] word $y \in \mathcal{B}_Y$ which is a primitive root of every $h(x)$, $x \in X$.

For a free monoid/group $\mathcal{A}$, the morphism $\mathrm{id}_{\mathcal{A}_X} : \mathcal{A}_X \to \mathcal{A}_X$ such that $\mathrm{id}_{\mathcal{A}_X}(x) = x$ for all $x \in X$ is the *identity morphism* on $\mathcal{A}_X$.

A morphism $h : \mathcal{A}_X \to \mathcal{B}_Y$ is *1-uniform* if, for every $x \in X$, $|h(x)| = 1$. It is a *renaming* if it is injective and 1-uniform. It is *non-erasing* if, for every $x \in X$, $|h(x)| \neq \varepsilon$, otherwise it is *erasing*.[5]

A *fixed point* of a morphism h is a word u such that $h(u) = u$, and we say that h *fixes* u. If h is not the identity morphism, then we say that u is a non-trivial fixed point (of h).

2.2.1 Automorphisms

For alphabets X, Y and free monoids (resp. groups) $\mathcal{A}_X, \mathcal{B}_Y$ generated by X and Y, a morphism $h : \mathcal{A}_X \to \mathcal{B}_Y$ is an *endomorphism* if $\mathcal{A}_X = \mathcal{B}_Y$ (i.e., if $X = Y$).[6] A bijective endomorphism is an *automorphism*. It follows from the relationship between the rank of a free group and its (free) generating sets that $h : \mathcal{A}_X \to \mathcal{A}_X$ is an automorphism if and only if $h(\mathcal{A}_X) = \mathcal{A}_X$. In particular, by definition, this must hold for h to be a surjective, and since $\{h(x) \mid x \in X\}$ is a generating set of $\mathcal{A}_X$ of size $\mathrm{rank}(\mathcal{A}_X)$, it must be a free generating set, implying that h is also injective. Similarly, h is an automorphism if and only if the set $\{h(x) \mid x \in X\}$ is a free generating set of $\mathcal{A}_X$. Consequently, the only automorphisms of a free monoid are renaming morphisms.

For every automorphism $h : \mathcal{A}_X \to \mathcal{A}_X$, there exists an inverse $h^{-1} : \mathcal{A}_X \to \mathcal{A}_X$ such that $h \circ h^{-1} = h^{-1} \circ h = \mathrm{id}_{\mathcal{A}_X}$. Consequently, the set of all automorphisms of a particular free monoid/group $\mathcal{A}_X$ forms a group under the operation of composition, which we shall refer to as $\mathrm{aut}(\mathcal{A}_X)$. If there exists $y \in \mathcal{A}_X$ such that,

[4]The only case in which the primitive root is not unique is when $h(x) = \varepsilon$ for all $x \in X$.

[5]In practice, the notion of non-erasing morphisms is less natural in the free group, since we can have "non-empty" words which are equivalent to ε. Hence we shall only make this distinction in the free monoid.

[6]Note that this does not necessarily imply that an endomorphism is surjective.

for every $x \in X$, $h(x) = yxy^{-1}$, then h is an *inner automorphism* generated by y. The inverse of h is given by $h^{-1}(x) = y^{-1}xy$. The only inner automorphism of a free monoid $\mathcal{M}$ is the identity $\mathrm{id}_{\mathcal{M}}$. An automorphism which is not an inner automorphism is an *outer automorphism.*

For a word $u \in \mathcal{A}_X$, if all morphisms fixing u are automorphisms of $\mathcal{A}_X$, then u is a *test word* of $\mathcal{A}_X$, or simply a test word. A *C-test word* is a word u such that for any two morphisms $g_1, g_2 : \mathcal{A}_X \to \mathcal{B}_Y$, $g_1(u) = g_2(u) \neq \varepsilon$ then $g_2 = g_1 \circ h$ for some inner automorphism $h : \mathcal{A}_X \to \mathcal{A}_X$.[7]

2.2.2 Ambiguity of Morphisms

Let X, Y be alphabets and let $\mathcal{A}_X, \mathcal{B}_Y$ be free monoids (resp. groups) generated by X and Y. For two morphisms $g, h : \mathcal{A}_X \to \mathcal{B}_Y$, g and h *agree* on a word $u \in \mathcal{A}_X$ if $g(u) = h(u)$. g and h are *distinct* on a subset X' of X if there exists $x \in X'$ such that $g(x) \neq h(x)$. If g and h are distinct on X then they are simply *distinct.* For a word $u \in \mathcal{A}_X$, and morphism $g : \mathcal{A}_X \to \mathcal{B}_Y$, g is *ambiguous* with respect to u if there exists a morphism $h : \mathcal{A}_X \to \mathcal{B}_Y$ such that g, h agree on u and are distinct. Otherwise g is *unambiguous* with respect to u. If there exists a morphism which is (un)ambiguous with respect to u then we say that u *possesses* an (un)ambiguous morphism. For example, if $g, h : \{x_1, x_2\}^* \to \Sigma^*$ are the morphisms given by $g(x_1) := \mathtt{abb}$, $g(x_2) := \mathtt{ababb}$, $h(x_1) := \mathtt{abbab}$ and $h(x_2) := \mathtt{abb}$, then for $u := x_1x_2x_1x_2$, we have:

$$g(u) = \underbrace{\mathtt{a\,b\,b}}_{g(x_1)}\underbrace{\mathtt{a\,b\,a\,b\,b}}_{g(x_2)}\underbrace{\mathtt{a\,b\,b}}_{g(x_1)}\underbrace{\mathtt{a\,b\,a\,b\,b}}_{g(x_2)}$$
$$\overbrace{\mathtt{a\,b\,b\,a\,b}}^{h(x_1)}\overbrace{\mathtt{a\,b\,b}}^{h(x_2)}\overbrace{\mathtt{a\,b\,b\,a\,b}}^{h(x_1)}\overbrace{\mathtt{a\,b\,b}}^{h(x_2)} = h(u)$$

The morphisms g and h are clearly distinct, so g is ambiguous with respect to u (and so is h).

For a word $u \in \mathcal{A}_X$, and morphism $g : \mathcal{A}_X \to \mathcal{B}_Y$, g is *ambiguous up to automorphism (resp. inner automorphism)* with respect to u if, for every morphism $h : \mathcal{A}_X \to \mathcal{B}_Y$ such that g and h agree on u, there exists an automorphism (resp. inner automorphism) $f : \mathcal{A}_X \to \mathcal{A}_X$ such that $h = g \circ f$.[8]

2.3 Patterns and Pattern Languages

In order to remain consistent with existing literature on ambiguity of morphisms, we shall borrow some terminology from the theory of pattern languages, which

[7] The original definition of a C-test word [38] is given slightly differently and uses notation which we do not introduce here. The fact that the two definitions are equivalent follows from Corollary 6 in Section 2.5.

[8] We introduce the idea of ambiguity up to automorphism and inner automorphism more thoroughly in Chapter 4.

allows us to distinguish between the images and pre-images of morphisms. In particular, we call a word a *pattern* if we intend to apply morphisms to it, and a *terminal word*, or simply a word, if we no longer intend to apply any morphisms. Similarly, we distinguish between the letters and alphabets of images and pre-images: letters which are no longer altered by a morphism are called *terminal symbols*, while we generally call other letters (i.e., non-terminal symbols) *variables.*

We shall generally use $\mathbb{N}$ (or subsets of $\mathbb{N}$) as our set(s) of variables, and $\Sigma := \{\mathtt{a}, \mathtt{b}\}$ as our set of terminal symbols (or *terminal alphabet*). A *pattern* is a word $\alpha \in (\mathbb{N} \cup \Sigma)^*$. We denote the set of variables occurring in α by $\mathrm{var}(\alpha)$. Hence, we have that $\mathrm{symb}(\alpha) \supseteq \mathrm{var}(\alpha)$. A morphism $\sigma : (\mathbb{N} \cup \Sigma)^* \to \Sigma^*$ is *terminal-preserving*, if $\sigma(\mathtt{a}_i) = \mathtt{a}_i$ for all terminal symbols $\mathtt{a}_i$. The *erasing pattern language* is the following set: $L_{\mathrm{E},\Sigma}(\alpha) := \{\sigma(\alpha) \mid \sigma : \mathrm{symb}(\alpha)^* \to \Sigma^*$ is a terminal-preserving morphism$\}$, while the *non-erasing pattern language* is the set: $L_{\mathrm{NE},\Sigma}(\alpha) := \{\sigma(\alpha) \mid \sigma : \mathrm{symb}(\alpha)^* \to \Sigma^*$ is a terminal-preserving, non-erasing morphism$\}$. A pattern is *terminal-free* if it does not contain any terminal symbols (i.e., $\mathrm{var}(\alpha) = \mathrm{symb}(\alpha)$). Hence a terminal-free pattern is a word in $\mathbb{N}^+$. A pattern language is *terminal-free* if it is generated by a terminal-free pattern. With the exceptions of Sections 4.4 and 6.2, we shall only consider terminal-free patterns, and hence for the majority of this thesis we shall simply refer to words in $\mathbb{N}^+$ as patterns, and we shall clearly state whenever we consider patterns which may consider terminal symbols.

We extend our definitions of patterns to the free group in the natural manner. A (general) pattern in the free group is a word in $\mathcal{F}_{\mathbb{N}\cup\Sigma}$, while a terminal-free pattern (usually just pattern) is a word in $\mathcal{F}_{\mathbb{N}}$. The free group pattern language of a pattern $\alpha \in \mathcal{F}_{\mathbb{N}\cup\Sigma}$ is the set:

$$L_\Sigma(\alpha) := \{\sigma(\alpha) \mid \sigma : \mathcal{F}_{\mathrm{symb}(\alpha)} \to \mathcal{F}_\Sigma \text{ is a terminal-preserving morphism}\}.^{9}$$

We shall often use σ and τ to denote morphisms mapping patterns to terminal symbols, while φ, ψ and ρ shall be used to denote morphisms between patterns. Moreover, where possible we shall distinguish between patterns and terminal words which are not stated explicitly by using $\alpha, \beta, \gamma, \delta, \eta, \mu, etc.$ to denote patterns and factors of patterns and $u, v, w, x, y, z, etc.$ for terminal words. For example we may define a pattern $\alpha := 1 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 3$ and a morphism $\sigma : \mathrm{var}(\alpha)^* \to \Sigma^*$ such that $\sigma(1), \sigma(2) := \mathtt{a}$ and $\sigma(3) := \mathtt{b}$. Then for $w := \sigma(\alpha)$, we have $w = \mathtt{aaabab}$.

Given patterns α, β and a set of variables $X \subset \mathrm{var}(\alpha)$, if β may be obtained from α by erasing all occurrences of variables in x, then β is a *subpattern* of

[9] We drop the subscript "E" as this is used in pattern languages in a free monoid to distinguish between erasing and non-erasing pattern languages and such a distinction does not make sense in the free group since there exist non-empty words which are equal to ε.

α. Two patterns $\alpha, \beta \in \mathbb{N}^*$ are *morphically coincident* if there exist morphisms $\varphi : \mathrm{var}(\alpha)^* \to \mathrm{var}(\beta)^*$ and $\psi : \mathrm{var}(\beta)^* \to \mathrm{var}(\alpha)^*$ such that $\varphi(\alpha) = \beta$ and $\psi(\beta) = \alpha$. We extend this to patterns in the free group $\mathcal{F}_{\mathbb{N}}$ in the obvious way. A pattern $\alpha \in \mathbb{N}^*$ is *morphically imprimitive* if it is morphically coincident to a pattern β such that $|\beta| < |\alpha|$, otherwise α is *morphically primitive*. We extend the idea of morphic (im)primitivity to a free group in a slightly less trivial way (cf. Section 5.5): a pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$ is morphically imprimitive if it is morphically coincident to a pattern $\beta \in \mathcal{F}_{\mathbb{N}}$ such that $|\mathrm{var}(\beta)| < |\mathrm{var}(\alpha)|$. It is not difficult to see that for patterns in the free monoid $\mathbb{N}^*$, these two definitions are equivalent. If $\mathrm{var}(\alpha) = \{1, 2, \ldots, n\}$, and the leftmost occurrence of each variable $x \in \mathbb{N}$ appears to the left of any variable y with $y > x$, then α is in canonical form. For example $1 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3$ is in canonical form while $2 \cdot 1 \cdot 2$ is not.

2.4 Equality-sets, the PCP, and the Dual PCP

Let X, Y be alphabets and let $\mathcal{A}_X, \mathcal{B}_Y$ be free monoids (resp. groups) generated by X and Y. For two distinct morphisms $g, h : \mathcal{A}_X \to \mathcal{B}_Y$, the set $\{u \in \mathcal{A}_X \mid g(u) = h(u)\}$ is called the *equality set* (or 'equalizer' in literature on free groups) of g and h, and is denoted $E(g, h)$. For a set of variables $\Delta \subset \mathbb{N}$, and two morphisms $\sigma, \tau : \Delta^* \to \Sigma^*$, the *Post Correspondence Problem* (PCP) is the problem of deciding whether the equality set $E(\sigma, \tau)$ is empty.

The *Dual Post Correspondence Problem* (Dual PCP) asks whether a pattern $\alpha \in \Delta^*$ belongs an equality set $E(\sigma, \tau)$, for morphisms $\sigma, \tau : \mathrm{var}(\alpha)^* \to \Sigma^*$ such that at least one of the morphisms is non-periodic. Hence a word satisfies the Dual PCP if it possesses a non-periodic ambiguous morphism. Words which do not satisfy the Dual PCP are called *periodicity forcing words*, since they force all morphisms which agree on them to be periodic.

We can extend the notion of periodicity forcing words to sets of words in the natural way. A set of patterns $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a *periodicity forcing set* if, for any two distinct morphisms $\sigma, \tau : (\mathrm{var}(\alpha_1) \cup \ldots \cup \mathrm{var}(\alpha_n))^* \to \Sigma^*$, such that σ, τ agree on all the patterns α_i, $1 \le i \le n$, both σ and τ are periodic.

2.5 Equations on Words

For a set of *unknowns* X, and alphabet $\Sigma := \{\mathtt{a}_1, \mathtt{a}_2, \ldots, \mathtt{a}_n\}$, a *word-equation* is a tuple (α, β), which we shall usually write as $\alpha = \beta$, such that $\alpha, \beta \in X^*$. Solutions to the equation are morphisms $\sigma : X^* \to \Sigma^*$ such that $\sigma(\alpha) = \sigma(\beta)$. We refer to the word $\sigma(\alpha)$ $(= \sigma(\beta))$ as the *solution-word*. For example, if $X := \{x_1, x_2, x_3\}$

is a set of unknowns, $\Sigma := \{\mathtt{a}, \mathtt{b}\}$, $\alpha := x_1 x_2$ and $\beta := x_3 x_3$, then the morphism $\sigma : X^* \to \Sigma^*$ given by $\sigma(x_1) := \mathtt{ab}$, $\sigma(x_2) := \mathtt{ababab}$ and $\sigma(x_3) := \mathtt{abab}$ is a solution to the word-equation $\alpha = \beta$:

$$\overbrace{\mathtt{a\,b}}^{\sigma(x_1)}\overbrace{\mathtt{a\,b\,a\,b\,a\,b}}^{\sigma(x_2)} = \overbrace{\mathtt{a\,b\,a\,b}}^{\sigma(x_3)}\overbrace{\mathtt{a\,b\,a\,b}}^{\sigma(x_3)}.$$

Similarly, an *equation in a free group* is a tuple (α, β), also written $\alpha = \beta$, where $\alpha, \beta \in \mathcal{F}_X$, and solutions are morphisms $\sigma : \mathcal{F}_X \to \mathcal{F}_\Sigma$. Usually we shall refer to word equations or equations in a free group simply as equations. An equation is *trivial* if α equals β (in which case all morphisms are solutions). Otherwise it is non-trivial.

Let S be a set of words which satisfy some non-trivial relation. Hence the free group $\mathcal{F}_S$ is generated by S, but *not* freely. It follows from the definitions that $|S| \geq \text{rank}(\mathcal{F}_S)$, however, recall that by Grushko's Theorem, if $|S| = \text{rank}(\mathcal{F}_S)$, then S generates $\mathcal{F}_\Sigma$ freely. Hence we must have $|S| > \text{rank}(\mathcal{F}_S)$. By the definition, this implies that there exists a generating set S' such that $|S'| = \text{rank}(\mathcal{F}_S)$. Hence the words in S may be produced from a set of $n < |S|$ words.

This conclusion is sometimes referred to as the *defect effect*, and is particularly useful when considering equations over two variables, resulting in the following folklore result.

Lemma 3. *Non-trivial equations in the free group in two unknowns have only periodic solutions.*

We also have a corresponding well-known result for word equations:

Lemma 4 (Lothaire [47])**.** *Non-trivial word-equations in two unknowns have only periodic solutions.*

In particular, since certain properties may be expressed by equations with two unknowns, we have the following corollary:

Corollary 5. *Let u, v be words (in a free group or free monoid). The following conditions are equivalent:*

1. *u and v satisfy a non-trivial equation,*

2. *u and v commute, and*

3. *u, v have the same primitive root.*[10]

[10]For this final statement to hold, we must consider all primitive words to be primitive roots of the empty word. Note that this fits with our definition given in Section 2.1.2.

Moreover, we have the following result concerning inner automorphisms of a free group. In particular, we note that for patterns α, β and the inner automorphism φ generated by β, we have $\varphi(\alpha) = \beta\alpha\beta^{-1}$ and therefore if $\sigma(\alpha) = \sigma \circ \varphi(\alpha)$, then $\sigma(\alpha), \sigma(\beta)$ satisfy a non-trivial equation in two unknowns.

Corollary 6. *Let u, v be words in a free group. Then $v = uvu^{-1}$ if and only if u, v share a primitive root. Consequently, for alphabets Δ_1, Δ_2, a pattern $\alpha \in \mathcal{F}_{\Delta_1}$, morphism $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ and inner automorphism $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$ generated by $\beta \in \mathcal{F}_{\Delta_1}$, we have $\sigma(\alpha) = \sigma \circ \varphi(\alpha)$ if and only if $\sigma(\alpha), \sigma(\beta)$ share a primitive root.*

Chapter 3

Related Literature

In the current chapter, we shall give a brief overview of related literature. We begin with literature which considers the ambiguity or unambiguity of morphisms explicitly, which so-far exists entirely within the context of a free monoid. We then consider pattern languages, for which ambiguity is a vital tool for understanding many combinatorial aspects – and indeed, where the notion of ambiguity of morphisms originates. We then consider equality sets and the Dual Post Correspondence Problem, which is essentially a question regarding ambiguity, and one which we address later in Chapter 6, before finally considering some related concepts in combinatorial group theory, mostly regarding automorphisms and test words for automorphisms – which play a key part in our considerations in Chapter 5.

3.1 Ambiguity of Morphisms in a Free Monoid

The topic of ambiguity was addressed for the first time by Freydenberger et al. [23], albeit using a subtly different terminology. In their paper, a morphic image w of a pattern α is ambiguous if there exist two distinct morphisms mapping α onto w. It is clear that this definition is equivalent to the one we use (where the morphism is called ambiguous, rather than the image), which is the preferred version in later publications, notably from Freydenberger, Reidenbach [21] onwards.

The main achievement of the initial paper by Freydenberger et al. [23] is a characterisation of whether, for a given pattern α, there exists an unambiguous injective morphism $\sigma : \text{var}(\alpha)^* \to \Sigma^*$. In particular, since $|\Sigma| = 2$ (and for a unary target alphabet the question of whether a morphism is ambiguous essentially boils down to a straightforward counting argument), the same result holds for any non-trivial target alphabet. The characterization is given in terms of fixed points as follows:

Theorem 7 (Freydenberger et al. [23])**.** *Let $\alpha \in \mathbb{N}^+$. There exists an injective* *morphism* $\sigma : \mathrm{var}(\alpha)^* \to \Sigma^*$ *which is unambiguous with respect to* α *if and only if* α *is fixed by a morphism which is not the identity on* $\mathrm{var}(\alpha)$.

Further analysis of ambiguity of morphisms – especially with respect to patterns with terminal symbols – lead to strong results on the learnability, and other properties of pattern languages (cf. Reidenbach [66], [68] and Freydenberger, Reidenbach [22]).

Although the proof of Theorem 7 is constructive – and provides a morphism σ_α^{SU} which is unambiguous with respect to α whenever such a morphism exists, it is worth noting that this morphism depends on α, and is tailor made to the pattern. In Freydenberger, Reidenbach [21], it is shown that for most patterns, the construction σ_α^{SU} is more complicated than necessary, and simpler unambiguous morphisms are given for large classes of patterns. One class of patterns for which the situation remains complicated, however, are the so-called SCRN patterns, or patterns which have the form $N^*(SC^*RN^*)^+$ where S, C, R, N are pairwise-disjoint subsets of $\mathbb{N}$. Specifically, it is noted that such patterns are the only patterns which require unambiguous morphisms to be *heterogeneous*[1].

A partition of $\mathbb{N}^*$ into morphically primitive and morphically imprimitive words is given in Reidenbach, Schneider [70]. Recall that a pattern is morphically primitive if it is not morphically coincident to a strictly shorter pattern (cf. Section 2.2). It is shown that morphic primitivity is also a characteristic condition for the existence of an injective unambiguous morphism along with some other natural concepts.

Theorem 8 (Reidenbach, Schneider [70]). *Let* $\alpha \in \mathbb{N}^+$. *The following statements are equivalent:*

(i) There exists an injective morphism which is unambiguous with respect to α.

(ii) α *is only fixed by the identity morphism.*

(iii) α *is morphically primitive.*

(iv) α *is succinct.*[2]

Furthermore, the authors give a concise combinatorial characterization of morphically primitive patterns with *imprimitivity factorizations*, as follows.

[1] A morphism is *homogeneous* if the images of the individual variables $\sigma(x)$ share a non-empty suffix and share a non-empty prefix (i.e., they all start with the same letter and end with the same letter). A morphism is *heterogeneous* if it is not homogeneous.

[2] A pattern is succinct if it is the (unique) shortest pattern describing a particular pattern language.

Definition 9. *Let $w \in \mathbb{N}^+$. An* imprimitivity *factorization (of w) is a mapping $f : \mathbb{N}^+ \to \mathbb{N}^n \times \mathbb{N}^{+n}, n \in \mathbb{N}$ such that, for $f(w) = (x, x_2, \ldots, x_n; v_1, v_2, \ldots, v_n)$, there exist $u_0, u_1, \ldots, u_n \in \mathbb{N}^*$ satisfying $w = u_0\, v_1\, u_1\, v_2\, u_2\, \ldots\, v_n\, u_n$ and*

(i) for every $i \in \{1, 2, \ldots, n\}$, $|v_i| \geq 2$,

(ii) for every $i \in \{0, 1, \ldots, n\}$ and every $j \in \{1, 2, \ldots, n\}$, $\mathrm{symb}(u_i) \cap \mathrm{symb}(v_j) = \emptyset$,

(iii) for every $i \in \{1, 2, \ldots, n\}$, $|v_i|_{x_i} = 1$ and if $x_i \in \mathrm{symb}(v_{i'})$, $i' \in \{1, 2, \ldots n\}$, then $v_i = v_{i'}$ and $x_i = x_{i'}$.

It is known from Head [30] that the existence of an imprimitivity factorization characterizes fixed points, so by Theorem 8, we have the following.

Theorem 10 (Reidenbach, Schneider [70])**.** *A pattern $\alpha \in \mathbb{N}^+$ is morphically primitive if and only if there does not exist an imprimitivity factorization of α.*

Bounds on the number of morphically primitive patterns of a given length are also discussed, leading to the interesting observation that most patterns are in fact morphically imprimitive. This is largely to do with the number of patterns for which at least one variable occurs only once. It is also worth noting that Holub [34] showed that morphic primitivity can be decided in polynomial (quadratic) time, and this has since been improved to linear time by Kociumaka et al. [45].

Schneider [77] initiated an examination of the ambiguity of non-injective, and in particular, erasing morphisms, and using a notion similar to imprimitivity factorisations, gave a characterisation for a pattern to possess an unambiguous morphism σ (injective or otherwise) in terms of the subpatterns of α. However, the result only holds in the case that the target alphabet is infinite. It is given as follows. For a set $X \subseteq \mathbb{N}$ of variables, we define $\pi_X : \mathbb{N}^* \to X^*$ to be the morphism such that $\pi_X(x) = x$ if $x \in X$ and $\pi_X(x) = \varepsilon$ otherwise, so that $\pi_X(\alpha)$ is the subpattern of α over the variables X.

Theorem 11 (Schneider [77])**.** *Let $\alpha \in \mathbb{N}^+$ and suppose Σ is an* infinite *alphabet. There exists a morphism $\sigma : \mathrm{var}(\alpha)^* \to \Sigma^*$ which is unambiguous with respect to α if and only if there exists a set $U \subseteq \mathrm{var}(\alpha)$ such that, for every subset $V \neq U$ of $\mathrm{var}(\alpha)$, we have $L_{E,\Sigma}(\pi_U(\alpha)) \not\subseteq L_{E,\Sigma}(\pi_V(\alpha))$.*

It is also worth mentioning that if a pattern possesses a non-erasing morphism which is unambiguous, then it possesses infinitely many, while on the other hand, the converse is not necessarily true as we see with the next example.

Example 12. *Let $\alpha := 1 \cdot 2 \cdot 1$. We have already seen in Chapter 1 that the morphisms mapping α to* a *and* b *are unambiguous. On the other hand, it is not difficult to see that all other morphisms (with the exception of the morphism erasing α completely, which is always unambiguous in a free monoid) are ambiguous with respect to α. Hence there are exactly two (or three if we include the erasing morphism) morphisms which are unambiguous with respect to α.*

An important observation to make from Example 12 is that the only unambiguous morphisms with respect to the pattern $\alpha := 1 \cdot 2 \cdot 1$ are non-injective, which due to the structure/information-preserving nature of unambiguity is surprising – and demonstrates the necessity to consider the (un)ambiguity non-injective and erasing morphisms.

In further research on the ambiguity of erasing morphisms by Reidenbach, Schneider [71], it is shown that, interestingly (and also rather surprisingly), the existence of (possibly erasing) unambiguous morphisms is dependent on the size of the target alphabet. More precisely, for a pattern $\alpha \in \mathbb{N}^*$, if $\mathrm{UNAMB}_\Sigma(\alpha)$ is the set of words $\mathrm{w} := \sigma(\alpha)$ such that $\sigma : \mathrm{var}(\alpha)^* \to \Sigma^*$ is a morphism which is unambiguous with respect to α, then:

Theorem 13 (Reidenbach, Schneider [71])**.** *There exist alphabets $\Sigma_k, \Sigma_{k+1}, \Sigma_{k+2}$ with $k, k+1, k+2$ letters respectively and a pattern $\alpha \in \mathbb{N}^+$ such that:*

(i) $|\,\mathrm{UNAMB}_{\Sigma_k}(\alpha)| = 0$*, and*

(ii) $|\,\mathrm{UNAMB}_{\Sigma_{k+1}}(\alpha)| \in \mathbb{N}$*, and*

(iii) $|\,\mathrm{UNAMB}_{\Sigma_{k+2}}(\alpha)|$ *is infinite.*

A characterization of whether a pattern possesses an unambiguous erasing morphism for (any) given target alphabet remains open, and is not even known to be decidable.

The unambiguity of non-erasing morphisms is explored further by Freydenberger et al. [20], which considers *weak unambiguity.* A morphism is weakly unambiguous if it is the only non-erasing morphism mapping a pattern to the corresponding image. Hence ambiguity implies weak unambiguity, but not vice-versa. The authors provide a concise characterization for patterns which possess a *length-increasing* weakly unambiguous morphism (i.e., one for which the morphic image is strictly longer than the pre-image), provided the target alphabet contains at least three letters.

Saarela [75] generalizes this result to patterns containing terminal symbols, and non-erasing morphisms which are not length-increasing (i.e., which map all variables to a single letter) are considered in Nevisi, Reidenbach [59].

The reliance of the results from Freydenberger et al. [20] and Saarela [75] on a minimum of three letters, along with Theorem 13, demonstrates the importance of target alphabet size when considering ambiguity. This motivates our choice of using a binary alphabet Σ for the majority of the thesis – since unary alphabets are often a simple case,[3] and any larger alphabets already include two symbols, and thus choosing a binary target alphabet is, usually, as general as possible.

3.2 Pattern Languages

Pattern languages were introduced in 1980 by Angluin [2], and were considered particularly from the point of view of finding a pattern best describing a set of strings – motivated largely (and unsurprisingly) by learning theory, while Shinohara [79] extended the definition to *erasing* pattern languages. For a general introduction, we recommend Mateescu, Salomaa [53].

While, for the majority of the current thesis we consider only terminal-free patterns, and hence just refer to them simply as "patterns", for the current section we shall distinguish between terminal-free patterns and (general) patterns, and shall therefore state explicitly when a pattern (language) is terminal-free. In particular, we recall that there are four main types of pattern language: erasing, non-erasing, terminal-free erasing, and terminal-free non-erasing.

In the remainder of this section we shall give an overview of some classical results on pattern languages from a language theoretic point of view. For a survey of results from an inductive inference point of view, we recommend Ng, Shinohara [60]. Firstly, it is well known that pattern languages are generally context-sensitive – although there are some which are context-free, and regular (cf. e.g., Jain et al. [40] and Reidenbach, Schmid [69]).[4] The membership problem – deciding whether a given word w belongs to a particular pattern language – is just the well-known *morphism problem* which is known to be NP-complete (cf. Ehrenfeucht, Rozenberg [16]). A thorough analysis of the morphism problem can be found, for example, in Fernau et al. [19], and Fernau, Schmid [18].

The inclusion problem is less straightforward. For terminal-free erasing pattern languages, we have the following characterization:

Theorem 14 (Jiang et al. [42])**.** *Let X be a set of variables, and let Σ be a terminal alphabet, $|\Sigma| \geq 2$. Let α, β be (terminal-free) patterns in X^*. Then $L_{E,\Sigma}(\alpha) \subseteq L_{E,\Sigma}(\beta)$ if and only if there exists a morphism $\varphi : \mathrm{var}(\beta)^* \to \mathrm{var}(\alpha)^*$ mapping β onto α.*

[3] One exception of note is our considerations in the second half of Section 5.6.

[4] For example, the language of any pattern containing each variable at most once is regular, while the language of e.g. xx is a classical context-free language.

Consequently, since the morphism problem is decidable, so is the inclusion problem for terminal-free erasing pattern languages. For terminal-free non-erasing pattern languages, the situation is still open. This distinction is due to phenomenon associated with pattern-unavoidability (cf. e.g., Jiang et al. [41]) which exist for non-erasing pattern languages but not for erasing ones. For example, it can be easily verified that every word over a two letter alphabet of length greater than 4 contains a square (i.e., a factor xx), we have that $L_{NE,\Sigma}(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \subset L_{NE,\Sigma}(1 \cdot 2 \cdot 2 \cdot 3)$ for e.g., $\Sigma := \{\mathtt{a}, \mathtt{b}\}$. Clearly there is no morphism mapping $1 \cdot 2 \cdot 2 \cdot 3$ onto $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$, so the characterization given in Theorem 14 cannot hold for the non-erasing terminal-free pattern languages. For general pattern languages, the inclusion problem is undecidable in both the erasing and non-erasing cases (cf. Jiang et al. [42], Freydenberger, Reidenbach [22]).

The equivalence problem is strongly related to the inclusion problem in the sense that the equivalence of two sets can be described as each one including the other. Consequently, from Theorem 14, we can infer the following characterization of when two terminal-free erasing pattern languages are the same.

Corollary 15 (Theorem 14)**.** *Let X be a set of variables, and let Σ be a terminal alphabet, $|\Sigma| \geq 2$. Let α, β be (terminal-free) patterns in X^*. Then $L_{E,\Sigma}(\alpha) = L_{E,\Sigma}(\beta)$ if and only if α and β are morphically coincident.*

For non-erasing pattern languages (terminal-free or otherwise) the equivalence problem is trivial: it can be observed without too much effort by simply comparing the shortest words in the language(s) that two (general, or terminal-free) pattern languages are equal if and only if their respective patterns are equal up to a renaming of the variables (cf. Angluin [2]).

This is a particularly intriguing result, as it means that for (general) non-erasing pattern languages, the inclusion problem is undecidable while the equivalence problem is decidable. The remaining case, namely the equivalence problem for (general) erasing pattern languages is still open, and conjectured to be decidable (cf. Jiang et al. [42]). A later conjecture of Ohlebusch and Ukkonen [63] was disproved by Reidenbach [67] (cf. also Freydenberger, Reidenbach [22]), essentially showing that one must take into account unintuitive combinatorial phenomena related to ambiguity of morphisms and pattern-avoidability.

Since Angluin's initial paper on the subject, it has been known that pattern languages are generally not closed under most of the usual language operations.

Theorem 16 (Angluin [2])**.** *The NE-pattern languages are not closed under union, intersection, complement, Kleene plus, morphism or inverse morphism.*

It is generally straightforward to show such properties. For example the fact that patterns are not closed under union follows easily from the existence of patterns with no variables (e.g., $\alpha_1 := \mathtt{ab}$ and $\alpha_2 := \mathtt{b}$). Since their languages are singletons, and since no pattern language contains exactly two elements, their union is clearly not again a pattern language. However, as with this example, many of the properties rely on the existence of terminal symbols, and moreover, tend to deal with special cases which do not necessarily reflect the full class of pattern languages. A more detailed analysis of closure properties of pattern languages is given in [8].

Finally, we point out that, recently, the notion of pattern languages has been extended to cover group-alphabets in a paper by Jain et al. [39].

3.3 Equality Sets and the (Dual) PCP

Equality sets were introduced by Salomaa [76] and Engelfriet, Rozenberg [17], and as we have already mentioned, they can be used to characterize the complexity classes P and NP (cf. Mateescu et al. [54]) and the recursively enumerable sets (cf. Culik II [4]). A particularly well studied application comes from the fact that the famous Post Correspondence Problem (PCP) can be expressed as the emptiness problem for equality sets.

Although the PCP is undecidable in general [64], it is known to be decidable, even in polynomial time when the morphisms have a binary domain alphabet (cf. Ehrenfeucht et al. [15] and Halava, Holub [28]). Currently, the smallest alphabet-size required known for which the problem is undecidable is 7 (cf. Matiyasevich, Sénizerguez [55]).[5] Hence for alphabet sizes 3–6, the decidabilty of the PCP is a long standing open problem.

A related problem, introduced by Culik II, Karhumäki [5] is the Dual PCP, which asks whether, for a given word α, there exists a non-trivial equality set $E(\sigma, \tau)$ such that $\alpha \in E(\sigma, \tau)$. Here, non-trivial implies that at least one of the morphisms σ, τ is non-periodic – a condition which is required in order to have a rich theory, and which also fits with motivation from the PCP since it is always decidable whether an equality set of two periodic morphisms is empty.

Hence, a word satisfies the Dual PCP (i.e., belongs to such an equality set) if it possesses a non-periodic ambiguous morphism. Words for which all non-periodic morphisms are unambiguous – which we shall study later, in Chapter 6 – are called *periodicity forcing*. The concept of periodicity forcing is also extended to sets of words in the natural way (cf. Chapter 2).

It is worth mentioning that a slightly different definition, requiring that both morphisms are non-periodic, is given in Harju, Karhumäki, [29]. Although we do

[5]There is a recent conference paper by Neary [58] with the claim that this may be improved to 5, but a full version of the paper has yet to be published.

not prove this, we expect that the definitions are equivalent, and in the remainder of the thesis, we shall use the original one given by Culik II, Karhumäki [5].

Since the PCP is also well understood for binary domain alphabets, it is perhaps not surprising that for the Dual PCP, this case is where the existing research is focused, and the situation for the Dual PCP is also reasonably well understood. Building on the original work by Culik II and Karhumäki, Holub [33], [32] characterized periodicity forcing sets containing at least two words for binary alphabets, and consequently, all ratio-imprimitive[6] periodicity forcing words.

Theorem 17 (Holub [33], [32]). *Let $\alpha, \beta \in \{1,2\}^+$ be distinct primitive patterns. Then $\{\alpha, \beta\}$ is a periodicity forcing set if and only if $\{\alpha, \beta\} = \{xy^n, y^n x\}$ where $n \in \mathbb{N}$ and $\{x, y\} = \{1, 2\}$.*

Corollary 18. *Let $\alpha \in \{1,2\}^*$ be a ratio-imprimitive pattern. Then α is periodicity forcing if and only if there exists $n \in \mathbb{N}$ such that $\alpha \in \{1 \cdot 2^n, 2^n \cdot 1\}^*$ or $\alpha \in \{1^n \cdot 2, 2 \cdot 1^n\}^*$.*

Further literature on (ratio-primitive) binary examples can be found in Hadravova, Holub [27], Czeizler et al. [6], and Karhumäki, Petra [44].

On the other hand, very little is known for larger alphabets. In Culik II, Karhumäki [5], it is shown that the problem is decidable in general. However the proof relies on Makanin's algorithm for solvability of (systems of) equations in a free semigroup [51], and although it is now known that the complexity of solving word equations is much lower than Makanin's algorithm originally indicated, the system of equations given in [5] is still sufficiently large that it is not feasible to classify examples in this way.

Finally, we note that in [12], an approach to classifying words which do/do not satisfy the Dual PCP is given via morphisms which preserve the property of being periodicity forcing. More precisely, criteria are given for a morphism φ such that if α is periodicity forcing, then so is $\varphi(\alpha)$. We explore this approach later in Section 6.1.

3.4 Automorphisms

One of the most studied and central topics in combinatorial group theory is the group of automorphisms $\mathrm{aut}(\mathcal{F}_n)$ of the free group with n generators. Early works by Nielson [62] and Whitehead [88, 87] tackled the question of whether two

[6]Recall that a word u is ratio-imprimitive if it has a proper prefix v with the same basic Parikh vector. It follows from the properties of morphisms of a free monoid that if two morphisms agree on u then they must agree on v and the corresponding suffix v'. Hence the agreement of two morphisms on u is reduced to the agreement on the set $\{v, v'\}$.

$n-$tuples of words are connected by an automorphism, and gave finite presentations for $\mathrm{aut}(\mathcal{F}_n)$, with several simplifications of these presentations coming later (cf. Rapaport [65], Higgins, Lyndon [31] and McCool [56]). In particular, an algorithm given by Whitehead has been the source of much interest since, and implies that it is decidable whether an endomorphism of a free group is an automorphism.

Theorem 19 (Whitehead [87, 88])**.** *Let $\mathcal{F}_n$ be the free group with n generators. It is decidable whether and endomorphism $\varphi : \mathcal{F}_n \to \mathcal{F}_n$ is an automorphism.*

Gersten [24] generalised the results of Whitehead by showing it was possible to compute, for two subgroups $\mathcal{F}_1, \mathcal{F}_2$ of a free group $\mathcal{F}$, whether there exists an automorphism $\varphi : \mathcal{F} \to \mathcal{F}$ such that $\varphi(\mathcal{F}_1) = \mathcal{F}_2$.

It should be noted, however, that while Whitehead's algorithm shows decidability in theory, in practice it is not necessarily feasible to use it due to complexity considerations.

The fixed subgroup of an automorphism $\varphi : \mathcal{F} \to \mathcal{F}$ is the set of all words $u \in \mathcal{F}$ such that φ fixes u. A survey on the fixed subgroups of automorphisms is given by Ventura [85]. Specifically, we note that the Scott Conjecture (cf. [14]), that the fixed subgroup of an automorphism of a free group is finitely generated, was solved (amongst others) by Gersten [25]. Using their theory of train-track maps, Bestvina, Handel [3] showed that not only is the fixed subgroup of an automorphism $\varphi : \mathcal{F} \to \mathcal{F}$ finitely generated, but that it has rank at most $\mathrm{rank}(\mathcal{F})$. Imrich, Turner [37], [36] showed that the same statement even holds for the fixed subgroup of an endomorphism, and Turner [84] showed that for endomorphisms the rank is strictly less than n.

Maslakova [52] showed that it is possible to compute a set of generators for the fixed subgroup of an automorphism φ of a free group $\mathcal{F}$, although the equivalent statement for endomorphisms remains open.

Considering a different approach to fixed points of automorphisms – asking which morphisms fix a specific (set of) word(s) rather than asking which words are fixed by a given (auto)morphism, we note a significant consequence of McCool [56]: that it is also possible to compute, for a subgroup $\mathcal{F}'$ of a free group $\mathcal{F}$, the group of automorphisms $\varphi : \mathcal{F} \to \mathcal{F}$ fixing each element of $\mathcal{F}'$. Ventura [86] gives a complementary algorithm for deciding whether a given subgroup $\mathcal{F}'$ of a free group $\mathcal{F}$ is the *fixed subgroup* of some finite family of automorphisms. The fixed subgroup of an automorphism $\varphi : \mathcal{F} \to \mathcal{F}$ is the set of all words $u \in \mathcal{F}$ such that φ fixes u. The fixed subgroup of a finite family of automorphisms is the intersection of the individual fixed subgroups.

A particularly relevant topic in this line of thinking is the subject of test words.

In [61], Nielson provides the following simple test for whether an endomorphism of $\mathcal{F}_2$ is an automorphism.

Theorem 20 (Nielson [61]). *Let $\varphi : \mathcal{F}_2 \to \mathcal{F}_2$ be a morphism. Then φ is an automorphism if and only if $\varphi(1^{-1} \cdot 2^{-1} \cdot 1 \cdot 2)$ is conjugate*[7] *to $1^{-1} \cdot 2^{-1} \cdot 1 \cdot 2$.*

As a direct consequence, since any word is conjugate with itself, we see that any morphism $\varphi : \mathcal{F}_2 \to \mathcal{F}_2$ fixing $1^{-1} \cdot 2^{-1} \cdot 1 \cdot 2$ is necessarily an automorphism. Thus, $1^{-1} \cdot 2^{-1} \cdot 1 \cdot 2$ is regarded as the first example of a test word (recall that a test word is one which is only fixed by automorphisms).

Further work on test words is given in a variety of papers (cf. e.g., [73, 80, 81, 72, 90, 89], before Turner [84] characterized test words in terms of retracts.

Theorem 21 (Turner [84]). *Let $\alpha \in \mathcal{F}_n$ for some $n \geq 2$. Then α is a test word if and only if it does not belong to any proper retract of $\mathcal{F}_n$.*

Unfortunately, this is a rather abstract characterization, and does not generally lead to the classification of specific examples, although the paper does also provide some necessary and sufficient conditions. It is also worth pointing out that it is theoretically decidable whether a word is a test word (cf. Cromerford [43]).

In [38], Ivanov introduces the notion of C-test words. A word is a C-test word is a word α such that if, for an alphabet Σ and two morphisms $\sigma, \tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$, $\sigma(\alpha) = \tau(\alpha) \neq \varepsilon$, then there exists a word $y \in \mathcal{F}_\Sigma$ such that $\sigma(x) = y \cdot \tau(x) \cdot y^{-1}$ for all $x \in \mathrm{var}(\alpha)$. It is straightforward to see that this is equivalent to asking that σ is unambiguous up to inner automorphism. It is also reasonably straightforward to see that the condition $\sigma(\alpha) \neq \varepsilon$ is necessary: since all periodic morphisms are ambiguous up to inner automorphism (cf. Proposition 26), and since there always exists a periodic morphism mapping α onto ε, if this condition is omitted then no C-test words exist. The main achievement of the paper is to provide examples of C-test words over every alphabet size.

Theorem 22 (Ivanov [38]). *There exist C-test words over any alphabet.*

A consequence of the paper is that there exist two words u, v in a free group $\mathcal{F}_n$, $n > 1$, such that any injective morphism is uniquely determined by its images under u and v. Hence any injective morphism is unambiguous with respect to the pair of words u and v, if we extend the definition of unambiguity to sets of words in the natural way.

Lee [46] improves this result by producing examples α of C-test words over any alphabet size which have the additional property that $\sigma(\alpha) = \varepsilon$ if and only if σ is periodic. Consequently, for such C-test words, which we shall refer to as C'-test

[7] two words x, y are conjugate if there exists z such that $x = zyz^{-1}$.

words, a morphism is unambiguous up to inner automorphism if and only if it is non-periodic. We shall exploit this fact later, in Section 6.2.

Finally, it is worth mentioning a classic result of Stallings [82], that for two subgroups $\mathcal{F}_1, \mathcal{F}_2$ of a free group $\mathcal{F}$, it is possible to compute a generating set for the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$. This is particularly relevant to equations in a free group since elements in the intersection satisfy a (possibly trivial) equation.

Chapter 4

Introducing Ambiguity in a Free Group

We begin the main part of this thesis by extending the existing notion of ambiguity of morphisms, which has so far belonged exclusively within the setting of free monoids, to free groups. Recall that a morphism g is ambiguous with respect to a word α if there exists another morphism h such that $g(\alpha) = h(\alpha)$, and such that $g(x) \neq h(x)$ for some $x \in \mathrm{var}(\alpha)$. Hence we ask whether there exist at least two distinct morphisms mapping one word to another.

It is, with some basic knowledge of free monoids and free groups, immediately clear that there are in fact more morphisms to consider when dealing with the latter. Of course, there are a countably infinite number of morphisms whether we consider free groups or monoids; however, given two sets of variables X and Y, and the corresponding free monoids $\mathcal{M}_X, \mathcal{M}_Y$ and free groups $\mathcal{F}_X, \mathcal{F}_Y$, for every morphism $g : \mathcal{M}_X \to \mathcal{M}_Y$, there exists a morphism $g' : \mathcal{F}_X \to \mathcal{F}_Y$ such that $g'(x) = g(x)$ for every $x \in X$. Thus, the set of morphisms between $\mathcal{M}_X$ and $\mathcal{M}_Y$ describes a strict subset of the morphisms between $\mathcal{F}_X$ and $\mathcal{F}_Y$. Put another way, the existence of inverses, and in particular, contractions, results in a vast expansion of combinatorial possibilities which in turn leads to "more" morphisms mapping to a wider range of morphic images. For example, there exists a morphism $g' : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ such that $g'(1 \cdot 2 \cdot 2 \cdot 1) = \mathtt{aabb}$ – for example, $g'(1) := \mathtt{a}$, $g'(2) := \mathtt{aba}^{-1}$ – while no such morphism $g : \{1,2\}^* \to \Sigma^*$ exists.

This has a simple but important implication when extending the idea of ambiguity. To start with, the above relationship implies that for every ambiguous morphism associated with an element of a free monoid, there is a related ambiguous morphism associated with the same element when viewed as belonging to a free group.

Example 23. *Let* $\alpha := 1 \cdot 1 \cdot 2 \cdot 2$*, and let* $g : \{1,2\}^* \to \{\mathtt{a},\mathtt{b}\}^*$ *be the morphism* *given by* $g(1) := \mathtt{aab}$ *and* $g(2) := \mathtt{a}$*. Then* g *is ambiguous with respect to* α*. This can be verified by observing that for the morphism* $h : \{1,2\}^* \to \{\mathtt{a},\mathtt{b}\}^*$ *given by* $h(1) := \mathtt{a}$ *and* $h(2) := \mathtt{baa}$*,* $h(\alpha) = g(\alpha)$*:*

$$\overbrace{\mathtt{a}\ \mathtt{a}\ \mathtt{b}}^{g(1)}\ \overbrace{\mathtt{a}\ \mathtt{a}\ \mathtt{b}}^{g(1)}\ \overbrace{\mathtt{a}}^{g(2)}\ \overbrace{\mathtt{a}}^{g(2)}$$
$$\underbrace{\mathtt{a}}_{h(1)}\ \underbrace{\mathtt{a}}_{h(1)}\ \underbrace{\mathtt{b}\ \mathtt{a}\ \mathtt{a}}_{h(2)}\ \underbrace{\mathtt{b}\ \mathtt{a}\ \mathtt{a}}_{h(2)}$$

Consequently, any morphism between $\mathcal{F}_{\{1,2\}}$ *and* $\mathcal{F}_\Sigma$ *which maps* α *to* $\mathtt{aabaabaa}$ *is ambiguous with respect to* α*, as both the morphisms* $g' : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ *and* $h' : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ *given by* $g'(x) = g(x)$ *and* $h'(x) = h(x)$ *for* $x \in \{1,\, 2\}$ *also map* α *to* $\mathtt{aabaabaa}$*:*

$$\overbrace{\mathtt{a}\ \mathtt{a}\ \mathtt{b}}^{g'(1)}\ \overbrace{\mathtt{a}\ \mathtt{a}\ \mathtt{b}}^{g'(1)}\ \overbrace{\mathtt{a}}^{g'(2)}\ \overbrace{\mathtt{a}}^{g'(2)}$$
$$\underbrace{\mathtt{a}}_{h'(1)}\ \underbrace{\mathtt{a}}_{h'(1)}\ \underbrace{\mathtt{b}\ \mathtt{a}\ \mathtt{a}}_{h'(2)}\ \underbrace{\mathtt{b}\ \mathtt{a}\ \mathtt{a}}_{h'(2)}.$$

Of course, the converse does not hold, and as we will see in the next example, if a morphism is ambiguous with respect to a given pattern in the free group, then even if a direct equivalent of that pattern and morphism exist in the free monoid, it does not necessarily need to be ambiguous in that setting.

Example 24. *Let* $\alpha = 1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2$ *and the morphism* $g : \{1,2,3\}^* \to \{\mathtt{a},\mathtt{b}\}^*$ *be given by* $g(1) := \mathtt{a}$*,* $g(2) := \mathtt{b}$*, and* $g(3) := \mathtt{ab}$*, so*

$$g(\alpha) = \overbrace{\mathtt{a}}^{g(1)}\ \overbrace{\mathtt{b}}^{g(2)}\ \overbrace{\mathtt{a}\ \mathtt{b}}^{g(3)}\ \overbrace{\mathtt{a}}^{g(1)}\ \overbrace{\mathtt{a}\ \mathtt{b}}^{g(3)}\ \overbrace{\mathtt{b}}^{g(2)}.$$

It is easily found by exhausting all possibilities that g *is unambiguous with respect to* α*. Since morphisms in free monoids generally increase length, there are only a limited number of possible alternatives, none of which produce the correct image. In the free group, however, this is not the case. Aside from the trivial extension of* g *to* g' *as demonstrated in the previous example, one such morphism mapping* α *onto* $g(\alpha)$ *is* $h' : \mathcal{F}_3 \to \mathcal{F}_\Sigma$ *given by* $h'(1) := \mathtt{ab}$ *and* $h'(2) := \mathtt{b}$*, and* $h'(3) := \mathtt{b}^{-1}\mathtt{ab}$*, so we have:*

$$\begin{aligned}
h_2'(\alpha) &= \overbrace{\mathtt{a}\ \mathtt{b}}^{h(1)}\ \overbrace{\mathtt{b}}^{h(2)}\ \overbrace{\mathtt{b}^{-1}\mathtt{a}\ \mathtt{b}}^{h(3)}\ \overbrace{\mathtt{a}\ \mathtt{b}}^{h(1)}\ \overbrace{\mathtt{b}^{-1}\mathtt{a}\ \mathtt{b}}^{h(3)}\ \overbrace{\mathtt{b}}^{h(2)} \\
&= \mathtt{a}\ \mathtt{b}\ \underbrace{\mathtt{b}\ \mathtt{b}^{-1}}_{=\varepsilon}\mathtt{a}\ \mathtt{b}\ \mathtt{a}\ \underbrace{\mathtt{b}\ \mathtt{b}^{-1}}_{=\varepsilon}\mathtt{a}\ \mathtt{b}\ \mathtt{b} \\
&= \underbrace{\mathtt{a}}_{g(1)}\ \underbrace{\mathtt{b}}_{g(2)}\ \underbrace{\mathtt{a}\ \mathtt{b}}_{g(3)}\ \underbrace{\mathtt{a}}_{g(1)}\ \underbrace{\mathtt{a}\ \mathtt{b}}_{g(3)}\ \underbrace{\mathtt{b}}_{g(2)} \\
&= g'(\alpha).
\end{aligned}$$

Thus, although the morphism g *is unambiguous with respect to* α *in the free monoid, it is ambiguous with respect to* α *in the free group.*

In the next example we see a pattern α and word w which exist both in the free group and the free monoid. However, while in the free monoid there does not exist a morphism mapping α to w, in the free group there are multiple morphisms mapping α to w.

Example 25. *Consider the pattern* $\alpha := 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1$, *and the morphism* $g : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ *such that* $g(1) := \mathtt{aba}$, $g(2) := \mathtt{a}^{-1}\mathtt{a}^{-1}$. *Then*

$$\begin{aligned} g(\alpha) &= \overbrace{\mathtt{a}\;\mathtt{b}\;\mathtt{a}}^{g(1)}\overbrace{\mathtt{a}^{-1}\mathtt{a}^{-1}}^{g(2)}\overbrace{\mathtt{a}\;\mathtt{b}\;\mathtt{a}}^{g(1)}\overbrace{\mathtt{a}^{-1}\mathtt{a}^{-1}}^{g(2)}\overbrace{\mathtt{a}\;\mathtt{b}\;\mathtt{a}}^{g(1)} \\ &= \mathtt{a}\;\mathtt{b}\;\mathtt{b}\;\mathtt{b}\;\mathtt{a}. \end{aligned}$$

We can see immediately that there does not exist a morphism $h : \{1, 2\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ *with* $h(\alpha) = \mathtt{abbba}$. *On the other hand, consider the morphism* $h' : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ *given by* $h'(1) := \mathtt{ab}^3\mathtt{a}$ *and* $h'(2) := \mathtt{a}^{-1}\mathtt{b}^{-3}\mathtt{a}^{-1}$. *Then*

$$\begin{aligned} h'(\alpha) &= \underbrace{\overbrace{\mathtt{a}\;\mathtt{b}^3\;\mathtt{a}}^{h'(1)}\overbrace{\mathtt{a}^{-1}\mathtt{b}^{-3}\mathtt{a}^{-1}}^{h'(2)}}_{\varepsilon}\underbrace{\overbrace{\mathtt{a}\;\mathtt{b}^3\;\mathtt{a}}^{h'(1)}\overbrace{\mathtt{a}^{-1}\mathtt{b}^{-3}\mathtt{a}^{-1}}^{h'(2)}}_{\varepsilon}\overbrace{\mathtt{a}\;\mathtt{b}^3\;\mathtt{a}}^{h'(1)} \\ &= \mathtt{a}\;\mathtt{b}^3\;\mathtt{a} \\ &= g(\alpha). \end{aligned}$$

Thus g is ambiguous with respect to α.

Hence, we can see that morphisms in the free group are, both generally and specifically speaking, more ambiguous. We can see that more morphisms will be ambiguous with respect to more patterns in the free group, and also that any morphism from the free monoid is at least as ambiguous with respect to a given word when (trivially) adapted to belong in free group. In fact, as we will demonstrate in the next part of this chapter, it turns out that in a free group, *every* morphism is ambiguous with respect to *every* pattern. While at first this seems to be bad news for ambiguity in free groups, closer inspection reveals that there are morphisms which are 'nearly unambiguous' – and that by refining our definition of ambiguity to account for a certain structure, we are again left with a rich theory.

4.1 All Morphisms are Ambiguous

In order to show that any given morphism is ambiguous with respect to every given pre-image in a free group, we utilize a certain structure which relies on composing the morphisms with specific inner automorphisms. While this trick, which we demonstrate in Example 28, works in the vast majority of cases, it fails in a few cases when our given morphism is periodic. In particular it fails when the primitive root of the morphism is the same as the primitive root of the pre-image. Thus we consider periodic morphisms separately in the following proposition.

Proposition 26. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ be a pattern with $|\operatorname{var}(\alpha)| > 1$, and let $\sigma : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be a periodic morphism. Then σ is ambiguous with respect to α.*

Proof. We shall construct a morphism $\tau : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ with $\tau \neq \sigma$ such that $\tau(\alpha) = \sigma(\alpha)$ as follows. Since σ is periodic, there exists a word $w \in \mathcal{F}_{\Sigma}$ such that $\sigma(z) = w^{n_z}$, $n_z \in \mathbb{Z}$, for every $z \in \operatorname{var}(\alpha)$. Let $x, y \in \operatorname{var}(\alpha)$ with $x \neq y$. Let $p := |\alpha|_x$ and $q := |\alpha|_y$. If $p = 0$, then let $\tau : \mathcal{F}_{\mathbb{N}} \to \mathcal{F}_{\Sigma}$ be the morphism given by $\tau(x) := w^{n_x+1}$ and $\tau(z) = \sigma(z)$ for all $z \neq x$. The case that $q = 0$ can be treated in the same way. Since $|\alpha|_x = 0$, every 'extra' occurrence of w in $\tau(\alpha)$ will be cancelled out by an extra occurrence of w^{-1} and vice versa. Hence, $\sigma(\alpha) = \tau(\alpha)$, and σ is ambiguous as required. Otherwise, $p \neq 0$ and $q \neq 0$. Let τ be the morphism given by $\tau(x) := w^{n_x+q}$, $\tau(y) := w^{n_y-p}$ and $\tau(z) = \sigma(z)$ otherwise. Let k be the number of occurrences of w in $\sigma(\alpha)$. Then $\tau(\alpha) = w^k w^{p \times q} w^{-q \times p} = w^k = \sigma(\alpha)$, and σ is ambiguous as required. □

Proposition 26 is not surprising, as periodic morphisms can be seen to preserve the least amount of structural information. Indeed, it can be verified with little effort that in the free monoid, nearly all periodic morphisms are ambiguous: a similar argument to the one given for Proposition 26 applies for all but the "shortest" morphisms (i.e., those mapping to the shortest images). It is noteworthy, however, that the range of images obtainable by the application of periodic morphisms is much larger in the free group than the free monoid. For instance, consider the pattern $\alpha := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$. Any word $w \in \mathcal{F}_{\Sigma}$ can be obtained as an image of α by applying the periodic morphism σ given by $\sigma(1) = w$ and $\sigma(2) = w^{-1}$. Thus, by Proposition 26, there exists an ambiguous morphism mapping α to w for all $w \in \mathcal{F}_{\Sigma}$, and consequently, since ambiguity is invariant between morphisms mapping onto the same image, all morphisms are ambiguous with respect to α. This is true even without using the inner automorphism construction we will use to prove the general case (see Example 28). By contrast, α is periodicity forcing (cf. Culik II, Karhumäki [5]) and thus has the largest possible set of unambiguous morphisms when considered in the free monoid (cf. Sections 3.3, 6.1).

In order to address non-periodic morphisms, we construct a second morphism by composing the first with a particular inner automorphism which fixes our pre-image α. The composition results in an identical morphism only when our first morphism is periodic. Since the inner automorphism fixes α, the composition and original morphism will agree on α, and are thus ambiguous. We can therefore extend Proposition 26 to cover all morphisms as claimed.

Theorem 27. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ be a pattern with $|\operatorname{var}(\alpha)| > 1$ and let $\sigma : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be a morphism. Then σ is ambiguous with respect to α.*

Proof. Assume that σ is non-periodic. The case that σ is periodic is covered by Proposition 26. Let $\varphi : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\operatorname{var}(\alpha)}$ be the inner automorphism given by $\varphi(x) = \alpha x \alpha^{-1}$ for each $x \in \operatorname{var}(\alpha)$. Then if $\alpha = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where $x_i \in \mathbb{N}$ and $p_i \in \{1, -1\}$ for $1 \leq i \leq n$, we have:

$$\begin{aligned}
\varphi(\alpha) &= \left(\alpha\, x_1\, \alpha^{-1}\right)^{p_1} \ \left(\alpha\, x_2\, \alpha^{-1}\right)^{p_2} \ \cdots \ \left(\alpha\, x_n\, \alpha^{-1}\right)^{p_n} \\
&= \alpha\, x_1^{p_1}\, \alpha^{-1}\, \alpha\, x_2^{p_2}\, \alpha^{-1} \ \cdots \ \alpha\, x_n^{p_n}\, \alpha^{-1} \\
&= \alpha\, x_1^{p_1}\, x_2^{p_2} \ \cdots \ x_n^{p_n} \alpha^{-1} \\
&= \alpha\, \alpha\, \alpha^{-1} \\
&= \alpha.
\end{aligned}$$

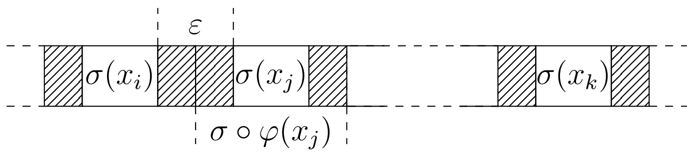
Thus $\sigma \circ \varphi(\alpha) = \sigma(\alpha)$. It remains to show that $\sigma \neq \sigma \circ \varphi$. Suppose to the contrary that $\sigma = \sigma \circ \varphi$. Then for each $x \in \operatorname{var}(\alpha)$, $\sigma(x) = \sigma(\alpha x \alpha^{-1})$. Recall from Corollary 6, that this implies $\sigma(x)$ and $\sigma(\alpha)$ share a primitive root. Thus there exists $w \in \mathcal{F}_{\Sigma}$ such that for every $x \in \operatorname{var}(\alpha)$, $\sigma(x) = w^n$ for some $n \in \mathbb{Z}$, and σ is periodic, which is a contradiction. Hence we have $\sigma \neq \sigma \circ \varphi$ and σ is ambiguous. □

We demonstrate this 'trick' of composition with a particular class of inner automorphisms with the next example. In the case of the free monoid, all inner automorphisms degenerate into the identity. As a result the two morphisms τ and σ in the construction become identical.

Example 28. *Let $\alpha := 1 \cdot 2 \cdot 2 \cdot 1$, and let $\sigma : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\Sigma}$ be the morphism given by $\sigma(1) :=$ $\mathtt{ab}$ and $\sigma(2) = \mathtt{a}$. Then $\sigma(\alpha) = \mathtt{abaaab}$. In order to obtain our second morphism, we compose σ with the inner automorphism φ given by $\varphi(1) = \alpha \cdot 1 \cdot \alpha^{-1}$ and $\varphi(2) = \alpha \cdot 2 \cdot \alpha^{-1}$. The result is a morphism $\tau = \sigma \circ \varphi$ such that $\tau(1) = \mathtt{aba^3babb^{-1}a^{-3}b^{-1}a^{-1}}$ and $\tau(2) = \mathtt{aba^3bab^{-1}a^{-3}b^{-1}a^{-1}}$. Thus*

$$\begin{aligned}
\tau(\alpha) &= \overbrace{\mathtt{a}\ \mathtt{b}\ \mathtt{a}^3\ \mathtt{b}}^{\sigma(\alpha)}\ \mathtt{a}\ \mathtt{b}\ \overbrace{\mathtt{b}^{-1}\mathtt{a}^{-3}\mathtt{b}^{-1}\mathtt{a}^{-1}}^{\sigma(\alpha)^{-1}}\overbrace{\mathtt{a}\ \mathtt{b}\ \mathtt{a}^3\ \mathtt{b}}^{\sigma(\alpha)}\ \mathtt{a}\ \overbrace{\mathtt{b}^{-1}\mathtt{a}^{-3}\mathtt{b}^{-1}\mathtt{a}^{-1}}^{\sigma(\alpha)} \\
&\quad \overbrace{\mathtt{a}\ \mathtt{b}\ \mathtt{a}^3\ \mathtt{b}}^{\sigma(\alpha)}\ \mathtt{a}\ \overbrace{\mathtt{b}^{-1}\mathtt{a}^{-3}\mathtt{b}^{-1}\mathtt{a}^{-1}}^{\sigma(\alpha)}\overbrace{\mathtt{a}\ \mathtt{b}\ \mathtt{a}^3\ \mathtt{b}}^{\sigma(\alpha)}\ \mathtt{a}\ \mathtt{b}\ \overbrace{\mathtt{b}^{-1}\mathtt{a}^{-3}\mathtt{b}^{-1}\mathtt{a}^{-1}}^{\sigma(\alpha)^{-1}} \\
&= \sigma(\alpha)\ \mathtt{a}\ \mathtt{b}\ \sigma(\alpha)^{-1}\sigma(\alpha)\ \mathtt{a}\ \sigma(1)^{-1}\ \sigma(\alpha)\ \mathtt{a}\ \sigma(\alpha)^{-1}\ \sigma(\alpha)\ \mathtt{a}\ \mathtt{b}\ \sigma(\alpha)^{-1} \\
&= \sigma(\alpha)\ \mathtt{a}\ \mathtt{b}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{b}\ \sigma(\alpha) \\
&= \sigma(\alpha)\ \sigma(\alpha)\ \sigma(\alpha)^{-1} \\
&= \sigma(\alpha).
\end{aligned}$$

We have mentioned previously that the fact that all periodic morphisms are ambiguous is intuitive, and fits with the idea that unambiguity is about structure and information preservation. However, this is not true of those morphisms which are ambiguous only because of composition with inner automorphisms. It is clear from the example that Theorem 27 does not fully reflect the combinatorial value of studying the ambiguity of morphisms. The structure is combinatorially trivial, and simply duplicates the original images so that they occur in additional positive and negative forms which then contract. Thus there is no presence of any real combinatorial "ambiguity" in the application of the morphism σ, and our second morphism τ tells us nothing about the relationship between the pre-image α and σ.



Furthermore, since inner automorphisms are so closely related to the identity morphism, the second morphism can be regarded as not only combinatorially close to the original, but algebraically so as well. A valuable study of ambiguity in free groups must therefore disregard such trivial structures.

Although more restricted versions already exist in the free monoid such as weak ambiguity (cf. Section 3.1), moderate ambiguity etc., these definitions do not smoothly generalize to the free group. More importantly, any reasonable generalizations would also not account sufficiently well for the inner automorphism construction we wish to avoid. We therefore use the next sections to introduce some types of (un)ambiguity which are natural and intuitive, but which also result in a theory that appropriately reflects the complexity of ambiguity of morphisms in a free group.

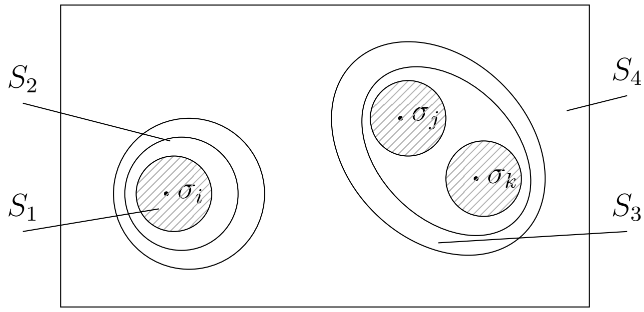


Figure 4.1: A visual representation of the relationship between morphisms σ and $\sigma \circ \varphi$ from Definition 30. S_4 is the set of all morphisms $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ for some pattern $\alpha \in \mathcal{F}_\mathbb{N}$. By $\sigma_i, \sigma_j, \sigma_k$, we denote some specific morphisms in S_4. The shaded areas (e.g., S_1) indicate the set of morphism which may be obtained through (pre-)composition with an inner automorphism (so $S_1 := \{\sigma_i \circ \varphi \mid \varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is an inner automorphism $\}$). The next boundary (e.g., S_2) indicates the set of morphisms which may be obtained through (pre-)composition with an automorphism (so $S_1 := \{\sigma_i \circ \varphi \mid \varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is an automorphism $\}$), while the final boundary indicates the set of morphisms which may be obtained through (pre-)composition with any endomorphism of $\mathcal{F}_{\mathrm{var}(\alpha)}$ (so $S_4 := \{\sigma_i \circ \varphi \mid \varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is a morphism $\}$). Hence the morphism σ_i is ambiguous up to inner automorphism with respect to α if there exists a second morphism (e.g., σ_k) *outside* the shaded area S_1 such that $\sigma_i(\alpha) = \sigma_k(\alpha)$. If no such morphism exists, then only a closely related set of morphisms may agree with σ_i on α, so σ_i is "nearly" unambiguous, or unambiguous up to inner automorphism.

4.2 Ambiguity up to Inner Automorphism

Our first type of unambiguity is also the strongest, in the sense that we disregard the minimum construction(s) necessary to produce a non-trivial (and sensible) theory.[1] More specifically, we have seen from Theorem 27, and Example 28 that it is necessary to ignore composition with (certain) inner automorphisms if we wish to have unambiguous non-periodic morphisms. With the following proposition from literature on C-test words, we are able to show that it is also sufficient to do so.

Proposition 29 (Ivanov [38]). *Let $\beta := (1^8 \cdot 2^8 \cdot 1^{-8} \cdot 2^{-8})^{100}$, and let α be the pattern:*

$$\beta \cdot 1 \cdot \beta^2 \cdot 1 \cdot \beta^3 \cdot 1^{-1} \cdot \beta^4 \cdot 1^{-1} \cdot \beta^5 \cdot 2 \cdot \beta^6 \cdot 2 \cdot \beta^7 \cdot 2^{-1} \cdot \beta^8 \cdot 2^{-1}.$$

Then α is a C-test word.

[1] Our statement here relies on the word "sensible": we can of course give an arbitrary definition of ambiguity which allows, e.g., for exactly one specific morphism to be unambiguous. Such a definition would, strictly speaking, lead to a stronger form of unambiguity, but is not a useful definition and does not fit with our existing ideas of what (un)ambiguity means.

By the definition of C-test words, a straightforward consequence of Proposition 29 is that all morphisms fixing the pattern α are equal to the the identity morphism composed with an inner automorphism. We can say that the identity morphism is *unambiguous up to inner automorphism* with respect to the pattern(s) α.

Definition 30 (Ambiguity up to Inner Automorphism)**.** *Let Δ_1, Δ_2 be alphabets and let $\alpha \in \mathcal{F}_{\Delta_1}$ be a pattern. Let $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ be a morphism. Then σ is* unambiguous up to inner automorphism *with respect to α if, for every morphism $\tau : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ with $\tau(\alpha) = \sigma(\alpha)$, there exists an inner automorphism $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$ such that $\tau = \sigma \circ \varphi$. Otherwise, σ is* ambiguous up to inner automorphism *with respect to α.*

Hence a morphism σ is unambiguous up to inner automorphism with respect to a pattern α if it is "nearly" unambiguous with respect to α, and only the most closely related morphisms may agree with σ on α (cf. Fig. 1).

Rather than specifically relating the inner automorphism φ to the individual pre-image, as we have been doing in the previous section, it is sensible to simply disregard all compositions with inner automorphisms as, for a (general) inner automorphism φ and morphism σ, the morphisms $\sigma \circ \varphi$ and σ will agree in a very restricted set of circumstances. More specifically, by Corollary 6, $\sigma \circ \varphi$ and σ agree on α if and only if φ is an inner automorphism generated by the primitive root of $\sigma(\alpha)$.

In particular, if σ is injective, then our definition is again optimal: the only inner automorphisms which may 'cause' ambiguity are the ones we must avoid (namely the one used in the proof of Theorem 27, and its powers).

Corollary 31 (Corollary 6)**.** *Let Δ_1, Δ_2 be alphabets and let $\alpha \in \mathcal{F}_{\Delta_1}$ and $\beta \in \mathcal{F}_{\Delta_2}$. Suppose that $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ is an injective morphism, and $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$ is the inner automorphism such that $\varphi(x) := \beta \cdot x \cdot \beta^{-1}$ for $x \in \Delta_1$. Then $\sigma \circ \varphi(\alpha) = \sigma(\alpha)$ if and only if α, β share a primitive root.*

Furthermore, it is worth noting that, although composition of morphisms is not always invertible, the set of inner automorphisms *is* closed under inverse. Thus the relation derived from the definition is symmetric as is required for any sensible definition of ambiguity.

Remark 32. *Let Δ_1, Δ_2 be alphabets, let $\alpha \in \mathcal{F}_{\Delta_1}$. Let $\sigma, \tau : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ be morphisms such that $\sigma(\alpha) = \tau(\alpha)$. Then σ is ambiguous up to inner automorphism with respect to α if and only if τ is.*

We now provide the following example of a morphism which is ambiguous up to inner automorphism.

Example 33. *Let $\alpha := 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. Let $\sigma, \tau : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\Sigma}$ be the morphisms given by $\sigma(1) := \mathtt{aa}$, $\sigma(2) := \mathtt{a}^{-4}\mathtt{ba}^4$, $\tau(1) := \mathtt{bb}$ and $\tau(2) := \mathtt{a}$. Then*

$$\begin{aligned}
\sigma(\alpha) &= \overbrace{\mathtt{a}\ \mathtt{a}}^{\sigma(1)}\ \overbrace{\mathtt{a}\ \mathtt{a}}^{\sigma(1)}\ \overbrace{\mathtt{a}^{-4}\,\mathtt{b}\ \mathtt{a}^4}^{\sigma(2)}\ \overbrace{\mathtt{a}^{-4}\,\mathtt{b}\ \mathtt{a}^4}^{\sigma(2)}\ \overbrace{\mathtt{a}^{-4}\,\mathtt{b}\ \mathtt{a}^4}^{\sigma(2)}\ \overbrace{\mathtt{a}^{-4}\,\mathtt{b}\ \mathtt{a}^4}^{\sigma(2)} \\
&= \underbrace{\mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}\ \mathtt{a}^{-4}}_{=\varepsilon}\,\mathtt{b}\ \underbrace{\mathtt{a}^4\,\mathtt{a}^{-4}}_{=\varepsilon}\,\mathtt{b}\ \underbrace{\mathtt{a}^4\,\mathtt{a}^{-4}}_{=\varepsilon}\,\mathtt{b}\ \underbrace{\mathtt{a}^4\,\mathtt{a}^{-4}}_{=\varepsilon}\,\mathtt{b}\ \mathtt{a}^4 \\
&= \underbrace{\mathtt{b}\ \mathtt{b}}_{\tau(1)}\ \underbrace{\mathtt{b}\ \mathtt{b}}_{\tau(1)}\ \underbrace{\mathtt{a}}_{\tau(2)}\ \underbrace{\mathtt{a}}_{\tau(2)}\ \underbrace{\mathtt{a}}_{\tau(2)}\ \underbrace{\mathtt{a}}_{\tau(2)} \\
&= \tau(\alpha).
\end{aligned}$$

In order to show that σ is ambiguous up to inner automorphism, we need to verify that $\tau \neq \sigma \circ \varphi$ for any inner automorphism $\varphi : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\{1,2\}}$. Let $\varphi : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\{1,2\}}$ be an arbitrary inner automorphism. Then there exists $x \in \mathcal{F}_{\{1,2\}}$ such that

$$\begin{aligned}
\sigma \circ \varphi(1) &= \sigma(x \cdot 1 \cdot x^{-1}) \\
&= \sigma(x)\ \sigma(1)\ \sigma(x)^{-1} \\
&= \sigma(x)\ \mathtt{a}\ \ \mathtt{a}\, \sigma(x)^{-1}.
\end{aligned}$$

However $|\tau(1)|_{\mathtt{b}} = 2$ and $|\sigma(x)\ \mathtt{a}\ \mathtt{a}\ \sigma(x)^{-1}|_{\mathtt{b}} = 0$. Thus $\tau(1) \neq \sigma \circ \varphi(1)$. Hence, there does not exist an inner automorphism φ such that $\tau = \sigma \circ \varphi$. Thus σ is ambiguous up to inner automorphism with respect to α.

We are also able to confirm that, as one might expect, all periodic morphisms remain ambiguous up to inner automorphism with respect to all patterns.

Proposition 34. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$, and let $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be a periodic morphism. Then σ is ambiguous up to inner automorphism with respect to α.*

Proof. Let $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be an inner automorphism. Then there exists $x \in \mathcal{F}_{\mathrm{var}(\alpha)}$ such that, for every $i \in \mathbb{N}$, $\varphi(i) = x \cdot i \cdot x^{-1}$. Because σ is periodic, we have that

$$\begin{aligned}
\sigma \circ \varphi(i) &= \sigma(x \cdot i \cdot x^{-1}) \\
&= \sigma(x)\ \sigma(i)\ \sigma(x^{-1}) \\
&= \sigma(x)\ \sigma(x^{-1})\ \sigma(i) \\
&= \sigma(i).
\end{aligned}$$

Hence, if σ is composed with an inner automorphism φ, the resulting morphism $\sigma \circ \varphi$ is identical to σ. We know from Proposition 26 that there is a second morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ with $\sigma(\alpha) = \tau(\alpha)$, and with $\sigma \neq \tau$. Thus τ is not the result of composing σ with an inner automorphism, and σ is ambiguous up to inner automorphism with respect to α. $\square$

Determining unambiguity however, is a far more difficult and lengthy task, so for an example which is unambiguous, we shall rely on Proposition 29 at this stage. Nevertheless, between Example 33 and Proposition 29, we know that ambiguity up to inner automorphism results in a non-trivial theory. Furthermore, we can demonstrate that it is non-trivial even when considering a single pattern.

Proposition 35. *There exists a pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$ and morphisms $\sigma_1, \sigma_2 : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that σ_1 is unambiguous up to inner automorphism with respect to α and σ_2 is ambiguous up to inner automorphism with respect to α.*

Proof. Let α be defined as in Proposition 29. Then by definition, the identity morphism is unambiguous up to inner automorphism with respect to α, and therefore it is trivial that the morphism $\sigma_1 : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ given by $\sigma_1(1) := \mathtt{a}$ and $\sigma_1(2) := \mathtt{b}$ is unambiguous up to inner automorphism with respect to α. On the other hand, by Proposition 34, the morphism $\sigma_2 : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ given by $\sigma_2(1), \sigma_2(2) := \mathtt{a}$ is ambiguous up to inner automorphism with respect to α. $\square$

Finally, we note that due to further results on C-test words, we are able to provide patterns for which all non-periodic morphisms are unambiguous up to inner automorphism.

Proposition 36. *Let Δ be a finite subset of $\mathbb{N}$. There exists a pattern α with $\mathrm{var}(\alpha) = \Delta$ such that every non-periodic morphism $\sigma : \mathcal{F}_{\Delta} \to \mathcal{F}_{\Sigma}$ is unambiguous up to inner automorphism with respect to α.*

Proof. Lee [46] shows that for any finite subset of $\mathbb{N}$, there exists a C-test word α with $\mathrm{var}(\alpha) = \Delta$, and such that for any morphism $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Sigma}$, if $\sigma(\alpha) = \varepsilon$ then σ is periodic. It follows from the definitions that every non-periodic morphism $\sigma : \mathcal{F}_{\Delta} \to \mathcal{F}_{\Sigma}$ is unambiguous up to inner automorphism with respect to α. $\square$

Such patterns form an analogy in the free group to so-called periodicity forcing words (cf. Section 3.3), both of which we shall study in further detail later, in Chapter 6.

4.3 Ambiguity up to Automorphism

In the same way that we consider ambiguity up to inner automorphism, it is possible to consider ambiguity up to automorphism, injective morphism, or indeed any other class of morphism. In the current work, we will focus on ambiguity

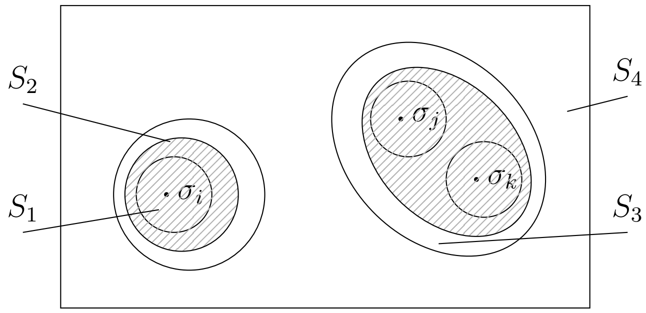


Figure 4.2: A visual representation of the relationship between morphisms σ and $\sigma \circ \varphi$ from Definition 37. For a detailed description of the figure, the reader is referred to Fig. 4.1. Here we see that instead, a morphism σ is unambiguous up to automorphism with respect to a pattern α if there exists another morphism outside the (larger) shaded area which agrees with σ on α. Hence we see that unambiguity up to automorphism is a weaker property than unambiguity up to inner automorphism.

up to inner automorphism, and up to automorphism as the first is the minimal restriction needed in order to obtain a rich theory, and thus the strongest form of unambiguity, and the later is arguably the most natural formulation in the context of a free group. We provide the complete formal definition for (un)ambiguity up to automorphism below.

Definition 37 (Ambiguity up to Automorphism)**.** *Let Δ_1, Δ_2 be alphabets and let $\alpha \in \mathcal{F}_{\Delta_1}$ be a pattern. Let $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ be a morphism. Then σ is* unambiguous up to automorphism *with respect to α if, for every morphism $\tau : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ with $\tau(\alpha) = \sigma(\alpha)$, there exists an automorphism $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$ such that $\tau = \sigma \circ \varphi$. Otherwise, σ is* ambiguous up to automorphism *with respect to α.*

As with inner automorphisms, all automorphisms are invertible, so we are guaranteed a symmetric relation when considering ambiguity up to automorphism defined in this way. It is worth noting, however, that the same is not true for e.g., injective morphisms, and so adapting the above definition for other classes of morphisms requires closer attention, or may lead to undesirable asymmetry in the relation.

Furthermore, since all inner automorphisms are automorphisms, unambiguity up to inner automorphism is a stronger property than unambiguity up to automorphism (cf. figure 4.2).

Proposition 38. *Let Δ_1, Δ_2 be alphabets. Let $\alpha \in \mathcal{F}_{\Delta_1}$ be a pattern and let $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ be a morphism. If σ is unambiguous up to inner automorphism with respect to α, then σ is unambiguous up to automorphism with respect to α.*

Proof. Since any inner automorphism is an automorphism, the statement follows directly from the definitions. □

Thus, we have from Proposition 29 an example of a morphism which is unambiguous up to automorphism. We see that the converse of Proposition 38 is not true in the next example which gives a morphism σ and pattern α such that, although σ is unambiguous up to automorphism with respect to α, it is ambiguous up to inner automorphism.

Example 39. *Let $\alpha := 1 \cdot 2 \cdot 1^{-1}2^{-1}$ and let $\sigma : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ be the morphism given by $\sigma(1) := \mathtt{a}$ and $\sigma(2) := \mathtt{b}$. Then, since α is a test word (cf. Section 3.4), it is only fixed by automorphisms. Consequently, if $\sigma \circ \varphi(\alpha) = \sigma(\alpha)$, then φ must be an automorphism on $\mathcal{F}_2$, and thus σ is unambiguous up to automorphism. Let $\tau : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(1) := \mathtt{ab}^{-1}$ and $\tau(2) := \mathtt{b}$. Then:*

$$\begin{aligned}\tau(\alpha) &= \overbrace{\mathtt{a}\ \mathtt{b}^{-1}}^{\tau(1)}\overbrace{\mathtt{b}}^{\tau(2)}\overbrace{\mathtt{b}\ \mathtt{a}^{-1}}^{\tau(1^{-1})}\overbrace{\mathtt{b}^{-1}}^{\tau(2^{-1})} \quad (\underbrace{\mathtt{b}^{-1}\ \mathtt{b}}_{\varepsilon})\\ &= \mathtt{a}\ \mathtt{b}\ \mathtt{a}^{-1}\mathtt{b}^{-1}\\ &= \sigma(\alpha).\end{aligned}$$

By the same arguments as in previous examples, it is clear that τ is not the result of composing σ with an inner automorphism. Thus σ is ambiguous up to inner automorphism.

Hence we see that our two definitions of ambiguity are not the same. It remains to observe an example of a morphism which is ambiguous up to automorphism. Of course, by the inverse of Proposition 38, any morphism which is ambiguous up to automorphism is also ambiguous up to inner automorphism.

Example 40. *Let $\alpha := 1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2$ and let $\sigma : \mathcal{F}_3 \to \mathcal{F}_\Sigma$ be the morphism given by $\sigma(1) := \mathtt{aa}$, $\sigma(2) := \mathtt{bb}$ and $\sigma(3) := \varepsilon$. Let $\tau : \mathcal{F}_3 \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(1) := \mathtt{aa}$, $\tau(2) := \mathtt{b}$, and $\tau(3) := \mathtt{b}$. Then:*

$$\sigma(\alpha) = \overbrace{\underbrace{\mathtt{a}\ \ \mathtt{a}}_{\tau(1)}}^{\sigma(1)}\overbrace{\underbrace{\mathtt{b}}_{\tau(2)}\ \underbrace{\mathtt{b}}_{\tau(3)}}^{\sigma(2)}\overbrace{\underbrace{\mathtt{a}\ \ \mathtt{a}}_{\tau(1)}}^{\sigma(1)}\overbrace{\underbrace{\mathtt{b}}_{\tau(3)}\ \underbrace{\mathtt{b}}_{\tau(2)}}^{\sigma(2)} = \tau(\alpha).$$

In order to show that τ is not the result of composing σ with some automorphism $\varphi : \mathcal{F}_3 \to \mathcal{F}_3$, we observe that

$$\sigma \circ \varphi(\mathcal{F}_3) = \sigma(\mathcal{F}_3) = \mathcal{F}_{\{\mathtt{aa},\mathtt{bb}\}}\,.$$

It is clear that $\mathtt{b} = \tau(2) \notin \mathcal{F}_{\{\mathtt{aa},\mathtt{bb}\}}$*, as any pattern in the latter will have an even number of occurrences of* $\mathtt{b}$*, while the former has exactly 1. It follows that* $\tau(2) \neq \sigma \circ \varphi(2)$*, so* $\tau \neq \sigma \circ \varphi$*, and hence that* σ *is ambiguous up to automorphism.*

Comparing the reasoning in Example 40 to that of, e.g., Example 33, we start to see that it can be noticeably harder to show ambiguity up to automorphism than to show ambiguity up to inner automorphism. Since inner automorphisms conform to a very strict form, equating (or rather, showing a non-equation between) a morphism τ and all possible compositions $\sigma \circ \varphi$ is far easier if φ is an inner automorphism. If φ may be an outer automorphism then more sophisticated reasoning is often required. We highlight the following useful remark, which describes reasoning we have introduced in Example 40 in more detail:

Remark 41. *Let* $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ *be a morphism, and let* $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$ *be an automorphism. Then* $\sigma(\mathcal{F}_{\Delta_1}) = \sigma \circ \varphi(\mathcal{F}_{\Delta_1})$*. Hence, if for morphisms* $\sigma, \tau : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$*, if* $\sigma(\mathcal{F}_{\Delta_1}) \neq \tau(\mathcal{F}_{\Delta_1})$*, then* σ *is not the result of composing* τ *with an automorphism. Moreover, it follows that if, for a pattern* α *and morphisms* $\sigma, \tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$*,* σ *is periodic while* τ *is non-periodic, then* σ *(and* τ*) are ambiguous up to automorphism with respect to* α*.*

Unlike ambiguity up to inner automorphism, as a direct result of the combinatorial richness of (outer) automorphisms, we have a surprising non-trivial situation when considering the ambiguity up to automorphism of periodic morphisms.

Proposition 42. *Let* $\alpha := 1 \cdot 1 \cdot 2 \cdot 2$*, and let* $\sigma : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\Sigma}$ *be the morphism given by* $\sigma(1) := \mathtt{a}^{-1}$ *and* $\sigma(2) := \mathtt{aa}$*. Then* σ *is unambiguous up to automorphism with respect to* α*.*

Proof. In order to prove our statement, we need the following observations. Firstly, that the set of morphisms $\tau : \mathcal{F}_2 \to \mathcal{F}_{\Sigma}$ such that $\sigma(\alpha) = \tau(\alpha)$ is given by

$$S := \{\tau \mid \tau(1), \tau(2) \in \mathcal{F}_{\{\mathtt{a}\}}, \text{ and } |\tau(1)| = -|\tau(2)| + 1\}.$$

We substantiate this claim, firstly by noting that any such morphism τ must be periodic, and furthermore have primitive root $\mathtt{a}$ (due to Lyndon, Schützenberger [49]), and secondly, observing that, given τ has primitive root $\mathtt{a}$, we must have $2|\tau(1)| + 2|\tau(2)| = 2$. The form given above is obtained by simply re-arranging this equation.

Our second claim is that, for any $k \in \mathbb{Z}$, the morphism $\varphi_k : \mathcal{F}_2 \to \mathcal{F}_2$ given by $\varphi_k(1) := 2^{-1} \cdot 1^{-k-1}$ and $\varphi_k(2) := 1^k \cdot 2$ is an automorphism. We verify this by observing that $\varphi_k(1^{-1} \cdot 2^{-1}) = 1$ and $\varphi_k((2 \cdot 1)^k \cdot 2) = 2$. Hence the image $\mathcal{F}_{\{1,2\}}$ under φ_k is exactly $\mathcal{F}_2$ and thus φ_k is an automorphism.

We can now prove our main statement as follows. Let $\tau \in S$, and let $k := -|\tau(2)| + 2$. Then we have

$$\begin{aligned} \sigma \circ \varphi_k(1) &= \sigma(2^{-1} \cdot 1^{-k-1}) & \sigma \circ \varphi_k(2) &= \sigma(1^k \cdot 2) \\ &= \mathsf{a}^{-2} \cdot \mathsf{a}^{-1(-k-1)} & &= \mathsf{a}^{-k} \cdot \mathsf{a}^2 \\ &= \mathsf{a}^{k-1} & &= \mathsf{a}^{-(-|\tau(2)|+2)+2} \\ &= \mathsf{a}^{-|\tau(2)|+1} & &= \mathsf{a}^{|\tau(2)|} \\ &= \tau(1), & &= \tau(2). \end{aligned}$$

Thus $\sigma \circ \varphi_k = \tau$. So, for any morphism τ such that $\tau(\alpha) = \sigma(\alpha)$, there exists an automorphism φ such that $\sigma \circ \varphi = \tau$ and σ is unambiguous up to automorphism with respect to α.

□

The example used in the proof of Proposition 42 is by no means an isolated case, and we will briefly discuss further examples in Chapter 5 (Section 5.6). For now, we provide a broad necessary condition showing that most periodic morphisms are in fact ambiguous up to automorphism like we might expect.

Proposition 43. *Let $n \in \mathbb{N}$, $n > 1$, and let $\sigma : \mathcal{F}_n \to \mathcal{F}_\Sigma$ be a periodic morphism with primitive root w. If there exists a pattern $\alpha \in \mathcal{F}_n$ such that σ is unambiguous up to automorphism with respect to α then $\sigma(\mathcal{F}_n) = \mathcal{F}_{\{w\}}$.*

Proof. Assume $n \geq 2$. W.l.o.g., suppose that the primitive root of σ is a. Firstly, for any given $\alpha \in \mathcal{F}_n$, we have that $\sigma(\alpha) = \tau(\alpha)$ for some periodic morphism $\tau : \mathcal{F}_n \to \mathcal{F}_\Sigma$ if and only if

$$x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = 0$$

where $x_i := |\sigma(i)| - |\tau(i)|$ and $y_i := |\alpha|_i$ for $1 \leq i \leq n$. Furthermore note that for any solution $x_1, x_2, ..., x_n$, and for any $1 \leq i < j \leq n$ there exists a solution $x_1, \ldots, x_i + p, \ldots, x_j + q, \ldots, x_n$ where $p = \frac{y_j}{\gcd(y_i, y_j)}$ and $q = \frac{-y_i}{\gcd(y_i, y_j)}$. Thus we have a morphism τ such that $|\sigma(l)| - |\tau(l)| = 0$ for $l \neq i, j$, and $|\sigma(i)| - |\tau(i)| = p$ and $|\sigma(j)| - |\tau(j)| = q$. So we have $|\tau(l)| = |\sigma(l)|$ for $l \neq i, j$, and $|\tau(i)| = \frac{-y_j}{\gcd(y_i, y_j)} + |\sigma(i)|$ and $|\tau(j)| = \frac{y_i}{\gcd(y_i, y_j)}$. Now, suppose that $\tau = \sigma \circ \varphi$ for some automorphism φ. Then

$$\tau(\mathcal{F}_n) = \sigma(\varphi(\mathcal{F}_n)) = \sigma(\mathcal{F}_n) = \{\mathsf{a}^{k \gcd\{|\sigma(x)| | 1 \leq x \leq n\}} \mid k \in \mathbb{Z}\}.$$

and

$$\tau(\mathcal{F}_n) = \{\mathsf{a}^{k \gcd\{|\tau(x)| | 1 \leq x \leq n\}} \mid k \in \mathbb{Z}\}.$$

However, p and q are co-prime, so $|\tau(i)| = -p + |\sigma(i)|$ and $|\tau(j)| = -q + |\sigma(j)|$ cannot both be multiples of $\gcd(|\sigma(i)|, |\sigma(j)|)$ unless $\gcd(|\sigma(i)|, |\sigma(j)|) = 1$. This also implies that $\gcd\{|\sigma(x) \mid 1 \leq x \leq n\} = 1$, and hence that $\sigma(\mathcal{F}_n) = \mathcal{F}_{\{\mathtt{a}\}}$. □

Turning our attention to morphisms which are, or may be, non-periodic, we address the following broad questions:

(1) Is there a morphism which is unambiguous up to automorphism with respect to every pattern?

(2) Is there a non-periodic morphism which is ambiguous up to automorphism with respect to every pattern?

We provide a negative answer to the first question with the following theorem, while the second question is answered – also in the negative – by Theorem 47. Note that for periodic morphisms, the second question has an affirmative answer due to Proposition 43.

Theorem 44. *Let Δ be a finite subset of $\mathbb{N}$. There exists a pattern α with $\mathrm{var}(\alpha) = \Delta$ such that, for every morphism $\sigma : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$, σ is ambiguous up to automorphism with respect to α.*

Proof. W. l. o. g. let $\Delta := \{1, 2, \ldots, n\}$. Let $\alpha := 1 \cdot 2 \cdot 3 \cdots n$. Let $\sigma : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ be an arbitrary morphism. Suppose first that σ is non-periodic. Let $\tau : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ be the morphism such that $\tau(1) := \sigma(\alpha)$ and $\tau(x) := \varepsilon$ for all $x \neq 1$. Then $\tau(\alpha) = \sigma(\alpha)$ and τ is periodic. By Remark 41, this is sufficient to show that σ is ambiguous up to automorphism.

Suppose instead that σ is periodic, with primitive root w and w. l. o. g. suppose that $\mathtt{b} \neq w$. Let $\tau : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ be the morphism such that $\tau(1) := \sigma(\alpha)\mathtt{b}$, $\tau(2) := \mathtt{b}^{-1}$ and $\tau(x) := \varepsilon$ for $x \notin \{1, 2\}$. Then τ is non-periodic, and by Remark 41, τ (and therefore σ) is ambiguous up to automorphism. □

Corollary 45. *Let Δ be a finite subset of $\mathbb{N}$. There is no morphism $\sigma : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ which is unambiguous up to automorphism with respect to every pattern $\alpha \in \mathcal{F}_\Delta$.*

And since ambiguity up to inner automorphism is a stronger condition than ambiguity up to automorphism, we can infer the same statement for the former.

Corollary 46. *Let Δ be a finite subset of $\mathbb{N}$. There is no morphism $\sigma : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ which is unambiguous up to inner automorphism with respect to every pattern $\alpha \in \mathcal{F}_\Delta$.*

These are important results for our endeavors in the next chapter, where we look for characterizations of those patterns for which there exists a morphism which is unambiguous up to (inner) automorphism. We do this by looking for morphisms which are unambiguous with respect to a pattern if and only if the identity morphism is. Corollaries 45 and 46 assert that no one morphism is sufficient for this. Furthermore, Theorem 44 along with Theorem 47 below and Proposition 36, demonstrate that such a characterization is non-trivial.

Theorem 47. *Let Δ be a finite subset of $\mathbb{N}$. There exists a pattern α with $\mathrm{var}(\alpha) = \Delta$ such that every non-periodic morphism $\sigma : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ is unambiguous up to automorphism with respect to α.*

Proof. Directly from Proposition 36 and Proposition 38. □

Corollary 48. *Let Δ be a finite subset of $\mathbb{N}$, and let $\sigma : \mathcal{F}_\Delta$ to $\mathcal{F}_\Sigma$ be a non-periodic morphism. There exists a pattern α with $\mathrm{var}(\alpha) = \Delta$ such that σ is unambiguous up to automorphism with respect to α.*

Finally we make the observation that while the only inner automorphism in the free monoid is the identity morphism (and thus ambiguity up to inner automorphism and standard ambiguity are equivalent in this setting), there are non-trivial outer automorphisms: namely permutations of the identity. While these are not combinatorially complex, they do have sufficient expressive power to create a disparity between standard ambiguity and ambiguity up to automorphism in the free monoid. In particular, we see that ambiguity up to automorphism is a strictly weaker notion than standard ambiguity.

Proposition 49. *There exists a pattern $\alpha \in \mathbb{N}^+$ and morphism $\sigma : \mathrm{var}(\alpha)^* \to \Sigma^*$ such that σ is ambiguous with respect to α, but is also unambiguous up to automorphism with respect to α.*

We can easily prove the proposition with e.g., the pattern $\alpha := 1 \cdot 2$ and the morphism $\sigma : \{1, 2\}^* \to \Sigma^*$ given by $\sigma(1) := \mathtt{a}$ and $\sigma(2) := \varepsilon$. Nevertheless, we choose to use a more interesting example to demonstrate the fact that this is a non-trivial consideration.

Example 50. *Let $\alpha := 1 \cdot 2 \cdot 1 \cdot 3 \cdot 3 \cdot 1 \cdot 4 \cdot 2 \cdot 4$. Let $\sigma : \{1, 2, 3, 4\}^* \to \Sigma^*$ be the morphism given by $\sigma(1) := \mathtt{a}$, $\sigma(2) := \mathtt{abba}$, $\sigma(3) := \mathtt{b}$ and $\sigma(4) := \varepsilon$. Let $\tau : \{1, 2, 3, 4\}^* \to \Sigma^*$ be the morphism given by $\tau(1) := \mathtt{a}$, $\tau(2) := \varepsilon$, $\tau(3) := \mathtt{b}$ and $\tau(4) := \mathtt{abba}$. Then*

$$\begin{aligned}\sigma(\alpha) &= \underbrace{\overbrace{\mathtt{a}}^{\sigma(1)}}_{\tau(1)} \overbrace{\underbrace{\mathtt{a}}_{\tau(1)}\underbrace{\mathtt{b}}_{\tau(3)}\underbrace{\mathtt{b}}_{\tau(3)}\underbrace{\mathtt{a}}_{\tau(1)}}^{\sigma(2)} \underbrace{\overbrace{\mathtt{a}}^{\sigma(1)}\overbrace{\mathtt{b}}^{\sigma(3)}\overbrace{\mathtt{b}}^{\sigma(3)}\overbrace{\mathtt{a}}^{\sigma(1)}}_{\tau(4)} \overbrace{\underbrace{\mathtt{a}\,\mathtt{b}\,\mathtt{b}\,\mathtt{a}}_{\tau(4)}}^{\sigma(2)} \\ &= \tau(\alpha)\end{aligned}$$

So σ is ambiguous. It can be shown by exhaustion that σ and τ are the only morphisms mapping α to $\sigma(\alpha)$. Note that σ and τ are essentially permutations of the same morphism. We have that $\tau = \sigma \circ \pi$ for the permutation $\pi : \{1, 2, 3, 4\}^* \to \{1, 2, 3, 4\}^*$ given by $\pi(1) := 1$, $\pi(2) := 4$, $\pi(3) := 3$ and $\pi(4) := 2$. Since π, like any permutation, is also an automorphism we have that σ is unambiguous up to automorphism with respect to α.

4.4 Ambiguity Within Classes of Morphisms

So far we have considered notions of ambiguity which require that, for a morphism to be ambiguous, the second morphism must be sufficiently unrelated (cf. Figs. 4.1, 4.2). Another natural question to ask is whether, given a particular subclass S of morphisms, there exist two morphisms $\sigma, \tau \in S$ which agree on a given pattern α. If there are, σ (and τ) can be said to be *S-ambiguous* with respect to α, and if for a morphism σ no such second morphism exists in S then σ can be said to be *S-unambiguous*. Moderate and weak ambiguity (cf. Section 3.1) from the free monoid are examples of such an approach. Technically speaking, the definitions considered in the two previous sections are also instances of this approach, although defining them in this context is less natural. For the question of S-ambiguity to be non-trivial (at least for non-periodic morphisms), the class S must not be closed under composition with inner automorphism.

Proposition 51. *Let Δ_1, Δ_2 be alphabets and let $S \subset \{\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}\}$. If S is closed under composition with inner automorphisms of Δ_1, then for every $\alpha \in \mathcal{F}_{\Delta_1}$, every non-periodic morphism $\sigma \in S$ is S-ambiguous with respect to α.*

Proof. If S is closed under composition with inner automorphisms, then, for every non-periodic morphism $\sigma \in S$ and any inner automorphism $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$, we have $\sigma \circ \varphi \in S$. In particular, for any given pattern $\alpha \in \mathcal{F}_\Delta$, we have that for the inner automorphism $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$ given by $\varphi(x) := \sigma(\alpha) \cdot x \cdot \sigma(\alpha)^{-1}$, the morphism $\sigma \circ \varphi \in S$. We have already seen in the proof of Theorem 27 that $\sigma \circ \varphi(\alpha) = \alpha$, and $\sigma \circ \varphi \neq \sigma$. □

In particular this rules out algebraically significant classes of morphisms, for example the class of automorphisms.

Corollary 52. *Let Δ be a finite subset of $\mathbb{N}$. Let $S = \mathrm{aut}(\mathcal{F}_\Delta)$, and let $\alpha \in \mathcal{F}_\Delta$. Then every non-periodic morphism is S-ambiguous with respect to α.*

One class of morphisms for which this paradigm works particularly well however, is substitutions, or *terminal-preserving morphisms*. For a set of variables

$X = \{x_1, x_2, \ldots\}$ and a set of terminal symbols $\Sigma = \{\mathtt{a}_1, \mathtt{a}_2, \ldots, \mathtt{a}_m\}$, we call a morphism $\sigma : \mathcal{F}_{(X\cup\Sigma)} \to \mathcal{F}_\Sigma$ *terminal-preserving* (or a substitution) if $\sigma(\mathtt{a}_i) = \mathtt{a}_i$ for every $\mathtt{a}_i \in \Sigma$ (for a more detailed introduction, see Chapters 2 and 3). Hence, the set of terminal-preserving morphisms can be seen as a subset S of $\{\sigma : \mathcal{F}_{X\cup\Sigma} \to \mathcal{F}_\Sigma\}$, and we can consider (un)ambiguity within S in the manner we have already described. Since we will always be explicit about the presence of terminal symbols, rather than saying a terminal-preserving morphism is S-ambiguous or S-unambiguous, we will simply refer to terminal-preserving morphisms as either ambiguous or unambiguous.

Example 53. *Let $\alpha := 1 \cdot \mathtt{b} \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 2 \cdot \mathtt{b} \cdot 3$, and let $\sigma, \tau : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_\Sigma$ be terminal-preserving morphisms such that $\sigma(1) := \mathtt{aab}$, $\sigma(2) := \varepsilon$ and $\sigma(3) := \mathtt{ab}$, and $\tau(1) := \mathtt{aa}$, $\tau(2) := \mathtt{b}$ and $\tau(3) := \mathtt{ab}$. Then*

$$\begin{aligned}\sigma(\alpha) &= \overbrace{\underbrace{\mathtt{a}\ \mathtt{a}}_{\tau(1)}\ \mathtt{b}}^{\sigma(1)}\ \underbrace{\mathtt{b}}_{\tau(2)}\ \overbrace{\underbrace{\mathtt{a}\ \mathtt{b}}_{\tau(3)}}^{\sigma(3)}\overbrace{\underbrace{\mathtt{a}\ \mathtt{b}}_{\tau(3)}}^{\sigma(3)}\overbrace{\underbrace{\mathtt{a}\ \mathtt{a}}_{\tau(1)}\underbrace{\mathtt{b}}_{\tau(2)}}^{\sigma(1)}\ \mathtt{b}\ \overbrace{\underbrace{\mathtt{a}\ \mathtt{b}}_{\tau(3)}}^{\sigma(3)} \\ &= \tau(\alpha)\end{aligned}$$

While terminal symbols are truly always preserved in morphic images in the free monoid: so that e.g., an occurrence $\mathtt{a}_1$ in the pre-image pattern implies a corresponding occurrence of $\mathtt{a}_1$ in the image, in the free group this is not the case. Instead, we note that terminal symbols may contract with other parts of the image, which can lead to some unintuitive instances, like the one in the following example.

Example 54. *Let $\alpha := 1 \cdot 1 \cdot \mathtt{a} \cdot \mathtt{a} \cdot 2 \cdot 2$, and let $\sigma : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_\Sigma$ be the terminal-preserving morphism such that $\sigma(1) := \mathtt{a}^{-1}$ and $\sigma(2) := \mathtt{b}$. Let $\tau : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_\Sigma$ be the terminal-preserving morphism such that $\tau(1) := \mathtt{b}$ and $\tau(2) := \mathtt{a}^{-1}$. Then*

$$\begin{aligned}\sigma(\alpha) &= \underbrace{\overbrace{\mathtt{a}^{-1}}^{\sigma(1)}\overbrace{\mathtt{a}^{-1}}^{\sigma(1)}\ \mathtt{a}\ \mathtt{a}}_{\varepsilon}\ \overbrace{\mathtt{b}}^{\sigma(2)}\overbrace{\mathtt{b}}^{\sigma(2)} \\ &= \mathtt{b}\ \mathtt{b} \\ &= \underbrace{\mathtt{b}}_{\tau(1)}\underbrace{\mathtt{b}}_{\tau(1)}\ \overbrace{\mathtt{a}\ \mathtt{a}\ \underbrace{\mathtt{a}^{-1}}_{\tau(2)}\underbrace{\mathtt{a}^{-1}}_{\tau(2)}}^{\varepsilon} = \tau(\alpha).\end{aligned}$$

Hence we see not only that σ is ambiguous with respect to α, but also that $\sigma(\alpha)$ has no occurrences of the terminal symbol $\mathtt{a}$ despite the fact that it occurs in α.

Perhaps even more unintuitive is the pattern $\alpha = 1 \cdot \mathtt{a} \cdot 1^{-1}$. Although we would refer to α as being a unary pattern, the terminal symbol $\mathtt{a}$ allows us to create an ambiguous structure using contractions, without mapping $\mathtt{a}$ to something else directly.

Example 55. *Let $\alpha := 1 \cdot \mathtt{a} \cdot 1^{-1}$, and let $\sigma_n : \mathcal{F}_{\{1\}\cup\Sigma} \to \mathcal{F}_\Sigma$ be the morphism given by $\sigma_n(1) := \mathtt{b} \cdot \mathtt{a}^n$. Then for $n, m \in \mathbb{Z}$ with $n \neq m$,*

$$\begin{aligned}
\sigma_n(\alpha) &= \overbrace{\mathtt{b}\ \ \mathtt{a}^n}^{\sigma_n(1)}\ \ \mathtt{a}\ \ \overbrace{\mathtt{a}^{-n}\,\mathtt{b}^{-1}}^{\sigma_n(1^{-1})} \quad \text{where } \underbrace{\mathtt{a}^n\ \mathtt{a}\ \mathtt{a}^{-n}}_{\mathtt{a}} \\
&= \mathtt{b}\ \ \mathtt{a}\ \ \mathtt{b}^{-1} \\
&= \underbrace{\mathtt{b}\ \ \mathtt{a}^m}_{\sigma_m(1)}\ \ \mathtt{a}\ \ \underbrace{\mathtt{a}^{-m}\,\mathtt{b}^{-1}}_{\sigma_m(1^{-1})} = \sigma_m(\alpha) \quad \text{where } \overbrace{\mathtt{a}^m\ \mathtt{a}\ \mathtt{a}^{-m}}^{\mathtt{a}}
\end{aligned}$$

Thus σ_n is ambiguous with respect to α.

Nevertheless, due to the fact that the set of terminal-preserving morphisms is not closed under inner automorphism, we are able to find instances of unambiguous morphisms. In fact, we can construct, for any pattern $\alpha \in \mathcal{F}_\mathbb{N}$ and morphism $\sigma : \mathcal{F}_\mathbb{N} \to \mathcal{F}_\Sigma$ such that σ is unambiguous up to inner automorphism with respect to α, a corresponding pattern α' (with terminal symbols) and terminal-preserving morphism $\sigma' : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_\Sigma$ such that σ is unambiguous with respect to α. We achieve this by replacing all occurrences of a variable x in α with the image $\sigma(x)$, and the result is that the inner-automorphism construction detailed in Theorem 27 (see also, Example 28), is no longer possible. For example if $\alpha := 1 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 3$, and $\sigma : \mathcal{F}_3 \to \mathcal{F}_\Sigma$ is the morphism given by $\sigma(1) := \mathtt{ab}$, $\sigma(2) := \mathtt{a}$ and $\sigma(3) := \mathtt{b}$, then we construct for example, the pattern $\alpha' := \mathtt{ab} \cdot 2 \cdot 3 \cdot 2 \cdot \mathtt{ab} \cdot 3$.

Proposition 56. *Let $\alpha \in \mathcal{F}_\mathbb{N}$, $|\operatorname{var}(\alpha)| > 1$, and $\sigma : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a non-periodic morphism. Let $x \in \operatorname{var}(\alpha)$ such that $\sigma(x)$, $\sigma(\alpha)$ do not share a primitive root. Let $\psi : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\operatorname{var}(\alpha)\cup\Sigma}$ be the morphism such that $\psi(x) := \sigma(x)$ and $\psi(y) := y$ for $y \in \operatorname{var}(\alpha)\backslash\{x\}$. Then if σ is unambiguous up to inner automorphism with respect to α, the terminal-preserving morphism σ' given by $\sigma'(y) = \sigma(y)$ for all $y \in \operatorname{var}(\alpha)\backslash\{x\}$ is unambiguous with respect to $\psi(\alpha)$.*

Proof. Suppose to the contrary that σ' is ambiguous with respect to $\psi(\alpha)$. Then there exists a terminal-preserving morphism $\tau' : \mathcal{F}_{\operatorname{var}(\alpha)\backslash\{x\}\cup\Sigma} \to \mathcal{F}_\Sigma$ such that $\tau \neq \sigma$ and $\sigma(\psi(\alpha)) = \tau(\psi(\alpha))$. Let $\tau : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_\Sigma$ be the morphism such that $\tau(y) := \tau'(y)$ for all $y \neq x$, and $\tau(x) = \sigma(x)$.

It follows that $\tau(\alpha) = \sigma(\alpha)$. Hence, if σ is unambiguous up to inner automorphism, we must have $\tau = \sigma \circ \varphi$ for some inner automorphism $\varphi : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\operatorname{var}(\alpha)}$. Moreover, since $\sigma \neq \tau$, we must have $\varphi \neq \mathrm{id}_{\mathcal{F}_{\operatorname{var}(\alpha)}}$. Let $\beta \in \mathcal{F}_{\operatorname{var}(\alpha)}$ such that

φ is generated by β (i.e., so that $\varphi(z) = \beta z \beta^{-1}$ for all $z \in \text{var}(\alpha)$). Note that $\beta \neq \varepsilon$. Additionally, we have that $\tau(\alpha) = \sigma \circ \varphi(\alpha) = \sigma(\alpha)$, so by Corollary 6, $\sigma(\alpha)$ and $\sigma(\beta)$ share a primitive root. Recall also that $\sigma(x) = \tau(x)$, so we have $\sigma(x) = \sigma(\beta)\sigma(x)\sigma(\beta)^{-1}$. By Corollary 6, this implies that $\sigma(\beta), \sigma(x)$ share a primitive root, and therefore $\sigma(x)$, $\sigma(\alpha)$ share a primitive root, which contradicts our definition of x. Hence σ' is unambiguous with respect to $\psi(\alpha)$ if σ is unambiguous up to inner automorphism with respect to α. □

We will revisit the ambiguity of terminal-preserving morphisms, alongside C-test words, and periodicity forcing words in Chapter 6, where we look for patterns with the least ambiguous images. As a consequence of Proposition 56, we know there exist morphisms which are not just unambiguous up to inner automorphism, but unambiguous in the strongest possible sense when considering patterns with terminal symbols, and thus they are a worthwhile subject when looking to restrict ambiguity as far as possible.

Chapter 5

Words with an Unambiguous Morphism

In the present chapter, we attempt to classify those words for which there is at least one unambiguous morphism. As we have seen in Chapter 3, this is a question which has been comprehensively answered for words in the free monoid, with many nice results and characterizations. Hence we will remain entirely within the setting of the free group until Chapter 6, where we consider words in both the free group and free monoid for which as many morphisms as possible are unambiguous – a topic for which much less is known in both the free group and free monoid.

Of course, we have a characterization already of words in the free group $\mathcal{F}_{\mathbb{N}}$ for which there exists an unambiguous morphism: Theorem 27 tells us that this set is trivially empty. We have also seen, however, that this broad theorem hides the true complexity and consequent reward for studying ambiguity of morphisms in a free group. Referring to our discussions in Sections 4.2, and 4.3, we will look to determine those words which possess a morphism which is unambiguous up to inner automorphism and up to automorphism respectively. We are particularly interested in injective unambiguous morphisms, and so we wish to answer the following question:

Question 1. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Does there exist an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that σ is unambiguous up to inner automorphism with respect to α?*

and the equivalent question for ambiguity up to automorphism:

Question 2. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Does there exist an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that σ is unambiguous up to automorphism with respect to α?*

For the free monoid $\mathbb{N}^*$, there exists a characterization of patterns for which there exists an unambiguous injective morphism given in terms of the ambiguity of the identity morphism: there exists an injective morphism which is unambiguous with respect to $\alpha \in \mathbb{N}^+$ if and only if α is fixed by a morphism which is not the identity – or in other words, if the identity morphism is ambiguous with respect to α.[1]

The premier objective of this chapter is to establish similar characterizations for patterns in $\mathcal{F}_{\mathbb{N}}$ which possess an an (injective) morphism which is unambiguous up to inner automorphism (resp. up to automorphism). In Section 5.4, we are able provide the following comprehensive answer to Question 1 which is a direct analogy to the case for the free monoid:

> **Theorem 101.** *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. There exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to inner automorphism with respect to α if and only if the identity morphism $\mathrm{id}_{\mathcal{F}_{\mathrm{var}(\alpha)}} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is unambiguous up to inner automorphism with respect to α.*

We also have a corresponding result for Question 2, with the same analogy. However it relies on the correctness of a conjecture regarding automorphisms.

> **Theorem 102.** *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. If Conjecture 99 is true, then there exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to automorphism with respect to α if and only if the identity morphism $\mathrm{id}_{\mathcal{F}_{\mathrm{var}(\alpha)}} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is unambiguous up to automorphism with respect to α.*

We will also address the general case, for which σ is not required to be injective – in particular for ambiguity up to automorphism. This is partly motivated by Proposition 42 in the previous chapter, which tells us there exist periodic morphisms which are unambiguous up to automorphism with respect to some patterns. Hence, we also ask the following question.

Question 3. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Does there exist a non-injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that σ is unambiguous up to automorphism with respect to α.*

The rest of this chapter is organized as follows: we begin in Section 5.1 by identifying some generic combinatorial structures which 'force' morphisms to be ambiguous, and are thus able to provide large classes of patterns for which all morphisms (resp. all injective morphisms) are ambiguous in both the cases of ambiguity up to automorphism and inner automorphism.

We begin our investigation into unambiguous morphisms in Section 5.2, where we lay the technical foundations for the proofs of our main theorems. Specifically, we construct a morphism $\sigma_{\alpha,\beta} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma_{\alpha,\beta}(\alpha)$ contains

[1] This is a paraphrasing of Theorem 8, refer to Chapter 3 for further details.

an "encoding" of α over the alphabet $\mathcal{F}_\Sigma$. This encoding forces any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ to have an associated morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\beta) = \alpha$.[2]

Then, in Section 5.3, we look to reverse the encoding process, and reconstruct τ from φ. In particular, we consider the possible morphisms τ given a specific morphism φ and establish conditions for which two different morphisms τ_1, τ_2 may be associated with the same morphism φ. It turns out that these conditions are sufficiently strong that, in the case that $\alpha = \beta$ (and hence φ becomes a morphism fixing α), if two such distinct morphisms τ_1, τ_2 exist, then α is fixed by a morphism which is not an inner automorphism and hence all morphisms are ambiguous up to inner automorphism with respect to α. We conjecture the corresponding result in the case of ambiguity up to automorphism.

We are then able to prove our main theorems addressing Questions 1 and 2 in Section 5.4, with the observation that if α is only fixed by (inner) automorphisms, then by the results of the previous section, there is only one τ mapping α to $\sigma_{\alpha,\alpha}(\alpha)$ for each morphism φ fixing α. Since we are able to show that the morphism $\tau = \sigma_{\alpha,\alpha} \circ \varphi$ will always be one such morphism, we can conclude that every morphism τ mapping α to $\sigma_{\alpha,\alpha}(\alpha)$ must be the result of composing $\sigma_{\alpha,\alpha}$ with an (inner) automorphism φ, and hence that if the identity morphism is unambiguous up to (inner) automorphism with respect to α then so is $\sigma_{\alpha,\alpha}$. It is straightforward that if the identity is ambiguous up to (inner) automorphism that all injective morphisms are ambiguous up to (inner) automorphism, and hence we obtain the characterizations as already described.

In Section 5.5, we consider the idea of morphic primitivity in a free group, establishing an appropriate generalization for the existing notion in a free monoid, and relating this both to the existing topic of test words, and to our theorems from Section 5.4.

We then consider the general case, addressing Question 3, in Section 5.6, where we provide classes of patterns for which there exist unambiguous non-injective morphisms. In particular, we focus on patterns for which no injective morphism is unambiguous, and for the binary case, we provide some insights regarding the ambiguity of periodic morphisms.

Finally, in Section 5.7 we are able to take advantage of our earlier construction $\sigma_{\alpha,\alpha}$ to provide some simple proofs of properties of terminal-free group pattern languages. Specifically, we are able to show that one pattern language includes another if and only if there exists a morphism from the latter to the former, and secondly that the union of two pattern languages is only ever again a pattern language in a trivial way.

[2]Our construction generalizes an existing one for the free monoid by Jiang et al. [42].

5.1 Some Ambiguous Structures

Before we address our main questions on the existence of unambiguous morphisms directly, we will first consider some structures which force morphisms to be ambiguous. Hence we get some broad, combinatorial, necessary conditions (given in a negated form) which must be satisfied for a pattern to possess a morphism (resp. injective morphism) which is unambiguous up to inner automorphism, or simply up to automorphism.

We begin with the following straightforward observation.

Proposition 57. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. If α is fixed by a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism then all injective morphisms are ambiguous up to automorphism with respect to α. Likewise if α is fixed by a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an inner automorphism, then all injective morphisms are ambiguous up to inner automorphism with respect to α.*

Proof. We prove the statement for ambiguity up to automorphism. The proof for ambiguity up to inner automorphism is a straightforward adaptation. Suppose there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism, and such that $\varphi(\alpha) = \alpha$. Let $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be an injective morphism. Note that $\sigma \circ \varphi(\alpha) = \sigma(\alpha)$. We shall now show that σ is ambiguous up to automorphism with respect to α by showing that, for $\tau := \sigma \circ \varphi$, there does not exist an automorphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\tau \circ \psi = \sigma$. In particular, suppose to the contrary that $\tau = \sigma \circ \varphi = \sigma \circ \psi$ for some automorphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$. Then due to the injectivity of σ, $\sigma(\psi(x)) = \sigma(\varphi(x))$ for all $x \in \mathrm{var}(\alpha)$ implies that $\psi(x) = \varphi(x)$ for all $x \in \mathrm{var}(\alpha)$, and hence that $\psi = \varphi$. However, since ψ is an automorphism and φ is not, this is a contradiction. Hence there is no automorphism ψ such that $\tau(= \sigma \circ \varphi) = \sigma \circ \psi$ and σ is ambiguous up to automorphism with respect to α. $\square$

We now define three classes IMP, $SCRN_{\mathcal{F}}$ and PER of patterns which, as we will soon establish, have ambiguity-inducing structures. The first, IMP is based on the existing notion of imprimitivity factorizations for patterns of the free monoid (cf. Reidenbach, Schneider [70]). The main feature of these imprimitivity factorizations is a consistent local neighborhood δ_x for certain variables $x \in \mathrm{var}(\alpha)$. More precisely, for each variable x, we have factors γ_x, γ'_x such that $x \notin \mathrm{var}(\gamma_x) \cup \mathrm{var}(\gamma'_x)$, and such that every occurrence of each x has γ_x to the left and γ'_x to the right. In the free group, as in the free monoid, this consistency results in certain variables being redudant in the structure of α – every possible morphic image can be reached without them (i.e., if they are erased) – and it is exactly this redundancy which induces ambiguity of morphisms.

Definition 58 (IMP). *We define the set $IMP \subset \mathcal{F}_{\mathbb{N}}$ as follows. Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Suppose that there exists a partition $\Delta_1, \Delta_2, \Delta_3$ of $\mathrm{var}(\alpha)$, $x_1, x_2, \ldots x_n \in \Delta_1$, $n \geq 1$, and $\beta_0, \beta_1, \ldots, \beta_n \in \mathcal{F}_{\Delta_3}$ such that:*

$$\alpha = \beta_0 \cdot \delta_{x_1}^{\pm 1} \cdot \beta_1 \cdot \delta_{x_2}^{\pm 1} \cdot \beta_2 \cdot \ldots \cdot \beta_{n-1} \cdot \delta_{x_n}^{\pm 1} \cdot \beta_n$$

where for each $x \in \Delta_1$, there exist $\gamma_x, \gamma'_x \in \mathcal{F}_{\Delta_2}$ with $\gamma_x \neq \varepsilon$ or $\gamma'_x \neq \varepsilon$ such that $\delta_x = \gamma_x \cdot x \cdot \gamma'_x$. Then $\alpha \in IMP$.

Because imprimitivity factorizations are used to define the morphically imprimitive patterns in the free monoid, many patterns belonging to IMP will share this simple and easy-to-identify structure.[3] Indeed, if a pattern is morphically imprimitive in the free monoid, that same pattern when viewed as belonging to the free group $\mathcal{F}_{\mathbb{N}}$ will belong to the class IMP.

Example 59. *The patterns $1 \cdot 2 \cdot 1$, $1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 2 \cdot 2$ and $1 \cdot 2 \cdot 3 \cdot 3 \cdot 2^{-1} \cdot 1^{-1}$ are all in IMP, while the patterns $1 \cdot 2 \cdot 2 \cdot 1$ and $1^3 \cdot 2^6 \cdot 3^9$ are not.*

However, due to the fact that the 'neighborhoods' γ_x, γ'_x may be involved in contractions, we get some examples of patterns in IMP which are less-obviously related to morphically imprimitive patterns in the free monoid.

Example 60. *Let $\alpha := 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 1^{-1} \cdot 4 \cdot 4$ and let $\Delta_1 := \{2, 3\}$, $\Delta_2 := \{1\}$, $\Delta_3 := \{4\}$. Let $\delta_2 := 1 \cdot 2 \cdot 1^{-1}$ and $\delta_3 := 1 \cdot 3 \cdot 1^{-1}$ (so that, $\gamma_2 = 1$, $\gamma'_2 = 1^{-1}$, $\gamma_4 = 1$ and $\gamma'_4 = 1^{-1}$). Then for $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \varepsilon$ and $\beta_5 = 4 \cdot 4$, we have:*

$$\overbrace{\varepsilon}^{\beta_1} \overbrace{\cdot \underbrace{1}_{\gamma_2} \cdot 2 \cdot \underbrace{1^{-1}}_{\gamma'_2} \cdot}^{\delta_2} \overbrace{\varepsilon}^{\beta_2} \overbrace{\cdot \underbrace{1}_{\gamma_2} \cdot 2 \cdot \underbrace{1^{-1}}_{\gamma'_2} \cdot}^{\delta_2} \overbrace{\varepsilon}^{\beta_3} \overbrace{\cdot \underbrace{1}_{\gamma_3} \cdot 3 \cdot \underbrace{1^{-1}}_{\gamma'_3} \cdot}^{\delta_3} \overbrace{\varepsilon}^{\beta_4} \overbrace{\cdot \underbrace{1}_{\gamma_3} \cdot 3 \cdot \underbrace{1^{-1}}_{\gamma'_3} \cdot}^{\delta_3} \overbrace{4 \cdot 4}^{\beta_5}$$

$$= 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 1^{-1} \cdot 4 \cdot 4 = \alpha.$$

Our second structure is also a generalization of an existing class of patterns in the free monoid, and is perhaps a particularly interesting one. So-called $SCRN$ patterns were introduced by Freydenberger, Reidenbach [21], and in the free monoid, permit an unambiguous injective morphism. However, they are the only patterns for which all unambiguous morphisms must be heterogeneous.[4] Consequently, they constitute a special case, for which it is arguably the most challenging to show the existence of an unambiguous morphism. We will see that in

[3] Recall from Chapter 3 that Holub [34] demonstrated that morphic primitivity can be decided in polynomial time, and this result has been strengthened to linear time by Kociumaka et al. [45].

[4] A morphism is *homogeneous* if the images of the individual variables $\sigma(x)$ share a non-empty suffix and share a non-empty prefix (i.e., they all start with the same letter and end with the same letter). A morphism is *heterogeneous* if it is not homogeneous.

the free group, however, this is no longer true, and that the $SCRN$ structure is one which, generally speaking, 'causes' ambiguity. This is not only an interesting difference between the free group and free monoid, but is also good news for our technical constructions in the following chapters, which are made considerably simpler by the fact that they do not need to be heterogeneous. We define our slightly more general set of patterns $SCRN_{\mathcal{F}}$, below.

Definition 61 ($SCRN_{\mathcal{F}}$). *We define the set $SCRN_{\mathcal{F}} \subset \mathcal{F}_{\mathbb{N}}$ as follows. Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Suppose there exists a partition S_0, C_0, R_0, N_0 of $\mathrm{var}(\alpha)$ such that $S_0 \cup R_0 \neq \emptyset$ and*

$$\alpha \in N^*(SCRN^*)^+$$

where $N = N_0 \cup N_0^{-1}$, $C = C_0 \cup C_0^{-1}$, $S = S_0 \cup R_0^{-1}$ and $R = R_0 \cup S_0^{-1}$. Then $\alpha \in SCRN_{\mathcal{F}}$.

Example 62. *Let $\alpha_1 := 1 \cdot 2 \cdot 2 \cdot 3 \cdot 4 \cdot 4 \cdot 1 \cdot 2 \cdot 3$, let $\alpha_2 := 1 \cdot 2 \cdot 2 \cdot 2 \cdot 1^{-1}$ and let $\alpha_3 := 1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2$. Then $\alpha_1, \alpha_2 \in SCRN_{\mathcal{F}}$, and $\alpha_3 \notin SCRN_{\mathcal{F}}$. For α_1, we have $N = \{4, 4^{-1}\}$, $S = \{1, 3^{-1}\}$, $C = \{2, 2^{-1}\}$ and $R = \{3, 1^{-1}\}$. For α_2, we have $N = \emptyset$, $C = \{2, 2^{-1}\}$, $S = \{1\}$ and $R = \{1^{-1}\}$. For α_3, there is no such partition of $\mathrm{var}(\alpha)$.*

Our third class is motivated by the fact that periodic morphisms, preserving the least structural information, are often the most ambiguous (although as we see in Section 5.6, this is not always the case). In particular, we consider a set PER such that, for any pattern $\alpha \in PER$ and for every $w \in \mathcal{F}_{\Sigma}$, there exists a periodic morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that $\sigma(\alpha) = w$. It is straightforward that in the case that there also exists a non-periodic morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that $\tau(\alpha) = w$, σ is ambiguous up to automorphism, and hence 'most' morphisms (in particular, all non-periodic morphisms) are ambiguous with respect to α. We give the definition as follows.

Definition 63 (PER). *Let PER be the set of patterns $\alpha \in \mathcal{F}_{\mathbb{N}}$ such that $|\mathrm{var}(\alpha)| > 1$ and $\gcd\{|\alpha|_{x_1}, |\alpha|_{x_2}, \dots, |\alpha|_{x_n}\} = 1$ where $\{x_1, x_2, \dots, x_n\} = \mathrm{var}(\alpha)$.*

Example 64. *The patterns $1 \cdot 2 \cdot 1 \cdot 2 \cdot 1$, $1 \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 2 \cdot 1$ and $1^4 \cdot 2^8 \cdot 3^{16} \cdot 4^{17}$ are all in PER while $1 \cdot 1 \cdot 2^4$, $1 \cdot 2 \cdot 1 \cdot 2^{-1}$ and $1^6 \cdot 2^8$ are not.*

As claimed, every pattern in PER can reach every possible morphic image with a periodic morphism, meaning we can expect that 'most' morphisms are ambiguous with respect to patterns in PER.

Proposition 65. *Let $\alpha \in PER$, and let $w \in \mathcal{F}_{\Sigma}$. Then there exists a periodic morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that $\sigma(\alpha) = w$.*

Proof. Let $\alpha \in \mathcal{F}_{\mathbb{N}}, w \in \mathcal{F}_{\Sigma}$, and let $\text{var}(\alpha) = \{x_1, x_2, \ldots, x_n\}$. Let $p_i := |\alpha|_{x_i}$ for $1 \leq i \leq n$. It is well known that if $\gcd\{p_1, p_2, \ldots, p_n\} = 1$, then there exist $q_1, q_2, \ldots q_n \in \mathbb{Z}$ such that

$$p_1q_1 + p_2q_2 + \ldots + p_nq_n = 1.$$

Let $\sigma : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be the morphism given by $\sigma(x_i) = w^{q_i}$. It follows from properties of morphisms that

$$\sigma(\alpha) = w^{p_1q_1+p_2q_2+\ldots+p_nq_n} = w.$$

□

We now consider the ambiguity of morphisms with respect to patterns in the classes $SCRN_{\mathcal{F}}, IMP$ and PER. Firstly, we see that for any pattern in each class, all injective morphisms are ambiguous up to automorphism (and therefore, by Proposition 38, also ambiguous up to inner automorphism).

Theorem 66. *Let $\alpha \in IMP \cup SCRN_{\mathcal{F}} \cup PER$. Then every injective morphism $\sigma : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ is ambiguous up to automorphism with respect to α.*

Proof. Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ and let $\sigma : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be an injective morphism. Suppose that $\alpha \in PER$. Then by Proposition 65, there exists a periodic morphism $\tau_{PER} : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that $\tau_{PER}(\alpha) = \sigma(\alpha)$. However, since σ is injective, and τ_{PER} is not, $\sigma(\mathcal{F}_{\text{var}(\alpha)}) \neq \tau_{PER}(\mathcal{F}_{\text{var}(\alpha)})$. It follows that $\tau_{PER} \neq \sigma \circ \varphi$ for any automorphism $\varphi : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\text{var}(\alpha)}$ and consequently that σ is ambiguous up to automorphism.

Suppose that $\alpha \in IMP$. Let $\Delta_1, \Delta_2, \Delta_3$ be defined according to Definition 58, and let δ_x, β_i also be defined accordingly for $x \in \Delta_1$ and $i \in \mathbb{N}_0$. Let $\tau_{IMP} : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be the morphism given by:

$$\tau_{IMP}(x) = \begin{cases} \sigma(x) & \text{if } x \in \Delta_3, \\ \varepsilon & \text{if } x \in \Delta_2, \\ \sigma(\delta_x) & \text{if } x \in \Delta_1. \end{cases}$$

Then $\sigma(\delta_x) = \tau_{IMP}(\delta_x)$ for all $x \in \Delta_1$, and $\sigma(\beta_i) = \tau_{IMP}(\beta_i)$ for $i \in \mathbb{N}$. Hence $\sigma(\alpha) = \tau_{IMP}(\alpha)$. By definition, we have $\Delta_2 \neq \varepsilon$, so there exists $y \in \Delta_2$ such that $\tau_{IMP}(y) = \varepsilon$ and τ_{IMP} is not injective. However, σ is injective so $\sigma(\mathcal{F}_{\Sigma}) \neq \tau_{IMP}(\mathcal{F}_{\Sigma})$ and thus $\tau_{IMP} \neq \sigma \circ \varphi$ for any automorphism $\varphi : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\text{var}(\alpha)}$. Consequently, σ is ambiguous up to automorphism with respect to α.

Finally, suppose that $\alpha \in SCRN_{\mathcal{F}}$. Let S_0, C_0, R_0, N_0 be defined according to Definition 61. Recall that by definition, $S_0 \cup R_0 \neq \emptyset$. W.l.o.g., let $R_0 \neq \emptyset$ and let $y \in R_0$. Let $\tau_{SCRN_{\mathcal{F}}} : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_\Sigma$ be the morphism given by:

$$\tau_{SCRN_{\mathcal{F}}}(x) = \begin{cases} \sigma(x) & \text{if } x \in N_0, \\ \sigma(x)\ \sigma(y) & \text{if } x \in S_0, \\ \sigma(y)^{-1}\ \sigma(x)\ \sigma(y) & \text{if } x \in C_0, \\ \sigma(y)^{-1}\sigma(x) & \text{if } x \in R_0. \end{cases}$$

We can infer from the Definition of $SCRN_{\mathcal{F}}$ that since $\alpha \in SCRN_{\mathcal{F}}$, there exist $\beta_0, \beta_1, \dots \beta_n \in \mathcal{F}_{N_0}$ and $\gamma_1, \gamma_2, \dots \gamma_n \in \mathcal{F}_{\mathbb{N}}$ such that

$$\alpha = \beta_0\ \gamma_1\ \beta_1 \dots \gamma_n\ \beta_n$$

where for $1 \leq i \leq n$, $\gamma_i = x_1 x_2 \dots x_{k_i}$ such that $x_1 \in S_0 \cup R_0^{-1}$, $x_{k_i} \in R_0 \cup S_0^{-1}$ and for $1 < j < k_i$, $x_j \in C_0 \cup C_0^{-1}$. Therefore, for $1 \leq i \leq n$,

$$\begin{aligned} \tau_{SCRN_{\mathcal{F}}}(\gamma_i) &= \sigma(x_1)\sigma(y)\sigma(y)^{-1}\sigma(x_2)\sigma(y) \dots \sigma(y)^{-1}\sigma(x_{k_i}) \\ &= \sigma(\gamma_i). \end{aligned}$$

Moreover, it is trivial that $\tau_{SCRN_{\mathcal{F}}}(\beta_i) = \sigma(\beta_i)$ for $0 \leq i \leq n$, and hence $\tau_{SCRN_{\mathcal{F}}}(\alpha) = \sigma(\alpha)$. Now, recall that $y \in R_0$. Then $\tau_{SCRN_{\mathcal{F}}}(y) = \varepsilon$ and thus $\tau_{SCRN_{\mathcal{F}}}$ is not injective. Since σ is injective, $\sigma(\mathcal{F}_{\text{var}(\alpha)}) \neq \tau_{SCRN_{\mathcal{F}}}(\mathcal{F}_{\text{var}(\alpha)})$. It follows that $\tau_{SCRN_{\mathcal{F}}} \neq \sigma \circ \varphi$ for any automorphism $\varphi : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\text{var}(\alpha)}$ and consequently that σ is ambiguous up to automorphism. □

We will see later in Section 5.6 that Theorem 66 does not always hold in the general case, where σ may be non-injective. However, we can say that every (possibly non-injective) morphism is ambiguous up to automorphism – and therefore also up to inner automorphism – for the following classes:

Theorem 67. *Let $\alpha \in IMP \cap PER$. Then every morphism $\sigma : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_\Sigma$ is ambiguous up to automorphism with respect to α.*

Proof. Let $\alpha \in IMP \cap PER$, and let $\Delta_1, \Delta_2, \Delta_3$, and $\delta_x, \gamma_x, \gamma'_x, x \in \Delta_1$ and $\beta_i, i \in \mathbb{N}_0$ be defined according to Definition 58. Let $\sigma : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a morphism. Firstly, suppose that $\sigma(\mathcal{F}_{\text{var}(\alpha)}) \neq \mathcal{F}_\Sigma$. Then there exists $w \in \mathcal{F}_\Sigma \setminus \sigma(\mathcal{F}_{\text{var}(\alpha)})$. Let $\tau : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_\Sigma$ be the morphism given by

$$\tau(x) = \begin{cases} \sigma(x) & \text{if } x \in \Delta_3, \\ w & \text{if } x \in \Delta_2, \\ w^{-p_x}\ \sigma(\delta_x)\ w^{-q_x} & \text{if } x \in \Delta_1 \end{cases}$$

where $p_x := \sum_{y \in \mathrm{var}(\gamma_x)} |\gamma_x|_y$ and $q_x := \sum_{y \in \mathrm{var}(\gamma'_x)} |\gamma'_x|_y$. Then $\tau(\delta_x) = \sigma(\delta_x)$ for all $x \in \Delta_1$, and $\sigma(\beta_i) = \tau(\beta_i)$ for $1 \leq i \leq n$. Hence $\sigma(\alpha) = \tau(\alpha)$. Now, for any automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$, we have $\sigma \circ \varphi(\mathcal{F}_{\mathrm{var}(\alpha)}) = \sigma(\mathcal{F}_{\mathrm{var}(\alpha)})$ and hence $w \notin \sigma \circ \varphi(\mathcal{F}_{\mathrm{var}(\alpha)})$. However, $w \in \tau(\mathcal{F}_{\mathrm{var}(\alpha)})$, so $\tau \neq \sigma \circ \varphi$. Thus σ is unambiguous up to automorphism as required.

Now suppose instead that $\sigma(\mathcal{F}_{\mathrm{var}(\alpha)}) = \mathcal{F}_\Sigma$. By Proposition 65, since $\alpha \in PER$ there exists a periodic morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\tau(\alpha) = \sigma(\alpha)$. It follows from the fact that τ is periodic that $\tau(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \mathcal{F}_\Sigma$. Thus for any automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$, $\tau(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \sigma \circ \varphi(\mathcal{F}_{\mathrm{var}(\alpha)}) = \sigma(\mathcal{F}_{\mathrm{var}(\alpha)}) = \mathcal{F}_\Sigma$. Hence $\tau \neq \sigma \circ \varphi$ and σ is ambiguous up to automorphism. □

In particular, we note that if a pattern contains at least two variables, one of which occurs exactly once, then it belongs to $IMP \cap PER$, and hence does not possess an unambiguous morphism of any kind.

Corollary 68. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ such that $\alpha = \beta_1\ x\ \beta_2$ where $x \in \mathbb{N} \cup \mathbb{N}^{-1}$, $x \notin \mathrm{var}(\beta_1) \cup \mathrm{var}(\beta_2)$ and $\beta_1 \neq \varepsilon$ or $\beta_2 \neq \varepsilon$. Then every morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ is ambiguous up to automorphism with respect to α.*

We can achieve a corresponding result for the class $SCRN_{\mathcal{F}}$ in a similar way.

Theorem 69. *Let $\alpha \in SCRN_{\mathcal{F}} \cap PER$. Then every morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ is ambiguous up to automorphism with respect to α.*

Proof. Let $\alpha \in SCRN_{\mathcal{F}} \cap PER$, and let S_0, C_0, R_0, N_0 be defined according to Definition 61. Note that $S_0 \cup R_0 \neq \emptyset$. W.l.o.g., let $S_0 \neq \emptyset$ and let $y \in S_0$. Suppose first that $\sigma(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \mathcal{F}_\Sigma$. Then there exists $w \in \mathcal{F}_\Sigma \setminus \sigma(\mathcal{F}_{\mathrm{var}(\alpha)})$. Let $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be the morphism given by

$$\tau(x) = \begin{cases} \sigma(x) & \text{if } x \in N_0, \\ \sigma(x)\ \sigma(y)^{-1}\ w & \text{if } x \in S_0, \\ w^{-1}\ \sigma(y)\ \sigma(x) & \text{if } x \in R_0 \\ w^{-1}\ \sigma(y)\ \sigma(x)\ \sigma(y)^{-1}\ w & \text{if } x \in C_0. \end{cases}$$

We can infer from the Definition of $SCRN_{\mathcal{F}}$ that since $\alpha \in SCRN_{\mathcal{F}}$, there exist $\beta_0, \beta_1, \ldots \beta_n \in \mathcal{F}_{N_0}$ and $\gamma_1, \gamma_2, \ldots \gamma_n \in \mathcal{F}_{\mathbb{N}}$ such that

$$\alpha = \beta_0\ \gamma_1\ \beta_1 \ldots \gamma_n\ \beta_n$$

where for $1 \leq i \leq n$, $\gamma_i = x_1 x_2 \ldots x_{k_i}$ such that $x_1 \in S_0 \cup R_0^{-1}$, $x_{k_i} \in R_0 \cup S_0^{-1}$ and for $1 < j < k_i$, $x_j \in C_0 \cup C_0^{-1}$. Therefore, for $1 \leq i \leq n$,

$$\begin{aligned}\tau_{SCRN_{\mathcal{F}}}(\gamma_i) &= \sigma(x_1)\sigma(y)^{-1}ww^{-1}\sigma(y)\sigma(x_2)\sigma(y)^{-1}ww^{-1}\ldots\sigma(y)\sigma(x_{k_i})\\ &= \sigma(\gamma_i).\end{aligned}$$

Moreover, it is trivial that $\tau(\beta_i) = \sigma(\beta_i)$ for $0 \leq i \leq n$. Hence $\tau(\alpha) = \sigma(\alpha)$. Now, recall that $y \in R_0$. Then $\tau(y) = \sigma(y)\sigma(y)^{-1}w = w$ and thus $w \in \tau(\mathcal{F}_{\mathrm{var}(\alpha)})$. However, for any automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$, we have $\sigma \circ \varphi(\mathcal{F}_{\mathrm{var}(\alpha)}) = \sigma(\mathcal{F}_{\mathrm{var}(\alpha)})$, so by the definition of w, $w \notin \sigma \circ \varphi(\mathcal{F}_{\mathrm{var}(\alpha)})$. Consequently $\tau \neq \sigma \circ \varphi$ for any automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$, and σ is ambiguous up to automorphism.

Now suppose instead that $\sigma(\mathcal{F}_{\mathrm{var}(\alpha)}) = \mathcal{F}_\Sigma$. By Proposition 65, since $\alpha \in PER$ there exists a periodic morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\tau(\alpha) = \sigma(\alpha)$. However, as τ is periodic, $\tau(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \mathcal{F}_\Sigma$. Thus for any automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$, $\tau(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \sigma \circ \varphi(\mathcal{F}_{\mathrm{var}(\alpha)}) = \sigma(\mathcal{F}_{\mathrm{var}(\alpha)}) = \mathcal{F}_\Sigma$. Hence $\tau \neq \sigma \circ \varphi$ and σ is ambiguous up to automorphism. □

We have so-far considered ambiguity up to automorphism since this automatically implies ambiguity up to inner automorphism. However, since the latter is a weaker notion, it is satisfied by considerably more (pairs of) morphisms. For our classes PER, IMP and $SCRN_{\mathcal{F}}$, we see that although some morphisms may be unambiguous up to automorphism with respect to some patterns in those classes (see Theorems 118 and 125 in Section 5.6), this is not the case when considering ambiguity up to inner automorphism.

Theorem 70. *Let $\alpha \in PER \cup IMP \cup SCRN_{\mathcal{F}}$. Then every morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ is ambiguous up to inner automorphism with respect to α.*

Proof. Let $\alpha \in \mathcal{F}_\mathbb{N}$ and let $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a morphism. By Proposition 26, if $\alpha \in PER$, then σ is ambiguous up to inner automorphism. Suppose $\alpha \in IMP$. Let $\Delta_1, \Delta_2, \Delta_3$, and $\delta_x, \gamma_x, \gamma'_x, x \in \Delta_1$ and $\beta_i, i \in \mathbb{N}_0$ be defined according to Definition 58. Let $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be the morphism given by

$$\tau(x) = \begin{cases} \sigma(x) & \text{if } x \in \Delta_3,\\ \sigma(x)\mathtt{a} & \text{if } x \in \Delta_2,\\ \tau(\gamma_x)^{-1}\,\sigma(\delta_x)\,\tau(\gamma'_x)^{-1} & \text{if } x \in \Delta_1.\end{cases}$$

Note that since $\gamma_x, \gamma'_x \in \mathcal{F}_{\Delta_2}$, the morphism τ is well defined. Moreover, $\sigma(\delta_x) = \tau(\delta_x)$ for all $x \in \Delta_1$, and $\sigma(\beta_i) = \tau(\beta_i)$ for $1 \leq i \leq n$. Hence $\sigma(\alpha) = \tau(\alpha)$. Recall that by definition, $\Delta_2 \neq \emptyset$ and thus there exists $x \in \Delta_2$ such that $\tau(x) = \sigma(x)\mathtt{a}$. For any inner automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$, we have that $|\sigma(x)|_\mathtt{a} = |\sigma \circ \varphi(x)|_\mathtt{a}$. Thus $\tau \neq \sigma \circ \varphi$ and σ is ambiguous up to inner automorphism.

Suppose instead that $\alpha \in SCRN_{\mathcal{F}}$, and let S_0, C_0, R_0, N_0 be defined according to Definition 61. Let $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be the morphism given by

$$\tau(x) = \begin{cases} \sigma(x) & \text{if } x \in N_0, \\ \sigma(x)\ \mathtt{a} & \text{if } x \in S_0, \\ \mathtt{a}^{-1}\ \sigma(y) & \text{if } x \in R_0, \\ \mathtt{a}^{-1}\ \sigma(y)\ \mathtt{a} & \text{if } x \in C_0. \end{cases}$$

We can infer from the Definition of $SCRN_{\mathcal{F}}$ that since $\alpha \in SCRN_{\mathcal{F}}$, there exist $\beta_0, \beta_1, \ldots \beta_n \in \mathcal{F}_{N_0}$ and $\gamma_1, \gamma_2, \ldots \gamma_n \in \mathcal{F}_{\mathbb{N}}$ such that

$$\alpha = \beta_0\ \gamma_1\ \beta_1 \ldots \gamma_n\ \beta_n$$

where for $1 \leq i \leq n$, $\gamma_i = x_1 x_2 \ldots x_{k_i}$ such that $x_1 \in S_0 \cup R_0^{-1}$, $x_{k_i} \in R_0 \cup S_0^{-1}$ and for $1 < j < k_i$, $x_j \in C_0 \cup C_0^{-1}$. Therefore, for $1 \leq i \leq n$,

$$\begin{aligned} \tau_{SCRN_{\mathcal{F}}}(\gamma_i) &= \sigma(x_1)\mathtt{a}\mathtt{a}^{-1}\sigma(x_2)\mathtt{a} \ldots \mathtt{a}^{-1}\sigma(x_{k_i}) \\ &= \sigma(\gamma_i). \end{aligned}$$

Moreover, it is trivial that $\tau(\beta_i) = \sigma(\beta_i)$ for $0 \leq i \leq n$. Hence $\tau(\alpha) = \sigma(\alpha)$. Recall that by definition, $S_0 \cup R_0 \neq \emptyset$ and thus there exists $x \in S_0 \cup R_0$ such that $\tau(x) = \sigma(x)\mathtt{a}$ or $\tau(x) = \mathtt{a}^{-1}\sigma(x)$. For any inner automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$, we have that $|\sigma(x)|_{\mathtt{a}} = |\sigma \circ \varphi(x)|_{\mathtt{a}}$. Thus $\tau(x) \neq \sigma \circ \varphi(x)$ and τ is not the result of composing σ with an inner automorphism. Hence σ ambiguous up to inner automorphism. □

Finally, we mention the following class of patterns for which all injective morphisms are ambiguous up to inner automorphism, which we prove as an application of Theorem 101.

Proposition 71. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$ such that $\mathrm{var}(\alpha) \cap \mathrm{var}(\beta) = \emptyset$. Suppose that $|\mathrm{var}(\alpha)| > |\mathrm{var}(\beta)| \geq 1$. Then every injective morphism is ambiguous up to inner automorphism with respect to any pattern*

$$\alpha^{p_1}\beta^{p_2} \cdots \alpha^{p_{n-1}}\beta^{p_n}$$

such that $p_i \in \mathbb{Z}$ for $1 \leq i \leq n$, and $p_j, p_{j+1} \neq 0$ for some j, $1 \leq j < n$.

Proof. Let $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be the morphism such that $\varphi(x) := \alpha x \alpha^{-1}$ if $x \in \mathrm{var}(\alpha)$, and $\varphi(x) = x$ otherwise. It is straightforward that φ is not an inner automorphism. Hence the identity morphism is ambiguous up to inner automorphism, and therefore, by Theorem 101, all injective morphisms are unambiguous up to inner automorphism. □

One interesting class of patterns which is largely covered by Proposition 71 is the set of so-called non-cross patterns – patterns for which all the variables occur in 'order' (e.g., $1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$). Since, for any non-cross pattern γ with at least three variables, γ satisfies the structure given in the proposition, we can conclude that if a non-cross pattern permits an injective morphism which is unambiguous up to inner automorphism, it must have at most two variables.

Corollary 72. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ be non-cross and suppose that $|\operatorname{var}(\alpha)| > 2$. Then every morphism $\sigma : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ is ambiguous up to inner automorphism with respect to α.*

5.2 A Morphic Encoding

Having established some classes of patterns for which "all" morphisms are ambiguous, we now consider explicitly patterns which do possess an unambiguous morphism. In the following sections we will consider specifically those patterns which possess an unambiguous injective morphism, and hence we address Questions 1 and 2. More precisely, we will present the necessary technical details required to prove the characterization given in Theorem 101 and the conjectured characterization given in Theorem 102, which we then present in Section 5.4.

We already know from Proposition 57 that if a pattern has an injective morphism $\sigma : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ which is unambiguous up to (inner) automorphism, then the identity morphism $\mathrm{id}_{\mathcal{F}_{\operatorname{var}(\alpha)}} : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\operatorname{var}(\alpha)}$ is unambiguous up to (inner) automorphism. Hence, to complete our characterizations, it remains to show that the converse also holds: that if the identity morphism is unambiguous up to (inner) automorphism with respect to α, then there exists an injective morphism $\sigma : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ which is unambiguous up to (inner) automorphism. The proofs of this statement for each case (ambiguity up to inner automorphism and ambiguity up to automorphism), and consequently Theorems 101 and 102, are largely the same, and consist mainly of two parts.

For the first part, which we address in the current section, we will construct an "encoding morphism" with the following, rather special property. For two patterns $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, we wish to construct $\sigma_{\alpha,\beta} : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that:

If $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ for some morphism $\tau : \mathcal{F}_{\operatorname{var}(\beta)} \to \mathcal{F}_{\Sigma}$, then there exists a morphism $\varphi_\tau : \mathcal{F}_{\operatorname{var}(\beta)} \to \mathcal{F}_{\operatorname{var}(\alpha)}$ mapping β onto α.

Hence the word $\sigma_{\alpha,\beta}(\alpha)$ essentially 'encodes' the pre-image α over the (binary) alphabet Σ.

$$\begin{tikzcd} & \sigma_{\alpha,\beta}(\alpha) & \\ \beta \arrow[ur, "\tau"] \arrow[rr, dashed, "\varphi_\tau"'] & & \alpha \arrow[ul, "\sigma_{\alpha,\beta}"'] \end{tikzcd}$$

For the second part of our proof(s), which we present in Section 5.3, we wish to assert that τ is as unambiguous as φ_τ, specifically in the case that $\alpha = \beta$ (and hence φ_τ is a morphism fixing α). More formally, we will prove the following statement:

If φ_τ is unambiguous up to inner automorphism with respect to α, then τ is unambiguous up to inner automorphism with respect to α.

We also conjecture the corresponding statement for ambiguity up to automorphism. Since φ_τ fixes α, if the identity morphism $\mathrm{id}_{\mathcal{F}_{\mathrm{var}(\alpha)}} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is unambiguous up to (inner) automorphism with respect to α, then so is φ_τ. Similarly, if $\sigma_{\alpha,\beta}$ is unambiguous up to (inner) automorphism with respect to α, then so is τ. Consequently, we can conclude that if the identity morphism is unambiguous up to inner automorphism with respect to α, then so is $\sigma_{\alpha,\alpha}$. Likewise, provided our conjecture is correct, the same holds for ambiguity up to automorphism. Along with Proposition 57, this is sufficient to prove Theorems 101 and 102.

Returning to the first part of our proof, namely the construction of $\sigma_{\alpha,\beta}$, we note that by considering the more general case that α is not necessarily equal to β, we do not introduce any noticable additional effort, and are able to make use of our construction again in Section 5.7 to easily prove some properties of terminal-free group pattern languages.

Our morphism $\sigma_{\alpha,\beta}$ is a generalization of a construction given by Jiang et al. [42]. In their paper, they introduce such an encoding morphism for patterns in the free monoid. As we will see in the remainder of this section, our task is more complicated due to the possible presence of contractions; however, it is interesting that our eventual construction of $\sigma_{\alpha,\beta}$ (see Definitions 83 and 87 in the next section) will be noticeably similar, and apart from some smaller details, it is the verification of the construction, rather than the construction itself which becomes more complex in the free group.

We base our construction on the following idea: for each variable $x \in \mathrm{var}(\alpha)$, the image $\sigma_{\alpha,\beta}(x)$ has a uniquely associated factor S_x which acts as an anchor, holding the place of x. If, for a set of anchors S, we can guarantee a direct correspondence between occurrences of each anchor S_x in the image $\sigma_{\alpha,\beta}(\alpha)$ and occurrences of S_x in the images $\tau(y)$, $y \in \mathrm{var}(\beta)$, then by replacing occurrences of the anchors S_x in the images $\tau(y)$ with their respective variables x, we get a morphism $\tau_S^{\mathrm{mod}} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ mapping β onto a word over $\mathrm{var}(\alpha) \cup \Sigma$ which has α as a subpattern. We can erase the letters in Σ from τ_S^{mod}, to obtain a morphism $\varphi_{\tau,S} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which maps β onto α.[5]

The success of our approach relies on every morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ having a set of anchors $S = \{S_x \mid x \in \mathrm{var}(\alpha)\}$ satisfying the following two (informal) properties.

(1) $\tau_S^{\mathrm{mod}}(\beta) = (\tau(\beta))_S^{\mathrm{mod}}$ *where* τ_S^{mod} *is obtained from* τ *by replacing occurrences of each anchor* $S_x \in S$ *with* x *in the individual images* $\tau(y), y \in \mathrm{var}(\beta)$*, and* $(\tau(\beta))_S^{\mathrm{mod}}$ *is obtained by replacing occurrences of each* S_x *with* x *in the full image* $\tau(\beta)$*; and*

(2) the anchors occur in the appropriate order in $\tau(\beta)$ $(= \sigma_{\alpha,\beta}(\alpha))$ *(i.e., so that* $(\tau(\beta))_S^{\mathrm{mod}}$ *has* α *as a subpattern).*

The second condition is perhaps the more obvious, and simply states that the anchors which we swap must occur in the same relative order as the variables in α (so that when swapping them for the variables, we get α as a subpattern instead of some other pattern). In other words, the anchors must occur in the image $\tau(\beta)$ in correspondence with occurrences of the variables in α. It is, for example no good choosing $\mathtt{a}$ as an anchor for the variable x if $\mathtt{a}$ occurs 100 times in $\tau(\beta)$ while x occurs twice in α. The first property is slightly more subtle. We produce a second morphism τ_S^{mod} by replacing instances of the anchors in the individual images of variables $\tau(x)$. In order to obtain our desired image $\tau_S^{\mathrm{mod}}(\beta)$, we must not disrupt the structure of the image when replacing our anchors for variables. We provide the following example to illustrate why the conditions are necessary to avoid such disruption.

Example 73. *Let* $\alpha, \beta := 1 \cdot 2 \cdot 2 \cdot 1$ *and let* $\sigma_{\alpha,\beta} : \mathcal{F}_3 \to \mathcal{F}_\Sigma$ *be the morphism given by* $\sigma_{\alpha,\beta}(1) := \mathtt{abaa}$*,* $\sigma_{\alpha,\beta}(2) := \mathtt{abba}$ *so that*

$$\sigma_{\alpha,\beta}(\alpha) = \mathtt{a\ b\ a\ a\ a\ b\ b\ a\ a\ b\ b\ a\ a\ b\ a\ a}.$$

let $\tau := \sigma_{\alpha,\beta}$*. Suppose that* $S_1 := \mathtt{b}$ *and* $S_2 := \mathtt{aaa}$*. Then replacing* S_1 *with* 1 *and* S_2 *with* 2 *in* $\tau(\beta)$ *will not yeild a word having* α *as a subword, since the occurrences of* $\mathtt{aaa}$ *and* $\mathtt{b}$ *do not occur in the appropriate order in the image (i.e., Condition (2)*

[5] Note that we refine our notation from φ_τ to $\varphi_{\tau,S}$ to accommodate the fact that our construction relies on the choice of anchors.

is not satisfied) and instead, the word $\mathtt{a} \cdot 2 \cdot 1 \cdot (2 \cdot \mathtt{aa})^3$ *is obtained. Moreover, if we choose* $S_1 := \mathtt{aba}$ *and* $S_2 := \mathtt{aabb}$*, then the factors occur in the correct order in the image, so replacing their occurrences in the image yeilds a word containing* α *as a subpattern as required, however* S_2 *occurs in the full image* $\tau(\beta)$*, but not the individual images* $\tau(1)$ *or* $\tau(2)$*, so Condition 1 cannot hold. In other words, the morphism* τ_S^{mod} *obtained by replacing occurrences of* S_1 *and* S_2 *with* 1 *and* 2 *in* $\tau(1)$ *and* $\tau(2)$ *does not produce a morphism mapping* β *to a word containing* α *as a sub-pattern, and thus we cannot derive a morphism* $\varphi_{\tau,S}$ *mapping* β *to* α*.*

Further details on what disruption can occur and how we can avoid it shall be given in Section 5.2.1.

We will see in Section 5.2.2 that Property (2) follows in a reasonably straightforward manner from our construction of $\sigma_{\alpha,\beta}$ (see Remark 84). On the other hand, Property (1) requires considerably more effort, and hence it is this that we will begin to address first. In order to be sufficiently precise, we formally define the notion of a replacement.[6]

5.2.1 Replacements

We introduce the following notation for replacing factors in both words and morphisms. By requiring that the factors to be replaced, u, are unbordered words, no two occurrences can overlap, and thus we get a well-defined function. For example, if we allow u to be a bordered word such as $\mathtt{aba}$ – then, for e.g., the word $w := \mathtt{ababa}$, the word obtained by replacing *all* occurrences of u with some word v is not well defined. We can either replace the left most occurrence, or the right most occurrence (resulting in two different outcomes), but not both.

Let X*,* Y *be alphabets. Let* u, w *be words in* $\mathcal{F}_X$ *and let* $v \in \mathcal{F}_Y$ *be an unbordered word. Denote by* $R[u \to v](w)$ *the word obtained by replacing all occurrences of* u *in* w *with* v*. For a set of variables* Δ *and a morphism* $\sigma : \mathcal{F}_\Delta \to \mathcal{F}_X$*, define the morphism* $R[u \to v](\sigma) : \mathcal{F}_\Delta \to \mathcal{F}_{X \cup Y}$ *such that* $R[u \to v](\sigma)(x) := R[u \to v]\sigma(x)$ *for each* $x \in \Delta$*.*

Then we have for example, that $R[\mathtt{aab} \to \mathtt{a}](\mathtt{aababb}) = \mathtt{aabb}$, and for the morphism $\sigma : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ given by $\sigma(1) := \mathtt{ab}$ and $\sigma(2) := \mathtt{aab}$, we have $R[\mathtt{a} \to \mathtt{aa}](\sigma)$ is the morphism $\sigma' : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ such that $\sigma'(1) = \mathtt{aab}$ and $\sigma'(2) = \mathtt{a}^4\mathtt{b}$. Note that since we regard a factor u^{-1} to be an occurrence of u, we have e.g., that $R[\mathtt{a} \to \mathtt{b}](\mathtt{abba}^{-1}) = \mathtt{bbbb}^{-1} = \mathtt{bb}$.

[6] It is worth pointing out that our replacement operation is essentially the same as operations carried out in the field of re-writing systems. However, since we don't need the more general (and therefore slightly more involved) notation used in that context, it is more convenient to use the brief notation we introduce here.

Hence, referring to Property (1), we wish to guarantee that for a morphism τ and anchor S_x,

$$R[S_x \to x](\tau)(\beta) = R[S_x \to x](\tau(\beta)). \tag{5.1}$$

Although Equation 5.1 only considers the replacement of a single anchor – while we will actually need to replace several (i.e., one for each variable) – we will see later that in our specific context this generalization is easily achieved. Hence, for the moment, we focus on providing a sufficient condition on τ, S_x such that Equation 5.1 is satisfied.

Intuitively, there are two reasons why Equation 5.1 might not hold for a potential anchor S_x. The first is that if we have some 'additional' or 'unexpected' occurrence of S_x in $R[S_x \to x](\tau(\beta))$, which rather than being directly produced by an occurrence of S_x in $\tau(y)$ for some $y \in \text{var}(\beta)$, is the product of two or more factors originating from different pre-image variables. This can happen if S_x occurs across two or more pre-image variables (i.e., it partially overlaps the image of some variable $\tau(y)$), as follows:

Example 74. *Let $\beta := 1 \cdot 2$, and let $\tau : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(1) :=$ $\mathtt{aa}$ and $\tau(2) := \mathtt{bb}$. Suppose that for $x = 1$, we choose $S_x := \mathtt{ab}$. Then since S_x does not occur in $\tau(1)$ or $\tau(2)$, we have $R[S_x \to x](\tau) = \tau$ and thus*

$$R[S_x \to x](\tau)(\beta) = \tau(\beta) = \mathtt{aabb}.$$

However, S_x does *occur in $\tau(\beta)$ and therefore*

$$R[S_x \to x](\tau(\beta)) = R[S_x \to x](\mathtt{aabb}) = \mathtt{a}\, x\, \mathtt{b} = \mathtt{a} \cdot 1 \cdot \mathtt{b}.$$

Hence $R[S_x \to x](\tau)(\beta) \neq R[S_x \to x](\tau(\beta))$.

Likewise, if there is an occurrence of S_x which is comprised of factors from two or more pre-image variables which become adjacent once contractions are removed (i.e., if the occurrence of S_x in $\tau(\beta)$ is not reduced):

Example 75. *Let $\beta := 1 \cdot 2$, and let $\tau : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(1) := \mathtt{ab}^{-1}$ and $\tau(2) := \mathtt{ba}$. Suppose that for $x = 1$ we choose $S_x := \mathtt{aa}$. Then since S_x does not occur in $\tau(1)$ or $\tau(2)$, we have $R[S_x \to x](\tau) = \tau$ and thus*

$$R[S_x \to x](\tau)(\beta) = \tau(\beta) = \mathtt{ab}^{-1}\mathtt{ba} = \mathtt{aa}.$$

However, S_x does occur in $\tau(\beta)$ and therefore

$$R[S_x \to x](\tau(\beta)) = R[S_x \to x](\mathtt{aa}) = x = 1.$$

Hence $R[S_x \to x](\tau)(\beta) \neq R[S_x \to x](\tau(\beta))$.

We see that if additional occurrences of S_x are produced – other than the ones contained entirely within the images of individual variables – we are unable to guarantee the necessary correspondence for Equation 5.1 to hold.

The second reason is that some occurrence of S_x in $\tau(y)$ may not correspond to an occurrence in $\tau(\beta)$ – if it is contracted (i.e., instead of producing an additional occurrence, we lose an occurrence). In fact, if it is fully contracted, this is not a problem, as the next example demonstrates. However if it is only partially contracted (i.e., it partially overlaps a maximal contraction), then replacing S_x with x can disrupt this contraction and alter the structure of the image, and thus meaning we lose the desired correspondence.

Example 76. *Let* $\beta := 1 \cdot 2$*, and let* $\tau_1 : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ *be the morphism given by* $\tau_1(1) := \mathtt{abb}$ *and* $\tau_1(2) := \mathtt{b}^{-1}\mathtt{a}$*. Suppose that for* $x = 1$*, we choose* $S_x := \mathtt{abb}$*. Then* $R[S_x \to x](\tau)$ *is the morphism* $\tau_1' : \mathcal{F}_2 \to \mathcal{F}_{\Sigma\cup\{1\}}$ *given by* $\tau_1'(1) := 1$ *and* $\tau_1'(2) := \mathtt{b}^{-1}\mathtt{a}$*. Hence*

$$R[S_x \to x](\tau_1)(\beta) = 1 \cdot \mathtt{b}^{-1}\mathtt{a}.$$

However,

$$R[S_x \to x](\tau_1(\beta)) = R[S_x \to x](\mathtt{abbb}^{-1}\mathtt{a}) = R[S_x \to x](\mathtt{aba}) = \mathtt{aba}.$$

Hence $R[S_x \to x](\tau_1)(\beta) \neq R[S_x \to x](\tau_1(\beta))$*. On the other hand, for* $\tau_2 : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ *such that* $\tau_2(1) := \mathtt{abb}$ *and* $\tau_2(2) := \mathtt{b}^{-1}\mathtt{b}^{-1}\mathtt{a}^{-1}\mathtt{b}$*, we have that* $R[S_x \to x](\tau_2)$ *is the morphism* $\tau_2' : \mathcal{F}_2 \to \mathcal{F}_{\Sigma\cup\{1\}}$ *such that* $\tau_2'(1) = 1$ *and* $\tau_2'(2) = 1^{-1}\mathtt{b}$*. Hence*

$$R[S_x \to x](\tau)(\beta) = \tau_2'(1 \cdot 2) = 1 \cdot 1^{-1} \cdot \mathtt{b} = \mathtt{b},$$

and

$$R[S_x \to x](\tau(\beta)) = R[S_x \to x](\mathtt{b}) = \mathtt{b}.$$

Hence $R[S_x \to x](\tau_2)(\beta) = R[S_x \to x](\tau_2(\beta))$.

Therefore, in order for an anchor S_x to satisfy Equation 5.1, each occurrence of S_x in the unreduced image of $\tau(\beta)$ should not be unreduced (Example 75), partially overlap the image of a variable (Example 74), or partially overlap a maximal contraction (Example 76). Formally, we shall say that if such an undesirable occurrence of S_x *does* exist, then S_x is split by τ in $\tau(\beta)$.

Definition 77. *Let* $\beta \in \mathcal{F}_\mathbb{N}$ *and let* $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ *be a morphism. Let* w *be the unreduced image* $\tau(\beta)$*. Let* $u \in \mathcal{F}_\Sigma$ *be an unbordered word. Then* u *is* split by τ in $\tau(\beta)$ *if:*

(i) there exists an unreduced occurrence of u in w, or

(ii) there exists an occurrence of u in w which partially overlaps with $\tau(x)$ for some $x \in \mathrm{var}(\beta)$, or

(iii) there exists an occurrence of u in w which partially overlaps with a maximal contraction.

We will now show that Definition 77 provides a sufficient condition (albeit in a negative form) for an anchor S_x to satisfy Equation 5.1. To prove this, we require the following lemmas regarding the combinatorics of replacements $R[u \to v](w)$. The first refers to Condition (ii), which, since contractions are not involved, is arguably the simplest. We will also use it in the proofs of Lemmas 79 and 80, and hence it takes a slightly more general form.

Lemma 78. *Let $w, u, v \in \mathcal{F}_\Sigma$ such that u is unbordered, and suppose that $w = w_1 w_2 \dots w_n$ for non-empty words $w_i \in \mathcal{F}_\Sigma$, and suppose that no occurrence of u in w partially overlaps a factor w_i. Then we have that*

$$R[u \to v](w_1) R[u \to v](w_2) \dots R[u \to v](w_n) = R[u \to v](w).$$

Proof. The statement follows from the definitions: since every occurrence of u is contained entirely inside a factor w_i, replacing all the factors of u in each w_i with v is equivalent to replacing all the factors of u in w. □

Our second lemma addresses the contractions, and is therefore relevant to Conditions (i) and (iii), although again we use it in a slightly more general way in the proof of Lemma 80.

Lemma 79. *Let $w, u, v \in \mathcal{F}_\Sigma$ be words such that u is unbordered and reduced, and w is unreduced. Suppose that $w = \varepsilon$. If no occurrence of u in w partially overlaps with a maximal contraction and every occurrence of u is reduced, then*

$$R[u \to v](w) = \varepsilon.$$

Proof. We first prove the following statement: Let w_1, $w_2 \in \mathcal{F}_\Sigma$ such that $w = w_1 x w_2$ where x is a maximal primary contraction. If no occurrence of u in w partially overlaps with x, then

$$R[u \to v](w) = R[u \to v](w_1 w_2).$$

To verify this claim, note that because no further maximal contractions occur in x, there exist reduced words x_1, x_2 such that $x = x_1 x_2$ and $x_1 = x_2^{-1}$. It follows that $R[u \to v](x_1) = R[u \to v](x_2^{-1}) = R[u \to v](x_2)^{-1}$. Furthermore, since u is reduced, there cannot be an occurrence of u in x which is contained partly in x_1 and partly in x_2 as it would then contain a contraction. Therefore by Lemma 78 we have that

$$\begin{aligned} R[u \to v](x) &= R[u \to v](x_1)R[u \to v](x_2) \\ &= R[u \to v](x_2)^{-1}R[u \to v](x_2) \\ &= \varepsilon. \end{aligned}$$

Now, if u does not partially overlap with x as we have assumed, but an occurrence of u partially overlaps with either w_1 or w_2, then that occurrence must have x as a factor. This contradicts the assumption that every occurrence of u is reduced. Hence we can assume that u does not partially overlap with w_1, x or w_2 and hence by Lemma 78, we have:

$$R[u \to v](w) = R[u \to v](w_1)\ R[u \to v](x)\ R[u \to v](w_2)$$

and therefore:

$$R[u \to v](w) = R[u \to v](w_1)R[u \to v](w_2).$$

By the same argument there cannot be an occurrence of u in w_1w_2 which partially overlaps either w_1 or w_2, since such an occurrence would imply an occurrence in w_1xw_2 which is unreduced. So we can again apply Lemma 78 and:

$$\begin{aligned} R[u \to v](w_1w_2) &= R[u \to v](w_1)R[u \to v](w_2) \\ &= R[u \to v](w). \end{aligned}$$

Hence we have proven our claim. In order to see that the statement of the lemma follows, simply observe that if we continuously remove the primary maximal contractions x from w, we eventually reach ε, and thus the statement can be reached by repeated applications of our claim. □

As claimed, using Lemmas 78 and 79, we are able to turn Definition 77 into a sufficient condition on τ and S_x such that Equation 5.1 is satisfied.

Lemma 80. *Let $\beta \in \mathcal{F}_{\mathbb{N}}$ and $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism. Let $u, v \in \mathcal{F}_\Sigma$ such that u is unbordered. If no occurrence of u is split by τ in $\tau(\beta)$, then*

$$R[u \to v](\tau)(\beta) = R[u \to v](\tau(\beta)).$$

Proof. Let w be the unreduced image $\tau(\beta)$. Suppose that u is not split by τ in $\tau(\beta)$ (and thus that Conditions (i), (ii) and (iii) of Definition 77 do not hold).

By Lemma 78, if Condition (ii) does not hold, then $R[u \to v](\tau)(\beta) = R[u \to v](w)$.

Hence it remains to show that $R[u \to v](w) = R[u \to v](\tau(\beta))$. To do this, let $w = w_0 z_1 w_1 \dots z_m w_m$ where each z_i is a maximal contraction and w_i is reduced (then we also have $w_0 w_1 \cdots w_m = \tau(\beta)$). If Condition (iii) of Definition 77 does not hold, then every occurrence of u is contained entirely within each w_i or z_i. Consequently, by Lemma 78,

$$\begin{aligned} R[u \to v](w) = R[u \to v](w_0) R[u \to v](z_1) R[u \to v](w_1) R[u \to v](z_2) \dots \\ \dots R[u \to v](z_m) R[u \to v](w_m). \end{aligned}$$

Furthermore, since Conditions (i) and (iii) do not hold, by Lemma 79,

$$R[u \to v](z_i) = \varepsilon$$

for $1 \le i \le m$. Thus

$$R[u \to v](w) = R[u \to v](w_0) R[u \to v](w_1) \dots R[u \to v](w_m).$$

Finally we conclude that no occurrence of u can partially overlap any w_i in $w_0 w_1 \dots w_m$, otherwise that occurrence of u must also contain some z_j as a factor in $w_0 z_1 w_1 \dots z_m w_m$, and would therefore contradict our assumption that Condition (i) does not hold. Hence by Lemma 78

$$\begin{aligned} R[u \to v](w_0) R[u \to v](w_1) \dots R[u \to v](w_m) &= R[u \to v](w_0 w_1 \dots w_m) \\ &= R[u \to v](\tau(\beta)), \end{aligned}$$

and therefore

$$R[u \to v](\tau)(\beta) = R[u \to v](w) = R[u \to v](\tau(\beta))$$

as claimed. □

Although Lemma 80 applies only to replacing a single anchor S_x, and is therefore not, in general, a sufficient means of showing that Property (1) is satisfied, we will see later that due to the design of our morphism $\sigma_{\alpha,\beta}$ (and more specifically the design of the anchors S_x), the necessary replacements are independent, and hence this generalization can be easily achieved. Hence we are now ready to begin our construction of $\sigma_{\alpha,\beta}$.

5.2.2 Construction of the Morphism $\sigma_{\alpha,\beta}$

As with the construction by Jiang et al. [42], we base our morphism $\sigma_{\alpha,\beta}$ on the idea of segments: factors $\mathtt{a}\mathtt{b}^i\mathtt{a}$ for $i \in \mathbb{N}$. The idea is that, by including enough segments unique to each $\sigma(x)$, $x \in \mathrm{var}(\alpha)$, we guarantee at least one will not be split by a contraction or variable transition in $\tau(\beta)$, and therefore may act as an anchor for x (by Lemma 80 we can replace it with x while maintaining the structure of the image). Due to the fact that Definition 77 – and therefore Lemma 80 – requires the anchors to be unbordered, we will rather use segments $s_i = \mathtt{a}\mathtt{b}^i\mathtt{a}^{-i}\mathtt{b}$.

One additional problem we have is that while in the free monoid it is trivial that any factor occurring in $\sigma(x)$ will occur in $\sigma(\alpha)$ if $x \in \alpha$, the same simple statement does not always hold true in the free group. Consider the morphism $\sigma : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ given by $\sigma(1) = \mathtt{ab}$ and $\sigma(2) := \mathtt{b}^{-1}$. Then $\sigma(1 \cdot 2) = \mathtt{abb}^{-1} = \mathtt{a}$, and hence the factors $\mathtt{b}$ in $\sigma(1)$ and $\mathtt{b}^{-1}$ in $\sigma(2)$ do not 'survive' in the reduced image $\sigma(1 \cdot 2)$. In order for our reasoning to work, we must guarantee that each segment s_i in $\sigma_{\alpha,\beta}$ does in fact survive in the image $\sigma(\alpha)$. To accomplish this, we will add 'contraction-blocking' factors μ as prefixes and suffixes to each $\sigma_{\alpha,\beta}(x)$, $x \in \mathrm{var}(\alpha)$. In particular, we will use factors $\mathtt{a}\mathtt{b}^i\mathtt{a}$ as no one is a prefix or suffix of the other. This is sufficient to stop any contractions from occurring beyond these factors, and hence we guarantee that each segment s_i survives. We define these blocking factors, as well as segments, formally below for ease of reference.

Definition 81. *For all $i \in \mathbb{N}$, let $\mu_i := \mathtt{a}\mathtt{b}^i\mathtt{a}$ and let $s_i := \mathtt{a}\mathtt{b}^i\mathtt{a}^{-i}\mathtt{b}$.*

Remark 82. *For any $i, j \in \mathbb{N}$ with $i \neq j$, we have $\mu_i\mu_j^{-1} = \mathtt{a}\mathtt{b}^{i-j}\mathtt{a}^{-1}$ (and likewise $\mu_i^{-1}\mu_j = \mathtt{a}^{-1}\mathtt{b}^{j-i}\mathtt{a}$) where $i - j \neq 0$.*

For our construction of $\sigma_{\alpha,\beta}$ we will map each variable to the appropriate blocking factor – which must be unique to that variable – then the string of segments s_i which form our potential anchors, and finally a second blocking factor. In order to avoid any combinatorial confusion between parts of the blocking factors and the segments, we ensure that the smallest segments are longer than the largest blocking factors. Hence we get a class of morphisms $\sigma_{k,p}$ as follows, where k is the number of distinct, unique segments per variable and p is the number of variables (and hence the minimum "length" of the segments).

Definition 83. *Let $k \in \mathbb{N}$ and let $\Delta = \{x_1, x_2, \ldots, x_j\}$ be a subset of $\mathbb{N}$ such that $x_i < x_{i+1}$ for $1 \leq i < j$. Let $\sigma_{k,\Delta} : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ be the morphism given by $\sigma_{k,\Delta}(x_i) := \mu_i \cdot s_{j+(i-1)k+1} \cdots s_{j+ik} \cdot \mu_i$ for $1 \leq i \leq j$.*

For example if $k := 3$ and $\Delta := \{1, 2, 3\}$, we have that $\sigma_{3,\Delta} : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ is the morphism given by:

$$\sigma_{3,\Delta}(1) = \mathtt{a\ b\ a\ \ a\ b^4a^{-4}b\ a\ b^5a^{-5}b\ a\ b^6a^{-6}b\ \ a\ b\ a},$$
$$\sigma_{3,\Delta}(2) = \mathtt{a\ b^2a\ \ a\ b^7a^{-7}b\ a\ b^8a^{-8}b\ a\ b^9a^{-9}b\ \ a\ b^2a},$$
$$\sigma_{3,\Delta}(3) = \mathtt{a\ b^3a\ \ a\ b^{10}a^{-10}b\ a\ b^{11}a^{-11}b\ a\ b^{12}a^{-12}b\ \ a\ b^3a},$$

Because the blocking factors μ_i severely restrict the manner in which any contractions may occur, it is straightforward to observe that, for a pattern $\alpha \in \mathcal{F}_\Delta$, the (reduced) image $\sigma_{k,\Delta}(\alpha)$ has the form

$$\mu_r^{p_1} U_1^{p_1} V_1 U_2^{p_2} V_2 \dots V_{n-1} U_n^{p_n} \mu_s^{p_n}$$

where $\alpha = x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$ and $r, s, x_i \in \mathbb{N}, p_i \in \{1, -1\}$ for $1 \leq i \leq n$; μ_r, μ_s are defined according to Definition 81; the factors U_i consist of k consecutive segments s_j which uniquely occur as factors of $\sigma_{k,p}(x_i)$; and $V_i = (\mathtt{ab}^{x_i}\mathtt{aab}^{x_{i+1}}\mathtt{a})^{p_i}$ if $p_i = p_{i+1}$ or $V_i = (\mathtt{ab}^{x_i - x_{i+1}}\mathtt{a}^{-1})^{p_i}$ otherwise.

Consequently, for a segment $s_i, i > |\Delta|$, all occurrences of s_i in $\sigma_{k,\Delta}(\alpha)$ occur exactly once in the factors U_i for which s_i is a factor of $\sigma(x_i)$, and nowhere else. Hence we may draw the following conclusion.

Remark 84. *Let $\alpha \in \mathcal{F}_\mathbb{N}$, let $\Delta := \mathrm{var}(\alpha)$, and let $k \geq 1$. For each $x \in \mathrm{var}(\alpha)$, let S_x be a segment s_i such that S_x is a factor of $\sigma_{k,\Delta}(x)$. Let $S := \{S_x \mid x \in \mathrm{var}(\alpha)\}$. Let W be the result of replacing each occurrence of S_x in $\sigma_{k,\Delta}(\alpha)$ with x. Then W has α as a subpattern.*

Thus, referring to Properties (1) and (2), we see that if $\tau(\beta) = \sigma_{k,\Delta}(\alpha)$, for some morphism τ, and $S = \{S_x \mid x \in \mathrm{var}(\alpha)\}$ is a set of anchors such that S_x is a segment, and a factor of $\sigma_{k,p}(x)$, then S satisfies Property (2). As we have briefly discussed in Section 5.2.1, we will rely on Lemma 80 to show that a set of anchors satisfies Property (1), and thus we wish to affirm that each S_x is not split by τ in $\tau(\beta)$. We therefore define a set of *anchor segments* for τ as follows.

Definition 85. *Let $\alpha, \beta \in \mathcal{F}_\mathbb{N}$, let $\Delta := \mathrm{var}(\alpha)$ and let $k \in \mathbb{N}$. Let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\beta) = \sigma_{k,\Delta}(\alpha)$. We say a* set of anchor segments *for τ is a set $S := \{S_{x_1}, S_{x_2}, \dots, S_{x_n}\}$ such that:*

(i) $\{x_1, x_2, \dots, x_n\} = \Delta$, *and*

(ii) $S_{x_i} = s_j$ *for some* $j \in \mathbb{N}, j > |\Delta|$, *and*

(iii) S_{x_i} *is a factor of* $\sigma_{k,\Delta}(x_i)$, *and*

(iv) S_{x_i} *is not split by* τ *in* $\tau(\beta)$.

While it is not true that, for any k, Δ, for every τ with $\tau(\beta) = \sigma_{k,\Delta}(\alpha)$ there exists a set of anchor segments as defined above, we are able to show that the same statement *is* true whenever k is "large enough". The main observation we need is that the number of maximal contractions occurring in $\tau(\beta)$, and hence the number of segments s_i which are split by τ in $\tau(\beta)$ is bounded by a function of $|\beta|$.[7]

Lemma 86. *Let* $w_1, w_2, \dots, w_n \in \mathcal{F}_\Sigma$ *be reduced words. Let* w *be the unreduced word* $w_1 w_2 \cdots w_n$. *Then there are at most* $\frac{n(n-1)}{2}$ *maximal contractions in* w.

Proof. For the purposes of our proof, we will classify the maximal contractions as follows: a primary maximal contraction is *degree*-1. In general, a maximal contraction u is *degree*-$k+1$ if it contains a maximal contraction $v \neq u$ such that v is degree-k, and all maximal contractions in u which are not equal to u are at most degree-k. For example, $\mathtt{aa}^{-1}$ is degree-1 because it is primary, while $\mathtt{aa}^{-1}\mathtt{bb}^{-1}$ is degree-2 because it contains maximal contractions of degree 1.

We will now 'count' the maximum possible number of maximal contractions of degree-m in w. We start with $m = 1$. Because each w_i is reduced, and because each primary contraction must contain a factor $\mathtt{aa}^{-1}$ for some $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$, we have at most $n-1$ primary maximal contractions (and therefore maximal contractions with degree-1) in w. Consider the word $w' = w_1' w_2' \dots w_n'$ obtained by removing all the primary contractions, where $w_1', w_2', \dots, w_n'$ are obtained by removing the corresponding parts of each primary contraction from $w_1, w_2, \dots w_n$ respectively. Note that this may result in w_i' being the empty word for some $i \in \mathbb{N}$. Moreover, any factor of a reduced word is also reduced, so each w_i' is reduced for $1 \leq i \leq n$.

Now consider the primary contractions in each w'. Note that each degree-2 maximal contraction in w must contain at least one of these. By the same logic as before, there can be at most $n-1$. However, we claim that there can be at most $n-2$. This is clearly the case if $w_i' = \varepsilon$ for some $i, 1 \leq i \leq n$ (since we then have $n-2$ or less 'transistions' where a factor $\mathtt{aa}^{-1}$ may occur, $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$). We now show that if $w_i \neq \varepsilon$ for all $i, 1 \leq i \leq n$, then a much stronger statement holds, that there are *no* contractions. In particular, if $w_i \neq \varepsilon$ for $1 \leq i \leq n$, then there exist $x_i, y_i, z_i \in \mathcal{F}_\Sigma$ for $1 \leq i \leq n$ such that $w_i = x_i y_i z_i$ (and hence the latter is reduced), $z_i x_{i+1}$ is a maximal contraction in w for $1 \leq i < n$, and $w_i' = y_i \neq \varepsilon$.

Now suppose to the contrary that we have a primary contraction in w'. Then, since any primary contraction contains a factor $\mathtt{aa}^{-1}$, $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$, and since each w_i' is reduced (and hence does not contain such a factor), we must have that for

[7] Our result improves a claim given by Makanin [50], however we expect that although we do not need it here, our bound may also be improved considerably.

some $k, 1 \leq k < n$, $w'_k = y_k$ has a suffix $\mathtt{a}$ and $w'_{k+1} = y_{k+1}$ has a prefix $\mathtt{a}^{-1}$ for some $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$. However, this contradicts the fact that the contraction $z_k x_{k+1}$ is a *maximal* contraction. Hence if there exist any primary contractions in w', at least one w'_i must be empty, and as we have already reasoned, there can be at most $n-2$ primary contractions in w'.

By repeating this argument, we see there are $n - p$ possible maximal contractions of degree p until $p = n$ and we have a reduced word $w^{(p)}$. Thus we have at most $\sum_{j=1}^{n-1} = \frac{n(n-1)}{2}$ distinct maximal contractions in w. □

As a consequence of Lemma 86, we see that for any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ with $\tau(\beta) = \sigma_{k,\mathrm{var}(\alpha)}(\alpha)$, we can infer that the number of segments s_i satisfying Conditions (i) and (iii) of Definition 77 is bounded by a (fixed) function of $|\beta|$. It is also clear that there are at most $|\beta| - 1$ segments satisfying Condition (ii) of the same definition, so the total number of segments which are split by τ in $\tau(\beta)$ is also bounded by some function of $|\beta|$. Hence, to guarantee the existence of anchor segments S_x satisfying Definition 85, we simply need to choose a value of k which is above this bound (and thus at least one segment occurring in each $\sigma_{k,\mathrm{var}(\alpha)}$ cannot be split by τ in $\tau(\beta)$).

We will see from Proposition 88 that the exact number required is $k := \frac{(3|\beta|+2)(|\beta|-1)}{2} + 1$. Hence we define the morphism $\sigma_{\alpha,\beta}$ as follows.

Definition 87. *For $\alpha, \beta \in \mathcal{F}_\mathbb{N}$ and let $k = \frac{(3|\beta|+2)(|\beta|-1)}{2} + 1$. Then we define $\sigma_{\alpha,\beta}$ to be the morphism $\sigma_{k,\mathrm{var}(\alpha)}$ as defined in Definition 83.*

The next proposition confirms that any morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ has at least one set of anchor segments satisfying Definition 85.

Proposition 88. *Let $\alpha, \beta \in \mathcal{F}_\mathbb{N}$ and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. Then there exists a set S of anchor segments for τ.*

Proof. Let seg_x be the set of segments s_i such that s_i is a factor of $\sigma(x)$ (i.e., so that $p + (x-1)k + 1 \leq i \leq p + xk$ where $p = |\operatorname{var}(\alpha)|$ and $k = \frac{(3|\beta|+2)(|\beta|-1)}{2} + 1$ as per Definition 87). Then clearly,

$$|seg_x| = k = \frac{(3|\beta|+2)(|\beta|-1)}{2} + 1.$$

Moreover it is easily determined (e.g., from Remark 84 or the preceeding discussion), that every segment $s \in seg_x$ occurs as a factor of $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. We have the following claim.

Claim 1. *Let $x \in \operatorname{var}(\alpha)$. Then there exists $s \in seg_x$ such that s is not split by τ in $\tau(\beta)$.*

Proof (Claim 1). We consider the maximum number of segments $s \in seg_x$ which may be split by τ in $\tau(\beta)$. We note that at most $|\beta|-1$ segments may satisfy Condition (ii) of Definition 77, since there are at most $|\beta|-1$ factors $\tau(x)\tau(y)$ of $\tau(\beta)$ with $x, y \in \mathbb{N} \cup \mathbb{N}^{-1}$. Similarly, by Lemma 86, there are at most $\frac{|\beta|(|\beta|-1)}{2}$ different maximal contractions in $\tau(\beta)$. Every segment $s \in seg_x$ satisfying Condition (i) of Definition 77 must contain a maximal contraction, and hence the condition may be satisfied by at most $\frac{|\beta|(|\beta|-1)}{2}$ different segments $s \in seg_x$. Finally, we consider the segments satisfying Condition (iii) of Definition 77. Note that an occurrence of $s \in seg_x$ partially overlaps a maximal contraction if it occurs partly inside the contraction and partly outside it. Hence it crosses the 'edge' of the contraction. Since there are two edges per maximal contraction, there can be at most two segments partially overlapping each maximal contraction, and hence at most $|\beta|(|\beta|-1)$ segments $s \in seg_x$ satisfying Condition (iii) of Definition 77. In total, we have at most

$$|\beta|(|\beta|-1) + \frac{|\beta|(|\beta|-1)}{2} + |\beta| - 1 = \frac{(3|\beta|+2)(|\beta|-1)}{2}$$

distinct segments $s \in seg_x$ which satisfy any of the conditions of Definition 77. Consequently, since $|seg_x| > \frac{(3|\beta|+2)(|\beta|-1)}{2}$, there exists $S_x \in seg_x$ which doesn't satisfy any of the conditions of Definition 77, and hence is not split by τ in $\tau(\beta)$. □

By Claim 1, we can define a set S containing exactly one segment S_x from each seg_x, $x \in \mathrm{var}(\alpha)$ such that S_x is not split by τ in $\tau(\beta)$. It follows that S satisfies Conditions (i) and (iv) of Definition 85. By definition, each S_x is a segment occurring as a factor of $\sigma_{\alpha,\beta}(x)$, so S also satisfies Conditions (ii) and (iii). Thus S is a set of anchor segments for τ as required. □

Thus it remains to show formally that the existence of a set of anchor segments S for τ satisfying Definition 81 is sufficient for the existence of a morphism $\varphi_{\tau,S} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β onto α. To this end we define the morphism $\varphi_{\tau,S} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ below, along with a morphism τ_S^{mod} which provides a convenient "intermediate" step between τ and $\varphi_{\tau,S}$ in both this section, and the next – where its main use will be.

Definition 89. *Let α, $\beta \in \mathcal{F}_{\mathbb{N}}$ and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. Let $\mathrm{var}(\alpha) = \{x_1, x_2, \ldots, x_n\}$, and let $S := \{S_{x_1}, S_{x_2}, \ldots, S_{x_n}\}$ be a set of anchor segments for τ in accordance with Definition 85. We define $\tau_S^{\mathrm{mod}} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ to be the morphism obtained by replacing each occurrence of S_{x_i} in each $\tau(y), y \in \mathrm{var}(\beta)$ with x_i. Furthermore, we define $\varphi_{\tau,S} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ to be the morphism obtained by erasing all the letters from Σ (i.e., $\mathtt{a}, \mathtt{b}$) from τ_S^{mod}.*

We present the following statement regarding the morphism τ_S^{mod} from which the fact that $\varphi_{\tau,S}(\beta) = \alpha$ follows effortlessly. We leave the statement in a more specific form than is immediately necessary, as we will require this additional detail later, in Section 5.3.

Proposition 90. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, and let $\tau : \mathcal{F}_{\text{var}(\beta)} \to \mathcal{F}_{\Sigma}$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. Let S be a set of anchor segments for τ. For each $S_x \in S$, let w_x be the prefix of $\sigma_{\alpha,\beta}(x)$ and w'_x be the suffix of $\sigma_{\alpha,\beta}(x)$ such that $\sigma_{\alpha,\beta}(x) = w_x S_x w'_x$. Then*

$$\tau_S^{\text{mod}}(\beta) = \left(w_{x_1} x_1 w'_{x_1}\right)^{p_1} \left(w_{x_2} x_2 w'_{x_2}\right)^{p_2} \cdots \left(w_{x_n} x_n w'_{x_n}\right)^{p_n}$$

such that $\alpha = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$, $x_i \in \mathbb{N}$, $p_i \in \{1, -1\}$.

Proof. Recall from Proposition 88, since $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$, there exists a set $S = \{S_{x_1}, S_{x_2}, \ldots, S_{x_m}\}$ of anchor segments for τ in accordance with Definition 85. By definition, S_1 is not split by τ in $\tau(\beta)$ and is unbordered, so by Lemma 80, we have that

$$R[S_{x_1} \to x_1](\tau)(\beta) = R[S_{x_1} \to x_1](\tau(\beta))$$

and since $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$, this implies

$$R[S_{x_1} \to x_1](\tau)(\beta) = R[S_{x_1} \to x_1](\sigma_{\alpha,\beta}(\alpha)).$$

Let $\tau^{(1)} := R[S_{x_1} \to x_1](\tau)$. Note that because S_{x_1} and S_{x_2} occur independently (i.e., they do not overlap) in each $\tau(x), x \in \text{var}(\beta)$, and also in $\tau(\beta)$, it follows from the fact that S_{x_2} is not split by τ in $\tau(\beta)$, that S_{x_2} is not split by $\tau^{(1)}$ in $\tau^{(1)}(\beta)$. In other words, replacing S_{x_1} with x_1 does not affect whether S_{x_2} satisfies any of the conditions of Definition 77. Hence, for $\tau^{(2)} := R[S_{x_2} \to x_2]\left(\tau^{(1)}\right)$, we have that

$$\tau^{(2)}(\beta) = R[S_{x_2} \to x_2]\big(\tau^{(1)}(\beta)\big) = R[S_{x_2} \to x_2]\Big(R[S_{x_1} \to x_1]\big(\sigma_{\alpha,\beta}(\alpha)\big)\Big).$$

By repeating the same logic we eventually have:

$$\tau^{(m)}(\beta) = R[S_{x_m} \to x_m]\Big(R[S_{x_{m-1}} \to x_{m-1}]\big(\ldots R[S_{x_1} \to x_1(\sigma_{\alpha,\beta}(\alpha))\big)\Big).$$

Clearly $\tau^{(m)} = \tau_S^{\text{mod}}$. Hence it remains to consider the other side of the equation. Note that by the definition of morphisms, we have that:

$$\sigma_{\alpha,\beta}(\alpha) = \left(w_{x_1} S_{x_1} w'_{x_1}\right)^{p_1} \left(w_{x_2} S_{x_2} w'_{x_2}\right)^{p_2} \ldots \left(w_{x_n} S_{x_n} w'_{x_n}\right)^{p_n}$$

such that $\alpha = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$, $x_i \in \mathbb{N}$, $p_i \in \{1, -1\}$ and $\sigma_{\alpha,\beta}(x_i) = w_{x_i} S_{x_i} w'_{x_i}$. Furthermore, by the construction of $\sigma_{\alpha,\beta}$ (see Remark 84), there are no other occurrences of each S_{x_i} in $\sigma_{\alpha,\beta}(\alpha)$. Hence the result of applying the replacements $R[S_{x_i} \to x_i]$ to the word $\sigma_{\alpha,\beta}(\alpha)$ we have exactly:

$$\tau_S^{\mathrm{mod}}(\beta) = \left(w_{x_1} x_1 w'_{x_1}\right)^{p_1} \left(w_{x_2} x_2 w'_{x_2}\right)^{p_2} \ldots \left(w_{x_n} x_n w'_{x_n}\right)^{p_n}$$

and our statement holds. □

Corollary 91. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ be a morphism. If $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$, then $\varphi_{\tau,S}(\beta) = \alpha$.*

Finally, we conclude this section with the following theorem which summarizes the achievement of our encoding morphism $\sigma_{\alpha,\beta}$.

Theorem 92. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. There exists a morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ if and only if there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\beta) = \alpha$.*

Proof. The 'if' statement is straightforward, since we can just take $\tau := \sigma_{\alpha,\beta} \circ \varphi$. The 'only if' statement is given in Corollary 91. □

We also include the following remark which is necessary for our considerations later, in Section 5.7.

Remark 93. *Our definition of $\sigma_{\alpha,\beta}$ relies only on the length of β and the set $\Delta := \mathrm{var}(\alpha)$: we choose $\sigma_{\alpha,\beta}$ based on $\sigma_{k,\Delta}$ where k is derived from $|\beta|$. All the results of this section also hold if, instead, we base $\sigma_{\alpha,\beta}$ on $\sigma_{k',\Delta}$ for any $k' \geq k$.*

5.3 Reversing the Encoding Process

Having fulfilled our aim in the previous section of constructing a morphism $\sigma_{\alpha,\beta}$ such that any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ mapping β onto $\sigma_{\alpha,\beta}(\alpha)$ "encodes" an associated morphism $\varphi_{\tau,S}$ mapping β onto α, we now turn our attention to the ambiguity of the morphisms τ. In particular, we wish to show that τ is as unambiguous as $\varphi_{\tau,S}$, or more precisely, that:

If $\varphi_{\tau,S}$ is unambiguous up to (inner) automorphism with respect to β, then τ is also unambiguous up to (inner) automorphism with respect to β.

In a rough sense, we wish to show that there are not "more" morphisms τ mapping β to $\sigma_{\alpha,\beta}$, than morphisms $\varphi_{\tau,S}$ mapping β to α. In order to achieve this, we ask whether it is possible to uniquely determine τ from $\varphi_{\tau,S}$, and thus we attempt to reverse – or decode – our encoding. Put another way, we ask whether

$$\begin{tikzcd}
\beta \arrow[r, "\varphi_{\tau_1,S_1}", "= \varphi_{\tau_2,S_2}"'] \arrow[rr, bend left, "\tau_1"] \arrow[rr, bend right, "\tau_2"'] & \alpha \arrow[r, "\sigma_{\alpha,\beta}"] & \sigma_{\alpha,\beta}(\alpha)
\end{tikzcd}$$

Figure 5.1: In order to reduce the ambiguity of τ to the ambiguity of $\varphi_{\tau,S}$, we need to consider the case that two different morphsims τ_1, τ_2 might exist mapping β onto $\sigma_{\alpha,\beta}(\alpha)$, but such that the respective morphisms φ_{τ_1,S_1} and φ_{τ_2,S_2} are identical, as shown in the diagram.

there exist distinct morphisms $\tau_1, \tau_2 : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ mapping β onto $\sigma_{\alpha,\beta}(\alpha)$ such that, for sets of anchor segments S_1, S_2, we have $\varphi_{\tau_1,S_1} = \varphi_{\tau_2,S_2}$ (cf. Fig. 5.1).

Proposition 95 provides a necessary condition for such morphisms τ_1, τ_2 to exist. In fact, it gives a considerably stronger statement, that we have one of the following two cases:

1. for every morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{\alpha,\beta}$, there exists a set of anchor segments S such that $\tau = \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$ (and hence that $\tau_1 = \tau_2 = \sigma_{\alpha,\beta} \circ \varphi_{\tau_1,S_1}$), or

2. there exists a morphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ mapping β onto α such that $\mathrm{symb}(\psi(x)) \cap \Sigma \neq \emptyset$ for some $x \in \mathrm{var}(\beta)$.

In the first case, it is not difficult to conclude that our desired statement holds:

Proposition 94. *Let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ and let S be a set of anchor segments for τ. If Case (1), above, holds then τ is ambiguous up to inner automorphism with respect to β if and only if $\varphi_{\tau,S}$ is ambiguous up to inner automorphism with respect to β. Moreover τ is ambiguous up to automorphism with respect to β if and only if $\varphi_{\tau,S}$ is ambiguous up to automorphism with respect to β.*

Proof. We shall prove the statement for ambiguity up to automorphism. The proof for ambiguity up to inner automorphism is a straightforward adaptation. Suppose that Case 1 holds. Then for every morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ mapping β onto $\sigma_{\alpha,\beta}(\alpha)$, there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ (namely $\varphi_{\tau,S}$) mapping β onto α such that $\tau = \sigma_{\alpha,\beta} \circ \varphi$. Moreover, for every morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β onto α we have that $\tau := \sigma_{\alpha,\beta} \circ \varphi$ is a morphism mapping β onto $\sigma_{\alpha,\beta}(\alpha)$. Hence the set of all morphisms mapping β onto $\sigma_{\alpha,\beta}(\alpha)$ is given by the set:

$$S := \{\sigma_{\alpha,\beta} \circ \varphi \mid \varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)} \text{ is a morphism such that } \varphi(\beta) = \alpha\}.$$

Now let $\tau_1 := \sigma_{\alpha,\beta} \circ \varphi_1$ and $\tau_2 := \sigma_{\alpha,\beta} \circ \varphi_2$ be morphisms in S such that $\varphi_1, \varphi_2 : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ are morphisms mapping β onto α. Note that, since $\sigma_{\alpha,\beta}$ is injective, for every $x \in \mathrm{var}(\beta)$, $\sigma_{\alpha,\beta}(\varphi_1(x)) = \sigma_{\alpha,\beta}(\varphi_2(x))$ if and only if $\varphi_1(x) = \varphi_2(x)$. Consequently, $\tau_1 = \tau_2$ if and only if $\varphi_1 = \varphi_2$, and by the same reasoning, $\tau_1 = \tau_2 \circ \psi$ for some automorphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\beta)}$ if and only if $\varphi_1 = \varphi_2 \circ \psi$.

Hence, for $\tau = \sigma_{\alpha,\beta} \circ \varphi$, there exists a second morphism τ' such that $\tau(\beta) = \tau'(\beta) = \sigma_{\alpha,\beta}(\alpha)$ (i.e., $\tau' \in S$) and $\tau' \neq \tau \circ \psi$ for any automorphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\beta)}$ if and only if there exists $\varphi' : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β onto α such that $\varphi' \neq \varphi \circ \psi'$ for any automorphism $\psi' : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\beta)}$. This is equivalent to saying that τ is ambiguous up to automorphism if and only if φ is, and our statement follows.

□

We shall see later that for the second case, the condition that $\psi(x)$ contains some letter from Σ which does not appear in $\psi(\beta) = \alpha$ allows us to construct many morphisms $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β onto α by replacing the letters in Σ with arbitrary factors in $\mathcal{F}_{\mathrm{var}(\alpha)}$ (cf. Example 96). In Proposition 97, we show that, at least when $\alpha = \beta$, this results in sufficient generality that φ (and hence all morphisms $\varphi_{\tau,S}$ also mapping β onto α) is ambiguous up to inner automorphism, and thus our target statement also holds in this case.

Proposition 95. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$ and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. Let S be a set of anchor segments for τ. If $\tau \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$, then there exists a morphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ such that*

(i) $\psi(\beta) = \alpha$, and

(ii) $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ for some $x \in \mathrm{var}(\beta)$.

Proof. We begin by constructing our morphism ψ. To this end, for each $x \in \mathrm{var}(\alpha)$, let $S_x \in S$ be the anchor segment associated with x. Then from the definition of $\sigma_{\alpha,\beta}$, there exist (unique) $v_x, v'_x \in \mathcal{F}_{\Sigma}$ such that $\sigma_{\alpha,\beta}(x) = v_x S_x v'_x$ and $v_x S_x v'_x$ is reduced. Let τ_S^{mod} be defined according to Definition 89, and let $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ be the morphism given by $\rho(\mathtt{a}) = \mathtt{a}$, $\rho(\mathtt{b}) = \mathtt{b}$ an $\rho(x) = v_x^{-1} x {v'_x}^{-1}$ for all $x \in \mathrm{var}(\alpha)$. We define our morphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ to be $\rho \circ \tau_S^{\mathrm{mod}}$. The following claim confirms that ψ satisfies Condition (i) of the proposition.

Claim 1. $\psi(\beta) = \alpha$.

Proof (Claim 1). Recall that $\psi = \rho \circ \tau_S^{\mathrm{mod}}$. By Proposition 90, we have

$$\tau_S^{\mathrm{mod}}(\beta) = \left(w_{x_1} x_1 w'_{x_1}\right)^{p_1} \left(w_{x_2} x_2 w'_{x_2}\right)^{p_2} \cdots \left(w_{x_n} x_n w'_{x_n}\right)^{p_n}$$

such that $\alpha = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where $x_i \in \mathbb{N}$ and $p_i \in \{1, -1\}$, and $\sigma_{\alpha,\beta}(x_i) = w_{x_i} S_{x_i} w'_{x_i}$ such that $w_{x_i} S_{x_i} w'_{x_i}$ is reduced. Now, since $\rho(\mathtt{a}) = \mathtt{a}$ and $\rho(\mathtt{b}) = \mathtt{b}$, and each $w_{x_i}, w'_{x_i} \in \mathcal{F}_\Sigma$, we have $\rho(w_{x_i}) = w_{x_i}$ and $\rho(w'_{x_i}) = w'_{x_i}$. Thus

$$\rho \circ \tau_S^{\mathrm{mod}}(\beta) = \left(w_{x_1} \rho(x_1) w'_{x_1}\right)^{p_1} \left(w_{x_2} \rho(x_2) w'_{x_2}\right)^{p_2} \cdots \left(w_{x_n} \rho(x_n) w'_{x_n}\right)^{p_n}.$$

Recall that $\rho(x) = v_x^{-1} x {v'_x}^{-1}$ where v_x, v'_x are defined such that $\sigma_{\alpha,\beta}(x) = v_x S_x v'_x$. By the construction of $\sigma_{\alpha,\beta}$ there is exactly one occurrence of S_{x_i} in $\sigma_{\alpha,\beta}(x_i)$, so we may conclude that $w_{x_i} = v_{x_i}$ and $w'_{x_i} = v'_{x_i}$ for $1 \le i \le n$. Hence $\rho(x_i) = w_{x_i}^{-1} x {w'_{x_i}}^{-1}$ and thus

$$\begin{aligned}
\rho \circ \tau_S^{\mathrm{mod}}(\beta) &= \left(w_{x_1} w_{x_1}^{-1} x_1 {w'_{x_1}}^{-1} w'_{x_1}\right)^{p_1} \left(w_{x_2} w_{x_2}^{-1} x_2 {w'_{x_2}}^{-1} w'_{x_2}\right)^{p_2} \cdots \\
&\qquad\qquad \cdots \left(w_{x_n} w_{x_n}^{-1} x_n {w'_{x_n}}^{-1} w'_{x_n}\right)^{p_n} \\
&= x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \\
&= \alpha.
\end{aligned}$$

Therefore we have $\psi(\beta) = \rho \circ \tau_S^{\mathrm{mod}}(\beta) = \alpha$ as claimed. □

In order to address the second condition, we present the following claim.

Claim 2. *Suppose there exists $x \in \mathrm{var}(\beta)$ such that $\tau(x)$ is reduced and $\tau(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. Then $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$.*

Proof (Claim 2). In order to prove our claim, we must consider more closely the structure of the morphisms τ, τ_S^{mod} and ψ. Firstly, recall that τ_S^{mod} is obtained from τ by replacing the anchor segments S_x with their respective variables x. Thus, there exist $u_0, u_1, \dots u_m \in \mathcal{F}_\Sigma$, $y_1, y_2, \dots y_m \in \mathbb{N}$,[8] and $p_1, p_2, \dots p_m \in \{1, -1\}$ such that

$$\tau(x) = u_0\ S_{y_1}^{p_1}\ u_1\ S_{y_2}^{p_2}\ u_2\ \dots\ S_{y_m}^{p_m}\ u_m$$

and

$$\tau_S^{\mathrm{mod}}(x) = u_0\ {y_1}^{p_1}\ u_1\ {y_2}^{p_2}\ u_2 \dots\ {y_m}^{p_m}\ u_m$$

where

$$y_1^{p_1}\ y_2^{p_2}\ \cdots\ y_m^{p_m} = \varphi_{\tau,S}(x).$$

Recall that $\psi = \rho \circ \tau_S^{\mathrm{mod}}$ where $\rho(\mathtt{a}) = \mathtt{a}$, $\rho(\mathtt{b}) = \mathtt{b}$, and $\rho(y) = v_x^{-1} x {v'_x}^{-1}$ such that $\sigma_{\alpha,\beta}(y) = v_y S_y v'_y$. Then since $u_i \in \mathcal{F}_\Sigma$ for $0 \le i \le m$, we have that $\rho(u_i) = u_i$ and

[8] Note that in the case no anchor segments S_x occur in $\tau(x)$, we simply have $\tau(x) = u_0 = \tau_S^{\mathrm{mod}}(x) = \psi(x)$, and the statement is straightforward.

hence:

$$\begin{aligned}\psi(x) &= \rho \circ \tau_S^{\mathrm{mod}}(x) \\ &= \rho\,(u_0\ {y_1}^{p_1}\ u_1\ {y_2}^{p_2}\ u_2\ \ldots\ {y_m}^{p_m}\ u_m) \\ &= u_0\ \rho(y_1)^{p_1}\ u_1\ \ldots\ \rho(y_m)^{p_m}\ u_m \\ &= w_0\ {y_1}^{p_1}\ w_1\ {y_2}^{p_2}\ w_2\ \ldots\ {y_m}^{p_m}\ w_m.\end{aligned}$$

where $w_0 = u_0 v_{y_1}^{-1}$ if $p_1 = 1$ and $w_0 = u_0 v'_{y_1}$ if $p_1 = -1$, likewise $w_m = {v'_{y_m}}^{-1} u_m$ if $p_m = 1$ and $v_{y_m} u_m$ if $p_m = -1$, and for $1 \leq i < m$,

$$w_i = \begin{cases} {v'}_{y_i}^{-1}\ u_i\ v_{y_{i+1}}^{-1} & \text{if } p_i = p_{i+1} = 1, \\ v_{y_i}\ u_i\ v_{y_{i+1}}^{-1} & \text{if } p_i = -1, p_{i+1} = 1, \\ {v'}_{y_i}^{-1}\ u_i\ v'_{y_{i+1}} & \text{if } p_i = 1, p_{i+1} = -1, \\ v_{y_i}\ u_i\ v'_{y_{i+1}} & \text{if } p_i = p_{i+1} = -1. \end{cases}$$

Now, because for each variable $y \in \mathrm{var}(\alpha)$, $\sigma_{\alpha,\beta}(y) = v_y S_y v'_y$, we have

$$\sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x) = \left(v_{y_1} S_{y_1} v'_{y_1}\right)^{p_1} \left(v_{y_2} S_{y_2} v'_{y_2}\right)^{p_2}\ \ldots\ \left(v_{y_m} S_{y_m} v'_{y_m}\right)^{p_m}.$$

It is clear that if $w_i = \varepsilon$ for $0 \leq i \leq m$, then $\tau(x) = \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. Consequently there exists i such that $w_i \neq \varepsilon$. If w_i is not contracted in $\psi(x)$ then since $w_i \in \mathcal{F}_\Sigma$, we have that $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ and our claim holds. Suppose instead that w_i *is* contracted. In particular it must be contracted (at least partially) with another factor w_j, $i \neq j$. This implies that for some k, $1 < k \leq m$, we have $w_k = \varepsilon$ and $y_{k-1}^{p_{k-1}} = (y_k^{p_k})^{-1}$. However this implies that $y_k = y_{k-1}$ and $p_k = -p_{k-1}$ and hence that that $w_k = v_{y_k}^{p_k} u_k v_{y_k}^{-p_k}$ or $w_k = {v'}_{y_k}^{-p_k} u_k {v'}_{y_k}^{p_k}$. In either case, this implies that $u_i = \varepsilon$. Consequently, we have that the factor $S_{y_{k-1}}^{p_{k-1}} u_i S_{y_k}^{p_k}$ of $\tau(x)$ also equals ε, which implies that $\tau(x)$ is not reduced. This is a contradiction and thus we have proven the claim. □

We are now ready to prove our statement. In particular, note that if there exists $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ such that for some $x \in \mathrm{var}(\beta)$, $\tau(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$ and $\tau(x)$ is *not* reduced, then for the morphism $\tau' : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau'(x)$ is the reduced version of $\tau(x)$, we still have $\tau'(\beta) = \tau(\alpha)$ and $\tau'(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. Hence we may assume w. l. o. g. that $\tau(x)$ is reduced for each $x \in \mathrm{var}(\beta)$. Therefore, if $\tau \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$, then there exists $x \in \mathrm{var}(\beta)$ such that $\tau(x)$ is reduced and $\tau(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. By Claim (2), this implies that the morphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ satisfies Condition (ii) of the Proposition, and by Claim (1), ψ also satisfies Condition (i), so the statement holds.

□

It would be a reasonable assumption (although we are not able to prove this in the most general case), that the existence of such a morphism ψ is sufficient to ensure that any morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β onto α is ambiguous – and therefore any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ mapping β onto $\sigma_{\alpha,\beta}(\alpha)$ is also ambiguous.[9] This is due to the existence of a letter $\mathtt{a} \in \mathcal{F}_{\Sigma}$ which occurs in $\psi(x)$ for some $x \in \mathrm{var}(\beta)$ but does not occur in $\psi(\beta)$ $(= \alpha)$. In particular, we may replace each occurrence of $\mathtt{a}$ in ψ with any factor $\eta \in \mathcal{F}_{\mathbb{N}}$ to obtain a morphism $\psi' : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\psi'(\beta) = \alpha$, and hence we have a combinatorially rich set of morphisms mapping β onto α. Formally, ψ' can be obtained by composing ψ with a morphism $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\rho(y) = y$ for all $y \in \mathrm{var}(\alpha)$.

Example 96. *Let $\beta := 1 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3$, let $\alpha := 1 \cdot 2^4 \cdot 3 \cdot 1 \cdot 2^2 \cdot 3$, and let $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ be the morphism given by $\psi(1) := 1 \cdot \mathtt{a}$, $\psi(2) := \mathtt{a}^{-1} \cdot 2^2 \cdot \mathtt{a}$ and $\psi(3) := \mathtt{a}^{-1} \cdot 3$. Then we have:*

$$\begin{aligned}\psi(\beta) &= 1 \cdot \mathtt{a} \cdot \mathtt{a}^{-1} \cdot 2 \cdot 2 \cdot \mathtt{a} \cdot \mathtt{a}^{-1} \cdot 2 \cdot 2 \cdot \mathtt{a} \cdot \mathtt{a}^{-1} \cdot 3 \cdot 1 \cdot \mathtt{a} \cdot \mathtt{a}^{-1} \cdot 2 \cdot 2 \cdot \mathtt{a} \cdot \mathtt{a}^{-1} \cdot 3 \\ &= 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \\ &= \alpha.\end{aligned}$$

Let $\gamma_{\mathtt{a}}$ be any pattern in $\mathcal{F}_{\mathrm{var}(\alpha)}$, and let $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be the morphism such that $\rho(x) := x$ for all $x \in \mathrm{var}(\alpha)$, $\rho(\mathtt{a}) := \gamma_{\mathtt{a}}$. Then we have

$$\begin{aligned}\rho \circ \psi(\beta) &= 1\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}2 \cdot 2\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}2 \cdot 2\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}3 \cdot 1\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}2 \cdot 2\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}3 \\ &= 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \\ &= \alpha.\end{aligned}$$

while, for example, $\rho \circ \psi(1) = 1 \cdot \gamma_{\mathtt{a}}$. Hence each possible $\gamma_{\mathtt{a}}$ results in a different morphism $\rho \circ \psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β onto α.

Unfortunately, as we will discuss in more depth later, it seems to be far from straightforward to prove that this combinatorial freedom directly results in the morphisms $\rho \circ \psi$ being ambiguous up to automorphism. For ambiguity up to inner automorphism, however, the situation is more manageable. In particular, for the case that $\alpha = \beta$, we can construct ρ such that $\rho \circ \psi$ is not an inner automorphism, and is therefore ambiguous up to inner automorphism with respect

[9] The conclusion that τ is also ambiguous can be obtained with a trivial generalization of the proof of Proposition 57 along with the fact that $\sigma_{\alpha,\beta}$ is injective.

to $\alpha = \beta$. This is due to the fact that it is easy to find a factor η such that for any inner automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α and $x \in \mathrm{var}(\alpha)$, η is not a factor of $\varphi(x)$. Hence we simply construct ρ such that $\rho(\mathtt{a})$ and $\rho(\mathtt{b})$ contain an occurrence of η, and such that at least one occurrence is not contracted – or partially contracted – in the image $\rho \circ \psi(x)$ for some $x \in \mathrm{var}(\beta)$.

Proposition 97. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Suppose there exists a morphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ such that:*

(i) $\psi(\alpha) = \alpha$, and

(ii) $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ for some $x \in \mathrm{var}(\alpha)$.

Then there exists a morphism $\psi' : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an inner automorphism, and such that $\psi'(\alpha) = \alpha$.

Proof. For the purposes of this proof, we will say a word w is *enclosed* by a variable z if (the reduced version of) w has a prefix in $\{z, z^{-1}\}$ and a suffix in $\{z, z^{-1}\}$. Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ and let $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathbb{N}\cup\Sigma}$ be a morphism satisfying Conditions (i) and (ii) of the proposition. In particular, we have a variable $x \in \mathrm{var}(\alpha)$ such that $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$. We will construct a morphism $\rho : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_{\mathbb{N}}$ such that $\rho(\alpha) = \alpha$ (and hence such that $\rho \circ \psi(\alpha) = \alpha$), and such that $\psi' = \rho \circ \psi$ is not an inner automorphism.

We define ρ as follows. W. l. o. g., suppose that $1, 2 \in \mathrm{var}(\alpha)$.[10] Let $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$ be a word which is enclosed by 1, such that neither η nor η^{-1} are a factor of $\alpha^n x \alpha^{-n}$ for any $n \in \mathbb{Z}$.[11] Let $\rho : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_{\mathbb{N}}$ be the morphism given by

$$\rho(y) := \begin{cases} 1^k \cdot 2 \cdot \eta \cdot 2 \cdot 1^k & \text{if } y = \mathtt{a}, \\ 1^{2k} \cdot 2 \cdot \eta \cdot 2 \cdot 1^{2k} & \text{if } y = \mathtt{b}, \\ y & \text{if } y \in \mathrm{var}(\alpha), \end{cases}$$

where $k = |\psi(x)| + 1$. Clearly, since $\mathtt{a}, \mathtt{b} \notin \mathrm{var}(\alpha)$, $\rho(\alpha) = \alpha$, and hence $\rho \circ \psi(\alpha) = \rho(\alpha) = \alpha$. Thus it remains to show that $\rho \circ \psi$ is not an inner automorphism. In particular, since the only inner automorphisms fixing α are those generated by α (see Corollary 6), it is sufficient to show that for some $y \in \mathrm{var}(\alpha)$, we have $\rho \circ \psi(y) \neq \alpha^n x \alpha^{-n}$ for any $n \in \mathbb{Z}$. We will choose $y = x$, and using the fact that $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$, prove that η or η^{-1} occurs as a factor of $\rho \circ \psi(x)$. It is trivial, of course, that η or η^{-1} occurs as a factor of the unreduced image $\rho \circ \psi(x)$, however, we must show that at least one occurrence of η is not partially

[10] The case that $|\mathrm{var}(\alpha)| = 1$ is trivial, since the only morphism fixing a unary pattern is the identity and thus the conditions (i) and (ii) of the proposition cannot be satisfied.

[11] $\eta := 1^{2|\alpha|}$ or $\eta := 1 \cdot 2^{2|\alpha|} \cdot 1$ would suffice.

or entirely contracted, but rather 'survives' and is hence a factor of the reduced word $\rho \circ \psi(x)$. Therefore we must consider all possible contractions occurring in $\rho \circ \psi(x)$. Firstly, we split $\psi(x)$ into factors μ which contain only variables from $\mathbb{N}$, and factors u only containing letters from Σ. More formally, we note that there exist $\mu_0, \mu_1, \dots \mu_m \in \mathcal{F}_{\mathbb{N}}$, $u_1, u_2, \dots u_m \in \mathcal{F}_\Sigma$ such that

$$\psi(x) = \mu_0 \ \ u_1 \ \ \mu_1 \ \ u_2 \ \dots \ u_m \ \ \mu_m,$$

where each $u_i \neq \varepsilon$, $1 \leq i \leq m$, and each $\mu_i \neq \varepsilon$ for $1 \leq i \leq m-1$. Since $\psi(x)$ contains at least one letter from Σ, we have $m > 0$ (i.e., at least one u_i exists). Moreover, since $\rho(x) = x$ for all $x \in \mathbb{N}$, we have $\rho(\mu_i) = \mu_i$, and hence:

$$\rho \circ \psi(x) = \mu_0 \ \ \rho(u_1) \ \ \mu_1 \ \ \rho(u_2) \ \dots \ \rho(u_m) \ \ \mu_m. \tag{5.2}$$

It follows from the fact that $\psi(x)$ is reduced that each μ_i is reduced for $0 \leq i \leq m$. We now consider the factors $\rho(u_i)$ with the following claim:

Claim 1. *Let $u_i \in \mathcal{F}_\Sigma$. Then there exist $p_i, p_i' \in \{k, -k, 2k, -2k\}$ and $q_i, q_i' \in \{1, -1\}$ such that*

$$\rho(u_i) = 1^{p_i} \cdot 2^{q_i} \cdot w_i \cdot 2^{q_i'} \cdot 1^{p_i'}$$

where w_i is reduced, contains η or η^{-1} as a factor, and is enclosed by 1.

Proof (Claim 1). Let $u_i = \mathtt{a}_1^{q_1}\mathtt{a}_2^{q_2} \dots \mathtt{a}_\ell^{q_\ell}$ such that $\mathtt{a}_j \in \Sigma$ and $q_j \in \{1, -1\}$ for $1 \leq j \leq \ell$. We remark that since u_i is reduced, if $\mathtt{a}_j = \mathtt{a}_{j+1}$ then $q_j = q_{j+1}$. For each j, we have $\rho(\mathtt{a}_j^{q_j}) = \gamma_j \ \eta^{q_j} \ \gamma_j'$ where γ_j, γ_j' depend on $\mathtt{a}_j$ and q_j. We can therefore write the following:

$$\rho(u_i) = \overbrace{\gamma_1 \ \ \eta^{q_1} \ \ \gamma_1'}^{\rho(\mathtt{a}_1^{q_1})} \ \overbrace{\gamma_2 \ \ \eta^{q_2} \ \ \gamma_2'}^{\rho(\mathtt{a}_2^{q_2})} \ \overbrace{\gamma_3 \ \ \dots \ \ \gamma_{\ell-1}'}^{\rho(\mathtt{a}_3^{q_3} \dots \mathtt{a}_{\ell-1}^{q_{\ell-1}})} \ \overbrace{\gamma_\ell \ \ \eta^{q_\ell} \ \ \gamma_\ell'}^{\rho(\mathtt{a}_\ell^{q_\ell})}.$$

From the definition of ρ, γ_j' is comprised of either 2 or 2^{-1}, followed by a series of 1s or 1^{-1}s, while γ_{j+1} is comprised of a series of 1s or 1^{-1}s followed by 2 or 2^{-1}. More precisely, we have

$$\gamma_j' \ \gamma_{j+1} = 2^{q_j} \cdot 1^r \cdot 2^{q_{j+1}}$$

where we can infer from the definition of ρ that $r = 0$ if and only if $\mathtt{a}_j = \mathtt{a}_{j+1}$ and $q_j = -q_{j+1}$. However, this would contradict the fact that u_i is reduced, so we may assume $r \neq 0$. It follows that

$$\rho(u_i) = \gamma_1 \cdot \eta^{q_1} \cdot \overbrace{2^{q_1} \cdot 1^{r_1} \cdot 2^{q_2}}^{\gamma_1'\gamma_2} \cdot \eta^{q_2} \cdot \overbrace{2^{q_2} \cdot 1^{r_2} \cdot 2^{q_3}}^{\gamma_2'\gamma_3} \cdot \ldots \cdot \overbrace{2^{q_{\ell-1}} 1^{r_{\ell-1}} \cdot 2^{q_\ell}}^{\gamma_{\ell-1}'\gamma_\ell} \cdot \eta^{q_\ell} \gamma_\ell'$$

such that $r_j \neq 0$ for $1 \leq j < \ell$. Recall that by definition, each η^{q_i} is reduced and is enclosed by 1. Hence the above word does not contain any contractions and is thus reduced. We can observe that our claim holds simply by taking

$$w_i := \eta^{q_1} \cdot 2^{q_1} \cdot 1^{r_1} \cdot 2^{q_2} \cdot \eta^{q_2} \cdot 2^{q_2} \cdot 1^{r_2} \cdot \ldots 1^{r_{\ell-1}} \cdot 2^{q_\ell} \cdot \eta^{q_\ell}$$

and noting that $\gamma_1 = 1^{p_i} \cdot 2^{q_i}$ and $\gamma'_\ell = 2^{q'_i} \cdot 1^{p'_i}$ for some $p_i, p'_i \in \{k, -k, 2k, -2k\}$ and $q_i, q'_i \in \{1, -1\}$. □

Now, by the application of Claim 1 to each $\rho(u_i)$ in (5.2), we can write

$$\rho \circ \psi(x) = \delta_0 \; w_1 \; \delta_1 \; w_2 \; \ldots w_m \; \delta_m \tag{5.3}$$

such $\delta_0 = \mu_0 \cdot 1^{p_1} \cdot 2^{q_1}$, $\delta_m = 2^{q'_m} \cdot 1^{p'_m} \cdot \mu_m$ and for $1 \leq i < m$,

$$\delta_i = 2^{q'_i} \cdot 1^{p'_i} \cdot \mu_i \cdot 1^{p_{i+1}} \cdot 2^{q_{i+1}}$$

where $p_j, p'_j q_j, q'_j$ and w_j are defined in accordance with Claim 1 for $1 \leq j \leq m$. In particular, each w_i contains a factor η or η^{-1}, is reduced and is enclosed by 1. We now claim that the reduced δ_i are non-empty and enclosed by 2.

Claim 2. *For each i, $1 \leq i < m$, the (reduced) word δ_i is enclosed by* 2.

Proof (Claim 2). Firstly, suppose that μ_i does not consist only of 1s or 1^{-1}s. Then there exist $s_1, s_2 \in \mathbb{Z}$ such that $\mu_i = 1^{s_1} \cdot v \cdot 1^{s_2}$ where v is non-empty, reduced, and does not start or end with 1 or 1^{-1}. Hence

$$\begin{aligned} \delta_i &= 2^{q'_i} \cdot 1^{p'_i} \cdot 1^{s_1} \cdot v \cdot 1^{s_2} \cdot 1^{p_{i+1}} \cdot 2^{q_{i+1}} \\ &= 2^{q'_i} \cdot 1^{p'_i+s_1} \cdot v \cdot 1^{p_{i+1}+s_2} \cdot 2^{q_{i+1}}. \end{aligned}$$

Note that the claim holds provided $p'_i + s_1 \neq 0$ and $p_{i+1} + s_2 \neq 0$. To see that this is true, we simply recall that $|p'_i| \geq k$ and $|p_{i+1}| \geq k$, and since $k > |\psi(x)| > |\mu_i|$ we have $k > |s_1|$ and $k > |s_2|$.

Now suppose instead that μ_i *does* consist only of 1s or 1^{-1}s. Recall that by definition, $\mu_i \neq \varepsilon$. Hence there exists $s \in \mathbb{Z}\backslash\{0\}$ such that $\mu_i = 1^s$, so:

$$\begin{aligned} \delta_i &= 2^{q'_i} \cdot 1^{p'_i} \cdot 1^s \cdot 1^{p_{i+1}} \cdot 2^{q_{i+1}} \\ &= 2^{q'_i} \cdot 1^{p'_i+s+p_{i+1}} \cdot 2^{q_{i+1}}. \end{aligned}$$

Again, since $|p'_i| \geq k$ and $|p_{i+1}| \geq k$ and $k > |\psi(x)| > |\mu| > s$, we have $p'_i + s + p_{i+1} \neq 0$ and the claim follows. □

We now consider δ_0 and δ_m. Recall that $\delta_0 = \mu_0 1^{p_1} 2^{q_1}$. Since $|\mu_0| < k$ we have $\mu_0 = v1^s$ for some $s < k$ and such that v is reduced and does not end with 1 or 1^{-1}. It follows that the reduced word δ_0 equals $v1^{p_1+s}2^{q_1}$. Since $p_1 \geq k > s$, and since v does not end with 1 or 1^{-1}, there are no further possible contractions. A symmetrical argument can be made for δ_m. Hence, recalling (5.3), we have

$$\rho \circ \psi(x) = \delta_0 \; w_1 \; \delta_1 \; w_2 \; \ldots w_m \; \delta_m$$

such that, by Claim 2, each (reduced) δ_i is non empty and enclosed by 2, and by Claim 1, each w_i is reduced, contains η or η^{-1} as a factor and is enclosed by 1. Hence there are no contractions occurring outside the δ_i factors, and at least one factor η or η^{-1} survives in the reduced word $\rho \circ \psi(x)$, so $\rho \circ \psi(x) \neq \alpha^n x \alpha^{-n}$ for any $n \in \mathbb{Z}$. Hence, $\psi' = \rho \circ \psi$ is not an inner automorphism generated by (a power of) α. Since $\psi'(\alpha) = \alpha$, it follows from Corollary 6 (Chapter 2) that ψ' is not an inner automorphism and hence the proof is complete. $\square$

Hence, when $\alpha = \beta$, we can infer from Propositions 95 and 97 that τ is "as unambiguous" as $\varphi_{\tau,S}$, at least when considering ambiguity up to inner automorphism.

Corollary 98. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. Let S be a set of anchor segments for τ and let $\varphi_{\tau,S}$ be defined according to Definition 89. If $\alpha = \beta$, then τ is unambiguous up to inner automorphism with respect to β if $\varphi_{\tau,S}$ is also unambiguous up to inner automorphism with respect to β.*

Proof. Suppose that $\alpha = \beta$ and that $\varphi_{\tau,S}$ is unambiguous up to inner automorphism with respect to $\beta = \alpha$. Since $\varphi_{\tau,S}$ maps α onto itself, we also have that the identity morphism is unambiguous up to inner automorphism with respect to α. By Proposition 97, this implies that there does not exist a morphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ such that:

(i) $\psi(\alpha) = \alpha$, and

(ii) $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ for some $x \in \mathrm{var}(\alpha)$.

Consequently, by Proposition 95, every morphism $\tau' : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ mapping α onto $\sigma_{\alpha,\alpha}(\alpha)$ must have the form $\sigma_{\alpha,\alpha} \circ \varphi_{\tau',S'}$ where S' is a set of anchor segments for τ' and $\varphi_{\tau',S'}$ is defined according to Definition 89. Consequently, by Proposition 94, our statement holds. $\square$

We expect that a stronger form of Proposition 97 holds, namely that if $\alpha = \beta$, and $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ is a morphism satisfying the two conditions of Proposition 95, then we are able to produce a morphism ψ' which is not an automorphism such that $\psi(\alpha)$. In particular, we expect that if ψ is a morphism fixing α such that $\mathtt{a} \in \mathrm{symb}(\psi(x))$ for some $\mathtt{a} \in \Sigma, x \in \mathrm{var}(\alpha)$, then ψ must have a sufficiently restricted form that we may construct a morphism ρ as in Example 96 such that $\rho \circ \psi$ is not an automorphism. However this seems to be considerably more complicated to prove, and thus we must leave the statement as a conjecture.

Conjecture 99. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. If there exists a morphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha) \cup \Sigma}$ such that:*

(i) $\psi(\alpha) = \alpha$, and

(ii) $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ for some $x \in \mathrm{var}(\beta)$.

Then there exists a morphism $\psi' : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism, and such that $\psi'(\alpha) = \alpha$.

Using the reasoning from the proof of Proposition 97, we are able to construct a morphism $\rho \circ \psi$ fixing α such that for any $\eta \in \mathcal{F}_{\mathbb{N}}$, there exists $x \in \mathrm{var}(\alpha)$, such that η occurs as a factor of $\rho \circ \psi(x)$. Hence we see that our conjecture holds in the following case.

Remark 100. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ and suppose that there exists $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$ such that for every automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ with $\varphi(\alpha) = \alpha$, η does not occur as a factor of $\varphi(x)$ for any $x \in \mathrm{var}(\alpha)$. Then Conjecture 99 holds for α.*

5.4 Characterizations for Injective Morphisms

We are now ready to prove our main theorem of the chapter, and thus characterize those patterns that possess an injective morphism that is unambiguous up to inner automorphism. The proof follows from the fact that, in the case that a pattern α is only fixed by inner automorphisms (a straightforward necessary condition which we give in Proposition 57), we can directly relate the ambiguity of $\sigma_{\alpha,\alpha}$ as defined in Section 5.2 to the ambiguity of a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α (by virtue of our results in Section 5.3). Since all morphisms fixing α must be inner automorphisms, φ is unambiguous up to inner automorphism and hence so is $\sigma_{\alpha,\alpha}$.

Theorem 101. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. There exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ which is unambiguous up to inner automorphism with respect to α if and only if the identity morphism $\mathrm{id}_{\mathcal{F}_{\mathrm{var}(\alpha)}} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is unambiguous up to inner automorphism with respect to α.*

Proof. The 'only if' direction is given by Proposition 57. Hence we consider the 'if' direction. Suppose the identity morphism is unambiguous up to inner automorphism with respect to α. Then any morphism fixing α must be an inner automorphism. Let $\sigma_{\alpha,\alpha} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be defined according to Definition 87. Note that $\sigma_{\alpha,\alpha}$ is injective. Let $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\alpha) = \sigma_{\alpha,\alpha}(\alpha)$. Let S be a set of anchor segments for τ, and let $\varphi_{\tau,S} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be defined according to Definition 89. Note that by Corollary 91, $\varphi_{\tau,S}(\alpha) = \alpha$, and hence $\varphi_{\tau,S}$ must be an inner automorphism. By Proposition 95, either $\tau = \sigma_{\alpha,\alpha} \circ \varphi_{\tau,S}$, or α is fixed by a morphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathbb{N}\cup\Sigma}$ such that, for some $x \in \mathrm{var}(\alpha)$, $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$. By Proposition 97, this implies that α is fixed by a morphism which is not an inner automorphism and that is a contradiction. Consequently, for any morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ with $\tau(\alpha) = \sigma_{\alpha,\alpha}(\alpha)$, we must have that $\tau = \sigma_{\alpha,\alpha} \circ \varphi_{\tau,S}$ for some inner automorphism $\varphi_{\tau,S}$. It follows that $\sigma_{\alpha,\alpha}$ is unambiguous up to inner automorphism with respect to α. □

We can apply nearly identical reasoning for the case of unambiguity up to automorphism, however we must rely on Conjecture 99 in the place of Proposition 97.

Theorem 102. *Let $\alpha \in \mathcal{F}_\mathbb{N}$. If Conjecture 99 is true, then there exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to automorphism with respect to α if and only if the identity morphism $\mathrm{id}_{\mathcal{F}_{\mathrm{var}(\alpha)}} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is unambiguous up to automorphism with respect to α.*

Proof. Identical to that of Theorem 101, mutatis mutandis. □

Corollary 103. *Let $\alpha \in \mathcal{F}_\mathbb{N}$. If Conjecture 99 is correct, then there exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that σ is unambiguous up to automorphism with respect to α if and only if α is a test word.*

Finally, we note that if $|\mathrm{var}(\alpha)| = 2$, then the morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ mapping one variable to $\mathtt{a}$ and the other to $\mathtt{b}$ is unambiguous up to automorphism with respect to α if and only if the identity morphism is also unambiguous up to automorphism with respect to α and hence our characterization holds trivially.

Remark 104. *Let $\alpha \in \mathcal{F}_\mathbb{N}$, $|\mathrm{var}(\alpha)| = 2$. Then there exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to automorphism with respect to α if and only if the identity morphism over* $\mathrm{var}(\alpha)$ *is also unambiguous up to inner automorphism.*

We conclude the section by remarking that although there has been plenty of research into those patterns which are/are not only fixed by automorphisms (cf. Section 3.4), the same is not true for patterns which are/are not fixed only by inner automorphisms (although the C-test words given by Ivanov [38] and Lee [46] are examples). Hence, for Theorem 101 to be practically useful, it is necessary to further investigate those patterns which are only fixed by inner automorphisms.

5.5 Morphic Primitivity in a Free Group

As we have already discussed at the start of this chapter, Theorems 101 and 102 are both direct analogies to the existing theorem for the free monoid: that a pattern in $\mathbb{N}^+$ possesses an unambiguous injective morphism if and only if it is a fixed point of a non-trivial (i.e., not the identity) morphism. Since the only automorphisms of the free monoid are renamings (permutations on the individual variables), the only automorphism (and hence the only inner automorphism) fixing a pattern is indeed the identity. Thus, when considered in the context of the free monoid, both our main theorems degenerate to exactly the same statement as described above.

However, there exists another characterization for the existence of an unambiguous injective morphism for patterns in the free monoid arising from the concept of morphically primitive words, defined as follows:

Definition 105. *Reidenbach, Schneider [70] Let $\alpha \in \mathbb{N}^+$ be a pattern. If there exists a pattern $\beta \in \mathbb{N}^+$ with $|\beta| < |\alpha|$ and morphisms φ, ψ such that $\varphi(\beta) = \alpha$ and $\psi(\alpha) = \beta$, then α is* morphically imprimitive*. Otherwise, α is* morphically primitive*.*

Reidenbach, Schneider [70] state that there exists an unambiguous injective morphism with respect to a pattern α if and only if α is morphically primitive. Of course, by virtue of characterizing the same phenomenon, the morphically primitive patterns are therefore exactly those which are not fixed by a non-trivial morphism. We will see later in this section, however, that this equivalence cannot hold for both the class of patterns to which Theorem 101 applies, and the class to which Theorem 102 applies.

Hence it is the purpose of this section to investigate an analogy to this second characterization of the existence of an unambiguous injective morphism via morphically primitive words in the free monoid. Our first observation is that this definition is less well suited to the free group, due to the prominent role of the length of the words α and β. We propose the following alternative definition which, in the free monoid, is equivalent to the existing one, and which fits more naturally with the free group.

Definition 106. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ be a pattern. If there exists a pattern $\beta \in \mathcal{F}_{\mathbb{N}}$ with $|\operatorname{var}(\beta)| < |\operatorname{var}(\alpha)|$ and morphisms φ, ψ such that $\varphi(\alpha) = \beta$ and $\psi(\beta) = \alpha$, then α is* morphically imprimitive*. Otherwise, α is* morphically primitive*.*

Our first observation is the (somewhat obvious) fact that morphic primitivity in a free group as defined above is a non-trivial property. We will also make use of the given morphically primitive example later, in the proof of Proposition 109.

Proposition 107. *The pattern* $\alpha_1 := 1 \cdot 2 \cdot 2 \cdot 1^{-1}$ *is morphically imprimitive. The pattern* $\alpha_2 := 1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}$ *is morphically primitive.*

Proof. Let $\varphi_1 : \mathcal{F}_2 \to \mathcal{F}_1$ be the morphism given by $\varphi_1(1) := \varepsilon$ and $\varphi_1(2) := 1$. Let $\psi_1 : \mathcal{F}_1 \to \mathcal{F}_2$ be the morphism given by $\psi_1(1) := 1 \cdot 2 \cdot 1^{-1}$. Let $\beta_1 := 1 \cdot 1$. Clearly $\varphi_1(\alpha_1) = \beta_1$ and $\psi_1(\beta_1) = \alpha_1$, so α_1 is morphically imprimitive.

In order to see that α_2 is morphically primitive, suppose to the contrary that there exists a pattern β_2 with fewer variables (i.e., exactly one, which we may assume w. l. o. g. is 1) and a morphism $\varphi_2 : \mathcal{F}_2 \to \mathcal{F}_1$ such that $\varphi_2(\alpha_2) = \beta_2$. Let $p, q \in \mathbb{Z}$ such that $\varphi_2(1) = 1^p$ and $\varphi_2(2) = 1^q$. Then

$$\varphi_2(\alpha) = 1^p \cdot 1^q \cdot 1^{-p} \cdot 1^{-q} = 1^{p+q-p-q} = \varepsilon.$$

Thus for any morphism $\psi_2 : \mathcal{F}_1 \to \mathcal{F}_2$, $\psi_2(\beta_2) = \psi_2(\varepsilon) = \varepsilon$ and hence $\psi_2(\beta_2) \neq \alpha_2$. Consequently α is morphically primitive. □

While the two definitions are equivalent in the free monoid, the following proposition asserts that they are not in equivalent in the free group.

Proposition 108. *There exist patterns* $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$ *and morphisms* $\varphi, \psi : \mathcal{F}_{\mathbb{N}} \to \mathcal{F}_{\mathbb{N}}$ *such that* $\varphi(\alpha) = \beta$, $\psi(\beta) = \alpha$, $|\alpha| > |\beta|$ *and* $|\operatorname{var}(\alpha)| < |\operatorname{var}(\beta)|$.

Proof. Let $\alpha := (1^3 \cdot 2)^2 \cdot 1^2$, let $\beta := (1 \cdot 2 \cdot 3)^2 \cdot 1^2$, let $\varphi : \mathcal{F}_2 \to \mathcal{F}_3$ be the morphism given by $\varphi(1) := 1$, and $\varphi(2) := 1^{-2} \cdot 2 \cdot 3$, and let $\psi : \mathcal{F}_3 \to \mathcal{F}_2$ be the morphism given by $\psi(1) := 1$, $\psi(2) := 1 \cdot 1 \cdot 2$ and $\psi(3) := \varepsilon$. Then $\varphi(\alpha) = \beta$ and $\psi(\beta) = \alpha$, $|\alpha| > |\beta|$ and $|\operatorname{var}(\alpha)| < |\operatorname{var}(\beta)|$. □

Significantly, we are able to show with little effort that the concept of morphic primitivity in a free group does not characterize those patterns that possess an injective morphism which is unambiguous up to inner automorphism:

Proposition 109. *There exists a morphically primitive pattern* α *and morphism* $\varphi : \mathcal{F}_{\operatorname{var}(\alpha)} \to \mathcal{F}_{\operatorname{var}(\alpha)}$ *such that* $\varphi(\alpha) = \alpha$ *and* φ *is not an inner automorphism.*

Proof. From Proposition 107, the pattern $1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}$ is morphically primitive. Let $\varphi : \mathcal{F}_2 \to \mathcal{F}_2$ be the morphism given by $\varphi(1) := 1 \cdot 2$ and $\varphi(2) := 2$. Then

$$\varphi(1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}) = 1 \cdot 2 \cdot 2 \cdot 2^{-1} \cdot 1^{-1} \cdot 2^{-1} = 1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1},$$

and it is clear that φ is not an inner automorphism. □

Remark 110. *It can easily be verified that the pattern $\alpha := 1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}$ is also morphically primitive according to the original (length-based) definition, and thus the above statement holds regardless of which definition we use.*

Thus, for the free group, unlike the free monoid, the set of morphically primitive patterns is not equal to the set of patterns with the minimal set of morphisms fixing them. Instead we have the following result.

Theorem 111. *Let α be a pattern. Then α is morphically primitive if and only if, for every morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\alpha) = \alpha$, φ is an automorphism.*

Proof. We start by showing that if α is morphically imprimitive, it is fixed by a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism. Let $\beta \in \mathcal{F}_{\mathbb{N}}$ with $|\mathrm{var}(\beta)| < |\mathrm{var}(\alpha)|$, and suppose there exist morphisms $\psi_1 : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\beta)}$ and $\psi_2 : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\psi_1(\alpha) = \beta$ and $\psi_2(\beta) = \alpha$. Clearly the morphism $\psi_2 \circ \psi_1$ fixes α. In order to show $\psi_2 \circ \psi_1$ is not an automorphism of $\mathcal{F}_{\mathrm{var}(\alpha)}$, consider the image $\psi_2 \circ \psi_1(\mathcal{F}_{\mathrm{var}(\alpha)})$. In particular, note that $\psi_1(\mathcal{F}_{\mathrm{var}(\alpha)}) \subseteq \mathcal{F}_{\mathrm{var}(\beta)}$ and hence $\psi_2(\psi_1(\mathcal{F}_{\mathrm{var}(\alpha)})) \subseteq \psi_2(\mathcal{F}_{\mathrm{var}(\beta)}) \subset \mathcal{F}_{\mathrm{var}(\alpha)}$. Thus $\psi_2 \circ \psi_1(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \mathcal{F}_{\mathrm{var}(\alpha)}$, and $\psi_2 \circ \psi_1$ is not an automorphism.

We now prove that if α is fixed by a morphism which is not an automorphism, then it is morphically imprimitive. The main step is to observe that if α is fixed by a morphism which is not an automorphism, then it is fixed by a morphism which is not injective.

Suppose α is fixed by a morphism which is not an automorphism of $\mathcal{F}_{\mathrm{var}(\alpha)}$. Then by definition it is not a test word. From Turner [84], this implies that α belongs to a proper retract R of $\mathcal{F}_{\mathrm{var}(\alpha)}$. By definition of a retract, there exists a morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that

1. $\sigma(u) = u$ for every $u \in R$, and
2. $\sigma(v) \in R$ for every $v \in \mathcal{F}_{\mathrm{var}(\alpha)}$.

Since $R \neq \mathcal{F}_{\mathrm{var}(\alpha)}$, there exists $w \notin R$ for some $w \in \mathcal{F}_{\mathrm{var}(\alpha)}$. By Condition (2), there exists $w' \in R$ such that $\sigma(w) = w'$. By Condition (1), $\sigma(w') = w'$. As $w \notin R$ and $w' \in R$, $w \neq w'$ and thus σ is not injective. Furthermore, by Condition (1), and due to the fact that $\alpha \in R$, we have $\sigma(\alpha) = \alpha$.

Thus α is fixed by some non-injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$. Let $x_1, x_2, \ldots, x_n$ be a free basis of the image $\sigma(\mathcal{F}_{\mathrm{var}(\alpha)})$ (note that since σ is not injective, $n < |\mathrm{var}(\alpha)|$). Then there exists a morphism $\rho : \mathcal{F}_{\{x_1, x_2, \ldots, x_n\}} \to \mathcal{F}_n$ such that $\rho(x_i) = i$. Let $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_n$ be the morphism $\rho \circ \sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_n$. Let ψ be the morphism such that $\psi(i) := x_i$. Then we have $\varphi(\alpha) = \beta$ for some $\beta \in \mathcal{F}_n$, and $\psi(\beta) = \alpha$. Consequently, α is morphically imprimitive. $\square$

We can therefore conclude that the set of morphically primitive patterns in $\mathcal{F}_{\mathbb{N}}$ is exactly the set of all patterns α such that α is a test word of $\mathcal{F}_{\mathrm{var}(\alpha)}$.

Corollary 112. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Then α is morphically primitive if and only if it is a test word of $\mathcal{F}_{\mathrm{var}(\alpha)}$.*

Moreover, we have the following observation.

Corollary 113. *The set of patterns for which there exists an injective morphism which is unambiguous up to inner automorphism is a strict subset of the set of morphically primitive patterns.*

Hence we can state the following corollary which, if Conjecture 99 is correct, is an analogue to Theorem 11 from Reidenbach, Schneider [70] (cf. also Chapter 3).[12]

Corollary 114. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. If Conjecture 99 is correct, the following statements are equivalent:*

(i) *α is morphically primitive.*

(ii) *If $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is a morphism such that $\varphi(\alpha) = \alpha$, then φ is an automorphism (i.e., α is a test word).*

(iii) *There exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that σ is unambiguous up to automorphism with respect to α.*

Proof. By Theorem 111, statements (i) and (ii) are equivalent. By Theorem 102, if Conjecture 99 is correct, then statements (ii) and (iii) are equivalent. □

Hence, subject to the correctness of the conjecture, we have a remarkable similarity between the ambiguity of injective morphisms in the free monoid and in the free group – provided we consider unambiguity up to automorphism, rather than unambiguity up to inner automorphism. The distinction between our two kinds of unambiguity is interesting in itself, but perhaps more interesting is the fact that while both forms of unambiguity share an analogous characterization to the free monoid in terms of fixed points of morphisms, only unambiguity up to automorphism shares the characterization in terms of morphic primitivity. Hence the most analogous – or closest – form of unambiguity in a free group to that in the free monoid (sharing both characterizations, instead of just one), is not in fact the strongest possible form, which is unambiguity up to inner automorphism.

[12] In Reidenbach, Schneider [70], the theorem also provides a fourth statement, that α is *succinct*, which we do not replicate/consider here. Nevertheless, succinctness in a free group appears to be an interesting avenue of research in itself and it would be interesting to know whether this fourth statement is also equivalent in the free group.

5.6 Non-Injective Unambiguous Morphisms

Section 5.4 provides a characterization for those patterns which possess an injective unambiguous morphism (although for unambiguity up to automorphism, the characterization relies on a conjecture). In the current section we consider the general case, and attempt to identify those patterns for which there exists a morphism (injective or otherwise) which is unambiguous. In particular, we look for patterns for which there exists an unambiguous non-injective morphism, but for which all injective morphisms are ambiguous.

Interestingly, contrary to our arguments concerning injective morphisms, it appears to be a the harder task to consider the unambiguity of non-injective morphisms up to inner automorphism than up to automorphism – whereas in the previous sections our arguments have been considerably simpler for the former. For the remainder of the current section, we will concentrate on unambiguity up to automorphism – although we still consider the case of ambiguity up to inner automorphism to be of interest. This emphasis is also partly motivated by Proposition 42, which tells us that surprisingly, periodic morphisms may be unambiguous up to automorphism, partly by the previous section, in which we see that in a specific sense, (un)ambiguity up to automorphism provides the closest analogy to unambiguity in the free monoid, and partly by Section 5.1, in which we identify classes of patterns for which we can show that all morphisms are unambiguous up to inner automorphism, but for which we can only show that all injective morphisms are unambiguous up to automorphism.

It is indeed these classes (specifically PER and IMP) which will form the basis of our investigation and we are able to show with Theorems 118 and 125 that they do indeed contain patterns which possess a morphism which is unambiguous up to automorphism. We begin with IMP, and for the sake of convenience, we recall the definition below.

Definition 58. *We define the set $IMP \subset \mathcal{F}_{\mathbb{N}}$ as follows. Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Suppose that there exists a partition $\Delta_1, \Delta_2, \Delta_3$ of* $\mathrm{var}(\alpha)$*, $x_1, x_2, \dots x_n \in \Delta_1$, $n \geq 1$, and $\beta_0, \beta_1, \dots, \beta_n \in \mathcal{F}_{\Delta_3}$ such that:*

$$\alpha = \beta_0 \cdot \delta_{x_1}^{\pm 1} \cdot \beta_1 \cdot \delta_{x_2}^{\pm 1} \cdot \beta_2 \cdot \ldots \cdot \beta_{n-1} \cdot \delta_{x_n}^{\pm 1} \cdot \beta_n$$

where for each $x \in \Delta_1$, there exist $\gamma_x, \gamma_x' \in \mathcal{F}_{\Delta_2}$ with $\gamma_x \neq \varepsilon$ or $\gamma_x' \neq \varepsilon$ such that $\delta_x = \gamma_x \cdot x \cdot \gamma_x'$. Then $\alpha \in IMP$.

We know already from Theorem 66 that all injective morphisms are ambiguous up to automorphism with respect to any pattern $\alpha \in IMP$. Moreover, from

Theorem 67 we know that there exist patterns in IMP for which all morphisms are ambiguous up to automorphism. However, we are able to provide the following sufficient condition for a pattern in IMP to possess a (non-injective) morphism which is unambiguous up to automorphism.

Proposition 115. *Let $\alpha \in IMP$. Let $\Delta_1, \Delta_2\Delta_3$ be defined according to Definition 58. Let $\pi_{\Delta_2} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Delta_1 \cup \Delta_3}$ be the morphism such that $\pi_{\Delta_2}(x) = x$ if $x \in \Delta_1 \cup \Delta_3$ and $\pi_{\Delta_2}(x) = \varepsilon$ if $x \in \Delta_2$. Suppose there exists a morphism $\sigma' : \mathcal{F}_{\Delta_1 \cup \Delta_3} \to \mathcal{F}_\Sigma$ such that*

(i) $\sigma'(\mathcal{F}_{\Delta_1 \cup \Delta_3}) = \mathcal{F}_\Sigma$, and

(ii) σ' is unambiguous up to automorphism with respect to $\pi(\alpha)$.

Then the morphism $\sigma := \sigma' \circ \pi_{\Delta_2}$ is unambiguous up to automorphism with respect to α.

Proof. Suppose that there exists a morphism $\sigma' : \mathcal{F}_{\Delta_1 \cup \Delta_3} \to \mathcal{F}_\Sigma$ satisfying Conditions (i) and (ii). Note that $\sigma(x) := \sigma'(x)$ if $x \in \Delta_1 \cup \Delta_3$ and $\sigma(x) := \varepsilon$ otherwise. Let $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\alpha) = \sigma(\alpha)$. We will show that there exists an automorphism $\varphi_\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\tau = \sigma \circ \varphi_\tau$.

Firstly, let $\tau' : \mathcal{F}_{\Delta_1 \cup \Delta_3} \to \mathcal{F}_\Sigma$ be the morphism given by $\tau'(x) = \tau(\delta_x)$ if $x \in \Delta_1$ and $\tau'(x) = \tau(x)$ if $x \in \Delta_3$ where $\delta_x = \gamma_x \cdot x \cdot \gamma'_x$ in accordance with Definition 58. Then

$$\tau'(\pi_{\Delta_2}(\alpha)) = \sigma'(\pi_{\Delta_2}(\alpha)).$$

Moreover, since σ' is unambiguous up to automorphism with respect to $\pi_{\Delta_2}(\alpha)$, there exists an automorphism $\varphi' : \mathcal{F}_{\Delta_1 \cup \Delta_3} \to \mathcal{F}_{\Delta_1 \cup \Delta_3}$ such that $\tau' = \sigma' \circ \varphi'$.

We are now ready to construct φ_τ. By Condition (i), for each $x \in \Delta_2$, there exists $\mu_x \in \mathcal{F}_{\Delta_1 \cup \Delta_3}$ such that $\sigma'(\mu_x) = \tau(x)$. For $x \in \Delta_2$, let $\varphi_\tau(x) := x\mu_x$. For $x \in \Delta_1$, recall from Definition 58 that for each $x \in \Delta_1$, there exist $\gamma_x, \gamma'_x \in \mathcal{F}_{\Delta_2}$ such that $\delta_x = \gamma_x x \gamma'_x$. For each $x \in \Delta_1$, let $\varphi_\tau(x) := \varphi_\tau(\gamma_x)^{-1}\varphi'(x)\varphi_\tau(\gamma'_x)^{-1}$. For $x \in \Delta_3$ let $\varphi_\tau(x) = \varphi'(x)$.

We claim first that $\tau = \sigma \circ \varphi_\tau$. Consider first variables $x \in \Delta_3$. Then

$$\tau(x) = \tau'(x) = \sigma' \circ \varphi'(x) = \sigma \circ \varphi'(x) = \sigma \circ \varphi_\tau(x).$$

Consider the variables $x \in \Delta_2$. Then

$$\sigma \circ \varphi_\tau(x) = \sigma(x)\ \sigma(\mu_x) = \sigma(\mu_x) = \sigma'(\mu_x) = \tau(x).$$

Finally, consider the variables $x \in \Delta_1$. Then

$$\sigma \circ \varphi_\tau(x) = \sigma(\varphi_\tau(\gamma_x)^{-1}\varphi'(x)\varphi_\tau(\gamma'_x)^{-1}).$$

Now, $\tau(\delta_x) = \tau(\gamma_x)\tau(x)\tau(\gamma'_x) = \tau'(x)$. Hence $\tau(x) = \tau(\gamma_x)^{-1}\tau'(x)\tau(\gamma'_x)^{-1}$. By the fact that $\sigma \circ \varphi(y) = \tau(y)$ for all $y \in \Delta_2$, and $\mathrm{var}(\gamma_x)$, $\mathrm{var}(\gamma'_x) \subseteq \Delta_2$, we have that $\sigma \circ \varphi_\tau(\gamma_x)^{-1} = \tau(\gamma_x)^{-1}$ and likewise $\sigma \circ \varphi_\tau(\gamma'_x)^{-1} = \tau(\gamma'_x)^{-1}$. Thus

$$\sigma(\varphi_\tau(\gamma_x)^{-1}\varphi'(x)\varphi_\tau(\gamma'_x)^{-1}) = \tau(\gamma_x)^{-1}\sigma(\varphi'(x))\tau(\gamma'_x)^{-1}.$$

Finally, we note that since $\varphi'(x) \in \mathcal{F}_{\Delta_1 \cup \Delta_3}$, we have $\sigma(\varphi'(x)) = \sigma'(\varphi'(x)) = \tau'(x)$, and therefore:

$$\sigma \circ \varphi(x) = \tau(\gamma_x)^{-1}\tau'(x)\tau(\gamma'_x)^{-1} = \tau(x).$$

Thus it remains only to show that φ_τ is an automorphism. To show this, we first claim that for every $x \in \Delta_1 \cup \Delta_3$, we have $\varphi'(x) \in \varphi_\tau(\mathcal{F}_{\mathrm{var}(\alpha)})$. Since φ' is an automorphism of $\mathcal{F}_{\Delta_1 \cup \Delta_3}$, this implies that $x \in \varphi_\tau(\mathcal{F}_{\mathrm{var}(\alpha)})$ for every $x \in \Delta_1 \cup \Delta_3$. For $x \in \Delta_3$ our claim is trivial. For $x \in \Delta_1$, recall that $\varphi_\tau(x) = \varphi_\tau(\gamma_x)^{-1}\varphi'(x)\varphi_\tau(\gamma'_x)^{-1}$. It follows that $\varphi_\tau(\gamma_x x \gamma'_x) = \varphi'(x)$. Thus it remains to show that $x \in \varphi_\tau(\mathcal{F}_{\mathrm{var}(\alpha)})$ for each $x \in \Delta_2$. Recall that $\varphi_\tau(x) = x\mu_x$ for some $\mu_x \in \mathcal{F}_{\Delta_1 \cup \Delta_3}$. Due to the fact that for every $y \in \Delta_1 \cup \Delta_3$, $y \in \varphi_\tau(\mathcal{F}_{\mathrm{var}(\alpha)})$, there exists some $\nu_x \in \mathcal{F}_{\Delta_1 \cup \Delta_3}$ such that $\varphi_\tau(\nu_x) = \mu_x^{-1}$, and thus $\varphi_\tau(x\nu_x) = x$. We have shown that for every $x \in \mathrm{var}(\alpha), x \in \varphi_\tau(\mathcal{F}_{\mathrm{var}(\alpha)})$. It follows that $\varphi_\tau(\mathcal{F}_{\mathrm{var}(\alpha)}) = \mathcal{F}_{\mathrm{var}(\alpha)}$ and hence φ_τ is an automorphism as required. □

Although we do not give a proof of this, we expect that most, if not all morphically primitive patterns will possess such a morphism σ', and hence that Proposition 115 applies to a large class of patterns. In particular, we note that if $\pi_{\Delta_2}(\alpha)$ is morphically primitive, then by Theorem 111, the identity morphism is unambiguous up to automorphism with respect to $\pi_{\Delta_2}(\alpha)$, and consquently if $\pi_{\Delta_2}(\alpha)$ has only two variables, then so is the morphism $\sigma' : \mathcal{F}_{\Delta_1 \cup \Delta_3} \to \mathcal{F}_\Sigma$ mapping one variable to $\mathtt{a}$ and the other to $\mathtt{b}$. Hence σ' satisfies the conditions of Proposition 115 and we have the following corollary.

Corollary 116. *Let $\alpha \in IMP$ and let $\Delta_1, \Delta_2, \Delta_3$ be defined according to Definition 58. Let π_{Δ_2} be defined as in Proposition 115. If $|\,\mathrm{var}(\pi(\alpha))| = 2$ and $\pi(\alpha)$ is morphically primitive, then the morphism $\sigma = \sigma' \circ \pi_{\Delta_2}$ is unambiguous up to automorphism with respect to α.*

Example 117. *Let $\alpha := 1 \cdot 2 \cdot 1 \cdot 3 \cdot 1^{-1} \cdot 1^{-1} \cdot 2^{-1} \cdot 3^{-1}$ and let $\Delta_1 := \{2, 3\}$, $\Delta_2 := \{1\}$ and $\Delta_3 := \emptyset$. Let $\delta_2 := 1 \cdot 2 \cdot 1$, $\delta_3 := 3 \cdot 1^{-1}$ (so that, i.e., $\gamma_2 := \gamma'_2 = 1$*

and $\gamma_3 := \varepsilon$, $\gamma'_3 := 1^{-1}$). Then for $\beta_0 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \varepsilon$, we have:

$$\begin{aligned}
&\beta_0\ \delta_2\ \beta_1\ \delta_3\ \beta_2\ \delta_2^{-1}\ \beta_3\ \delta_3^{-1}\ \beta_4 \\
&= \varepsilon \cdot 1 \cdot 2 \cdot 1 \cdot \varepsilon \cdot 3 \cdot 1^{-1} \cdot \varepsilon \cdot 1^{-1} \cdot 2^{-1} \cdot 1^{-1} \cdot \varepsilon \cdot 1 \cdot 3^{-1} \\
&= 1 \cdot 2 \cdot 1 \cdot 3 \cdot 1^{-1} \cdot 1^{-1} \cdot 3^{-1} \cdot 3^{-1} \\
&= \alpha,
\end{aligned}$$

so $\alpha \in IMP$. Let $\pi_{\Delta_2} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Delta_1 \cup \Delta_3}$ be defined according to Proposition 115. Then

$$\pi_{\Delta_2}(\alpha) = 2 \cdot 3 \cdot 2^{-1} \cdot 3^{-1},$$

and from Proposition 107 we can infer that $\pi_{\Delta_2}(\alpha)$ is morphically primitive. Consequently, by Theorem 111, it is only fixed by automorphisms. It is therefore straightforward that the morphism $\sigma' : \mathcal{F}_{\Delta_1 \cup \Delta_3} \to \mathcal{F}_\Sigma$ such that $\sigma'(2) = \mathtt{a}$ and $\sigma'(3) = \mathtt{b}$ is unambiguous up to automorphism with respect to $\pi_{\Delta_2}(\alpha)$. Thus, by Proposition 115, the morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma(1) := \varepsilon$, $\sigma(2) := \mathtt{a}$ and $\sigma(3) := \mathtt{b}$ is unambiguous up to automorphism with respect to α.

As an immediate consequence of Proposition 115, we get the following Theorem, confirming that while injectivity and unambiguity are conceptually similar (see Chapter 1 for more detail on this), if our goal is purely to find an unambiguous morphism, then, for certain patterns, injectivity is a property which must be avoided.

Theorem 118. *There exists a pattern $\alpha \in \mathcal{F}_\mathbb{N}$ such that all injective morphisms are ambiguous up to automorphism with respect to α, and there exists a morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that σ is unambiguous up to automorphism with respect to α.*

Proof. Take, e.g., $\alpha := 1{\cdot}3{\cdot}2{\cdot}3^{-1}{\cdot}1^{-1}{\cdot}2^{-1}$. Let $\Delta_1 := \{1\}$,$\Delta_2 := \{3\}$ and $\Delta_3 := \{2\}$. Let $\gamma_1 := \varepsilon$ and $\gamma'_1 := 3$. Then by Definition 58, we can see that $\alpha \in IMP$. Hence by Theorem 66, all injective morphisms are ambiguous up to automorphism with respect to α. However, note that if $\pi_{\Delta_2} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Delta_1 \cup \Delta_3}$ is defined as in Proposition 115 to be the morphism erasing 3 and acting as the identity on $1, 2$, then $\pi_{\Delta_2}(\alpha) = 1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}$, which by Proposition 107 is morphically primitive. Hence by Theorem 111, the identity morphism is unambiguous up to automorphism with respect to $\pi_{\Delta_2}(\alpha)$. It is therefore trivial that the morphism $\sigma' : \mathcal{F}_{\Delta_1 \cup \Delta_3} \to \mathcal{F}_\Sigma$ given by $\sigma'(1) := \mathtt{a}$ and $\sigma' := \mathtt{b}$ is unambiguous up to automorphism with respect to α. Clearly $\sigma'(\mathcal{F}_{\Delta_1 \cup \Delta_2}) = \mathcal{F}_\Sigma$, and hence by Proposition 115, there exists a morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to automorphism with respect to α. $\square$

For the second part of our investigation into unambiguous non-injective morphisms, we consider those morphisms which are the least injective – namely periodic morphisms, and we will focus in particular on patterns with two variables. In particular, we note that a morphism $\sigma : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ is non-injective if and only if it is periodic. Remark 104 tells us that a pattern in $\mathcal{F}_2$ possesses an injective morphism which is unambiguous up to automorphism if and only if it is only fixed by automorphisms – and is thus a test word. Since binary test words are easily identified (cf. Section 3.4), if we wish to determine whether a pattern in $\mathcal{F}_2$ possesses (any) unambiguous morphism, then we must simply consider the ambiguity of periodic morphisms with respect to patterns in $\mathcal{F}_2$ – and in particular, those which are not test words. It is therefore perhaps not so surprising that there exist patterns (in $\mathcal{F}_2$) for which the only unambiguous morphisms are periodic; however, we expect that this is also true for some patterns over more than two variables, and hence that in order to fully understand ambiguity in the general case we must consider periodic morphisms.

We begin with the following straightforward observation.

Remark 119. *Let $\alpha \in \mathcal{F}_\mathbb{N}$ and let $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a periodic morphism. If there exists a non-periodic morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma(\alpha) = \tau(\alpha)$ then since $\sigma(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \tau(\mathcal{F}_{\mathrm{var}(\alpha)})$, there is no automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\sigma = \tau \circ \varphi$, and thus σ is ambiguous up to automorphism with respect to α.*

Hence since we will (only) consider morphisms $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma(\mathcal{F}_{\mathrm{var}(\alpha)}) \subseteq \mathcal{F}_{\{\mathtt{a}\}}$, we introduce the following set of patterns.

Definition 120. *Let $\Lambda \subset \mathcal{F}_\mathbb{N}$ be the set of patterns for which there exists a non-periodic morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma(\alpha) \in \mathcal{F}_{\{\mathtt{a}\}}$.*

Due to a famous result of Lyndon and Schützenberger, we can easily offer examples of patterns not in Λ.

Proposition 121. *[Lyndon, Schützenberger [49]] Let x, y, z be unknowns and let $n, m, p > 1$. Then all solutions to the equation $x^n y^m = z^p$ in a free group are periodic.*

Proposition 122. *Let $\alpha := 1^n \cdot 2^m$ for $n, m > 1$. Then $\alpha \notin \Lambda$.*

Proof. Suppose to the contrary that $\alpha \in \Lambda$, so there exists a non-periodic morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma(\alpha) = \mathtt{a}^k$ for some $k \in \mathbb{Z}$. Now if $k = 0$, then σ satisfies a non-trivial equation in two unknowns, and therefore must be periodic (cf. Lemma 3). Suppose that $|k| \geq 1$. Let $\tau : \mathcal{F}_\Sigma \to \mathcal{F}_\Sigma$ be the morphism such that $\tau(\mathtt{a}) = \mathtt{a}^2$ and $\tau(\mathtt{b}) = \mathtt{b}$. Note that since σ is non-periodic, $\tau \circ \sigma$ is non-periodic.

Moreover, $\tau \circ \sigma(1^n \cdot 2^m) = \mathtt{a}^p$ for $|p| > 1$. It follows that $\tau \circ \sigma$ corresponds to a solution to an equation of the form given in Proposition 121, and thus must be periodic which is a contradiction, and thus $\alpha \notin \Lambda$. $\square$

We now provide two classes of patterns in $\mathcal{F}_2$ which possess a periodic morphism which is unambiguous up to automorphism.

Proposition 123. *Let $k, n \in \mathbb{Z}$. If $\alpha \in \mathcal{F}_2 \backslash \Lambda$ and $\mathrm{P}(\alpha) = \mathrm{P}\left((1 \cdot 2^k)^n\right)$, then there exists a periodic morphism which is unambiguous up to automorphism with respect to α.*

Proof. Let $\sigma : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ be the morphism given by $\sigma(1) := \mathtt{a}$ and $\sigma(2) := \varepsilon$. Since $\alpha \notin \Lambda$, all morphisms $\tau : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ with $\sigma(\alpha) = \tau(\alpha)$ are periodic. Thus the set of all morphisms which agree with σ on α is given by:

$$S := \{\tau_m : \mathcal{F}_2 \to \mathcal{F}_\Sigma \mid \tau_m(1) := \mathtt{a}^{1-mk}, \tau_m(2) := \mathtt{a}^m, m \in \mathbb{Z}\}.$$

For each $m \in \mathbb{Z}$ let $\varphi_m : \mathcal{F}_2 \to \mathcal{F}_2$ be the morphism given by $\varphi_m(1) := 1 \cdot (2 \cdot 1^m)^{-k}$ and $\varphi_m := 2 \cdot 1^m$. Then $\sigma \circ \varphi_m(1) = \mathtt{a}^{1-km}$ and $\sigma \circ \varphi_m(2) = \mathtt{a}^m$. Thus for any $\tau_m \in S$, we have $\tau_m = \sigma \circ \varphi_m$. It therefore remains to show that φ_m is an automorphism. Let $\beta := 1 \cdot 2^{-k}$. Then $\varphi_m(\beta) = 1$. Furthermore, $\varphi_m(2 \cdot \beta^{-m}) = 2$. Thus $\varphi(\mathcal{F}_2) = \mathcal{F}_2$ and φ_m is an automorphism as required.

Hence we have that, for any $\tau : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ with $\tau(\alpha) = \sigma(\alpha)$, $\tau = \sigma \circ \varphi_m$ for some automorphism φ_m, so σ is unambiguous up to automorphism with respect to α. $\square$

Our first class, while general, does not contain any patterns in PER. However, we are able to provide a second, similar class which contains many patterns from PER.

Proposition 124. *Let $k, n \in \mathbb{Z}$. If $\alpha \in \mathcal{F}_2 \backslash \Lambda$ and $\mathrm{P}(\alpha) = p\left((1^{k-1} \cdot 2^k)^n\right)$, then there exists a periodic morphism which is unambiguous up to automorphism with respect to α.*

Proof. Let $\sigma : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ be the morphism given by $\sigma(1) := \mathtt{a}$ and $\sigma(2) := \mathtt{a}$. Since $\alpha \notin \Lambda$, all morphisms $\tau : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ with $\sigma(\alpha) = \tau(\alpha)$ are periodic. Furthermore, since $k-1$ and k are co-prime, the set of all morphisms which agree with σ on α is given by:

$$S := \{\tau_m : \mathcal{F}_2 \to \mathcal{F}_\Sigma \mid \tau_m(1) := \mathtt{a}^{1-mk}, \tau_m(2) := \mathtt{a}^{1+(k-1)m}, m \in \mathbb{Z}\}.$$

For each $m \in \mathbb{Z}$ let $\varphi_m : \mathcal{F}_2 \to \mathcal{F}_2$ be the morphism given by $\varphi_m(1) := 1 \cdot (2 \cdot 1^{m-1})^{-k}$ and $\varphi_m(2) := 1 \cdot (2 \cdot 1^{m-1})^{k-1}$. Then $\sigma \circ \varphi_m(1) = \mathtt{a}^{1-km}$ and $\sigma \circ \varphi_m(2) =$

$\mathtt{a}^{1+(k-1)m}$. Thus for any $\tau_m \in S$, we have $\tau_m = \sigma \circ \varphi_m$. It therefore remains to show that φ_m is an automorphism. Let $\beta := 2 \cdot (2^{-1} \cdot 1)^{k-1}$. Then $\varphi_m(\beta) = 1$. Furthermore, $\varphi_m(1^{-1} \cdot 2 \cdot \beta^{1-m}) = 2$. Thus $\varphi(\mathcal{F}_2) = \mathcal{F}_2$ and φ_m is an automorphism as required.

Hence we have that, for any $\tau : \mathcal{F}_2 \to \mathcal{F}_\Sigma$ with $\tau(\alpha) = \sigma(\alpha)$, $\tau = \sigma \circ \varphi_m$ for some automorphism φ_m, so σ is unambiguous up to automorphism with respect to α. □

Hence we can observe that there exist patterns in PER for which there exists a morphism which is unambiguous up to automorphism. In fact we have the following, stronger statement, which confirms that, if we wish to determine whether a pattern permits an unambiguous morphism, we must account for periodic morphisms.

Theorem 125. *There exists a pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$ such that there exists a periodic morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to automorphism with respect to α, and for which every non-periodic morphism is ambiguous up to automorphism with respect to α.*

Proof. Let $\alpha := 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2$. Then $\alpha \in PER$, and hence by Theorem 66, all injective morphisms are ambiguous up to automorphism with respect to α. Since all binary non-periodic morphisms are injective, this implies that any morphism which is unambiguous up to automorphism is periodic. To show that such an unambiguous morphism exists, we note that by Proposition 122, $\alpha \notin \Lambda$ and hence by Proposition 124, there exists a (periodic) morphism which is unambiguous up to automorphism with respect to α. □

Our interest in binary patterns is driven by Remark 104: since we already have a characterization of when there exists an unambiguous morphism up to automorphism with respect to a binary pattern, it remains only to consider non-injective patterns in the binary case. However, we expect that periodic morphisms have relevance for the study of unambiguity up to automorphism for patterns over all alphabets. Hence we note that our reasoning can be easily generalized to patterns over larger alphabets. We provide the following example.

Proposition 126. *Let $n, k \in \mathbb{Z}$. If $\alpha \in \mathcal{F}_{\mathbb{N}} \setminus \Lambda$ and $\mathrm{P}(\alpha) = \mathrm{P}\left((1 \cdot 2^k)^n\right)$, then there exists a periodic morphism which is unambiguous up to automorphism with respect to α.*

Proof. W. l. o. g. let $\mathrm{var}(\alpha) = \{1, 2, \ldots, p\}$ for some $p \geq 2$.[13] Let $\sigma : \mathcal{F}_p \to \mathcal{F}_\Sigma$ be the morphism given by $\sigma(1) := \mathtt{a}$ and $\sigma(i) := \varepsilon$ for $1 < i \leq p$. Since $\alpha \notin \Lambda$,

[13] Note that we can have, e.g., $\mathrm{var}(\alpha) = \{1, 2, 3\}$ if 3 and 3^{-1} both appear in α an equal number of times, such as in the pattern $1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 3^{-1}$.

all morphisms $\tau : \mathcal{F}_p \to \mathcal{F}_\Sigma$ with $\sigma(\alpha) = \tau(\alpha)$ are periodic. Thus all morphisms $\tau : \mathcal{F}_p \to \mathcal{F}_\Sigma$ such that $\tau(\alpha) = \sigma(\alpha)$ have the form:

$$\tau(x) := \begin{cases} \mathtt{a}^{1-m_2k} & \text{if } x = 1, \\ \mathtt{a}^{m_x} & \text{if } 2 \leq x \leq p, \end{cases}$$

where $m_x \in \mathbb{Z}$ for $2 \leq x \leq p$. We now show that for any such morphism τ, there exists an automorphism $\varphi : \mathcal{F}_p \to \mathcal{F}_p$ such that $\tau = \sigma \circ \varphi$. In particular, we define φ as follows:

$$\varphi(x) := \begin{cases} 1 \cdot (2 \cdot 1^{m_2})^{-k} & \text{if } x = 1, \\ x \cdot 1^{m_x} & \text{if } 2 \leq x \leq p. \end{cases}$$

Now, we have

$$\sigma \circ \varphi(1) = \sigma(1 \cdot (2 \cdot 1^{m_2})^{-k}) = \mathtt{a} \cdot (\mathtt{a}^{m_2})^{-k} = \mathtt{a}^{1-m_2k},$$

and for $2 \leq x \leq p$, we have:

$$\sigma \circ \varphi(x) = \sigma(x \cdot 1^{m_x}) = \mathtt{a}^{m_x}.$$

Thus $\tau = \sigma \circ \varphi$. It therefore remains to show that φ is an automorphism of $\mathcal{F}_p$. We note that, for $\beta := 1 \cdot 2^{-k}$:

$$\varphi(\beta) = 1,$$
$$\varphi(x \cdot \beta^{-m_x}) = x$$

for $2 \leq x \leq p$.Thus $\varphi(\mathcal{F}_p) = \mathcal{F}_p$, and φ is therefore an automorphism. Moreover, we have shown that, for any $\tau : \mathcal{F}_p \to \mathcal{F}_\Sigma$ with $\tau(\alpha) = \sigma(\alpha)$, $\tau = \sigma \circ \varphi$ for some automorphism φ, so σ is unambiguous up to automorphism with respect to α. □

So for instance, we see that Proposition 126 applies to e.g., the pattern $\alpha := (1^2 \cdot 2^4)^2 \cdot 3 \cdot (1^2 \cdot 2^4)^4 \cdot 3^{-1}$. In fact, referring to the morphism we use in the proof, it is not difficult to produce such patterns α over any set of variables for which the morphism mapping one variable to $\mathtt{a}$ and the rest to ε is unambiguous up to automorphism. Such a morphism is as non-injective as possible, with the exception of the morphism erasing all variables which is always ambiguous up to automorphism.

5.7 Application: Properties of Pattern Languages

In the final part of this chapter, we take advantage of our construction from Section 5.2 to provide some simple proofs of properties of terminal-free pattern languages over a group alphabet. Firstly, we are able to characterize when two such languages satisfy a subset relation. It is unsurprising that our construction leads to this result, as it is a generalization of the construction of Jiang et al. [42] whose purpose was exactly to prove the equivalent statement for pattern languages in a free monoid.

Theorem 127. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. Then $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$ if and only if there exists a morphism $\varphi : \mathcal{F}_{\mathbb{N}} \to \mathcal{F}_{\mathbb{N}}$ such that $\varphi(\beta) = \alpha$.*

Proof. Suppose firstly that there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\beta) = \alpha$. By definition, for every $w \in L_\Sigma(\alpha)$, there exists $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma(\alpha) = w$. Clearly, $\sigma \circ \varphi(\beta) = w$, so for every $w \in L_\Sigma(\alpha)$, we have $w \in L_\Sigma(\beta)$ and thus $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$.

Now suppose that $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$. Then since $\sigma_{\alpha,\beta}(\alpha) \in L_\Sigma(\alpha)$ where $\sigma_{\alpha,\beta}$ is defined according to Definition 87, there exists a morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. By Theorem 92, this implies the existence of $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\beta) = \alpha$. □

Of course our characterization of the inclusion problem for group pattern languages automatically provides a characterisation for the equivalence problem.

Corollary 128. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. Then $L_\Sigma(\alpha) = L_\Sigma(\beta)$ if and only if there exist morphisms $\varphi, \psi : \mathcal{F}_{\mathbb{N}} \to \mathcal{F}_{\mathbb{N}}$ such that $\varphi(\beta) = \alpha$ and $\psi(\alpha) = \beta$.*

Slightly less straightforward is our characterization of when the union of two pattern languages is again a pattern language. It is known that pattern languages in a free monoid are generally not closed under the traditional language-theoretic operations such as union, intersection and complement (cf. Section 3.2 and [8]) so we expect that normally the union of two (group) pattern languages is not again a (group) pattern language. In actual fact, we are able to show that the union is again a pattern language only in the trivial case that one is a subset of the other.

Theorem 129. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. Then there exists $\gamma \in \mathcal{F}_{\mathbb{N}}$ satisfying $L_\Sigma(\alpha) \cup L_\Sigma(\beta) = L_\Sigma(\gamma)$ if and only if $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$ and $L_\Sigma(\beta) = L_\Sigma(\gamma)$, or $L_\Sigma(\beta) \subseteq L_\Sigma(\alpha)$ and $L_\Sigma(\alpha) = L_\Sigma(\gamma)$.*

Proof. The 'if' direction is trivial. We consider the 'only if' direction. Suppose that $L_\Sigma(\alpha) \cup L_\Sigma(\beta) = L_\Sigma(\gamma)$. Let $\sigma_{\gamma,\alpha}$ and $\sigma_{\gamma,\beta}$ be defined according to Definition 87. In particular, note that there exist $k_1, k_2 \in \mathbb{N}$ such that $\sigma_{\gamma,\alpha} = \sigma_{k_1,\mathrm{var}(\gamma)}$ (as defined in Definition 83) and $\sigma_{\gamma,\beta} = \sigma_{k_2,\mathrm{var}(\gamma)}$. Let $k := \max(k_1, k_2)$. Note that for the union relation to hold, there exists a morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\tau(\alpha) = \sigma_{k,\mathrm{var}(\gamma)}(\gamma)$ or a morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{k,\mathrm{var}(\gamma)}(\gamma)$. W.l.o.g. suppose that $\tau(\alpha) = \sigma_{k,\mathrm{var}(\gamma)}(\gamma)$ for some morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$. By Theorem 92 and Remark 93, this implies that there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\gamma)}$ such that $\varphi(\alpha) = \gamma$. It follows from Theorem 127 that $L_\Sigma(\gamma) \subseteq L_\Sigma(\alpha)$. It is clear that $L_\Sigma(\alpha) \subseteq L_\Sigma(\gamma)$ and $L_\Sigma(\alpha) \subseteq L_\Sigma(\gamma)$, and hence by Corollary 128 $L_\Sigma(\alpha) = L_\Sigma(\gamma)$, and our statement holds. □

Remark 130. *We can apply the same reasoning using either the morphism from Jiang et al. [42], or by adapting our own morphism $\sigma_{\alpha,\beta}$ to the free monoid in order to prove an equivalent statement to Theorem 129 for terminal-free erasing pattern languages in a free monoid (i.e., over Σ^*).*[14]

[14] This result is presented for the pattern languages in a free monoid in [8].

Chapter 6

Words with Maximal Unambiguity

So far, we have discussed the existence of unambiguous morphisms – addressing the question of whether, for a pattern α, we may construct a morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that σ is unambiguous[1] with respect to α.

In the present chapter, we investigate patterns for which not only one, but (almost) all morphisms are unambiguous. We have seen already that most[2] periodic morphisms are ambiguous with respect to any given pattern. Therefore we ask whether all non-periodic morphisms are unambiguous, and thus, roughly speaking, whether a pattern has the maximal amount of unambiguous morphisms.

While the questions we have considered in the previous chapter have already been thoroughly answered in the free monoid (cf. Chapter 3) – and consequently we have so-far remained entirely within the context of free groups – this is not the case for our focus in the current chapter, and we shall therefore begin with patterns in the free monoid $\mathbb{N}^+$. In fact, this forms the bulk of our remaining exposition. Specifically, in Section 6.1, we shall consider so-called *periodicity forcing words* – words which do not satisfy the *Dual Post Correspondence Problem* defined, as follows:

(Culik II, Karhumäki [5]). *For a given word w, does w belong to an equality set $E(g, h)$ for two morphisms g, h, where at least one morphism is non-periodic?*

Hence, a pattern does not satisfy the Dual PCP if and only if all ambiguous morphisms are periodic.[3] For example, the word $1 \cdot 2 \cdot 2 \cdot 1$ belongs to $\mathrm{E}(g, h)$

[1] Since we have been considering this question in the free group, we ask that σ is unambiguous up to inner automorphism or up to automorphism (cf. Chapter 4).

[2] This statement follows from basic number theory for the free monoid. For the free group we have seen that all periodic morphisms are ambiguous up to inner automorphism (cf. Proposition 34), and most periodic morphisms are ambiguous up to automorphism (cf Proposition 43).

[3] There are actually two subtly different definitions of the Dual PCP in literature (which we

where g, $h : \{1,2\}^* \to \{1,2\}^*$ are the morphisms given by:

$$g(x) := \begin{cases} 1 \cdot 2 \cdot 1 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \end{cases} \quad \text{and} \quad h(x) := \begin{cases} 1 & \text{if } x = 1, \\ 2 \cdot 1 \cdot 2 & \text{if } x = 2. \end{cases}$$

Thus $1 \cdot 2 \cdot 2 \cdot 1$ satisfies the Dual PCP; and is not a periodicity forcing word. In contrast, the word $1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ does not satisfy the Dual PCP (cf. Culik II, Karhumäki [5]), and hence is a periodicity forcing word – although proving this requires a little more effort.

In general, despite being closely linked to the extensively studied Post Correspondence Problem (PCP), the satisfaction of the Dual PCP as a property of words has not been widely investigated – particularly for words over more than two letters, and although it is decidable in theory, this relies on Makanin's algorithm for solving (systems) of word equations and is not practical in general.[4] Moreover, while examples of words which do satisfy the Dual PCP (sometimes called equality words) are easy to find, this is not true for periodicity forcing words. We therefore focus our attention on generating examples of periodicity forcing words, and try to determine what examples of periodicity forcing words may look like, how prevalent they are, and whether they are as scarce as one might initially expect.

Perhaps not surprisingly, there exists a similar notion to that of periodicity forcing words in the free group – although it comes from a different context. So-called C-test words are words α such that for all morphisms $\sigma, \tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$, if $\sigma(\alpha) = \tau(\alpha) \neq \varepsilon$, then there exists an inner automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\tau = \sigma \circ \varphi$. In Lee [46], a class of C-test words α is presented with the additional property that $\sigma(\alpha) = \varepsilon$ for some morphism σ if and only if σ is periodic.

Hence, for such C-test words α, all non-periodic morphisms are unambiguous up to inner automorphism (and therefore also up to automorphism) with respect to α. In Section 6.2, we first show that some of our methods used for generating examples of periodicity forcing words can be effortlessly adapted to produce new examples of (general) C-test words. Then, as an application of C-test words, we prove several results on the ambiguity of terminal-preserving morphisms (cf. Section 4.4). In particular, we show that for the set of variables $\mathbb{N}$ and set of terminal symbols Σ there exist patterns $\alpha \in \mathcal{F}_{\mathbb{N} \cup \Sigma}$ such that every terminal-preserving morphism is unambiguous with respect to α, as well as patterns for which only some, or no morphisms are unambiguous. We are also able to adapt

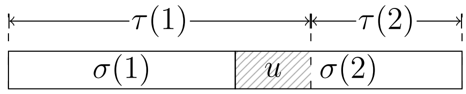
expect are actually identical, although we do not have a proof of this). We give the original one here. For a more detailed introduction to the Dual PCP, refer to Chapter 3.

[4] Although the complexity of solving word equations has been shown to be considerably simpler than Makanin's algorithm suggests, this does not necessarily translate immediately to Dual PCP, as we explain in Section 3.3.

our reasoning to show that, surprisingly, for any terminal-preserving morphism $\sigma : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_{\Sigma}$, there exists a pattern $\alpha \in \mathcal{F}_{\mathbb{N}\cup\Sigma}$ such that σ is the only morphism such that $\sigma(\alpha) = \varepsilon$. Hence we see not only that morphisms which completely erase patterns may be unambiguous (in stark contrast to the terminal-free case, where such a morphism is always ambiguous, even up to automorphism), but also that such a property does not depend at all on the morphism, but only on the pre-image pattern α.

6.1 Periodicity Forcing Words

It is not difficult to find examples of words which do satisfy the Dual PCP (i.e., words which are not periodicity forcing). For instance, given any morphism $\sigma : \{1, 2\}^* \to \{\mathtt{a},\mathtt{b}\}^*$, there exists another morphism $\tau : \{1, 2\}^* \to \{\mathtt{a},\mathtt{b}\}^*$ such that $\sigma(1 \cdot 2) = \tau(1 \cdot 2)$. The second morphism may be constructed from the first by simply transferring a prefix u of $\sigma(2)$ to the end of $\sigma(1)$ or vice versa.



Since this construction is possible for all morphisms, σ may be chosen to be non-periodic, so the word $\alpha := 1 \cdot 2$ satisfies the Dual PCP. The same is true for words with the same, or similar structural features – such as the word $1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 2 \cdot 2$. A substantial generalization of such structures exists in the form of morphically imprimitive words, all of which satisfy the Dual PCP (cf. Sections 3.1 and 3.3). For convenience, we shall refer to the set of patterns over $\mathbb{N}$ which satisfy the Dual PCP as DPCP:

Let DPCP $:= \{\alpha \in \mathbb{N}^+ |$ *there exist a non-periodic morphism* $\sigma : \mathrm{var}(\alpha)^* \to \Sigma^*$ *and a morphism* $\tau : \mathrm{var}(\alpha)^* \to \Sigma^*$ *such that* $\sigma(\alpha) = \tau(\alpha)\}$. *Consequently, the set of periodicity forcing words is given by* DPCP$^\neg$.

In many cases however, whether or not a word is in DPCP is much less obvious. For example, consider $1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 \cdot 1$. In [5], Culik II and Karhumäki demonstrate that it belongs to the non-trivial equality set $\mathrm{E}(\sigma, \tau)$ where $\sigma : \{1, 2\}^* \to \{\mathtt{a},\mathtt{b}\}^*$ is the morphism given by

$$\sigma(1) = \mathtt{aabaa},\ \sigma(2) = \mathtt{baaaab},$$

and $\tau : \{1, 2\}^* \to \{\mathtt{a},\mathtt{b}\}^*$ is the morphism given by

$$\tau(1) = \mathtt{a},\ \tau(2) = \mathtt{baaaabaabaaaab}.$$

Notice that σ is non-periodic, so the Dual PCP is satisfied by $1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 \cdot 1$.

$$\overbrace{\mathtt{a\,a\,b\,a\,a}}^{\sigma(1)}\overbrace{\mathtt{a\,a\,b\,a\,a}}^{\sigma(1)}\overbrace{\mathtt{b\,a\,a\,a\,a\,b}}^{\sigma(2)}\overbrace{\mathtt{b\,a\,a\,a\,a\,b}}^{\sigma(2)}\overbrace{\mathtt{a\,a\,b\,a\,a}}^{\sigma(1)}\overbrace{\mathtt{a\,a\,b\,a\,a}}^{\sigma(1)}$$

$$\underbrace{\mathtt{a}}_{\tau(1)}\underbrace{\mathtt{a}}_{\tau(1)}\underbrace{\mathtt{b\,a\,a\,a\,a\,b\,a\,a\,b\,a\,a\,a\,a\,b}}_{\tau(2)}\underbrace{\mathtt{b\,a\,a\,a\,a\,b\,a\,a\,b\,a\,a\,a\,a\,b}}_{\tau(2)}\underbrace{\mathtt{a}}_{\tau(1)}\underbrace{\mathtt{a}}_{\tau(1)}$$

Now, consider the (very) similar word $1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$. It seems clear that the second word is structurally very close to the first; however, it is shown by Czeizler et al. in [6] that the word equation $u^2v^3u^2 = w^2x^3w^2$ is only satisfied if u, v, w, x are repetitions of the same word. This is equivalent to the statement that all morphisms which agree on $1^2 \cdot 2^3 \cdot 1^2$ are periodic, and thus the word $1^2 \cdot 2^3 \cdot 1^2$ is not in DPCP. The paper is a good example of how difficult it can be to solve certain word equations and hence determine the boundary between words in, and not in DPCP.

In the remainder of this section, we consider two approaches for producing periodicity forcing words which bypass some of the difficulties associated with solving word equations. The first is based on morphisms which preserve the property of being periodicity forcing (i.e., morphisms φ such that if $\alpha \in \mathrm{DPCP}^{\neg}$, then $\varphi(\alpha) \in \mathrm{DPCP}^{\neg}$). It turns out that such morphisms are easily found, and as a result we can immediately produce a wealth of periodicity forcing words, especially over two variables. There are two major advantages to this approach: the first is that a single morphism may be applied iteratively to many pre-images to produce large classes of examples, and the second is that it is easy to use such morphisms to produce ratio-primitive examples – for which it is usually much harder to classify using more direct word-equations based approaches. Recall that a pattern is ratio-imprimitive (i.e., not ratio-primitive) if it has a proper prefix with the same basic Parikh vector as the whole pattern. In particular, this allows the agreement of morphisms to be broken down to the agreement on both the prefix and complementary suffix, often meaning that the corresponding word equations become easier to solve. We shall give more detail on classifying ratio-imprimitive examples in Section 6.1.3.

In Section 6.1.1, we are able to show that there exist ratio-primitive periodicity forcing words over any set of variables Δ. We also show a more surprising level of generality, that periodicity forcing words exist which have any given prefix/suffix/factor. Given the number and variety of morphisms which preserve the property of being periodicity forcing, it is possible to span large parts of the set $\mathrm{DPCP}^{\neg}$ simply by applying these morphisms to existing examples. In Section 6.1.2, we examine this phenomenon in more detail. Specifically, $\mathrm{DPCP}^{\neg}$ is divided into those words which may be reached by a non-trivial morphism from other elements of the set, and those which cannot. The latter form a "prime" subset of DPCP$^{\neg}$ from which all periodicity forcing words may be generated. We show firstly that this set of prime words is non-empty, and secondly that it is sufficient to span the entire set. As a by-product of our investigation, we are also able to provide bounds on the length of the shortest periodicity forcing words over a specific alphabet which is, in itself, of fundamental interest to understanding the Dual PCP.

Of course, the existence of prime periodicity forcing words means by definition that not all periodicity forcing words may be obtained as morphic images of others. Hence we must also consider alternative methods, and in Section 6.1.3, we do this using periodicity forcing sets, generalizing a technique established by Culik II, Karhumäki [5]. Similarly to our "morphic" approach, we are able to produce wide classes of patterns over all alphabet sizes – although it is an inherent property of the technique that the examples we generate are all ratio-imprimitive. Nevertheless, we are also able to gain further insights into the set of prime periodicity forcing words and surprisingly, we see that unlike the property of being periodicity forcing itself, being a prime periodicity forcing word can vary between powers (i.e., there exists a pattern α and $k \in \mathbb{N}$ such that α is prime while α^k is not).

Finally, as an application of periodicity forcing words, we consider the closure under intersection of pattern languages. We are able to show that, like many other similar standard language operations, the terminal-free E pattern languages are not closed under intersection.

6.1.1 Preserving Ambiguity of Non-Periodic Morphisms under Composition

We begin our investigation into periodicity forcing words with the following lemma[5] which establishes a criterion for a morphism $\varphi : {\Delta_1}^* \to {\Delta_2}^*$ such that if α is a periodicity forcing word with $\mathrm{var}(\alpha) = \Delta_1$, then $\varphi(\alpha)$ is also a periodicity forcing word. In actual fact, the criterion guarantees that if $\varphi(\alpha) \in$ DPCP, then $\alpha \in$ DPCP – from which our desired statement follows. To achieve this, we require that the morphism φ preserves the properties of *distinctness* and *periodicity* under composition. Specifically, if $\sigma, \tau : {\Delta_2}^* \to \Sigma^*$ are distinct, we wish to guarantee that $\sigma \circ \varphi$ and $\tau \circ \varphi$ are distinct, and if σ is non-periodic then so is $\sigma \circ \varphi$.

If, for such a morphism φ, $\varphi(\alpha)$ possesses an ambiguous, non-periodic morphism $\sigma : {\Delta_2}^* \to \Sigma^*$, then we can show also that α possesses an unambiguous, non-periodic morphism. If $\alpha \notin$ DPCP, this is a contradiction and so we must

[5]The lemma was already published in the current form by the author prior to the start of the period of study for this thesis, and as such, the author does not take credit for the lemma in the context of the current thesis. The same is true of Proposition 141.

have $\varphi(\alpha) \notin \mathrm{DPCP}$. More precisely, we reason that if there exist $\tau : {\Delta_2}^* \to \Sigma^*$ such that σ, τ are distinct and $\sigma(\varphi(\alpha)) = \tau(\varphi(\alpha))$ – then the morphisms $\sigma \circ \varphi$ and $\tau \circ \varphi$ are distinct and $\sigma \circ \varphi$ is non-periodic. By the basic properties of morphisms, $\sigma \circ \varphi(\alpha) = \tau \circ \varphi(\alpha)$, so it follows that $\sigma \circ \varphi$ is non-periodic and ambiguous with respect to α. Since periodicity forcing words are those which do not possess an ambiguous non-periodic morphism, this means that $\varphi(\alpha)$ is periodicity forcing whenever α is periodicity forcing.

$$\begin{tikzcd} \alpha \arrow[r, "\varphi"] \arrow[rr, bend left=60, "\sigma' \circ \varphi"] \arrow[rr, bend right=60, "\tau' \circ \varphi"'] & \beta \arrow[r, bend left, "\sigma'"] \arrow[r, bend right, "\tau'"'] & w \end{tikzcd}$$

By identifying morphisms which satisfy these conditions, we are able to produce new examples of periodicity forcing words from existing ones. The primary advantage of this approach is that unlike with other methods (such as described in Section 6.1.3), it is straightforward to construct examples which are ratio-primitive. We shall also see that, for relatively little effort, we are able to establish large classes of examples, for many of which it would be far more difficult to classify using more direct approaches based on word equations.

Lemma 131 ([12]). *Let Δ_1, Δ_2 be sets of variables. Let $\varphi : {\Delta_1}^* \to {\Delta_2}^*$ be a morphism such that, for every $x \in \Delta_2$, there exists a $y \in \Delta_1$ satisfying $x \in \mathrm{var}(\varphi(y))$, and*

(i) *for every non-periodic morphism $\sigma : {\Delta_2}^* \to \{\mathtt{a}, \mathtt{b}\}^*$, $\sigma \circ \varphi$ is non-periodic, and*

(ii) *for all distinct morphisms σ, $\tau : {\Delta_2}^* \to \{\mathtt{a}, \mathtt{b}\}^*$, where at least one is non-periodic, $\sigma \circ \varphi$ and $\tau \circ \varphi$ are distinct.*

Then, for any $\alpha \in \mathrm{DPCP}^\neg$ with $\mathrm{var}(\alpha) = \Delta_1$, $\varphi(\alpha) \in \mathrm{DPCP}^\neg$.

By a straightforward application of the definitions, Condition (ii) is satisfied if and only if the set $S := \{\varphi(x) \mid x \in \Delta_1\}$ is periodicity forcing. Moreover, since a set of patterns commutes if and only if each pair of patterns in the set commutes, by Corollary 5, the morphism $\sigma \circ \varphi : \Delta_1 \to \{\mathtt{a}, \mathtt{b}\}^*$ is periodic if and only if the set $\{\sigma(\varphi(x)) \mid x \in \Delta_1\}$ commutes. Hence, condition (i) is satisfied if and only if the set S is *commutativity forcing*, defined as follows:

A set S of patterns from ${\Delta_2}^*$ is commutativity forcing if, for every morphism $\sigma : \Delta_2^* \to \Sigma^*$ such that the set $\{\sigma(\beta) \mid \beta \in S\}$ commutes, the set $\{\sigma(x) \mid x \in \Delta_2\}$ also commutes (and hence σ is periodic).

Therefore, put formally, we have the following remark on morphisms satisfying Lemma 131.[6]

Remark 132. *Let Δ_1, Δ_2 be sets of variables and let $\varphi : {\Delta_1}^* \to {\Delta_2}^*$ be a morphism. Then φ satisfies Condition (i) (resp. Condition (ii)) of Lemma 131 if and only if the set $\{\varphi(x)) \mid x \in \Delta_1\}$ is commutativity forcing (resp. periodicity forcing).*

It is surprisingly easy to produce sets which are both periodicity forcing and commutativity forcing. For example, it is straightforward to show that the following set satisfies an even stronger property: that two morphisms agree on every pattern in the set if and only if they are identical.

Example 133. *Let $S := \{1, 1\cdot 2, 1\cdot 2\cdot 3\}$. Let $\sigma, \tau : \{1, 2, 3\}^* \to \Sigma^*$ be morphisms, and suppose that $\sigma(\beta) = \tau(\beta)$ for every $\beta \in S$. Then, since $\sigma(1) = \tau(1)$, and $\sigma(1 \cdot 2) = \tau(1 \cdot 2)$, we have $\sigma(2) = \tau(2)$ and likewise we can infer that $\sigma(3) = \tau(3)$. Thus S is a periodicity forcing set and, by similar reasoning we can infer that S is also commutativity forcing. Consequently, for instance, the morphism $\varphi : \{1, 2, 3\}^* \to \{1, 2, 3\}^*$ given by $\varphi(1) := 1$, $\varphi(2) := 1 \cdot 2$ and $\varphi(3) := 1 \cdot 2 \cdot 3$ satisfies the conditions of Lemma 131.*

It is just as straightforward to show that e.g., all 1-uniform morphisms, projections and the Fibonacci morphism(s)[7], as well as many other intuitive classes of morphism satisfy the conditions of Lemma 131. On the other hand, no periodic morphism φ satisfies Condition (i) of the lemma, since for a periodic morphism φ and non-periodic morphism σ, the morphism $\sigma \circ \varphi$ is periodic. Nevertheless, this is not a problem, as it is easy to see that periodic morphisms are of limited interest for our current investigations:

Remark 134. *Let Δ_1, Δ_2 be sets of variables and let $\varphi : {\Delta_1}^* \to {\Delta_2}^*$ be a periodic morphism (so that for some $\gamma \in {\Delta_2}^*$, $\varphi(x) \in \{\gamma\}^*$ for all $x \in \Delta_1$). Then for any pattern $\alpha \in {\Delta_1}^+$, $\varphi(\alpha) = \gamma^k$ for some $k \in \mathbb{N}_0$. Since a word is periodicity forcing if and only if its primitive root is periodicity forcing, $\varphi(\alpha)$ is periodicity forcing if and only if $k > 0$ and γ is periodicity forcing.*

Thus we cannot use periodic morphisms φ to produce 'new' examples. On the other hand, for non-periodic morphisms – at least in the case that $|\Delta_2| = 2$, we shall see that Lemma 131 is characteristic in the sense that for a periodicity forcing word α, $\varphi(\alpha)$ is periodicity forcing if and only if φ satisfies the two conditions of the lemma. Firstly, we note that such morphisms always satisfy Condition (i) of the lemma:

[6] A more thorough analysis may be found in [12].
[7] This insight is attributed to P. C. Bell.

Proposition 135. *Let Δ_1, Δ_2 be sets of variables such that $|\Delta_2| = 2$ and let $\varphi : {\Delta_1}^* \to {\Delta_2}^*$ be a non-periodic morphism. Then φ satisfies Condition (i) of Lemma 131.*

Proof. Let $S := \{\varphi(x) \mid x \in \Delta_1\}$, and recall from Remark 132 that φ satisfies Condition (i) of the lemma if and only if S is commutativity forcing. Suppose to the contrary that S is not commutativity forcing, and hence there exists a non-periodic morphism $\sigma : {\Delta_2}^* \to \Sigma^*$ and word $w \in \Sigma^*$ such that $\sigma(\varphi(x)) \in \{w\}^*$ for all $x \in \Delta_1$. Consequently, there exist $k_1, k_2, \dots, k_n \in \mathbb{N}_0$ such that

$$\sigma(\varphi(x_1))^{k_1} = \sigma(\varphi(x_2))^{k_2} = \cdots = \sigma(\varphi(x_n))^{k_n}$$

where $\Delta_1 = \{x_1, x_2, \dots, x_n\}$. This is equivalent to the statement that the system of word equations

$$\varphi(x_1)^{k_1} = \varphi(x_2)^{k_2} = \cdots = \varphi(x_n)^{k_n}$$

has a non-periodic solution. However, since $|\Delta_2| = 2$, the above system of equations is over two unknowns (i.e., the two variables in Δ_2). Thus by Lemma 4, it only has a non-periodic solution if each equation is trivial (i.e., $\varphi(x_i)^{k_i} = \varphi(x_j)^{k_j}$ for all $1 \leq i < j \leq n$). However this implies that φ is periodic which is a contradiction. $\square$

Proposition 136. *Let Δ_1, Δ_2 be sets of variables such that $|\Delta_2| = 2$. Let α be a periodicity forcing word with $\mathrm{var}(\alpha) = \Delta_1$ and let $\varphi : {\Delta_1}^* \to {\Delta_2}^*$ be a non-periodic morphism. Then $\varphi(\alpha)$ is periodicity forcing if and only if φ satisfies the conditions of Lemma 131.*

Proof. The if direction is given by Lemma 131 itself. Thus we consider the 'only if' direction. Suppose that φ does not satisfy both conditions of the lemma. By Proposition 135, φ satisfies Condition (i), so we may assume that φ does not satisfy Condition (ii). Then by Remark 132, the set $S := \{\varphi(x) \mid x \in \Delta_1\}$ is not periodicity forcing. Consequently, there exist two morphisms $\sigma, \tau : {\Delta_2}^* \to \Sigma^*$ such that $\sigma \neq \tau$, and σ is non-periodic, such that $\sigma(\varphi(x)) = \tau(\varphi(x))$ for all $x \in \Delta_1$. Hence $\sigma \circ \varphi = \tau \circ \varphi$ and therefore $\sigma(\varphi(\alpha)) = \tau(\varphi(\alpha))$. Since σ is non-periodic, and $\sigma \neq \tau$, this implies that $\varphi(\alpha)$ is not a periodicity forcing word. $\square$

It follows from Propositions 135 and 136 that for alphabets Δ_1, Δ_2 with $|\Delta_2| = 2$, a non-periodic morphism $\varphi : \Delta_1^* \to \Delta_2^*$ satisfies the conditions of Lemma 131 if and only if the set $\{\varphi(x) \mid x \in \Delta_1\}$ is periodicity forcing. In particular, recalling Theorem 17 from Section 3.3, we get the following:

Corollary 137. *Let Δ_1, Δ_2 be sets of variables with $|\Delta_2| = 2$ and let $\varphi : {\Delta_1}^* \to {\Delta_2}^*$ be a non-periodic morphism. Then φ satisfies the conditions of the Lemma 131 if and only if $\{\varphi(x) \mid x \in \Delta_1\} \neq \{xy^n, y^n x\}$ for any $n \in \mathbb{N}$ and $\{x, y\} = \Delta_2$.*

Example 138. *Let $\varphi_1 : \{1,2\}^* \to \{1,2\}^*$ be the morphism given by $\varphi_1(1) := 1 \cdot 2$, $\varphi_1(2) := 2 \cdot 1 \cdot 1$, let $\varphi_2 : \{1,2\}^* \to \{1,2\}^*$ be the morphism given by $\varphi_2(1) := 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1$ and $\varphi_2(2) := 1 \cdot 1$, and let $\varphi_3 : \{1,2\}^* \to \{1,2\}^*$ be the morphism given by $\varphi_3(1) := 1 \cdot 2^{10}$ and $\varphi_3(2) := 2^{10} \cdot 1$. Then φ_1 and φ_2 both satisfy the conditions of Lemma 131, while φ_3 does not.*

Hence we have a simple and effective characterization for all patterns over two variables which are the images of periodicity forcing words under a non-periodic morphism. In particular we have the following statement. Note that we strengthen one direction slightly by observing that a pattern belonging to $\{1 \cdot 2^n, 2^n \cdot 1\}^* \cup \{2 \cdot 1^n, 1^n \cdot 2\}^*$ for any $n \in \mathbb{N}$ is not periodicity forcing (cf. Corollary 18).

Corollary 139. *Let $\alpha \in \mathbb{N}$ be a periodicity forcing word, and let $\varphi : \mathrm{var}(\alpha)^* \to \{1,2\}^*$ be a non-periodic morphism. Then $\varphi(\alpha)$ is periodicity forcing if and only if $\varphi(\alpha) \notin \{1 \cdot 2^n, 2^n \cdot 1\}^* \cup \{2 \cdot 1^n, 1^n \cdot 2\}^*$ for any $n \in \mathbb{N}$.*

To demonstrate the wealth of patterns to which Corollary 139 applies, we provide a small selection of the more simple morphic images of the periodicity forcing word $\alpha = 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$.

Example 140. *The patterns $1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1$, $1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 2$, $1 \cdot 2 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2$, $1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2$, $1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1$, $1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1$ and $1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1$ are all morphic images of the periodicity forcing word $\alpha = 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$, and thus, by Corollary 139, are also all periodicity forcing words. On the other hand, the pattern $1 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 1$ is also a morphic image of α, but belongs to $\{1 \cdot 2, 2 \cdot 1\}^*$ and is consequently not a periodicity forcing word. In fact, the only binary morphic images of α which are not periodicity forcing words, are those mapped to by the morphisms $\varphi_n : \{1,2\}^* \to \{1,2\}^*$ such that $\varphi_n(1) := 1 \cdot 2^n$ and $\varphi_n(2) := 2^n \cdot 1$ and their renamings, of which there are 2 of length 11, and 4 of lengths 16,21,26,31, etc. This is in contrast to other morphic images of α, of which there are a far greater number.*

One class of morphisms for which it is considerably harder to find examples satisfying the conditions of Lemma 131 is for those which increase the number of variables (i.e., such that $|\mathrm{var}(\varphi(\alpha))| > |\mathrm{var}(\alpha)|$). Nevertheless, we have the following construction from [12].

Proposition 141 ([12]). *Let* $\varphi_n : \{1, 2, ..., n\}^* \to \{1, 2, ...n, n+1\}^*$ *be the morphism such that* $\varphi_n(1) := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ *and* $\varphi_n(x) := 1 \cdot (x+1) \cdot 1 \cdot 1 \cdot (x+1) \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1$ *for* $2 \leq x \leq n$*. Then* φ_n *satisfies the conditions of Lemma 131.*

It is, of course, a straightforward consequence that we can construct periodicity forcing words over arbitrary alphabets, and perhaps more interestingly, we are able to construct examples which are ratio-primitive, and therefore less suited to classification/identification by the more direct word-equations based methods.

Theorem 142. *Let* Δ *be a finite subset of* $\mathbb{N}$*. Then there exists a ratio-primitive periodicity forcing word* β *such that* $\mathrm{var}(\beta) = \Delta$*.*

Proof. W. l. o. g. we shall assume that $\Delta = \{1, 2, \ldots, n\}$. The case that $|\Delta| = 1$ is trivial since all patterns over a single variable are periodicity forcing. for the case that $|\Delta| = 2$ recall that $\alpha_2 := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ is periodicity forcing, and clearly ratio-primitive. For $n \geq 2$, let $\varphi_n : \{1, 2, ..., n\}^* \to \{1, 2, ...n, n+1\}^*$ be the morphism such that $\varphi_n(1) := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ and $\varphi_n(x) := 1 \cdot (x+1) \cdot 1 \cdot 1 \cdot (x+1) \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1$ for $2 \leq x \leq n$. By Proposition 141, the morphisms φ_n satisfy the conditions of Lemma 131, so the patterns $\alpha_k := \varphi_{k-1}(\alpha_{k-1})$, $k \geq 3$ are all periodicity forcing. The fact that each α_k is ratio-primitive may be observed as follows. Firstly, it is clear that for each k, the set of vectors $\{\mathrm{P}(\varphi_{k-1}(x)) \mid 1 \leq x \leq k-1\}$ is linearly independent. It follows from basic linear algebra that if α_{k-1} is ratio-primitive, then α_k is also ratio-primitive. Since α_2 is ratio-primitive, our statement follows by induction. $\square$

For example, for $\Delta = \{1, 2, 3\}$, we have that $\varphi_2 : \{1, 2\}^* \to \{1, 2, 3\}^*$ is the morphism such that $\varphi_2(1) := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ and $\varphi_2(2) := 1 \cdot 3 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1$, and for e.g., $\alpha := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$, the pattern $\beta = \varphi_2(\alpha)$ is as follows:

$$1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1.$$

A further immediate consequence is that we are also able to construct, for any pattern β, a periodicity forcing set containing β. For example, if $\beta' \in \mathrm{DPCP}^{\neg}$ and $\mathrm{var}(\beta) = \mathrm{var}(\beta')$, then $\{\beta, \beta'\}$ is periodicity forcing. More generally, the addition of a periodicity forcing word over an appropriate alphabet is sufficient to turn any finite set of patterns into a periodicity forcing set. Thus we have a high degree of freedom when producing sets which are periodicity forcing, and therefore also morphisms satisfying Lemma 131. In particular, we are able to construct, for any given pattern β, a morphism φ and pre-image α' such that the pattern $\alpha := \varphi(\alpha')$ is periodicity forcing and contains β as a factor, prefix or suffix.

Recall from Remark 132 that in order to guarantee that φ satisfies the conditions given in Lemma 131, the set $\{\varphi(x) \mid x \in \mathrm{var}(\alpha)\}$ must not only be periodicity forcing, but also commutativity forcing – i. e. every morphism σ such that the words $\sigma(\varphi(x))$, $x \in \mathrm{var}(\alpha)$, commute is periodic. A construction satisfying this condition is given in the next proposition.

Proposition 143. *Let α_0 be a pattern, and let $n := \lceil \log_2(|\mathrm{var}(\alpha_0)|) \rceil$. There exist patterns α_1, α_2, ..., α_n with $\mathrm{P}(\alpha_0) = \mathrm{P}(\alpha_1) = \cdots = \mathrm{P}(\alpha_n)$ such that $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ is commutativity forcing.*

Proof. Consider the case that $|\mathrm{var}(\alpha_0)| = 2^n$. The case that this is not true may easily be adapted.[8] W. l. o. g. let α_0 be in canonical form, and note that this implies that α_0 can be expressed as $\gamma_1 \cdot \gamma_2 \cdots \gamma_m$ where $m = |\mathrm{var}(\alpha_0)|$, and $\gamma_i := i \cdot \beta_i$ for some pattern $\beta_i \in \{1, 2,..., i\}^*$. For $i \leq n$, let α_i be the pattern obtained from α_0 by 'swapping' adjacent factors consisting of 2^{i-1} consecutive patterns γ_j, i.e.,

$$\alpha_1 = \gamma_2 \cdot \gamma_1 \cdot \gamma_4 \cdot \gamma_3 \cdots \gamma_{m-1} \cdot \gamma_m$$

$$\alpha_2 = \gamma_3 \cdot \gamma_4 \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_{m-1} \cdot \gamma_m \cdot \gamma_{m-3} \cdot \gamma_{m-2}$$

$$\vdots$$

$$\alpha_n = \gamma_{\frac{m}{2}+1} \cdot \gamma_{\frac{m}{2}+2} \cdots \gamma_m \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_{\frac{m}{2}}$$

Note that $\mathrm{P}(\alpha_0) = \mathrm{P}(\alpha_1) = \cdots = \mathrm{P}(\alpha_n)$, so for any morphism σ, we have that

$$|\sigma(\alpha_0)| = |\sigma(\alpha_1)| = \cdots = |\sigma(\alpha_n)|.$$

Thus, the system of word equations

$$\alpha_i \alpha_j = \alpha_j \alpha_i$$

for all i, j with $0 \leq i < j \leq n$ is equivalent to the simpler system

$$\alpha_0 = \alpha_1 = \cdots = \alpha_n.$$

It is now shown that all solutions to the above system of word equations are periodic. Let $\sigma : \{1, 2, ..., n\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ be an arbitrary solution, and consider

[8] One way to see that this holds, is to increase the number of variables in α_0 to 2^n by e.g., simply adding them on the end, and noting that if the resulting set of patterns α_0, α_1, α_2, ..., α_n are commutativity forcing and have the same Parikh vector, then the same is true for the set of patterns α'_0, α'_1, α'_2, ..., α'_n obtained by removing the "extra" variables.

the equality $\alpha_0 = \alpha_1$. This is equivalent to

$$\sigma(\gamma_1) \cdot \sigma(\gamma_2) \cdots \sigma(\gamma_m) = \sigma(\gamma_2) \cdot \sigma(\gamma_1) \cdots \sigma(\gamma_m) \cdot \sigma(\gamma_{m-1}).$$

By comparing the prefix of length $|\sigma(\gamma_1)| + |\sigma(\gamma_2)|$ on either side, $\sigma(\gamma_1)\sigma(\gamma_2) = \sigma(\gamma_2)\sigma(\gamma_1)$. By Corollary 5, it follows that there exists a primitive word $w_1 \in \{\mathtt{a}, \mathtt{b}\}^*$ such that $\sigma(\gamma_1)$, $\sigma(\gamma_2) \in \{w_1\}^*$. A similar argument may be made for the next, and indeed every pair of patterns γ_j, γ_{j+1} where $j < m$ is odd. Thus, for $1 \leq i \leq \frac{m}{2}$, there exists a primitive word $w_i \in \{\mathtt{a}, \mathtt{b}\}^*$ such that $\sigma(\gamma_{2i-1})$, $\sigma(\gamma_{2i}) \in \{w_i\}^*$. Moreover, by the equation $\alpha_1 = \alpha_2$, it is possible to employ the same argument to determine that for $1 \leq i \leq \frac{m}{4}$, the words w_{2i-1} and w_{2i} are equal. By continuing this argument for each successive equality $\alpha_j = \alpha_{j+1}$, it follows that $w_1 = w_2 = \cdots = w_{\frac{m}{2}}$, so there exists a primitive word $w \in \{\mathtt{a}, \mathtt{b}\}^*$ such that $\sigma(\gamma_i) \in \{w\}^*$ for all $1 \leq i \leq m$.

Since $\gamma_1 \in 1^+$, this implies $\sigma(1) \in \{w\}^*$. Assume that $\sigma(1)$, $\sigma(2)$, ..., $\sigma(r) \in \{w\}^*$ for some $1 \leq r < m$. Then since $\sigma(\gamma_{r+1}) \in \{w\}^*$ and $\gamma_{r+1} \in \{1, 2, ..., r+1\}^+$ and $r + 1 \in \text{var}(\gamma_{r+1})$, by Lemma 4, $\sigma(r + 1) \in \{w\}^*$. Thus, by induction, $\sigma(x) \in \{w\}^*$ for all $1 \leq x \leq m$, and σ is periodic. □

It is now possible to show that for any given pattern β, there exists a periodicity forcing word with β as a factor.

Theorem 144. *For any pattern $\beta \in \mathbb{N}^+$, there exists a pattern $\alpha \notin$ DPCP such that β is a factor/prefix/suffix of α.*

Proof. It is known from [12] that there exists a pattern $\beta_1 \notin$ DPCP such that $\text{var}(\beta) = \text{var}(\beta_1)$. By Proposition 143, there exist patterns β_2, β_3, ..., β_n with $\text{P}(\beta) = \text{P}(\beta_2) = \cdots = \text{P}(\beta_n)$ such that the set $\{\beta, \beta_2, \cdots, \beta_n\}$[9] is commutativity forcing. Since $\text{var}(\beta) = \text{var}(\beta_i)$ for $1 \leq i \leq n$, it follows that the augmented set $\{\beta, \beta_1, \beta_2, \cdots, \beta_n\}$ is commutativity forcing. Furthermore, since β_1 is periodicity forcing, the set is also periodicity forcing. Thus the morphism $\varphi : \{1, 2, ..., n + 1\}^* \to \text{var}(\beta)^*$ given by $\varphi(i) := \beta_i$ for $1 \leq i \leq n$ and $\varphi(n + 1) := \beta$ satisfies both conditions of Lemma 131. From [12], there exists a pattern $\alpha' \notin$ DPCP such that $\text{var}(\alpha') = \{1, 2, ..., n + 1\}$, and by Lemma 131, $\alpha := \varphi(\alpha') \notin$ DPCP. Since $\beta = \varphi(n + 1)$ and $n + 1 \in \text{var}(\alpha')$, β is a factor of α as required. The case that β is a prefix (resp. suffix) of α can be shown simply by instead using a renaming of α' for which the variable $n + 1$ occurs at as a prefix (resp. suffix). □

[9] We note that this construction also works for β_1 (so we get the set $\{\beta_1, \beta_2, \ldots \beta_n\}$), however, since periodicity forcing words are generally long, it is more efficient for e.g., the next example, if we construct the patterns $\beta_2, \beta_3, \ldots, \beta_n$ around the factor β instead.

Example 145 demonstrates how φ, and therefore α may be constructed in the case that $\beta = 1 \cdot 1 \cdot 2 \cdot 3$.

Example 145. *Let* $\beta := 1 \cdot 1 \cdot 2 \cdot 3$. *Let* $\beta_1 := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1$. *By [12] (Proposition 32),* $\beta_1 \notin \mathrm{DPCP}$. *By Proposition 143, there exist patterns* β_2, β_3 *such that* $\mathrm{P}(\beta) = \mathrm{P}(\beta_2) = \mathrm{P}(\beta_3)$, *and* $\{\beta, \beta_2, \beta_3\}$ *is a commutativity forcing set. In particular, using the construction given in the proof of Proposition 143 we obtain* $\beta_2 := 2 \cdot 3 \cdot 1 \cdot 1$, *and* $\beta_3 := 3 \cdot 1 \cdot 1 \cdot 2$. *It is easy to verify that these patterns satisfy the condition, as any morphism* σ *will map* β, β_2 *and* β_3 *to words of the same length. Thus,*

$$\begin{aligned} \sigma(311)\sigma(2) &= \sigma(2)\sigma(311) \\ &= \sigma(11)\sigma(23) = \sigma(23)\sigma(11) \\ &= \sigma(112)\sigma(3) = \sigma(3)\sigma(112) \end{aligned}$$

implying that $\sigma(1)$, $\sigma(2)$ *and* $\sigma(3)$ *commute, and hence* σ *is periodic. It follows that the extended set* $\{\beta_1, \beta, \beta_2, \beta_3\}$ *is commutativity forcing. Hence the morphism* $\varphi : \{1, 2, 3, 4\}^* \to \{1, 2, 3\}^*$ *given by* $\varphi(i) := \beta_i$ *for* $1 \leq i \leq 3$ *and* $\varphi(4) := \beta$ *satisfies Condition (ii) of Lemma 131. Since* $\beta_1 \in \mathrm{DPCP}^\neg$, *the set of patterns* $\{\beta, \beta_1, \beta_2, \beta_3\}$ *is also periodicity forcing and thus* φ *satisfies Condition (ii) of the Lemma.*

Let α' *be a pattern in* $\mathrm{DPCP}^\neg$ *with* $\mathrm{var}(\alpha) = \{1, 2, 3, 4\}$. *Then by Lemma 131,* $\alpha := \varphi(\alpha') \in \mathrm{DPCP}^\neg$. *Moreover it is clear that* β *appears as a factor of* α.

Using this approach, it is straightforward to choose the morphisms φ and pre-images α' such that $\alpha = \varphi(\alpha')$ is ratio-primitive. We conclude the current section by noting that many examples of periodicity forcing words may be produced by applying morphisms to existing ones, and by doing so we are able to produce a surprisingly wide range of both ratio-primitive and ratio-imprimitive patterns. In the following section, we continue this focus on morphisms between periodicity forcing words, but with an emphasis on the set $\mathrm{DPCP}^\neg$ as a whole.

6.1.2 Prime PFWs and a Morphic Structure of $DPCP^\neg$

One consequence of our constructions of morphisms satisfying Lemma 131 is that for every periodicity forcing word α, there exists another (fundamentally different) periodicity forcing word which is a morphic image of α.

Corollary 146 ([12])**.** *Let* $\alpha \in \mathrm{DPCP}^\neg$. *Then there exists a morphism* $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ *which is not a renaming morphism, such that* $\varphi(\alpha) \in \mathrm{DPCP}^\neg$.

Although this statement is itself fairly easily obtained, and comes as no surprise, it is worth noting the richness and variety in such morphisms φ (which are characterized in [12]), and therefore also in the subsequent patterns $\varphi(\alpha)$ which can be obtained through the application of morphisms. Thus an obvious question arises: is every periodicity forcing word the morphic image of another?

As we have briefly mentioned in the introduction of this chapter, the answer is trivially affirmative if φ is permitted to be a renaming morphism (such as the identity), or if α can be unary (every pattern is a morphic image of $\alpha :=$ 1). However, if we restrict α and φ to avoid these trivial instances, the answer is no longer clear. In fact, a negative answer is provided by Proposition 151 below. Hence, the partition of periodicity forcing words into those which are morphic images of another, and those which are not, is non-trivial. We will call the latter *prime*. It is reasonable to expect that these prime periodicity forcing words are sufficient, given the appropriate set of morphisms, to generate the full set. However such a conclusion is far from immediate, since we must account for potentially (two-sided) infinite chains of morphic (pre-)images. Thus we ask whether we can "trace" any periodicity forcing word back, via its morphic pre-images to eventually reach a prime periodicity forcing word, or whether some periodicity forcing words have infinitely many (non-trivial) morphic pre-images. We show later (cf. Theorem 152) that such two-sided infinite chains cannot exist, and consequently that every periodicity forcing word is the morphic image of a prime periodicity forcing word.

The proofs of these results rely on a lower bound for the length of periodicity forcing words, given relative to the alphabet size. This bound is achieved by considering patterns belonging to the equality sets of (pairs of) "nearly periodic morphisms" σ – of the form

$$\sigma(x) := \begin{cases} \mathtt{a}^r\ \mathtt{ba}^s & \text{if } x = y, \\ \mathtt{a}^{p_x} & \text{otherwise,} \end{cases}$$

where y is some fixed variable, and r, s, $p_x \in \mathbb{N}_0$. It is apparent that the equality set of two morphisms σ_1 and σ_2 of this type is determined by a system of linear Diophantine equations, and in the case that y is the same for both morphisms, it is possible to infer a strong sufficient condition for a pattern to belong to such an equality set. Since the morphisms are non-periodic, any such pattern is not periodicity forcing.

Proposition 147. *Let α be a pattern, and let $n := |\operatorname{var}(\alpha)|$. Suppose that $|\alpha|_x < n$ for some $x \in \operatorname{var}(\alpha)$. Then $\alpha \in$ DPCP.*

Proof. Consider a pattern α such that $\operatorname{var}(\alpha) = \{x_1, x_2, ..., x_n\}$, and $|\alpha|_{x_i} < n$ for some $i \leq n$. W.l.o.g. let $i := n$. Then there exists a $k \in \mathbb{N}$ such that $|\alpha|_{x_n} = n-k$, and α can be written as $\beta_1 \cdot x_n \cdot \beta_2 \cdot x_n \cdot ... \cdot \beta_{n-k} \cdot x_n \cdot \beta_{n-k+1}$ for some patterns β_1, β_2, ..., $\beta_{n-k+1} \in \{x_1, x_2, ..., x_{n-1}\}^*$.

Consider the morphisms σ, $\tau : \{x_1, x_2, ..., x_n\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ given by

$$\sigma(x_i) := \begin{cases} \mathtt{a}^{r_1}\ \mathtt{b}\mathtt{a}^{s_1} & \text{if } i = n, \\ \mathtt{a}^{p_i} & \text{otherwise,} \end{cases} \quad \text{and } \tau(x_i) := \begin{cases} \mathtt{a}^{r_2}\ \mathtt{b}\mathtt{a}^{s_2} & \text{if } i = n, \\ \mathtt{a}^{q_i} & \text{otherwise,} \end{cases}$$

for some $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n, r_1, r_2, s_1, s_2 \in \mathbb{N}_0$. Clearly, σ and τ are non-periodic, provided $p_i \neq 0$ and $q_j \neq 0$ for some i, j respectively. For $1 \leq i < n$ let $t_i := p_i - q_i$, let $r := r_2 - r_1$, and let $s := s_2 - s_1$. Then $\sigma(\alpha) = \tau(\alpha)$ if and only if the following system of equations is satisfied:

$$t_1|\beta_1|_{x_1} + t_2|\beta_1|_{x_2} + \cdots + t_{n-1}|\beta_1|_{x_{n-1}} = r$$

$$t_1|\beta_2|_{x_1} + t_2|\beta_2|_{x_2} + \cdots + t_{n-1}|\beta_2|_{x_{n-1}} = r + s$$

$$\vdots$$

$$t_1|\beta_{n-k}|_{x_1} + t_2|\beta_{n-k}|_{x_2} + \cdots + t_{n-1}|\beta_{n-k}|_{x_{n-1}} = r + s$$

$$t_1|\beta_{n-k+1}|_{x_1} + t_2|\beta_{n-k+1}|_{x_2} + \cdots + t_{n-1}|\beta_{n-k+1}|_{x_{n-1}} = s.$$

Since r, s, t_1, ..., $t_{n-1} \in \mathbb{N}_0$ depend on the definition of σ and τ, they may be chosen freely, and $|\beta_i|_{x_j}$ for $1 \leq i \leq n-k+1$ and $1 \leq j \leq n-1$ depend on α so they are fixed. Notice that σ and τ are distinct if and only if $s \neq 0$ or $r \neq 0$ or there exists an i such that $t_i \neq 0$. Thus, σ and τ can be chosen such that they are distinct, non-periodic and agree on α if there exists a non-trivial solution (t_1, t_2, ..., t_{n-1}).

Let $f_{i,j} := |\beta_i|_{x_j} - |\beta_1|_{x_j} - |\beta_{n-k+1}|_{x_j}$ for $1 \leq i < n$ and $2 \leq n-k$. Then our system can be written as follows:

$$t_1 f_{2,1} + t_2 f_{2,2} + \cdots + t_{n-1} f_{2,n-1} = 0$$

$$t_1 f_{3,1} + t_2 f_{3,2} + \cdots + t_{n-1} f_{3,n-1} = 0$$

$$\vdots$$

$$t_1 f_{n-k,1} + t_2 f_{n-k,2} + \cdots + t_{n-1} f_{n-k,n-1} = 0$$

This is a system of $n-k-1$ homogeneous equations in $n-1$ unknowns with integer coefficients, $k \geq 1$, and therefore there exists a non-trivial integer solution (t_1, t_2, ... t_{n-1}). Since r and s can be chosen freely, such a solution is always a solution to the first system for some integers r, s. Thus σ and τ can be chosen such that they are distinct, non-periodic and agree on α. Consequently, $\alpha \in \mathrm{DPCP}$ whenever $|\alpha|_x < n$ for some $x \in \mathrm{var}(\alpha)$. $\square$

It follows that, for a periodicity forcing word with n variables, each variable must occur at least n times, implying the next corollary which provides a lower bound on the length of the shortest periodicity forcing word for any alphabet size.

Corollary 148. *Let* $\alpha \notin \mathrm{DPCP}$*, and let* $n := |\mathrm{var}(\alpha)|$*. Then* $|\alpha| \geq n^2$*.*

Since periodicity forcing words can be obtained as concatenations of words in a particular type of periodicity forcing set (see Section 6.1.3), it is possible to infer a corresponding upper bound from results by Holub, Kortelainen [35]. The authors provide a concise test set (containing at most $5n$ words, each of length n) for the set S_n consisting of all permutations of the word $x_1 \cdot x_2 \cdots x_n$. Although it is stated in [35] that S_n itself is not periodicity forcing, it can be verified using results from [35] and Culik II, Karhumäki [5] that the augmented set ${S_n}' := S_n \cup \{x_1 \cdot x_1 \cdot x_2 \cdot x_2 \cdots x_n \cdot x_n\}$ is. Given a test set T_n for S_n, a test set for ${S_n}'$ is clearly $T_n \cup \{x_1 \cdot x_1 \cdot x_2 \cdot x_2 \cdots x_n \cdot x_n\}$. Thus, there exists a test set for ${S_n}'$ containing at most $5n$ words of length n and one word of length $2n$. The periodicity forcing word resulting from concatenating these words is at most $5n^2 + 2n$ letters long.

Proposition 149. *Let* α_n *be a shortest pattern in* $\mathrm{DPCP}^{\neg}$ *such that* $|\mathrm{var}(\alpha)| = n$*. Then* $n^2 \leq |\alpha| \leq 5n^2 + 2n$*.*

The lower bounds are particularly useful when considering prime elements of $\mathrm{DPCP}^{\neg}$, which we define formally below.

Definition 150. *Let* $\alpha \in \mathrm{DPCP}^{\neg}$ *be a pattern with* $|\mathrm{var}(\alpha)| \geq 2$*. Then* α *is said to be a* prime element of $\mathrm{DPCP}^{\neg}$ *(or simply* prime*) if for every pattern* $\beta \in \mathrm{DPCP}^{\neg}$ *with* $|\mathrm{var}(\beta)| > 1$*, and every morphism* $\varphi : \mathrm{var}(\beta)^* \to \mathrm{var}(\alpha)^*$*,* $\varphi(\beta) = \alpha$ *implies that* φ *is a renaming morphism.*

Showing that a pattern satisfies Definition 150 is, in general, a highly non-trivial task, since all morphisms must be accounted for with respect to every pattern $\beta \in \mathrm{DPCP}^{\neg}$. However, due to Proposition 147, it is possible to provide a relatively simple example:

Proposition 151. *The pattern* $\alpha := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ *is a prime element of* $\mathrm{DPCP}^{\neg}$*.*

Proof. It is known from Culik II, Karhumäki [5] that α is periodicity forcing. Assume that $\beta \in \mathrm{DPCP}^{\neg}$ is a pattern, and that $\varphi : \mathrm{var}(\beta)^* \to \mathrm{var}(\alpha)^*$ is a morphism such that $\varphi(\beta) = \alpha$. Due to the fact that $|\alpha|_2 = 2$, there exists a variable $x \in \mathrm{var}(\beta)$ such that $|\beta|_x \leq 2$. Hence, by Proposition 147, $|\mathrm{var}(\beta)| = 2$. Since α is primitive, φ is non-erasing and thus $|\beta| \leq 5$. Furthermore, all periodicity forcing words of length at most 5 are given by Culik II, Karhumäki [5], so it is possible to determine by inspection that no non-renaming morphism exists which maps any of these patterns to α, and thus Definition 150 is satisfied. $\square$

By the same argument, the patterns $1 \cdot 2 \cdot 1 \cdot 2 \cdot 2$, $1 \cdot 1 \cdot 2 \cdot 1 \cdot 2$ and $1 \cdot 2 \cdot 2 \cdot 1 \cdot 2$ are also prime.

As mentioned earlier, while Proposition 151 settles the question of whether every periodicity forcing word is the morphic image of another in a non-trivial way, the negative answer induces a second question: what is the smallest subset of $\mathrm{DPCP}^{\neg}$ required to span the full set via the application of morphisms? Clearly such a subset is strict (this follows from Corollary 146), and must be a superset of the set of prime elements of $\mathrm{DPCP}^{\neg}$.

In order to answer this question, it is necessary to determine whether there exist infinite chains of patterns

$$\cdots \to \beta_i \to \beta_{i+1} \to \beta_{i+2} \to \cdots \to \beta_{i+n} \to \cdots$$

where each β_i is the morphic image of β_{i-1}. By Corollary 146, all such chains can continue indefinitely in one direction. Theorem 152 below confirms that any such chain must terminate in the other. Note that for convenience when proving the theorem, the order of the indices of the patterns β_i has been reversed.

Theorem 152. *There does not exist an infinite sequence of periodicity forcing words $S := \beta_0, \beta_1, \beta_2, \cdots$ such that for every $i > 1$,*

- *there exists a morphism φ_i satisfying $\beta_{i-1} = \varphi_i(\beta_i)$ and*
- *φ_i is not a renaming morphism.*

Proof. Assume to the contrary that a such a sequence S exists which is infinite. For any $i, j \in \mathbb{N}_0$ with $i > j$, let $\psi_{i,j} := \varphi_{j+1} \circ \varphi_{j+2} \circ \cdots \circ \varphi_i$, so that $\psi_{i,j}(\beta_i) = \beta_j$. We will need to use the following results from Reidenbach, Schneider [70]: firstly that if two patterns are morphically coincident, then they are either the same (up to renaming) or at least one is morphically imprimitive and therefore not periodicity forcing, and secondly that if a pattern is fixed by a non-trivial morphism (not the identity), it is morphically imprimitive. We now prove some further preliminary claims.

Claim 1. *No patterns β_i, β_j, $i \neq j$, in the sequence S are renamings of each other.*

Proof: Claim 1. Assume to the contrary that, for some i, $j \in \mathbb{N}_0$ with $i > j$, β_i is a renaming of β_j. Let σ be the renaming morphism such that $\sigma(\beta_j) = \beta_i$. If $i = j+1$, then $\varphi_i(\beta_i) = \beta_j$. Thus, $\sigma \circ \varphi_i(\beta_i) = \beta_i$. However, since φ_i is not a renaming morphism, $\sigma \circ \varphi_i$ is not the identity, and β_i is morphically imprimitive. If $i > j+1$, then $\varphi_i(\beta_i) = \beta_{i-1}$, and $\psi_{i-1,j}(\beta_{i-1}) = \beta_j$. This implies $\sigma \circ \psi_{i-1,j}(\beta_{i-1}) = \beta_i$. Thus, at least one of β_i, β_{i-1} is morphically imprimitive. □

Our second claim provides a bound on the number of variables occurring in the patterns β_i.

Claim 2. *There exists $n \in \mathbb{N}$ such that every pattern in S has at most n variables.*

Proof: Claim 2. Let $n := |\beta_0|$. Let $i \in \mathbb{N}$ be arbitrary and consider the morphism $\psi_{i,0}$ mapping β_i to β_0. In particular, consider the subset of $\mathrm{var}(\beta_i)$ of variables which are not erased by $\psi_{i,0}$. Clearly the subset contains at least one variable x. Furthermore, $|\beta_i|_x \leq n$. By Proposition 147, it follows that $|\operatorname{var}(\beta_i)| \leq n$. □

Note that we can replace any β_i with one of its renamings, and S will still satisfy the criteria of the theorem. Thus, by assuming that the patterns of the sequence are in canonical form, we can assume that there exists a finite alphabet Δ such that each $\beta_i \in \Delta^*$. We now give our final preliminary claim.

Claim 3. *Any infinite subsequence of S also satisfies the conditions of the theorem.*

Proof: Claim 3. Let $S' = \beta_{p_0}, \beta_{p_1}, \beta_{p_2}, \ldots$ be an infinite subsequence of S. Then, for every $p_i > 1$, there exists a morphism φ'_{p_i} satisfying $\beta_{p_{i-1}} = \varphi'_{p_i}(\beta_{p_i})$ (simply take $\varphi' = \psi_{p_i, p_{i-1}}$). Furthermore, by Claim 1, each φ'_{p_i} cannot be a renaming morphism. Thus S' satisfies the conditions of the theorem. □

We are now ready to prove the theorem, which we do by deriving from S an infinite subsequence S_k which satisfies the conditions for the theorem whenever S does. Thus, by showing S_k does not satisfy the conditions, we obtain a contradiction and our assumption that S is infinite cannot hold.

Let $\delta_{i,0}$ be the subpattern of β_i whose variables are not erased by $\psi_{i,0}$. Since each $\delta_{i,0}$ contains only variables from a finite alphabet Δ, and must have length at most $|\beta_0|$, the set $\{\delta_{i,0} \mid i \in \mathbb{N}\}$ contains only finitely many different patterns. In particular, at least one such pattern $\delta_{i,0}$ must occur as a subpattern of infinitely many different patterns β_j. Let this pattern be δ_0. By Claim 3, the sequence S_0

$$\begin{tikzcd}
\beta_0^{(k)} & \beta_1^{(k)} \arrow[l, "\varphi_0^{(k)}"'] & \beta_2^{(k)} \arrow[l, "\varphi_1^{(k)}"'] & \beta_3^{(k)} \arrow[l, "\varphi_2^{(k)}"'] & \beta_4^{(k)} \arrow[l, "\varphi_3^{(k)}"'] & \beta_5^{(k)} \arrow[l, "\varphi_4^{(k)}"'] \\
 & \delta_0^{(k)} \arrow[ul, dashed] & \delta_0^{(k)} & \delta_0^{(k)} & \delta_0^{(k)} & \delta_0^{(k)} \\
 & & \delta_1^{(k)} \arrow[uul, dashed] & \delta_1^{(k)} & \delta_1^{(k)} & \delta_1^{(k)} \\
 & & & \delta_2^{(k)} \arrow[uuul, dashed] & \delta_2^{(k)} & \delta_2^{(k)} \\
 & & & & \delta_3^{(k)} \arrow[uuuul, dashed] & \delta_3^{(k)} \\
 & & & & & \delta_4^{(k)} \arrow[uuuuul, dashed]
\end{tikzcd}$$

Figure 6.1: Depiction of the first 5 patterns of the sequence S_k. Each pattern $\beta_i^{(k)}$ has its subpatterns $\delta_j^{(k)}$ listed below. Solid arrows indicate the morphisms which are explicitly given in the definition of the sequence, while the dashed arrows represent the implicit non-erasing morphisms from the subpatterns. Note that for clarity, the dotted arrows are omitted for all but the leftmost occurrence of each $\delta_i^{(k)}$.

obtained by removing all patterns after β_0 which do not have δ_0 as a subpattern still satisfies the criteria of the theorem. Note that S_0 is also still infinite. We will call the patterns of the modified sequence $\beta_0^{(0)}$, $\beta_1^{(0)}$, $\beta_2^{(0)}$ etc., and define the morphisms $\varphi_i^{(0)}$ and $\psi_{i,j}^{(0)}$ accordingly.

Similarly let $\delta_{i,1}^{(0)}$ be the subpattern of $\beta_i^{(0)}$ whose variables are not erased by $\psi_{i,1}^{(0)}$. By the same reasoning as above, there exists some infinitely occurring subpattern $\delta_1^{(0)}$, so we can produce an infinite subsequence S_1 of S_0 containing only the patterns $\beta_0^{(0)}$, $\beta_1^{(0)}$ and $\beta_i^{(0)}$ with $\delta_1^{(0)}$ as a subpattern when $i > 1$.

By repeating this process $k > 2^{|\Delta+1|}$ times, we have an infinite sequence S_k for which each pattern $\beta_i^{(k)}$, $i > k$ contains $\delta_0^{(k)}$, $\delta_1^{(k)}$, ..., $\delta_k^{(k)}$ as subpatterns (see Figure 6.1). Note that by definition, each $\beta_i^{(k)}$ is a (non-erasing) morphic image of $\delta_i^{(k)}$.

However, $\beta_i^{(k)}$ can only have finitely many (at most $2^{|\Delta|} - 1$) different, non-empty subpatterns. Thus there exist p, q, r such that $\delta_p^{(k)} = \delta_r^{(k)}$ for some $p > q > r$. Note that $\delta_r^{(k)}$ is a sub-pattern of $\beta_q^{(k)}$, since $q \geq r+1$. Furthermore, there exists a morphism $\psi_{p,q}^{(k)}$ from $\beta_p^{(k)}$ to $\beta_q^{(k)}$. However, since $\delta_r^{(k)}(= \delta_p^{(k)})$ is a subpattern of $\beta_q^{(k)}$, there exists a morphism from $\beta_q^{(k)}$ to $\beta_p^{(k)}$ (see Figure 6.2). This implies they are morphically coincident, and since, by Claim 1, they are not renamings of each other, at least one must be morphically imprimitive. This contradicts the assumption that all patterns are periodicity forcing, and thus completes the proof. $\square$

Consequently every periodicity forcing word is either a prime element of DPCP$^\neg$ or the morphic image of a prime element of DPCP$^\neg$, and the set DPCP$^\neg$ is spanned by one-sided infinite chains of the form

$$\beta_0 \to \beta_1 \to \cdots \beta_n \to \cdots$$

$$\begin{tikzcd}
\beta_q^{(k)} \arrow[d] & \beta_p^{(k)} \arrow[l, "\psi_{p,q}^{(k)}"'] & \beta_{p+1}^{(k)} \arrow[l, dashed, "\varphi_p^{(k)}"'] \arrow[d, dashed] \\
\delta_r^{(k)} (= \delta_p^{(k)}) \arrow[ur] & & \delta_p^{(k)} \arrow[ul, dashed]
\end{tikzcd}$$

Figure 6.2: Diagram showing morphic coincidence of $\beta_p^{(k)}$ and $\beta_q^{(k)}$. Morphisms are indicated by arrows, where the solid arrows indicate which morphisms responsible for the coincidence 'loop'.

where each β_i is the morphic image of β_{i-1} and β_0 is prime.

Corollary 153. *Let α be a periodicity forcing word. Then α is either prime, or the morphic image of a prime periodicity forcing word.*

Since a characterization of morphisms which map periodicity forcing words to periodicity forcing words is given in [10], Theorem 152 provides a strong insight into the structure of DPCP$^\neg$.

By definition, it is not possible to use morphisms to generate prime periodicity forcing words, so alternative methods must be used to find them. This is is investigated in the next section.

6.1.3 Periodicity Forcing Sets: A Divide and Conquer Approach

While Section 6.1.2 provides motivation for the further study of generating periodicity forcing words with morphisms, it also demonstrates the need for other methods, since prime patterns can clearly not be obtained in this way. In [5], Culik II and Karhumäki show that this may be done using periodicity forcing sets. Indeed, patterns not in DPCP are essentially periodicity forcing sets with a cardinality of 1. However, it is generally easier to construct periodicity forcing sets with higher cardinalities, as more patterns result in a more restricted class of pairs of morphisms which agree on every pattern. This is precisely the advantage gained when using morphisms to generate periodicity forcing words.

It follows from their basic properties that the agreement of two morphisms on a ratio-imprimitive pattern can be reduced to the agreement of those morphisms on a set of two (or more) shorter patterns. In particular, if $\alpha = \beta_1 \cdot \beta_2 \cdot ... \cdot \beta_n$, where $\mathrm{P}(\beta_1) = \mathrm{P}(\beta_2) = \cdots = \mathrm{P}(\beta_n)$, then $\alpha \notin$ DPCP if and only if $\{\beta_1, \beta_2, ..., \beta_n\}$ is a periodicity forcing set.

Hence, given a periodicity forcing set of patterns with the same basic Parikh vector, it is possible to construct periodicity forcing words by concatenating all the patterns in the set. It is the focus of the present section to investigate periodicity forcing sets which have this additional property and use them to obtain periodicity forcing words which may be prime.

We will give constructions (Theorem 156 and Theorem 160) which allow new periodicity forcing sets to be formed from existing ones. In particular, since strong sufficient conditions are known for a set of patterns over two variables to be periodicity forcing (see, e.g., Holub [33]), we will provide constructions which increase the alphabet size. We recall the following concise example from Culik II, Karhumäki [5] which will be used later on.

Lemma 154 (Culik II, Karhumäki [5])**.** *The set* $\{1 \cdot 2,\ 1 \cdot 1 \cdot 2 \cdot 2\}$ *is periodicity forcing.*

Note that by the reasoning above, we can infer that the patterns $1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 2$ and $1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 \cdot 2$ are periodicity forcing.

Our constructions are based on the substitution of individual variables with patterns. For example, consider the set $\{\alpha \cdot \beta,\ \alpha \cdot \alpha \cdot \beta \cdot \beta\}$ for some patterns α, β. We can immediately conclude for any σ, τ which agree on both patterns of the set, that they are either identical over α and β (i.e. $\sigma(\alpha) = \tau(\alpha)$ and $\sigma(\beta) = \tau(\beta)$), or they are periodic over α and β (i.e., $\sigma(\alpha)$, $\tau(\alpha)$, $\sigma(\beta)$, $\tau(\beta) \in \{w\}^*$ for some word w). Since any morphic image of α (resp. β) is also a morphic image of 1 (resp. 2), the existence of σ and τ not adhering to one of these cases would be in direct contradiction to Lemma 154.

Note however that the set $\{\alpha \cdot \beta,\ \alpha \cdot \alpha \cdot \beta \cdot \beta\}$ is not necessarily periodicity forcing. For example, it may be the case that a morphism σ is periodic over α and β, but not their individual variables. In general, additional patterns will be required in order to achieve to turn the original set into a periodicity forcing one. These additional patterns will be formed by splitting a pattern $\gamma = \gamma_1 \cdot \gamma_2$ and inserting some other pattern δ, obtaining $\gamma_1 \cdot \delta \cdot \gamma_2$. Thus in the case described above, we have that $\sigma(\gamma_1 \cdot \delta \cdot \gamma_2)$ is of the form $w^{k_1} \cdot u \cdot w^q \cdot v \cdot w^{k_2}$ where $uv = w$. Thus, we will use the following technical lemma when considering the agreement of two such morphisms on $\gamma_1 \cdot \delta \cdot \gamma_2$.

Lemma 155. *Let w be a primitive word, and let u, u', v, v' be words such that u,$v \neq \varepsilon$ and $u \cdot v = u' \cdot v' = w$. Then for any k_1, k_2, k_3, k_4, q_1, $q_2 \in \mathbb{N}_0$ with $q_1 \neq 0$ or $q_2 \neq 0$, the equation*

$$w^{k_1} \cdot u \cdot w^{q_1} \cdot v \cdot w^{k_2} = w^{k_3} \cdot u' \cdot w^{q_2} \cdot v' \cdot w^{k_4} \tag{6.1}$$

only has solutions in the case that $k_1 = k_3$, $k_2 = k_4$, $q_1 = q_2$, $u = u'$ and $v = v'$.

Proof. Firstly, suppose that $q_1 = 0$. Then equality (6.1) can be reduced to $w^{(k_1+k_2+1)-(k_3+k_4)} = u' \cdot w^{q_2} \cdot v'$. In this case is well known and easily proved that

u, v and w commute and thus that the statement of the Lemma holds. Hence we assume $q_1 \neq 0$. Symmetrically, we can also assume that $q_2 \neq 0$, and by the same reasoning, that u', $v' \neq \varepsilon$.

W. l. o. g. let $|u| \geq |u'|$. Then since $u \cdot v = u' \cdot v'$, there exist words c, d, e such that $u = cd$, $v = e$, $u' = c$ and $v' = de$. Note that this implies $w = cde$. Hence equality (6.1) can be expressed as

$$(cde)^{k_1} \cdot cd \cdot (cde)^{q_1} \cdot e \cdot (cde)^{k_2} = (cde)^{k_3} \cdot c \cdot (cde)^{q_2} \cdot de \cdot (cde)^{k_4}.$$

If $d = \varepsilon$, then unless $k_1 = k_3$, $k_2 = k_4$ and $q_1 = q_2$, the equation is non-trivial and in two unknowns – namely c and e, so by Lemma 4, c and e commute and w is imprimitive. Hence $c \neq \varepsilon$, $d \neq \varepsilon$ and $e \neq \varepsilon$.

The equation can be divided into three distinct cases, according to the sign of $k_1 - k_3$. In each case, it is shown that whenever the equation is non-trivial, w must be imprimitive, which is a contradiction.

If $k_1 > k_3$, by comparing the prefix of each side of length $(k_3 + 1)|cde| + |c|$,

$$(cde)^{k_3} \cdot (cde) \cdot c = (cde)^{k_3} \cdot c \cdot (cde).$$

Therefore

$$(cde) \cdot c = c \cdot (cde).$$

so c and cde commute. Since c, d, $e \neq \varepsilon$, $|w| > |c|$. Thus, w is imprimitive, which is a contradiction.

If $k_1 < k_3$, by comparing the prefix of length $(k_3 + q_2)|cde| + |c| + |d|$, there exist n, $m \in \mathbb{N}_0$ such that

$$(cde)^{k_1} \cdot cd \cdot (cde)^{q_1 - n} \cdot (ecd)^m = (cde)^{k_3} \cdot c \cdot (cde)^{q_2} \cdot d,$$

and therefore

$$cd \cdot (cde)^{q_1 - n} \cdot (ecd)^m = (cde)^{k_3 - k_1} \cdot c \cdot (cde)^{q_2} \cdot d,$$

where $m \leq k_2$, $0 \leq n < q_1$ and $m = 0$ if $n \neq 0$. Notice that $k_3 - k_1 \geq 1$. If $m = 0$ then by comparing the suffix of length $|d| + |e|$ of either side, d and e commute. By Corollary 5, equality (6.1) becomes a non-trivial equation in two unknowns, so c, d, e commute. If $m \geq 2$, then by comparing the suffix of length $2|d| + |c| + |e|$, $decd = cded$. Thus $dec = cde$ and de, c commute. If $m = 1$, then

$$cd \cdot (cde)^{q_1} \cdot ec = (cde)^{k_3 - k_1} \cdot c \cdot (cde)^{q_2};$$

so $q_1 \geq q_2$ (since $k_3 - k_1 \geq 1$). It follows that

$$(cde)^{q_2} \cdot ec = ec \cdot (cde)^{q_2},$$

and hence cde, ec commute. Since $|cde| > |ec|$, it follows that $cde^r = ec^s$ for some $r > s > 0$. Thus if $k_1 < k_3$, w is not primitive, which is a contradiction.

If $k_1 = k_3$, then

$$d \cdot (cde)^{q_1} \cdot e \cdot (cde)^{k_2} = (cde)^{q_2} \cdot de \cdot (cde)^{k_4},$$

so w is imprimitive, providing a contradiction as required. □

We now present our first of two constructions for producing new periodicity forcing sets from existing ones. Note that both constructions can easily be used to produce sets of patterns which share the same basic Parikh vector. Thus we can use the following theorems to generate periodicity forcing words which are not necessarily obtainable using the methods from [10]. The construction relies on 'splitting' one variable y into two (so each occurrence of y becomes, e. g., y_1y_2) in each pattern. New patterns are then introduced to force the periodicity of y_1 and y_2. Although the theorem appears very technical, it is relatively simple to apply, as Example 157 shall demonstrate.

Theorem 156. *Let $\Delta := \{x_1, x_2, ..., x_n\}$ be a set of variables, and let $y \notin \Delta$ be a variable. Let $\Pi := \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a periodicity forcing set such that $\bigcup_{i=1}^m \mathrm{var}(\alpha_m) = \Delta$. Let $\varphi : \Delta^* \to (\Delta \cup \{y\})^*$ be the morphism given by $\varphi(x_n) := x_n \cdot y$ and $\varphi(x_i) := x_i$ for $1 \leq i < n$. Let $t \in \mathbb{N}$, and for $1 \leq i \leq t$, let $\beta_i := x_n \cdot \gamma_i \cdot y$ for some pattern γ_i. Let $\beta_{t+1} := x_1 \cdot x_1 \cdot x_2 \cdot x_2 \cdots x_n \cdot x_n \cdot y \cdot y$. If*

(i) $\gamma_1, \gamma_2, ..., \gamma_t$ are patterns such that $\mathrm{var}(\gamma_1) = \mathrm{var}(\gamma_2) = \cdots = \mathrm{var}(\gamma_t) = \Delta \backslash \{x_n\}$, and

(ii) the set $\{\gamma_1, \gamma_2, \cdots, \gamma_t\}$ is commutativity forcing,

then the set $\{\varphi(\alpha_1), \varphi(\alpha_2), ..., \varphi(\alpha_m), \beta_1, \beta_2, ..., \beta_{t+1}\}$ is periodicity forcing.

Proof. Let $\sigma, \tau : (\Delta \cup \{y\})^* \to \{\mathtt{a}, \mathtt{b}\}^*$ be two distinct morphisms which agree on the set $\varphi(\alpha_2), ..., \varphi(\alpha_m)\}$. Then since $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ is a periodicity forcing set, we have one of the following cases:

(1) $\sigma(\varphi(x_i)) = \tau(\varphi(x_i))$ for $1 \leq i \leq n$, or

(2) there exists a primitive word $w \in \{\mathtt{a,b}\}^*$ such that $\sigma(\varphi(x_i)), \tau(\varphi(x_i)) \in \{w\}^*$ for $1 \leq i \leq n$.

Consider first Case 1. It follows from the definition of φ that $\sigma(x_n \cdot y) = \tau(x_n \cdot y)$, and $\sigma(x_i) = \tau(x_i)$ for $1 \leq i < n$. Furthermore, $\sigma(\beta_{t+1}) = \tau(\beta_{t+1})$. Then σ and τ must agree on $x_n \cdot x_n \cdot y \cdot y$. However, by Lemma 154 $\{x_n \cdot y,\ x_n \cdot x_n \cdot y \cdot y\}$ is a periodicity forcing set, so there exists a $w \in \{\mathtt{a,b}\}^*$ and $k_1,\ k_2,\ k_3,\ k_4 \in \mathbb{N}_0$ such that $\sigma(x_n) = w^{k_1}$, $\tau(x_n) = w^{k_3}$, $\sigma(y) = w^{k_2}$, $\tau(y) = w^{k_4}$. Due to the fact that $\sigma(\beta_i) = \tau(\beta_i)$ for $1 \leq i \leq t$,

$$w^{k_1} \cdot \sigma(\gamma_1) \cdot w^{k_2} = w^{k_3} \cdot \tau(\gamma_1) \cdot w^{k_4}$$

$$w^{k_1} \cdot \sigma(\gamma_2) \cdot w^{k_2} = w^{k_3} \cdot \tau(\gamma_2) \cdot w^{k_4}$$

$$\vdots$$

$$w^{k_1} \cdot \sigma(\gamma_t) \cdot w^{k_2} = w^{k_3} \cdot \tau(\gamma_t) \cdot w^{k_4}.$$

Note that since $\sigma(\varphi(x_i)) = \tau(\varphi(x_i))$ for $1 \leq i \leq n$ and $\gamma_i \in \{x_1,\ x_2,\ ...,\ x_{n-1}\}^*$ for $1 \leq i \leq t$, it follows that $\sigma(\gamma_i) = \tau(\gamma_i)$ for $1 \leq i \leq t$. Unless $k_1 = k_3$, and $k_2 = k_4$ (in which case σ and τ are not distinct), each equation is non-trivial and in two variables (w and $\sigma(\gamma_i)$), so by Lemma 4, $\sigma(\gamma_i) \in \{w\}^*$ for $1 \leq i \leq t$. Thus the words $\sigma(\gamma_i)$ commute. However, by Condition (ii) of the proposition, this implies that there exists a primitive word $w' \in \{\mathtt{a,b}\}^*$ such that $\sigma(x_i) \in \{w'\}^*$ for $1 \leq i < n$. It follows from Lemma 4 that $w' = w$, so σ is periodic. The same holds for τ.

Consider Case 2. Then there exist $k_1,\ k_2,\ ...\ k_n,\ l_1,\ l_2,\ ...\ l_n \in \mathbb{N}_0$ and a word $w \in \{\mathtt{a,b}\}^+$ such that $\sigma(x_i) = w^{k_i}$ and $\tau(x_i) = w^{l_i}$ for $1 \leq i < n$, and $\sigma(x_n \cdot y) = w^{k_n}$, $\tau(x_n \cdot y) = w^{l_n}$. If $k_n = l_n = 0$, then σ and τ are periodic. Otherwise there exist u, v, u', v' and $q_1,\ q_2,\ q_3,\ q_4 \in \mathbb{N}_0$ such that $\sigma(x_n) = w^{q_1} \cdot u$, $\sigma(y) = v \cdot w^{q_2}$, $\tau(x_n) = w^{q_3} \cdot u'$ and $\tau(y) = v' \cdot w^{q_4}$, with $uv = u'v' = w$. Note that if $u = \varepsilon$ or $v = \varepsilon$, σ is periodic. Since $\sigma(\beta_1) = \tau(\beta_1)$,

$$\sigma(x_n) \cdot \sigma(\gamma_1) \cdot \sigma(y) = \tau(x_n) \cdot \tau(\gamma_1) \cdot \tau(y),$$

so

$$w^{q_1} \cdot u \cdot w^{s_1} \cdot v \cdot w^{q_2} = w^{q_3} \cdot u' \cdot w^{s_2} \cdot v' \cdot w^{q_4}$$

for some $s_1,\ s_2 \in \mathbb{N}_0$. If $s_1 = s_2 = 0$, then $\sigma(x_n \cdot y) = \tau(x_n \cdot y)$, and if $\sigma(\beta_{t+1}) = \tau(\beta_{t+1})$, $\sigma(x_n \cdot x_n \cdot y \cdot y) = \tau(x_n \cdot x_n \cdot y \cdot y)$, so by Lemma 154, σ and τ must be periodic over $\{x_n,\ y\}$ (i.e., $\sigma(x)$, $\sigma(y)$, $\tau(x)$, $\tau(y) \in \{z\}^*$ for some word $z \in \{\mathtt{a}, \mathtt{b}\}^*$). Since they are empty over all other variables, they are periodic over Δ. Otherwise, by Lemma 155, u, v, u', $v' \in \{w\}^*$, which again implies that σ and τ are periodic.

Thus, there exist no two non-periodic morphisms which agree on the set $\{\varphi(\alpha_1),\ ...,\ \varphi(\alpha_m),\ \beta_1,\ \beta_2,\ ...,\ \beta_{t+1}\}$. Hence it is periodicity forcing as required. $\square$

Example 157. *Let* $\Delta := \{1, 2\}$*,* $y := 3$*, and* $\Pi := \{1 \cdot 2, 1 \cdot 1 \cdot 2 \cdot 2\}$*. Then* $\varphi : \{1, 2\}^* \to \{1, 2, 3\}^*$ *is the morphism given by* $\varphi(1) = 1$ *and* $\varphi(2) = 2 \cdot 3$*. Let* $\gamma_1 := 1$*,* $\beta_1 := 2 \cdot 1 \cdot 3$ *and* $\beta_2 := 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3$*. Then by Theorem 156, we have that the set* $\Pi' := \{1 \cdot 2 \cdot 3, 1 \cdot 1 \cdot 2 \cdot 3 \cdot 2 \cdot 3, 2 \cdot 1 \cdot 3, 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3\}$ *is periodicity forcing. Since all the patterns have the same basic Parikh vector, we can conclude that, for example, the pattern* $1 \cdot 2 \cdot 3 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ *is periodicity forcing.*

We can then use Π' *to again apply the theorem. This time we have* $y := 4$ *and* $\Delta := \{1, 2, 3\}$*. By Proposition 143, possible choices for* γ_1 *and* γ_2 *are* $1 \cdot 2$ *and* $2 \cdot 1$*. Thus, by applying the theorem, we can conclude that the set* $\Pi'' := \{1{\cdot}2{\cdot}3{\cdot}4, 1{\cdot}1{\cdot}2{\cdot}3{\cdot}4{\cdot}2{\cdot}3{\cdot}4, 2{\cdot}1{\cdot}3{\cdot}4, 1{\cdot}1{\cdot}2{\cdot}2{\cdot}3{\cdot}4{\cdot}3{\cdot}4, 3{\cdot}1{\cdot}2{\cdot}4, 3{\cdot}2{\cdot}1{\cdot}4, 1{\cdot}1{\cdot}2{\cdot}2{\cdot}3{\cdot}3{\cdot}4{\cdot}4\}$ *is periodicity forcing, and again we can concatenate the patterns to form a periodicity forcing word.*

Our second method relies on inserting a new variable repeatedly into occurrences of a single pattern not in DPCP. It is relatively simple to establish a set of patterns with the same basic Parikh vectors in this way. The following definition is given to provide a notation for inserting a new variable x at a specified place in a pattern α.

Definition 158. *Let* α *be a pattern and let* $x \in \mathrm{var}(\alpha)$ *be a variable. Let* $\mathrm{pre}_x(\alpha)$ *be the prefix of* α *up to, and including the first occurrence of* x*. Let* $\mathrm{suf}_x(\alpha)$ *be the suffix of* α *starting after (not including) the first occurrence of* x*.*

Note that $\mathrm{pre}_x(\alpha) \cdot \mathrm{suf}_x(\alpha) = \alpha$, so the pattern $\mathrm{pre}_x(\alpha) \cdot y \cdot \mathrm{suf}_x(\alpha)$ is the pattern obtained by inserting the variable y into the pattern α directly after the first occurrence of x.

The following lemma produces periodicity forcing sets which will form the basis of our construction. Although the patterns in these sets do not have the same basic Parikh vectors, it is expanded in Theorem 160 to provide a construction with patterns that do, and thus can be used to produce periodicity forcing words.

Lemma 159. *Let* $\alpha \notin \mathrm{DPCP}$ *be a pattern, and let* $x \notin \mathrm{var}(\alpha)$ *be a variable. Let* β_z *denote the pattern* $\mathrm{pre}_z(\alpha) \cdot x \cdot \mathrm{suf}_z(\alpha)$ *for any* $z \in \mathrm{var}(\alpha)$*. Then the set* $\{\alpha, x\} \cup \{\beta_y \mid y \in \mathrm{var}(\alpha)\}$ *is periodicity forcing.*

Proof. Let $\sigma, \tau : (\mathrm{var}(\alpha) \cup \{x\})^* \to \{\mathtt{a,b}\}^*$ be distinct morphisms, let y be arbitrary, and consider the equation $\sigma(\beta_y) = \tau(\beta_y)$. If $\sigma(\alpha) = \tau(\alpha)$, by properties of DPCP, there must exist a word $w \in \{\mathtt{a,b}\}^*$ such that $\sigma(z) \in \{w\}^*$ for every $z \in \mathrm{var}(\alpha)$. Therefore, there exist $p, q, r, s \in \mathbb{N}_0$ such that $\sigma(\beta_y) = \tau(\beta_y)$ if and only if

$$w^p \cdot \sigma(x) \cdot w^q = w^r \cdot \tau(x) \cdot w^s.$$

Note that y can be chosen such that $p \neq r$ whenever σ, τ are distinct, by taking the leftmost variable such that $\sigma(y) \neq \tau(y)$. Furthermore, because $\sigma(x) = \tau(x) = u$ for some word $u \in \{\mathtt{a},\mathtt{b}\}^*$, by Lemma 4, u and w must commute, so σ and τ must be periodic to agree on every pattern in $\{\alpha, x\} \cup \{\beta_y \mid y \in \mathrm{var}(\alpha)\}$ as required. □

Note that in the following theorem, the set $\{x, \alpha\}$ from Lemma 159 is replaced with a set containing patterns with the same basic Parikh vector as the others. More specifically, the new set is formed by substituting the variables 1 and 2 in the example from Lemma 154 for x and α. Using the set from Lemma 154 is not the only possibility, however. The construction is easily generalized to use any periodicity forcing set of patterns with the appropriate basic Parikh vector.

Theorem 160. *Let* $\alpha \notin \mathrm{DPCP}$ *and let* $x \notin \mathrm{var}(\alpha)$. *Then the set* $\Pi := \{x \cdot \alpha, x \cdot x \cdot \alpha \cdot \alpha\} \cup \{\mathrm{pre}_y(\alpha) \cdot x \cdot \mathrm{suf}_y(\alpha) \mid y \in \mathrm{var}(\alpha)\}$ *is periodicity forcing.*

Proof. Let $\sigma, \tau : (\mathrm{var}(\alpha) \cup \{x\})^* \to \{\mathtt{a},\mathtt{b}\}^*$ be distinct morphisms which agree on every pattern in Π. Then they agree on $x \cdot \alpha$ and $x \cdot x \cdot \alpha \cdot \alpha$, so by Lemma 154, either

(1) $\sigma(x) = \tau(x)$ and $\sigma(\alpha) = \tau(\alpha)$, or

(2) $\sigma(x)$, $\tau(x)$, $\sigma(\alpha)$, $\tau(\alpha) \in \{w\}^*$ for some primitive word w.

If Case 1 holds, then σ and τ agree on $\Pi \cup \{\alpha, x\}$. Since this is a superset of the set $\{x, \alpha\} \cup \{\mathrm{pre}_y(\alpha) \cdot x \cdot \mathrm{suf}_y(\alpha) \mid y \in \mathrm{var}(\alpha)\}$, which by Lemma 159 is periodicity forcing, σ and τ are periodic. Consider Case 2 and assume to the contrary that σ is non-periodic. Then there exists a $y \in \mathrm{var}(\alpha)$ such that $\sigma(y) \notin \{w\}^*$. Let y be the first such variable to occur in α, and consider the equation

$$\sigma(\mathrm{pre}_y(\alpha) \cdot x \cdot \mathrm{suf}_y(\alpha)) = \tau(\mathrm{pre}_y(\alpha) \cdot x \cdot \mathrm{suf}_y(\alpha)).$$

Clearly, $\sigma(\mathrm{pre}_y(\alpha)) = w^{k_1} \cdot u$ for some word $u \notin \{w\}^*$ and $k_1 \in \mathbb{N}_0$. It follows that $\sigma(\mathrm{suf}_y(\alpha)) = v \cdot w^{k_2}$ for some word $v \notin \{w\}^*$ and $k_2 \in \mathbb{N}_0$ with $u \cdot v = w$. Furthermore, there exist words u', v' such that $\tau(\mathrm{pre}_y(\alpha)) = w^{k_3} \cdot u'$ and $\tau(\mathrm{suf}_y(\alpha)) = v' \cdot w^{k_4}$ for some $k_3, k_4 \in \mathbb{N}_0$ with $u' \cdot v' = w$. Let $\sigma(x) = w^{q_1}$ and $\tau(x) = w^{q_2}$ for some numbers q_1, q_2. Then

$$w^{k_1} \cdot u \cdot w^{q_1} \cdot v \cdot w^{k_2} = w^{k_3} \cdot u' \cdot w^{q_2} \cdot v' \cdot w^{k_4}.$$

Note that if both q_1 and q_2 are 0, then $\sigma(x) = \tau(x) = \varepsilon$, meaning $\sigma(\alpha) = \tau(\alpha)$; so σ must be periodic, which is a contradiction. Thus it is assumed that $q_1 > 0$ or $q_2 > 0$, and by Lemma 155, $k_1 = k_3$, $k_2 = k_4$, $q_1 = q_2$, $u = u'$, and $v = v'$. Therefore σ and τ are not distinct, which is a contradiction. A symmetrical argument can be made for when τ is non-periodic. Thus σ and τ must be periodic to agree on every element in Π, so Π is a periodicity forcing set. $\square$

By applying Theorem 160 to $\alpha := 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ and $x := 3$, and concatenating the patterns in the resulting set, we obtain, for example, the periodicity forcing word

$$3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 1 \cdot 1 \cdot 2,$$

which appears to be a good candidate for being prime. We can also conclude the following from Theorem 160:

Proposition 161. *Let $\beta = \alpha^k$ for some pattern α and number $k \geq |\operatorname{var}(\alpha)| + 3$. Then β is not a prime element of* $\mathrm{DPCP}^{\neg}$.

Proof. Let α be a pattern and let $x \notin \operatorname{var}(\alpha)$. By Theorem 160, the set $\Pi := \{x \cdot \alpha, x \cdot x \cdot \alpha \cdot \alpha\} \cup \{\operatorname{pre}_y(\alpha) \cdot x \cdot \operatorname{suf}_y(\alpha) \mid y \in \operatorname{var}(\alpha)\}$ is periodicity forcing. Furthermore, every pattern in Π has the same basic Parikh vector. Thus any concatenation of patterns in Π such that every pattern is included at least once is not in DPCP. Let $\beta = \gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_k$ be such a pattern with $\gamma_i \in \Pi$ for $1 \leq i \leq k$. Notice that $k \geq |\Pi|$, and $|\Pi| = 3 + |\operatorname{var}(\alpha)|$. Let $\varphi : (\operatorname{var}(\alpha) \cup \{x\})^* \to \operatorname{var}(\alpha)^*$ be the morphism given by $\varphi(x) := \varepsilon$ and $\varphi(y) := y$ for every $y \in \operatorname{var}(\alpha)$. Clearly $\varphi(\gamma_i) = \alpha$ for $1 \leq i \leq k$, so $\varphi(\beta) = \alpha^k$, and $\beta = \alpha^k$ is not prime as required. $\square$

This is an interesting result since the properties associated with the Dual PCP are, due to the nature of morphisms, generally consistent for repetitions of the same word. It can also be interpreted that, as a result of the proposition, the majority of periodicity forcing words are not prime.

6.1.4 Application: Intersection of Pattern Languages

As we have discussed already in Section 3.2, existing results on the closure properties of pattern languages generally rely on the use of terminal symbols. In Section 5.7 in the previous chapter, we characterised when the union of two terminal-free group pattern languages was again a terminal-free group pattern languages (and hence also showed that the terminal-free group pattern languages are not closed under union). We remarked that the same reasoning and results apply to terminal-free (erasing) pattern languages in a free monoid as well, and we present the characterisation in this form in [8].

In the current section we will exploit the link between periodicity forcing words and the solutions to certain word equations to prove that the terminal-free E-pattern languages (in a free monoid) are not closed under intersection. More precisely, for a restricted class of pairs of patterns, we are able to provide a characterization of those pairs of pattern languages where the intersection is again a terminal-free E-pattern language, and, using a construction based on periodicity forcing words, we show that for this class, the situation is non-trivial (i.e., there exist both positive and negative examples). We proceed by considering the link between word equations and intersections of pattern-languages.

If, for a word equation $\alpha = \beta$, the words α and β are over disjoint sets of unknowns, then the set of solutions $\sigma : (\mathrm{var}(\alpha) \cup \mathrm{var}(\beta))^* \to \Sigma^*$ corresponds exactly to the set of pairs of morphisms $\tau_1 : \mathrm{var}(\alpha)^* \to \Sigma^*$, $\tau_2 : \mathrm{var}(\beta)^* \to \Sigma^*$ such that $\tau_1(\alpha) = \tau_2(\beta)$. Thus, it also exactly describes the intersection $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$. Furthermore, such an intersection is invariant under renamings of α and of β, so any intersection of E-pattern languages can be described in this way. We shall next characterize when the intersection of two terminal-free E-pattern languages is again a terminal-free E-pattern language in the restricted case that the corresponding word equation permits only periodic solutions (see Proposition 164). Note that, for α and β over disjoint alphabets, such solutions always exist. Before we can present this characterization, we remark on a an immediate consequence of Lemma 4: that no non-empty word can have two distinct primitive roots and consequently that the primitive root of a periodic morphism will always be the primitive root of its images.

Remark 162. *Let u be a primitive word, and suppose that u^n is a solution-word for some word equation which permits only periodic solutions. Then the corresponding solution σ has u as a primitive root. Furthermore, this means one can replace all occurrences of u in the definition of σ with a single terminal symbol* $\mathtt{a}$*, and thus* $\mathtt{a}^n$ *will also be a solution.*

Now, we are able to prove that if the erasing pattern language of a terminal-free pattern γ equals the intersection of two terminal-free erasing pattern languages (where the word equation constructed from the corresponding patterns only permits periodic solutions), then the erasing pattern language of γ is equal to the erasing pattern language of some pattern 1^k. This result constitutes one half of the desired characterization and is stated separately, since we shall use it again later.

Proposition 163. *Let Σ be an alphabet, $|\Sigma| \geq 2$. Let α, β be patterns over disjoint sets of variables, and suppose that the word equation $\alpha = \beta$ permits only periodic solutions. Furthermore, suppose that $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$ is a terminal-free E-pattern language $L_{\mathrm{E},\Sigma}(\gamma)$ for some $\gamma \in X^+$. Then $L_{\mathrm{E},\Sigma}(\gamma) = L_{\mathrm{E},\Sigma}(1^k)$ for some $k \in \mathbb{N}$.*

Proof. Let δ be the primitive root of γ with $\gamma = \delta^k$. It follows that for some primitive word $u \in \Sigma^+$, the word u^k is a solution-word. By Remark 162, this implies that $\mathtt{a}^k$ is also a solution-word, and thus that $\mathtt{a}^k \in L_{\mathrm{E},\Sigma}(\gamma)$. Consequently, $|\delta|_x = 1$ for some $x \in \mathrm{var}(\gamma)$, and thus $L_{\mathrm{E},\Sigma}(\gamma) = L_{\mathrm{E},\Sigma}(1^k)$. □

It is easy to see that the number k in Proposition 163 is the length of the shortest non-empty solution-word to the corresponding equation. This, in particular, means that if the word equation $\alpha = \beta$ permits only periodic solutions and $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta) = L_{\mathrm{E},\Sigma}(1^k)$, then it is necessary that every solution-word $u := \mathtt{a}^l$ to the equation satisfies that l is a multiple of k. The next proposition states that this necessary condition is also a sufficient one:

Proposition 164. *Let Σ be an alphabet, $|\Sigma| \geq 2$ and $\mathtt{a} \in \Sigma$. Let α, β be terminal-free patterns over disjoint sets of variables, and suppose that the word equation $\alpha = \beta$ permits only periodic solutions. Let w be the shortest non-empty solution-word. Then $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$ is a terminal-free E-pattern language if and only if, for every solution-word $u := \mathtt{a}^k$ to the equation, k is a multiple of $|w|$.*

Proof. The *only if* direction holds due to Proposition 163. Let α, $\beta \in X^+$ and suppose that the word equation $\alpha = \beta$ permits only periodic solutions. By Remark 162, there exists a $p \in \mathbb{N}$ such that $\mathtt{a}^p$ is a shortest non-empty solution-word. Clearly, since $\mathtt{a}^p$ is a solution-word, w^p is also a solution-word, for any word $w \in \Sigma^*$. Thus, if there does not exist a solution word $\mathtt{a}^k$, where $k \neq p \times q$ for some $q \in \mathbb{N}_0$, the set of solution words is exactly $\{w^{p \times q} \mid w \in \Sigma^*,\ q \in \mathbb{N}_0\} = L_{\mathrm{E},\Sigma}(1^p)$. This proves the *if* direction and the statement. □

We shall now utilize Proposition 164 in the following way. For example patterns α and β for which $\alpha = \beta$ permits only periodic solutions, we show that $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$ does not satisfy the conditions of the characterization of Proposition 164 and therefore $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$ cannot be a terminal-free E-pattern language.

Proposition 165. *Let Σ be an alphabet, $|\Sigma| \geq 2$, and let $\alpha := 1 \cdot 2 \cdot 1^2 \cdot 2 \cdot 1^3 \cdot 2^2$ and $\beta := 3 \cdot 4^2 \cdot 3^2 \cdot 4^6 \cdot 3^3$. Then $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$ is not a terminal-free E-pattern language.*

Proof. Note that since $\mathrm{var}(\alpha) \cap \mathrm{var}(\beta) = \emptyset$, the set $L := L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$ is equivalent to the set $\{\sigma(\alpha) \mid \sigma$ is a solution to the word equation $\alpha = \beta\}$. Thus, consider the equation

$$\overbrace{1 \cdot 2 \cdot 1 \cdot 1 \cdot 2}^{u} \cdot \overbrace{1 \cdot 1 \cdot 1 \cdot 2 \cdot 2}^{v} = \overbrace{\cdot 3 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 4 \cdot 4}^{w} \cdot \overbrace{4 \cdot 4 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 3}^{x}. \tag{6.2}$$

Since u and v contain the same number of each variable, and likewise for w and x, it is possible to conclude that for any solution σ, $|\sigma(u)| = |\sigma(v)| = |\sigma(w)| =$

$|\sigma(x)|$. Therefore $\sigma(u) = \sigma(w)$ and $\sigma(v) = \sigma(x)$; so the equation is equivalent to the following system of word equations:

$$1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 = 3 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 4 \cdot 4$$

$$1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 3\,,$$

which, by the substitution $5 := 44$, is equivalent to the system:

$$1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 = 3 \cdot 5 \cdot 3 \cdot 3 \cdot 5 \tag{6.3}$$

$$1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 = 5 \cdot 5 \cdot 3 \cdot 3 \cdot 3 \tag{6.4}$$

$$4 \cdot 4 = 5\,.$$

Note that by Lemma 4, this substitution does not alter the periodicity of solutions: any solution which is periodic over 1, 2, 3, 4 must also be periodic over all the variables 1, 2, 3, 4, and 5. Similarly, any solution which is periodic over 1, 2, 3 and 5, will also be periodic over all the variables 1, 2, 3, 4 and 5. From the fact that $1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$ is a periodicity forcing word (see Culik II, Karhumäki [5]), Equation (6.3) has only solutions

1. which are periodic over 1, 2, 3, and 5 (and therefore, also 4), or
2. such that $\sigma(1) = \sigma(3)$ and $\sigma(2) = \sigma(5) = \sigma(44)$.

Clearly, any solution which adheres to the first case corresponds to a periodic solution of Equation (6.2). Consider a solution which adheres to the second case. By substituting 1 for 3 and 2 for 5 in Equation (6.4), we obtain the equation $1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 = 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1$, which is a non-trivial equation in two unknowns. Thus, by Lemma 4, any solution will be periodic over 1 and 2. Since $1 = 3$ and $2 = 5$ any solution will also be periodic over 1, 2, 3, 5 (and therefore also 4). Consequently, all solutions to Equation (6.2) are periodic. The shortest solutions are clearly $\mathtt{a}^6$, for $\mathtt{a} \in \Sigma$. However, $\mathtt{a}^8$ is also a solution. Thus, by Proposition 164, the intersection is not a terminal-free E-pattern language. □

It is even possible to give a much stronger statement, showing the extent to which the 'pattern-language mechanism' is incapable of handling this seemingly uncomplicated set of solutions.

Corollary 166. *Let Σ be an alphabet, $|\Sigma| \geq 2$, and let $\alpha := 1 \cdot 2 \cdot 1^2 \cdot 2 \cdot 1^3 \cdot 2^2$ and $\beta := 3 \cdot 4^2 \cdot 3^2 \cdot 4^6 \cdot 3^3$. Then $L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$ is not a finite union of terminal-free E-pattern languages.*

Proof. By extending the proof of Proposition 163, any such union would, without loss of generality, be generated by patterns of the form 1^k, $k \in \mathbb{N}$. Assume to the contrary that $\{1^{k_1}, 1^{k_2}, \ldots, 1^{k_n}\}$ is a finite set of patterns whose languages cover $L := L_{\mathrm{E},\Sigma}(\alpha) \cap L_{\mathrm{E},\Sigma}(\beta)$. Note that the case that $n = 1$ is covered by Proposition 165, so we may assume $n \geq 2$. Furthermore, for every even k with $k > 26$, $\mathtt{a}^k \in L$. Also note that every k_i is 6 or larger. Thus, the word $\mathtt{a}^p$ is contained in L, where $p := 2 + (k_1 \times k_2 \times \cdots \times k_n)$. Clearly, p is not a multiple of any k_i, and therefore $\mathtt{a}^p$ is not in any language $L_{\mathrm{E},\Sigma}(1^{k_i})$. This is a contradiction and thus proves the statement. □

It is worth noting that the approach above can be used to show that, for $\alpha' := 1 \cdot 2 \cdot 1^2 \cdot 2^2 \cdot 1^3 \cdot 2^3$ and $\beta' := 3 \cdot 4^2 \cdot 3^2 \cdot 4^7 \cdot 3^3$, $L_{\mathrm{E},\Sigma}(\alpha') \cap L_{\mathrm{E},\Sigma}(\beta')$ equals $L_{\mathrm{E},\Sigma}(1^6)$. This demonstrates that the intersection of two E-pattern languages can in some cases be expressed as an E-pattern language, and therefore that the problem of whether the intersection of two E-pattern languages form an E-pattern language is nontrivial. However it is worth pointing out that a characterization of this situation is probably very difficult to acquire due to the challenging nature of finding solution-sets of word equations.

6.2 C-Test Words and Terminal-preserving Morphisms

Having considered patterns in the free monoid with "maximal unambiguity", i.e., periodicity forcing words, we now consider their counterpart in the free group. Like periodicity forcing words, C-test words are patterns in a free group for which most morphisms are unambiguous (up to inner automorphism, cf. Chapter 4). Recall that a pattern α is a C-test word if, for any morphisms $\sigma, \tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$, $\sigma(\alpha) = \tau(\alpha) \neq \varepsilon$, then there exists an inner automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\sigma = \tau \circ \varphi$.[10] In Lee [46], a class of C-test words α is given with the additional property that if $\sigma(\alpha) = \varepsilon$, then σ is periodic. We shall call such patterns C'-test words.[11] Hence, for C'-test words, a morphism is ambiguous up to inner automorphism if and only if it is periodic, and thus they form an analogy to the periodicity forcing words in the free group. We note that examples of both C-test words and C'-test words exist over all alphabet sizes.

Proposition 167 (Ivanov [38],[46]). *Let Δ be a finite subset of* $\mathbb{N}$. *Then there*

[10]This definition is not the original one but is easily shown to be equivalent. We refer to Chapters 2, 3 for more details.

[11]We expect that this property is actually a property of all C-test words, although we do not have a proof of this.

exists a C-test word α_1 *and* C'*-test word* α_2 *such that* $\mathrm{var}(\alpha_1) = \mathrm{var}(\alpha_2) = \Delta$.

Moreover, unsurprisingly, we see that if a C'-test word extends trivially to the free monoid (i.e., it has no negative occurrences of variables), then it is also a periodicity forcing word.

Proposition 168. *Let* $\alpha \in \mathcal{F}_{\mathbb{N}}$ *be a* C'*-test word. If* $\alpha \in \mathbb{N}^+$ *then* α *is a periodicity forcing word.*

Proof. Suppose to the contrary that $\alpha \in \mathbb{N}^+$ and α is not a periodicity forcing word. Then there exists a non-periodic morphism $\sigma : \mathrm{var}(\alpha)^* \to \Sigma^*$ and morphism $\tau : \mathrm{var}(\alpha)^* \to \Sigma^*$ such that $\sigma(\alpha) = \tau(\alpha)$ and $\sigma(x) \neq \tau(x)$ for some $x \in \mathrm{var}(\alpha)$. However, this also implies that the morphisms $\sigma', \tau' : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ defined such that $\sigma'(x) := \sigma(x)$ and $\tau'(x) := \tau(x)$ for all $x \in \mathrm{var}(\alpha)$ also agree on α. It is straightforward that σ' is also non-periodic, and that $\sigma' \neq \tau'$, so α is not a C'-test word. This is a contradiction, so α must be a periodicity forcing word as required. □

Unfortunately, all examples of C-test words from the literature have negative occurrences of variables and hence are not in $\mathbb{N}^+$ so our proposition is only applicable in theory (it is at least reasonable to expect that C-test words exist in $\mathbb{N}^+$). Considerably more useful in practice is the following lemma that generalizes Lemma 131 from the previous section to the free group. We see that like for patterns in the free monoid, morphisms satisfying the conditions of the lemma preserve the distinctness and non-periodicity of morphisms under composition, and hence preserve the property of being a C'-test word.

Lemma 169. *Let* Δ_1, Δ_2 *be sets of variables and let* $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ *be a morphism, and let* α *be a* C'*-test word such that* $\mathrm{var}(\alpha) = \Delta_1$ *and* $\mathrm{var}(\varphi(\alpha)) = \Delta_2$. *Suppose that*

(i) *for every non-periodic morphism* $\sigma : \mathcal{F}_{\Delta_2} \to \mathcal{F}_{\Sigma}$, $\sigma \circ \varphi$ *is non-periodic, and*

(ii) *for all morphisms* σ, $\tau : \mathcal{F}_{\Delta_2} \to \mathcal{F}_{\Sigma}$ *such that* $\sigma \neq \tau \circ \psi$ *for all inner automorphisms* $\psi : \mathcal{F}_{\Delta_2} \to \mathcal{F}_{\Delta_2}$ *and where at least one is non-periodic,* $\sigma \circ \varphi \neq \tau \circ \varphi \circ \psi$ *for any inner automorphism* $\psi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$.

Then $\varphi(\alpha)$ *is a* C'*-test word.*

Proof. Let α be a C'-test word with $\mathrm{var}(\alpha) = \Delta_1$ and $\mathrm{var}(\varphi(\alpha)) = \Delta_2$. Assume to the contrary that $\varphi(\alpha)$ is not a C'-test word. Then there exist morphisms $\sigma, \tau : \mathcal{F}_{\Delta_2} \to \mathcal{F}_{\Sigma}$ such that σ is non-periodic, $\sigma(\varphi(\alpha)) = \tau(\varphi(\alpha))$ and $\sigma \neq \tau \circ \psi$ for any inner automorphism $\psi : \mathcal{F}_{\Delta_2} \to \mathcal{F}_{\Delta_2}$. By Condition (i), $\sigma \circ \varphi$ is non-periodic and by Condition (ii), there does not exist an inner automorphism $\psi : \mathcal{F}_{\Delta_2} \to \mathcal{F}_{\Delta_2}$ such that $\sigma \circ \varphi = \tau \circ \varphi \circ \psi$. Therefore it follows that α is not a C'-test word, which is a contradiction, so $\varphi(\alpha)$ must be a C'-test word as claimed. □

The notions of periodicty forcing sets and commutativity forcing sets can be also extended to the free group in the obvious way, and it is clear that Remark 132 also holds for our free group-version of the lemma, and although showing sets of patterns from $\mathcal{F}_{\mathbb{N}}$ are periodicity forcing and/or commutativity forcing will generally be a more challenging task than for sets of patterns in $\mathbb{N}$, it is still straightforward for many simple examples, such as the one given in Example 133 for which the same reasoning applies when interpreted in the free group.

Moreover, since examples of C-test words exist over all alphabets, there is less need to produce morphisms which increase the number of variables which are generally the most difficult. We shall therefore not continue to try and produce new examples, but rather focus on an interesting application of C-test words to an earlier consideration from Chapter 4. Recall from Proposition 56 (Chapter 4) that we can produce (completely) unambiguous morphisms in the free group if we allow terminal symbols. Put another way, there exist (completely) unambiguous terminal-preserving morphisms in the free group. In the following propositions we investigate this further, and in particular, show that there exist patterns (with terminal symbols) such that all terminal-preserving morphisms are unambiguous, patterns for which only some terminal-preserving morphisms are unambiguous and patterns for which all terminal-preserving morphisms are ambiguous. We also show, rather surprisingly, that the morphism erasing a pattern can be unambiguous, and more generally that for any morphism there exists a pattern which can only be erased by that morphism.

We begin by showing that there exist patterns for which all terminal-preserving morphisms are unambiguous.

Proposition 170. *For any finite $\Delta \subset \mathbb{N}$, there exists a pattern $\alpha \in \mathcal{F}_{\mathbb{N}\cup\Sigma}$ with* $\mathrm{var}(\alpha) = \Delta$ *such that all terminal-preserving morphisms $\sigma : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_{\Sigma}$ are unambiguous with respect to α.*

Proof. Let $x, y \in \mathbb{N} \setminus \Delta$. Let $\Delta' := \Delta \cup \{x, y\}$. We recall from Lee [46] that there exists a C'-test word α' with $\mathrm{var}(\alpha) = \Delta'$.

Let α be the pattern obtained by replacing all occurrences of x and y in α' with $\mathtt{a}$ and $\mathtt{b}$ respectively. We now claim that all terminal-preserving morphisms are unambiguous with respect to α. Suppose to the contrary that $\sigma(\alpha) = \tau(\alpha)$ for some distinct terminal-preserving morphisms σ, $\tau : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\Sigma}$. Let σ', $\tau' : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be terminal-preserving morphisms such that $\sigma'(x) := \mathtt{a}$, $\sigma'(y) := \mathtt{b}$, and $\sigma'(z) := \sigma(z)$ for all $z \in \Delta\backslash\{x, y\}$. Define τ' in the same way.

Note that $\sigma'(\alpha') = \tau'(\alpha')$. However, it is clear from the fact that $\sigma'(x) = \mathtt{a}$ and $\sigma'(y) = \mathtt{b}$, that σ' is non-periodic. Moreover, since σ and τ are distinct, so are σ' and τ'. Finally, we note that since $\sigma'(x), \sigma'(y)$ do not share a primitive root, by Corollary 6, there does not exist an inner automorphism $\varphi : \mathcal{F}_{\text{var}(\alpha')} \to \mathcal{F}_{\text{var}(\alpha')}$ such that $\sigma' \circ \varphi = \tau'$, and hence α' is not a C'-test word, which is a contradiction. Thus every morphism is unambiguous with respect to α. $\square$

Of course as a result, there are no terminal-preserving morphisms which are ambiguous with respect to all patterns. Note that this is in contrast to ambiguity up to inner automorphisms, for terminal-free patterns, where periodic morphisms are always ambiguous up to inner automorphism.

Corollary 171. *For any finite $\Delta \subset \mathbb{N}$, every terminal-preserving morphism $\sigma : \mathcal{F}_{\Delta\cup\Sigma} \to \mathcal{F}_\Sigma$ is unambiguous with respect to some pattern α with $\text{var}(\alpha) = \Delta$.*

The following theorem establishes the classification of patterns we have already mentioned into those for which all terminal-preserving morphisms are ambiguous, for which only some are, and for which none are.

Theorem 172. *There exist patterns α_1, α_2, $\alpha_3 \in \mathcal{F}_{\mathbb{N}\cup\Sigma}$ such that*

(i) All terminal-preserving morphisms are ambiguous with respect to α_1,

(ii) there exist ambiguous and unambiguous terminal-preserving morphisms with respect to α_2, and

(iii) all terminal-preserving morphisms are unambiguous with respect to α_3.

Proof. The third statement is given by Proposition 170. The first statement follows from, e.g., Theorem 67 since patterns in $\mathcal{F}_\mathbb{N}$ are also in $\mathcal{F}_{\mathbb{N}\cup\Sigma}$. Thus we concentrate on the second statement. Let α' be a C'-test word (with at least three variables), and let $x \in \text{var}(\alpha')$. Let α_2 be the pattern obtained by replacing all occurrences of x with $\mathtt{a}$ in α'. We shall prove that, for a terminal-preserving morphism $\sigma : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_\Sigma$, σ is ambiguous with respect to α_2 if and only if it is non-periodic, and hence that the second statement holds. The case that σ is periodic may easily be shown using the same reasoning as Proposition 26, since $|\text{var}(\alpha')| \geq 3$ implies $|\text{var}(\alpha_2)| \geq 2$. Hence we consider the case that σ, τ are non-periodic. In particular, let $\tau : \mathcal{F}_{\mathbb{N}\cup\Sigma} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\alpha_2) = \sigma(\alpha_2)$. Let σ', $\tau' : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_\Sigma$ be morphisms such that $\sigma'(x) = \mathtt{a}$ and $\sigma'(z) = \sigma(z)$ for all $z \in \Delta\backslash\{x\}$. Define τ' in the same way. Note that $\sigma'(\alpha') = \tau'(\alpha')$. By definition, this implies that either σ', τ' are periodic (and hence σ and τ are also periodic which contradicts our earlier assumption), or $\sigma' = \tau' \circ \varphi$ for some inner automorphism $\varphi : \mathcal{F}_{\text{var}(\alpha')} \to \mathcal{F}_{\text{var}(\alpha')}$. However, since $\sigma'(x) = \tau'(x) = \mathtt{a}$, by Corollary 6, this implies that φ is the inner automorphism generated by $\mathtt{a}^k$ for some $k \in \mathbb{Z}$. Consequently, $\sigma'(\alpha) = \tau' \circ \varphi(\alpha) = \mathtt{a}^k\tau'(\alpha)\mathtt{a}^k$. However, $\sigma'(\alpha') = \tau'(\alpha')$, so we have $\tau'(\alpha') = \mathtt{a}^k\tau'(\alpha)\mathtt{a}^k$ implying that either $k = 0$, in which case φ is the identity morphism, $\sigma' = \tau'$ and $\sigma = \tau$, or $\tau'(\alpha') \in \mathcal{F}_{\{\mathtt{a}\}}$, and thus that τ' is periodic which is again a contradiction. Therefore, $\sigma = \tau$ whenever $\sigma(\alpha_2) = \tau(\alpha_2)$ and thus σ is unambiguous. □

Finally, we consider the ambiguity of terminal-preserving morphisms erasing patterns. In particular we note the contrast to the terminal-free case, where it is straightforward that any morphism erasing any pattern is ambiguous up to automorphism, and therefore inner automorphism. For terminal-free patterns, this means that the erasing morphism is even "more ambiguous" than periodic morphisms which are always ambiguous up to inner automorphism, but not always up to automorphism (cf. Section 5.6). As the following theorem shows, it is not only the case that the morphism erasing a pattern may be unambiguous, but in fact, for any given terminal-preserving morphism σ, there exists a a pattern α with terminal symbols such that σ is the *only* morphism erasing α. In other words, $\sigma(\alpha) = \varepsilon$ and σ is ambiguous with respect to α.

Theorem 173. *For any finite $\Delta \subset \mathbb{N}$ and terminal-preserving morphism $\sigma : \mathcal{F}_{\Delta\cup\Sigma} \to \mathcal{F}_\Sigma$, there exists a pattern $\alpha \in \mathcal{F}_{\Delta\cup\Sigma}$ such that $\sigma(\alpha) = \varepsilon$ and σ is unambiguous.*

Proof. By Proposition 170, there exists a pattern $\alpha' \in \mathcal{F}_{\Delta\cup\Sigma}$ such that all terminal-preserving morphisms (including σ) are unambiguous with respect to α'. Let $\alpha := \alpha'\sigma(\alpha')^{-1}$. Since $\sigma(\alpha') \in \mathcal{F}_\Sigma$, the fact that σ is unambiguous with respect to α' implies that σ is unambiguous with respect to α. Moreover, we have that, since σ is terminal-preserving, $\sigma(\alpha) = \sigma(\alpha')\sigma(\alpha')^{-1} = \varepsilon$. □

Of course, by simpy adding "more" terminal symbols either to the start or the end of the patterns, it possible to generalise our previous theorem so that, for any terminal-preserving morphism σ and terminal word w, there exists a pattern α such that σ is the *only* morphism mapping α to w.

Corollary 174. *For any finite $\Delta \subset \mathbb{N}$, terminal-preserving morphism $\sigma : \mathcal{F}_{\Delta\cup\Sigma} \to \mathcal{F}_\Sigma$ and word $w \in \mathcal{F}_\Sigma$, there exists a pattern $\alpha \in \mathcal{F}_{\mathbb{N}\cup\Sigma}$ such that $\sigma(\alpha) = w$ and σ is unambiguous with respect to α.*

This is a particularly interesting result, since it implies that if we are not restricted in our choice of pre-image, the morphism and image are not necessarily able to "cause" ambiguity. In fact, we can choose a periodic morphism and structurally trivial image (such as $\mathtt{aa}\cdots\mathtt{a}$) and it is still possible to find a pattern such that we may uniquely decode the morphism from the pre-image, image pair. Alternatively, we can choose a morphism which maps each variable to e.g., $\mathtt{a}$, and a seemingly completley unrelated image, e.g., $\mathtt{bb} \cdots \mathtt{b}$ and again there exists a pattern for which the morphism firstly maps the pattern onto the "unrelated" image, and secondly is the only one to do so.

Chapter 7

Conclusions and Open Problems

In the present thesis, we have considered the ambiguity of morphisms in both the contexts of the free group and the free monoid, with particular emphasis on morphisms which are *unambiguous*. We have considered patterns for which at least one morphism is unambiguous (Chapter 5), and for which all non-periodic morphisms are unambiguous (Chapter 6). We have also proven some properties of pattern languages as applications of our results (cf. Sections 5.7 and 6.1.4).

In Chapter 4, we have generalized the notion of (un)ambiguity of morphisms, which has been introduced already for free monoids, to free groups. We have shown that for the most straightforward generalization, the extra combinatorial possibilities in a free group (in particular, the existence of non-trivial inner automorphisms) are sufficient to cause every morphism to be ambiguous (cf. Theorem 27). Put another way, for any pattern α and word w, either there is no morphism mapping α onto w, or there exist at least two morphisms mapping α onto w. This is in contrast to the situation in the free monoid, for which there exist many unambiguous morphisms.

However, as we have discussed in Chapter 4, this statement does not reflect the true value of studying a general notion of ambiguity of morphisms, and it remains a sensible question to ask which morphic images of a word preserve the most information about the associated morphisms. To this end we have considered two natural extensions of the definition of unambiguity. The first, unambiguity up to inner automorphism, is the most restrictive in the sense that fewer morphisms are unambiguous. While a morphism between a pattern α and word w is ambiguous if there exists a second (distinct) morphism mapping α onto w, it is ambiguous up to inner automorphism if there is a second morphism mapping α onto w which is not the result of composing the first with an inner automorphism. Since inner automorphisms are very close, both algebraically and combinatorially speaking, to the identity morphism, pairs of morphisms which *are* related by composition with an inner automorphism can be considered to be especially similar.

Our second notion is ambiguity up to automorphism and is largely the same definition, with the only difference that we replace inner automorphisms with automorphisms. Since automorphisms act as the identity on the algebraic structure as a whole (but not necessarily the individual elements),[1] they are also a set of morphisms which may be viewed as closely related to the identity. Hence two morphisms related by composition with an automorphism may also be considered closely related. Moreover, since all inner automorphisms are automorphisms but not all automorphisms are inner automorphisms, our second version is a strictly less restrictive version of the first, in the sense that it allows for more unambiguous morphisms.

We have shown that both "new" versions of (un)ambiguity are non-trivial: that both unambiguous and ambiguous morphisms exist in both cases (cf. Proposition 35, Example 39), and furthermore, that there exist patterns for which all non-periodic morphisms are unambiguous (cf. Proposition 36), and patterns for which all morphisms are ambiguous (cf. Theorem 44). For ambiguity up to inner automorphism, we have seen that all periodic morphisms are ambiguous with respect to any pattern (cf. Proposition 34), which was to be expected and in keeping with our intuitive notion of (un)ambiguity. On the other hand, we have seen that, surprisingly, there exist periodic morphisms which are unambiguous up to automorphism with respect to some patterns (cf. Proposition 42) – a phenomenon which has been discussed in more detail in Section 5.6.

Motivated by the observation that for some – but not all – patterns α, all morphisms are ambiguous up to (inner) automorphism with respect to α, we have attempted to classify patterns according to whether or not they possess an unambiguous morphism.

Since this question has already been thoroughly addressed in the free monoid (at least for the original, "simple" definition of unambiguity which is also equivalent in the free monoid to ambiguity up to inner automorphism), in Chapter 5, we have focused on answering it for patterns in the free group and in particular, we have tried to establish equivalent results to those known already for the free monoid (cf. Section 3.1).

To achieve this, we firstly considered whether for a given pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$, there exists an *injective* morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ which is unambiguous up to (inner) automorphism (cf. Questions 1 and 2).

For ambiguity up to inner automorphism, we were able to replicate an existing characterization for words in a free monoid which is given in terms of non-trivial fixed points. In particular, Theorem 101 asserts that there exists an injective

[1] So for a free group $\mathcal{F}$ and automorphism φ we have $\varphi(\mathcal{F}) = \mathcal{F}$, but we may have $\varphi(x) \neq x$ for some $x \in \mathcal{F}$.

morphism which is unambiguous up to inner automorphism with respect to a given pattern if and only if the identity morphism is ambiguous up to inner automorphism with respect to that pattern (i.e., it is fixed by a morphism which is not an inner automorphism). Since the only inner automorphism in the free monoid is the identity, our characterization is the same as the existing result for the free monoid that a pattern possesses an injective unambiguous morphism if and only if it is fixed by a morphism which is not the identity.

The majority of our reasoning involved in the proof of Theorem 101 also applies in the case of ambiguity up to automorphism, with the exception of Proposition 97, which states that if a pattern α is fixed by a morphism ψ such that for some $x \in \mathrm{var}(\alpha)$, $\mathrm{symb}(\psi(x)) \not\subseteq \mathrm{var}(\alpha)$, then α is fixed by a morphism which is not an inner automorphism.

We have provided a conjecture that the corresponding statement when considering ambiguity up to automorphism: that if α is fixed by such a morphism ψ, then α is fixed by a morphism which is not an automorphism, also holds (cf. Conjecture 99), and subject to the correctness of the conjecture, we have shown that a pattern possesses an injective morphism which is unambiguous up to automorphism if and only if it is only fixed by automorphisms (i.e., if and only if it is a test word). Hence we have the following obvious open question:

Open Question 1. *Is Conjecture 99 correct?*

Furthermore, by Remark 100, we can reduce this to the following question. Note that it follows from Theorem 111 and Proposition 57 that all injective morphisms are ambiguous with respect to morphically imprimitive patterns, so we only need to consider those which are morphically primitive.

Open Question 2. *Given a (morphically primitive) pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$, does there exist a factor $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$ such that, for every automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α, η does not occur as a factor of $\varphi(x)$ for any $x \in \mathrm{var}(\alpha)$?*

We provide the following two comments concerning Open Question 2. Firstly, we note that for any such factor η there exists an automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ with η as a factor of $\varphi(x)$ for some $x \in \mathrm{var}(\alpha)$. For example, we may simply take the inner automorphism generated by η so that $\varphi(x) = \eta \cdot x \cdot \eta^{-1}$.[2] It is worth pointing out however, that these inner automorphisms only fix patterns sharing a primitive root with η, and therefore do not all fix a single pattern α so are not sufficient to provide an answer to the question.

Secondly, we point out that there exist patterns α such that, for every $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$, there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α and such that η

[2]Note that for at least one $x \in \mathrm{var}(\alpha)$, $\eta \cdot x \cdot \eta^{-1}$ is reduced provided η is reduced, and so the fact that η is a factor is not lost through unwanted contractions.

occurs as a factor of $\varphi(x)$ for some $x \in \mathrm{var}(\alpha)$. For example, let $\alpha := 1{\cdot}2{\cdot}3{\cdot}1{\cdot}2{\cdot}2{\cdot}3$. For $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$, let $\varphi_\eta : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be the morphism given by $\varphi(1) := 1 \cdot \eta^{-1}$, $\varphi(2) := \eta \cdot 2 \cdot \eta^{-1}$ and $\varphi(3) := \eta \cdot 3$. Then

$$\begin{aligned}\varphi(\alpha) &= \overbrace{1 \cdot \eta^{-1}}^{\varphi(1)}\overbrace{\eta \cdot 2 \cdot \eta^{-1}}^{\varphi(2)}\overbrace{\eta \cdot 3}^{\varphi(3)}\ \overbrace{1 \cdot \eta^{-1}}^{\varphi(1)}\overbrace{\eta \cdot 2 \cdot \eta^{-1}}^{\varphi(2)}\overbrace{\eta \cdot 2 \cdot \eta^{-1}}^{\varphi(2)}\overbrace{\eta \cdot 3}^{\varphi(3)}\\ &= 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3 = \alpha.\end{aligned}$$

However, all such examples known to the author a morphically imprimitive (in the free group sense), and thus do not possess an injective morphism which is unambiguous up to automorphism.

Therefore, while our two comments provide an insight into the complexity of our open questions, they are not sufficient to disprove our conjecture. To the contrary, the structures involved seem to indicate a necessity that any pattern fixed by morphisms with "arbitrary" factors appearing in the images, must have some inherent ambiguous structure causing morphic imprimitivity, and hence support the conjecture.

In Section 5.5, we have considered another existing characterization of patterns in the free monoid which possess an unambiguous injective morphism. In addition to the characterization given via fixed points, it is known that a pattern in the free monoid permits an unambiguous injective morphism if and only if it is morphically primitive. Interestingly, when generalising this notion to the free group, we have seen that the same characterization holds (subject to the correctness of Conjecture 99) for ambiguity up to automorphism – that a pattern possesses an injective morphism which is unambiguous up to automorphism if and only if it is morphically primitive (cf. Corollary 114). However, the same characterization does not hold for ambiguity up to inner automorphism (cf. Corollary 113). Therefore, in a sense, unambiguity up to automorphism is a "closer fit" – or better analogy – to unambiguity in a free monoid than ambiguity up to inner automorphism.

Although we have not generally discussed algorithmic complexity considerations in the current thesis, it is worth noting that deciding morphic primitivity for patterns in the free monoid can be achieved surprisingly efficiently: even in linear time (cf. Holub [34], Kociumaka et al. [45]). It is perhaps therefore also worth asking whether a similar result holds for patterns in the free group.

Open Question 3. *How efficiently can it be decided whether a given $\alpha \in \mathcal{F}_{\mathbb{N}}$ is morphically primitive?*

Having considered the existence of injective unambiguous morphisms, in Section 5.6 we have addressed the question of whether for a given pattern there exists a non-injective morphism which is unambiguous up to (inner) automorphism. Although we have not given a characterization for this case, we were able to show that there exist patterns for which all injective morphisms are ambiguous, but which possess a non-injective (or even periodic) unambiguous (up to automorphism) morphism (cf. Theorems 118 and 125). This is a rather paradoxical result that, in some cases, in order to infer more information about a morphism σ from the pattern α and image $\sigma(\alpha)$, it is necessary to choose a morphism which from the classical point of view, preserves less information about the pre-image. Although this is a rather strange statement, it is not wholly unsurprising, since the same statement exists in the context of a free monoid. Moreover, although we have demonstrated the result for ambiguity up to automorphism, for ambiguity up to inner automorphism it remains open as to whether the same statement holds.

Open Question 4. *Does there exist a pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$ which possesses a morphism which is unambiguous up to inner automorphism with respect to α, but such that every injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ is ambiguous up to inner automorphism with respect to α?*

By Proposition 34, all periodic morphisms are ambiguous up to inner automorphism with respect to any pattern α (provided $|\mathrm{var}(\alpha)| \neq 1$), so the stronger version of this paradox cannot hold for ambiguity up to inner automorphism. On the other hand, it seems to be a highly non-trivial and interesting topic to consider the unambiguity up to automorphism of periodic morphisms since, as is evident from Section 5.6, a certain combinatorial understanding of the set of automorphisms is necessary. Accordingly we highlight the following open question:

Open Question 5. *Given a pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$, does there exist a periodic morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ such that σ is unambiguous up to automorphism with respect to α?*

In Chapter 6 we have considered patterns for which all non-periodic morphisms are unambiguous. Most of the results were given in the context of a free monoid, on periodicity forcing words, and hence the Dual PCP.

We have shown using morphisms which preserve the property of being periodicity forcing, that there exist ratio-primitve examples over all alphabet sizes (cf. Theorem 142), and that examples with arbitrary prefixes/suffixes/factors may be constructed (cf. Theorem 144). Moreover, we have complemented existing research on the binary case, for which the majority of known examples are ratio-imprimitive, by describing large classes of ratio-primitive periodicity forcing words over a two-letter alphabet (cf. e.g., Corollary 139).

We have introduced a prime subset of the set of all periodicity forcing words ($\mathrm{DPCP}^{\neg}$), allowing the set to be described as chains of morphic images, and it has been shown that this subset is non-empty and that $\mathrm{DPCP}^{\neg}$ can be exactly generated by it along with morphisms characterized in [10] (cf. Corollary 153). Identifying prime periodicity forcing words remains a challenging task however, and we leave not only a characterization, but also strong sufficient conditions open for further research.

Open Question 6. *What is the set of prime periodicity forcing words?*

We have also shown that the shortest periodicity forcing words over an alphabet of size n must have length between n^2 and $5n^2 + 2n$ (cf. Proposition 149). Improving this bound seems to be a challenging but interesting task which we leave open, even for $n = 3$.

Open Question 7. *What is the length of the shortest periodicity forcing word over* 3 *letters? More generally, what is the length of the shortest periodicity forcing word over* n *letters?*

Finally, we have seen that the set of periodicity forcing words is perhaps much larger and more varied than originally thought. It is clear that most patterns are not periodicity forcing, since a pattern must be morphically primitive to be periodicity forcing, and most patterns are not morphically primitive (cf. Reidenbach, Schneider [70]). However, this fact relies on an abundance of patterns containing a variable which occurs only once, which are, from certain perspectives at least, of less interest since all morphic images can be reached simply by erasing all the other variables. Therefore, we ask whether the majority of morphically primitive patterns are periodicity forcing.

Open Question 8. *Are most morphically primitive patterns periodicity forcing?*

A positive answer would have the intriguing implication for the ambiguity of non-periodic morphisms in a free monoid that unambiguous morphisms are in a significant majority, at least for morphically primitive patterns, and that ambiguity is actually a reasonably rare phenomenon, occurring mostly due to structural features caused by morphic imprimitivity which may be easily and efficiently identified.

Of course, the answer is negative if only "short" patterns are considered, since, as we have just mentioned, patterns α with length less than $|\operatorname{var}(\alpha)|^2$ are not periodicity forcing. However, for longer words, the situation is less clear. In the case that $|\operatorname{var}(\alpha)| = 2$, for example, our results from Section 6.1 indicate an abundance of periodicity forcing words as the length increases, and a higher number may be produced as the morphic image of shorter examples.

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