Analysing university closed book examinations using two frameworks

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Assessment influences students' approaches to learning and conveys to the learners what the exam-setters value. Frameworks have been developed in order to understand and analyse the demands of the assessment tasks. In this paper, two frameworks are used to analyse one undergraduate closed book examination in abstract algebra. The analysis of the tasks resulting from the two frameworks are presented and discussed. Finally, some aspects regarding the applicability of the frameworks are highlighted and further steps are suggested.

Keywords: Undergraduate, closed book examinations, abstract algebra.

INTRODUCTION

Research highlights the strong relationship between assessment demands and students' approaches to learning (Ramsden, 1983; Trigwell & Prosser, 1991). Assessment also conveys what lecturers consider significant about their subject (Smith et al., 1996; van de Watering et al., 2008), thus it is important to examine and understand the demands of assessment.

In mathematics departments in the United Kingdom, the predominant method of summative assessment is the closed book examinations (Iannone & Simpson, 2011). This paper focuses on one closed book examination from a Year 2 course in pure mathematics. Three tasks from this examination are analysed using two frameworks: Mathematical Assessment Task Hierarchy (MATH) developed by Smith and colleagues (1996) and the framework introduced by Tang, Morgan and Sfard in 2012. The analysis of the tasks using both frameworks is presented and discussed. This will allow us to better understand the potential of each and to improve their use for specific research questions.

In what follows we first present the two frameworks and introduce the context of our study. Afterwards, we analyse in detail one of the examination tasks and offer an overview of the analysis of the rest. We then discuss the results, comment on the applicability of the frameworks and make some suggestions for further research.

THE FRAMEWORKS

Different frameworks exist offering ways of analysing the tasks used in assessment. One of the most common is Bloom's taxonomy of educational objectives (Bloom et al., 1956). This taxonomy examines the different educational objectives the educators set for their students and assists them in developing balanced assessments. Different adaptations of this taxonomy have been used in mathematics. One of those adaptations tailored specifically for undergraduate mathematics closed book examinations is offered by a team of mathematicians and mathematics educators (Smith et al., 1996). Smith and colleagues (1996) introduce the MATH taxonomy aiming to assist lecturers in constructing examinations demanding a range of knowledge and skills. They distinguish between eight categories of knowledge and skills and they propose three groups: A, B and C (Table 1). In solving Group A tasks, the students are asked to recall factual knowledge and fact systems, comprehend factual knowledge and be able to use basic procedures. Students have to display the ability to transfer information and apply information or methods in new situations when attempting to answer tasks belonging to Group B. Finally, in answering tasks from Group C students are asked to justify and interpret a result, offer conjectures and comparisons and evaluate results. Smith et al. argue that examinations should have items from all the groups and note that items from Group A could guide students to adopt surface learning approaches (Ramsden, 1992) whereas items from Group B and

Group A	Group B	Group C
Factual knowledge and fact systems	Information transfer	Justification and interpretation
Comprehension of factual knowledge	Application to new situations	Implications, conjectures and comparisons
Routine use of procedures		Evaluation

Table 1: The MATH taxonomy

C might help them foster deep learning approaches (Ramsden, 1992).

While the MATH taxonomy was developed to assist in the creation of examinations assessing a range of knowledge and skills, the framework introduced by Tang, Morgan and Sfard (2012) was developed to characterise the discourse of school mathematics. Aiming to examine whether the nature of students' participation in the mathematical discourse changed in the last thirty years, they focused on analysing the public examinations in the UK taken at age 16 (GCSE – General Certificate of Secondary Education). This framework draws on Systemic Functional Linguistics (Halliday, 1978; Morgan, 2006) and Sfard's theory (2008) of commognition. The framework explores the mathematical discourse that the student engages with when reading and responding to an examination task. This analysis "allows a subtle characterisation of the nature of mathematics and of student mathematical activity construed through the forms of language used" (Morgan & Tang, 2012, p. 242). The framework has two components: mathematics and the student. The mathematics component characterises the mathematical discourse the student is expected to engage in when reading and solving the task. This is further distinguished using Sfard's theory (2008) in four categories: vocabulary and syntax, visual mediators, routines and endorsed narratives. The student component investigates the relation between the examination task and the student. More specifically, it examines the positioning of the student in relation to the exam-setter, the presence of human beings, in the task, engaged in everyday or mathematical activities, and finally the decisions and directions shaping the student's response.

METHODOLOGY

For the purpose of the paper, we analyse examination tasks from an abstract algebra course in a mathematics department in the UK. This is a compulsory course and focuses on linear algebra in the autumn term and on group and ring theory in the spring term. Our data consists of the coursework tasks, the examinations tasks focused on group and ring theory and their model solutions, produced for departmental use, and was collected for a doctoral study (Ioannou, 2012). The examination accounted for 80% of the course's final grade and selected coursework tasks accounted for the remaining 20%. This examination, which is the focus of our analysis, had six tasks: three on linear algebra and the remaining three on group and ring theory. The examination lasted three hours and the students had to respond to five of the tasks. Notes were not permitted in the examination and the students were told that they could use the general theorems without proof unless stated otherwise. Non-programmable calculators were permitted during the examinations. In what follows we present a detailed analysis of task 4 (Figure 1) followed by an overview of the analysis of the other tasks on group and ring theory (Figure 2).

4)

(1) Describe the group S of rotational symmetries of a solid cube in $\mathbb{R}^3$. List the possible axes of rotation and angles of rotation, and hence show that $|S| = 24$. Let l be an axis passing through the centres of a pair of opposite faces of the cube and T be the set of rotations in S which send l to itself. Prove that T is a subgroup of S and $|T| = 8$.

(2) Suppose G is a group and H a subgroup of G. Prove that the relation $\sim$ on G given by $g_1 \sim g_2$ if and only if $g_1^{-1}g_2 \in H$ is an *equivalence relation*, saying carefully what this means. In the case where G is a finite group, prove that all equivalence classes have $|H|$ elements.

(3) State Lagrange's Theorem, and use (2) to give a proof of this.

Figure 1: Examination task 4

ANALYSIS USING THE MATH TAXONOMY

Task 4 consists of three subtasks. In (4.1) the students have to describe the 24 elements of the group of rotational symmetries of a solid cube, listing all the possible axes and angles of rotation. This is classified as information transfer since the students have to visualize the cube, identify the axes and angles and describe the elements. Afterwards, in the same subtask, the students are asked to examine whether a specific set of rotational symmetries is a subgroup and to prove that the order of the subgroup is 8. Here, the students are asked to select from the 24 elements the ones satisfying the criteria of sending l to itself. This is categorised as evaluation.

In the second subtask (4.2) students have to examine whether the given relation is an equivalence relation. In the MATH taxonomy the process of deciding whether the conditions of a definition are satisfied belongs to different categories depending on the definition of the concept. If the definition is considered simple it belongs in the comprehension category and if "understanding [*the definition*] requires a significant change in the students' mode of thought or mathematical knowledge" (Smith et al., 1996, p. 69) it is considered a conceptual definition and belongs to the information transfer. Here, the concept of the equivalence relation is considered a conceptual definition and thus classified as information transfer. Then, the students need to comment on the form of the equivalence classes defined by this relation, which are actually the left cosets, and this is classified as comprehension. Finally, the subtask asks to prove that if the group G is a finite group then the equivalence classes formed from the relation have the same elements. The students have to examine whether these left cosets have the same order as the subgroup H by defining a bijective function. This is categorised as comprehension, as the students have to show understanding of the equivalence classes' concept and define the bijective function.

The students, at subtask (4.3), must state Lagrange's theorem and prove this theorem using the knowledge demonstrated previously in subtask (4.2). This subtask is classified as factual knowledge and fact systems and justifying and interpreting.

The skills and knowledge needed to respond to tasks 5 and 6 (Figure 2) are classified as factual knowledge

5)

(1) Suppose G is a group.

(a) What does it mean to say that a subgroup N of G is a *normal* subgroup? If N is a normal subgroup of G, explain how to make the set G/N of left cosets of N in G into a group.

(b) State the First Isomorphism Theorem for groups, defining the terms *kernel* and *image* in your statement.

(c) Suppose H is a cyclic group. By defining a suitable homomorphism $\phi : (\mathbb{Z}, +) \to H$, or otherwise, prove that $H \cong \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{Z}$.

(2) Let R be the ring $\mathbb{Z}[\sqrt{-7}] = \{m + n\sqrt{-7} : m, n \in \mathbb{Z}\}$. In the following you may use the fact that the function $N : R \to \mathbb{Z}$ given by $N(m + n\sqrt{-7}) = m^2 + 7n^2$ satisfies $N(ab) = N(a)N(b)$ for all $a, b \in R$.

(a) Prove that the only units in R are ± 1.

(b) Give two different factorizations of 8 into irreducibles in R and deduce that R is not a unique factorization domain. You should justify carefully any assertions you make about irreducibility of various elements of R.

6)

(1) Suppose R is a commutative ring. What is meant by saying that a subset $I \subseteq R$ is an *ideal* of R? Suppose $f \in R$ and let $fR = \{fr : r \in R\}$ be the *principal ideal* generated by f. Prove that this *is* an ideal of R. Prove that, for $f, g \in R$

$$fR \subseteq gR \Leftrightarrow g \text{ divides } f \text{ in } R.$$

(2) Prove that the element $f = x^3 + x^2 + x - 1$ in the polynomial ring $R = \mathbb{F}_3[x]$ is an irreducible (where $\mathbb{F}_3$ denotes the field with 3 elements). Explain why the ideal $I = fR$ is a maximal ideal in R. What does this imply about the quotient ring R/I? Let $h = x^4 - x^2 + 1 \in R$. Is hR a maximal ideal in R? Justify your answer.

Figure 2: Examination tasks 5 and 6

and fact systems, information transfer, application to new situations and justification and interpretation.

ANALYSIS USING THE TANG, MORGAN AND SFARD FRAMEWORK

First, we present the analysis regarding the student component. Considering the student and exam-setter relationship, we examine whether the students are given commands or asked to examine mathematical questions. In this task, the students are given imperative instructions. Regarding the directions, the students are directed to present the group S in a specific way ("list…rotation"), they are also given directions, by implication ("and hence show") on how to prove that the order of the group is 24. Also, they are given directions, by instruction in subtask (4.3) as it specifically states that they should use the facts proved in (4.2) to prove Lagrange's theorem. Regarding the depth and accuracy of the expected response, the students are directed in (4.2) ("saying carefully what this means"). Examining the decisions the students have to make, we observe that in (4.1) the students can decide whether or not to provide visual representations of the rotational symmetries of the cube. Finally, in our analysis of these tasks we do not consider the presence of the human beings as the tasks do not offer descriptions of human beings engaged in mathematical or everyday activities.

In our analysis regarding the mathematics component we focus on the routines, due to space limitation. The routines discuss the patterns observed in the discursants' activity when attempting to construct and substantiate narratives endorsed by the mathematical community (Sfard, 2008). Here, according to the framework, we examine the form of student engagement and the areas of mathematics involved. The imperatives, present in the tasks, are analysed in order to examine the student's engagement which is distinguished in: engagement in material processes, construing the student's role as a 'scribbler'; and engagement in mental processes, construing the student as a 'thinker' (Rotman, 1988). Here, the students are asked to engage in material actions ("describe", "list", "state", "use") as well as in mental activity ("show", "let", "prove", "suppose"). Relating to the areas of mathematics involved we see that the students have to engage with concepts from set and group theory, but also to demonstrate knowledge from geometry when asked on the rotational symmetries of the cube. Finally, we offer a categorisation of the routines, using Sfard's theory, into: construction, resulting in new endorsable narratives; substantiation, assisting in the decision to endorse a previously constructed narrative; and recall, bringing to mind previously endorsed narratives (Sfard, 2008, p. 225). In this task, the students are requested to engage in a construction routine when asked to describe the elements of the group S and in a substantiation routine when examining which of the elements of S send l to itself. In (4.2), the students have to engage in a substantiation routine, as they have to verify the definition of the equivalence relation. Then they have to prove that all the equivalence classes have $|H|$ elements engaging in a construction routine in order to construct one of the classes and the mapping of the elements; and next in a substantiation routine where they have to examine that the mapping is bijective. In the last subtask, the students are required to state and prove Lagrange's theorem using (4.2). Consequently, the students have to engage in a recall and a substantiation routine.

In tasks 5 and 6 the students' actions are pre-shaped since they are given explicit or implicit directions regarding the presentation, the depth and accuracy of their response and the methods. There are only a few instances where the students can decide on the method and the degree of accuracy. The tasks involve number theory, set theory, group and ring theory. We also note that the student's role is construed both as scribbler and thinker. Finally, the students are asked to engage in construction, substantiation and recall routines.

DISCUSSION AND SUGGESTIONS FOR FURTHER RESEARCH

The aim of this paper was to uncover the potential of two frameworks in analysing examination tasks. To this aim we discuss the results obtained from the analysis of three examination tasks and we offer here some reflections on the application of the frameworks.

The MATH taxonomy highlights the nature of the skills needed to respond correctly to the task. The students are asked to demonstrate their knowledge of the basic concepts and theorems used in the course. They are required not only to remember but to show their understanding of them too. In order to solve these tasks the students have to demonstrate factual knowledge and fact systems, comprehension of factual knowledge, information transfer, application in new situations, justification and interpretation and evaluation. The analysis using the Tang and colleagues (2012) framework highlights that most of the students' actions are pre-shaped with the implicit or explicit directions given to them, allowing them to be autonomous in very few cases. Furthermore, the student's role is interpreted as both a scribbler and a thinker engaging with material actions and mental activity. The students are asked to engage with concepts from the following mathematical areas: geometry, number theory, set theory, group and ring theory. Finally, the students engage in recall, substantiation and construction routines in different parts of the tasks.

Both frameworks deal with the concept of familiarity through the categories of recall routine or factual knowledge and factual systems. We should point out that familiarity, a highly contextual and subjective concept, is not clearly defined in either one. A task may be considered as familiar to some students and thus require them to engage in a memory retrieval procedure, while the same task presented to students exposed to different teaching material might require them to engage in mathematical activities of a different nature. Also, familiarity can be different for the individual students belonging to the same teaching group as each one engages differently with the given material. In our analysis we classified a task as belonging to the categories above when it required stating a definition or a theorem. However, we should note that some parts of the tasks were given to the students as coursework although the model solutions of these tasks were not made available to them, as these were the ones assessed in the coursework. Here we should report that there is a framework which examines the concept of familiarity more rigorously. Bergqvist (2007) used a framework developed by Lithner (2008) regarding the reasoning expected of the students. She analysed examination tasks from a Swedish university and categorised them into tasks requiring imitative or creative reasoning. A task was classified as demanding imitative reasoning if it asked for a fact or a theory item, for which the students were clearly informed that it might be requested in the examinations; or the task occurred at least three times in the textbooks of the course. Note that the context of our study is different from the one in Sweden, as in the United Kingdom the students mostly rely on their lecture notes and not on textbooks.

In our analysis of the tasks using the MATH taxonomy we were unable to exclusively classify one task into one of the eight categories. Furthermore, we experienced difficulties when trying to position the tasks in some categories as we didn't have clear instructions regarding the effect of the background information on the classification of the tasks. However, both of these issues are related to the origin of the taxonomy. The taxonomy was developed to assist lecturers in creating balanced examinations and not as a research tool as illustrated in the quote below:

> [I]t is not our aim to be able to uniquely characterize every conceivable assessment task. Rather, the aim of the descriptors is to assist in writing examination questions, and to allow the examiner's judgement, objectives and experience to determine the final evaluation of an assessment task. (Smith et al., 1996, p. 68)

Some interesting aspects of the tasks are pointed out using the Tang and colleagues (2012) framework. More specifically by examining the areas of mathematics involved we gain information regarding students' engagement with mathematical concepts from other mathematical areas than the one that the course focuses on. This aspect is not highlighted in the MATH taxonomy as the focus is on the activity and not on the areas of mathematics. To be more specific, the tasks analysed here are assessment tasks from an abstract algebra course, but in order to solve them the students have to display knowledge of geometry (4.1), number theory (5 and 6) and set theory, which are not the focus of this course. This emphasises the nature of the mathematics involved; the prior knowledge expected from the students and also examines the students' ability to draw on different areas of mathematics.

The level of guidance given to the students and the degree of their autonomy when solving a task is also highlighted by the Tang and colleagues (2012) framework. As it examines the directions given to the student; and the complexity of the response expected of the students, namely the decisions they have to make. Examining a task in this respect might provide some information on the exam-setters' perceptions of their students, though this would need to be confirmed with interviews with the exam-setters. The explicit or implicit directions on the method may display what the exam-setters value or think that their students would be able to manage better. Similarly, by examining the directions regarding the presentation of the response we gain information on the exam-setters' perception of the depth and the accuracy of students' responses. For example we have three instances where the students are explicitly asked to provide a response with a certain degree of accuracy and depth ("saying carefully what this means" (4.2), "you should justify carefully" (5.2.b) and "justify your answer" (6.2)). On the other hand, we have the decisions the students had to make in these tasks on the degree of accuracy (5.1.a) or the presentation of their response (4.1). It would be interesting to examine how the accuracy and the depth of the students' responses in this case are assessed by their examiners. Finally, there were some decisions the students had to make regarding the method of solution. One of them was explicitly stated: "By defining ... or otherwise" (5.1.c), but having in mind the students' knowledge of the subject their methods of solution are limited to the specific methods they have encountered in the course.

The classification process of the examination tasks is highly subjective as a response to the task is taken into account. In order to position the task in a specific category in the MATH taxonomy and in order to examine the routines using Sfard's theory (2008) we have to consider a possible solution to the task. The final classification depends on the researchers' choice of response and their opinion of that response (Jolliffe & Ponsford, 1989). In our attempt to reduce the subjectivity of our classification we took into account the model solutions produced by the lecturer of the course for departmental use (Figure 3).

Choosing this response for our analysis we investigate the lecturer's expectation of the students' solution and not the actual solutions produced by the students. In order to examine the actual routines the students engage in, for the Tang and colleagues (2012) framework, and the skills and knowledge, for the MATH taxonomy, we need to examine the solutions produced by the individual students. Follow up interviews with the students are also necessary as different students can employ different routines and different skills and knowledge, depending on their background, to arrive to the same solution.

In conclusion, our analysis of the same examination tasks using two different frameworks highlights some interesting aspects regarding the two frameworks and their applicability. We should note that in our analysis of the Tang and colleagues (2012) framework from the mathematics component we took into account only the routines aspect. We intend to explore the results from the other aspects of the framework namely vocabulary and syntax, visual mediators and endorsed narratives. Finally, in the following stages of this research, we aim to seek the views of the lecturers, who

(1) S consists of the following rotations:
   (i) Axis of rotation through a pair of opposite vetrices; angle of rotation $\pm 2\pi/3$, number of this type: $4 * 2 = 8$.
   (ii) Axis of rotation through a pair of opposite faces; angle of rotation $\pi/2, \pi, 3\pi/2$, number of these: $3 * 3 = 9$.
   (iii) Axis of rotation through mid-point of a pair of opposite edges; angle of rotation π, number of these: 6.

   Together with the identity, this gives a total of $1 + 8 + 9 + 6 = 24$ rotational symmetries. T consists of 4 rotations (including the identity) of type (ii) with axis l, together with rotations about axes perpendicular to l through an angle π which 'invert' l. There are two of these of type (ii) and two of type (iii). Thus $|T| = 8$.

(2) Need to show that $\sim$ is:
   Symmetric: if $g_1 \sim g_2$ then ${g_1}^{-1} g_2 \in H$ so as $H \leq G$, $({g_1}^{-1} g_2)^{-1} = {g_2}^{-1} g_1 \in H$ i.e. $g_2 \sim g_1$.
   Reflexive: $g_1 \sim g_1$ as ${g_1}^{-1} g_1 = e \in H$.
   Transitive: if $g_1 \sim g_2$ and $g_2 \sim g_3$ then ${g_1}^{-1} g_2, {g_2}^{-1} g_3 \in H$. As H is a subgroup $({g_1}^{-1} g_2) * ({g_2}^{-1} g_3) = {g_1}^{-1} g_3 \in H$ so $g_1 \sim g_3$.
   An equivalence class is of the form $\{g_2 : g_1 \sim g_2\}$ for some $g_1 \in G$ and by definition this is $\{g_1 h : h \in H\}$
   The map $H \longrightarrow \{g_1 h : h \in H\}$, $h \longrightarrow g_1 h$ is bijective so the number of elements in the set is $|H|$.

(3) Lagrange's Theorem: If G is a finite group and H a subgroup of G then $|H|$ divides $|G|$. The equivalence classes in (2) partition G: every element of G lives in a unique equivalence class. All classes have $|H|$ elements, so $|G| = |H| x$ number of equivalence classes. Thus $|H|$ divides $|G|$.

Figure 3: Model solution of task 4

set the tasks, and the views of the students solving the tasks and relate these findings with the results from the frameworks.

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