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Exact closed-form Fractional Spectral Moments for linear fractional oscillators excited by a white noise

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ABSTRACT

In the last decades the research community has shown an increasing interest in the engineering applications of fractional calculus, which allows to accurately characterize the static and dynamic behaviour of many complex mechanical systems, e.g. the non-local or non-viscous constitutive law. In particular, fractional calculus has gained considerable importance in the random vibration analysis of engineering structures provided with viscoelastic damping. In this case, the evaluation of the dynamic response in the frequency domain presents significant advantages, once a probabilistic characterization of the input is provided. On the other hand, closed-form expressions for the response statistics of dynamical fractional systems are not available even for the simplest cases. Taking advantage of the Residue Theorem, in this paper the exact expressions of the spectral moments of integer and complex orders (i.e. fractional spectral moments) of linear fractional oscillators driven by acceleration time histories obtained as samples of stationary Gaussian white noise processes are determined.

1 Introduction

There is an increasing amount of research on the use of fractional operators to describe viscoelastic properties of materials in structural dynamics [1]. In fact, it has been shown that fractional integrals and derivatives are suited to mathematically model constitutive equations of viscoelastic materials, returning creep and relaxation functions whose general shapes well fit experimental data [2–9].

In civil and mechanical engineering, when fractional operators are used to model dissipative forces in dynamic systems, the latter are indicated with the term fractional oscillators. Several numerical methods to integrate the equations of motion of fractional systems have been proposed. A comprehensive review of papers dealing with the dynamic behaviour of fractional linear and nonlinear systems, single and multi-degrees-of-freedom, including vibration of rods, beams, plates and shells, among others, can be found in [1] and [10].

Among the various studies on fractional oscillators in literature, Rüdinger [11] proposes their use as viscoelastic tuned mass dampers to reduce vibrations of systems excited by an external white noise. The fractional oscillator is constituted by a mass linked to the main structure through a linear spring placed in parallel to a viscoelastic damping element exerting a force

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proportional to the fractional derivative of the relative displacements of the damper mass and the primary structure. Recently, a variation of this vibration absorber with no linear spring has been proposed as Fractional Tuned Mass Damper in [12–15]. In the initial approaches to this dynamic problem, numerical integration methods have been used to characterise the system stochastic response to the white noise excitation.

However, given the stationary nature of the problem, a complete description of the random response process can be actually done in terms of Power Spectral Density (PSD) and its derived parameters. Among the latter, the spectral moments (SMs), defined by VanMarcke [16] as the moments of the one-sided PSD with respect to the frequency origin, have significant importance since they represent statistics of the response and allow determining characteristics of the stochastic response process that can be used for the probabilistic assessment of structural failure and to determine the distribution of the response peaks or barrier crossing rates in reliability analysis. Moreover, the SMs are related to geometric parameters of the PSD, such as the central frequency and the bandwidth factor, that can be used as distinctive indicators between narrowband and broadband processes [17]. Additional information on SMs can be found, for example, in [18–20].

Although the SMs return important information on the nature of a stochastic process, they do not allow fully describing the PSD or the correlation of the process itself [21]. Cottone and Di Paola [22] proposed an extension of the traditional integer order SMs to complex order moments, by defining the so-called Fractional Spectral Moments (FSMs) as the Mellin transform of the one-sided PSD of a stochastic process. The advantage of these complex quantities is that they are able to reconstruct both the PSD and the correlation functions, and, therefore, can be seen as an alternative representation of the process itself [22, 23]. Additionally, Cottone and Di Paola [24] have shown that FSMs can be used as coefficients of a time series defined in terms of the Riesz fractional derivatives of a white noise for the digital simulation of realizations of a stochastic process with assigned PSD.

In this paper, the SMs and FSMs of a fractional oscillator excited by a Gaussian stationary white noise are evaluated in exact closed-form solution. The latter require the computation of an improper integral, not accessible with standard integration techniques. A powerful tool for solving not only real improper integrals, but also a wide range of problems arising in applied mathematics and engineering, is the Residue Theorem. The latter implies that the mathematical description of real phenomena can benefit from their translation in the complex domain. The theory of residues is applicable to various mathematical fields, such as theory of equations, theory of numbers, matrix analysis, evaluation of real definite or improper integrals, summation of finite and infinite series, expansions of functions into infinite series and products, ordinary and partial differential equations, mathematical and theoretical physics, finite differences and difference equations [25, 26].

Herein, the Residue Theorem is initially used to compute the integer order SMs of a classic linear oscillator excited by a white noise, showing how the technique returns exact expressions well-known in literature. Then, the exact closed-form FSMs for the same system are determined. It is demonstrated that the integer order SMs can be obtained as limit values of these FSMs. To the best of the authors' knowledge, these expressions have never been presented in literature. Finally, using a similar methodology, the SMs and FSMs for a fractional oscillator with no linear spring and excited by a white noise are determined in exact form, as well. Also for this case, the authors' are not aware of previous publications reporting such expressions. The proposed formulas have been validated by comparison with results obtained through numerical integration. To conclude, a sensitivity analysis of the FSMs with respect to the parameters involved in the definition of the fractional oscillator is reported.

2 Theoretical background

The evaluation of the statistical properties of a system excited by a stochastic process can be, in general, a very difficult task. For the stationary case, the full characterization of a process in the frequency domain is achieved by determining its PSD function. Several significant statistics, such as the variance and its time derivatives, the central frequency, the bandwidth parameter, first-passage problems and the estimation of statistical distribution of peaks can be determined from the SMs.

The SMs, firstly introduced by Vanmarcke [16], can be defined as the moments of the unilateral process PSD function with respect to the frequency origin. Therefore, the generic m-th order SM of a stochastic process is defined as:

$$\lambda_{m,X} = \int_0^{+\infty} \omega^m G_X(\omega)\, d\omega, \quad m \in \mathbb{N} \tag{1}$$

where $G_X(\omega)$ is the one-sided PSD function of the process $X(t)$.

Recently, the FSMs have been proposed as the extension of the concept of SMs to complex order [24]. The FSMs are defined as the moments of order γ of the one-sided PSD function of the process:

$$\lambda_{\gamma,X} = \int_0^{+\infty} \omega^\gamma G_X(\omega)\, d\omega, \quad \gamma \in \mathbb{C} \tag{2}$$

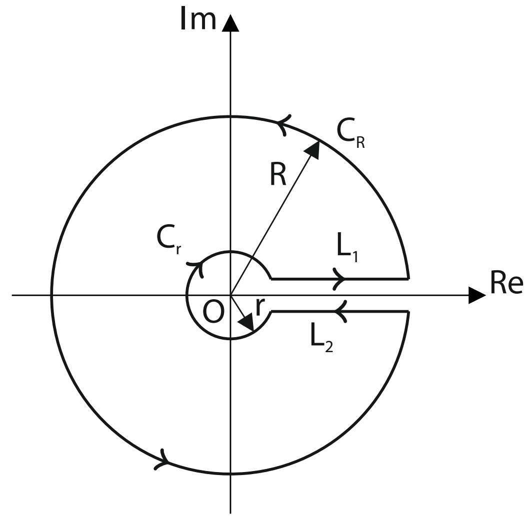

Fig. 1. Keyhole domain

Closed-form solutions for the integrals in eqs. (1) and (2) are available, to the best of the authors' knowledge, only for few particular cases of stochastic processes, and numerical integration is often required for their evaluation, restricting their use in stochastic analysis. In this paper, the authors propose the use of the Residue Theorem in order to solve the integrals in eqs. (1) and (2) and to determine exact analytic expressions for both SMs and FSMs for single-degree-of-freedom (SDOF) oscillators excited by stationary white noise processes.

2.1 Integer order SM

In several cases of practical interest, such as white noise excitations or when the seismic stochastic input is modelled by a Tajimi-Kanai filter, the SMs in eq. (1) belong to the following class of integrals:

$$I_m = \int_0^{+\infty} f(\omega)\, d\omega = \int_0^{+\infty} \frac{\omega^m}{P(\omega)} d\omega, \quad m \in \mathbb{N} \tag{3}$$

in which $P(\omega)$ is a polynomial of degree N. Generally, the integrand function is not symmetrical with respect to the real axis, so it is not possible to relate I_m to a whole real line integral. Herein, the authors propose to consider a new function $g(z) = f(z) Log(z)$, defined in the complex domain, and to compute its integral along the boundary Γ of an opportunely selected keyhole shaped domain (see Fig. 1):

$$I_\Gamma = \int_\Gamma g(z)\, dz = \int_\Gamma f(z) Log(z)\, dz \tag{4}$$

The keyhole contour $\Gamma = L_1 \cup C_R \cup L_2 \cup C_r$ has the following parametrization:

1. L_1: $z = \omega e^{i\theta}$, $r \leq \omega \leq R$ and $\theta \to 0$;
2. C_R: $z = Re^{i\theta}$, $0 < \theta < 2\pi$;
3. L_2: $z = \omega e^{\iota(2\pi-\theta)}$, $r \leq \omega \leq R$ and $\theta \to 0$;
4. C_r: $z = re^{\iota\theta}$, $0 < \theta < 2\pi$.

Thus, the contour integral has to be computed as the sum of the following 4 contributions:

$$\int_\Gamma g(z)\, dz = \int_{C_R} g(z)\, dz + \int_{C_r} g(z)\, dz + \int_{L_1} g(z)\, dz + \int_{L_2} g(z)\, dz \tag{5}$$

It has to be stressed that the function $Log(z) = \ln|z| + i\theta$ is a multi–valued function in the complex domain and its value depends on the branch cut used to compute its imaginary part ($\theta = \arg(z)$). If, as usually done, the principal value of $Log(z)$ is computed (i.e. the non positive real axis is used as branch cut, $-\pi \leq \theta < \pi$), then a discontinuity is introduced in the keyhole shaped domain. Instead, to have a single-valued continuous function in the considered domain, the non negative real axis has to be used as branch cut ($0 \leq \theta < 2\pi$).
For $m+1 < N$, it can be demonstrated that, along C_R and C_r, the following limits hold true:

$$\lim_{R\to+\infty}\int_{C_R}\frac{z^m Log(z)}{P(z)}dz = 0; \qquad \lim_{r\to 0}\int_{C_r}\frac{z^m Log(z)}{P(z)}dz = 0 \tag{6}$$

Along the line segments L_1 and L_2 the integrals assumes the following values:

$$\int_{L_1} g(z)\, dz = \lim_{\theta\to 0}\int_r^R f(\omega e^{i\theta})\,(\ln|\omega| + i\theta)\, d\omega \tag{7}$$

$$\int_{L_2} g(z)\, dz = \lim_{\theta\to 0}\int_R^r f\left(\omega e^{i(2\pi-\theta)}\right)[\ln|\omega| + i(2\pi-\theta)]\, d\omega \tag{8}$$

Taking the limits $r \to 0$ and $R \to \infty$ in eqs.(7)-(8), substituting them and (6) into eq. (5) and taking into account eq. (4):

$$\begin{aligned}\lim_{\substack{r\to 0\\ R\to+\infty}}\int_\Gamma f(z)\, Log(z)\, dz &= \int_0^{+\infty} f(\omega)\ln|\omega| d\omega - \int_0^{+\infty} f(\omega)(\ln|\omega| + 2\pi i)\, d\omega = \\ &= -2\pi i\int_0^{+\infty} f(\omega)\, d\omega = -2\pi i\, I_m\end{aligned} \tag{9}$$

Hence, denoting $z_k, (k = 1, 2, \ldots, N)$ the simple zeros of the function $P(z)$ and applying the Residue Theorem to the integral I_Γ:

$$I_m = -\sum_{k=1}^{N} Residue\,[f(z)\, Log(z), z_k] \tag{10}$$

2.2 Fractional spectral moments

In this section a technique, based again on Residue Theorem, is proposed to evaluate the FSMs in the form:

$$I_\gamma = \int_0^{+\infty}\frac{\omega^\gamma}{P(\omega)}d\omega, \quad \gamma \in \mathbb{C}\setminus\mathbb{N} \tag{11}$$

where $P(\omega)$ is, also in this case, a polynomial of degree N. Again, the same keyhole shaped domain and parametrization are considered. However, in this case, the function $f(z) = z^\gamma/P(z)$ is integrated along the keyhole contour $\Gamma = L_1 \cup C_R \cup L_2 \cup C_r$. Note that z^γ is a multi–valued function in the complex domain, since

$$z^\gamma = e^{\gamma Log(z)} = e^{\gamma(\ln|z| + i\theta)} = |z|^\gamma e^{i\gamma\theta} \tag{12}$$

Moreover, the origin as well as $z = +\infty$ are branch points for the function $f(z)$. Therefore, the positive real axis is chosen as branch cut ($0 \leq \theta < 2\pi$). Then:

$$\int_0^{+\infty}\frac{\omega^\gamma}{P(\omega)}d\omega = \lim_{\substack{r\to 0\\ R\to+\infty}}\int_\Gamma f(z)\, dz \tag{13}$$

If $\mathrm{Re}[\gamma]+1<N$, it can be demonstrated that:

$$\lim_{R\to+\infty}\int_{C_R} f(z)\,dz=0; \qquad \lim_{r\to 0}\int_{C_r} f(z)\,dz=0 \tag{14}$$

Under these hypotheses, it follows:

$$\int_{\Gamma} f(z)\,dz=\int_{L_1} f(z)\,dz+\int_{L_2} f(z)\,dz \tag{15}$$

and, taking into account the parametrization of L_1 and L_2, the following relationships are obtained:

$$\int_{L_1} f(z)\,dz=\lim_{\theta\to 0}\int_r^R f\left(\omega e^{i\theta}\right)e^{i\theta}d\omega=\int_r^R f(\omega)\,d\omega \tag{16}$$

$$\int_{L_2} f(z)\,dz=\lim_{\theta\to 0}\int_r^R f\left(\omega e^{i(2\pi-\theta)}\right)e^{i(2\pi-\theta)}d\omega=-e^{2\pi i\gamma}\int_r^R f(\omega)\,d\omega \tag{17}$$

Thus, taking the limits $R\to+\infty$ and $r\to 0$, substitution of eqs.(16) and (17) into eq.(15) leads to:

$$\int_{\Gamma} f(z)\,dz=\left(1-e^{2\pi i\gamma}\right)\int_0^{+\infty} f(\omega)\,d\omega=\left(1-e^{2\pi i\gamma}\right)\,I_\gamma \tag{18}$$

Recalling that $z_k,(k=1,\ldots,N)$ are the simple zeros of $P(z)$, according to the Residue Theorem, eq. (18) may be rewritten as:

$$I_\gamma=\frac{2\pi i}{1-e^{2\pi i\gamma}}\sum_{k=1}^{N} Residue\left[f(z),z_k\right] \tag{19}$$

3 SMs and FSMs of a SDOF system excited by a white noise

Exact closed-form expressions for the SMs and FSMs for two SDOF linear systems, namely the classic linear oscillator and the fractional oscillator, both subjected to a white noise Gaussian random accelerations are here determined using the methodology proposed in the previous section.
In the first case, all the forces involved in the dynamic equilibrium of the system are related to integer-order derivatives of the response displacements $x(t)$. In this case, in fact, the equation of motion is retained in the form:

$$\ddot{x}(t)+2\zeta\omega_0\dot{x}(t)+\omega_0^2x(t)=f(t) \tag{20}$$

where ζ is the viscous damping ratio, ω_0 is the natural circular frequency, $f(t)$ is a sample of the white noise process and upper dots means time derivatives.
The second system under consideration, depicted in Fig. 2 is a fractional oscillator, in which a mass is supported by a *springpot*, i.e. a viscoelastic link whose exerted force is proportional to the fractional derivative of the response displacements $x(t)$. The equation of motion of this linear fractional oscillator can be expressed as:

$$\ddot{x}(t)+\eta\left({}^CD_{0^+}^{\beta}x\right)(t)=f(t) \tag{21}$$

being β the order of the fractional derivative, $\eta>0$ a real coefficient and the fractional operator $\left({}^CD_{0^+}^{\beta}x\right)(t)$ the so-called *Caputo's fractional derivative* of order β of the displacements $x(t)$, defined as:

$$\left({}^CD_{0^+}^{\beta}x\right)(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t\left(t-\bar{t}\right)^{-\beta}\dot{x}(\bar{t})\,d\bar{t} \tag{22}$$

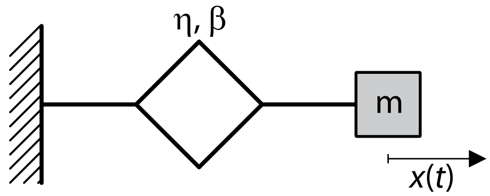

Fig. 2. Fractional oscillator

Since both systems in eq. (20) and eq. (21) are linear time-invariant systems, the PSD of the system response can easily be determined by rewriting the equation of motions in the frequency domain. In fact, by taking the Fourier transform of both sides of eqs. (20) and (21), the PSD of the response is obtained as:

$$G_X(\omega) = |H(\omega)|^2 G_0 \tag{23}$$

where $G_X(\omega)$ is the one-sided PSD function of the response process, G_0 is the white noise intensity, and $H(\omega)$ is the transfer function of the oscillator which, in turn, assumes the following expressions for classic and fractional oscillators, respectively:

$$H_c(\omega) = \frac{1}{\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega}; \qquad H_f(\omega) = \frac{1}{-\omega^2 + \eta(i\omega)^\beta} \tag{24}$$

3.1 SMs and FSMs for classic SDOF oscillators

In this case, the m-order SM is expressed as:

$$\lambda_{m,X} = \int_0^{+\infty} \frac{\omega^m G_0}{\left(\omega_0^2 - \omega^2\right)^2 + (2\zeta\omega_0\omega)^2} d\omega \tag{25}$$

The singularities of the integrand function in eq.(25) are:

$$z_k = \pm\omega_0\left(\sqrt{1-\zeta^2} \pm i\zeta\right) \qquad k = 1, \ldots, 4 \tag{26}$$

In this case, non-divergent SMs exist only for $m < 3$. Using eq. (10), the following expressions for the SMs are obtained:

$$\lambda_{0,X} = \frac{\pi G_0}{4\,\omega_0^3\,\zeta}; \quad \lambda_{1,X} = G_0 \frac{\pi - 2\alpha}{4\,\omega_0^2\,\zeta\sqrt{1-\zeta^2}}; \quad \lambda_{2,X} = \frac{\pi G_0}{4\,\omega_0\,\zeta} \tag{27}$$

in which $\alpha = \arctan\left(\zeta/\sqrt{1-\zeta^2}\right)$. It should be stressed that these closed-form expressions for the SMs are well-known in literature, and here they are reported to demonstrate the ease and correctness of the methodology.
The proposed approach can also be applied for the evaluation of γ-order FSMs ($-1 < \mathrm{Re}[\gamma] < 3$ and $\gamma \neq 0, 1, 2$), defined as:

$$\lambda_{\gamma,X} = \int_0^{+\infty} \frac{\omega^\gamma G_0}{\left(\omega_0^2 - \omega^2\right)^2 + (2\zeta\omega_0\omega)^2} d\omega \tag{28}$$

The singularities in eq. (28) are the same reported in eq. (26) and, after some algebraic manipulation, the exact closed-form FSMs are obtained as:

$$\lambda_{\gamma,X} = \frac{\pi G_0}{4\,\zeta\,\omega_0^{3-\gamma}} \sec\left(\frac{\pi\gamma}{2}\right)\left[\cos\left(\alpha\gamma - \frac{\pi\gamma}{2}\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\alpha\gamma - \frac{\pi\gamma}{2}\right)\right] \tag{29}$$

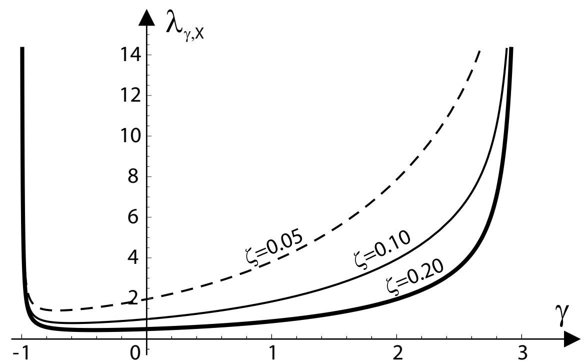

Fig. 3. FSMs for classic linear oscillators with varying damping ratios

To the best of the authors' knowledge, eq. (29) has never been presented in literature. It is worth noting that the well-known expression in eq. (27) can be directly obtained by taking the limit of eq. (29) as γ approaches 0, 1 and 2, respectively. Moreover, the proposed methodology applies for integrand functions with simple singularities.
Fig. 3 reports γ-order FSMs against γ for three selected values of the damping ratio ζ, considering $\omega_0 = 2\,\mathrm{rad/s}$ and unitary value of G_0. These FMSs are determined for $\gamma \in \mathbb{R}$, but for complex-order moments similar results can be obtained.

3.2 SMs and FSMs for fractional SDOF oscillators

For the fractional oscillator depicted in Fig. 2, the FSMs are defined as the following integral:

$$\lambda_{\gamma,X} = \int_0^{+\infty} \frac{\omega^\gamma G_0}{\omega^4 - 2\cos\left(\beta\frac{\pi}{2}\right)\eta\omega^{\beta+2} + \eta^2\omega^{2\beta}}\, d\omega, \tag{30}$$

where $\beta \in (0,1)$ is the fractional derivative order. The integral in eq. (30) can be written, by considering the substitution of variables $t = \eta\omega^{\beta-2}$, as follows:

$$\lambda_{\gamma,X} = \frac{G_0\eta^{\frac{3-\gamma}{\beta-2}}}{2-\beta} \int_0^{+\infty} \frac{t^h}{t^2 - 2\cos\left(\beta\frac{\pi}{2}\right)t + 1} dt \tag{31}$$

where $h = (\gamma - \beta - 1)/(\beta - 2)$. The domain of existence of the integral in eq. (30) is depicted in Fig. 4, i.e. the FSMs for this system exists only if $2\beta - 1 < \mathrm{Re}[\gamma] < 3$.
The singularities z_k of the integrand function in eq. (31) are determined as:

$$z_k = \eta^{\frac{1}{\beta-2}} e^{\pm i\frac{\pi\beta}{2(2-\beta)}} \qquad k = 1,2 \tag{32}$$

for every $\gamma \neq \beta + 1$. Since $h \in \mathbb{C}$, the evaluation of FSMs is performed by applying eq. (19), resulting in the following exact closed-form expressions:

$$\lambda_{\gamma,X} = \frac{\pi G_0 \eta^{\frac{\gamma-3}{\beta-2}}}{\beta - 2} \csc(\pi h) \frac{\sin\left[\pi h \frac{\beta-2}{2}\right]}{\sin\left(\frac{\pi}{2}\beta\right)} \tag{33}$$

In the particular case $\gamma = \beta + 1$, i.e. $h = 0$ and the FSMs are determined by eq. (10), which leads to

$$\lambda_{(\gamma=\beta+1),X} = \frac{\pi G_0}{2\eta} \csc\left(\frac{\pi\beta}{2}\right) \tag{34}$$

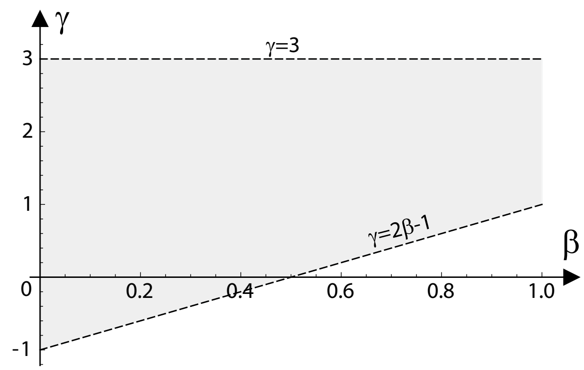

Fig. 4.   Existence dominion for FSMs of fractional oscillators

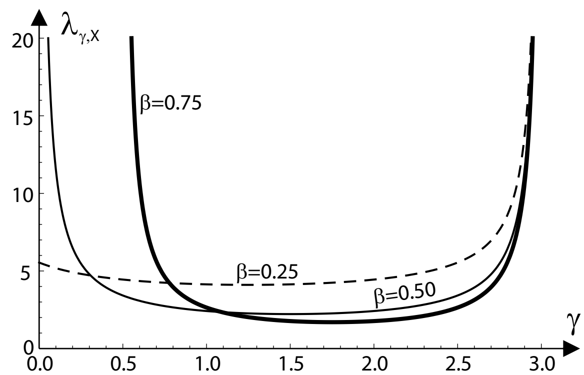

Fig. 5.   Real-order FSMs for fractional oscillators

Fig. 5 shows γ-order FSMs for three selected β and for unitary G_0 and η. The FSMs reported in Fig. 5 are restricted to $\gamma \in \mathbb{R}$, although similar results can be obtained for complex-order moments.
As specific cases of eq.(33), the SMs from order zero to two can be derived in exact form as:

$$\lambda_{0,X} = \frac{\pi G_0 \eta^{\frac{3}{\beta-2}}}{\beta-2} \csc\left(\pi \frac{\beta+1}{\beta-2}\right) \cot\left(\pi \frac{\beta}{2}\right) \tag{35}$$

$$\lambda_{1,X} = \frac{\pi G_0 \eta^{\frac{2}{\beta-2}}}{\beta-2} \csc\left(\pi \frac{\beta}{\beta-2}\right) \tag{36}$$

$$\lambda_{2,X} = \frac{\pi G_0 \eta^{\frac{1}{\beta-2}}}{\beta-2} \csc\left(\frac{\pi(1-\beta)}{\beta-2}\right) \cot\left(\frac{\pi\beta}{2}\right) \tag{37}$$

It is worth reminding that $\lambda_{0,X}$ and $\lambda_{2,X}$ are equal to the variances of the displacements and velocities of the fractional system, respectively. Moreover, these SMs can be used straightforwardly to determine the central frequency and bandwidth factor of the response process.

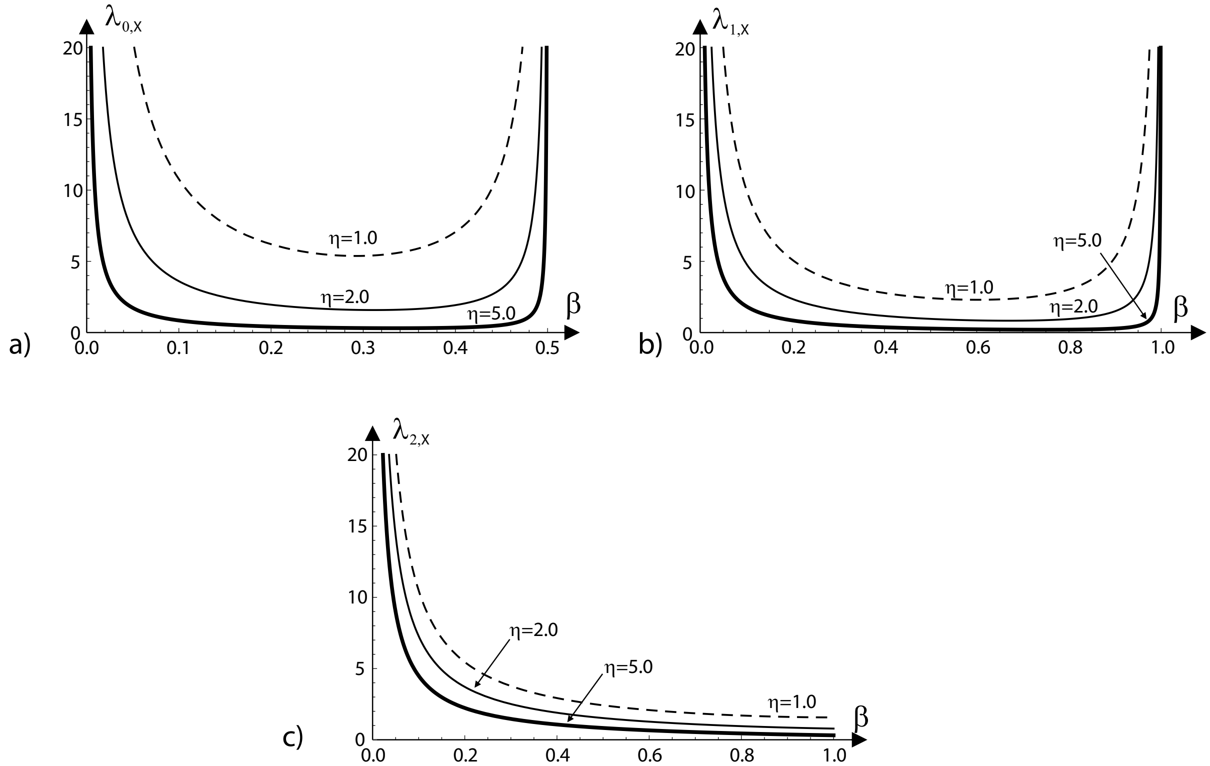

Fig. 6. SMs of order 0, 1 and 2 for fractional oscillators

The SMs in eqs. (35)–(37) are shown in Fig. 6 against the fractional derivative order β, for three selected values of the coefficient η. It is worth to remark that $\lambda_{0,X}$ in eq. (35) exists only for $0<\beta<1/2$, while $\lambda_{1,X}$ and $\lambda_{2,X}$ converge at least for $0<\beta<1$.

4 Concluding remarks

This paper presents novel exact closed-form solutions for the spectral moments (SMs) of integer and complex order, i.e. fractional spectral moments (FSMs), for two distinct SDOF linear systems, excited by a white noise process. The mathematical problem has been approached using the Residue Theorem. In particular, the integrand function has been extended to the complex domain, and its line integral has been performed on the boundary of an opportunely defined keyhole shaped domain.

A classic dynamic system, with linear stiffness and viscous damping, has been considered at first. For this case, the SMs of integer order are well-known in literature and have been evaluated to show the ease of the mathematical approach. Then, the extended exact formula for the FSMs has been obtained, and it has been shown that their limit value return exactly the integer order SMs.

In the second case, a fractional oscillator has been analysed. The latter is constituted by a mass, linked to the ground through a viscoelastic element (*springpot*) exerting a force proportional to the fractional derivative of the mass displacements. Also for this system, the FSMs have been evaluated in exact closed-form, while the integer order SMs have been obtained as their particular cases.

To the best of the authors' knowledge, the exact solutions for the FSMs of classic and fractional oscillators have been unknown till now, as well as the SMs for the fractional case. It is worth stressing that these results also imply the knowledge of the exact solutions for the variance of the displacements and velocities of the fractional system, while the exact form of the SMs of classic dynamic systems allow to analytically approach problems as first passage or barrier cross rates. Moreover, in both cases, the FSMs act as alternative complete representation of the power spectral density of the system response and can be adopted for digital simulation techniques.

The mathematical approach used to determine both the SMs and FSMs is absolutely general and could be applied to other stationary excitations. In fact, further studies are currently in progress to determine similar exact SMs and FSMs for systems excited by different stationary stochastic processes, as well as extending the results to multi-degrees of freedom systems with and without fractional elements.

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