Isospectral Orbifold Lens Spaces

by

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Abstract

Spectral theory is the study of Mark Kac's famous question [K], "can one hear the shape of a drum?" That is, can we determine the geometrical or topological properties of a manifold by using its Laplace Spectrum? In recent years, the problem has been extended to include the study of Riemannian orbifolds within the same context. In this thesis, on the one hand, we answer Kac's question in the negative for orbifolds that are spherical space forms of dimension higher than eight. On the other hand, for the three-dimensional and four-dimensional cases, we answer Kac's question in the affirmative for orbifold lens spaces, which are spherical space forms with cyclic fundamental groups.

We also show that the isotropy types and the topology of the singularities of Riemannian orbifolds are not determined by the Laplace spectrum. This is done in a joint work with E. Stanhope and D. Webb by using P. Berard's generalization of T. Sunada's theorem to obtain isospectral orbifolds.

Finally, we construct a technique to get examples of orbifold lens spaces that are not isospectral, but have the same asymptotic expansion of the heat kernel. There are several examples of such pairs in the manifold setting, but to the author's knowledge, the examples developed in this thesis are among the first such examples in the orbifold setting.

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I dedicate this thesis to my parents whose love, sacrifice and prayers have made it possible for me to overcome difficulties and challenges throughout my life.

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Contents

Chapter 1

Introduction

Many years ago, Mark Kac [K] posed a model inverse problem which has attracted the attention and energy of many mathematicians: Do the Dirichlet eigenvalues of a bounded domain determine its geometry, or, more famously, Can one hear the shape of a drum? Exercising their penchant for generalization,mathematicians recast the question in a mathematically natural setting: Does the eigenvalue spectrum of the Laplacian on a compact Riemannian manifold M (with suitable boundary conditions if $\partial M \neq \emptyset$) determine its geometry? Kac's original question was answered in the negative in 1992 by Gordon, Webb and Wolpert [GWW] who constructed examples of non-isometric isospectral plane domains. We call two compact Riemannian manifolds M_1 and M_2 isospectral if the Laplacians on functions have the same eigenvalue spectrum, and define the isospectral set of M to be the set of all Riemannian manifolds for which the spectrum of the Laplacian on functions equals that of M. As pointed out, geometry is not in general a spectral invariant. The first example of nonisometric isospectral manifolds was found in 1964 by John Milnor, who exhibited two distinct but isospectral 16-dimensional manifolds. This was followed by the construction in the 1980s and 1990s of many different examples of nonisometric but isospectral manifolds. Among these are discrete families of isospectral manifolds, continuous families of isospectral manifolds, isospectral plane domains, and even isospectral conformally equivalent manifolds. In general, there are three known methods to construct or discover these examples of nonisometric isospectral manifolds:

(1) *Explicit Construction*: Examples constructed by explicit computations include isospectral at manifolds with surprising spectral properties ([MR], [MR2], [MR3]), the first examples [Sz1] of isospectral manifolds with boundary having different local geometry(these partially motivated and were later reinterpreted by the torus action method below) and the first examples of pairs of isospectral metrics on balls and spheres [Sz2].

(2) *Representation-Theoretic Construction*: Representation theoretic methods, especially the celebrated Sunada technique [Su], have provided the most systematic and widely used methods for constructing isospectral manifolds with the same local, but different global, geometry.

(3) *Torus Actions*: This method generally produces isospectral manifolds with different local geometry.

For a more complete overview of these methods see [Go].

The problem is to characterize the isospectral set of a given Riemannian manifold M. There are two natural approaches to this problem. First, one can form spectral invariants such as the heat trace, the wave trace, or the determinant of the Laplacian and compute them geometrically in order to obtain geometric invariants of the spectrum: this approach has its roots in Selberg's trace formula for a compact surface [Se] and its natural expression in such key developments as Duistermaat-Guillemin trace formula [DG] and the computation of heat invariants by Gilkey and others (see, for example, [G1]). A second and complementary approach is to use techniques of group theory or Lie theory to construct families of manifolds with the same spectrum but distinct geometry. A key development in this approach was the celebrated paper of Sunada [Su] in which he showed how to reduce the construction of isospectral manifolds to an exercise in group theory.

To any compact Riemannian manifold (M, g) (with or without boundary), we can associate a second-order partial differential operator, the Laplace operator Δ, defined by $\Delta\ (f) = div(grad(f))$ for $f \in L^2(M, g)$. Sometimes it is also written as Δ_g if we want to emphasize which metric the Laplace operator is associated with. The set of eigenvalues of Δ (the spectrum of Δ, or of M),which we will write as $spec(\Delta)$ or $spec(M, g)$, then forms a discrete sequence $\lambda_0 \leq \lambda_1 \leq ....$ For simplicity, we will assume that M is a closed connected Riemannian manifold; this will, for example imply that the smallest eigenvalue, λ_0, occurs with multiplicity 1. Note that the Laplacian also acts on p-forms in addition to functions via the definition Δ = -($d\delta$ + δd), where δ is the adjoint of d with respect to the Riemannian structure on the manifold. This aspect of the Laplacian will not be treated in this thesis, the focus being the ordinary Laplacian acting on functions or 0-forms. With that in mind, there are two broad questions that are at the heart of spectral geometry:

i What can we say about the spectrum of M given the geometry?

ii What can we say about the geometry of M given the spectrum?

The former is the *direct problem* while the latter is the *inverse problem* . We can also generalize these problems for spaces that have singular points.

A smooth n-dimensional orbifold is a topological space that is locally modeled on an orbit space of $\mathbb{R}^n$ under the action of a finite group of diffeomorphisms. Riemannian orbifolds are spaces that are locally modelled on quotients of Riemannian manifolds by finite groups of isometries. An orbifold is *good* if it is a global quotient of a closed manifold by a finite group. Otherwise, it is a *bad* orbifold. Orbifolds have wide applicability, for example, in the study of 3-manifolds and in string theory [ALR], [DHVW]. In Chapter 2, which will have all the background material, we will formally define the notion of a Riemannian orbifold and review the fundamental ideas necessary for studying orbifold geometry. We will also define the Laplace Beltrami operator on an orbifold and state some of the results from the spectral theory of manifolds that carry over to the orbifold setting.

The tools of spectral geometry can be transferred to the setting of Riemannian orbifolds by using their well-behaved local structure (see [Ch], [S1] and [S2]). As in the manifold setting, the spectrum of the Laplace operator of a compact Riemannian orbifold is a sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 ... \uparrow \infty$, where each eigenvalue is repeated according to its finite multiplicity. We say that two orbifolds are isospectral if their Laplace spectra agree.

The literature on inverse spectral problems on orbifolds is less developed than that for manifolds. Examples of isospectral orbifolds include pairs with boundary ([BW] and [BCDS]); isospectral flat 2-orbifolds ([DR]); arbitrarily large finite families of isospectral orbifolds ([BSW]); isospectral orbifolds with different maximal isotropy orders ([RSW]); isospectral deformation of metrics on an orbifold quotient of a nil-manifold ([PS1]); and isospectral orbifold lens spaces ([Ba]).

Orbifolds began appearing sporadically in the spectral theory literature in the early 1990s and have received more concentrated attention in the last few years. Farsi [F] showed that the spectrum of an orbifold determines its volume by proving that Weyl's asymptotic formula holds for orbifolds. Dryden and Strohmaier [DS] showed that, for a compact and negatively curved two-dimensional orbifold,the Laplace spectrum determines both the length spectrum and the orders of the singular points and vice versa; on the other hand, Doyle and Rossetti [DR] gave(disconnected) examples of isospectral flat two-dimensional orbifolds with different length spectra and orders of singular points.

For Riemannian manifolds, the asymptotic expansion of the heat kernel can be used to relate the geometry of the manifold to its spectrum. From the so-called heat invariants appearing in the asymptotic expansion, one can tell the dimension, the volume, and various quantities involving the curvature of the manifold. In the case of a good Riemannian orbifold (i.e., an orbifold arising as the orbit space of a manifold under the action of a discrete group of isometries), Donnelly [D] proved the existence of the heat kernel and constructed the asymptotic expansion for the heat trace. In [DGGW], Dryden, Gordon, Greenwald, Webb and Zhu generalized Donnelly's work to the case of general compact orbifolds. We will discuss the results in [D] and [DGGW] in Chapter 6 in more detail.

A very interesting question in the spectral geometry of orbifolds is how much one can hear about the structure of the singular set. For example, we can ask about the isotropy types of the singular strata. As noted before, on the positive side, Dryden, Gordon, Greenwald and Webb [DGGW] have shown that the Laplace spectrum determines the number and type of singularities in two-dimensional orbifolds with positive Euler characteristic. In [RSW] Rossetti, Schueth and Weilandt constructed pairs of compact Riemannian orbifolds which are isospectral for the Laplace operator on functions such that the maximal isotropy order of singular points in one of the orbifolds is higher than in the other. In one type of examples, isospectrality is shown to arise from a version of the Sunada Theorem [Su] which also implies isospectrality on p-forms; here the orbifolds are quotients of certain compact normal homogeneous spaces. In another type of examples, the orbifolds are quotients of Euclidean $\mathbb{R}^3$ and are shown to be isospectral on functions using dimension formulas for the eigenspaces developed in [MR]. In the latter type of examples the orbifolds are not isospectral on 1-forms. Along the way the authors also gave several additional examples of isospectral orbifolds which do not have maximal isotropy groups of different size but other interesting properties.

In addition, Stanhope [S2] proved the following result for orbifold singular sets:

Theorem 1.0.1. *Only finitely many isotropy types may arise in a family of isospectral orbifolds whose members have a uniform lower bound on Ricci curvature.*

However, the author along with Stanhope and Webb [BSW] constructed arbitrarily large (finite) families of isospectral orbifolds with different isotropy types, showing that Theorem 1.0.1 is not true in general. Indeed, we construct arbitrarily large families of isospectral orbifolds with different isotropy types: given an odd prime P and an integer $m \geq 1$, we constructed an $(m+1)$-element family of isospectral $(P^{3m}-1)$-dimensional orbifolds, each with points of distinct isotropy. The orbifolds in these families are quotients of the round sphere by properly discontinuous orthogonal actions. The author's contribution was to construct an example to show that that some topological properties of the singular set of an orbifold are not spectrally determined. Chapter 3 of this thesis will contain this result.

A related question is whether a manifold could be isospectral to an orbifold with non-trivial isotropy. In [GR], Gordon and Rossetti showed that whenever two isospectral good orbifolds share a common Riemannian cover, their respective singular sets are either both trivial or both non-trivial. This means that a good orbifold with non-trivial isotropy could not be isospectral to a manifold that shares the same Riemannian cover.

In the study of inverse isospectral problem, spherical space forms provide a rich and important set of orbifolds with interesting results. For the 2-dimensional case, it is known [DGGW] that the spectrum determines the spherical orbifolds of constant curvature $R > 0$. In [L], Lauret found examples in dimensions 5 through 8 of orbifold lens spaces (spherical orbifold spaces with cyclic fundamental groups) that are isospectral but not isometric. For dimension 9 and higher, the author proved the existence of isospectral orbifold lens spaces that are non-isometric [Ba]. Chapters 4 of this thesis will contain this result as a corollary to Theorem 4.5.5. We will conclude Chapter 4 with an example demonstrating the results.

For 3-dimensional manifold lens spaces Ikeda and Yamamoto (see [I1], [IY] and [Y])proved that the spectrum determines the lens space. In [I2], Ikeda further proved that for general 3-dimensional manifold spherical space forms, the spectrum determines the space form. In the manifold case, it is also known that even dimensional spherical space forms are only the canonical sphere and the real projective space. For orbifold spherical space forms this is not the case. In this thesis we limit our study to orbifold spherical space forms where the fundamental group is cyclic, i.e. the space forms are lens spaces. In Chapter 5 of this thesis we will develop our proofs for two of our main results:

Theorem 5.1.1 Two three-dimensional isospectral orbifold lens spaces are isometric.

Theorem 5.2.1 Two four-dimensional isospectral orbifold lens spaces are isometric.

The above two results will complete the classification of the inverse spectral problem on orbifold lens spaces in all dimensions.

As mentioned earlier, a major tool in determining the things that can be heard is the asymptotic expansion of the heat kernel. It is known that two isospectral manifolds(or orbifolds) will have the same asymptotic expansion of the trace of the heat kernel. The converse, however, is not true. There are many examples of pairs of non-isospectral manifolds having the same finite cover which have the same asymptotic expansion of the trace of the heat kernel. For example, the asymptotic expansion of the trace of the heat kernel for a flat 2-dimensional torus T and a Klein bottle K are given by $\frac{\sqrt{VolT}}{4\pi t}$ and $\frac{\sqrt{VolK}}{4\pi t}$, respectively. That means if the two manifolds have the same volume they will have the same asymptotic expansion even when they are not isospectral [RS]. To the author's knowledge, no such examples are known for orbifolds. In Chapter 6, we will use techniques developed by [D] and [DGGW] to prove a result that will allow us to create examples of orbifold lens spaces that are not isospectral, but have the same asymptotic expansion of the trace of the heat kernel.

Chapter 2

Orbifolds

An orbifold is a generalization of a manifold which is locally modelled on $\mathbf{R}^n$ modulo the action of a finite group. This allows orbifolds to possess singular sets. With this generalization many of the mathematical tools used in the study of manifolds can be defined for the study of orbifolds as well. In this chapter we will define some of these tools that will be needed throughout this thesis. The definitions we use are the ones used by Stanhope [S1] and E. Dryden, C. Gordon, S. Greenwald, D. Webb and Zhu in [DGGW].

2.1 Smooth Orbifolds

Definition 2.1.1. *Let X be a Hausdorff topological space. For an open set U in X, an* orbifold coordinate chart *over U is a triple $(U, \widetilde{U}/\Gamma, \pi)$ such that:*

1. *$\widetilde{U}$ is a connected open subset of $\mathbf{R}^n$,*

2. *Γ is a finite group of diffeomorphisms acting effectively on $\widetilde{U}$, possibly with fixed point sets, and*

3. $\pi : \widetilde{U} \to U$ *is a continuous map which induces a homeomorphism* ϕ *between* $\widetilde{U}/\Gamma$ *and* U*, for which* $\phi \circ \pi \circ \gamma = \phi \circ \pi$ *for all* $\gamma \in \Gamma$.

Next we define the concept of an embedding between orbifold charts. We assume that U and U' are open subsets of a Hausdorff space X. Let $(U, \widetilde{U}/\Gamma, \pi)$ and $(U', \widetilde{U}'/\Gamma', \pi')$ be charts over U and U', respectively.

Definition 2.1.2. *An* embedding *between orbifold charts is an injection*

$$\lambda : (U, \widetilde{U}/\Gamma, \pi) \hookrightarrow (U', \widetilde{U}'/\Gamma', \pi')$$

that consists of an open embedding

$$\tilde{\lambda} : \widetilde{U} \hookrightarrow \widetilde{U}'$$

and an injective homomorphism $f : \Gamma \hookrightarrow \Gamma'$*, such that the following diagram commutes:*

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "\tilde{\lambda}"] \arrow[d, "\pi"'] & \widetilde{U}' \arrow[d, "\pi'"] \\
\widetilde{U}/\Gamma \arrow[d, "\phi"'] & \widetilde{U}'/\Gamma' \arrow[d, "\phi'"] \\
U \arrow[r, "\lambda"] & U'
\end{tikzcd}$$

and the embedding is equivariant with respect to f, that is, for all $\gamma \in \Gamma$ *and* $x \in \tilde{U}$, $\tilde{\lambda}(\gamma(x)) = f(\gamma)(\tilde{\lambda}(x))$.

Definition 2.1.3. *A* smooth orbifold $(X, \mathcal{A})$ *consists of a Hausdorff topological space* X *together with an atlas of charts* $\mathcal{A}$ *satisfying the following conditions:*

1. *For any pair of charts* $(U, \widetilde{U}/\Gamma, \pi)$ *and* $(U', \widetilde{U}'/\Gamma', \pi')$ *in* $\mathcal{A}$ *with* $U \subset U'$ *there exists an embedding* $\lambda : (U, \widetilde{U}/\Gamma, \pi) \hookrightarrow (U', \widetilde{U}'/\Gamma', \pi')$.

2. *The open sets $U \subset X$ for which there exists a chart $(U, \widetilde{U}/\Gamma, \pi)$ in $\mathcal{A}$ form a basis of open sets in X.*

For the remainder of this thesis we will denote an orbifold $(X, \mathcal{A})$ simply by O.

Definition 2.1.4. *Let O be an orbifold and x be a point in O. Let $(U, \widetilde{U}/\Gamma, \pi)$ be a coordinate chart about x, and let $\tilde{x}$ be a point in $\widetilde{U}$ such that $\pi(\tilde{x}) = x$. Let $\Gamma^U_{\tilde{x}}$ denote the isotropy group of $\tilde{x}$ under the action of Γ. It can be shown that $\Gamma^U_{\tilde{x}}$ is independent of both the choice of lift and the choice of chart [Br]. Therefore, $\Gamma^U_{\tilde{x}}$ can be denoted by Γ_x. We call Γ_x the* isotropy group of x.

Definition 2.1.5. *Let O be an orbifold. A point $x \in O$ is said to be* singular *if Γ_x is non-trivial.*

We will denote the set of all singular points in O by Σ_O.

Note that the definition of an orbifold is a generalization of the definition of a V-manifold introduced by I. Satake [Sat]. A V-manifold is an orbifold that requires the singular set to have co-dimension ≥ 2.

We say that an orbifold is *good* if it is the orbit space of a manifold M under the smooth action of a discrete group Γ. Otherwise, it is said to be *bad*. A good orbifold O is denoted by M/Γ. We note that every point in an orbifold has a neighborhood that is a good orbifold. Further, any manifold can be viewed as a good orbifold for which all points have trivial isotropy.

Example 2.1.6. *The order 4 cyclic group generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ acts on $\mathbf{R}^2$ by 90° rotations leaving the origin fixed. If we denote this group by G, then $\mathbf{R}^2/G$ gives us a cone. This is an example of a good orbifold.*

Example 2.1.7. *Not all orbifolds are good. The $\mathbf{Z}_n$-teardrop is an example of a bad orbifold. Topologically it is homeomorphic to $\mathbb{S}^2$, and its singular set consists of* *an isolated cone point, which is locally homeomorphic to* $\mathbf{R}^2/\mathbf{Z}_n$*, where* $\mathbf{Z}_n$ *acts by rotations. It has been shown that the* $\mathbf{Z}_n$*-teardrop cannot be covered by a manifold (see [Sc]).*

Note: Henceforth, we will assume that the underlying space X of an orbifold O is always a second countable topological space.

2.2 Riemannian Orbifolds

A Riemannian structure on an orbifold is an assignment of a Riemannian metric on the orbifold. Just like in the manifold case, this is done by obtaining Riemannian metrics locally on coordinate charts, and then patching them up via a partition of unity.

Definition 2.2.1. *A map* $f : O \to \mathbf{R}$ *is called a* smooth function *on* O *if on each chart* $(U, \widetilde{U}/\Gamma, \pi)$*, its lift* $\tilde{f} = f \circ \pi$ *is a smooth function on* $\widetilde{U}$*.*

Definition 2.2.2. *Let* $\{U_\alpha\}$ *be a locally finite covering of an orbifold* O *that is subordinate to the orbifolds's covering of coordinate charts, i.e.* U_α*'s are coordinate charts with associated groups* Γ_α*'s. Let* $\{V_\alpha\}$ *be an open covering of* O *such that each* V_α *has compact closure, and* $\overline{V}_\alpha \subset U_\alpha$*. We define* Γ_α*-invariant functions on each* $\widetilde{U}_\alpha$ *as follows:*

$$\tilde{\lambda}_\alpha = \frac{1}{|\Gamma_\alpha|} \sum_{\gamma_\alpha \in \Gamma_\alpha} \tilde{\lambda} \circ \gamma_\alpha$$

where $\tilde{\lambda}$ *is a function on* $\widetilde{U}_\alpha$*.* $\tilde{\lambda}_\alpha$ *is assumed to be positive on* $\widetilde{V}_\alpha$ *and vanishes off of* V_α*.* $\tilde{\lambda}_\alpha$*'s define functions* λ_α*'s on* O *which are positive on* V_α *and zero elsewhere. We obtain a partition of unity by setting* $\mu_\alpha = \frac{\lambda_\alpha}{\Sigma \lambda_\alpha}$*.*

Definition 2.2.3. *Let* $(U, \widetilde{U}/\Gamma, \pi)$ *be a coordinate chart for an orbifold* O*.*

1. *Given a tensor field $\tilde{w}$ on $\widetilde{U}$ and $\gamma \in \Gamma$, we get a new tensor field $\tilde{w}^\gamma = \gamma^*(\tilde{w})$ on $\widetilde{U}$. We obtain a Γ-invariant tensor field on $\widetilde{U}$ defined as:*

$$\tilde{w}^\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \tilde{w}^\gamma.$$

   Such a Γ-invariant tensor field on $\widetilde{U}$ gives a tensor field w on U.

   We obtain a tensor field on O by patching together local tensor fields on charts using a partition of unity.

2. *We define a* smooth tensor *field on the orbifold O to be one that lifts to smooth tensor fields of the same type in all local covers.*

Definition 2.2.4. *Let O be an orbifold. A* Riemannian Structure *on O is an assignment to each orbifold chart $(U, \widetilde{U}/\Gamma, \pi)$ of a Γ-invariant Riemannian metric $g^{\widetilde{U}}$ on $\widetilde{U}$ satisfying the compatibility condition that each embedding λ appearing in Definition 2.1.3 is isometric. Every orbifold admits Riemannian structures.*

If $O = M/\Gamma$ is a good orbifold, we can obtain a Riemannian metric on O by specifying a Riemannian metric on M that is invariant under the action of Γ. Thus, locally Riemannian orbifolds look like the quotient of a Riemannian manifold by a finite group of isometries.

It is also known that by suitably choosing the coordinate charts of an orbifold we can assume that the local group actions are by finite subgroups of $O(n)$ (see [S1]).

2.3 Spectral Geometry on Orbifolds

Let O be a Riemannian orbifold and let f be a smooth function on O. By definition, the lift of f on each chart $(U, \widetilde{U}/\Gamma, \pi)$ is a smooth function $\tilde{f} = \pi^* f$ on $\widetilde{U}$. Let g_{ij}

denote the Γ-invariant metric on $\widetilde{U}$ given by the Riemannian structure on O and let $e = \sqrt{det(g_{ij})}$. Let $\widetilde{\Delta}$ denote the Laplacian on $\widetilde{U}$. On $\widetilde{U}$, $\widetilde{\Delta}\tilde{f}$ is given in the usual way as

$$\widetilde{\Delta}\tilde{f} = \sum_{i,j=1}^{n} \frac{1}{e} \cdot \frac{\partial}{\partial \tilde{x}^i}(g^{ij}\frac{\partial f}{\partial \tilde{x}^j}e).$$

We define the Laplacian on U by

$$\widetilde{\Delta}\tilde{f} = \Delta f \circ \pi.$$

That is, on local charts, the Laplacian acts on f by simply acting on its lift $\tilde{f}$ on the local cover. We say that λ is an eigenvalue of Δ if $\Delta f = \lambda f$ for some non-zero function f on O.

The following results about the eigenvalues of a Riemannian orbifold are known (see [Ch]).

Theorem 2.3.1. *Let O be a closed Riemannian orbifold.*

1. *The set of eigenvalues consists of an infinite sequence* $0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 < \bar{\lambda}_3 \ldots \uparrow \infty$.

2. *Each eigenvalue $\bar{\lambda}_i$ has finite multiplicity. We write $0 \leq \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3 < \ldots \uparrow \infty$ where each eigenvalue is repeated according to its multiplicity.*

3. *There exists an orthogonal basis of $L^2(O)$ (the space of square-integrable functions on O) composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3 \ldots$ where*

$$\Delta\phi_i = \bar{\lambda}_i\phi_i.$$

Note: For background information on orientability and integration on an orbifold, see [S1].

The *spectrum* of the Laplacian on O, denoted by $Spec(O)$, is defined by the sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$ in Theorem 2.3.1.

Definition 2.3.2. *Two compact connected Riemannian orbifolds O_1 and O_2 are said to be* isospectral *to each other if $Spec(O_1) = Spec(O_2)$.*

Chapter 3

Orbifold Isotropy Types

An interesting question in the spectral geometry of orbifolds is whether one can have isospectral orbifolds whose singularities have different isotropy types. On the positive side, E. Dryden, C. Gordon, S. Greenwald and D. Webb [DGGW] have shown that the Laplace spectrum determines the number and type of singularities in two-dimensional orbifolds with positive Euler characteristic. The definition of Euler characteristic is taken as follows [T]:

Definition 3.0.3. *When an orbifold O has a cell-division of X_O such that each open cell is in the same stratum of the singular locus (i.e., the group associated to the interior points of a cell is constant), then the Euler characteristic $\chi(O)$ is defined by the formula*

$$\chi(O) = \sum_{c_i} (-1)^{dim(c_i)} \frac{1}{|\Gamma(c_i)|},$$

where c_i ranges over cells and $|\Gamma(c_i)|$ is the order of the group $\Gamma(c_i)$ associated to each cell.

As is clear from this definition, the Euler characteristic is not always an integer. In addition, Stanhope [S2] proved the following result:

Theorem 3.0.1. *Only finitely many isotropy types may arise in a family of isospectral orbifolds whose members have a uniform lower bound on Ricci curvature.*

In this chapter we construct arbitrarily large (finite) families of isospectral orbifolds with different isotropy types, showing that the former result is not true in general. This work is an exposition of the author's joint publication with E.Stanhope and D.Webb [BSW] in 2006. The author's contribution is the construction of the example in Section 3.3.

Given an odd prime P and an integer $m \geq 1$, we construct an $(m+1)$-element family of isospectral $(P^{3m}-1)$-dimensional orbifolds, each with points of distinct isotropy. The orbifolds in these families are quotients of the round sphere by properly discontinuous orthogonal actions. By studying a related example, we see in Section 3.3 that some topological properties of the singular set of an orbifold are not spectrally determined.

3.1 Sunada's Theorem

Most known examples of isospectral, non-isometric manifolds are constructed using a group-theoretic method of Sunada [Su]. A generalization of Sunada's Theorem due to Bérard allows the use of the Sunada technique to obtain isospectral orbifolds. In this section we briefly review the algebraic background for Sunada's Theorem and provide some examples. We then state Bérard's version of Sunada's Theorem, used in the next section to construct our isospectral families of orbifolds.

Definition 3.1.1. *Two subgroups Γ_1 and Γ_2 of a finite group G are said to be almost conjugate if each G-conjugacy class $[g]_G$ intersects Γ_1 and Γ_2 in the same number of elements.*

Remark 3.1.2. *The condition that subgroups Γ_1 and Γ_2 of a finite group G be almost conjugate is equivalent to requiring that the representations of G induced from the trivial one-dimensional representations of Γ_1 and Γ_2 (these induced representations are just the linear permutation representations of G determined by Γ_1 and Γ_2) be equivalent as linear representations of G, i.e., that $(1_{\Gamma_1}) \uparrow_{\Gamma_1}^{G} \cong (1_{\Gamma_2}) \uparrow_{\Gamma_2}^{G}$. This fact follows easily from the formula for the character of an induced representation (see [K0] or [CF], page 362). Thus the use of Sunada's method to produce examples of isospectral manifolds that are not isometric is based upon the existence of permutation representations that are inequivalent as G-sets but nevertheless give rise to equivalent linear representations of G. The existence of almost conjugate but not conjugate subgroups was first used by Gassman [Ga] to construct nonisomorphic algebraic number fields with the same zeta function.*

Example 3.1.3. *[Br] Let p be an odd prime, and let G be the symmetric group on p^3 letters. Let $E = \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$ be the p-elementary group of order p^3, and let H be the Heisenberg group over the field $\mathbf{Z}_p$:*

$$H = \left\{ \begin{bmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbf{Z}_p \right\}$$

the unique nonabelian group of order p^3 in which every nonidentity element has order p. View H and E as subgroups of G via the natural action of each group on itself by left-multiplication. Then E and H are almost conjugate in G, as follows from the fact that two permutations are conjugate in the symmetric group if they have the same cycle structure. These groups are not isomorphic, as E is abelian while H is not.

Example 3.1.4. *[Br] Let $m \geq 1$, let G be the symmetric group on p^{3m} letters, and let $\{H_i\}_{0 \leq i \leq m}$ be the collection of subgroups of G defined as follows. Let H denote* *the mod-p Heisenberg group, E the abelian p-elementary group $E = \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$. Let $H_i = H^i \times E^{m-i}$, where H^i is the product of i copies of H and E^j is the product of j copies of E. As in Example 3.1.3, the subgroups H_i of G are pairwise almost conjugate. They are pairwise nonisomorphic: the center of H_i is a $\mathbf{Z}_p$-vector space of dimension $i + 3(m-i) = 3m - 2i$, since the center of H is $\mathbf{Z}_p$.*

With these examples in mind we now state Bérard's generalized version of Sunada's Theorem. In Sunada's original statement of this theorem, the actions by Γ_1 and Γ_2 were required to be fixed point free.

Theorem 3.1.5. *[B] Let H_1 and H_2 be almost conjugate subgroups of a finite group G. Let (M, g) be a compact Riemannian manifold on which G acts faithfully on the left by isometries. Then the orbit spaces $H_1\backslash M$ and $H_2\backslash M$ are isospectral as orbifolds:*

$$\mathrm{spec}(H_1\backslash M, g) = \mathrm{spec}(H_2\backslash M, g).$$

3.2 Construction

We turn now to the construction of isospectral orbifolds with different isotropy types. We show that given an odd prime number p and an integer $m \geq 1$ there exists an $(m+1)$-element family of isospectral $(p^{3m}-1)$-dimensional orbifolds each containing a point of isotropy type found in no other orbifold family. We construct these families by arranging for the groups from Example 3.1.4 to act isometrically on the round sphere, and then applying Theorem 3.1.5.

Let $\mathbf{E}^n$ denote the vector space $\mathbf{R}^n$ together with the standard Euclidean inner product, and let $\{e_0, e_1, \ldots, e_{n-1}\}$ be the standard orthonormal basis for $\mathbf{E}^n$. Also, let $\mathbf{S}^n$ denote the n-dimensional unit sphere. The symmetric group G on p^{3m} letters from Example 3.1.4 acts orthogonally on the Euclidean space $\mathbf{E}^{p^{3m}}$ by permuting the elements $e_0, e_1, \ldots, e_{p^{3m}-1}$ of the standard basis. Restricting this action to the unit sphere $\mathbf{S}^{p^{3m}-1}$, G acts on $\mathbf{S}^{p^{3m}-1}$ by isometries. By Theorem 3.1.5 we conclude that the orbit spaces are isospectral as orbifolds:

$$\operatorname{spec}(H_0\backslash\mathbf{S}^{p^{3m}-1}) = \operatorname{spec}(H_1\backslash\mathbf{S}^{p^{3m}-1}) = \cdots = \operatorname{spec}(H_m\backslash\mathbf{S}^{p^{3m}-1}).$$

Because each group H_i fixes the two unit vectors $\pm\frac{1}{\sqrt{p^{3m}}}(1, 1, \ldots, 1) \in \mathbf{E}^{p^{3m}}$, the corresponding points in the orbifold $H_i\backslash\mathbf{S}^{p^{3m}-1}$ have full isotropy G. Since the groups H_i are pairwise nonisomorphic, it follows that each orbifold in our isospectral family has a pair of points with isotropy type not found in any other member of the family. Finally, note that because these orbifolds all have constant sectional curvature equal to one, they satisfy the bound on Ricci curvature needed for Theorem 3.0.1; thus Theorem 3.0.1 cannot be improved from a finiteness assertion to a bound.

3.3 An Example

In this section we show that in a pair of isospectral orbifolds, the underlying topological spaces of the singular strata may differ. Thus the topological nature of the singular strata is not spectrally determined.

We examine the nature of the singular set in a specific example. Consider the algebraic setting of Example 3.1.3 with $p = 3$; thus G is the symmetric group S_{27} of all permutations of the set $\{0, 1, \ldots, 26\}$, H_0 is the abelian group $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$, and H_1 is the mod-3 Heisenberg group. The groups H_0 and H_1 are the only groups of order 27 all of whose nonidentity elements have order three (see [DF], page 183). The group G acts orthogonally on the Euclidean space $\mathbf{E}^{27}$ of dimension 27 by permuting the standard orthonormal basis vectors $e_0, e_1, \ldots, e_{26}$, hence G acts on the unit sphere $\mathbf{S}^{26} \subseteq \mathbf{R}^{27}$.

Then the quotient orbifolds $\mathcal{O}_0 = H_0\backslash\mathbf{S}^{26}$ and $\mathcal{O}_1 = H_1\backslash\mathbf{S}^{26}$ are isospectral, by Theorem 3.1.5. Let $\Delta_\pm = \pm\frac{1}{\sqrt{27}}(1,1,\ldots,1) \in \mathbf{S}^{26} \subseteq \mathbf{E}^{27}$, the "poles" left fixed by the entire symmetric group.

For $i = 0$ or 1, the only nontrivial proper subgroups of H_i are isomorphic either to $\mathbf{Z}_3$ or to $\mathbf{Z}_3 \times \mathbf{Z}_3$, so the only possible nontrivial isotropy types are $\mathbf{Z}_3$, $\mathbf{Z}_3 \times \mathbf{Z}_3$, and the full group H_i. A point of the orbifold $\mathcal{O}_i$ is nonsingular if its isotropy group is trivial. The points whose isotropy is $\mathbf{Z}_3$ will be called *mild* singularities; those points with isotropy $\mathbf{Z}_3 \times \mathbf{Z}_3$ will be called *moderate* singularities, while those points whose isotropy is the full group H_i (the poles $\Delta_\pm$) will be called *wild* singularities. For $e = 0, 1, 2, 3$, let $\mathcal{O}_i^{(e)} = \{x \in \mathcal{O}_i : \#\Gamma_x = 3^e\}$, the set of points in $\mathcal{O}_i$ whose isotropy groups have order 3^e; thus $\mathcal{O}_i^{(0)}$ is the set of nonsingular points, while $\mathcal{O}_i^{(3)}$ is the set of wild singularities. We will be interested in the sets $\mathcal{O}_i^{(2)}$ of moderate singularities; specifically, we will show that $\mathcal{O}_0^{(2)}$ and $\mathcal{O}_1^{(2)}$ are not homeomorphic, so the underlying topological space of the moderate singular stratum is not spectrally determined.

We begin by considering the singular strata $\mathcal{O}_i^{(1)}$ in the orbifolds $\mathcal{O}_i$ $(i = 0, 1)$ consisting of the mild singular points; such singular points are represented by points in the sphere $\mathbf{S}^{26}$ fixed by a single nonidentity element $h \in H_i$; that is, their isotropy group is the cyclic group $\langle h\rangle = \{1, h, h^{-1}\} \cong \mathbf{Z}_3$.

For the sake of brevity, we adopt the following notation for elements of the groups H_0 and H_1. An element of $H_0 = \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$ is a triple (a, b, c) of elements $a, b, c \in \mathbf{Z}_p$; we view the element $h = (a, b, c)$ as the ternary representation of an integer n_g satisfying $0 \leq n_g < 27$, and we denote g by the integer n_g. Thus, for example, the element $(1, 1, 2)$ is denoted by 14. Similarly, it is easily checked that the mod-3 Heisenberg group H_1 is isomorphic to the unique nontrivial semidirect product $(\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes_\alpha \mathbf{Z}_3$, where the action $\alpha : \mathbf{Z}_3 \to \mathrm{Aut}(\mathbf{Z}_3 \times \mathbf{Z}_3) \cong \mathrm{GL}_2(\mathbf{Z}_3)$ of $\mathbf{Z}_3$ on $\mathbf{Z}_3 \times \mathbf{Z}_3$ is given by

$$\alpha(c) = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

Thus the multiplication in H_1, consistent with matrix multiplication, is given by

$$\left(\begin{bmatrix} a \\ b \end{bmatrix}, c \right) \left(\begin{bmatrix} a' \\ b' \end{bmatrix}, c' \right) = \left(\begin{bmatrix} a + a' + cb' \\ b + b' \end{bmatrix}, c + c' \right).$$

As in the case of H_0, the element $g = \left(\begin{bmatrix} a \\ b \end{bmatrix}, c \right) \in H_1$ is denoted by the integer n_g (where $0 \leq n_g < 27$) whose ternary representation is (a, b, c).

It is easy to compute the left-translation actions of H_0 and H_1 upon themselves as permutations of $\{0, 1, \ldots, 26\}$. For example, the element $g = (0, 1, 1) \in H_1$, denoted by 4 according to our convention above, corresponds to the permutation

$$(0\,4\,17)(1\,5\,15)(2\,3\,16)(6\,19\,23)(7\,20\,21)(8\,18\,22)(9\,13\,26)(10\,14\,24)(11\,12\,25) \tag{3.1}$$

in the symmetric group S_{27}. We will denote the fixed-point set for the H_i action on the sphere $\mathbf{S}^{26}$ of the element $g \in H_i$ by $\sum_i^g$. For example, in the case of the element $g = (0, 1, 1) \in H_1$ considered above, it is clear that the fixed point set is the collection of unit vectors $(x_0, x_1, \ldots, x_{26}) \in \mathbf{E}^{27}$ satisfying the conditions

$$x_0 = x_4 = x_{17},\ x_1 = x_5 = x_{15},\ x_2 = x_3 = x_{16},\ \ldots, x_{11} = x_{12} = x_{25}$$

imposed by the requirement that the vector be invariant under the permutation whose cycle decomposition was written out above in (3.1). For simplicity, we denote this fixed point set with the same notation as the cycle decomposition of g; thus

$$\textstyle\sum_1^4 = [0\,4\,17][1\,5\,15][2\,3\,16][6\,19\,23][7\,20\,21][8\,18\,22][9\,13\,26][10\,14\,24][11\,12\,25]$$

Also, $\sum_1^4 = \sum_1^{17}$, since the elements $4 = (0, 1, 1)$ and $17 = (1, 2, 2)$ are inverses in H_1. As another example of our notation

$$[0\,1\,2\,9\,10\,11\,18\,19\,20][3\,4\,5\,12\,13\,14\,21\,22\,23][6\,7\,8\,15\,16\,17\,24\,25\,26]$$

denotes the set of unit vectors $(x_0, x_1, \ldots, x_{26}) \in \mathbf{E}^{27}$ satisfying the conditions

$$x_0 = x_1 = x_2 = x_9 = x_{10} = x_{11} = x_{18} = x_{19} = x_{20},$$

$$x_3 = x_4 = x_5 = x_{12} = x_{13} = x_{14} = x_{21} = x_{22} = x_{23}$$

and

$$x_6 = x_7 = x_8 = x_{15} = x_{16} = x_{17} = x_{24} = x_{25} = x_{26}.$$

There are twenty-six nonidentity elements of H_0 or H_1; each has the same fixed-point set as its inverse. Thus there are thirteen fixed-point sets to consider for each group. Each fixed-point set in $\mathbf{E}^{27}$ is a nine-dimensional linear subspace; thus the fixed-point sets in the sphere $\mathbf{S}^{26}$ are eight-dimensional, since the extra constraint that the vector be of unit length is also imposed.

Consider first the group $H_0 = \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$. It is straightforward to compute the thirteen fixed-point sets $\sum_0^j$ of nontrivial elements $j \in H_0$, that is, the points in the unit sphere $\mathbf{S}^{26}$ representing singular points of the quotient orbifold $\mathcal{O}_0$ whose isotropy is nontrivial. They are given by

$$\begin{aligned}
\textstyle\sum_0^1 &= [0\,1\,2][3\,4\,5][6\,7\,8][9\,10\,11][12\,13\,14][15\,16\,17][18\,19\,20][21\,22\,23][24\,25\,26],\\
\textstyle\sum_0^3 &= [0\,3\,6][1\,4\,7][2\,5\,8][9\,12\,15][10\,13\,16][11\,14\,17][18\,21\,24][19\,22\,25][20\,23\,26],\\
\textstyle\sum_0^4 &= [0\,4\,8][1\,5\,6][2\,3\,7][9\,13\,27][10\,14\,15][11\,12\,16][18\,22\,26][19\,23\,24][20\,21\,25],\\
\textstyle\sum_0^5 &= [0\,5\,7][1\,3\,8][2\,4\,6][9\,14\,16][10\,12\,17][11\,13\,15][18\,23\,25][19\,21\,26][20\,22\,24],\\
\textstyle\sum_0^9 &= [0\,9\,18][1\,10\,19][2\,11\,20][3\,12\,21][4\,13\,22][5\,14\,23][6\,15\,24][7\,16\,25][8\,17\,26],\\
\textstyle\sum_0^{10} &= [0\,10\,20][1\,11\,18][2\,9\,19][3\,13\,23][4\,14\,21][5\,12\,22][6\,16\,26][7\,17\,24][8\,15\,25],\\
\textstyle\sum_0^{11} &= [0\,11\,19][1\,9\,20][2\,10\,18][3\,14\,22][4\,12\,23][5\,13\,21][6\,17\,25][7\,15\,26][8\,16\,24],\\
\textstyle\sum_0^{12} &= [0\,12\,24][1\,13\,25][2\,14\,26][3\,15\,18][4\,16\,19][5\,17\,20][6\,9\,21][7\,10\,22][8\,11\,23],\\
\textstyle\sum_0^{13} &= [0\,13\,26][1\,14\,24][2\,12\,25][3\,16\,20][4\,17\,18][5\,15\,19][6\,10\,23][7\,11\,21][8\,9\,22],\\
\textstyle\sum_0^{14} &= [0\,14\,25][1\,12\,26][2\,13\,24][3\,17\,19][4\,15\,20][5\,16\,18][6\,11\,22][7\,9\,23][8\,10\,21],\\
\textstyle\sum_0^{15} &= [0\,15\,21][1\,16\,22][2\,17\,23][3\,9\,24][4\,10\,25][5\,11\,26][6\,12\,18][7\,13\,19][8\,14\,20],\\
\textstyle\sum_0^{16} &= [0\,16\,23][1\,17\,21][2\,15\,22][3\,10\,26][4\,11\,24][5\,9\,25][6\,13\,20][7\,14\,18][8\,12\,19],\\
\textstyle\sum_0^{17} &= [0\,17\,22][1\,15\,23][2\,16\,21][3\,11\,25][4\,9\,26][5\,10\,24][6\,14\,19][7\,12\,20][8\,13\,18]. \qquad (3.2)
\end{aligned}$$

Since the fixed point set of a group element coincides with that of its inverse, we also have $\sum_0^2 = \sum_0^1$, $\sum_0^6 = \sum_0^3$, $\sum_0^8 = \sum_0^4$, $\sum_0^7 = \sum_0^5$, $\sum_0^{18} = \sum_0^9$, $\sum_0^{20} = \sum_0^{10}$, $\sum_0^{19} = \sum_0^{11}$, $\sum_0^{24} = \sum_0^{12}$, $\sum_0^{26} = \sum_0^{13}$, $\sum_0^{25} = \sum_0^{14}$, $\sum_0^{21} = \sum_0^{15}$, $\sum_0^{23} = \sum_0^{16}$, and $\sum_0^{22} = \sum_0^{17}$.

Now consider the group H_1. We next compute the thirteen fixed-point sets $\sum_1^j$ of nontrivial group elements $j \in H_1$, that is, the points in the unit sphere $\mathbf{S}^{26}$ representing singular points of the quotient orbifold $\mathcal{O}_1$ whose isotropy is nontrivial; they are given by

$$\begin{aligned}
\textstyle\sum_1^1 &= [0\,1\,2][3\,13\,23][4\,14\,21][5\,12\,22][6\,25\,17][7\,26\,15][8\,24\,16][9\,10\,11][18\,19\,20],\\
\textstyle\sum_1^3 &= [0\,3\,6][1\,4\,7][2\,5\,8][9\,12\,15][10\,13\,16][11\,14\,17][18\,21\,24][19\,22\,25][20\,23\,26],\\
\textstyle\sum_1^4 &= [0\,4\,17][1\,5\,15][2\,3\,16][6\,19\,23][7\,20\,21][8\,18\,22][9\,13\,26][10\,14\,24][11\,12\,25],\\
\textstyle\sum_1^5 &= [0\,5\,25][1\,3\,26][2\,4\,24][6\,11\,13][7\,9\,14][8\,10\,12][15\,20\,22][16\,18\,23][17\,19\,21],\\
\textstyle\sum_1^7 &= [0\,7\,23][1\,8\,21][2\,6\,22][3\,10\,17][4\,11\,15][5\,9\,16][12\,19\,26][13\,20\,24][14\,18\,25],\\
\textstyle\sum_1^8 &= [0\,8\,13][1\,6\,14][2\,7\,12][3\,20\,25][4\,18\,26][5\,19\,24][9\,17\,22][10\,15\,23][11\,16\,21],\\
\textstyle\sum_1^9 &= [0\,9\,18][1\,10\,19][2\,11\,20][3\,12\,21][4\,13\,22][5\,14\,23][6\,15\,24][7\,16\,25][8\,17\,26],\\
\textstyle\sum_1^{10} &= [0\,10\,20][1\,11\,18][2\,9\,19][3\,22\,14][4\,23\,12][5\,21\,13][6\,7\,8][15\,16\,17][24\,25\,26],\\
\textstyle\sum_1^{11} &= [0\,11\,19][1\,9\,20][2\,10\,18][3\,5\,4][6\,26\,16][7\,24\,17][8\,25\,15][12\,14\,13][21\,23\,22],\\
\textstyle\sum_1^{12} &= [0\,12\,24][1\,13\,25][2\,14\,26][3\,15\,18][4\,16\,19][5\,17\,20][6\,9\,21][7\,10\,22][8\,11\,23],\\
\textstyle\sum_1^{14} &= [0\,14\,16][1\,12\,17][2\,13\,15][3\,8\,19][4\,6\,20][5\,7\,18][9\,23\,25][10\,21\,26][11\,22\,24],\\
\textstyle\sum_1^{15} &= [0\,15\,21][1\,16\,22][2\,17\,23][3\,9\,24][4\,10\,25][5\,11\,26][6\,12\,18][7\,13\,19][8\,14\,20],\\
\textstyle\sum_1^{22} &= [0\,22\,26][1\,23\,24][2\,21\,25][3\,7\,11][4\,8\,9][5\,6\,10][12\,16\,20][13\,17\,18][14\,15\,19]. \qquad (3.3)
\end{aligned}$$

Also, $\sum_1^2 = \sum_1^1$, $\sum_1^6 = \sum_1^3$, $\sum_1^{17} = \sum_1^4$, $\sum_1^{25} = \sum_1^5$, $\sum_1^{23} = \sum_1^7$, $\sum_1^{13} = \sum_1^8$, $\sum_1^{18} = \sum_1^9$, $\sum_1^{20} = \sum_1^{10}$, $\sum_1^{19} = \sum_1^{11}$, $\sum_1^{24} = \sum_1^{12}$, $\sum_1^{16} = \sum_1^{14}$, $\sum_1^{21} = \sum_1^{15}$, and $\sum_1^{26} = \sum_1^{22}$.

Next, we consider points in the sphere $\mathbf{S}^{26}$ having isotropy $\mathbf{Z}_3 \times \mathbf{Z}_3$ for the H_i-action. Let $\sum_i^{j,k}$ denote the intersection of sets $\sum_i^j$ and $\sum_i^k$. It is easy to see that the points of $\mathbf{S}^{26}$ of isotropy type $\mathbf{Z}_3 \times \mathbf{Z}_3$ for the H_i-action are just the points of

$$\bigcup_{j,k\in H_i,<j>\neq<k>} (\textstyle\sum_i^{j,k} -\{\Delta_\pm\}).$$

Indeed, let x denote such a point. Then x is not one of the poles (which have fully isotropy H_i), and it is fixed by two elements $j,k \in H_i$ that generate distinct cyclic subgroups $\langle j\rangle \neq \langle k\rangle \subseteq H_i$, so manifestly $x \in \sum_i^{j,k}$, i.e.,

$$x \in \bigcup_{j,k\in H_i, <j>\neq<k>} (\textstyle\sum_i^{j,k} - \{\Delta_\pm\}).$$

Conversely, any point of

$$\bigcup_{j,k\in H_i, <j>\neq<k>} (\textstyle\sum_i^{j,k} - \{\Delta_\pm\})$$

has isotropy $\langle j,k \rangle$ (for some $j,k \in H_i$) properly larger than $\mathbf{Z}_3$ but properly contained in H_i, so the only possibility for its isotropy is $\mathbf{Z}_3 \times \mathbf{Z}_3$. Thus, the points in the sphere $\mathbf{S}^{26}$ having isotropy $\mathbf{Z}_3 \times \mathbf{Z}_3$ for the H_i-action are readily determined by computing the pairwise intersections $\sum_i^{j,k} = \sum_i^j \cap \sum_i^k$ of the above fixed-point sets, discarding the poles, and taking the union.

We begin with the case of H_0. As is clear from (3.4) below, each double intersection $\sum_0^{j,k}$ is given by twenty-four independent linear conditions on the twenty-seven components $x_0, x_1, \ldots, x_{26}$, together with the condition that $(x_0, x_1, \ldots, x_{26})$ be a unit vector; thus each $\sum_0^{j,k}$ is the intersection of a three-dimensional linear subspace of $\mathbf{R}^{27}$ with the unit sphere $\mathbf{S}^{26}$, hence is a 2-sphere; in particular, each $\sum_0^{j,k} - \{\Delta_\pm\}$ is a 2-sphere with a pair of antipodal points removed, and hence is connected. The thirteen distinct double intersections $\sum_0^{j,k}$ are easily computed; they are given by

$$
\begin{aligned}
\textstyle\sum_0^{1,3} &= [0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8][9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17][18\ 19\ 20\ 21\ 22\ 23\ 24\ 25\ 26],\\
\textstyle\sum_0^{1,9} &= [0\ 1\ 2\ 9\ 10\ 11\ 18\ 19\ 20][3\ 4\ 5\ 12\ 13\ 14\ 21\ 22\ 23][6\ 7\ 8\ 15\ 16\ 17\ 24\ 25 26],\\
\textstyle\sum_0^{1,12} &= [0\ 1\ 2\ 12\ 13\ 14\ 24\ 25\ 26][3\ 4\ 5\ 15\ 16\ 17\ 18\ 19\ 20][6\ 7\ 8\ 9\ 10\ 11\ 21\ 22\ 23],\\
\textstyle\sum_0^{1,15} &= [0\ 1\ 2\ 15\ 16\ 17\ 21\ 22\ 23][3\ 4\ 5\ 9\ 10\ 11\ 24\ 25\ 26][6\ 7\ 8\ 12\ 13\ 14\ 18\ 19\ 20],\\
\textstyle\sum_0^{3,9} &= [0\ 3\ 6\ 9\ 12\ 15\ 18\ 21\ 24][1\ 4\ 7\ 10\ 13\ 16\ 19\ 22\ 25][2\ 5\ 8\ 11\ 14\ 17\ 20\ 23\ 26],\\
\textstyle\sum_0^{3,10} &= [0\ 3\ 6\ 10\ 13\ 16\ 20\ 23\ 26][1\ 4\ 7\ 11\ 14\ 17\ 18\ 21\ 24][2\ 5\ 8\ 9\ 12\ 15\ 19\ 22\ 25],\\
\textstyle\sum_0^{3,11} &= [0\ 3\ 6\ 11\ 14\ 17\ 19\ 22\ 25][1\ 4\ 7\ 9\ 12\ 15\ 20\ 23\ 26][2\ 5\ 8\ 10\ 13\ 16\ 18\ 21\ 24],\\
\textstyle\sum_0^{4,9} &= [0\ 4\ 8\ 9\ 13\ 17\ 18\ 22\ 26][1\ 5\ 6\ 10\ 14\ 15\ 19\ 23\ 24][2\ 3\ 7\ 11\ 12\ 16\ 20\ 21\ 25],\\
\textstyle\sum_0^{4,10} &= [0\ 4\ 8\ 10\ 14\ 15\ 20\ 21\ 25][1\ 5\ 6\ 11\ 12\ 16\ 18\ 22\ 26][2\ 3\ 7\ 9\ 13\ 17\ 19\ 23\ 24],\\
\textstyle\sum_0^{4,11} &= [0\ 4\ 8\ 11\ 12\ 16\ 19\ 23\ 24][1\ 5\ 6\ 9\ 13\ 17\ 20\ 21\ 25][2\ 3\ 7\ 10\ 14\ 15\ 18\ 22\ 26],\\
\textstyle\sum_0^{5,9} &= [0\ 5\ 7\ 9\ 14\ 16\ 18\ 23\ 25][1\ 3\ 8\ 10\ 12\ 17\ 19\ 21\ 26][2\ 4\ 6\ 11\ 13\ 15\ 20\ 22\ 24],\\
\textstyle\sum_0^{5,10} &= [0\ 5\ 7\ 10\ 12\ 17\ 20\ 22\ 24][1\ 3\ 8\ 11\ 13\ 15\ 18\ 23\ 25][2\ 4\ 6\ 9\ 14\ 16\ 19\ 21\ 26],\\
\textstyle\sum_0^{5,11} &= [0\ 5\ 7\ 11\ 13\ 15\ 19\ 21\ 26][1\ 3\ 8\ 9\ 14\ 16\ 20\ 22\ 24][2\ 4\ 6\ 10\ 12\ 17\ 18\ 23\ 25]. \qquad (3.4)
\end{aligned}
$$

Next, we compute the locus of points whose isotropy under the H_1 action is $\mathbf{Z}_3 \times \mathbf{Z}_3$; as above, this is easily carried out by computing the pairwise intersections $\sum_1^{j,k} = \sum_1^j \cap \sum_1^k$ of the above fixed-point sets. In this case, most of the pairs of fixed-point sets intersect only in the poles; i.e., for most pairs $j, k \in H_1$, we have $\sum_1^{j,k} = \{\Delta_\pm\} = \{\pm\frac{1}{\sqrt{27}}(1, 1, \ldots, 1)\}$. There are only four larger double intersections, given by

$$
\begin{aligned}
\textstyle\sum_1^{1,9} &= [0\ 1\ 2\ 9\ 10\ 11\ 18\ 19\ 20][3\ 4\ 5\ 12\ 13\ 14\ 21\ 22\ 23][6\ 7\ 8\ 15\ 16\ 17\ 24\ 25\ 26],\\
\textstyle\sum_1^{3,9} &= [0\ 3\ 6\ 9\ 12\ 15\ 18\ 21\ 24][1\ 4\ 7\ 10\ 13\ 16\ 19\ 22\ 25][2\ 5\ 8\ 11\ 14\ 17\ 20\ 23\ 26],\\
\textstyle\sum_1^{4,8} &= [0\ 4\ 8\ 9\ 13\ 17\ 18\ 22\ 26][1\ 5\ 6\ 10\ 14\ 15\ 19\ 23\ 24][2\ 3\ 7\ 11\ 12\ 16\ 20\ 21\ 25],\\
\textstyle\sum_1^{5,7} &= [0\ 5\ 7\ 9\ 14\ 16\ 18\ 23\ 25][1\ 3\ 8\ 10\ 12\ 17\ 19\ 21\ 26][2\ 4\ 6\ 11\ 13\ 15\ 20\ 22\ 24]. \qquad (3.5)
\end{aligned}
$$

In both the cases, $i = 0$ and $i = 1$, it is easily verified that two distinct double intersections intersect only in the poles: $\sum_i^{j,k} \cap \sum_i^{r,s} = \{\Delta_\pm\}$ unless $\sum_i^{j,k} = \sum_i^{r,s}$.

For $i = 0, 1$ recall that $\mathcal{O}_i^{(2)}$ denotes the set of moderate singular points in $\mathcal{O}_i$, i.e., those whose isotropy is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$. Thus, by 3.4 and 3.5, $\mathcal{O}_i^{(2)}$ is the set of points in $\mathcal{O}_i$ with a representative in the set

$$\mathbf{S}_i^{(2)} := \bigcup_{j,k \in H_i, <j> \neq <k>} \left(\sum_i^{j,k} - \{\Delta_\pm\}\right) \subseteq \mathbf{S}^{26}.$$

As noted above, $\mathbf{S}_i^{(2)}$ is the disjoint union of subsets of $\mathbf{S}^{26}$, each homeomorphic to a 2-sphere with a pair of antipodal points removed. Now (3.4) shows that $\mathbf{S}_0^{(2)}$ has thirteen connected components, while (3.5) shows that $\mathbf{S}_1^{(2)}$ has only four connected components.

We now consider whether two different components $\sum_i^{j,k} - \{\Delta_\pm\}$ and $\sum_i^{r,s} - \{\Delta_\pm\}$ of the set of points $\mathbf{S}_i^{(2)} \subseteq \mathbf{S}^{26}$ might be identified via the H_i-action and hence might represent the same connected component of the moderate singular stratum $\mathcal{O}_i^{(2)}$ in the quotient orbifold $\mathcal{O}_i$. For $i = 0$, this cannot occur. Indeed, if this were so, some group element $g \in H_0$ would carry $\sum_i^{j,k} - \{\Delta_\pm\}$ to $\sum_i^{r,s} - \{\Delta_\pm\}$, so their isotropy groups $\langle j, k\rangle$ and $\langle r, s\rangle$ would be conjugate in H_0; however, since H_0 is abelian, all conjugations are trivial so this is impossible. Thus, $\mathcal{O}_0^{(2)}$ has exactly thirteen connected components. However, $\mathcal{O}_1^{(2)}$ has at most four connected components, since $\mathbf{S}_1^{(2)}$ has only four connected components.

Thus the sets $\mathcal{O}_0^{(2)}$ and $\mathcal{O}_1^{(2)}$ of moderate singular points have different numbers of connected components in the two orbifolds $\mathcal{O}_0$ and $\mathcal{O}_1$, so one cannot hear the underlying topology of the set of points of a given isotropy type.

Chapter 4

Orbifold Lens Spaces

In this chapter we will generalize the idea of manifold lens spaces to orbifold lens spaces. Note that lens spaces are special cases of *spherical space forms*, which are *connected complete Riemannian manifolds of positive constant sectional curvature 1.* An n-dimensional spherical space form can be written as $\mathbb{S}^n/G$ where G is a finite subgroup of the orthogonal group $O(n+1)$. In fact, the definition of spherical space forms can be generalized to allow G to have fixed points making $\mathbb{S}^n/G$ an orbifold. Manifold lens spaces are spherical space forms where the n-dimensional sphere S^n of constant curvature 1 is acted upon by a cyclic group of fixed point free isometries on S^n. We will generalize this notion to orbifolds by allowing the cyclic group of isometries to have fixed points.

Our goal in this chapter is to construct examples of isospectral orbifold lens spaces that are not isometric. The results in this chapter are an exposition of the author's work published in 2011[Ba].

4.1 Orbifold Lens Spaces Generating Functions

In this section we will reproduce the background work developed by Ikeda in [I1] and [I2] for manifold spherical space forms. We will note that with slight modifications the results are valid for orbifold spherical space forms. This is the background work we will need to develop our results for orbifold lens spaces.

We will first consider general $2n-1$ dimensional lens spaces. Let q be a positive integer. Set

$$q_0 = \begin{cases} \frac{q-1}{2} & \text{if } q \text{ is odd,} \\ \frac{q}{2} & \text{if } q \text{ is even.} \end{cases}$$

Throughout this chapter we assume that $q_0 \geq 4$.

For $n \leq q_0$, let $p_1, \ldots, p_n$ be n integers. Note, if $g.c.d.(p_1, \ldots, p_n, q) \neq 1$, we can divide all the $p_i's$ and q by this gcd to get a case where the $gcd = 1$. So, without loss of generality, we can assume $g.c.d.(p_1, \ldots, p_n, q) = 1$. We denote by g the orthogonal matrix given by

$$g = \begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_n/q) \end{pmatrix},$$

where $R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$. Then g generates a cyclic subgroup $G = \{g^l\}_{l=1}^{q}$ of order q of the special orthogonal group $SO(2n)$ since $\det g = 1$. Note that g has eigenvalues γ^{p_1}, γ^{-p_1},γ^{p_2}, γ^{-p_2},..., γ^{p_n}, γ^{-p_n}, where γ is a primitive q-th root of unity. We define the lens space $L(q : p_1, \ldots, p_n)$ as follows:

$$L(q : p_1, \ldots, p_n) = S^{2n-1}/G.$$

Note that if $gcd(p_i, q) = 1 \ \forall i$, $L(q : p_1, \ldots, p_n)$ is a smooth manifold; Ikeda and Yamamoto have answered Kac's question in the affirmative for 3-dimensional manifold lens spaces ([IY], [Y]). To get an orbifold in this setting with non-trivial singularities, we must have $gcd(p_i, q) > 1$ for some i. In such a case $L(q : p_1, \ldots, p_n)$ is a good smooth orbifold with S^{2n-1} as its covering manifold. Let π be the covering projection of S^{2n-1} onto S^{2n-1}/G

$$\pi : S^{2n-1} \to S^{2n-1}/G.$$

Since the round metric of constant curvature one on S^{2n-1} is G-invariant, it induces a Riemannian metric on S^{2n-1}/G. Henceforth, the term "lens space" will refer to this generalized definition. Ikeda proved the following result for manifold spherical space forms. We note that the proof doesn't require the groups to be fixed-point free, and reproduce the result for orbifold spherical space forms:

Lemma 4.1.1. *Let $\mathbb{S}^n/G$ and $\mathbb{S}^n/G'$ be spherical space forms for any integer $n \geq 2$. Then $\mathbb{S}^n/G$ is isometric to $\mathbb{S}^n/G'$ if and only if G is conjugate to G' in $O(n+1)$.*

Proof. If ϕ is an isometry of $\mathbb{S}^n/G$ onto $\mathbb{S}^n/G'$, then there exists an isometry $\tilde{\phi}$ of $\mathbb{S}^n$ onto itself which covers ϕ since $\mathbb{S}^n$ is the universal cover for spherical space forms. Now $\tilde{\phi}$ is an element of $O(n+1)$ and it gives a conjugation between G and G'. Conversely, if $\tilde{\phi} \in O(n+1)$ such that $\tilde{\phi}G\tilde{\phi}^{-1} = G'$, then $\tilde{\phi}$ induces an isometry ϕ of $\mathbb{S}^n/G$ onto $\mathbb{S}^n/G'$ so that $\phi\pi(x) = \pi'(\tilde{\phi}x)$ for any $x \in \mathbb{S}^n$ where π and π' are projections maps from $\mathbb{S}^n$ onto $\mathbb{S}^n/G$ and $\mathbb{S}^n/G'$ respectively. $\square$

Note that if we have a lens space $\mathbb{S}^{2n-1}/G = L(q : p_1, \ldots, p_n)$, with $G =< g >$, permuting the p_i's doesn't change the underlying group G; similarly, if we multiply all the p_i's by some number $\pm l$ where $gcd(l, q) = 1$, that simply means we have mapped the generator g to the generator g^l, and so we still have the same group G. Also note that if two lens spaces $\mathbb{S}^{2n-1}/G = L(q : p_1, \ldots, p_n)$ and $\mathbb{S}^{2n-1}/G' = L(q : s_1, \ldots, s_n)$ are isometric, then by the above lemma G and G' must be conjugate. So, the lift of the isometry on $\mathbb{S}^{2n-1}$ maps a generator, g of G to a generator g'^l of G'. This means that the eigenvalues of g and g'^l are the same, which means that each p_i is equivalent to some ls_j or $-ls_j$ (mod q). These facts give us the following corollary for lemma 4.1.1

Corollary 4.1.2. *Let $L = L(q : p_1, \ldots, p_n)$ and $L' = L(q : s_1, \ldots, s_n)$ be lens spaces. Then L is isometric to L' if and only if there is a number l coprime with q and there are numbers $e_i \in \{-1, 1\}$ such that $(p_1, \ldots, p_n)$ is a permutation of $(e_1 ls_1, \ldots, e_n ls_n)$ (mod q).*

Assume we have a spherical space form $\mathbb{S}^m/G$ for any integer $m \geq 2$. For any $f \in C^\infty(\mathbb{S}^m/G)$, we define the Lapacian on the spherical space form as $\widetilde{\Delta}(\pi^* f) = \pi^*(\Delta f)$. We now construct the spectral generating function associated with the Laplacian on $\mathbb{S}^{2n-1}/G$ analogous to the construction in the manifold case (see [I1], [I2] and [IY]). Let $\tilde{\Delta}$, Δ and Δ_0 denote the Laplacians of S^{2n-1}, $\mathbb{S}^{2n-1}/G$ and $\mathbf{R}^{2n}$, respectively.

Definition 4.1.3. *For any non-negative real number λ, we define the* eigenspaces $\widetilde{E}_\lambda$ *and E_λ as follows:*

$$\widetilde{E}_\lambda = \{f \in C^\infty(S^{2n-1}) \mid \widetilde{\Delta} f = \lambda f\},$$
$$E_\lambda = \{f \in C^\infty(\mathbb{S}^{2n-1}/G) \mid \Delta f = \lambda f\}.$$

The following lemma follows from the definitions of Δ and smooth function.

Lemma 4.1.4. *(i) For any $f \in C^\infty(\mathbb{S}^{2n-1}/G)$, we have $\widetilde{\Delta}(\pi^* f) = \pi^*(\Delta f)$.*

(ii) For any G-invariant function F on S^{2n-1}, there exists a unique function $f \in C^\infty(\mathbb{S}^n/G)$ such that $F = \pi^* f$.

Proof. The natural projection $\pi : S^{2n-1} \to S^{2n-1}/G$ induces the injective map $\pi^* : C^\infty(S^{2n-1}/G) \to C^\infty(S^{2n-1})$. Now (i) follows from the definition of the Laplacian on an orbifold as defined in Section 2.3. Also, since F is G-invariant it lies in the image of $\pi^*(C^\infty(S^{2n-1}/G))$. We define $f = (\pi^*)^{-1}(F)$. This proves (ii). $\square$

Corollary 4.1.5. *Let $\left(\widetilde{E}_\lambda\right)_G$ be the space of all G-invariant functions of $\widetilde{E}_\lambda$. Then* $\dim(E_\lambda) = \dim(\widetilde{E})_G$.

Proof. By the above lemma, we can see that if $f \in E_\lambda$, then there exists a unique $F \in \widetilde{E}_\lambda$ such that F is G-invariant and $F = \pi^* f$. Conversely, for any G-invariant eigenfunction $F \in \widetilde{E}_\lambda$, there exists a unique eigenfunction $f \in E_\lambda$ such that $F = \pi^* f$. Both of these facts follow from the above lemma and the fact that $\pi^* : C^\infty(S^{2n-1}/G) \to C^\infty(S^{2n-1})$ is an injection.

Now these facts imply that there is a one-to-one correspondence between functions in E_λ and functions in $\left(\widetilde{E}_\lambda\right)_G$. Therefore, $\dim(E_\lambda) = \dim\left(\widetilde{E}_\lambda\right)_G$. $\square$

Let Δ_0 be the Laplacian on $\mathbf{R}^{2n}$ with respect to the flat Kähler metric. Set $r^2 = \sum_{i=1}^{2n} x_i^2$, where $(x_1, x_2, \dots, x_{2n})$ is the standard coordinate system on $\mathbf{R}^{2n}$. For $k \geq 0$, let P^k denote the space of complex valued homogeneous polynomials of degree k on $\mathbf{R}^{2n}$. Let H^k be the subspace of P^k consisting of harmonic polynomials on $\mathbf{R}^{2n}$,

$$H^k = \left\{ f \in P^k \,\middle|\, \Delta_0 f = 0 \right\}.$$

Each orthogonal transformation of $\mathbf{R}^{2n}$ canonically induces a linear isomorphism of P^k.

Proposition 4.1.6. *The space H^k is $O(2n)$-invariant, and P^k has the direct sum decomposition: $P^k = H^k \oplus r^2 P^{k-2}$.*

The injection map $i : S^{2n-1} \to \mathbf{R}^{2n}$ induces a linear map $i^* : C^\infty(\mathbf{R}^{2n}) \to C^\infty(S^{2n-1})$. We denote $i^*(H^k)$ by $\mathcal{H}^k$.

Proposition 4.1.7. *$\mathcal{H}^k$ is an eigenspace of $\widetilde{\Delta}$ on S^{2n-1} with eigenvalue $k(k+2n-2)$ and $\sum_{k=0}^{\infty} \mathcal{H}^k$ is dense in $C^\infty(S^{2n-1})$ in the uniform convergence topology. Moreover, $\mathcal{H}^k$ is isomorphic to H^k. That is, $i^* : H^k \xrightarrow{\simeq} \mathcal{H}^k$.*

For proofs of these propositions, see [BGM].

Now Corollary 4.1.5 and Proposition 4.1.7 imply that if we denote by $\mathcal{H}^k_G$ be the space of all G-invariant functions in $\mathcal{H}^k$, then

$$\dim E_{k(k+2n-2)} = \dim \mathcal{H}^k_G.$$

Moreover, for any integer k such that $\dim \mathcal{H}^k_G \neq 0$, $\bar{\lambda}_k = k(k+2n-2)$ is an eigenvalue of Δ on $\mathbb{S}^{2n-1}/G$ with multiplicity equal to $\dim \mathcal{H}^k_G$, and no other eigenvalues appear in the spectrum of Δ.

Definition 4.1.8. *Let O be a closed compact Riemannian orbifold with the Laplace spectrum, $0 \le \bar{\lambda}_1 < \bar{\lambda}_2 < \bar{\lambda}_3 \ldots \uparrow \infty$. For each $\bar{\lambda}_k$, let the eigenspace be*

$$E_{\bar{\lambda}_k} = \left\{ f \in C^\infty(O) \,\middle|\, \Delta f = \bar{\lambda}_k f \right\}.$$

We define the spectrum generating function associated to the spectrum of the Laplacian on O as

$$F_O(z) = \sum_{k=0}^{\infty} \left(\dim E_{\bar{\lambda}_k} \right) z^k.$$

In terms of spherical space forms, the definition becomes

Definition 4.1.9. *The* generating function $F_G(z)$ *associated to the spectrum of the Laplacian on $\mathbb{S}^n/G$ is the generating function associated to the infinite sequence*

$\{\dim \mathcal{H}_G^k\}_{k=0}^{\infty}$, *i.e.*,

$$F_G(z) = \sum_{k=0}^{\infty} \left(\dim \mathcal{H}_G^k\right) z^k.$$

By Corollary 4.1.5, Proposition 4.1.7 and subsequent discussion, we know that the generating function determines the spectrum of $\mathbb{S}^n/G$. This fact gives us the following proposition:

Proposition 4.1.10. *Let $\mathbb{S}^n/G$ and $\mathbb{S}^n/G'$ be two spherical space forms. Let $F_G(z)$ and $F_{G'}(z)$ be their respective spectrum generating functions. Then $\mathbb{S}^n/G$ is isospectral to $\mathbb{S}^n/G'$ if and only if $F_G(z) = F_{G'}(z)$.*

Our first goal is to find an alternative expression for $F_G(z)$ that will allow us to compare $F_G(z)$ and $F_{G'}(z)$.

If G is a finite subgroup of $O(2n)$ with orientation preserving action on S^{2n-1} then G is a subgroup of $SO(2n)$. In the following we will consider orientation-preserving group actions.

The following theorem, proved for manifold spherical space forms in [I1] and [I2], holds true for the orbifold spherical space forms as well.

Theorem 4.1.11. *Let G be a finite subgroup of $SO(2n)$, and let S^{2n-1}/G be a spherical space form with spectrum generating function $F_G(z)$. Then, on the domain $\{z \in \mathbb{C} \mid |z| < 1\}$, $F_G(z)$ converges to the function*

$$F_G(z) = \frac{1}{|G|} \sum_{g \in G} \frac{1 - z^2}{det(I_{2n} - gz)}.$$

where $|G|$ denotes the order of G and I_{2n} is the $2n \times 2n$ identity matrix.

Proof. We reproduce the proof from [I1] and [I2] here to show that the hypothesis that G acts freely is not used. Let χ_k and $\tilde{\chi}_k$ be the characters of the natural representations of $SO(2n)$ on H^k and P^k, respectively. Then we have (see [FH], pp. 10-18)

$$\dim \mathcal{H}_G^k = \frac{1}{|G|} \sum_{g \in G} \chi_k(g). \tag{4.1}$$

By Proposition 4.1.6, we get

$$\chi_k(g) = \tilde{\chi}_k(g) - \tilde{\chi}_{k-2}(g), \tag{4.2}$$

where we put $\tilde{\chi}_{-t} = 0$ for $t > 0$.

If an element $g \in SO(2n)$ is conjugate to an element $g' \in SO(2n)$ in $O(2n)$, then

$$\tilde{\chi}_k(g') = \tilde{\chi}_k(g),\ k \geq 0. \tag{4.3}$$

Let g be an element in G of order q. Set $\gamma = e^{2\pi i/q}$ and let $\gamma^{p_1}, \bar{\gamma}^{p_1}, ..., \gamma^{p_n}, \bar{\gamma}^{p_n}$ be the eigenvalues of g, then g is conjugate to the element

$$g' = \begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_n/q) \end{pmatrix}$$

in $SO(2n)$.

Let $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ be the standard Euclidean coordinates on $\mathbf{R}^{2n}$. Set $z_j = x_j + iy_j$, where $i = \sqrt{-1}$ $\quad (j = 1, 2, \ldots n)$. Then we can view the space P^k having a basis consisting of all monomials of the form

$$z^I \cdot \bar{z}^J = (z_1)^{i_1} \cdots (z_n)^{i_n} \cdot (\bar{z}_1)^{j_1} \cdots (\bar{z}_n)^{j_n},$$

where $i_1, \ldots, i_n, j_1, \ldots, j_n \geq 0$ and $i_1 + \cdots + i_n + j_1 + \cdots j_n = k$. We denote $i_1 + \cdots + i_n + j_1 + \cdots j_n = k$ by $I_n + J_n = k$. Then for any monomial $z^I \cdot \bar{z}^J$, we will have

$$g'\left(z^I \cdot \bar{z}^J\right) = \gamma^{i_1 p_1 + \cdots + i_n p_n - j_1 p_1 - \cdots - j_n p_n} \left(z^I \cdot \bar{z}^J\right).$$

So,

$$\tilde{\chi}_k(g') = \sum_{I_n + J_n = k} \gamma^{i_1 p_1 + \cdots + i_n p_n - j_1 p_1 - \cdots - j_n p_n}. \tag{4.4}$$

Then,

$$\begin{aligned}
F_G(z) &= \sum_{k=0}^{\infty} \left(\dim \mathcal{H}_G^k\right) z^k \\
&= \sum_{k=0}^{\infty} \frac{1}{|G|} \sum_{g \in G} \chi_k(g) z^k && \text{by (4.2)} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{k=0}^{\infty} \chi_k(g) z^k \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{k=0}^{\infty} \left(\tilde{\chi}_k(g) - \tilde{\chi}_{k-2}(g)\right) z^k && \text{by (4.3)} \\
&= \frac{(1 - z^2)}{|G|} \sum_{g \in G} \sum_{k=0}^{\infty} \tilde{\chi}_k(g) z^k \\
&= \frac{(1 - z^2)}{|G|} \sum_{g \in G} \sum_{k=0}^{\infty} \tilde{\chi}_k(g') z^k
\end{aligned}$$

$$= \frac{(1-z^2)}{|G|} \sum_{g\in G} \sum_{k=0}^{\infty} \Big(\sum_{I_n+J_n=k} \gamma^{i_1p_1+\cdots+i_np_n-j_1p_1-\cdots-j_np_n} \Big) z^k \qquad \text{by (4.2)}$$

$$= \frac{(1-z^2)}{|G|} \sum_{g\in G} \sum_{k=0}^{\infty} \sum_{I_n+J_n=k} (\gamma^{p_1}z)^{i_1} \cdots (\gamma^{p_n}z)^{i_n} \left(\gamma^{-p_1}z\right)^{j_1} \cdots \left(\gamma^{-p_n}z\right)^{j_n}$$

$$= \frac{(1-z^2)}{|G|} \sum_{g\in G} \prod_{i=1}^{n} \left(1+\gamma^{p_i}z+\gamma^{2p_i}z^2+\cdots\right) \left(1+\gamma^{-p_i}z+\gamma^{-2p_i}z^2+\cdots\right).$$

On the domain $\left\{z \in \mathbb{C} \,\middle|\, |z|<1\right\}$, the power series

$$\left(1+\gamma^{p_i}z+\gamma^{2p_i}z^2+\cdots\right)$$

converges to $\frac{1}{(1-\gamma^{p_i}z)}$. So, the product

$$\prod_{i=1}^{n} \left(1+\gamma^{p_i}z+\gamma^{2p_i}z^2+\cdots\right) \left(1+\gamma^{-p_i}z+\gamma^{-2p_i}z^2+\cdots\right)$$

converges to

$$\frac{1}{\prod_{i=1}^{n}(1-\gamma^{p_i}z)(1-\gamma^{-p_i}z)}.$$

Now if we denote by $E(g)$ to be the set of eigenvalues of g, then we write

$$F_G(z) = \frac{(1-z^2)}{|G|} \sum_{g\in G} \frac{1}{\prod_{\gamma\in E(g)}(1-\gamma z)} = \frac{(1-z^2)}{|G|} \sum_{g\in G} \frac{1}{\det(I_{2n}-gz)}.$$

$\square$

We denote the generating function for a lens space $L = L(q : p_1, \ldots, p_n)$ by $F_q(z :$

$p_1, ..., p_n)$.

Corollary 4.1.12. *Let $L(q : p_1, \dots, p_n)$ be a lens space and $F_q(z : p_1, \dots, p_n)$ the generating function associated to the spectrum of $L(q : p_1, \dots, p_n)$. Then, on the domain $\{z \in \mathbb{C} \mid |z| < 1\}$,*

$$F_q(z : p_1, \dots, p_n) = \frac{1}{q} \sum_{l=1}^{q} \frac{1 - z^2}{\prod_{i=1}^{n} (z - \gamma^{p_i l})(z - \gamma^{-p_i l})},$$

where γ is a primitive q-th root of unity.

Proof. In the notation of the Theorem 4.1.11, we get

$$\dim \mathcal{H}_G^k = \frac{1}{|G|} \sum_{g \in G} \chi_k(g) = \frac{1}{q} \sum_{l=1}^{q} \chi_k(g^l). \tag{4.5}$$

So

$$\begin{aligned} F_q(z : p_1, \dots, p_n) &= \frac{(1 - z^2)}{|G|} \sum_{g \in G} \frac{1}{\prod_{i=1}^{n} (1 - \gamma^{p_i} z)(1 - \gamma^{-p_i} z)} \\ &= \frac{(1 - z^2)}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n} (z - \gamma^{p_i l})(z - \gamma^{-p_i l})}, \end{aligned}$$

since multiplying through by $1 = (-\gamma^{-p_i l})(-\gamma^{p_i l})$ gives
$(1 - \gamma^{p_i l} z)(1 - \gamma^{-p_i l} z) = (z - \gamma^{-p_i l})(z - \gamma^{p_i l})$. $\square$

Remark: By the Theorem 4.1.11 and unique analytic continuation, we can consider the generating function to be a meromorphic function on the whole complex plane $\mathbb{C}$ with poles on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

From this remark we have,

Corollary 4.1.13. *Let S^{2n-1}/G and S^{2n-1}/G' be two spherical space forms. If there is a one to one mapping ϕ of G onto G' such that the set $E(g)$ = the set $E(\phi(g)), \forall g \in$*

G, then S^{2n-1}/G is isospectral to S^{2n-1}/G'.

Proof. The proof follows from the fact that

$$\prod_{\gamma\in E(g)}(1-\gamma z)=\prod_{\gamma\in E(g)}(z-\gamma)=\det(I_{2n}-gz).$$

□

Corollary 4.1.14. *Let S^{2n-1}/G and S^{2n-1}/G' be two isospectral spherical space forms. Then $|G|=|G'|$.*

Proof. $F_G(z)$ can be considered as a meromorphic function on the whole complex plane. $F_G(z)$ has a pole of order $2n-1$ at $z=1$. Note that $\gamma^{p_i l}=1$ iff $p_i l\equiv 1(\text{mod } q)$. Since $gcd(p_1,p_2,...,p_n;q)=1$, this is only true for all p_i at once for $l=q$. In particular, this implies that for this corollary to hold, we don't need to have a manifold quotient, nor is this required in the proof of Theorem 4.1.11.

$$\begin{aligned}
\text{We take } &\lim_{z\to 1}(1-z)^{2n-1}F_G(z)\\
&=\lim_{z\to 1}(1-z)^{2n-1}\ \frac{1}{|G|}\sum_{g\in G}\frac{(1-z^2)}{\prod_{\gamma\in E(g)}(1-\gamma z)}\\
&=\frac{1}{|G|}\lim_{z\to 1}(1-z)^{2n}\sum_{g\in G}\frac{(1+z)}{\prod_{\gamma\in E(g)}(1-\gamma z)}\\
&=\frac{1}{|G|}\lim_{z\to 1}(1+z)\sum_{g\in G}\frac{(1-z)^{2n}}{\prod_{\gamma\in E(g)}(1-\gamma z)}\\
&=\frac{2}{|G|}(\underbrace{0+0+\cdots+0}_{2n-1\text{ times}}+1)=\frac{2}{|G|}.
\end{aligned}$$

Since the spaces are isospectral, this implies that

$$\lim_{z\to 1}(1-z)^{2n-1}F_G(z)=\lim_{z\to 1}(1-z)^{2n-1}F_{G'}(z),$$

which gives $|G| = |G'|$. $\square$

4.2 Formulation of Generating Function

In order to use the generating function to find isospectral non-isometric lens spaces, we need to formulate it in ways that are useful for certain types of lens spaces. The formulation of the generating functions used by Ikeda ([I1]) for manifold lens spaces needs to be changed to allow for singular points when we are dealing with orbifold lens spaces. In this section we will develop forms of the generation function that will be used to find isospectral pairs of non-isometric orbifold lens spaces.

4.2.1 Preliminaries

Let q be a positive integer that is *not prime*. Set

$$q_0 = \begin{cases} \frac{q-1}{2} & \text{if } q \text{ is odd,} \\ \frac{q}{2} & \text{if } q \text{ is even.} \end{cases}$$

Throughout this chapter we assume that $q_0 \geq 4$.

For any positive integer n with $2 \leq n \leq q_0 - 2$, we denote by $\widetilde{I}(q,n)$ the set of n-tuples $(p_1, \ldots, p_n)$ of integers. We define a subset $\widetilde{I}_0(q,n)$ of $\widetilde{I}(q,n)$ as follows:

$$\widetilde{I}_0(q,n) =$$
$$\left\{ (p_1, \ldots, p_n) \in \widetilde{I}(q,n) \,\middle|\, p_i \not\equiv \pm p_j \pmod q, 1 \leq i < j \leq n,\ g.c.d.(p_1, \ldots, p_n, q) = 1 \right\}.$$

We introduce an equivalence relation in $\widetilde{I}(q,n)$ as follows: $(p_1, \ldots, p_n)$ is equivalent to $(s_1, \ldots, s_n)$ if and only if there is a number l prime to q and there are numbers

$e_i \in \{-1, 1\}$ such that $(p_1, \ldots, p_n)$ is a permutation of $(e_1 l s_1, \ldots, e_n l s_n) \pmod q$. This equivalence relation also defines an equivalence relation on $\widetilde{I}_0(q, n)$.

We set $I(q, n) = \widetilde{I}(q, n)/\sim$ and $I_0(q, n) = \widetilde{I}_0(q, n)/\sim$. Let $k = q_0 - n$. We define a map w of $I_0(q, n)$ into $I_0(q, k)$ as follows: For any element $(p_1, \ldots, p_n) \in \widetilde{I}_0(q, n)$, we choose an element $(q_1, \ldots, q_k) \in \widetilde{I}_0(q, k)$ such that the set of integers

$$\Big\{p_1, -p_1, \ldots, p_n, -p_n, q_1, -q_1, \ldots, q_k, -q_k\Big\}$$

forms a complete set of incongruent residues $\pmod q$. Then we define

$$w([p_1, \ldots, p_n]) = [q_1, \ldots, q_k].$$

Suppose there is another set $(s_1, \ldots, s_k) \in \widetilde{I}_0(q, k)$ such that the set of integers

$$\Big\{p_1, -p_1, \ldots, p_n, -p_n, s_1, -s_1, \ldots, s_k, -s_k\Big\}$$

forms a complete set of incongruent residues $\pmod q$. Suppose there is no l prime to q such that $(q_1, \ldots, q_k)$ is congruent to a permutation of $(l e_1 s_1, \ldots, l e_k s_k) \pmod q$. That means that for any given l prime to q, there is at least one q_i which is incongruent to $l e_j s_j \pmod q$ for $j = 1, 2, \ldots, k$ and $e_j \in \{-1, 1\}$. This would mean that the number of incongruent residues $\pmod q$ is greater than q. This is not possible. Therefore $(s_1, \ldots, s_k)$ must be equivalent to $(q_1, \ldots, q_k)$. Therefore, w is well-defined.

With similar arguments, it is easy to see that w is one-to-one and onto. Assume

$$w([p_1, \ldots, p_n]) = w([r_1, \ldots, r_n]) = [q_1, \ldots, q_k].$$

This means that

$$\{p_1, -p_1, \ldots, p_n, -p_n, q_1, -q_1, \ldots, q_k, -q_k\}$$

and

$$\{r_1, -r_1, \ldots, r_n, -r_n, q_1, -q_1, \ldots, q_k, -q_k\}$$

are both complete sets of incongruent residues $(\bmod\, q)$.

If there is no l prime to q such that $(p_1, \ldots, p_n)$ is congruent to a permutation of $(le_1r_1, \ldots, le_nr_n)$ $(\bmod\, q)$, where $e_i \in \{-1, 1\}$, then that would mean that there are more than q incongruent residues $(\bmod\, q)$. This is not possible. Therefore, $[p_1, \ldots, p_n] = [r_1, \ldots, r_n]$, and w is one-to-one.

Now, given a $[q_1, \ldots, q_k] \in I_0(q, k)$, there are exactly k of the $\{q_1, -q_1, \ldots, q_k, -q_k\}$ $(\bmod\, q)$ that are less than or equal to q_0. Now since $q_0 = n + k$, we can choose the other n integers to be $p_1, \ldots, p_n$ so that the set

$$\Big\{p_1, -p_1, \ldots, p_n, -p_n, q_1, -q_1, \ldots, q_k, -q_k\Big\}$$

forms a complete set of incongruent residues $(\bmod\, q)$. Thus, we can have $[p_1, \ldots, p_n] \in I_0(q, n)$ that maps onto $[q_1, \ldots, q_k]$. So w is onto. This gives us a bijection

$$w : I_0(q, n) \xrightarrow{\sim} I_0(q, k). \tag{4.6}$$

The following proposition is similar to a result in [I1]:

Proposition 4.2.1. *Let $I_0(q, n)$ be as above. Then,*

$$|I_0(q, n)| \geq \frac{1}{q_0}\binom{q_0}{n}$$

where

$$\binom{q_0}{n} = \begin{cases} 1 & \textit{if } q_0 n = 0, \\ \frac{q_0!}{n!(q_0-n)!} & \textit{otherwise.} \end{cases}$$

Proof. Let $I_0(q,n)$ be as above. Consider a subset $\widetilde{I}'_0(q,n)$ of $\widetilde{I}_0(q,n)$ as follows:

$$\widetilde{I}'_0(q,n) = \Big\{(p_1,\dots,p_n) \in \widetilde{I}_0(q,n) \,\Big|\, \text{at least one of the } p_i \text{ is co-prime to } q\Big\}.$$

It is easy to see that the equivalence relation on $\widetilde{I}_0(q,n)$ induces an equivalence relation on $\widetilde{I}'_0(q,n)$. Since we eliminate classes where none of the p_i's is co-prime to q, we get

$$|I_0(q,n)| \geq |I'_0(q,n)|,$$

where $I'_0(q,n) = \widetilde{I}'_0(q,n)/\sim$. Now consider a subset $\widetilde{I}''_0(q,n)$ of $\widetilde{I}'_0(q,n)$ as follows:

$$\widetilde{I}''_0(q,n) = \Big\{(p_1,\dots,p_n) \in \widetilde{I}'_0(q,n) \,\Big|\, 1 = p_1 < \cdots < p_n \leq q_0\Big\}.$$

Then it is easy to see that any element of $\widetilde{I}'_0(q,n)$ has an equivalent element in $\widetilde{I}''_0(q,n)$. On the other hand, for any equivalence class in $I'_0(q,n)$, the number of elements in $\widetilde{I}''_0(q,n)$ which belong to that class is at most n. Hence we have:

$$|I_0(q,n)| \geq |I'_0(q,n)| \geq \frac{1}{n}\,\big|\,\widetilde{I}''_0(q,n)\,\big| \;= \frac{1}{n}\binom{q_0-1}{n-1} = \frac{1}{q_0}\binom{q_0}{n}.$$

This proves the proposition. ∎

Lemma 4.2.2. *Let $q = p^m$ or $q = p_1 \cdot p_2$, where p, p_1, p_2 are primes. Let D be the set of all non-zero integers mod q that are not co-prime to q. Then $|D|$ is even if q is odd and $|D|$ is odd if q is even.*

Proof. For $q = p^m$:

If q is odd, then p is an odd prime. $\frac{q}{p} = p^{m-1}$ which is an odd number. Therefore the number of elements in D, $(p^{m-1} - 1)$ is even.

If q is even, then $p = 2$. $\frac{q}{p} = 2^{m-1}$ is even. So the number of elements in D, $(2^{m-1} - 1)$, is odd.

For $q = p_1 \cdot p_2$. $(p_1 \neq p_2)$:

If q is odd, then both p_1 and p_2 are odd primes. The number of elements in D is $(\frac{q}{p_1} + \frac{q}{p_2} - 2) = (p_2 + p_1 - 2)$ which is even since $p_1 + p_2$ is even.

If q is even, then one of the p_i's is 2 and the other is an odd prime. Assume $p_1 = 2$. So, the number of elements in D is $(\frac{q}{p_1} + \frac{q}{p_2} - 2) = (p_2 + p_1 - 2) = (p_2 + 2 - 2) = p_2$, which is odd.

This proves the lemma. $\square$

We will say that $|D| = 2r$ if $|D|$ is even; and $|D| = 2r - 1$ if $|D|$ is odd, where r is some positive integer. It is easy to see that if $|D|$ is even, then exactly r members of D are less than q_0. If $|D|$ is odd, then $r - 1$ members of D are strictly less than q_0 and one member of D is equal to q_0 (recall that for even q, we set $q_0 = q/2$, and for odd q, we set $q_0 = (q-1)/2$).

With these results we now obtain a better lower bound for $|I_0(q, n)|$.

Proposition 4.2.3. *Let $I_0(q, n)$, $I_0'(q, n)$, $\widetilde{I}_0'(q, n)$ and $\widetilde{I}_0''(q, n)$ be as in Proposition 4.2.1. Let $k = q_0 - n$. Then*

$$|I_0(q, n)| \geq \sum_{t=u}^{r} \frac{1}{n-t} \binom{q_0 - 1 - r}{n - 1 - t} \binom{r}{t},$$

where $u = r - k$ if $r > k$ and $u = 0$ if $r \leq k$, and r is as defined above.

Proof. The number of ways in which we can assign values to the p_i's in

$$(1 = p_1, p_2, \ldots, p_n) \in \widetilde{I}''_0(q, n)$$

such that t of the p_i's are *not* co-prime to q is

$$\binom{q_0 - 1 - r}{n - 1 - t}\binom{r}{t}.$$

On the other hand for any equivalence class in $I'_0(q, n)$ with t of the p_i's not being co-prime to q, the number of elements which belong to that class is at most $n - t$. So the number of such possible classes is at least

$$\frac{1}{n-t}\binom{q_0 - 1 - r}{n - 1 - t}\binom{r}{t}.$$

Now if $r > k$, this would mean that $n > q_0 - r$, or $n - 1 > q_0 - 1 - r$. This means that t cannot take any values less than $r - k$, since that would mean that we are choosing $(n - 1 - t)$, a number larger than $(q_0 - 1 - r)$ from $q_0 - 1 - r$ and that is not possible. So, the smallest value for t in this case can be $r - k$.

On the other hand, if $r \le k$, then $n \le q_0 - r$, or $n - 1 \le q_0 - 1 - r$. This means that it is possible for us to choose n-tuples in $\widetilde{I}''_0(q, n)$ with all values being co-prime to q. Thus, the smallest value for t would be 0 in this case.

It is obvious that the maximum value t can take is r since $(1, p_2, \ldots, p_n)$ cannot have more than r values that are not co-prime to q. Now, adding up all the degrees for different values of t we get

$$|I_0(q, n)| \ge |I'_0(q.n)| \ge \sum_{t=u}^{r} \frac{1}{n-t}\binom{q_0 - 1 - r}{n - 1 - t}\binom{r}{t},$$

where $u = 0$ if $r \leq k$ and $u = r - k$ if $r > k$.

This proves the proposition. $\square$

Definition 4.2.4. *(i) Let q be a positive integer and γ a primitive q-th root of 1. We denote by $\mathbf{Q}(\gamma)$ the q-th cyclotomic field over the rational number field $\mathbf{Q}$ and denote by $\Phi_q(z)$ the q-th cyclotomic polynomial*

$$\Phi_q(z) = \sum_{t=0}^{q-1} z^t.$$

Let A be the set of residues mod q that are co-prime to q. We define a map $\psi_{q,k}$ of $I_0(q,k)$ into $\mathbf{Q}(\gamma)[z]$ as follows:

For any equivalence class in $I_0(q,k)$, we take an element $(q_1, \ldots, q_k)$ of $\widetilde{I}_0(q,k)$ which belongs to that class. We define

$$\psi_{q,k}([q_1, \ldots, q_k])(z) = \sum_{l \in A} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l}).$$

This polynomial in $\mathbf{Q}(\gamma)[z]$ is independent of the choice of elements which belong to the class $[q_1, \ldots, q_k]$. Therefore, the map is well-defined.

(ii) Given $q = p^m$, we define

$$B_j = \left\{ x \in \mathbf{Z}^+ : p^j \mid x, p^{j+1} \nmid x \right\}.$$

We define the maps $\alpha_{q,k}^{(j)}$ of $I_0(q,k)$ into $\mathbf{Q}(\gamma)[z]$ as follows:

For any equivalence class in $I_0(q,k)$, we take an element $(q_1, \ldots, q_k)$ of $\widetilde{I}_0(q,k)$

which belongs to that class. We define

$$\alpha^{(j)}_{q,k}([q_1, \ldots, q_k])(z) = \sum_{l \in B_j} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l}).$$

These polynomials are also independent of the choice of the elements which belong to the class $[q_1, \ldots, q_k]$. Therefore the maps are well defined.

(iii) *Now assume $q = p_1 \cdot p_2$. We define the following sets of numbers that are not co-prime to q.*

$$B = \left\{ xp_1 \,\middle|\, x = 1, 2, \ldots, (p_2 - 1) \right\} \text{ and } C = \left\{ xp_2 \,\middle|\, x = 1, 2, \ldots, (p_1 - 1) \right\}.$$

We define maps $\alpha_{q,k}$ and $\beta_{q,k}$ as follows:

For any equivalence class in $I_0(q,k)$, we take an element $(q_1, \ldots, q_k)$ of $\widetilde{I}_0(q,k)$ which belongs to that class. We define

$$\alpha_{q,k}([q_1 . \ldots, q_k])(z) = \sum_{l \in B} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l})$$

and

$$\beta_{q,k}([q_1 . \ldots, q_k])(z) = \sum_{l \in C} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l}).$$

These polynomials in $\mathbf{Q}(\gamma)[z]$ are again independent of the choice of the elements which belong to $[q_1 . \ldots, q_k]$; so these maps are also well defined.

Since $(z - \gamma^{q_i l})(z - \gamma^{-q_i l}) = (\gamma^{q_i l} z - 1)(\gamma^{-q_i l} z - 1)$, the following proposition is easy to see.

Proposition 4.2.5. *If we put*

$$\begin{aligned}
\psi_{q,k}([q_1,\dots,q_k])(z) &= \sum_{i=0}^{2k}(-1)^i\, a_i z^{2k-i},\\
\alpha_{q,k}^{(j)}([q_1,\dots,q_k])(z) &= \sum_{i=0}^{2k}(-1)^i\, b_{i,j} z^{2k-i},\\
\alpha_{q,k}([q_1,\dots,q_k])(z) &= \sum_{i=0}^{2k}(-1)^i\, b_i z^{2k-i},\\
\beta_{q,k}([q_1,\dots,q_k])(z) &= \sum_{i=0}^{2k}(-1)^i\, c_i z^{2k-i},
\end{aligned}$$

then we have

1. $a_i = a_{2k-i}$, $b_{i,j} = b_{(2k-i),j}$, $b_i = b_{2k-i}$ *and* $c_i = c_{2k-i}$.

2. $a_0 = |A|$, $b_{0,j} = |B_j|$, $b_0 = |B|$ *and* $c_0 = |C|$.

4.2.2 Generating Functions and Isospectrality

Now let $\widetilde{\mathcal{L}}(q,n)$ be the family of all $(2n-1)$-dimensional lens spaces with fundamental groups of order q, and let $\widetilde{\mathcal{L}}_0(q,n)$ be the subfamily of $\widetilde{\mathcal{L}}(q,n)$ defined by:

$$\widetilde{\mathcal{L}}_0(q,n) = \left\{L(q:p_1,\dots,p_n) \in \widetilde{\mathcal{L}}(q,n) \,\middle|\, p_i \not\equiv \pm p_j \pmod{q}, 1 \le i < j \le n\right\}.$$

The set of isometry classes of $\widetilde{\mathcal{L}}(q,n)$ is denoted by $\mathcal{L}(q,n)$, and the set of isometry classes of $\widetilde{\mathcal{L}}_0(q,n)$ is denoted by $\mathcal{L}_0(q,n)$.

By Proposition 4.1.2, the map

$$L(q:p_1,\dots,p_n) \mapsto (p_1,\dots,p_n)$$

of $\widetilde{\mathcal{L}}_0(q,n)$ $\big[$resp. $\widetilde{\mathcal{L}}(q,n)\big]$ onto $\widetilde{I}_0(q,n)$ $\big[$resp. $\widetilde{I}(q,n)\big]$ induces a one-to-one map between $\mathcal{L}_0(q,n)$ and $I_0(q,n)$ $\big[$resp. $\mathcal{L}(q,n)$ and $I(q,n)\big]$.

The above fact, together with Proposition 4.2.3, gives us the following:

Proposition 4.2.6. *Retaining the notations as above, we get*

$$|\mathcal{L}_0(q,n)| \geq \sum_{t=u}^{r} \frac{1}{n-t}\binom{q_0-1-r}{n-1-t}\binom{r}{t},$$

where $u = r-k$ if $r > k$, and $u = 0$ if $r \leq k$; r is the number of residues(modq) that are not co-prime to q and are less than or equal to q_0.

Note that by Proposition 4.2.1, we also get that

$$|\mathcal{L}_0(q,n)| \geq \frac{1}{q_0}\binom{q_0}{n}.$$

Next, we will re-formulate the generating function $F_q(z : p_1, \ldots, p_n)$ in a form that will help us find isospectral pairs that are non-isometric (see Proposition 2.2.12 in [Ba]).

Proposition 4.2.7. *Let $L(q : p_1, \ldots, p_n)$ be a lens space belonging to $\widetilde{\mathcal{L}}_0(q,n)$, $k = q_0 - n$, and let w be the map of $I_0(q,n)$ onto $I_0(q,k)$ defined in section 4.2.1. Then*

(i) If $q = P^m$, where P is a prime, we have

$$F_q(z : p_1, \ldots, p_n) = \frac{1}{q}\left\{ \frac{(1-z^2)}{(1-z)^{2n}} + \frac{\psi_{q,k}(w([p_1, \ldots, p_n]))(z)(1-z^2)}{\Phi_q(z)} + \sum_{j=1}^{m-1} \frac{\alpha_{q,k}^{(j)}(w([p_1, \ldots, p_n]))(z)(1-z^2)}{(\Phi_{P^{m-j}}(z))^{P^j}(1-z)^{P^j-1}} \right\}.$$

(ii) If $q = P_1 \cdot P_2$, where P_1 and P_2 are primes, we have

$$F_q(z : p_1, \dots, p_n) = \frac{1}{q}\Bigg\{ \frac{(1-z^2)}{(1-z)^{2n}} + \frac{\psi_{q,k}(w([p_1, \dots, p_n]))(z)(1-z^2)}{\Phi_q(z)} + \frac{\alpha_{q,k}(w([p_1, \dots, p_n]))(z)(1-z^2)}{(\Phi_{P_2}(z))^{P_1}(1-z)^{P_1-1}} + \frac{\beta_{q,k}(w([p_1, \dots, p_n]))(z)(1-z^2)}{(\Phi_{P_1}(z))^{P_2}(1-z)^{P_2-1}} \Bigg\},$$

where $\psi_{q,k}$, $\alpha^{(j)}_{q,k}$, $\alpha_{q,k}$ and $\beta_{q,k}$ are as defined in definition 4.2.4 and $\Phi_t(z) = \sum_{v=0}^{t-1} z^v$.

Proof. We choose integers $q_1, \dots, q_k$ such that the set of integers

$$\{p_1, -p_1, \dots, p_n, -p_n, q_1, -q_1, \dots, q_k, -q_k\}$$

forms a complete set of residues mod q.

(i) We write

$$F_q(z : p_1, \dots, p_n) = \frac{1}{q}\Bigg[\sum_{l \in A} \frac{(1-z^2)}{\prod_{i=1}^{n}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})} + \sum_{j=1}^{m-1} \sum_{l \in B_j} \frac{(1-z^2)}{\prod_{i=1}^{n}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})} \Bigg\}.$$

Now, for any $l \in A$, we have

$$\frac{1}{\prod_{i=1}^{n}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})} = \frac{\prod_{i=1}^{k}(z - \gamma^{q_i l})(z - \gamma^{-q_i l})}{\Phi_q(z)}.$$

For $l \in B_j$, we have

$$\frac{1}{\prod_{i=1}^{n}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})} = \frac{\prod_{i=1}^{k}(z - \gamma^{q_i l})(z - \gamma^{-q_i l})}{(\Phi_{P^{m-j}}(z))^{P^j}(1-z)^{P^j - 1}}.$$

Now, (i) follows from these facts.

(ii) We write

$$F_q(z:p_1,\dots,p_n)=\frac{1}{q}\Bigg[\sum_{l\in A}\frac{(1-z^2)}{\prod_{i=1}^{n}(z-\gamma^{p_il})(z-\gamma^{-p_il})}$$
$$+\sum_{l\in B}\frac{(1-z^2)}{\prod_{i=1}^{n}(z-\gamma^{p_il})(z-\gamma^{-p_il})}+\sum_{l\in C}\frac{(1-z^2)}{\prod_{i=1}^{n}(z-\gamma^{p_il})(z-\gamma^{-p_il})}\Bigg].$$

Again, for $l\in A$,

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_il})(z-\gamma^{-p_il})}=\frac{\prod_{i=1}^{k}(z-\gamma^{q_il})(z-\gamma^{-q_il})}{\Phi_q(z)}.$$

For $l\in B$, we have

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_il})(z-\gamma^{-p_il})}=\frac{\prod_{i=1}^{k}(z-\gamma^{q_il})(z-\gamma^{-q_il})}{(\Phi_{P_2}(z))^{P_1}(1-z)^{P_1-1}}.$$

For $l\in C$, we have

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_il})(z-\gamma^{-p_il})}=\frac{\prod_{i=1}^{k}(z-\gamma^{q_il})(z-\gamma^{-q_il})}{(\Phi_{P_1}(z))^{P_2}(1-z)^{P_2-1}}.$$

Now, (ii) follows from these facts.

□

From Proposition 4.1.10 and Proposition 4.2.7, we get the following proposition

Proposition 4.2.8. *Let $L=L(q:p_1,\dots,p_n)$ and $L'=L(q:s_1,\dots,s_n)$ be lens spaces belonging to $\tilde{\mathcal{L}}_0(q,n)$. Let $k=q_0-n$.*

(i) If $q = P^m$, then L is isospectral to L' if

$$\begin{aligned} &\psi_{q,k}(w([p_1,\dots,p_n])) = \psi_{q,k}(w([s_1,\dots,s_n])) \\ \textit{and} \qquad &\alpha_{q,k}^{(j)}(w([p_1,\dots,p_n])) = \alpha_{q,k}^{(j)}(w([s_1,\dots,s_n])) \end{aligned}$$

for $j = 1,\dots,m-1$.

(ii) If $q = P_1 \cdot P_2$, then L is isospectral to L' if

$$\begin{aligned} &\psi_{q,k}(w([p_1,\dots,p_n])) = \psi_{q,k}(w([s_1,\dots,s_n])), \\ &\alpha_{q,k}(w([p_1,\dots,p_n])) = \alpha_{q,k}(w([s_1,\dots,s_n])) \\ \textit{and} \qquad &\beta_{q,k}(w([p_1,\dots,p_n])) = \beta_{q,k}(w([s_1,\dots,s_n])) \end{aligned}$$

By applying Proposition 4.2.6 and Proposition 4.2.8 we will obtain our main Theorem 4.3.5 in this chapter for odd-dimensional lens spaces. Next, in Theorem 4.4.5 we will extend the results to obtain even-dimensional pairs of lens spaces corresponding to every pair of odd-dimensional lens spaces.

4.3 Odd-Dimensional Lens Spaces

From the results in the previous sections we get the following diagrams:
For $q = P^m$,

$$\mathcal{L}_0(q,n) \xrightarrow{\sim} I_0(q,n) \xrightarrow[w]{\sim} I_0(q,k) \xrightarrow[\tau_{q,k}^{(m)}]{} Q^m(\gamma)[z], \tag{4.7}$$

where $\tau_{q,k}^{(m)} = (\psi_{q,k}, \alpha_{q,k}^{(1)}, \dots, \alpha_{q,k}^{(m-1)})$, and $Q^m(\gamma)[z]$ denotes m copies of the field of rational polynomials $Q(\gamma)[z]$.

For $q = P_1 \cdot P_2$,

$$\mathcal{L}_0(q,n) \xrightarrow{\sim} I_0(q,n) \underset{w}{\xrightarrow{\sim}} I_0(q,k) \xrightarrow[\mathcal{S}^{(3)}_{q,k}]{} Q^3(\gamma)[z], \tag{4.8}$$

where $\mathcal{S}^{(3)}_{q,k} = (\psi_{q,k}, \alpha_{q,k}, \beta_{q,k})$.

Now, from Proposition 4.2.8, if $\tau^{(m)}_{q,k}$ [resp. $\mathcal{S}^{(3)}_{q,k}$] is not one-to-one, then we will have non-isometric lens spaces having the same generating function. This would give us our desired results.

The following two propositions will give us the possible number of expressions for $\tau^{(m)}_{q,k}$ and $\mathcal{S}^{(3)}_{q,k}$ for the case when $k = 2$. But before we get to the propositions, we first calculate the values for the required coefficients of $\psi_{q,2}$, $\alpha^{(j)}_{q,2}$, $\alpha_{q,2}$ and $\beta_{q,2}$.

Recall from Proposition 4.2.5 that if we set

$$\begin{aligned}
\psi_{q,k}([q_1,\ldots,q_k])(z) &= \sum_{i=0}^{2k} (-1)^i\, a_i z^{2k-i},\\
\alpha^{(j)}_{q,k}([q_1,\ldots,q_k])(z) &= \sum_{i=0}^{2k} (-1)^i\, b_{i,j} z^{2k-i},\\
\alpha_{q,k}([q_1,\ldots,q_k])(z) &= \sum_{i=0}^{2k} (-1)^i\, b_i z^{2k-i},\\
\beta_{q,k}([q_1,\ldots,q_k])(z) &= \sum_{i=0}^{2k} (-1)^i\, c_i z^{2k-i}
\end{aligned}$$

then we will have $a_i = a_{2k-i}$, $b_{i,j} = b_{(2k-i),j}$, $b_i = b_{2k-i}$ and $c_i = c_{2k-i}$. We will also have $a_0 = |A|$, $b_{0,j} = |B_j|$, $b_0 = |B|$ and $c_0 = |C|$.

Recall also that,

$$\psi_{q,k}([q_1,\ldots,q_k])(z) = \sum_{l\in A}\ \prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l}),$$
$$\alpha_{q,k}^{(j)}([q_1,\ldots,q_k])(z) = \sum_{l\in B_j}\prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l}),$$
$$\alpha_{q,k}([q_1,\ldots,q_k])(z) = \sum_{l\in B}\ \prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l}),$$
$$\beta_{q,k}([q_1,\ldots,q_k])(z) = \sum_{l\in C}\ \prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l}),$$

where

$$A = \left\{x \pmod q : (x,q)=1\right\},$$
$$B_j = \left\{x\in\mathbf{Z}^+ : p^j \mid x, p^{j+1}\nmid x\right\},$$
$$B = \left\{xp_1 : x=1,\ldots,(p_2-1)\right\},$$
$$C = \left\{xp_2 : x=1,\ldots,(p_1-1)\right\}$$

From these definitions we can calculate the values of the various coefficients of $\psi_{q,k}$, $\alpha_{q,k}^{(j)}$, $\alpha_{q,k}$ and $\beta_{q,k}$.

First we will find coefficients for z and z^2 for any given k, and from that we can find the values for when $k=2$.

From the definitions of $\psi_{q,k}([q_1,\ldots,q_k])$ it is easy to see that

$$a_1 = \sum_{i=1}^{k}\sum_{l\in A}\gamma^{q_i l} + \sum_{i=1}^{k}\sum_{l\in A}\gamma^{-q_i l} = 2\sum_{i=1}^{k}\sum_{l\in A}\gamma^{q_i l}.$$

Similarly,

$$\begin{aligned} b_{1,j} &= 2\sum_{i=1}^{k}\sum_{l\in B_j}\gamma^{q_i l}, \\ b_1 &= 2\sum_{i=1}^{k}\sum_{l\in B}\gamma^{q_i l}, \\ c_1 &= 2\sum_{i=1}^{k}\sum_{l\in C}\gamma^{q_i l}. \end{aligned}$$

Also,

$$\begin{aligned} a_2 &= \sum_{l\in A}\Big[k + \sum_{1\le i<t\le k}\gamma^{(q_i+q_t)l} + \sum_{1\le i<t\le k}\gamma^{-(q_i+q_t)l} + \sum_{1\le i<t\le k}\gamma^{(q_i-q_t)l} + \sum_{1\le i<t\le k}\gamma^{-(q_i-q_t)l}\Big] \\ &= k\,|A| + 2\sum_{l\in A}\sum_{1\le i<t\le k}\gamma^{(q_i+q_t)l} + 2\sum_{l\in A}\sum_{1\le i<t\le k}\gamma^{(q_i-q_t)l}. \end{aligned}$$

Similarly,

$$\begin{aligned} b_{2,j} &= k\,|B_j| + 2\sum_{l\in B_j}\sum_{1\le i<t\le k}\gamma^{(q_i+q_t)l} + 2\sum_{l\in B_j}\sum_{1\le i<t\le k}\gamma^{(q_i-q_t)l}, \\ b_2 &= k\,|B| + 2\sum_{l\in B}\sum_{1\le i<t\le k}\gamma^{(q_i+q_t)l} + 2\sum_{l\in B}\sum_{1\le i<t\le k}\gamma^{(q_i-q_t)l}, \\ c_2 &= k\,|C| + 2\sum_{l\in C}\sum_{1\le i<t\le k}\gamma^{(q_i+q_t)l} + 2\sum_{l\in C}\sum_{1\le i<t\le k}\gamma^{(q_i-q_t)l}, \end{aligned}$$

where $|A|$, $|B_j|$, $|B|$ and $|C|$ are cardinalities of A, B_j, B and C, respectively.

In a similar fashion we can find values of coefficients of higher powers of z when $k > 2$. These coefficients will contain terms that include higher sums and differences of the various q_i's in the powers of γ.

We notice that the values of a_1, $b_{1,j}$, b_1 and c_1 are dependent upon where the various q_i belong - in A, $B_{i,j}$, B or C. Similarly, a_2, $b_{2,j}$, b_2 and c_2 are dependent upon where the various $q_i + q_t$ and $q_i - q_t$ belong - in A, $B_{i,j}$, B or C. The same would be true for coefficients of higher powers of z for $k > 2$. That is, the coefficients always depend upon where the various sums and differences of the various q_i's reside. Therefore, we conclude that the maps $\tau_{q,k}^{(m)}$ and $\mathcal{S}_{q,k}^{(3)}$ as defined in (4.7) and (4.8) are dependent on where the various q_i's and their higher order sums and differences reside (we assume that q and k are fixed). This means that the number of $\tau_{q,k}^{(m)}$ [resp. $\mathcal{S}_{q,k}^{(3)}$] we will get will depend on the number of cases we will get for various q_i's as well as various higher sums and differences of the various q_i's belonging to A or B_j's [resp. A, B or C].

At the end of this chapter we will look at an example where we will actually calculate the values of the various coefficients shown above. Here we will prove two propositions (see Proposition 3.1.2 and Proposition 3.1.3 respectively in [Ba]) that will give us upper bounds on the number of expressions for $\tau_{q,k}^{(j)}$ and $\mathcal{S}_{q,k}^{(3)}$, respectively, where $k = 2$.

Proposition 4.3.1. *Let p be an odd prime and let $q = p^m$ where m is an integer greater than 1. Let $q_0 = \frac{q-1}{2}$. Let $k = 2$ and $n = q_0 - 2$. Then the number of expressions that $\tau_{q,2}^{(j)}$ can have is at most m^2.*

Proof. We will find the number of $\tau_{q,2}^{(j)}$ by considering the following cases:

Case 1: $q_1, q_2 \in B_j \quad (j = 1, 2, \ldots, (m-1))$, where $B_j = \left\{x \in \mathbf{Z}^+ : p^j \mid x, p^{j+1} \nmid x\right\}$.

We will consider all of the possibilities one by one, i.e., $q_1, q_2 \in B_1$, then $q_1, q_2 \in B_2$, and so on. When $q_1, q_2 \in B_1$, then we have the following possibilities for

$q_1 + q_2$ and $q_1 - q_2$:

$$
\begin{aligned}
& q_1 + q_2 \in B_1 \text{ and } q_1 - q_2 \in B_1 & \\
\text{or } & q_1 + q_2 \in B_1 \text{ and } q_1 - q_2 \in B_2 & \text{(or vice versa)} \\
\text{or } & q_1 + q_2 \in B_1 \text{ and } q_1 - q_2 \in B_3 & \text{(or vice versa)} \\
& \vdots & \\
\text{or } & q_1 + q_2 \in B_1 \text{ and } q_1 - q_2 \in B_{m-1} & \text{(or vice versa)}
\end{aligned}
$$

This means that there are at most $(m-1)$ different possibilities for $\tau_{q,2}^{(j)}$.

Now when $q_1, q_2 \in B_2$, we will get the following $(m-2)$ different possibilities:

$$
\begin{aligned}
& q_1 + q_2 \in B_2 \text{ and } q_1 - q_2 \in B_2 & \\
\text{or } & q_1 + q_2 \in B_2 \text{ and } q_1 - q_2 \in B_3 & \text{(or vice versa)} \\
\text{or } & q_1 + q_2 \in B_2 \text{ and } q_1 - q_2 \in B_4 & \text{(or vice versa)} \\
& \vdots & \\
\text{or } & q_1 + q_2 \in B_2 \text{ and } q_1 - q_2 \in B_{m-1} & \text{(or vice versa)}
\end{aligned}
$$

Note that the case where $q_1 + q_2 \in B_2$ and $q_1 - q_2 \in B_1$ will not occur since B_1 contains only multiples of p whereas B_2 contains multiples of p^2. So, whereas it is possible that one of $q_1 + q_2$ or $q_1 - q_2$ is xp^r, $r > 2$, neither $q_1 + q_2$ or $q_1 - q_2$ will ever be a multiple like xp, where $x \neq p^t$ for any t. The same reasoning will apply when we consider other B_j's.

So, now we have $(m-2)$ different possibilities for $\tau_{q,2}^{(j)}$. Proceeding in this manner for the remaining B_j's one by one we will get one less equation than the previous time, until we get to the case where $q_1, q_2 \in B_{(m-1)}$. Here there is only one possibility that $q_1 \pm q_2 \in B_{(m-1)}$. So $\tau_{q,2}^{(j)}$ will have one equation at the most.

Now adding all the possibilities for $\tau_{q,2}^{(j)}$ we get

$$(m-1) + (m-2) + \cdots + 3 + 2 + 1 = \frac{m(m-1)}{2} \qquad \text{expressions}$$

at the most.

<u>Case 2:</u> $q_1 \in B_j$ and $q_2 \in B_t$, $B_j \neq B_t$.

Notice that if $j < t$, then $q_1 \pm q_2$ will always belong to B_j. To see this, assume $q_1 \in B_j$ and $q_2 \in B_t$. Then $q_1 = xp^j$ and $q_2 = yp^t$ for some x and y.

$$\begin{aligned} \Rightarrow q_1 \pm q_2 &= xp^j \pm yp^t = xp^j \pm yp^r p^j \quad (\text{where } r + j = t) \\ &= (x \pm yp^r) \cdot p^j \in B_j \text{ since } x \pm yp^r \neq p^s \text{ for any numbers.} \end{aligned}$$

So, we again view all the possibilities starting with $q_1 \in B_1$ and $q_2 \in B_j \quad (j = 2, 3, \ldots, (m-1))$ (or vice versa) $q_1 \pm q_2 \in B_1$ always. This would give us $(m-2)$ possible expressions for $\tau_{q,2}^{(j)}$.

Next we consider the case where $q_1 \in B_2$ and $q_2 \in B_j \quad (j = 3, 4, \ldots, (m-1))$. This will give us $(m-3)$ possibilities for $\tau_{q,2}^{(j)}$. We keep proceeding in this manner until we reach the case where $q_1 \in B_{(m-2)}$ and $q_2 \in B_{(m-1)}$, where we get just one possibility. Now adding all these we get a maximum number of possible expressions for $\tau_{q,2}^{(j)}$ in this case:

$$(m-2) + (m-3) + \cdots + 3 + 2 + 1 = \frac{(m-1)(m-2)}{2}.$$

<u>Case 3:</u> $q_1 \in B_j$ and $q_2 \in A$, or vice versa.

Here we note that $q_1 \pm q_2$ always belongs to A. Therefore, in this case we will get $(m-1)$ possible expressions for $\tau_{q,2}^{(j)}$, one each for the case where $q_1 \in A$ and $q_2 \in B_j \quad (j = 1, 2, \ldots, (m-1))$, or vice versa.

Case 4: $q_1, q_2 \in A$.

We will get 1 possible equation if $q_1 \pm q_2 \in A$. Then we will get 1 possible equation each for the case when $q_1 + q_2 \in A$ and $q_1 - q_2 \in B_j$ (or vice versa) for $j = 1, 2, \ldots, (m-1)$. There are no other possibilities in this case. So the maximum number of possible expressions for $\tau_{q,2}^{(j)}$ in this case will be $m-1+1 = m$.

Case 1 though Case 4 are the only possible cases that occur for $k = 2$. Adding up the numbers of all possible expressions for $\tau_{q,2}^{(j)}$ from each case we get the maximum number of possible expressions that $\tau_{q,2}^{(j)}$ can have:

$$\frac{m(m-1)}{2} + \frac{(m-1)(m-2)}{2} + (m-1) + m$$
$$= \frac{m^2 - m + m^2 - 3m + 2 + 2m - 2 + 2m}{2} = \frac{2m^2}{2} = m^2.$$

□

Proposition 4.3.2. *Let $q = p_1 \cdot p_2$, where p_1, p_2 are distinct odd primes. Let $q_0 = \frac{q-1}{2}$. Let $k = 2$ and $n = q_0 - 2$. Then the number of possible expressions for $\mathcal{S}_{q,2}^{(3)}$ is at most 11.*

Proof. As in the previous proposition, we prove this result by considering all the possible cases for q_1 and q_2 (where $q_1 \pm q_2$ is not congruent to $0 (\mathrm{mod}\ q)$).

Case 1 $q_1, q_2 \in B \quad$ (or $q_1, q_2 \in C$), where $B = \{xp_1 \mid x = 1, \ldots, (p_2 - 1)\}$ and $C = \{xp_2 \mid x = 1, \ldots, (p_1 - 1)\}$.

Then $q_1 \pm q_2 \in B$ (or $q_1 \pm q_2 \in C$, respectively). There are no other possibilities for this case.

Case 2: $q_1 \in B$ and $q_2 \in C$ (or vice versa).

We have just one possible equation in this case, when $q_1 \pm q_2 \in A$.

Case 3: $q_1 \in A, q_2 \in B$ or $q_1 \in A, q_2 \in C$ (or vice versa).

We will get one equation each when $q_1 \pm q_2 \in A$. Then we will get one possible equation for the case when $q_1 \in A, q_2 \in B$, and $q_1 + q_2 \in A, q_1 - q_2 \in C$, (or vice versa).

We will get one more possible equation for the case when $q_1 \in A, q_2 \in C$, and $q_1 + q_2 \in A, q_1 - q_2 \in B$ (or vice versa).

So, in this case we get a possible 4 expressions for $\mathcal{S}_{q,2}^{(3)}$.

Case 4: $q_1, q_2 \in A$.

We will get one possible equation where $q_1 \pm q_2 \in A$. We get another possible equation where $q_1 + q_2 \in A$ and $q_1 - q_2 \in B$ (or vice versa). We get a third possible equation where $q_1 + q_2 \in A$ and $q_1 - q_2 \in C$ (or vice versa). We get a fourth possible equation where $q_1 + q_2 \in B$ and $q_1 - q_2 \in C$ (or vice versa).

So, we get a total of 4 possible expressions for $\mathcal{S}_{q,2}^{(3)}$ in this case.

Case 1 through Case 4 are the only possible cases than can occur for $k = 2$. Adding up the number of all possible expressions for $\mathcal{S}_{q,2}^{(3)}$ from each case we get the maximum number of possible expressions for $\mathcal{S}_{q,2}^{(3)}$:

$$2 + 1 + 4 + 4 = 11.$$

$\square$

It is important to note that in the above propositions the number of possible expressions is the *maximum* number of expressions that can happen. It is possible that for a given $q = p^m$ or $q = p_1 \cdot p_2$ not all the expressions will occur. We will see this in an example later.

We now prove two similar propositions (see Proposition 3.1.4 and Proposition 3.1.5 respectively in [Ba]) for even q of the form 2^m and $2p$, where m is a positive integer and p is a prime.

Proposition 4.3.3. *Let $q = 2^m$ where $m \geq 3$. Let $q_0 = \frac{q}{2}$, i.e., $q_0 = 2^{m-1}$. Let $k = 2$ and $n = q_0 - 2$. Then the number of possible expressions that $\tau_{q,2}^{(j)}$ can have is at most $(m-1)^2$.*

Proof. We proceed as in the previous propositions.

<u>Case 1:</u> $q_1, q_2 \in B_j \quad (j = 1, 2, \ldots, (m-3))$, where $B_j = \left\{x \in \mathbf{Z}^+ : 2^j \mid x, 2^{j+1} \nmid x\right\}$.

We first note that the cases where $q_1, q_2 \in B_{m-2}$ or B_{m-1} will not occur: B_{m-1} has only one element, namely 2^{m-1}; B_{m-2} has just two elements, 2^{m-2} and $3 \cdot 2^{m-2}$, so if we were to take these two elements and add them we would get 2^m, which violates our definition of $I_0(q, 2)$ and $\mathcal{L}_0(q, 2)$.

Now when $q_1, q_2 \in B_j$, then one of the $q_1 + q_2$ or $q_1 - q_2$ will belong to B_{j+1} and the other will belong to B_t for $t > j + 1$.

To see this assume $q_1, q_2 \in B_j$. Since B_j only contains powers of 2^j with odd coefficients, we can assume that $q_1 = (2u-1)2^j$ and $q_2 = (2v-1)2^j$ for some numbers u, v.

Now $q_1 + q_2 = (u+v-1)2^{j+1}$ and $q_1 - q_2 = (u-v)2^{j+1}$. If one of the $(u+v-1)$ or $(u-v)$ is odd then we know that one of the $q_1 + q_2$ or $q_1 - q_2$ will belong to B_{j+1}. Assume both are even, i.e., $u + v - 1 = 2x$ and $u - v = 2y$. Adding the two equations we get $(2u-1) = 2(x+y)$. This is not possible since the number on the left is odd and the number on the right is even. Therefore one of them is odd.

A similar argument shows that both $u+v-1$ and $u-v$ cannot be odd. Therefore one of them is even. Which means that one of the q_1+q_2 or q_1-q_2 belongs to B_t for $t > j+1$ since it gets at least one extra power of 2.

Now, starting with $q_1, q_2 \in B_1$, we get $(m-3)$ possible expressions for $\tau_{q,2}^{(j)}$ where we get one equation each for q_1+q_2 (alt. q_1-q_2) in B_2 and q_1-q_2 (alt. q_1+q_2) in B_t for $t = 3, 4, \ldots, m-1$.

Then considering $q_1, q_2 \in B_2$, we get $(m-4)$ possible expressions for $\tau_{q,2}^{(j)}$ where we get one equation each for $q_1+q_2 \in B_3$ (alt. $q_1-q_2 \in B_3$) and $q_1-q_2 \in B_t$ (alt. $q_1+q_2 \in B_t$). Continuing in this manner until we get to the point where $q_1, q_2 \in B_{m-3}$, where we get just one equation such that $q_1+q_2 \in B_{m-2}$ (alt. $q_1-q_2 \in B_{m-2}$) and $q_1-q_2 \in B_{m-1}$ (alt. $q_1+q_2 \in B_{m-1}$).

So, in this case, the total number of possible expressions for $\tau_{q,2}^{(j)}$ are:

$$(m-3)+(m-4)+\cdots+3+2+1 = \frac{(m-2)(m-3)}{2}.$$

<u>Case 2:</u> $q_1 \in B_j$ and $q_2 \in B_t$, where $B_j \neq B_t$.

We can assume that $j < t$. This would mean that $q_1 \pm q_2 \in B_j$ always. So, as in Case 2 of Proposition 4.3.1, we get that the total number of expressions for $\tau_{q,2}^{(j)}$ will be $\frac{(m-1)(m-2)}{2}$.

<u>Case 3:</u> $q_1 \in B_j$ and $q_2 \in A$ (or vice versa).

We notice that $q_1 \pm q_2 \in A$ always. So, just like in Case 3 of Proposition 4.3.1, we will get that the total number of possible expressions for $\tau_{q,2}^{(j)}$ will be $(m-1)$.

Case 4: $q_1, q_2 \in A$.

In this case one of $q_1 + q_2$ or $q_1 - q_2$ will belong to B_1 and the other will belong to one of the B_j for $j > 1$.

To see this, assume $q_1 = 2u - 1$ and $q_2 = 2v - 1$ for some numbers u and v. Thus, $q_1 + q_2 = 2(u + v - 1)$, which is even, and $q_1 - q_2 = 2(u - v)$, which is even.

As in the argument for Case 1 above, we get that exactly one of the $u + v - 1$ and $u - v$ is odd and the other is even. Since B_1 contains odd multiples of 2 , we will get that one of the $q_1 + q_2$ or $q_1 - q_2$ will be in B_1. Since one of $u + v - 1$ and $u - v$ is even one of the $q_1 + q_2$ or $q_1 - q_2$ will get at least one additional power of 2, which would mean that it belongs to a B_t where $t > 1$.

Therefore, for this case we will get $(m - 2)$ possible expressions for $\tau_{q,2}^{(j)}$, one each for the case when $q_1 + q_2 \in B_1$ (alt. $q_1 - q_2 \in B_1$) and $q_1 - q_2 \in B_t$ (alt. $q_1 + q_2 \in B_t$) for $t = 2, 3, \ldots, m - 1$.

Now, adding up all the possible expressions from the four cases above we get the maximum number of possible expressions for $\tau_{q,2}^{(j)}$:

$$\begin{aligned}
&\frac{(m-2)(m-3)}{2} + \frac{(m-1)(m-2)}{2} + (m-1) + (m-2) \\
&= \frac{m^2 - 5m + 6 + m^2 - 3m + 2 + 2m - 2 + 2m - 4}{2} \\
&= m^2 - 2m + 1 = (m-1)^2.
\end{aligned}$$

$\square$

Our next proposition gives us the maximum number of expressions for $\mathcal{S}_{q,2}^{(3)}$ when $q = 2p$ for some prime p.

Proposition 4.3.4. *Let $q = 2p$ where p is an odd prime. Let $q_0 = \frac{q}{2} = p$. Let $k = 2$ and $n = q_0 - 2$. Then the number of possible expressions for $\mathcal{S}_{q,2}^{(3)}$ is at most* 6.

Proof. As before we will analyze the different possible cases. Note that in this situation we have $B = \{2, 4, 6, \ldots, 2(p-1)\}$ and $C = \{p\}$.

Case 1: $q_1, q_2 \in B$. We will have $q_1 \pm q_2 \in B$ always.

Notice that in this case q_1, q_2 cannot belong to C since C has only one element. So we get 1 possible equation in this case for $\mathcal{S}_{q,2}^{(3)}$.

Case 2: $q_1 \in B, q_2 \in C$. In this case $q_1 \pm q_2 \in A$ always.

So, we get 1 possible equation in this case for $\mathcal{S}_{q,2}^{(3)}$.

Case 3: $q_1 \in A, q_2 \in B$ or $q_1 \in A, q_2 \in C$.

When $q_1 \in A$ and $q_2 \in C$, then $q_1 \pm q_2 \in B$ always. So, we get 1 possible equation for $\mathcal{S}_{q,2}^{(3)}$. When $q_1 \in A, q_2 \in B$, we will get 1 possible equation for the situation when $q_1 \pm q_2 \in A$. We will get another possible equation for $\mathcal{S}_{q,2}^{(3)}$ where $q_1 + q_2 \in A$ (alt. $q_1 - q_2 \in A$) and $q_1 - q_2 \in C$ (alt. $q_1 + q_2 \in C$).

So, there are a total of 3 possible expressions for $\mathcal{S}_{q,2}^{(3)}$ in this case.

Case 4: $q_1, q_2 \in A$. Then $q_1 \pm q_2 \in B$ always.

So, we get 1 possible equation for this case.

Now, adding up all the possible expressions from the above four cases we get the maximum number of possible expressions for $\mathcal{S}_{q,2}^{(3)}$ to be $1 + 1 + 3 + 1 = 6$. $\square$

With these four propositions, we are now ready for our first main theorem (see Theorem 3.1.6 in [Ba]).

Theorem 4.3.5. *(i) Let $p \geq 5$ (alt. $p \geq 3$) be an odd prime and let $m \geq 2$ (alt. $m \geq 3$) be any positive integer. Let $q = p^m$. Then there exist at least two $(q-6)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order p^m - which are isospectral but not isometric.*

(ii) Let p_1, p_2 be odd primes such that $q = p_1 \cdot p_2 \geq 33$. Then there exists at least two $(q-6)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order $p_1 \cdot p_2$ - which are isospectral but not isometric.

(iii) Let $q = 2^m$ where $m \geq 6$ be any positive integer. Then there exist at least two $(q-5)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order 2^m - which are isospectral but not isometric.

(iv) Let $q = 2p$, where $p \geq 7$ is an odd prime. Then there exist at least two $(q-5)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order $2p$ - which are isospectral but not isometric.

Proof. We first recall from Proposition 4.2.6 that

$$|\mathcal{L}_0(q,n)| \geq \sum_{t=r-2}^{r} \frac{1}{n-t} \binom{q_0 - 1 - r}{n-1-t} \binom{r}{t}$$

for $k = 2$ and $r > 2$. This means that for $k = 2$ and $r > 2$ we have,

$$
\begin{aligned}
|\mathcal{L}_0(q,n)| &\geq \frac{1}{n-(r-2)}\binom{q_0-r-1}{n-1-(r-2)}\binom{r}{r-2} \\
&\quad + \frac{1}{n-(r-1)}\binom{q_0-r-1}{(n-1)-(r-1)}\binom{r}{r-1} + \frac{1}{n-r}\binom{q_0-r-1}{n-r-1}\binom{r}{r} \\
&= \frac{1}{q_0-2-r+2}\binom{q_0-r-1}{q_0-2-1-r+2}\binom{r}{r-2} \\
&\quad + \frac{1}{q_0-2-r+1}\binom{q_0-r-1}{q_0-2-1-r+1}\binom{r}{r-1} \\
&\quad + \frac{1}{q_0-2-r}\binom{q_0-r-1}{q_0-2-r-1} \qquad\qquad \text{since } n = q_0-2 \\
&= \frac{1}{q_0-r}\binom{q_0-r-1}{q_0-r-1}\binom{r}{r-2} + \frac{1}{q_0-r-1}\binom{q_0-r-1}{q_0-r-2}\binom{r}{r-1} \\
&\quad + \frac{1}{q_0-r-2}\binom{q_0-r-1}{q_0-r-3}\binom{r}{r} \\
&= \frac{1}{q_0-r}\cdot 1 \cdot \frac{r(r-1)}{2} + \frac{1}{(q_0-r-1)}\cdot (q_0-r-1)\cdot r \\
&\quad + \frac{1}{(q_0-r-2)}\cdot\frac{(q_0-r-1)(q_0-r-2)}{2}\cdot 1 \\
&= \frac{r(r-1)}{2(q_0-r)} + r + \frac{(q_0-r-1)}{2}.
\end{aligned} \tag{4.9}
$$

Since

$$
|\mathcal{L}_0(q,n)| \geq \frac{r(r-1)}{2(q_0-r)} + r + \frac{(q_0-r-1)}{2}
$$

for $k = 2$ and $r > 2$, it is sufficient for us to show that (4.9) is greater than the number of possible expressions in each case to establish the existence of isospectral pairs for non-isometric lens spaces.

(i) For $q = p^m$, we have a total of m^2 possible expressions for $\tau_{q,2}^{(j)}$ from Proposition 4.3.1. So, we will have isospectrality when (4.9) is greater than or equal to

m^2+1. That is

$$
\begin{aligned}
& \frac{r(r-1)}{2(q_0-r)} + r + \frac{(q_0-r-1)}{2} \geq m^2+1 \\
\Rightarrow \quad & r(r-1) + 2r(q_0-r) + (q_0-r)(q_0-r-1) \geq 2(q_0-r)(m^2+1) \\
\Rightarrow \quad & r^2 - r + (q_0-r)[2r + q_0 - r - 1 - 2m^2 - 2] \geq 0 \\
\Rightarrow \quad & (r^2-r) + (q_0-r)[q_0 + r - 2m^2 - 3] \geq 0 \\
\Rightarrow \quad & r^2 - r + q_0^2 + q_0 r - q_0 2m^2 - 3q_0 - q_0 r - r^2 + 2m^2 r + 3r \geq 0 \\
\Rightarrow \quad & q_0(q_0 - 2m^2 - 3) + 2r(m^2+1) \geq 0 \\
\Rightarrow \quad & -q_0[(2m^2+3) - q_0] \geq -2r(m^2+1) \\
\Rightarrow \quad & q_0[(2m^2+3) - q_0] \leq 2r(m^2+1). \qquad (4.10)
\end{aligned}
$$

So for any given m, we can choose p big enough so that $2m^2+3 \leq q_0$. This would guarantee isospectrality. We can calculate r by $r = (\frac{p^{m-1}-1}{2})$ in this case. Now if $p \geq 5$, $q_0 \geq \frac{5^m-1}{2} > 2m^2+3$ for all $m \geq 2$. This is easy to see since $5^m > 4m^2+7$ for $m \geq 2$ as the left hand side grows exponentially greater than the right hand side. So, for all $p \geq 5$ and all $m \geq 2$, (4.10) will be true and we will get isospectral pairs of dimension $(q-6) = 2n-1$. Now for $q = 3^m$, we have $3^m > 4m^2+7$ for $m \geq 4$. So we will have isospectrality. We check cases $m = 2$ and $m = 3$.

When $m = 2$, $q = 9$, $r = 1$, $q_0 = 4$. So L.H.S. of (4.10) gives $4[2(4)+3-4] = 4(7) = 28$ and R.H.S. of (4.10) gives $2(1)(4+1) = 10$. So the sufficiency condition is not satisfied.

When $m = 3$, $q = 27$, $r = 4$, $q_0 = 13$. L.H.S. of (4.10) gives $13[2(9)+3-13] = 13[8] = 104$ and R.H.S. of (4.10) gives $2(4)[9+1] = 8(10) = 80$. So the sufficiency condition is not satisfied.

However, when we check individually all the possible expressions for these cases we realize that they are less than m^2.

For $q = 3^2$, the only two expressions are for the cases when $q_1 \in A$, $q_2 \in B_1$, $q_1 \pm q_2 \in A$, and $q_1, q_2 \in A$, $q_1 + q_2 \in A$, $q_1 - q_2 \in B_1$. No other possible expressions exist.

However, there are only two classes in $\mathcal{L}_0(q,n)$, i.e.,$|\mathcal{L}_0(q,n)| = 2$. The two classes are

$$[1,2] = \left\{ (p_1, p_2) \in \widetilde{\mathcal{L}}_0(q,2) \,\middle|\, p_1, p_2 \in A \right\},$$
$$[1,3] = \left\{ (p_1, p_2) \in \widetilde{\mathcal{L}}_0(q,2) \,\middle|\, p_1 \in A, p_2 \in B_1 \text{ (alt. } p_1 \in B_1, p_2 \in A) \right\},$$

where $n = 2$, $A = \{1, 2, 4, 5, 7, 8\}$ and $B_1 = \{3, 6\}$. Therefore, we do not obtain isospectral pairs.

For $q = 3^3$, there are 7 expressions (instead of $3^2 = 9$ possible expressions). The case where $q_1, q_2, q_1 \pm q_2 \in B_1$ and the case where $q_1, q_2 \in B_2$ do not occur. This gives us 2 less expressions than the estimated number of 9. But the number of classes is greater than or equal to

$$\frac{4(4-1)}{2(13-4)} + 4 + \frac{13-4-1}{2} = \frac{2}{3} + 4 + 4 = 8\frac{2}{3} > 7 \qquad \text{(from 4.9).}$$

This means we will have non-isometric isospectral lens spaces. This gives us our result that for $p \geq 3$ and $m \geq 3$, we will get isospectral pairs that are non-isometric.

(ii) For $q = p_1 \cdot p_2$, $r = \frac{p_1+p_2-2}{2}$.

From (4.9) and Proposition 4.3.2 we get the following sufficiency condition:

$$\begin{aligned}
&\frac{r(r-1)}{2(q_0-r)}+r+\frac{(q_0-r-1)}{2}\geq 12 \\
&\Rightarrow q_0(25-q_0)\leq 24r
\end{aligned} \tag{4.11}$$

From this we get that for $q_0 \geq 25$, we will always find non-isometric, isospectral lens spaces because (4.11) will always be satisfied. We now check for cases where $q = 2q_0 + 1 < 51$.

For $q < 51$, and $q = p_1 \cdot p_2$ with p_1, p_2 being odd primes, there are only the following possibilities:

(a) $q = 3 \cdot 7 = 21$; $B = \{3, 6, 9, 12, 15, 18\}$, $C = \{7, 14\}$.

In this case we have 9 instead of 11 possible expressions. The case where $q_1, q_2 \in C = \{7, 14\}$ is not possible, and the case where $q_1, q_2 \in A$ and $q_1 \pm q_2 \in A$ is also not possible since then $q_2 \equiv -q_1 (\bmod\ q)$. Therefore, we get 2 less expressions. Now for isospectrality we use (4.9):

$$\frac{4(4-1)}{2(10-4)}+4+\frac{(10-4-1)}{2}=7\frac{1}{2},$$

which is not greater than 9. So the isospectrality condition is not met.

(b) $q = 3 \cdot 5 = 15$. In this case we have 7 instead of 11 expressions. Here $B = \{3, 6, 9, 12\}$ and $C = \{5, 10\}$. In this case, the following cases do not occur: $q_1, q_2 \in C$; $q_1 \in A, q_2 \in C$, $q_1 \pm q_2 \in A$; $q_1, q_2, q_1 \pm q_2 \in A$; $q_1, q_2 \in A$, $q_1 + q_2 \in A$, $q_1 - q_2 \in C$. So we get 4 less expressions than 11. To check for isospectrality we use (4.9) and get $\frac{3(3-1)}{2(7-3)} + 3 + \frac{(7-3-1)}{2} = 5\frac{1}{4}$, which is less than 7. So the isospectrality condition is not satisfied.

For (a) and (b) it can be easily shown that $|\mathcal{L}_0(q,n)|$ is equal to 9 and 7 respectively. This means that there are no isospectral pairs in these cases.

(c) $q = 3 \cdot 11 = 33$. $B = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$ and $C = \{11, 22\}$. Here $q_0 = 16$ and $r = 6$. We check for isospectrality using (4.11):

$$q_0(25 - q_0) = 16(25 - 16) = 144$$

$$24r = 24(6) = 144$$

So (4.11) is satisfied.

(d) $q = 5 \cdot 7 = 35$, $B = \{5, 10, 15, 20, 25, 30\}$ and $C = \{7, 14, 21, 28\}$. Here $q_0 = 17$ and $r = 5$. Using (4.11) we get

$$q_0(25 - q_0) = 17(25 - 17) = 138$$

$$24r = 24(5) = 120$$

So (4.11) is not satisfied. However, we notice that in this case the actual number of expressions is 10 instead of 11. So, we use (4.9) to check for isospectrality. Plugging in $r = 5$ and $q_0 = 17$ into (4.9) we get

$$\frac{5(4)}{2(12)} + 5 + \frac{11}{2} = 11\frac{1}{3} > 10$$

This implies that $\mathcal{S}^{(3)}_{q,2}$ is not one-one and therefore, we will have non-isometric isospectral lens spaces in this case.

(e) Finally, we check $q = 3 \cdot 13 = 39$.

Here $q_0 = 19$ and $r = 7$. Using (4.11) we see

$$q_0(25 - q_0) = 19(25 - 19) = 114$$

$$24r = 24(7) = 168$$

So (4.11) is satisfied and we will have isospectral pairs in this case.

(a)-(e) are all the cases of $q = p_1 \cdot p_2 < 51$, where p_1, p_2 are odd primes. Combining these results with the fact that for $q \geq 51$, (4.11) will always be satisfied, we prove (iii).

(iii) Let $q = 2^m$. We use Proposition 4.3.3 and (4.9) to get a sufficiency condition for isospectrality:

$$\frac{r(r-1)}{2(q_0 - r)} + r + \frac{(q_0 - r - 1)}{2} \geq (m-1)^2 + 1$$

Here $q_0 = \frac{2^m}{2} = 2^{m-1}$ and $2r = 2^{m-1}$. Therefore, $q_0 = 2r$ in this case. Simplifying the above inequality, we get

$$q_0[(2m^2 - 4m + 5) - q_0] \leq 2r(m^2 - 2m + 2).$$

But since $q_0 = 2r$, we get

$$(m^2 - 2m + 3) \leq q_0$$

If $m \geq 6$, then $m^2 - 2m + 3 < 2^{m-1}$. Further, the right hand side grows exponentially bigger than the left hand side as m grows. For $m = 3$, 4 and 5, the actual number of expressions for $\tau_{q,2}^{(j)}$ are 4, 9 and 16 respectively. Further, it can be easily shown that for $m = 3$, 4 and 5, $|\mathcal{L}_0(q,n)|$ is 4, 9 and 16 respectively. Therefore, for $m = 3$, 4 and 5 we do not get isospectrality. This gives us (iii).

(iv) Using Proposition 4.3.4 and (4.9) we get the sufficiency condition for isospectrality for $q = 2p$, where p is an odd prime ≥ 7. Note that in this case $q_0 = \frac{q}{2} = p$ and $r = \frac{q+2}{4}$. Now for isospectrality we should have

$$
\begin{aligned}
&\frac{r(r-1)}{2(q_0 - r)} + r + \frac{(q_0 - r - 1)}{2} \geq 7 \\
\Rightarrow\ & q_0(15 - q_0) \leq 14r \\
\Rightarrow\ & p(15 - p) \leq 7(p+1) \\
\Rightarrow\ & 0 \leq p^2 - 8p + 7
\end{aligned}
$$

$$
\text{or } (p-1)(p-7) \geq 0 \tag{4.12}
$$

Since (4.12) is positive whenever $p \geq 7$, we will have isospectrality. When $q = 2 \cdot 5 = 10$, then $|\mathcal{L}_0(q,n)| = 6 =$ number of expressions for $\mathcal{S}^{(3)}_{q,2}$. So, we do not get isospectral pairs when $p = 5$. This proves (iv).

□

4.4 Even Dimensional Lens Spaces

Recall that in the manifold case, the only even dimensional spherical space forms are the sphere S^{2n} and the real projective space $P^{2n}(\mathbf{R})$, and these two are not isospectral (see [I2]).

Recall that lens spaces are spherical space forms where the acting group, G, is a finite cyclic group. Since we allow G to have fixed points, we are not limited to the sphere and the real projective space. We will see that given two odd-dimensional isospectral non-isometric orbifold lens spaces as in Section 4.3, we can modify our construction slightly to get isospectral non-isometric pairs of even dimensional orbifold lens spaces.

Let $L = L(q : p_1, \ldots, p_n) = S^{2n-1}/G$ and $L' = [L(q : s_1, \ldots, s_n) = S^{2n-1}/G'$ be two isospectral non-isometric orbifold lens spaces as obtained in the previous section, where $G = \langle g \rangle$, $G' = \langle g' \rangle$

$$g = \begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_n/q) \end{pmatrix}$$

and

$$g' = \begin{pmatrix} R(s_1/q) & & 0 \\ & \ddots & \\ 0 & & R(s_n/q) \end{pmatrix}$$

We define

$$\tilde{g}_{1+} = \begin{pmatrix} R(p_1/q) & & & 0 \\ & \ddots & & \\ & & R(p_n/q) & \\ 0 & & & 1 \end{pmatrix}$$

and

$$\tilde{g}'_{1+} = \begin{pmatrix} R(s_1/q) & & & 0 \\ & \ddots & & \\ & & R(s_n/q) & \\ 0 & & & 1 \end{pmatrix}$$

where $\tilde{g}_{1+}$ and $\tilde{g}'_{1+}$ are extensions of g and g' to orthogonal transformations of $O(2n+1)$ by adding a 1 in the $(2n+1, 2n+1)$ entries of g and g' respectively. Let $\tilde{G}_{1+} = \langle \tilde{g}_{1+} \rangle$ and $\tilde{G}'_{1+} = \langle \tilde{g}'_{1+} \rangle$. Then $\tilde{G}_{1+}$ and $\tilde{G}'_{1+}$ are cyclic groups of order q. We define lens spaces $\tilde{L}_{1+} = S^{2n}/\tilde{G}_{1+}$ and $\tilde{L}'_{1+} = S^{2n}/\tilde{G}'_{1+}$. We denote $\tilde{L}_{1+} = L(q : p_1, \ldots, p_n, 0)$ and $\tilde{L}'_{1+} = L(q : s_1, \ldots, s_n, 0)$. We now prove a proposition similar to Proposition 4.1.2.

Proposition 4.4.1. *Let $\tilde{L}_{1+}$ and $\tilde{L}'_{1+}$ be as defined above. Then $\tilde{L}_{1+}$ is isometric to $\tilde{L}'_{1+}$ iff there is a number l coprime with q and there are numbers $e_i \in \{-1, 1\}$ such that $(p_1, \ldots, p_n)$ is a permutation of $(els_1, \ldots, els_n) \mod q$.*

Proof. Let $\tilde{L}_{1+}$ and $\tilde{L}'_{1+}$ be isometric. Now any isometry of $\tilde{L}_{1+}$ and $\tilde{L}'_{1+}$ lifts to an orthogonal transformation that conjugates $\tilde{G}_{1+}$ and $\tilde{G}'_{1+}$.
$(\Rightarrow)$ If $\tilde{g}_{1+}$ is a generator of $\tilde{G}_{1+}$, then the orthogonal transformation will take $\tilde{g}_{1+}$ to a generator $\tilde{g}'^{l}_{1+}$ of $\tilde{G}'_{1+}$. The eigenvalues of $\tilde{g}_{1+}$ and $\tilde{g}'_{1+}$ are the same and the eigenvalue 1 is mapped to the eigenvalue 1 of $\tilde{g}'^{l}_{1+}$. For the remaining eigenvalues, each p_i is equivalent to some ls_j or ls_j modulo q. Thus, $(p_1, \ldots, p_n)$ is a permutation of $(le_1 s_1, \ldots, le_n s_n) \pmod q$, where $e_i \in \{-1, 1\}$ for $i = 1, \ldots, n$.

Conversely, let $(p_1, \ldots, p_n)$ be a permutation of $(le_1 s_1, \ldots, le_n s_n) \pmod q$. By Proposition 4.1.2, we know that L and L' are isometric. Let ϕ be an isometry between L and L'. This isometry lifts to $\tilde{\phi}$, an isometry of S^{2n-1} onto itself. $\tilde{\phi}$ is an orthogonal transformation that conjugates G and G'. This orthogonal transformation can be extended to an orthogonal transformation of $O(2n)$ by adding a 1 in the $(2n+1, 2n+1)$ entry. This is an isometry of S^{2n} which induces an isometry of $\tilde{L}_{1+}$ onto $\tilde{L}'_{1+}$ □

The following lemma follows directly from this proposition.

Lemma 4.4.2. *Let L, L', $\tilde{L}_{1+}$ and $\tilde{L}'_{1+}$ be as defined above. Then L is isometric to L' iff $\tilde{L}_{1+}$ is isometric to $\tilde{L}'_{1+}$.*

Now let $\widetilde{\mathcal{L}}_0^{1+}(q,n,0)$ be the family of all $2n$-dimensional orbifold lens spaces that are obtained in the manner described above. Let $\mathcal{L}_0^{1+}(q,n,0)$ denote the set of isometry classes of $\widetilde{\mathcal{L}}_0^{1+}(q,n,0)$.

Then from 4.7 and 4.8 we get the following diagrams:

For $q = p^m$

$$\mathcal{L}_0^{1+}(q,n,0) \xrightarrow{\sim} \mathcal{L}_0(q,n) \xrightarrow{\sim} I_0(q,n) \underset{w}{\xrightarrow{\sim}} I_0(q,k) \xrightarrow[\tau_{q,k}^{(m)}]{} Q^m(\gamma)[z] \tag{4.13}$$

and for $q = p_1 \cdot p_2$

$$\mathcal{L}_0^{1+}(q,n,0) \xrightarrow{\sim} \mathcal{L}_0(q,n) \xrightarrow{\sim} I_0(q,n) \underset{w}{\xrightarrow{\sim}} I_0(q,k) \xrightarrow[\mathcal{S}_{q,k}^{(3)}]{} Q^3(\gamma)[z] \tag{4.14}$$

where $\tau_{q,k}^{(m)}$ and $\mathcal{S}_{q,k}^{(3)}$ are as defined for 4.7 and 4.8 respectively. Now we can view $\mathbf{R}^{2n+1}$ as a subspace of $\mathbf{C}^{n+1}$, where

$$\mathbf{R}^{2n+1} = \Big\{(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, 0) \in \mathbf{C}^{n+1}, z_i = x_i + \sqrt{-1}y_i \\ \text{for } i = 1, \ldots, n \text{ and } z_{n+1} = x_{n+1}\Big\}$$

Theorem 4.4.3. *Let*

$$\tilde{L}_{1+} = L(q : p_1, \ldots, p_n, 0) \in \mathcal{L}_0^{1+}(q,n,0)$$

and let $F_q(z : p_1, \ldots, p_n, 0)$ be the generating function associated with the spectrum of $\tilde{L}_{1+}$. Then on the domain $\{z \in \mathbf{C} \,|\, |z| < 1\}$,

$$F_q(z : p_1, \ldots, p_n, 0) = \frac{1}{q}\sum_{l=1}^{q} \frac{(1+z)}{\prod_{i=1}^{n}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})}$$

Proof. Recall the definitions of Δ_0, r^2, P^k, H^k, $\mathcal{H}^k$ and $\mathcal{H}^k_G$ from Section 4.1. We extend the definitions for $\mathbf{R}^{2n+1}$. That is, let Δ_0 be the Laplacian on $\mathbf{R}^{2n+1}$ with respect to the Flat Riemannian metric; $r^2 = \sum_{i=1}^{2n+1} x_i^2$, where $(x_1, \ldots, x_{2n+1})$ is the standard coordinate system on $\mathbf{R}^{2n+1}$; for $k \geq 0$, P^k is the space of complex valued homogenous polynomials of degree k in $\mathbf{R}^{2n+1}$; H^k is the subspace of P^k consisting of harmonic polynomials on $\mathbf{R}^{2n+1}$; $\mathcal{H}^k$ is the image of $i^* : C^\infty(\mathbf{R}^{2n+1}) \longrightarrow C^\infty(S^{2n})$ where $i : S^{2n} \longrightarrow \mathbf{R}^{2n+1}$ is the natural injection; and $\mathcal{H}^k_{\widetilde{G}}$ is the space of all $\widetilde{G}$-invariant functions of $\mathcal{H}^k$.

Then from Proposition 4.1.6 and Proposition 4.1.7, we get that H^k is $O(2n+1)$-invariant; P^k has the direct sum decomposition $P^k = H^k \oplus r^2 P^{k-2}$; $\mathcal{H}^k$ is an eigenspace of $\widetilde{\Delta}$ on S^{2n} with eigenvalue $k(k+2n-1)$; $\sum_{k=0}^{\infty} \mathcal{H}^k$ is dense in $C^\infty(S^{2n})$ in the uniform convergence topology and $\mathcal{H}^k$ is isomorphic to H^k.

Using this,along with the results implied by Corollary 4.1.5 and Proposition 4.1.7, where $\dim E_{k(k+2n-1)} = \dim \mathcal{H}^k_{\tilde{G}_{1+}}$, we get

$$F_q(z : p_1, \ldots, p_n, 0) = \sum_{k=0}^{\infty} (\dim \mathcal{H}^k_{\tilde{G}_{1+}}) z^k.$$

Now $\tilde{G}_{1+}$ is contained in $SO(2n+1)$.

Let χ_k and $\tilde{\chi}_k$ be the characters of the natural representations of $SO(2n+1)$ on H^k and P^k respectively. Then

$$\dim \mathcal{H}^k_{\widetilde{G}_{1+}} = \frac{1}{\left|\widetilde{G}_{1+}\right|} \sum_{\tilde{g}_{1+} \in \widetilde{G}} \chi_k(\tilde{g}_{1+}) = \frac{1}{q} \sum_{l=1}^{q} \chi_k(\tilde{g}^l_{1+}) \tag{4.15}$$

Proposition 4.1.6 gives

$$\chi_k(\tilde{g}^l_{1+}) = \tilde{\chi}_k(\tilde{g}^l_{1+}) - \tilde{\chi}_{k-2}(\tilde{g}^l_{1+}) \tag{4.16}$$

where $\tilde{\chi}_t = 0$ for $t > 0$.

We can view the space P^k as having a basis consisting of all monomials of the form

$$z^I \cdot \bar{z}^J \cdot z_{n+1}^t = (z_1)^{i_1} \cdots (z_n)^{i_n} \cdot (\bar{z}_1^{j_1}) \cdots (\bar{z}_n^{j_n}) \cdot (z_{n+1})^t \tag{4.17}$$

where $z_i = x_i + \sqrt{-1}y_i$ for $i = 1, \dots, n$ and $z_{n+1} = x_{n+1}$ with $(x_1, y_1, \dots, x_n, y_n, x_{n+1})$ being the standard euclidean coordinates on $\mathbf{R}^{2n+1}$; and $i_1, \dots, i_n, j_1, \dots, j_n, t \geq 0$ such that $i_1 + \cdots + i_n + j_1 + \cdots + j_n + t = k$ (denoted by $I_n + J_n + t = k$). Now $\tilde{g}_{1+}^l$ has eigenvalues $\gamma^{p_1 l}, \bar{\gamma}^{p_1 l}, \dots, \gamma^{p_n l}, \bar{\gamma}^{p_n l}, 1$, where $\gamma = e^{2\pi i/q}$ is a primitive q-th root of unity. So, for any monomial $z^I \cdot \bar{z}^J \cdot z_{n+1}^t$

$$\begin{aligned}
\tilde{g}_{1+}^l(z^I \cdot \bar{z}^J \cdot z_{n+1}^t) &= \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l}(z^I \cdot \bar{z}^J \cdot z_{n+1}^t) \\
\Rightarrow F_q(z : p_1, \dots, p_n, 0) &= \sum_{k=0}^{\infty} (\dim \mathcal{H}_{\tilde{G}_{1+}}^k) z^k \\
&= \frac{1}{q} \sum_{k=0}^{\infty} \sum_{l=1}^{q} \chi_k(\tilde{g}_{1+}^l) z^k && \text{by (4.15)} \\
&= \frac{1}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} (\tilde{\chi}_k(\tilde{g}_{1+}^l) - \tilde{\chi}_{k-2}(\tilde{g}_{1+}^l)) z^k && \text{by (4.16)} \\
&= \frac{(1 - z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \tilde{\chi}_k(\tilde{g}_{1+}^l) z^k \\
&= \frac{(1 - z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \sum_{I_n + J_n + t = k} \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l} z^k \\
= \frac{(1 - z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} &\sum_{I_n + J_n + t = k} (\gamma^{p_1 l} z)^{i_1} \cdots (\gamma^{p_n l} z)^{i_n} (\gamma^{-p_1 l} z)^{j_1} \cdots (\gamma^{-p_n l} z)^{j_n} (z)^t \\
= \frac{(1 - z^2)}{q} \sum_{l=1}^{q} \prod_{i=1}^{n} &\Big(\sum_{k=0}^{\infty} (\gamma^{p_i l} z)^k\Big)\Big(\sum_{k=0}^{\infty} (\gamma^{-p_i l} z)^k\Big)\Big(\sum_{k=0}^{\infty} z^k\Big).
\end{aligned}$$

On the domain $\{z \in \mathbf{C} \mid |z| < 1\}$, the power series

$$\prod_{i=1}^{n}(1+\gamma^{p_i l}z+\gamma^{2p_i l}z^2+\cdots)(1+\gamma^{-p_i l}z+\gamma^{-2p_i l}z^2+\cdots)(1+z+z^2+\cdots)$$

converges to

$$\frac{1}{\prod_{i=1}^{n}(1-\gamma^{p_i l}z)(1-\gamma^{-p_i l}z)(1-z)}$$

So,

$$F_q(z:p_1,\ldots,p_n,0)=\frac{(1+z)}{q}\sum_{l=1}^{q}\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})}$$

□

Corollary 4.4.4. *When $L(q:p_1,\ldots,p_n)$ and $L(q:s_1,\ldots,s_n)$ have the same generating function, then $L(q:p_1,\ldots,p_n,0)$ and $L(q:s_1,\ldots,s_n,0)$ also have the same generating function*

Proof. This follows from the fact that

$$F_q(z:p_1,\ldots,p_n,0)=\frac{1}{(1-z)}\ F_q(z:p_1,\ldots,p_n)$$

. □

The above corollary shows that just like the generating function $F_q(z:p_1,\ldots,p_n)$, the new generating function $F_q(z:p_1,\ldots,p_n,0)$ is dependent on $\tau_{q,k}^{(j)}$ [resp. $\mathcal{S}_{q,k}^{(3)}$] for $q=P^m$ [resp. $q=P_1\cdot P_2$]. Therefore, for every pair of isospectral, non-isometric odd-dimensional orbifold lens spaces that we obtained in the previous section, we will have a corresponding pair of isospectral, non-isometric even-dimensional orbifold lens spaces. Thus we have the following theorem.

Theorem 4.4.5. *(i) Let $P \geq 5$ (alt. $P \geq 3$) be an odd prime and let $m \geq 2$ (alt. $m \geq$*

3) *be and positive integer. Let* $q = P^m$. *Then there exist at least two* $(q-5)$*-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order* P^m *- which are isospectral but not isometric.*

(ii) *Let* P_1, P_2 *be odd primes such that* $q = P_1 \cdot P_2 \geq 33$. *Then there exist at least two* $(q-5)$*-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order* $P_1 \cdot P_2$ *- which are isospectral but not isometric.*

(iii) *Let* $q = 2^m$ *where* $m \geq 6$ *be any positive integer. Then there exist at least two* $(q-4)$*-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order* 2^m *- which are isospectral but no isometric.*

(iv) *Let* $q = 2P$, *where* $P \geq 7$ *is an odd prime. Then there exist at least two* $(q-4)$*-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order* $2P$ *- which are isospectral but not isometric.*

4.5 Lens Spaces for General Integers

The techniques used in Section 4.4 can be further generalized to generate infinitely many families of pairs of isospectral non-isometric orbifold lens spaces of any dimension greater than 8.

$$\text{Let} \qquad L = L(q : p_1, \ldots, p_n) = S^{2n-1}/G \qquad \text{and}$$
$$L' = L(q : s_1, \ldots, s_n) = S^{2n-1}/G'$$

be two isospectral non-isometric orbifold lens spaces as obtained in Section 4.3 where $G = \langle g \rangle$, $G' = \langle g' \rangle$.

$$g = \begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_n/q) \end{pmatrix}$$

and

$$g' = \begin{pmatrix} R(s_1/q) & & 0 \\ & \ddots & \\ 0 & & R(s_n/q) \end{pmatrix}$$

We define

$$\tilde{g}_{W+} = \begin{pmatrix} R(p_1/q) & & & 0 \\ & \ddots & & \\ & & R(p_n/q) & \\ 0 & & & I_W \end{pmatrix}$$

and

$$\tilde{g}'_{W+} = \begin{pmatrix} R(s_1/q) & & & 0 \\ & \ddots & & \\ & & R(s_n/q) & \\ 0 & & & I_W \end{pmatrix}$$

where I_W is the $W \times W$ identity matrix for some integer W. We can define $\tilde{G}_{W+} = \langle \tilde{g}_{W+} \rangle$ and $\tilde{G}'_{W+} = \langle \tilde{g}'_{W+} \rangle$. Then $\tilde{G}_{W+}$ and $\tilde{G}'_{W+}$ are cyclic groups of order q. We define lens spaces $\tilde{L}_{W+} = S^{2n+W-1}/\tilde{G}_{W+}$ and $\tilde{L}'_{W+} = S^{2n+W-1}/\tilde{G}'_{W+}$. Then, like Proposition 4.4.1 and Lemma 4.4.2, we get:

Proposition 4.5.1. *Let $\tilde{L}_{W+}$ and $\tilde{L}'_{W+}$ be as defined above. Then $\tilde{L}_{W+}$ is isometric to $\tilde{L}'_{W+}$ iff there is a number l coprime with q and there are numbers $e_i \in \{-1, 1\}$*

such that $(p_1, \ldots, p_n)$ *is a permutation of* $(e_1 l s_1, \ldots, e_n l s_n) \pmod q$.

Lemma 4.5.2. *Let* L, L', $\tilde{L}_{W+}$ *and* $\tilde{L}'_{W+}$ *be as defined above. Then* L *is isometric to* L' *iff* $\tilde{L}_{W+}$ *is isometric to* $\tilde{L}'_{W+}$.

Similar to Theorem 4.4.3, we get the following theorem (see Theorem 3.2.3 in [Ba]):

Theorem 4.5.3. *Let* $\widetilde{\mathcal{L}}_0^{W+}(q, n, 0)$ *be the family of all* $(2n+W-1)$*-dimensional orbifold lens spaces with fundamental groups of order* q *that are obtained in the manner described above. Let* $\tilde{L}_{W+} \in \mathcal{L}_0^{W+}(q, n, 0)$ *(where* $\mathcal{L}_0^{W+}(q, n, 0)$ *denotes the set of isometry classes of* $\widetilde{\mathcal{L}}_0^{W+}(q, n, 0)$*). Let* $F_q^{W+}(z : p_1, \ldots, p_n, 0)$ *be the generating function associated to the spectrum of* $\tilde{L}_{W+}$. *Then on the domain* $\{z \in \mathbf{C} \mid |z| < 1\}$,

$$F_q^{W+}(z : p_1, \ldots, p_n, 0) = \frac{(1+z)}{(1-z)^{W-1}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n} (z - \gamma^{p_i l})(z - \gamma^{-p_i l})}$$

Proof. The proof of this result is similar to the proof of Theorem 4.4.3. The definitions for Δ_0, r^2, P^k, H^k, $\mathcal{H}^k$ and $\mathcal{H}_G^k$ are analogous for $\mathbf{R}^{2n+W}$.

If W is even, then expression (4.17) for our present case becomes (for monomials forming a basis for P^k):

$$z^I \cdot \bar{z}^J = (z_1)^{i_1} \cdots (z_{n+v})^{i_{n+v}} \cdot (\bar{z}_1)^{j_1} \cdots (\bar{z}_{n+v})^{j_{n+v}},$$

where $W = 2v$ and where $I_{n+v} + J_{n+v} = i_1 + \cdots + i_{n+v} + j_1 + \cdots + j_{n+v} = k$ and $i_1, \ldots, i_{n+v}, j_1, \ldots, j_{n+v} \geq 0$. Then,

$$\tilde{g}_{W+}^l (z^I \cdot \bar{z}^J) = \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l} (z^I \cdot \bar{z}^J).$$

If W is odd, (say $W = 2u+1$), then we get for basis of P^k

$$z^I \cdot \bar{z}^J \cdot z_{n+2u+1}^t = (z_1)^{i_1} \cdots (z_{n+u})^{i_{n+u}} \cdot (\bar{z}_1)^{j_1} \cdots (\bar{z}_{n+u})^{j_{n+u}} \cdot (z_{n+2u+1})^t,$$

where $z_{n+2u+1} = x_{n+W}$ where $(x_1, y_1, \ldots, x_{n+W-1}, y_{n+W-1}, x_{n+W})$ is the standard euclidean coordinate system on $\mathbf{R}^{2n+W}$ with $z_i = x_i + iy_i$ for $i = 1, 2, \ldots, n+W-1$, and $i_1, \ldots, i_{n+u}, j_1, \ldots, j_{n+u}$, $t \geq 0$ and $i_1 + \cdots + i_{n+u} + j_1 + \cdots + j_{n+u} + t = k = I_{n+u} + J_{n+u} + t$. So, in that case

$$\tilde{g}_{W+}^l(z^I \cdot \bar{z}^J \cdot z_{n+2u+1}) = \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l}(z^I \cdot \bar{z}^J \cdot z_{n+2u+1}).$$

So, for W even case, we will get

$$\begin{aligned} F_q^{W+}(z : p_1, \ldots, p_n, 0) &= \frac{1}{q} \sum_{k=0}^{\infty} \sum_{l=1}^{q} \chi_k(\tilde{g}_{W+}^l) z^k \\ &= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \tilde{\chi}_k(\tilde{g}_{W+}^l) z^k \\ &= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \sum_{I_{n+v}+J_{n+v}=k} \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l} z^k \end{aligned}$$

$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \sum_{I_{n+v}+J_{n+v}=k} (\gamma^{p_1 l} z)^{i_1} \cdots (\gamma^{p_n l} z)^{i_n} (\gamma^{-p_1 l} z)^{j_1} \cdots$$
$$(\gamma^{-p_n l} z)^{j_n} \cdot z^{i_{n+1} + \cdots + i_{n+v} + j_{n+1} + \cdots + j_{n+v}}$$

$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \prod_{i=1}^{n} (\sum_{k=0}^{\infty} (\gamma^{p_i l} z)^k)(\sum_{k=0}^{\infty} (\gamma^{-p_i l} z)^k)(\sum_{k=0}^{\infty} z^k)^W$$

$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(1-\gamma^{p_i l} z)(1-\gamma^{-p_i l} z)(1-z)^W} \quad \text{on } \{z \in \mathbf{C} \,\big|\, |z| < 1\}$$

$$= \frac{(1+z)}{(1-z)^{W-1}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})}$$

For W odd case, we get by similar calculations,

$$F_q^{W+}(z : p_1, \ldots, p_n) = \frac{(1-z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \sum_{I_{n+u}+J_{n+u}+t=k} \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l} z^k$$

$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \sum_{I_{n+u}+J_{n+u}+t=k} (\gamma^{p_1 l} z)^{i_1} \cdots (\gamma^{p_n l} z)^{i_n} (\gamma^{-p_1 l} z)^{j_1} \cdots$$

$$(\gamma^{-p_n l} z)^{j_n} \cdot z^{i_{n+1} + \cdots + i_{n+u} + j_{n+1} + \cdots + j_{n+u} + t}$$

$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \prod_{i=1}^{n} (\sum_{k=0}^{\infty} (\gamma^{p_i l} z)^k)(\sum_{k=0}^{\infty} (\gamma^{-p_i l} z)^k)(\sum_{k=0}^{\infty} z^k)^W$$

$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(1-\gamma^{p_i l} z)(1-\gamma^{-p_i l} z)(1-z)^W} \quad \text{on } \{z \in \mathbf{C} \,\big|\, |z| < 1\}$$

$$= \frac{(1+z)}{(1-z)^{W-1}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} \quad \text{as before.}$$

□

Corollary 4.5.4. *When $L(q : p_1, \ldots, p_n)$ and $L(q : s_1, \ldots, s_n)$ have the same generating function, then $\tilde{L}_{W+}$ and $\tilde{L}'_{W+}$ (as defined above) also have the same generating function*

Proof. This follows from the fact that

$$F_q^{W+}(z : p_1, \dots, p_n, 0) = \frac{1}{(1-z)^W}\; F_q(z : p_1, \dots, p_n)$$

. ∎

The above results give us the following theorem and corollary (see Theorem 3.2.5 and Corollary 3.2.6 in [Ba]).

Theorem 4.5.5. (i) *Let $P \geq 5$ (alt. $P \geq 3$) be any odd prime and let $m \geq 2$ (alt. $m \geq 3$) be any positive integer. Let $q = P^m$. Then there exist at least two $(q+W-6)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order P^m - which are isospectral but not isometric.*

(ii) Let P_1, P_2 be two odd primes such that $q = P_1 \cdot P_2 \geq 33$. Then there exist at least two $(q+W-6)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order $P_1 \cdot P_2$ - which are isospectral but not isometric.

(iii) Let $q = 2^m$ where $m \geq 6$ is any positive integer. Then there exist at least two $(q+W-5)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order 2^m - which are isospectral but not isometric.

(iv) Let $q = 2P$, where $P \geq 7$ is an odd prime. Then there exist at least two $(q+W-5)$-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order $2P$ - which are isospectral but not isometric.

Corollary 4.5.6. (i) *Let $x \geq 19$ be any integer. Then there exist at least two x-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental groups of order 25 - which are isospectral but not isometric.*

(ii) *Let $x \geq 27$ be any integer. Then there exist at least two x-dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental group of order* 33 *- which are isospectral but not isometric.*

(iii) *Let $x \geq 59$ be any integer. Then there exist at least two x dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental group of order* 64 *- which are isospectral but not isometric.*

(iv) *Let $x \geq 9$ be any integer. Then there exist at least two x dimensional orbifold lens spaces - with non-trivial singular sets and with fundamental group of order* 14 *- which are isospectral but not isometric.*

Proof. (i) Let $q = 25$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (i) of the theorem.

(ii) Let $q = 33$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (ii) of the theorem.

(iii) Let $q = 64$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (iii) of the theorem.

(iv) Let $q = 14$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (iv) of the theorem.

□

4.6 An Example

In [Ba] we showed several examples of isospectral non-isometric orbifold lens spaces. Here we just show one example to demonstrate how the construction works.

Example 4.6.1. *Let $q = 5^2 = 25$, $q_0 = \frac{q-1}{2} = 12$, $n = 10$, $k = 2$,*
$A = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24\}$, $B_1 = \{5, 10, 15, 20\}$. *Let $w([p_1, \ldots, p_{10}]) = [q_1, q_2]$. Let $\gamma = e^{2\pi i/25}$ and $\lambda = e^{2\pi i/5}$.* $a_0 = |A| = 20$, $b_{0,1} = |B_1| = 4$.

Now,

$$\sum_{l\in A}\gamma^l = -1-(\gamma^5+\gamma^{10}+\gamma^{15}+\gamma^{20}) = -1-(\lambda^1+\lambda^2+\lambda^3+\lambda^4) = -1-(-1) = 0,$$
$$\sum_{l\in B_1}\gamma^l = \gamma^5+\gamma^{10}+\gamma^{15}+\gamma^{20} = \lambda^1+\lambda^2+\lambda^3+\lambda^4 = -1,$$
$$\sum_{l\in A}\lambda^l = 5(\lambda^1+\lambda^2+\lambda^3+\lambda^4) = 5(-1) = -5,$$
$$\sum_{l\in B_1}\lambda^l = \lambda^5+\lambda^{10}+\lambda^{15}+\lambda^{20} = 4.$$

Note that for $q = P^2$, we will always have (using similar calculations as above):

$$\sum_{l\in A}\gamma^l = 0\,,\ \sum_{l\in B_1}\gamma^l = -1\,,\ \sum_{l\in A}\lambda^l = -P \ \textit{and} \ \sum_{l\in B_1}\lambda^l = (P-1). \tag{4.18}$$

<u>*Case 1:*</u> *$q_1, q_2 \in B_1$ and $q_1 \pm q_2 \in B_1$.*

$$\begin{aligned}
a_1 &= 4\sum_{l\in A}\lambda^l \quad (\textit{since } q_1, q_2 \in B_1)\\
&= 4(-5) = -20.\\
b_{1,1} &= 4\sum_{l\in B_1}\lambda^l = 4(4) = 16.\\
a_2 &= 2(20) + 4\sum_{l\in A}\lambda^l \quad (\textit{since } q_1 \pm q_2 \in B_1)\\
&= 40 + 4(-5) = 40 - 20 = 20.\\
b_{2,1} &= 2(4) + 4\sum_{l\in B_1}\lambda^l = 8 + 4(4) = 24.
\end{aligned}$$

So,

$$\psi_{25,2}([q_1,q_2])(z) = 20z^4+20z^3+20z^2+20z+20,$$
$$\alpha^{(1)}_{25,2}([q_1,q_2])(z) = 4z^4-16z^3+24z^2-16z+4.$$

This corresponds to the case where

$$[p_1,\ldots,p_{10}] = [1,2,3,4,6,7,8,9,11,12],$$

which corresponds to a manifold lens spaces.

<u>*Case 2:*</u> *Since there is only one B_1 this case does not occur.*

<u>*Case 3:*</u> *$q_1 \in B_1$ and $q_2 \in A$ (alt. $q_1 \in A, q_2 \in B_1$). $q_1 \pm q_2 \in A$ always.*

In this case $a_0 = 20$, $b_{0,1} = 4$.

$$\begin{aligned}
a_1 &= 2\sum_{l\in A}\lambda^l + 2\sum_{l\in A}\gamma^l \quad (\textit{since } q_1, -q_1 \in B_1 \textit{ and } q_2, -q_2 \in A)\\
&= 2(-5)+2(0) = -10.\\
b_{1,1} &= 2\sum_{l\in B_1}\lambda^l + 2\sum_{l\in B_1}\gamma^l = 2(4)+2(-1) = 8-2 = 6.\\
a_2 &= 2(20) + 4\sum_{l\in A}\gamma^l \quad (\textit{since } q_1 \pm q_2 \in A)\\
&= 40+4(0) = 40.\\
b_{2,1} &= 2(4) + 4\sum_{l\in B_1}\gamma^l = 8+4(-1) = 4.
\end{aligned}$$

So,

$$\psi_{25,2}([q_1,q_2])(z) = 20z^4 + 10z^3 + 40z^2 + 10z + 20$$
$$\alpha^{(1)}_{25,2}([q_1,q_2])(z) = 4z^4 - 6z^3 + 4z^2 - 6z + 4$$
$$\textit{corresponding to} \quad [p_1,\dots,p_{10}] = [1,2,3,4,5,6,7,8,9,11]$$
$$\textit{and to} \quad [s_1,\dots,s_{10}] = [1,2,3,4,6,7,8,9,10,11]$$
$$\textit{and} \quad [p_1,\dots,p_{10}] \neq [s_1,\dots,s_{10}].$$

So, we get two isospectral non-isometric orbifolds:

$$L_1 = L(25 : 1,2,3,4,5,6,7,8,9,11)$$

and

$$L_2 = L(25 : 1,2,3,4,6,7,8,9,10,11).$$

We denote by $\sum_i$ *the singular set of* L_i*.*

Then $\sum_1 = \left\{(0,0,\dots,x_9,x_{10},0,0,\dots,0) \in S^{19} \,\middle|\, x_9^2 + x_{10}^2 = 1\right\}$

and $\sum_2 = \left\{(0,0,\dots,x_{17},x_{18},0,0) \in S^{19} \,\middle|\, x_{17}^2 + x_{18}^2 = 1\right\}$ *with isotropy groups* $\langle g_1^5\rangle$ *and* $\langle g_2^5\rangle$*, where*

$$g_1^5 = \begin{pmatrix} R(5p_1/25) & & \mathit{0} \\ & \ddots & \\ \mathit{0} & & R(5p_{10}/25) \end{pmatrix} = \begin{pmatrix} R(p_1/5) & & \mathit{0} \\ & \ddots & \\ \mathit{0} & & R(p_{10}/5) \end{pmatrix}$$

and

$$g_2^5 = \begin{pmatrix} R(s_1/5) & & \mathit{0} \\ & \ddots & \\ \mathit{0} & & R(s_{10}/5) \end{pmatrix}$$

where g_1 *and* g_2 *are generators of* G_1 *and* G_2*, respectively with* $L_1 = S^{19}/G_1$ *and* $L_2 = S^{19}/G_2$*.* $\sum_1$ *and* $\sum_2$ *are homeomorphic to* S^1*. We denote the two isotropy groups by* $H_1 = \langle g_1^5 \rangle$ *and* $H_2 = \langle g_2^5 \rangle$*.*

Case 4: *(a)* $q_1, q_2 \in A$ *and* $q_1 \pm q_2 \in A$*. So,*

$$\begin{aligned} a_1 &= 4\sum_{l\in A}\gamma^l = 4(0) = 0, \\ b_{1,1} &= 4\sum_{l\in B_1}\gamma^l = 4(-1) = -4, \\ a_2 &= 2(20) + 4\sum_{l\in A}\gamma^l = 40 + 4(0) = 40, \\ b_{2,1} &= 2(4) + 4\sum_{l\in B_1}\gamma^l = 8 + 4(-1) = 4. \end{aligned}$$

So,

$$\begin{aligned} \psi_{25,2}([q_1, q_2])(z) &= 20z^4 + 40z^2 + 20, \\ \alpha_{25,2}^{(1)}([q_1, q_2])(z) &= 4z^4 + 4z^3 + 4z^2 + 4z + 4. \end{aligned}$$

corresponding to

$$L_3 = L(25:1,2,3,4,5,6,7,8,9,10) = S^{19}/G_3,\ \textit{where } G_3 = \langle g_3 \rangle,$$

$$L_4 = L(25:1,2,3,4,5,6,7,8,10,11) = S^{19}/G_4,\ \textit{where } G_4 = \langle g_4 \rangle,$$

and

$$L_5 = L(25:1,2,3,4,5,6,7,10,11,12) = S^{19}/G_5,\ \textit{where } G_5 = \langle g_5 \rangle.$$

The isotropy groups for L_3, L_4 and L_5 are $\langle g_3^5 \rangle$, $\langle g_4^5 \rangle$ and $\langle g_5^5 \rangle$, respectively. $\sum_3$, $\sum_4$ and $\sum_5$ are all homeomorphic to S^3. So, here we get 3 isospectral orbifold lens spaces that are non-isometric.

(b) $q_1, q_2 \in A$ and $q_1 + q_2 \in B_1$, $q_1 - q_2 \in A$ (alt. $q_1 + q_2 \in A$, $q_1 - q_2 \in B_1$). And $a_1 = 0$, $b_{1,1} = -4$ as in (a).

$$\begin{aligned} a_2 &= 2(20) + 2\sum_{l \in A} \gamma^l + 2\sum_{l \in A} \lambda^l \\ &= 40 + 2(0) + 2(-5) = 40 - 10 = 30, \\ b_{2,1} &= 2(4) + 2\sum_{l \in B_1} \gamma^l + 2\sum_{l \in B_1} \lambda^l \\ &= 8 + 2(-1) + 2(4) = 8 - 2 + 8 = 14. \end{aligned}$$

So,

$$\psi_{25,2}([q_1, q_2])(z) = 20z^4 + 30z^2 + 20,$$

$$\alpha_{25,2}^{(1)}([q_1, q_2])(z) = 4z^4 + 4z^3 + 14z^2 + 4z + 4$$

corresponding to

$$\begin{aligned} &L_6 = L(25:1,2,3,4,5,6,7,9,10,11) = S^{19}/G_6, \ \textit{where } G_6 = \langle g_6 \rangle \\ \textit{and} \quad &L_7 = L(25:1,2,3,4,5,6,7,8,10,11) = S^{19}/G_7, \ \textit{where } G_7 = \langle g_7 \rangle \end{aligned}$$

Then, again, $\sum_6$ *and* $\sum_7$ *are homeomorphic to* S^3*, and* L_6 *and* L_7 *have isotropy groups* $\langle g_6^5 \rangle$ *and* $\langle g_7^5 \rangle$.

Chapter 5

3-Dimensional and 4-Dimensional Lens Spaces

Recall that in Chapter 1 we stated the results of [DGGW], which showed that all 2-dimensional orbifold spherical space forms are determined by their spectrum, and [L], which showed examples in dimensions 5, 6, 7, and 8 of pairs of isospectral orbifold lens spaces that are not isometric. Then, in Chapter 4 of this thesis, we proved our results for higher dimensional lens spaces and showed that for every dimension greater than 8 there exist pairs of isospectral non-isometric orbifold lens spaces (Corollary 4.5.6). In this chapter, we will study orbifold lens spaces in dimensions 3 and 4, and prove (Theorem 5.1.1 and Theorem 5.2.1 respectively) that in these two dimensions, the spectrum determines the geometry of an orbifold lens space. Then in Theorem 5.3.6, we will prove that an orbifold lens space cannot be isospectral to a spherical space form with non-cyclic fundamental group. The results in this and the following chapter have not been published yet.

5.1 3-Dimensional Orbifold Lens Spaces

For 3-dimensional manifold lens spaces, it is known that if two lens spaces are isospectral then they are also isometric ([IY] and [Y]). We will generalize this result to the orbifold case.

Using the notation adopted in the previous chapter, we write the two isospectral lens spaces as $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. Now there are only five possibilities:

Case 1 Both L_1 and L_2 are manifolds. In this case $gcd(p_i, q) = 1 = gcd(s_i, q)$ for $i = 1, 2$.

Case 2 One of the two lens spaces, say L_1 is a manifold, while the other, L_2 is an orbifold with non-trivial isotropy groups. This means that $gcd(p_1, q) = gcd(p_2, q) = 1$, while at least one of s_1 or s_2 is not coprime to q.

Case 3 Both L_1 and L_2 are orbifolds with non-trivial isotropy groups so that exacly one of p_1 or p_2 is coprime to q and exactly one of s_1 or s_2 is coprime to q.

Case 4 Both L_1 and L_2 are orbifolds with non-trivial isotropy groups, but in one case, say for L_1, exactly one of p_1 or p_2 is coprime to q, while for the other lens space, L_2 neither s_1 nor s_2 is coprime to q.

Case 5 None of p_1, p_2, s_1 and s_2 is coprime to q.

With these five cases in mind, we will prove our main theorem:

Theorem 5.1.1. *Given two 3-dimensional lens spaces $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. If L_1 is isospectral to L_2, then the two lens spaces are isometric.*

Proof. We will consider each case separately:

Case 1 In this case L_1 and L_2 are both manifolds. Ikeda and Yamamoto proved this case (see [IY] and [Y]).

Case 2 We know that whenever two isospectral good orbifolds share a common Riemannian cover, their respective singular sets are either both trivial or both non-trivial [GR]. Therefore, for orbifold lens spaces we can't have a situation where two lens spaces are isospectral, but one has a trivial singular set while the other has a non-trivial singular set. So this case is not possible.

Case 3 By multiplying the entries of L_1 and L_2 by appropriate numbers coprime to q we can rewrite $L_1 = L(q : 1, x)$ and $L_2 = L(q : 1, y)$, where x and y are not coprime to q. Let $F_1(z)$ [resp. $F_2(z)$] be the generating function associated to the spectrum of L_1 [resp.L_2]. Let γ be a primitive q-th root of unity.

Then

$$\lim_{z\to\gamma}(z-\gamma)F_1(z) = \lim_{z\to\gamma}(z-\gamma)F_2(z).$$

Now,

$$\begin{aligned}
&\lim_{z\to\gamma}(z-\gamma)F_1(z)\\
&= \lim_{z\to\gamma}\frac{1}{q}\sum_{l=1}^{q}\frac{(z-\gamma)(1-z^2)}{(1-\gamma^l z)(1-\gamma^{-l}z)(1-\gamma^{xl}z)(1-\gamma^{-xl}z)}\\
&= \frac{2}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})}.
\end{aligned}$$

The last equality follows from the fact that the solution to the congruence $l+1 \equiv 0 (\mathrm{mod}\ q)$ [resp. $-l+1 \equiv 0(\mathrm{mod}\ q)$] is $l = q-1$ [resp. $l = 1$], and that the congruences $xl+1 \equiv 0(\mathrm{mod}\ q)$ and $-xl+1 \equiv 0(\mathrm{mod}\ q)$ have no solutions since x is not coprime to q.

Since

$$\lim_{z\to\gamma}(z-\gamma)F_1(z) = \lim_{z\to\gamma}(z-\gamma)F_2(z),$$

we get

$$\begin{aligned}
&\frac{2}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})} = \frac{2}{q(1-\gamma^{-y+1})(1-\gamma^{y+1})},\\
&\implies \frac{1}{[1-(\gamma^{-x+1}+\gamma^{x+1})+\gamma^2]} = \frac{1}{[1-(\gamma^{-y+1}+\gamma^{y+1})+\gamma^2]},\\
&\implies (\gamma^{-x+1}+\gamma^{x+1}) = (\gamma^{-y+1}+\gamma^{y+1}).
\end{aligned}$$

Since $\gamma \neq 0$, we get

$$\begin{aligned}
&(\gamma^{-x}+\gamma^{x}) = (\gamma^{-y}+\gamma^{y}),\\
&\implies (\frac{1}{\gamma^x}+\gamma^x) = (\frac{1}{\gamma^y}+\gamma^y),\\
&\implies (\frac{1+\gamma^{2x}}{\gamma^x}) = (\frac{1+\gamma^{2y}}{\gamma^y}),\\
&\implies (\gamma^y+\gamma^{2x+y}) = (\gamma^x+\gamma^{x+2y}),\\
&\implies (\gamma^y-\gamma^{x+2y}) = (\gamma^x-\gamma^{2x+y}),\\
&\implies \gamma^y(1-\gamma^{x+y}) = \gamma^x(1-\gamma^{x+y}),\\
&\implies (\gamma^y-\gamma^x)(1-\gamma^{x+y}) = 0,\\
&\implies (\gamma^y-\gamma^x) = 0 \text{ or } (1-\gamma^{x+y}) = 0,\\
&\implies x \equiv y (\text{mod } q) \text{ or } x \equiv -y (\text{mod } q).
\end{aligned}$$

In either case, by Corollary 4.1.2 we get that L_1 and L_2 are isometric.

Case 4 By multiplying the entries of L_1 by appropriate numbers coprime to q we can rewrite $L_1 = L(q : 1, x)$, where x is not coprime to q, and $L_2 = L(q : s_1, s_2)$, where s_1 and s_2 are not coprime to q. Let $F_1(z)$ [resp. $F_2(z)$] be the generating function associated to the spectrum of L_1 [resp.L_2].

Then

$$\lim_{z\to\gamma}(z-\gamma)F_1(z) = \lim_{z\to\gamma}(z-\gamma)F_2(z).$$

Now,

$$\begin{aligned}
&\lim_{z\to\gamma}(z-\gamma)F_1(z) \\
&= \lim_{z\to\gamma}\frac{1}{q}\sum_{l=1}^{q}\frac{(z-\gamma)(1-z^2)}{(1-\gamma^{l}z)(1-\gamma^{-l}z)(1-\gamma^{xl}z)(1-\gamma^{-xl}z)} \\
&= \frac{2}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})}.
\end{aligned}$$

Since

$$\lim_{z\to\gamma}(z-\gamma)F_1(z) = \lim_{z\to\gamma}(z-\gamma)F_2(z),$$

we get

$$\begin{aligned}
&\frac{2}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})} \\
&= \lim_{z\to\gamma}\frac{1}{q}\sum_{l=1}^{q}\frac{(z-\gamma)(1-z^2)}{(1-\gamma^{s_1l}z)(1-\gamma^{-s_1l}z)(1-\gamma^{s_2l}z)(1-\gamma^{-s_2l}z)}.
\end{aligned}$$

But the congruences $s_1l+1\equiv 0(\text{mod } q)$, $-s_1l+1\equiv 0(\text{mod } q)$, $s_2l+1\equiv 0(\text{mod } q)$ and $-s_2l+1\equiv 0(\text{mod } q)$, have no solutions since s_1 and s_2 are not coprime to q. So the above equation becomes

$$\frac{2}{q(1-\gamma^{-x+1})(1-\gamma^{x+1})} = 0,$$

which is a contradiction. So this case is not possible.

Case 5 This is the hardest of all the cases. We will need to prove a few lemmas to prove this case. Let $L_1 = L(q : ax, by)$ and $L_2 = L(q : cu, dv)$ be the two isospectral lens spaces with fundamental group of order q. Here $gcd(ax, q) = x$, $gcd(by, q) = y$, $gcd(cu, q) = u$ and $gcd(dv, q) = v$. By multiplying the entries of L_1 and L_2 by appropriate numbers coprime to q we can rewrite $L_1 = L(q : x, py)$ and $L_2 = L(q : u, sv)$. We will also assume that $gcd(x, py) = 1 = gcd(u, sv)$ because if say $gcd(x, py) = e > 0$, then we could divide x, py and q by e and get a lens space with fundamental group of order q/e instead of q, which is a contradiction.

We will need two lemmas to prove the theorem for Case 5:

Lemma 5.1.2. *Suppose $L_1 = L(q : x, py)$ and $L_2 = L(q : u, sv)$ are two isospectral lens orbifolds where $gcd(x, q) = x$, $gcd(py, q) = y$, $gcd(u, q) = u$ and $gcd(sv, q) = v$. Then either $u = x$ and $v = y$, or $u = y$ and $v = x$.*

Note: If $u = x$ and $v = y$, then $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$; if $u = y$ and $v = x$, then $L_1 = L(q : x, py)$ and $L_2 = L(q : y, sx) = L(q : s^{-1}y, x) = L(q : x, s^{-1}y)$. In either case, this implies that we can write $L_1 = L(q : x, py)$ and $L_2 = L(q : x, s'y)$ where $s' = s$ or $s' = s^{-1}$.

We now prove the lemma:

Proof. We denote $q_{/x} = \frac{q}{x}$ and $q_{/y} = \frac{q}{y}$. Then

$$\lim_{z \to \gamma^x} (z - \gamma^x) F_1(z) = \lim_{z \to \gamma^x} \frac{1}{q} \sum_{l=1}^{q} \frac{(z - \gamma^x)(1 - z^2)}{(1 - \gamma^{xl} z)(1 - \gamma^{-xl} z)(1 - \gamma^{pyl} z)(1 - \gamma^{-pyl} z)}$$

Note that the only non-zero terms in this limit will be the ones where $xl + x \equiv 0 (\text{mod } q)$ or $-xl + x \equiv 0 (\text{mod } q)$, which gives $l = tq_{/x} - 1$ or $l = tq_{/x} + 1$ for $t \in \{1, ..., x\}$. Also note that for such a t, we have

$$\frac{1}{(1-\gamma^{py(tq_{/x}-1)+x})(1-\gamma^{-py(tq_{/x}-1)+x})} = \frac{1}{(1-\gamma^{py[(x-t)q_{/x}+1]+x})(1-\gamma^{-py[(x-t)q_{/x}+1]+x})}.$$

These two facts give

$$0 \neq \frac{2}{q}\sum_{t=1}^{x}\frac{1}{(1-\gamma^{py(tq_{/x}-1)+x})(1-\gamma^{-py(tq_{/x}-1)+x})} = \lim_{z\to\gamma^x}(z-\gamma^x)F_1(z).$$

Since

$$\lim_{z\to\gamma^x}(z-\gamma^x)F_1(z) = \lim_{z\to\gamma^x}(z-\gamma^x)F_2(z),$$

we get

$$\begin{aligned}0 \neq \frac{2}{q}\sum_{t=1}^{x}\frac{1}{(1-\gamma^{py(tq_{/x}-1)+x})(1-\gamma^{-py(tq_{/x}-1)+x})} &= \lim_{z\to\gamma^x}(z-\gamma^x)F_2(z)\\ &= \lim_{z\to\gamma^x}\frac{1}{q}\sum_{l=1}^{q}\frac{(z-\gamma^x)(1-z^2)}{(1-\gamma^{ul}z)(1-\gamma^{-ul}z)(1-\gamma^{svl}z)(1-\gamma^{-svl}z)}.\end{aligned}$$

So there must be an l such that

$$ul + x \equiv 0 (\text{mod } q),$$

or

$$-ul + x \equiv 0 (\text{mod } q),$$

or

$$svl + x \equiv 0 (\text{mod } q),$$

or

$$-svl + x \equiv 0(\text{mod } q).$$

Recall that $u|q$. Then $ul + x \equiv 0(\text{mod } q)$ or $-ul + x \equiv 0(\text{mod } q)$ imply that $u|x$. Similarly, since $v|q$, we can show that if $svl + x \equiv 0(\text{mod } q)$ or $-svl + x \equiv 0(\text{mod } q)$ then $v|x$. So either $u|x$ or $v|x$.

Now by multiplying the elements of L_1 by an appropriate number we can rewrite $L_1 = L(q : y, p'x)$. Then applying the same argument as above where we swap the roles of x and y, we get either $u|y$ or $v|y$.

Suppose $u|x$. Then since $gcd(x, y) = 1$ we can't have $u|y$. Similarly, if $v|x$, then we can't have $v|y$. Therefore, either $u|x$ and $v|y$, or $v|x$ and $u|y$ since if u or v divide both, then it contradicts $gcs(q, x, py) = 1$.

We can swap the roles of L_1 and L_2 and repeat the above arguments again to get either $x|u$ and $y|v$, or $y|u$ and $x|v$.

If $u|x$ and $v|y$, and at the same time $x|v$ and $y|u$, then $x|y$, which contradicts the fact that $gcd(q, x, y) = 1$. So, the only possibilities are:

i. $u|x$, $v|y$, $x|u$ and $y|v$. This means $x = u$ and $y = v$.

ii. $v|x$, $u|y$, $x|v$ and $y|u$. This means $x = v$ and $y = u$.

□

REMARK: From now on, we can write the two lens spaces as $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$. Further, If q is odd, we can also assume that both s and p are odd since if one of them, say p, is even then we can replace the lens space with $L(q : x, (q-p)y)$ which is isometric to L_1 and the coefficient $q - p$ is odd. Also, if q is even, then both x and py (resp.sy) can't be even simultaneously since $gcd(x, py)$(resp. sy); from now on, without loss of generality, if q is even we will assume that x is even and py (resp. sy) is odd since if py (resp. sy) is even and x is odd, then we can multiply the entries of the lens spaces by an appropriate number to re-write it as $L_1 = L(q : y, p'x)$ (resp. $L_2 = L(q : y, s'x)$).

Lemma 5.1.3. *Suppose, q is an integer. Given two isospectral lens spaces $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$ as above. Suppose $gcd(py+x, q) = d_1$, $gcd(py-x, q) = e_1$, $gcd(sy + x, q) = d_2$, and $gcd(sy - x, q) = e_2$. Then*

(i) $gcd(d_1, e_1) = 1 = gcd(d_2, e_2)$.

(ii) *Either* $d_1 = d_2$ *and* $e_1 = e_2$, *OR* $d_1 = e_2$ *and* $e_1 = d_2$

Note We will use this lemma in our proof showing the conjugation map between the cyclic groups defining L_1 and L_2.

Proof. (i) Suppose $d = gcd(d_1, e_1)$. So $d|(py + x)$ and $d|(py - x)$. This means $d|2py$ and $d|2x$. But, $gcd(x, py) = 1$. That means $d = 1$ or $d = 2$. Now, if q is even, then x is even and py is odd, that means $py + x$ and $py - x$ are both odd. So, d must be odd, and hence $d = 1$. If q is odd, then since d divides q, which is odd, d can't be even; so again $d = 1$. Using a similar argument we can also prove that $gcd(d_2, e_2) = 1$.

(ii) We first suppose that $d_1 = e_1 = 1$. Suppose $F_1(z)$ and $F_2(z)$ are the respective generating functions associated to L_1 and L_2. Then, if $d_2 > 1$, $F_2(z)$ will have a pole of order 2 at γ^{xq/d_2} and we get

$$\lim_{z \to \gamma^{xq/d_2}} (z - \gamma^{xq/d_2})^2 F_2(z) = \frac{2}{q(1 - \gamma^{-q(sy-x)/d_2})},$$

since the only non-zero terms will be for $l = \frac{q}{d_2}$ and $l = q - \frac{q}{d_2}$.

Since

$$\lim_{z \to \gamma^{xq/d_2}} (z - \gamma^{xq/d_2})^2 F_2(z) = \lim_{z \to \gamma^{xq/d_2}} (z - \gamma^{xq/d_2})^2 F_1(z),$$

we get

$$\frac{2}{q(1 - \gamma^{-q(sy-x)/d_2})} = 0,$$

which is a contradiction. Therefore, $d_2 = 1$. With a similar argument we can show that $e_2 = 1$.

Now assume $d_1 > 1$. Then, as in the above case, $F_1(z)$ will have a pole of order 2 at γ^{xq/d_1}, and we get

$$\lim_{z \to \gamma^{xq/d_1}} (z - \gamma^{xq/d_1})^2 F_1(z) = \frac{2}{q(1 - \gamma^{-q(py-x)/d_1})},$$

Since $F_2(z) = F_1(z)$, $F_2(z)$ also has a pole of order 2 at γ^{xq/d_1}. This means that either $(sy+x)\frac{q}{d_1} \equiv 0(\text{mod } q)$ or $(sy-x)\frac{q}{d_1} \equiv 0(\text{mod } q)$. This means that either d_1 divides d_2 or d_1 divides e_2.

Suppose d_1 divides d_2. Now, since $d_2 \geq d_1 > 1$, we can apply the same argument as above and get that either d_2 divides d_1 or d_2 divides e_1. If d_2 divides d_1, then $d_1 = d_1$. If, d_2 divides e_1, that means d_1 divides e_1. But this contradicts (i)above. Therefore, $d_1 = d_2$ in this case.

If $e_1 = 1$ and $e_2 > 1$ then applying a similar argument as above, we can show that either e_2 divides d_1 or e_2 divides e_1. But if e_2 divides d_1, that means e_2 divides d_2, which again contradicts (i). So, e_2 must divide e_1, and we get a contradiction for $e_2 > 1$.

If $e_1 > 1$, then applying the same argument as before, we can show that either

e_1 divides d_2 or e_1 divides e_2. Again, if e_1 divides d_2, then e_1 divides d_1, and we get a contradiction. So we must have that e_1 divides e_2. Now reversing the argument as before we can show that e_2 divides e_1, and therefore $e_2 = e_1$

If, on the other hand, d_1 divides e_2, then we can apply the same arguments as before to show that $d_1 = e_2$ and $e_1 = d_2$.

This completes the proof of the lemma.

$\square$

NOTE If we are in the situation where $gcd(py + x, q) = gcd(sy - x, q)$ and $gcd(py - x, q) = gcd(sy + x, q)$, then, writing the second lens space as $L_2 = L(q : x, -sy) = L(q : x, s'y)$, we can ensure that $gcd(py + x, q) = gcd(s'y + x, q)$ and $gcd(py - x, q) = gcd(s'y - x, q)$. Therefore, without loss of generality, from now on, we will always assume that $gcd(py + x, q) = gcd(sy + x, q)$ and $gcd(py - x, q) = gcd(sy - x, q)$.

5.1.1 Finite Subgroups of SO(4)

We now have two isospectral lens spaces $L_1 = L(q : x, py)$ and $L_2 = L(q : x, sy)$ where the respective cyclic groups are G and G'. We will now show that these two groups are conjugate to each other and that, according to Lemma 4.1.1, will prove that L_1 and L_2 are isometric. In order to do this we will use the classification of finite subgroups of $SO(4)$ in [MS]. It is convenient to use the relationship of $SO(4)$ to quaternions for this.

Recall that the quaternion algebra $\mathbb{H}$ is given by [MS]

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = -1, ij = k = -ji\}$$
$$:= \{z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}.$$

As usual the 'conjugate' of a quaternion $q = a + bi + cj + dk$ is $\bar{q} = a - bi - cj - dk$, and its 'real part' is $Re(q) = a$. It is then easy to check that

$$|q|^2 = q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2.$$

Any non-zero quaternion has a two-sided multiplicative inverse given by $q^{-1} = \bar{q}/|q|^2$. We will consider the 3-sphere as the set of unit quaternions (the quaternions of length 1) as follows:

$$\mathbb{S}^3 = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\} = \{z_1 + z_2 j \mid |z_1|^2 + |z_2|^2 = 1\}$$

The product in $\mathbb{H}$ induces a group structure on $\mathbb{S}^3$. For each pair (p, q) of elements of $\mathbb{S}^3$, the function

$$\Phi_{p,q} : \mathbb{H} \to \mathbb{H}$$

with $\Phi_{p,q}(h) = phq^{-1}$ leaves invariant the length of quaternions. We can, therefore, define a homomorphism of groups:

$$\Phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$$

such that $\Phi(p, q) = \Phi_{p,q}$.

The homomorphism Φ is surjective with kernel of $\{(1, 1), (-1, -1)\}$. The homomorphism Φ gives a 1-1 correspondence between finite subgroups of $SO(4)$ and finite subgroups of $\mathbb{S}^3 \times \mathbb{S}^3$ containing the kernel of Φ. Moreover two subgroups are conjugated in $SO(4)$ iff the corresponding groups in $\mathbb{S}^3 \times \mathbb{S}^3$ are conjugated [MS]. So to prove that two finite subgroups of $SO(4)$ are conjugate, we prove that the corresponding subgroups of $\mathbb{S}^3 \times \mathbb{S}^3$ are conjugate.

Let G be a finite subgroup of $\mathbb{S}^3 \times \mathbb{S}^3$. For $i = 1, 2$, we denote by

$$\pi_i : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$$

the two projections. We use the following notations as used in [MS]:

$$L = \pi_1(G),$$

$$L_K = \pi_1((\mathbb{S}^3 \times \{1\}) \cap G),$$

$$R = \pi_2(G),$$

$$R_K = \pi_2((\{1\} \times \mathbb{S}^3) \cap G).$$

The projections π_1 and π_2 induce isomorphisms (see [MS] and [DV]) given respectively by

$$\bar{\pi}_1 : G/(L_K \times R_K) \to L/L_K$$

and

$$\bar{\pi}_2 : G/(L_K \times R_K) \to R/R_K.$$

From these we get an isomorphism

$$\phi_G : L/L_K \to R/R_K$$

such that $\phi_G = \bar{\pi}_2 \circ \bar{\pi}_1^{-1}$.

Conversely, given two finite subgroups L and R of $\mathbb{S}^3$, with two normal subgroups L_K and R_K such that there exists an isomorphism $\phi : L/L_K \to R/R_K$, we can define a subgroup G of $\mathbb{S}^3 \times \mathbb{S}^3$ such that

$$L = \pi_1(G),$$

$$L_K = \pi_1((\mathbb{S}^3 \times \{1\}) \cap G),$$

$$R = \pi_2(G),$$

$$R_K = \pi_2((\{1\} \times \mathbb{S}^3) \cap G).$$

and $\phi = \phi_G$. As a result, the subgroups of $\mathbb{S}^3 \times \mathbb{S}^3$ can be uniquely identified by 5-tuples (L, L_K, R, R_K, ϕ) [MS].

For classifying subgroups of $\mathbb{S}^3 \times \mathbb{S}^3$ upto conjugation, Du Val [DV], Mecchia and Seppi [MS] used the following result:

Proposition 5.1.4. *Let G and G' two subgroups of $\mathbb{S}^3 \times \mathbb{S}^3$ described respectively by (L, L_K, R, R_K, ϕ) and $(L', L'_K, R', R'_K, \phi')$. The groups G and G' are conjugated in $\mathbb{S}^3 \times \mathbb{S}^3$ if and only if there exist two inner automorphisms, α and β, of $\mathbb{S}^3$ such that $\alpha(L) = L'$, $\beta(R) = R'$, $\alpha(L_K) = L'_K$, $\beta(R_K) = R'_K$ and $\phi = \bar{\beta}^{-1}\phi'\bar{\alpha}$, where $\bar{\alpha}$ and $\bar{\beta}$ are the maps induced by α and β on the factors L/L_K and R/R_K.*

Up to conjugation the finite cyclic subgroups of $\mathbb{S}^3$ are the following:

$$C_n = \{\cos(\frac{2\alpha\pi}{n}) + i\sin(\frac{2\alpha\pi}{n}) \mid \alpha = 0, ..., n-1\}$$

and up to conjugation the cyclic subgroups of $\mathbb{S}^3 \times \mathbb{S}^3$ containing $(-1,-1)$ are [MS]:

Type 1: $(C_{2mr}, C_{2m}, C_{2nr}, C_{2n}, \phi_s)$, with $gcd(s, r) = 1$ and

$$\phi_s : C_{2mr}/C_{2m} \to C_{2nr}/C_{2n}$$

given by $\phi_s(e^{i\pi/mr}C_{2m}) = e^{is\pi/nr}C_{2n}$. In this case $|\Phi(G)| = 2mnr$.

Type 2: $(C_{mr}, C_m, C_{nr}, C_n, \phi_s)$, with $gcd(s,r) = 1 = gcd(2,n) = gcd(2,m)$, $gcd(2,r) = 2$ and

$$\phi_s : C_{mr}/C_m \to C_{nr}/C_n$$

given by $\phi_s(e^{i2\pi/mr}C_m) = e^{i2s\pi/nr}C_n$. In this case $|\Phi(G)| = mnr/2$.

Coming back to the proof of our main theorem, we let the isometry group acting on L_1 be denoted by $G =< g >$, where

$$g = \begin{pmatrix} \cos\frac{2x\pi}{q} & \sin\frac{2x\pi}{q} & 0 & 0 \\ -\sin\frac{2x\pi}{q} & \cos\frac{2x\pi}{q} & 0 & 0 \\ 0 & 0 & \cos\frac{2py\pi}{q} & \sin\frac{2py\pi}{q} \\ 0 & 0 & -\sin\frac{2py\pi}{q} & \cos\frac{2py\pi}{q} \end{pmatrix}.$$

Similarly, the isometry group acting on L_2 is denoted by $G' =< g' >$, where

$$g' = \begin{pmatrix} \cos\frac{2x\pi}{q} & \sin\frac{2x\pi}{q} & 0 & 0 \\ -\sin\frac{2x\pi}{q} & \cos\frac{2x\pi}{q} & 0 & 0 \\ 0 & 0 & \cos\frac{2sy\pi}{q} & \sin\frac{2sy\pi}{q} \\ 0 & 0 & -\sin\frac{2sy\pi}{q} & \cos\frac{2sy\pi}{q} \end{pmatrix}.$$

Using the definition of the homomorphism Φ, we can calculate the pre-images of the two generators g and g' in $\mathbb{S}^3\times\mathbb{S}^3$; they are $(e^{-\frac{i2\pi(x+py)}{2q}}, e^{\frac{i2\pi(x-py)}{2q}})$ and $(e^{-\frac{i2\pi(x+sy)}{2q}}, e^{\frac{i2\pi(x-sy)}{2q}})$ respectively.

In order to prove our result we first have to show that the pre-images of the two groups, G and G' can't lie in the two different types described above.

For odd q it is easy to see since in that case both pre-images will be of Type 2, since Type 1 are the groups that, under the image on Φ have order *2mnr*, which is even and can't be equal to q.

Now suppose q is even. We first notice that by the definition of the subgroups L, L_K, R, and R_K for the cyclic subgroups of $\mathbb{S}^3\times\mathbb{S}^3$, we get that for G, the order of L_K is $gcd((py-x),q)$ and the order of R_K is $gcd((py+x),q)$. Similarly, for G', the order of L_K is $gcd((sy-x),q)$ and the order of R_K is $gcd((sy+x),q)$. We denote $\Phi^{-1}G$ and $\Phi^{-1}G'$ by (L, L_K, R, R_K, ϕ_s) and $(L', L'_K, R', R'_K, \phi_{s'})$ respectively. Now, from Lemma 5.1.3 and our subsequent note, we know that $gcd(py+x,q) = gcd(sy+x,q)$ and $gcd(py-x,q) = gcd(sy-x,q)$. So, $|L_K| = |L'_K|$ and $|R_K| = |R'_K|$. Now it is obvious that $\Phi^{-1}G$ and $\Phi^{-1}G'$ are of the same Type.

I: *q is odd.* If $gcd(py+x,q) = gcd(sy+x,q) = d_1$ and $gcd(py-x,q) = gcd(sy-x,1) = d_2$. We will denote q/d_i by q_i. In this case, the two subgroups will correspond to the subgroups $(C_{2q_1}, C_{d_2}, C_{2q_2}, C_{d_1}, \phi_t)$ (where $t \equiv (py-x)w(\text{mod } 2q_2)$ for some number w coprime to $2q_2$) and $(C_{2q_1}, C_{d_2}, C_{2q_2}, C_{d_1}, \phi_{t'})$ (where $t' \equiv (sy-x)w'(\text{mod } 2q_2)$ for some number w' coprime to $2q_2$). We now need to find inner-automorphisms α and β according to Proposition 5.1.4 such that the following diagram commutes:

$$\begin{array}{ccc} C_{2q_1}/C_{d_2} & \xrightarrow{\phi_t} & C_{2q_2}/C_{d_1} \\ \bar{\alpha}\downarrow & & \uparrow\bar{\beta}^{-1} \\ C_{2q_1}/C_{d_2} & \xrightarrow{\phi_{t'}} & C_{2q_2}/C_{d_1} \end{array}$$

We need to find inner-automorphisms of α and β of $\mathbb{S}^3$ such that $\alpha(e^{\frac{i2\pi}{2q_1}}) = e^{\frac{i2\pi}{2q_1}}$ and $\beta(e^{\frac{i2\pi t}{2q_2}}) = e^{\frac{i2\pi t'}{2q_2}}$. The definition of α is obvious as we can define $\alpha = Id$, which conjugates every element to itself.

Now, to define β we recall some facts from the identification of the quaternions with the euclidean 4-space or $\mathbb{R}^4$. Recall that any quaternion $q = a+bi+cj+dk$ can be written as $q = (a, V)$ where the *pure quaternion* $V = bi + cj + dk$ can be identified with the point (b, c, d) of the subspace $\mathbb{R}^3$ of $\mathbb{R}^4$. It is also known that any conjugation in the quaternions is equivalent to a rotation in $\mathbb{R}^3$. We notice that the points $e^{i2\pi t/2q_2}$ and $e^{i2\pi t'/2q_2}$ lie on the same unit circle in the complex plane $\mathbb{C}$ as $2q_2$ roots of unity. So, we can view these points in $\mathbb{R}^3$ as $(\cos 2\pi t/2q_2, \sin 2\pi t/2q_2, 0)$ and $(\cos 2\pi t'/2q_2, \sin 2\pi t'/2q_2, 0)$. This means that we can find a rotation of $\mathbb{R}^3$ that maps $(\cos 2\pi t/2q_2, \sin 2\pi t/2q_2, 0)$ to $(\cos 2\pi t'/2q_2, \sin 2\pi t'/2q_2, 0)$. From the above facts about quaternions, we know that such a rotation will be a conjugation in the quaternions and hence an inner-automorphism in $\mathbb{S}^3$. Indeed, if we now view $(\cos 2\pi t/2q_2, \sin 2\pi t/2q_2, 0)$ and $(\cos 2\pi t'/2q_2, \sin 2\pi t'/2q_2, 0)$ as *pure-quaternions* and write them as $0 + \cos 2\pi t/2q_2 i + \sin 2\pi t/2q_2 j + 0k$ and $0 + \cos 2\pi t'/2q_2 i + \sin 2\pi t'/2q_2 j + 0k$, respectively, then it is easy to see that the unit quaternion $\cos 2\pi(t' - t)/4q_2 + 0i + 0j + \sin 2\pi(t' - t)/4q_2 k$ conjugates $0 + \cos 2\pi t/2q_2 i + \sin 2\pi t/2q_2 j + 0k$ to $0 + \cos 2\pi t'/2q_2 i + \sin 2\pi t'/2q_2 j + 0k$.

Now, by Proposition 5.1.4, we have the two groups G and G' as conjugates in $SO(4)$ and therefore, the corresponding orbifold lens spaces are isometric.

II: *q is even, x is even, and py (resp. sy) is odd:* In this case again the two groups will of of Type 2, and the proof will go exactly as it did for I above.

III: *q is even, x is odd, and py (resp. sy) is odd:* In this case, the two groups will be of Type 1, and with slight modifications it can be shown that the proof will go exactly as it did for I and II.

This completes our proof for Case 5.

□

5.2 4 Dimensional Orbifold Lens Spaces

It is known that in the manifold case, even dimensional spherical space forms are only the sphere and the real projective spaces [I2]. It is also known that the sphere $\mathbb{S}^n$ is not isospectral to the real projective space $P^n(\mathbb{R})$ [BGM].

In the orbifold case, there are many even dimensional spherical space forms with fixed points. We will focus on the 4-dimensional orbifold lens spaces. In [L], Lauret has classified cyclic subgroups of $SO(2n+1)$ up to conjugation. According to this classification, any cyclic subgroup G of $SO(2n+1)$ is represented by $G =< \gamma >$ where $\gamma = diag(R(\frac{2\pi p_1}{q}), ..., R(\frac{2\pi p_n}{q}), 1)$ and $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Suppose $n = 2$. Let

$$\tilde{g}_1 = \begin{pmatrix} R(p_1/q) & & 0 \\ & R(p_2/q) & \\ 0 & & 1 \end{pmatrix}$$

and

$$\tilde{g}_2 = \begin{pmatrix} R(s_1/q) & & 0 \\ & R(s_2/q) & \\ 0 & & 1 \end{pmatrix}.$$

Suppose there are 4-dimensional orbifold lens spaces $O_1 = \mathbb{S}^4/\tilde{G}_1$ and $O_2 = \mathbb{S}^4/\tilde{G}_2$, where $\tilde{G}_1 =< \tilde{g}_1 >$ and $\tilde{G}_2 =< \tilde{g}_2 >$.

Theorem 5.2.1. *Given O_1, O_2, $\tilde{G}_1$ and $\tilde{G}_2$ as above. If O_1 and O_2 are isospectral then they are isometric.*

Proof. From Theorem 4.4.3 we know that on the domain $\{z \in \mathbf{C} \,\big|\, |z| < 1\}$, the spectrum generating functions of O_1 and O_2, respectively, are,

$$F_q(z : p_1, p_2, 0) = \frac{1}{q} \sum_{l=1}^{q} \frac{(1+z)}{\prod_{i=1}^{2}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})}$$

and

$$F_q(z : s_1, s_2, 0) = \frac{1}{q} \sum_{l=1}^{q} \frac{(1+z)}{\prod_{i=1}^{2}(z - \gamma^{s_i l})(z - \gamma^{-s_i l})}$$

.

Notice that $F_q(z : p_1, p_2) = (1-z)F_q(z : p_1, p_2, 0)$ and $F_q(z : s_1, s_2) = (1 - z)F_q(z : s_1, s_2, 0)$, where $F_q(z : p_1, p_2)$ and $F_q(z : s_1, s_2)$ are respectively the spectrum generating functions for the 3-dimensional orbifold lens spaces $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. This means that if O_1 and O_2 are isospectral then L_1 and L_2 are also isospectral.

Now, from Theorem 5.1.1, we know that L_1 and L_2 are isometric. By Lemma 4.4.2 we know that L_1 is isometric to L_2 iff O_1 is isometric to O_2. This proves the theorem. $\square$

5.2.1 Some Higher Dimensional Orbifold Lens Spaces

We can generalize the above results to obtain pairs of higher dimensional orbifold lens spaces which may be distinguished by their spectra. In Chapter 4 we saw examples of orbifold lens spaces in dimensions 9 and higher. Also, in [L] there exist examples of orbifold lens spaces in dimensions 5 through 8 where the spectrum doesn't determine the orbifold. We will now prove some results to demonstrate that in every dimension there exist pairs of orbifold lens spaces where the spectrum does determine the lens space.

We first prove that for all odd dimensions ≥ 5, there exist pairs of orbifold lens spaces that may be distinguished by their spectra. Suppose $n \geq 3$. Let

$$\tilde{g}_1 = \begin{pmatrix} R(p_1/q) & & 0 \\ & R(p_2/q) & \\ 0 & & I_{2n-4} \end{pmatrix}$$

and

$$\tilde{g}_2 = \begin{pmatrix} R(s_1/q) & & 0 \\ & R(s_2/q) & \\ 0 & & I_{2n-4} \end{pmatrix},$$

where I_{2n-4} is the $2n-4$ by $2n-4$ identity matrix.

Suppose there are $m = 2n-1$-dimensional orbifold lens spaces $O_1 = \mathbb{S}^m/\tilde{G}_1$ and $O_2 = \mathbb{S}^m/\tilde{G}_2$, where $\tilde{G}_1 =< \tilde{g}_1 >$ and $\tilde{G}_2 =< \tilde{g}_2 >$.

Proposition 5.2.2. *All distinct orbifolds of the form O_1 and O_2 as defined above have distinct spectra.*

Proof. From Theorem 4.5.3 we know that on the domain $\{z \in \mathbf{C} \,\big|\, |z| < 1\}$, the spectrum generating functions of O_1 and O_2, respectively, are,

$$F_q^{2n-4}(z : p_1, p_2, 0) = \frac{(1+z)}{(1-z)^{2n-5}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{2}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})}$$

and

$$F_q^{2n-4}(z : s_1, s_2, 0) = \frac{(1+z)}{(1-z)^{2n-5}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{2}(z - \gamma^{s_i l})(z - \gamma^{-s_i l})}.$$

We note that $F_q(z : p_1, p_2) = (1-z)^{2n-4} F_q^{2n-4}(z : p_1, p_2, 0)$ and $F_q(z : s_1, s_2) = (1-z)^{2n-4} F_q^{2n-4}(z : s_1, s_2, 0)$, where $F_q(z : p_1, p_2)$ and $F_q(z : s_1, s_2)$ are respectively the spectrum generating functions for the 3-dimensional orbifold lens spaces $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. This means that if O_1 and O_2 are isospectral then L_1 and L_2 are also isospectral.

Now, from Theorem 5.1.1, we know that L_1 and L_2 are isometric. By Lemma 4.5.2 we know that L_1 is isometric to L_2 iff O_1 is isometric to O_2. This proves the theorem. □

We now prove that for all even dimensions ≥ 6, there exist pairs of orbifold lens spaces that may be distinguished by their spectra.

Suppose $n \geq 3$. Let

$$\tilde{g}_1 = \begin{pmatrix} R(p_1/q) & & 0 \\ & R(p_2/q) & \\ 0 & & I_{2n-3} \end{pmatrix}$$

and

$$\tilde{g}_2 = \begin{pmatrix} R(s_1/q) & & 0 \\ & R(s_2/q) & \\ 0 & & I_{2n-3} \end{pmatrix},$$

where I_{2n-3} is the $2n-3$ by $2n-3$ identity matrix.

Suppose there are $m = 2n$-dimensional orbifold lens spaces $O_1 = \mathbb{S}^m/\tilde{G}_1$ and $O_2 = \mathbb{S}^m/\tilde{G}_2$, where $\tilde{G}_1 =< \tilde{g}_1 >$ and $\tilde{G}_2 =< \tilde{g}_2 >$.

Proposition 5.2.3. *All distinct orbifolds of the form O_1 and O_2 as defined above have distinct spectra.*

Proof. From Theorem 4.5.3 we know that on the domain $\left\{z \in \mathbf{C} \,\middle|\, |z| < 1\right\}$, the spectrum generating functions of O_1 and O_2, respectively, are,

$$F_q^{2n-3}(z : p_1, p_2, 0) = \frac{(1+z)}{(1-z)^{2n-4}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{2}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})}$$

and

$$F_q^{2n-3}(z : s_1, s_2, 0) = \frac{(1+z)}{(1-z)^{2n-4}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{2}(z - \gamma^{s_i l})(z - \gamma^{-s_i l})}.$$

As before, we note that $F_q(z : p_1, p_2) = (1-z)^{2n-3} F_q^{2n-3}(z : p_1, p_2, 0)$ and $F_q(z : s_1, s_2) = (1-z)^{2n-3} F_q^{2n-3}(z : s_1, s_2, 0)$, where $F_q(z : p_1, p_2)$ and $F_q(z : s_1, s_2)$ are respectively the spectrum generating functions for the 3-dimensional orbifold lens spaces $L_1 = L(q : p_1, p_2)$ and $L_2 = L(q : s_1, s_2)$. This means that if O_1 and O_2 are isospectral then L_1 and L_2 are also isospectral.

Now, from Theorem 5.1.1, we know that L_1 and L_2 are isometric. By Lemma 4.5.2 we know that L_1 is isometric to L_2 iff O_1 is isometric to O_2. This proves the theorem. $\square$

5.3 Lens Spaces and Other Spherical Space Forms

One question still remains: Is an orbifold lens space ever isospectral to an orbifold spherical space form which has non-cyclic fundamental group?

Our final result in this chapter is to prove that an orbifold lens space cannot be isospectral to a general spherical space form with non-cyclic fundamental group. We will use some results from [I2] noting that in some cases his assumption that the acting group is fixed-point free is not used in certain proofs, and therefore, the results hold true for orbifolds.

Definition 5.3.1. *Let G be finite group, and let G_k be the subset of G consisting of all elements of order k in G. Let $\sigma(G)$ denote the set consisting of orders of elements in G. Then we have*

$$G = \cup_{k \in \sigma(G)} G_k \text{ (disjoint union)}$$

The following lemma is proved in [I2] for fixed-point free subgroups of $SO(2n)$, but we note that the proof doesn't require this condition and reproduce the proof from [I2].

Lemma 5.3.2. *Let G be a finite subgroup of $SO(2n)$ ($n \geq 2$). Then the subset G_k is divided into the disjoint union of subsets $C_k^1, ..., C_k^{i_k}$ such that each $C_k^t (t = 1, 2, ..., i_k)$ consists of all generic elements of some cyclic subgroup of order k in G.*

Proof. For any $g \in G_k$, we denote by A_g the cyclic subgroup of G generated by g. Now, for $g, g' \in G_k$ the cyclic group $A_g \cap A_{g'}$ is of order k if and only if $A_g = A_{g'}$. Now the lemma follows from this observation immediately.

□

We now state another lemma (see [I2] for proof) that will be used to prove our result.

Lemma 5.3.3. *Let g be an element in $SO(2n)$ $(n \geq 2)$ and of order q $(q \geq 3)$. Set $\gamma = e^{2\pi\sqrt{-1}/q}$. Assume g has eigenvalues γ, γ^{-1}, γ^{p_1}, γ^{-p_1},..., γ^{p_k}, γ^{-p_k} with multiplicities $l, l, i_1, i_1, ..., i_k, i_k$, respectively, where $p_1, ..., p_k$ are integers prime to q with $p_i \not\equiv \pm p_j (mod q)$ (for $1 \leq i < j \leq k$), $p \not\equiv \pm l (mod q)$ (for $i = 1, , k$) and $l + i_1 + ... + i_k = n$. Then the Laurent expansion of the meromorphic function $\frac{1-z^2}{det\,(1_{2n} - gz)}$ at $z = \gamma$ is*

$$\frac{1}{(z-\gamma)^l} \frac{(\sqrt{-1})^{n+l}\gamma^l}{2^{n-l}(1-\gamma^2)^{n-1}} \prod_{j=1}^{k} \{\cot \frac{\pi}{q}(p_j + 1) - \cot \frac{\pi}{q}(p_j - 1)\}^{i_j} + \text{ lower order terms.}$$

The following proposition is proved by Ikeda for a group G that acts freely. However, we note that the proposition is true even if G does not act freely since the proof does not use the property that G acts freely.

Proposition 5.3.4. *Let G be a finite subgroup of $SO(2n)$ $(n \geq 2)$, and let $k \in \sigma(G)$. We define a positive integer k_0 by*

$$\begin{aligned} k_0 &= 2n - 1 \text{ if } k = 1 \text{ or } 2, \\ &= max_{g \in G_k}\{max.\ of\ multiplicities\ of\ eigenvalues\ of\ g\} \text{ if } k \geq 3. \end{aligned}$$

Then the generating function $F_G(z)$ has a pole of order k_0 at any primitive k-th root of 1.

Proof. At $z = 1$, we notice that for $g = I_{2n} \in G_1$, we get

$$\lim_{z \to 1} (1-z)^{2n-1} F_G(z) = \frac{2}{|G|},$$

as g has eigenvalue 1 with multiplicity $2n$. So, $F_G(z)$ has a pole of order $2n - 1$ at $z = 1$.

At $z=-1$ we notice that for $g=-I_{2n}\in G_2$, we get

$$\lim_{z\to 1}(1+z)^{2n-1}F_G(z)=\frac{2}{|G|},$$

as g has eigenvalue -1 with multiplicity $2n$. Also, for any other $g'\in G_2$, the eigenvalue -1 has multiplicity at most $2n$. So $F_G(z)$ has a pole of order $2n-1$ at $z=-1$ as well.

We now assume $k\geq 3$. Now let $G_k, C_k^1, ..., C_k^{i_k}$ be as in Lemma 5.3.2. Then we have

$$\begin{aligned}|G|F_G(z) &= \sum_{g\in G_k}\frac{1-z^2}{det(I_{2n}-gz)}+\sum_{g\in G-G_k}\frac{1-z^2}{det(I_{2n}-gz)}\\ &= \sum_{j=1}^{i_k}\sum_{g\in G_k}\frac{1-z^2}{det(I_{2n}-gz)}+\sum_{g\in G-G_k}\frac{1-z^2}{det(I_{2n}-gz)}\end{aligned}\tag{5.1}$$

Set $\gamma=e^{2\pi\sqrt{-1}/k}$. For any primitive k-th root γ^t of 1, where t is an integer prime to k, let

$$\frac{a_{k_0}(t)}{(z-\gamma^t)^{k_0}}+\frac{a_{k_0-1}(t)}{(z-\gamma^t)^{k_0-1}}+...+\frac{a_1(t)}{(z-\gamma^t)}$$

be the principal part of the Laurent expansion of $F_G(z)$ at $z=\gamma^t$. Then each coefficient $a_i(t)$ is an element in the k-th cyclotomic field $\mathbb{Q}(\gamma)$ over the rational number field $\mathbb{Q}$. The automorphisms σ_t of $\mathbb{Q}(\gamma)$ defined by

$$\gamma\to\gamma^t$$

transforms $a_i(1)$ to $a_i(t)$ by Equation (5.1). Hence, it is sufficient to show that the generating function $F_G(z)$ has a pole of order k_0 at $z-\gamma$, that is, to show that $a_{k_0}(1)\neq 0$.

Note that if $0<b<a<\pi$, then $\cot a-\cot b<0$. Now the proposition follows immediately from Lemma 5.3.3 and Equation (5.1). $\square$

From Proposition 5.3.4, we get

Corollary 5.3.5. *Let $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ be two isospectral orbifold spherical space forms. Then $\sigma(G) = \sigma(G')$.*

We now prove our result

Theorem 5.3.6. *Let $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ be two (orbifold) spherical space forms. Suppose G is cyclic and G' is not cyclic. Then $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ cannot be isospectral.*

Proof. By Corollary 4.1.14, we already know that if $|G| \neq |G'|$ then $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ cannot be isospectral. So let us assume that $|G| = |G'| = q$.

Suppose $\mathbb{S}^{2n-1}/G$ and $\mathbb{S}^{2n-1}/G'$ are isospectral. If G is cyclic then it has an element of order q. Now, by Corollary 5.3.5, G' must also have an element of order q, but since $|G'| = q$, that implies that G' is cyclic, which is not true by assumption, and we arrive at a contradiction. This proves the theorem. $\square$

Chapter 6

Heat Kernel for Orbifold Lens Spaces

In the previous chapter we used the spectrum to determine the geometry of three and four dimensional orbifold lens spaces. It is also known that if two orbifolds (manifolds) have the same spectrum, then their respective asymptotic expansions of the heat kernel will also be the same. The question arises whether we can prove the results in Chapter 5 by using the coefficients of the asymptotic expansion of the heat kernel? In this chapter, we show that the equality of the heat coefficients for two orbifolds is not enough to determine their geometry. In other words, we cannot obtain the results of the previous chapter only from these coefficients.

6.1 Heat Kernel

In the mathematical study of heat conduction and diffusion, a heat kernel is the fundamental solution to the heat equation on a specified domain with appropriate boundary conditions. It is also one of the main tools in the study of the spectrum of the Laplace operator, and is thus of some auxiliary importance throughout mathematical physics. The heat kernel represents the evolution of temperature in a region whose boundary is held fixed at a particular temperature (typically zero), such that an initial unit of heat energy is placed at a point at time t = 0.

Definition 6.1.1. *Let M be a Riemannian manifold. A heat kernel, or alternatively, a fundamental solution to the heat equation, is a function*

$$K : (0, \infty) \times M \times M \to M \tag{6.1}$$

that satisfies

1. *$K(t, x, y)$ is C^1 in t and C^2 in x and y;*
2. *$\partial K/\partial t + \Delta_2(K) = 0$, where Δ_2 is the Laplacian with respect to the second variable (i.e., the first space variable);*
3. *$\lim_{t\to 0^+} \int_M K(t, x, y)f(y)dy = f(x)$ for any compactly supported function f on M.*

The heat kernel exists and is unique for compact Riemannian manifolds. Its importance stems from the fact that the solution to the heat equation

$$\frac{\partial u}{\partial t} + \Delta(u) = 0,$$

$$u : [0, \infty) \times M \to \mathbb{R},$$

(where Δ is the Laplacian with respect to the second variable) with initial condition $u(0, x) = f(x)$ is given by

$$u(t, x) = \int_M K(t, x, y)f(y)dy. \tag{6.2}$$

If $\{\lambda_i\}$ is the spectrum of M and $\{\zeta_i\}$ are the associated eigenfunctions (normalized so that they form an orthonormal basis of $L^2(M)$), then we can write

$$K(t,x,y) = \sum_i e^{-\lambda_i t}\zeta_i(x)\zeta_i(y).$$

From this, it is clear that the heat trace,

$$Z(t) = \sum_i e^{-\lambda_i t},$$

is a spectral invariant. The heat trace has an asymptotic expansion as $t \to 0+$:

$$Z(t) = (4\pi t)^{dim(M)/2} \sum_{j=1}^{\infty} a_j t^j,$$

where the a_j are integrals over M of universal homogeneous polynomials in the curvature and its covariant derivatives ([MP], see [G] or [CPR] for details). The first few of these are

$$a_0 = vol(M),$$

$$a_1 = \frac{1}{6}\int_M \tau,$$

$$a_2 = \frac{1}{360}\int_M (5\tau^2 - 2|\rho|^2 - 10|R|^2),$$

where $\tau = \sum_{a,b=1}^{dim(M)} R_{abab}$ is the scalar curvature, $\rho = \sum_{c=1}^{dim(M)} R_{acbc}$ is the Ricci tensor, and R is the curvature tensor. The dimension, the volume, and the total scalar curvature are thus completely determined by the spectrum. If M is a surface, then the Gauss-Bonnet Theorem implies that the Euler characteristic of M is also a spectral invariant.

The importance of $K(t,x,y)$ in studying the spectrum of Δ derives largely from the following theorem by Minakshisundaram [MP]:

Theorem 6.1.2. *The fundamental solution of the heat equation has an asymptotic expansion in a neighbourhood of the diagonal in $M \times M$:*

$$K(t,x,y) \sim (4\pi t)^{dim(M)/2} e^{-r^2/4t} (\sum_{j=0}^{\infty} u_j(x,y) t^j) \text{ as } t \downarrow 0,$$

where $r = d(x,y)$ is the Riemannian distance from x to y.

The $u_j(x,y)$ are smooth functions. Fix x and suppose y is in some normal coordinate neighbourhood w^i of x. Then the $u_j(x,y)$ are given recursively by

$$u_0(x,y) = \theta^{-1/2}, \tag{6.3}$$

$$u_j(x,y) = -r^{-j}\theta^{-1/2} \int_0^r \theta^{1/2}(x,y_s)\Delta_y(u_{j-1}(x,y))s^{j-1}ds, \tag{6.4}$$

where the integration is along the geodesic y_s joining x to y, and θ is defined by $dvol = \theta dw$. It is well known that $\theta = (det(g_{ij}))^{1/2}$ where $g_{ij} = g(\partial/\partial w^i, \partial/\partial w^j)$.

A more in depth study of the heat trace can yield more information. It is known for example that if M is a closed, connected Riemannian manifold of dimension $n \leq 6$, and if M has the same spectrum as the n-sphere $\mathbb{S}^n$ with the standard metric (resp. $\mathbb{R}^n$), then M is in fact isometric to $\mathbb{S}^n$ (resp. $\mathbb{R}^n$). More on this can be found in [CPR]. There are other invariants besides those mentioned above. For generic closed Riemannian manifolds for example, the geodesic length spectrum - the set of lengths of closed geodesics - is a spectral invariant [C].

6.1.1 Heat Trace Results for Orbifolds

In the case of a *good* Riemannian orbifold, Donnelly [D] proved the existence of the heat kernel and also proved the following results:

Theorem 6.1.3. *Let $f : M \to M$ be an isometry of a manifold M, with fixed point set Ω. Then there is an asymptotic expansion as $t \downarrow 0$*

$$\sum_{\lambda} Tr(f_{\lambda}{}^{\sharp})e^{t\lambda} \approx \sum_{N\in\Omega}(4\pi t)^{-n/2}\sum_{k=0}^{\infty} t^k \int_N b_k(f,a)dvol_N(a),$$

where N is a subset of Ω (and a submanifold of M), λ is an eigenvalue of Δ, $f_{\lambda}{}^{\sharp}$ is a linear map from λ-eigenspace to itself induced by f, and the functions $b_k(f,a)$ depend only on the germ of f and the Riemannian metric of M near the points $a \in N$.

Theorem 6.1.4. *The coefficients $b_k(f,a)$ are of the form $b_k(f,a) = |detB|b_k^{'}(f,a)$ where $b_k^{'}(f,a)$ is an invariant polynomial in the components of $B = (I-A)^{-1}$ (where A is defined in the following remarks no. 2) and the curvature tensor R and its covariant derivatives at a. In particular,*

$$b_0(f,a) = |detB| \tag{6.5}$$

$$b_1(f,a) = |detB|(\frac{\tau}{6} + \frac{1}{6}\rho_k k + \frac{1}{3}R_{iksh}B_{ki}B_{h3} + \frac{1}{3}R_{ikth}B_{kt}B_{hi} - R_{k\alpha h\alpha}B_{ks}B_{hs}). \tag{6.6}$$

We will summarise the tools used by Donnelly to prove the above results in the following remarks:

Remarks and Notation I:

1. Suppose x is a point in a normal coordinate system on the Riemannian manifold M. Suppose $N \subset M$ is a totally geodesic submanifold and let $\pi : (TN)^{\perp} \to N$ be the normal bundle of N. Denote by F the fiber of $\pi : (UN) \to N$ such that

$\pi(x) = a \in N$, then the factor $\psi(x)$ by which the exponential map blows up the volume, i.e. such that $d\mathrm{vol}_{UN}(x) = \psi(x)exp*(d\mathrm{vol}_M(x))$ has the Taylor series expansion

$$\psi(x) = 1 - \frac{1}{2}R_{i\alpha j\alpha}x^i x^j - \frac{1}{6}R_{ikjk}x^i x^j + O(x^3).$$

2. A denotes the endomorphism induced by f on the fiber of the normal bundle over $x \in N$ which is a connected sub-manifold of dimension n in Ω.

3. Donnelly used the fact that $det(I - A) \neq 0$ to make the change of variables $\bar{x} = x - f(x)$. Then using the classical Morse Lemma, he found a smooth coordinate change so that

$$d(\bar{x} + f(x), f(x))^2 = \sum_{i=1}^{s} y_i^2 = ||y||^2.$$

With this change of coordinates, the Taylor series expansion for $\psi(x)$ becomes

$$\psi(x) = 1 - \frac{1}{2}R_{k\alpha h\alpha}B_{ks}B_{ht}y^s y^t - \frac{1}{6}R_{kihi}B_{ks}B_{ht}y^s y^t + O(y^3), \tag{6.7}$$

and the absolute value of the Jacobian determinant of this change of variables has the Taylor series expansion

$$|J(\bar{x}, y)| = 1 + \frac{1}{6}(R_{ikih}B_{ks}B_{ht} + R_{iksh}B_{ki}B_{ht} + R_{ikth}B_{ks}B_{hi})y^s y^t + O(y^3). \tag{6.8}$$

Also, the Taylor series expansion for $u_0(f(x), x)$ is given by

$$u_0(f(x), x) = 1 + \frac{1}{12}\rho_{kh}y^k y^h + O(y^3). \tag{6.9}$$

4. It is shown that

$$b_k(f,a) = \sum_{j=0}^{k} \frac{1}{j!} \Delta_y^j (|det(B) u_i(f(x),x)| J(\bar{x},y)|\psi(x))(0), \qquad (6.10)$$

where

$$\triangle_y = \sum \frac{\partial^2}{\partial y_i^2}.$$

In [DGGW] Donnelly's work is extended to general compact orbifolds, where the heat invariants are expressed in a form that clarifies the asymptotic contributions of each part of the singular set of the orbifold. We will summarise the construction used in [DGGW] in the following remarks before stating their main theorem.

Remarks and Notation II:

1. An Orbifold O was identified with the orbit space $F(O)/O(n)$, where $F(O)$ - a smooth manifold - is the orthonormal frame bundle of O and *O(n)* is the orthogonal group, acting smoothly on the right and preserving the fibers. It can be shown that the action of *O(n)* on the frame bundle *F(O)* gives rise to a (Whitney) stratification of O. The strata are connected components of the isotropy equivalence classes in O. The set of regular points of O intersects each connected component O_0 of O in a single stratum that constitutes an open dense submanifold of O_0. We refer to the strata of O as O-strata.

2. If $(\tilde{U}, G_U, \pi_U)$ is an orbifold chart on O, then it can be shown that the action of G_U on $\tilde{U}$ gives rise to stratifications both of $\tilde{U}$ and of U. These are referred to as $\tilde{U}$-strata and U-strata, respectively.

3. Let O be a Riemannian orbifold and $(\tilde{U}, G_U, \pi_U)$ an orbifold chart. Let $\tilde{N}$ be a $\tilde{U}$-stratum in $\tilde{U}$. Then it can be shown that all the points in $\tilde{N}$ have the

same isotropy group in G_U; this group is referred to as the isotropy group of $\tilde{N}$, denoted $Iso(\tilde{N})$.

4. Given a $\tilde{U}$-stratum $\tilde{N}$, denote by $Iso^{max}(\tilde{N})$ the set of all $\gamma \in Iso(\tilde{N})$ such that $\tilde{N}$ is open in the fixed point set $Fix(\gamma)$ of γ. For $\gamma \in G_U$, it can be shown that each component W of the fixed point set $Fix(\gamma)$ of γ (equivalently, the fixed point set of the cyclic group generated by γ) is a manifold stratified by a collection of $\tilde{U}$-strata, and the strata in W of maximal dimension are open and their union has full measure in W. In particular, the union of those $\tilde{U}$-strata $\tilde{N}$ for which $\gamma \in Iso^{max}(\tilde{N})$ has full measure in $Fix(\gamma)$.

5. Let γ be an isometry of a Riemannian manifold M and let $\Omega(\gamma)$ denote the set of components of the fixed point set of γ. Each element of $\Omega(\gamma)$ is a submanifold of M. For each non-negative integer k, Donnelly [D] defined a real-valued function (cited above), which we temporarily denote $b_k((M,\gamma),.)$, on the fixed point set of γ. For each $W \in \Omega(\gamma)$, the restriction of $b_k((M,\gamma),.)$ to W is smooth. Two key properties of the b_k are:

   (a) *Locality.* For $a \in W$, $b_k((M,\gamma),a)$ depends only on the germs at a of the Riemannian metric of M and of the isometry γ. In particular, if U is a γ-invariant neighborhood of a in M, then $b_k((M,\gamma),a) = b_k((U,\gamma),a)$.

   (b) *Universality.* If M and M' are Riemannian manifolds admitting the respective isometries γ and γ', and if $\sigma : M \to M'$ is an isometry satisfying $\sigma \circ \gamma = \gamma' \circ \sigma$, then $b_k((M,\gamma),x) = b_k((M',\gamma'),\sigma(x))$ for all $x \in Fix(\gamma)$.

   In view of the locality property, we will usually delete the explicit reference to M and rewrite these functions as $b_k(\gamma,.)$, as they are written in [D].

6. Let O be an orbifold and let $(\tilde{U}, G_U, \pi_U)$ be an orbifold chart. Let $\tilde{N}$ be a

$\tilde{U}$-stratum and let $\gamma \in Iso^{max}(\tilde{N})$. Then $\tilde{N}$ is an open subset of a component of $Fix(\gamma)$ and thus, $b_k(\gamma, .)(= b_k((\tilde{U}, \gamma), .))$ is smooth on $\tilde{N}$ for each nonnegative integer k. Define a function $b_k(\tilde{N}, .)$ on $\tilde{N}$ by

$$b_k(\tilde{N}, x) = \sum_{\gamma \in Iso^{max}(\tilde{N})} b_k(\gamma, x).$$

Definition 6.1.5. *Let O be a Riemannian orbifold and let N be an O-stratum.*

(i) For each nonnegative integer k, define a real-valued function $b_k(N, .)$ by setting $b_k(N, p) = b_k(\tilde{N}, \tilde{p})$ where $(\tilde{U}, G_U, \pi_U)$ is any orbifold chart about p, $\tilde{p} \in {\pi_U}^{-1}(p)$, and $\tilde{N}$ is the $\tilde{U}$-stratum through $\tilde{p}$.

(ii) The Riemannian metric on O induces a Riemannian metric - and thus a volume element - on the manifold N. Set

$$I_N := (4\pi t)^{-dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N, x) dvol_N(x),$$

where $dvol_N$ is the Riemannian volume element.

(iii) Set

$$I_0 = (4\pi t)^{-dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k,$$

where the $a_k(O)$ (which we will usually write simply as a_k) are the familiar heat invariants. In particular, $a_0 = vol(O)$, $a_1 = \frac{1}{6} \int_O \tau(x) dvolO(x)$, and so forth. Observe that if O is finitely covered by a Riemannian manifold M (say, $O = G \backslash M$) then $a_k(O) = \frac{1}{|G|} a_k(M)$.

We now state the theorem that [DGGW] proved:

Theorem 6.1.6. *Let O be a Riemannian orbifold and let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the spectrum of the associated Laplacian acting on smooth functions on O. The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of O is asymptotic as $t \to 0^+$ to*

$$I_0 + \sum_{N \in S(O)} \frac{I_N}{|Iso(N)|},$$

where $S(O)$ is the set of all O-strata, $|Iso(N)|$ is the order of the isotropy at each $p \in N$, and $Iso(p)$ is the conjugacy class of subgroups of $O(n)$. This asymptotic expansion is of the form

$$(4\pi t)^{-dim(O)/2} \sum_{j=0}^{\infty} c_j t^{j/2}$$

for some constants c_j .

Using the above theorem, [DGGW] proved the following results for surfaces:

1. Within the class of all footballs (bad or good) and all teardrops, the spectrum distinguishes footballs from teardrops and determines the orders of the cone points. Roughly speaking, a *pq*-football is topologically homeomorphic to a 2-sphere with two isolated cone points of order p and q respectively; locally the singular points are homeomorphic to $\mathbb{R}^2/\mathbb{Z}_p$ and $\mathbb{R}^2/\mathbb{Z}_q$ respectively. A p-teardrop is topologically homeomorphic to a 2-sphere with a single cone point of order p, which is locally homeomorphic to $\mathbb{R}^2/\mathbb{Z}_p$.

2. The spectrum distinguishes teardrops and footballs from triangular pillows with positive Euler characteristic. Roughly speaking, a triangular pillow is a two dimensional orbifold with three isolated cone points where the orbifold is locally covered by $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$ and the group action is either by a cyclic group or a dihedral group. For definitions and notation for footballs, teardrops and triangular pillows see [Co].

3. The spectrum distinguishes triangular pillows with positive Euler characteristic from triangular pillows with negative Euler characteristic.

4. The spectrum distinguishes teardrops from triangular pillows with negative Euler characteristic.

5. If C is the class consisting of all closed orientable 2-orbifolds with non-negative Euler characteristic, then the spectrum distinguishes the elements of C from smooth oriented closed surfaces.

6. Within the class of all closed 2-orbifolds with non-negative Euler characteristic, the spectrum distinguishes whether the orbifold has zero or positive Euler characteristic.

7. Within the class of closed 2-orbifolds of constant nonzero curvature R or -R, the spectrum determines the sign of the curvature - that is, whether the orbifold is spherical or hyperbolic.

8. Within the class of spherical 2-orbifolds of constant curvature $R > 0$, the spectrum determines the orbifold.

It is also known [Sa] that

$$u_2(a, a) = \frac{1}{120}\{\tau^2 + 2|\rho|^2\},$$

and the Taylor series expansion for $u_1(f(x), x)$ is given by

$$u_1(f(x), x) = W + \frac{1}{2}W_k y^k + \frac{1}{3}[W_{kh} + \frac{1}{6}W\rho_{kh}]y^k y^h + O(y^3),$$

where $W = \frac{1}{6}\tau$, $W_k = \frac{1}{6}\tau_{;k}$ and

$$W_{kh} = \frac{1}{5!}\{9\tau_{;kh} + 3\rho_{kh;uu} + \frac{5}{3}\tau\rho_{kh} - 4\rho_{ku}\rho_{hu} + 2\rho_{uv}R_{kuhv} + 2R_{kuvw}R_{huvw}\}.$$

6.2 Three Dimensional Lens Spaces

We define the normal coordinates for a three-sphere as follows [Iv]: Consider a three-sphere of radius r,

$$\mathbb{S}^3(r) = \{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : (v_1)^2 + (v_2)^2 + (v_3)^2 + (v_4)^2 = r^2\},$$

and let (R, ψ, θ, ϕ) be the spherical coordinates in $\mathbb{R}^4$ where $R \in (0, \infty)$, $\psi \in [0, 2\pi]$, $\theta \in (0, \pi]$ and $\phi \in (0, \pi]$. These coordinates are connected with the standard coordinate system (u_1, u_2, u_3, u_4) in $\mathbb{R}^4$ by the following equations:

$$\begin{aligned} u_1 &= R \sin\psi \sin\theta \cos\phi, \\ u_2 &= R \sin\psi \sin\theta \sin\phi, \\ u_3 &= R \sin\psi \cos\theta, \\ u_4 &= R \cos\psi. \end{aligned} \tag{6.11}$$

The equation of $\mathbb{S}^3(r)$ in these coordinates is $R^2 = r^2$. The functions $x_1 = \psi$, $x_2 = \theta$, and $x_3 = \phi$ provide an internal coordinate system on $\mathbb{S}^3(r)$ (without one point) in which the metric g induced on $\mathbb{S}^3(r)$ from $\mathbb{E}^3$ has components g_{ij} such that

$$(g_{ij}) = \begin{pmatrix} r^2 & & \text{\huge 0} \\ & r^2 \sin^2 \psi & \\ \text{\huge 0} & & r^2 \sin^2 \psi \sin^2 \theta \end{pmatrix}.$$

g induces on $\mathbb{S}^3(r)$ a Riemannian connection $\bigtriangledown$. Using the formula

$$\Gamma^m_{ij} = \frac{1}{2} g^{ml} [\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}],$$

we can calculate the Christoffel symbols, which are as follows:

$\Gamma^2_{21} = \Gamma^2_{12} = \cot\psi$, $\Gamma^3_{31} = \Gamma^3_{13} = \cot\psi$, $\Gamma^3_{32} = \Gamma^3_{23} = \cot\theta$, $\Gamma^1_{22} = -\sin\psi\cos\psi$, $\Gamma^1_{33} = -\sin\psi\cos\psi\sin^2\theta$, $\Gamma^2_{33} = -\sin\theta\cos\theta$. All the other symbols are zero.

Now let $\gamma : [0, 2\pi] \to \mathbb{S}^3(r)$ be a path in $\mathbb{S}^3(r)$ such that $x_i \circ \gamma = \pi/2$ for $i = 1, 2$ and $x_3 \circ \gamma = id|_{[0,2\pi]}$. Since $\cos\pi/2 = \cot\pi/2 = 0$ and $\sin\pi/2 = 1$ we have $\Gamma^i_{jk}|_{\gamma([0,2\pi])} = 0$, and consequently, if we take $R = r = 1$, we get $g_{ij} = \delta^j_i$. Therefore, the coordinate system $\{x_1, x_2, x_3\}$ and the frame $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3\}$ are normal for $\bigtriangledown$ along the path γ.

From the Equations (6.11) it is clear that the set $\gamma([0, 2\pi])$ is a circle obtained by intersecting $\mathbb{S}^3(r)$ with the $(v_1, v_2)-$plane $\{v \in \mathbb{R}^4 : v_i(p) = 0 \text{ for} i \geq 3\}$ in $\mathbb{R}^4$. In fact, we have

$$\gamma([0, 2\pi]) = \{(v_1, v_2, 0, 0) \in \mathbb{R}^4 : v_1^2 + v_2^2 = r^2\} = \mathbb{S}^1(r) \times (0, 0).$$

It is clear if C is a circle on $\mathbb{S}^3(r)$ obtained by intersecting $\mathbb{S}^3(r)$ by a 2-plane through its origin then there are coordinates on $\mathbb{S}^3(r)$ normal along C for the Riemannian connection considered above.

We will assume $r = 1$. Then, using the above normal coordinate system, and the formulas

$$R^i_{jlm} = \partial_l \Gamma^i_{mj} - \partial_m \Gamma^i_{lj} + \Gamma^k_{mj}\Gamma^i_{lk} - \Gamma^k_{lj}\Gamma^i_{km},$$

$$R_{abcd} = g_{aj} R^j_{bcd}$$

we calculate the values of the curvature as follows:

$$R_{1212} = R_{\psi\theta\psi\theta} = \sin^2\psi,$$
$$R_{1313} = R_{\psi\phi\psi\phi} = \sin^2\psi \sin^2\theta,$$
$$R_{2323} = R_{\theta\phi\theta\phi} = \sin^4\psi \sin^2\theta.$$

All other values are zero. The values of the Ricci tensor, calculated by $\rho_{ab} = R^c_{acb}$, are as follows:

$$\rho_{11} = \rho_{\psi\psi} = 2,$$
$$\rho_{22} = \rho_{\theta\theta} = 2\sin^2\psi,$$
$$\rho_{33} = \rho_{\phi\phi} = 2\sin^2\psi \sin^2\theta.$$

All other values are zero. We then calculate the scalar curvature as follows:

$$\tau = g^{\psi\psi}\rho_{\psi\psi} + g^{\theta\theta}\rho_{\theta\theta} + g^{\phi\phi}\rho_{\phi\phi} = 6.$$

Since τ is constant all its covariant derivatives, $\tau_{;j}$ are zero. Using $\rho_{ab;m} = \partial_m \rho_{ab} - \rho_{lb}\Gamma^l_{ma} - \rho_{al}\Gamma^l_{mb}$, we also calculate all the covariant derivatives of the Ricci tensor, which turn out to be zero as well.

Let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$ be the standard basis in $\mathbb{R}^4$. We define the following two subsets:

$$N_a = \Big\{(x, y, 0, 0) : x^2 + y^2 = 1\Big\} \subset \mathbb{R}^4 \text{ and } N_b = \Big\{(0, 0, z, w) : z^2 + w^2 = 1\Big\} \subset \mathbb{R}^4.$$

The tangent space $T_{e_1}\mathbb{S}^3$, has basis vectors $\{e_2, e_3, e_4\}$ such that $\{e_2\}$ is a basis for $T_{e_1}N_a$ and $\{e_3, e_4\}$ is a basis for $T_{e_1}N_a^{\perp}$. Similarly, the tangent space $T_{e_4}\mathbb{S}^3$, has basis vectors $\{e_1, e_2, e_3\}$ such that $\{e_3\}$ is a basis for $T_{e_4}N_b$ and $\{e_1, e_2\}$ is a basis for $T_{e_4}N_b^{\perp}$. We will now calculate the values for $b_0(f, a)$, $b_1(f, a)$ and $b_2(f, a)$. Suppose $O = \mathbb{S}^3/G$ is an orbifold lens space where $G =< \gamma >$ and

$$\gamma = \begin{pmatrix} \cos\frac{2\hat{p_1}\pi}{q} & \sin\frac{2\hat{p_1}\pi}{q} & 0 & 0 \\ -\sin\frac{2\hat{p_1}\pi}{q} & \cos\frac{2\hat{p_1}\pi}{q} & 0 & 0 \\ 0 & 0 & \cos\frac{2\hat{p_2}\pi}{q} & \sin\frac{2\hat{p_2}\pi}{q} \\ 0 & 0 & -\sin\frac{2\hat{p_2}\pi}{q} & \cos\frac{2\hat{p_2}\pi}{q} \end{pmatrix},$$

where $\hat{p_1} \not\equiv \pm\hat{p_2} \pmod q$. Suppose $gcd(\hat{p_1}, q) = q_1$ and $gcd(\hat{p_2}, q) = q_2$, so that $\hat{p_1} = p_1q_1$, $\hat{p_2} = p_2q_2$ and $q = \hat{\alpha}q_1 = \hat{\beta}q_2$. Suppose $gcd(\hat{\alpha}, \hat{\beta}) = g$ so that $\hat{\alpha} = \alpha g$, $\hat{\beta} = \beta g$ and $gcd(\alpha, \beta) = 1$. This means we can write γ as

$$\gamma = \begin{pmatrix} \cos\frac{2p_1\pi}{\alpha g} & \sin\frac{2p_1\pi}{\alpha g} & 0 & 0 \\ -\sin\frac{2p_1\pi}{\alpha g} & \cos\frac{2p_1\pi}{\alpha g} & 0 & 0 \\ 0 & 0 & \cos\frac{2p_2\pi}{\beta g} & \sin\frac{2p_2\pi}{\beta g} \\ 0 & 0 & -\sin\frac{2p_2\pi}{\beta g} & \cos\frac{2p_2\pi}{\beta g} \end{pmatrix}.$$

Now

$$\gamma^{\hat{\alpha}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{2p_2\pi\alpha}{\beta} & \sin\frac{2p_2\pi\alpha}{\beta} \\ 0 & 0 & -\sin\frac{2p_2\pi\alpha}{\beta} & \cos\frac{2p_2\pi\alpha}{\beta} \end{pmatrix}$$

fixes N_a, and

$$\gamma^{\hat{\beta}} = \begin{pmatrix} \cos\frac{2p_1\pi\beta}{\alpha} & \sin\frac{2p_1\pi\beta}{\alpha} & 0 & 0 \\ -\sin\frac{2p_1\pi\beta}{\alpha} & \cos\frac{2p_1\pi\beta}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

fixes N_b.

Note that since the group action is transitive and the fixed point sets are $\mathbb{S}^1$, the functions $b_k(.,.)$ are constant along these fixed circles. Therefore, it suffices to consider just a single point in these fixed point sets to calculate the values of the functions. We will choose the points $e_1 \in N_a$ and $e_4 \in N_b$ to calculate the values of functions.

We have, in the notation of the Theorem 6.1.6, $\tilde{N}_a \cong \mathbb{S}^1 \times \{(0,0)\}$ and $\tilde{N}_b \cong \{(0,0)\} \times \mathbb{S}^1$.

Also, $Iso_{N_a} = \{1, \gamma^{\hat{\alpha}}, \gamma^{2\hat{\alpha}}, ...\gamma^{(\beta-1)\hat{\alpha}}\}$, $|Iso_{N_a}| = \beta$, $Iso_{N_b} = \{1, \gamma^{\hat{\beta}}, \gamma^{2\hat{\beta}}, ...\gamma^{(\alpha-1)\hat{\beta}}\}$ and $|Iso_{N_b}| = \alpha$.

We now use Theorem 6.1.6 to calculate the heat trace asymptotic for O using the formula $I_0 + \frac{I_{N_a}}{\beta} + \frac{I_{N_b}}{\alpha}$ where

$$I_0 = (4\pi t)^{-dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k = (4\pi t)^{-dim(O)/2} \sum_{k=0}^{\infty} \frac{1}{|G|} a_k(\mathbb{S}^3) t^k$$

$$= \frac{(4\pi t)^{-3/2}}{q} \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{4k!} t^k = \frac{(4t)^{-3/2}}{4q\pi} \sum_{k=0}^{\infty} \frac{t^k}{k!} = \frac{t^{-3/2}}{32q\pi} e^t,$$

and for $i \in a, b$,

$$\begin{aligned} I_{N_i} &= (4\pi t)^{-dim(N_i)/2} \sum_{k=0}^{\infty} t^k \int_{N_i} b_k(N_i, x) dvol_{N_i}(x) \\ &= \frac{(\pi t)^{-1/2}}{2} \sum_{k=0}^{\infty} t^k \int_{\tilde{N}_i} b_k(\tilde{N}_i, x) dvol_{\tilde{N}_i}(x), \text{ since } \tilde{N}_i \to N_i \text{ is trivial in this case} \\ &= \frac{(\pi t)^{-1/2}}{2} \sum_{k=0}^{\infty} t^k 2\pi b_k(\tilde{N}_i, x) \text{ (for any choice of } x \text{ by homogeneity)} \\ &= \sqrt{\pi} t^{-1/2} \sum_{k=0}^{\infty} t^k b_k(\tilde{N}_i, x) \text{ , where } b_k(\tilde{N}_i, x) = \sum_{\gamma \in Iso^{max} \tilde{N}_i} b_k(\gamma, x). \end{aligned}$$

Now for $a = e_1$ and $r \in \{1, 2, ...(\beta - 1)\}$,

$$A_{\gamma^r \hat{\alpha}}(a) = \begin{pmatrix} \cos \frac{2p_2\pi\alpha r}{\beta} & \sin \frac{2p_2\pi\alpha r}{\beta} \\ \\ -\sin \frac{2p_2\pi\alpha r}{\beta} & \cos \frac{2p_2\pi\alpha r}{\beta} \end{pmatrix},$$

$$I - A_{\gamma^r \hat{\alpha}}(a) = \begin{pmatrix} 1 - \cos \frac{2p_2\pi\alpha r}{\beta} & -\sin \frac{2p_2\pi\alpha r}{\beta} \\ \\ \sin \frac{2p_2\pi\alpha r}{\beta} & 1 - \cos \frac{2p_2\pi\alpha r}{\beta} \end{pmatrix},$$

$$B_{\gamma^r \hat{\alpha}}(a) = (I - A_{\gamma^r \hat{\alpha}}(a))^{-1} = \frac{1}{4\sin^2 \frac{p_2\pi\alpha r}{\beta}} \begin{pmatrix} 1 - \cos \frac{2p_2\pi\alpha r}{\beta} & -\sin \frac{2p_2\pi\alpha r}{\beta} \\ \\ \sin \frac{2p_2\pi\alpha r}{\beta} & 1 - \cos \frac{2p_2\pi\alpha r}{\beta} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -\cot \frac{p_2\pi\alpha r}{\beta} \\ \\ \cot \frac{p_2\pi\alpha r}{\beta} & 1 \end{pmatrix}.$$

So, $|det B_{\gamma^r \hat{\alpha}}(a)| = \frac{1}{4}(1 + \cot^2 \frac{p_2\pi\alpha r}{\beta}) = \frac{1}{4\sin^2 \frac{p_2\pi\alpha r}{\beta}}$.

Similarly we can show that for $b = e_4$ and $r \in \{1, 2, ...(\alpha - 1)\}$,

$$B_{\gamma^r \hat{\beta}}(b) = \frac{1}{2} \begin{pmatrix} 1 & -\cot \frac{p_1\pi\beta r}{\alpha} \\ \\ \cot \frac{p_1\pi\beta r}{\alpha} & 1 \end{pmatrix},$$

and $|det B_{\gamma^r \hat{\beta}}(b)| = \frac{1}{4}(1 + \cot^2 \frac{p_1\pi\beta r}{\alpha}) = \frac{1}{4\sin^2 \frac{p_1\pi\beta r}{\alpha}}$.

We will now calculate $b_i(\tilde{N}_j, .)$ for $i = 0, 1, 2$ and $j = a, b$:

$$b_0(\gamma^{r\hat{\alpha}}, a) = |detB_{\gamma^{r\hat{\alpha}}}(a)| = \frac{1}{4}(1 + \cot^2 \frac{p_2\pi\alpha r}{\beta}) = \frac{1}{4\sin^2 \frac{p_2\pi\alpha r}{\beta}}.$$

So,

$$\begin{aligned}
b_0(\tilde{N}_a, a) &= \sum_{f \in Iso^{max}\tilde{N}_a} b_0(f, a) \\
&= \sum_{r=1}^{\beta-1} b_0(\gamma^{r\hat{\alpha}}, a) \\
&= \sum_{r=1}^{\beta-1} \frac{1}{4}(1 + \cot^2 \frac{p_2\pi\alpha r}{\beta}) \\
&= \sum_{r=1}^{\beta-1} \frac{1}{4}(1 + \cot^2 \frac{\pi r}{\beta}) \text{ , since } gcd(p_2\alpha, \beta) = 1 \\
&= \sum_{r=1}^{\beta-1} \frac{1}{4\sin^2 \frac{\pi r}{\beta}} \\
&= \frac{\beta^2 - 1}{12} \text{ , by lemma 5.4 in [DGGW].}
\end{aligned}$$

We can similarly show that

$$b_0(\tilde{N}_b, b) = \sum_{r=1}^{\alpha-1} \frac{1}{4}(1 + \cot^2 \frac{\pi r}{\alpha}) = \frac{\alpha^2 - 1}{12}.$$

We will now calculate $b_1(\tilde{N}_a, a)$ and $b_1(\tilde{N}_b, b)$. Note that for both $B_{\gamma^{r\hat{\alpha}}}(a)$ and $B_{\gamma^{r\hat{\beta}}}(b)$, $B_{13} = B_{23} = B_{31} = B_{32} = B_{33} = 0$. Using the formula in Theorem 6.1.4, we get

$$\begin{aligned} b_1(\gamma^{r\hat{\alpha}}, a) = \frac{|det(B_{\gamma^{r\hat{\alpha}}}(a))|}{3} \Big\{ & R_{1212}\Big[2 - \frac{1}{4}(\cot\theta_r - \cot\theta_r)^2 - (\frac{1}{2} + \frac{1}{2})^2 - 2((\frac{1}{4} + \frac{1}{4})\Big] \\ & + R_{1313}\Big[2 - (\frac{1}{2} + 0)^2 - 2(\frac{1}{4} + 0) - 3(\frac{1}{4}\cot^2\theta_r + 0)\Big] \\ & + R_{2323}\Big[2 - (\frac{1}{2} + 0)^2 - 2(\frac{1}{4} + 0) - 3(\frac{1}{4}\cot^2\theta_r + 0)\Big]\Big\}, \end{aligned}$$

which gives

$$\begin{aligned} b_1(\gamma^{r\hat{\alpha}}, a) &= \frac{1}{12}(1 + \cot^2\theta_r)\Big\{R_{1313}\Big(2 - \frac{3}{4} - \frac{3}{4}\cot^2\theta_r)\Big) + R_{2323}\Big(2 - \frac{3}{2} - \frac{3}{4}\cot^2\theta_r)\Big)\Big\} \\ &= \frac{1}{12}(1 + \cot^2\theta_r)(R_{1313} + R_{2323})[2 - \frac{3}{4}(1 + \cot^2\theta_r)] \\ &= (R_{1313} + R_{2323})\Big[\frac{1}{6}(1 + \cot^2\theta_r) - \frac{1}{16}(1 + \cot^2\theta_r)^2\Big] \\ &= (R_{1313} + R_{2323})\Big[\frac{1}{6\sin^2\theta_r} - \frac{1}{16\sin^2\theta_r}\Big], \end{aligned}$$

where $\theta_r = \frac{p_2\pi\alpha r}{\beta}$.

So,

$$\begin{aligned} b_1(\tilde{N}_a, a) &= \sum_{r=1}^{\beta-1} b_1(\gamma^{r\hat{\alpha}}, a) \\ &= \sum_{r=1}^{\beta-1}(R_{1313} + R_{2323})\Big[\frac{1}{6\sin^2\frac{p_2\pi\alpha r}{\beta}} - \frac{1}{16\sin^2\frac{p_2\pi\alpha r}{\beta}}\Big] \\ &= (R_{1313} + R_{2323})\Big[\frac{1}{6}\sum_{r=1}^{\beta-1}\frac{1}{\sin^2\frac{\pi r}{\beta}} - \frac{1}{16}\sum_{r=1}^{\beta-1}\frac{1}{\sin^4\frac{\pi r}{\beta}}\Big], \end{aligned}$$

since $gcd(p_2\alpha, \beta) = 1$.

Also, $\sum_{r=1}^{\beta-1} \frac{1}{\sin^2 \frac{\pi r}{\beta}} = \frac{\beta^2-1}{3}$ and $\sum_{r=1}^{\beta-1} \frac{1}{\sin^4 \frac{\pi r}{\beta}} = \frac{\beta^4+10\beta^2-11}{45}$ (see [DGGW]). So we get

$$\begin{aligned} b_1(\tilde{N}_a, a) &= (R_{1313} + R_{2323})\Big(\frac{\beta^2-1}{18} - \frac{\beta^4+10\beta^2-11}{720}\Big) \\ &= -(R_{1313} + R_{2323})\frac{(\beta^2-29)(\beta^2-1)}{720}. \end{aligned}$$

We can similarly show that

$$\begin{aligned} b_1(\tilde{N}_b, b) &= (R_{1313} + R_{2323})\Big(\frac{\alpha^2-1}{18} - \frac{\alpha^4+10\alpha^2-11}{720}\Big) \\ &= -(R_{1313} + R_{2323})\frac{(\alpha^2-29)(\alpha^2-1)}{720}. \end{aligned}$$

We will now calculate $b_2(\tilde{N}_a, a)$ and $b_2(\tilde{N}_b, b)$ using (6.10):

$$\begin{aligned} b_2(f,c) = |det(B)|\Big[&u_2(c,c) + \Delta_y(u_1(f(x),x)\psi(x)|J(\tilde{x},y)|)(0) \\ &+ \frac{1}{2}\Delta_y^2(u_0(f(x),x)\psi(x)|J(\tilde{x},y)|)(0)\Big], \end{aligned}$$

where $\psi(x)$, $|J(\bar{x},y)|$ and $u_0(f(x),x)$are taken from (6.7), (6.8) and (6.9) respectively.

Note that

$$\begin{aligned} &\Delta_y^2(u_0(f(x),x)\psi(x)|J(\bar{x},y)|)(0) \\ =&\Delta_y^2(1 + \frac{1}{12}\rho_{kh}y^k y^h + \frac{1}{6}R_{iksh}B_{ki}B_{ht}y^s y^t \\ &+ \frac{1}{6}R_{ikth}B_{ks}B_{hi}y^s y^t - \frac{1}{2}R_{k\alpha h\alpha}B_{ks}B_{ht}y^s y^t + O(y^3))(0) = 0. \end{aligned}$$

We now calculate $\Delta_y(u_1(f(x),x)\psi(x)|J(\tilde{x},y)|)(0)$, which is

$$\begin{aligned}\Delta_y\Big(&\Big[W+\frac{1}{2}W_k y^k+\frac{1}{3}(W_{kh}+\frac{1}{6}W\rho_{kh})y^k y^h+O(y^3)\Big]\\ &\Big[1-\frac{1}{6}R_{kihi}B_{ks}B_{ht}y^s y^t-\frac{1}{2}R_{k\alpha h\alpha}B_{ks}Bhty^s y^t+O(y^3)\Big]\\ &\Big[1+\frac{1}{6}(R_{ikih}B_{ks}B_{ht}+R_{iksh}B_{ki}B_{ht}+R_{ikth}B_{ks}B_{hi})\Big]\Big)(0),\end{aligned}$$

where $W=\frac{1}{6}\tau$, $W_k=\frac{1}{6}\tau_{;k}$ and

$$W_{kh}=\frac{1}{5!}\{9\tau_{;kh}+3\rho_{kh;uu}+\frac{5}{3}\tau\rho_{kh}-4\rho_{ku}\rho_{hu}+2\rho_{uv}R_{kuhv}+2R_{kuvw}R_{huvw}\}.$$

For lens spaces, we have $\tau=6$, and all the covariant derivatives of τ and ρ_{kh} are zero. So, in this case, we will have $W=1$, $W_k=0$ and

$$W_{kh}=\frac{1}{5!}\{\frac{5}{3}\tau\rho_{kh}-4\rho_{ku}\rho_{hu}+2\rho_{uv}R_{kuhv}+2R_{kuvw}R_{huvw}\}.$$

This gives

$$\begin{aligned}&\Delta_y(u_1(f(x),x)\psi(x)|J(\tilde{x},y)|)(0)\\ &=\Delta_y\Big(\Big[1+\frac{1}{3}(W_{kh}+\frac{1}{6}\rho_{kh})y^k y^h+O(y^3)\Big]\\ &\qquad\Big[1-\frac{1}{6}R_{kihi}B_{ks}B_{ht}y^s y^t-\frac{1}{2}R_{k\alpha h\alpha}B_{ks}Bhty^s y^t+O(y^3)\Big]\\ &\qquad\Big[1+\frac{1}{6}(R_{ikih}B_{ks}B_{ht}+R_{iksh}B_{ki}B_{ht}+R_{ikth}B_{ks}B_{hi})\Big]\Big)(0)\\ &=\Delta_y\Big(1+\frac{1}{3}\Big\{\frac{1}{5!}\Big[10\rho_{kh}-4\rho_{ku}\rho_{hu}+2\rho_{uv}R_{kuhv}+2R_{kuvw}R_{huvw}\Big]+\frac{1}{6}\rho_{kh}\Big\}y^k y^h\\ &\qquad+\frac{1}{6}R_{iksh}B_{ki}B_{ht}y^s y^t+\frac{1}{6}R_{ikth}B_{ks}B_{hi}y^s y^t-\frac{1}{2}R_{k\alpha h\alpha}B_{ks}B_{ht}y^s y^t+O(y^3)\Big)(0)\\ &=\frac{1}{3}\Big\{\frac{1}{5!}\Big[20\rho_{kk}-8\rho_{kk}^2+4\rho_{\alpha\alpha}R_{k\alpha k\alpha}+R_{k\alpha k\alpha}^2\Big]\\ &\qquad+\frac{1}{3}\rho_{kk}+R_{iksh}B_{ki}B_{hs}+R_{ikth}B_{kt}B_{hi}-3R_{k\alpha h\alpha}B_{ks}B_{hs}\Big\}.\end{aligned}$$

So, for $a = e_1$ and $r \in \{1, 2, ...\beta - 1\}$,

$$\begin{aligned} b_2(\gamma^{r\hat{\alpha}}, a) =& |det(B_{\gamma^{r\hat{\alpha}}}(a))| \Big[\frac{1}{120}\tau^2 + \frac{1}{60}|\rho|^2 + \frac{1}{18}\rho_{kk} - \frac{1}{45}\rho_{kk}^2 + \frac{1}{90}\rho_{\alpha\alpha}R_{k\alpha k\alpha} \\ &+ \frac{1}{90}R_{k\alpha k\alpha}^2 + \frac{1}{9}\rho_{kk} + \frac{1}{3}R_{iksh}B_{ki}B_{hs} + \frac{1}{3}R_{ikth}B_{kt}B_{hi} - R_{k\alpha h\alpha}B_{ks}B_{hs} \Big] \end{aligned}$$

gives

$$\begin{aligned} b_2(\gamma^{r\hat{\alpha}}, a) =& |det(B_{\gamma^{r\hat{\alpha}}}(a))| \Big\{ \frac{1}{120}\tau^2 + \frac{1}{60}|\rho|^2 + \frac{1}{18}(\rho_{11} + \rho_{22} + \rho_{33}) \\ &- \frac{1}{90}\Big[2\rho_{11}^2 + 2\rho_{22}^2 + 2\rho_{33}^2 - (\rho_{11} + \rho_{22})R_{1212} - (\rho_{11} + \rho_{33})R_{1313} \\ &- (\rho_{22} + \rho_{33})R_{2323} + 2R_{1212}^2 + 2R_{1313}^2 + 2R_{2323}^2 \Big] \\ &- \frac{1}{3}\Big[R_{1212}[(B_{12} + B_{21})^2 + 2(B_{11}^2 + B_{22}^2) \\ &+ (B_{11} + B_{22})^2] - R_{1313}(B_{11}^2 + B_{12}^2) - R_{2323}(B_{21}^2 + B_{22}^2) \Big] \Big\}. \end{aligned}$$

We denote by Q the following expression:

$$\begin{aligned} &\Big\{ \frac{1}{120}\tau^2 + \frac{1}{60}|\rho|^2 + \frac{1}{18}(\rho_{11} + \rho_{22} + \rho_{33}) \\ &- \frac{1}{90}\Big[2\rho_{11}^2 + 2\rho_{22}^2 + 2\rho_{33}^2 - (\rho_{11} + \rho_{22})R_{1212} - (\rho_{11} + \rho_{33})R_{1313} \\ &- (\rho_{22} + \rho_{33})R_{2323} + 2R_{1212}^2 + 2R_{1313}^2 + 2R_{2323}^2 \Big] \Big\}. \end{aligned}$$

Using this, we can now write

$$b_2(\gamma^{r\hat{\alpha}}, a) = |det(B_{\gamma^{r\hat{\alpha}}}(a))| \Big\{ Q - \frac{1}{3}[2R_{1212} - \frac{1}{4}(1 + \cot^2 \frac{p_2 \pi \alpha r}{\beta})(R_{1313} + R_{2323})] \Big\},$$

and

$$
\begin{aligned}
b_2(\tilde{N}_a, a) =& \sum_{r=1}^{\beta-1} b_2(\gamma^{r\hat{\alpha}}, a) \\
=& \sum_{r=1}^{\beta-1} |det(B_{\gamma^{r\hat{\alpha}}}(a))| \Big\{ Q - \frac{1}{3}[2R_{1212} - \frac{1}{4}(1+\cot^2 \frac{p_2\pi\alpha r}{\beta})(R_{1313}+R_{2323})] \Big\} \\
=& Q \sum_{r=1}^{\beta-1} \frac{1}{4}(1+\cot^2 \frac{p_2\pi\alpha r}{\beta}) - \frac{2}{3} R_{1212} \sum_{r=1}^{\beta-1} \frac{1}{4}(1+\cot^2 \frac{p_2\pi\alpha r}{\beta}) \\
&+ \frac{(R_{1313}+R_{2323})}{3} \sum_{r=1}^{\beta-1} \frac{1}{16}(1+\cot^2 \frac{p_2\pi\alpha r}{\beta})^2 \\
=& \frac{Q}{4} \sum_{r=1}^{\beta-1} (1+\cot^2 \frac{\pi r}{\beta}) - \frac{2}{12} R_{1212} \sum_{r=1}^{\beta-1} (1+\cot^2 \frac{\pi r}{\beta}) \\
&+ \frac{(R_{1313}+R_{2323})}{48} \sum_{r=1}^{\beta-1} (1+\cot^2 \frac{\pi r}{\beta})^2 \\
=& \frac{Q}{4} \sum_{r=1}^{\beta-1} \frac{1}{\sin^2 \frac{\pi r}{\beta}} - \frac{2}{12} R_{1212} \sum_{r=1}^{\beta-1} \frac{1}{\sin^2 \frac{\pi r}{\beta}} + \frac{(R_{1313}+R_{2323})}{48} \sum_{r=1}^{\beta-1} \frac{1}{\sin^4 \frac{\pi r}{\beta}} \\
=& \frac{3Q - 2R_{1212}}{12} \Big(\frac{\beta^2 - 1}{3} \Big) + \frac{R_{1313}+R_{2323}}{48} \Big(\frac{\beta^4 + 10\beta^2 - 11}{45} \Big).
\end{aligned}
$$

Similarly, we can show that for $b = e_4$, for instance,

$$
b_2(\tilde{N}_b, b) = \frac{3Q - 2R_{1212}}{12} \Big(\frac{\alpha^2 - 1}{3} \Big) + \frac{R_{1313}+R_{2323}}{48} \Big(\frac{\alpha^4 + 10\alpha^2 - 11}{45} \Big).
$$

Using Theorem 6.1.6 we now calculate the first few coefficients of the asymptotic expansion as follows:

$$
\begin{aligned}
& I_0 + \frac{I_{N_a}}{|Iso(N_a)|} + \frac{I_{N_b}}{|Iso(N_b)|} \\
=& \frac{t^{-3/2}}{32q\pi}e^t + \frac{(\pi t)^{-1/2}}{\beta}\Big[t^0\pi b_0(\tilde{N}_a, a) + t^1\pi b_1(\tilde{N}_a, a) + t^2\pi b_2(\tilde{N}_a, a) + ...\Big] \\
& + \frac{(\pi t)^{-1/2}}{\alpha}\Big[t^0\pi b_0(\tilde{N}_b, b) + t^1\pi b_1(\tilde{N}_b, b) + t^2\pi b_2(\tilde{N}_b, b) + ...\Big] \\
=& \frac{t^{-3/2}}{32q\pi}(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + ...) + \Big(\frac{b_0(\tilde{N}_a, a)}{\beta} + \frac{b_0(\tilde{N}_b, b)}{\alpha}\Big)\sqrt{\pi}t^{-1/2} \\
& + \Big(\frac{b_1(\tilde{N}_a, a)}{\beta} + \frac{b_1(\tilde{N}_b, b)}{\alpha}\Big)\sqrt{\pi}t^{1/2} + \Big(\frac{b_2(\tilde{N}_a, a)}{\beta} + \frac{b_2(\tilde{N}_b, b)}{\alpha}\Big)\sqrt{\pi}t^{3/2} + ...
\end{aligned}
$$

From this, the coefficient of $t^{-3/2}$ is $\frac{1}{32q\pi}$;

the coefficient of $t^{-1/2}$ is

$$
\frac{1}{32q\pi} + \frac{b_0(\tilde{N}_a, a)}{\beta}\sqrt{\pi} + \frac{b_0(\tilde{N}_b, b)}{\alpha}\sqrt{\pi} = \frac{1}{32q\pi} + \frac{\sqrt{\pi}}{12\beta}(\beta^2 - 1) + \frac{\sqrt{\pi}}{12\alpha}(\alpha^2 - 1);
$$

the coefficient of $t^{1/2}$ is

$$
\frac{1}{64q\pi} - \frac{\sqrt{\pi}(R_{1313} + R_{2323})[\alpha(\beta^2 - 29)(\beta^2 - 1) + \beta(\alpha^2 - 29)(\alpha^2 - 1)]}{720\alpha\beta};
$$

and the coefficient of $t^{3/2}$ is

$$
\begin{aligned}
\frac{1}{192q\pi} + \sqrt{\pi}\Big\{ & \frac{3Q - 2R_{1212}}{36}\Big[\frac{(\alpha^2 - 1)}{\alpha} + \frac{(\beta^2 - 1)}{\beta}\Big] \\
& + \frac{R_{1313} + R_{2323}}{2160}\Big[\frac{\alpha^4 + 10\alpha^2 - 11}{\alpha} + \frac{\beta^4 + 10\beta^2 - 11}{\beta}\Big]\Big\}.
\end{aligned}
$$

The above results show that the coefficients are dependent on α, β and the curvature tensor and its covariant derivatives. Since all lens spaces are finitely covered by $\mathbb{S}^3$, the parts of the coefficients that consist of the curvature tensor and its covariant derivatives will be the same for all lens spaces. The only difference will therefore be in the terms containing α and β. We can rewrite

$$b_0(\tilde{N}_a, a) = \sum_{r=1}^{\beta-1} \frac{1}{4}(1 + \cot^2 \frac{p_2\pi\alpha r}{\beta}) = \sum_{r=1}^{\beta-1} \frac{1}{4} + \sum_{r=1}^{\beta-1} \frac{1}{4} \cot^2 \frac{p_2\pi\alpha r}{\beta},$$

$$b_0(\tilde{N}_b, b) = \sum_{r=1}^{\alpha-1} \frac{1}{4}(1 + \cot^2 \frac{p_1\pi\beta r}{\alpha}) = \sum_{r=1}^{\alpha-1} \frac{1}{4} + \sum_{r=1}^{\alpha-1} \frac{1}{4} \cot^2 \frac{p_1\pi\beta r}{\alpha},$$

$$\begin{aligned}
b_1(\tilde{N}_a, a) &= \sum_{r=1}^{\beta-1} (R_{1313} + R_{2323}) \Big[\frac{1}{6}(1 + \cot^2 \frac{p_2\alpha\pi r}{\beta}) - \frac{1}{16}(1 + \cot^2 \frac{p_2\alpha\pi r}{\beta})^2\Big] \\
&= \sum_{r=1}^{\beta-1} \frac{5(R_{1313} + R_{2323})}{48} + \sum_{r=1}^{\beta-1} \Big(\frac{R_{1313} + R_{2323}}{24}\Big) \cot^2 \frac{p_2\alpha\pi r}{\beta} \\
&\quad - \sum_{r=1}^{\beta-1} \Big(\frac{R_{1313} + R_{2323}}{16}\Big) \cot^4 \frac{p_2\alpha\pi r}{\beta}, \\
b_1(\tilde{N}_b, b) &= \sum_{r=1}^{\alpha-1} (R_{1313} + R_{2323}) \Big[\frac{1}{6}(1 + \cot^2 \frac{p_1\beta\pi r}{\alpha}) - \frac{1}{16}(1 + \cot^2 \frac{p_1\beta\pi r}{\alpha})^2\Big] \\
&= \sum_{r=1}^{\alpha-1} \frac{5(R_{1313} + R_{2323})}{48} + \sum_{r=1}^{\alpha-1} \Big(\frac{R_{1313} + R_{2323}}{24}\Big) \cot^2 \frac{p_1\beta\pi r}{\alpha} \\
&\quad - \sum_{r=1}^{\alpha-1} \Big(\frac{R_{1313} + R_{2323}}{16}\Big) \cot^4 \frac{p_1\beta\pi r}{\alpha},
\end{aligned}$$

$$
\begin{aligned}
b_2(\tilde{N}_a, a) =& \frac{Q}{4}\sum_{r=1}^{\beta-1}(1+\cot^2\frac{p_2\alpha\pi r}{\beta}) - \frac{2}{12}R_{1212}\sum_{r=1}^{\beta-1}(1+\cot^2\frac{p_2\alpha\pi r}{\beta}) \\
&+ \frac{(R_{1313}+R_{2323})}{48}\sum_{r=1}^{\beta-1}(1+\cot^2\frac{p_2\alpha\pi r}{\beta})^2 \\
=& \sum_{r=1}^{\beta-1}\frac{12Q-8R_{1212}+R_{1313}+R_{2323}}{48} \\
&+ \sum_{r=1}^{\beta-1}\Big(\frac{6Q+4R_{1212}+R_{1313}+R_{2323}}{24}\Big)\cot^2\frac{p_2\alpha\pi r}{\beta} \\
&+ \Big(\frac{R_{1313}+R_{2323}}{48}\Big)\cot^4\frac{p_2\alpha\pi r}{\beta}, \\
b_2(\tilde{N}_b, b) =& \frac{Q}{4}\sum_{r=1}^{\alpha-1}(1+\cot^2\frac{p_1\beta\pi r}{\alpha}) - \frac{2}{12}R_{1212}\sum_{r=1}^{\alpha-1}(1+\cot^2\frac{p_1\beta\pi r}{\alpha}) \\
&+ \frac{(R_{1313}+R_{2323})}{48}\sum_{r=1}^{\alpha-1}(1+\cot^2\frac{p_1\beta\pi r}{\alpha})^2 \\
=& \sum_{r=1}^{\alpha-1}\frac{12Q-8R_{1212}+R_{1313}+R_{2323}}{48} \\
&+ \sum_{r=1}^{\alpha-1}\Big(\frac{6Q+4R_{1212}+R_{1313}+R_{2323}}{24}\Big)\cot^2\frac{p_1\beta\pi r}{\alpha} \\
&+ \Big(\frac{R_{1313}+R_{2323}}{48}\Big)\cot^4\frac{p_1\beta\pi r}{\alpha}.
\end{aligned}
$$

Note that each $b_j(\tilde{N}_a, a)$, $(j = 0, 1, 2)$ is of the form

$$
b_j(\tilde{N}_a, a) = \sum_{r=1}^{\beta-1}\sum_{i=1}^{A_j} C^a_{ij}(R)\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta},
$$

where A_j is the finite number of monomials in the powers of $\cot\frac{p_2\alpha\pi r}{\beta}$, and for each i, $C^a_{ij}(R)$ are constant functions in terms of the curvature tensor and its covariant derivatives of the covering space, i.e. the sphere. Since $gcd(p_2\alpha, \beta) = 1$, and we are summing over r as it ranges from 1 to $\beta-1$, we can write

$$b_j(\tilde{N}_a, a) = \sum_{r=1}^{\beta-1}\sum_{i=1}^{A_j} C^a_{ij}(R)\cot^{\lambda_i}\frac{\pi r}{\beta}.$$

Similarly, since $gcd(\alpha, p_1\beta) = 1$, we can write

$$b_j(\tilde{N}_b, b) = \sum_{r=1}^{\alpha-1}\sum_{i=1}^{A_j} C^b_{ij}(R)\cot^{\lambda_i}\frac{\pi r}{\alpha}.$$

More generally, for any k, the functions $b_k(\gamma^{r\hat{\alpha}}, a)$ and $b_k(\gamma^{r\hat{\beta}}, a)$ are universal polynomials in the components of the curvature tensor, its covariant derivatives and the elements of $B_{\gamma^{r\hat{\alpha}}}(a)$ and $B_{\gamma^{r\hat{\beta}}}(b)$ respectively. Since the elements of $B_{\gamma^{r\hat{\alpha}}}(a)$ are $B_{11} = B_{22} = 1/2$, $B_{12} = -\frac{1}{2}\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta}$ and $B_{21} = \frac{1}{2}\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta}$, every $b_k(\gamma^{r\hat{\alpha}}, a)$ will be of the form $\sum_{i=1}^{A_j} C^a_{ij}(R)\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta}$. This means that for each k, we will have,

$$b_k(\tilde{N}_a, a) = \sum_{r=1}^{\beta-1}\sum_{i=1}^{A_k} C^a_{ik}(R)\cot^{\lambda_i}\frac{\pi r}{\beta},$$

and similarly,

$$b_k(\tilde{N}_b, b) = \sum_{r=1}^{\alpha-1}\sum_{i=1}^{A_k} C^b_{ik}(R)\cot^{\lambda_i}\frac{\pi r}{\alpha}.$$

This observation gives us the following lemma:

Lemma 6.2.1. *Given two orbifold lens spaces $O_1 = \mathbb{S}^3/G_1$ and $O_2 = \mathbb{S}^3/G_2$, such that $G_1 =< \gamma_1 >$ and $G_2 =< \gamma_2 >$ where*

$$\gamma_1 = \begin{pmatrix} e^{\frac{2\hat{p_1}\pi i}{q}} & 0 \\ \\ 0 & e^{\frac{2\hat{p_2}\pi i}{q}} \end{pmatrix}$$

with $\hat{p_1} \not\equiv \pm\hat{p_2}\,(mod\, q)$*,* $gcd(\hat{p_1}, q) = q_{11}$*,* $gcd(\hat{p_2}, q) = q_{21}$*,* $\hat{p_1} = p_1 q_{11}$*,* $\hat{p_2} = p_2 q_{21}$*,* $q = \hat{\alpha_1} q_{11} = \hat{\beta}_1 q_{21}$*,* $gcd(\hat{\alpha_1}, \hat{\beta}_1) = g_1$*,* $\hat{\alpha_1} = \alpha_1 g_1$*,* $\hat{\beta}_1 = \beta_1 g_1$*, and*

$$\gamma_2 = \begin{pmatrix} e^{\frac{2\hat{s_1}\pi i}{q}} & 0 \\ & \\ 0 & e^{\frac{2\hat{s_2}\pi i}{q}} \end{pmatrix},$$

with $\hat{s_1} \not\equiv \pm\hat{s_2}\,(mod\, q)$*,* $gcd(\hat{s_1}, q) = q_{12}$*,* $gcd(\hat{s_2}, q) = q_{22}$*,* $\hat{s_1} = s_1 q_{12}$*,* $\hat{s_2} = s_2 q_{22}$*,* $q = \hat{\alpha_2} q_{12} = \hat{\beta}_2 q_{22}$*,* $gcd(\hat{\alpha_2}, \hat{\beta}_2) = g_2$*,* $\hat{\alpha_2} = \alpha_2 g_2$*,* $\hat{\beta}_2 = \beta_2 g_2$*.*
Then $O_1 = \mathbb{S}^3/G_1$ *and* $O_2 = \mathbb{S}^3/G_2$ *will have the exact same asymptotic expansion of the heat kernel if* $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$*.*

This lemma gives us a tool to find examples of 3-dimensional orbifold lens spaces that are non-isometric (hence non-isospectral) but have the exact same asymptotic expansion of the heat kernel.

Example 6.2.2. *Suppose* $q = 195$*, and consider the two lens spaces* $O_1 = L(195 : 3, 5)$ *and* $O_2 = L(195 : 6, 35)$*. Since there is no integer* l *coprime to* 195 *and no* $e_i \in \{1, -1\}$ *such that* $\{e_1 l3, e_2 l5\}$ *is a permutation of* $\{6, 35\}(mod\ q)$*,* O_1 *and* O_2 *are not isometric (and hence non-isospectral). However, in the notation of the lemma above,* $\hat{p_1} = 3$*,* $\hat{p_2} = 5$*,* $\hat{s_1} = 6$*,* $\hat{s_2} = 35$*,* $gcd(\hat{p_1}, q) = 3 = gcd(\hat{s_1}, q)$*,* $gcd(\hat{p_2}, q) = 5 = gcd(\hat{s_2}, q)$ *and* $q = 195 = 3 \times 65 = 5 \times 39$*. So,* $\hat{\alpha_1} = \hat{\alpha_2} = 65$ *and* $\hat{\beta}_1 = \hat{\beta}_2 = 39$*, with* $gcd(\hat{\alpha_i}, \hat{\beta}_i) = 13$ *(for* $i = 1, 2$*) giving* $\alpha_1 = \alpha_2 = 5$ *and* $\beta_1 = \beta_2 = 3$*. Therefore,* $O_1 = L(195 : 3, 5)$ *and* $O_2 = L(195 : 6, 35)$ *have the exact same asymptotic expansion.*

6.3 Four Dimensional Lens Spaces

We define the normal coordinates for a four-sphere as follows [Iv]: Consider a four-sphere of radius r,

$$\mathbb{S}^4(r) = \{(v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5 : (v_1)^2 + (v_2)^2 + (v_3)^2 + (v_4)^2 + (v_5)^2 = r^2\},$$

and let $(R, \psi, \theta, \phi, t)$ be the spherical coordinates in $\mathbb{R}^5$ where $R \in (0, \infty)$, $\psi \in (0, \pi]$, $\theta \in (0, \pi]$, $\phi \in (0, \pi]$ and $t \in [0, 2\pi]$. These coordinates are connected with the standard coordinate system $(u_1, u_2, u_3, u_4, u_5)$ in $\mathbb{R}^5$ by the following equations:

$$\begin{aligned} u_1 &= R\sin\psi\sin\theta\sin\phi\sin t, \\ u_2 &= R\sin\psi\sin\theta\sin\phi\cos t, \\ u_3 &= R\sin\psi\sin\theta\cos\phi, \\ u_4 &= R\sin\psi\cos\theta, \\ u_5 &= R\cos\psi. \end{aligned} \tag{6.12}$$

The equation of $\mathbb{S}^4(r)$ in these coordinates is $R^2 = r^2$. The functions $x_1 = \psi$, $x_2 = \theta$, $x_3 = \phi$ and $x_4 = t$ provide an internal coordinate system on $\mathbb{S}^4(r)$ (without one point) in which the metric g induced on $\mathbb{S}^4(r)$ from $\mathbb{E}^3$ has components g_{ij} such that

$$(g_{ij}) = \begin{pmatrix} r^2 & & & 0 \\ & r^2\sin^2\psi & & \\ & & r^2\sin^2\psi\sin^2\theta & \\ 0 & & & r^2\sin^2\psi\sin^2\theta\sin^2\phi \end{pmatrix}.$$

g induces on $\mathbb{S}^3(r)$ a Riemannian connection $\bigtriangledown$. Using the formula

$$\Gamma^m_{ij} = \frac{1}{2} g^{ml}[\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}],$$

we can calculate the Christoffel symbols, which are as follows:

$\Gamma^2_{21} = \Gamma^2_{12} = \Gamma^3_{31} = \Gamma^3_{13} = \Gamma^4_{41} = \Gamma^4_{14} = \cot\psi$, $\Gamma^3_{32} = \Gamma^3_{23} = \Gamma^4_{42} = \Gamma^4_{24} = \cot\theta$, $\Gamma^1_{22} = -\sin\psi\cos\psi$, $\Gamma^1_{33} = -\sin\psi\cos\psi\sin^2\theta$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{44} = -\sin\phi\cos\phi$, $\Gamma^1_{44} = -\sin\psi\cos\psi\sin^2\theta\sin^2\phi$, $\Gamma^2_{44} = -\sin\theta\cos\theta\sin^2\phi$, $\Gamma^4_{43} = \cot\phi$. All the other symbols are zero.

Now let $\gamma : [0, 2\pi] \to \mathbb{S}^4(r)$ be a path in $\mathbb{S}^4(r)$ such that $x_i \circ \gamma = \pi/2$ for $i = 1, 2, 3$ and $x_4 \circ \gamma = id|_{[0,2\pi]}$. Since $\cos\pi/2 = \cot\pi/2 = 0$ and $\sin\pi/2 = 1$ we have $\Gamma^i_{jk}|_{\gamma([0,2\pi])} = 0$, and consequently, if we take $R = r = 1$, we get $g_{ij} = \delta^j_i$. Therefore, the coordinate system $\{x_1, x_2, x_3, x_4\}$ and the frame $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial x_4\}$ are normal for $\bigtriangledown$ along the path γ.

From the equations (6.12) it is clear that the set $\gamma([0, 2\pi])$ is a circle obtained by intersecting $\mathbb{S}^4(r)$ with the $(u_1, u_2)-$plane $\{v \in \mathbb{R}^5 : u_i(p) = 0 \text{ for} i \geq 3\}$ in $\mathbb{R}^5$. In fact, we have

$$\gamma([0, 2\pi]) = \{(v_1, v_2, 0, 0, 0) \in \mathbb{R}^5 : v_1^2 + v_2^2 = r^2\} = \mathbb{S}^1(r) \times (0, 0, 0).$$

It is clear if C is a circle on $\mathbb{S}^4(r)$ obtained by intersecting $\mathbb{S}^4(r)$ by a 2-plane through its origin then there are coordinates on $\mathbb{S}^4(r)$ normal along C for the Riemannian connection considered above.

We will assume $r = 1$. Then, using the above normal coordinate system, and the formulas

$$R^i_{jlm} = \partial_l \Gamma^i_{mj} - \partial_m \Gamma^i_{lj} + \Gamma^k_{mj}\Gamma^i_{lk} - \Gamma^k_{lj}\Gamma^i_{km},$$

$$R_{abcd} = g_{aj} R^j_{bcd},$$

we calculate the values of the curvature tensor as follows:

$$
\begin{aligned}
R_{1212} = R_{\psi\theta\psi\theta} &= \sin^2\psi, \\
R_{1313} = R_{\psi\phi\psi\phi} &= \sin^2\psi\sin^2\theta, \\
R_{1414} = R_{\psi t\psi t} &= \sin^2\psi\sin^2\theta\sin^2\phi, \\
R_{2323} = R_{\theta\phi\theta\phi} &= \sin^4\psi\sin^2\theta, \\
R_{2424} = R_{\theta t\theta t} &= \sin^4\psi\sin^2\theta\sin^2\phi, \\
R_{3434} = R_{\phi t\phi t} &= \sin^4\psi\sin^4\theta\sin^2\phi.
\end{aligned}
$$

All other values are zero. The values of the Ricci tensor, calculated by $\rho_{ab} = R^c_{acb}$, are as follows:

$$
\begin{aligned}
\rho_{11} = \rho_{\psi\psi} &= 3, \\
\rho_{22} = \rho_{\theta\theta} &= 3\sin^2\psi, \\
\rho_{33} = \rho_{\phi\phi} &= 3\sin^2\psi\sin^2\theta, \\
\rho_{44} = \rho_{tt} &= 3\sin^2\psi\sin^2\theta\sin^2\phi.
\end{aligned}
$$

All other values are zero. We then calculate the scalar curvature as follows:

$$
\tau = g^{\psi\psi}\rho_{\psi\psi} + g^{\theta\theta}\rho_{\theta\theta} + g^{\phi\phi}\rho_{\phi\phi} + g^{tt}\rho_{tt} = 12.
$$

Since τ is constant all its covariant derivatives, $\tau_{;j}$ are zero. Using $\rho_{ab;m} = \partial_m\rho_{ab} - \rho_{lb}\Gamma^l_{ma} - \rho_{al}\Gamma^l_{mb}$, we also calculate all the covariant derivatives of the Ricci tensor, which turn out to be zero as well.

Let $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$, $e_3 = (0,0,1,0,0)$, $e_4 = (0,0,0,1,0)$ and

$e_5 = (0,0,0,0,1)$ be the standard basis in $\mathbb{R}^5$. We define the following two subsets:

$$N_a = \Big\{(x,y,0,0,v) : x^2+y^2+v^2 = 1\Big\} \subset \mathbb{R}^5$$

and

$$N_b = \Big\{(0,0,z,w,v) : z^2+w^2+v^2 = 1\Big\} \subset \mathbb{R}^5.$$

The tangent space $T_{e_1}\mathbb{S}^4$, has basis vectors $\{e_2,e_3,e_4,e_5\}$ such that $\{e_2,e_5\}$ is a basis for $T_{e_1}N_a$ and $\{e_3,e_4\}$ is a basis for $T_{e_1}N_a^{\perp}$. Similarly, the tangent space $T_{e_4}\mathbb{S}^4$, has basis vectors $\{e_1,e_2,e_3,e_5\}$ such that $\{e_3,e_5\}$ is a basis for $T_{e_4}N_b$ and $\{e_1,e_2\}$ is a basis for $T_{e_4}N_b^{\perp}$.

Suppose $O = \mathbb{S}^4/G$ is an orbifold lens space where $G =< \gamma >$ and

$$\gamma = \begin{pmatrix} \cos\frac{2\hat{p_1}\pi}{q} & \sin\frac{2\hat{p_1}\pi}{q} & 0 & 0 & 0 \\ -\sin\frac{2\hat{p_1}\pi}{q} & \cos\frac{2\hat{p_1}\pi}{q} & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{2\hat{p_2}\pi}{q} & \sin\frac{2\hat{p_2}\pi}{q} & 0 \\ 0 & 0 & -\sin\frac{2\hat{p_2}\pi}{q} & \cos\frac{2\hat{p_2}\pi}{q} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\hat{p_1} \not\equiv \pm\hat{p_2} \pmod q$. Suppose $gcd(\hat{p_1},q) = q_1$ and $gcd(\hat{p_2},q) = q_2$, so that $\hat{p_1} = p_1q_1$, $\hat{p_2} = p_2q_2$ and $q = \hat{\alpha}q_1 = \hat{\beta}q_2$. Suppose $gcd(\hat{\alpha},\hat{\beta}) = g$ so that $\hat{\alpha} = \alpha g$, $\hat{\beta} = \beta g$ and $gcd(\alpha,\beta) = 1$. This means we can write γ as

$$\gamma = \begin{pmatrix} \cos\frac{2p_1\pi}{\alpha g} & \sin\frac{2p_1\pi}{\alpha g} & 0 & 0 & 0 \\ -\sin\frac{2p_1\pi}{\alpha g} & \cos\frac{2p_1\pi}{\alpha g} & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{2p_2\pi}{\beta g} & \sin\frac{2p_2\pi}{\beta g} & 0 \\ 0 & 0 & -\sin\frac{2p_2\pi}{\beta g} & \cos\frac{2p_2\pi}{\beta g} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now

$$\gamma^{\hat{\alpha}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{2p_2\pi\alpha}{\beta} & \sin\frac{2p_2\pi\alpha}{\beta} & 0 \\ 0 & 0 & -\sin\frac{2p_2\pi\alpha}{\beta} & \cos\frac{2p_2\pi\alpha}{\beta} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

fixes N_a, and

$$\gamma^{\hat{\beta}} = \begin{pmatrix} \cos\frac{2p_1\pi\beta}{\alpha} & \sin\frac{2p_1\pi\beta}{\alpha} & 0 & 0 & 0 \\ -\sin\frac{2p_1\pi\beta}{\alpha} & \cos\frac{2p_1\pi\beta}{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

fixes N_b.

Note that since the group action is transitive and the fixed point sets are $\mathbb{S}^2$, the functions $b_k(.,.)$ are constant along these fixed spheres. Therefore, it suffices to consider just a single point in these fixed point sets to calculate the values of the functions. We will choose the points $e_1 \in N_a$ and $e_4 \in N_b$ to calculate the values of functions.

We have, in the notation of the Theorem 6.1.6, $\tilde{N_a} \cong \mathbb{S}^2 \times \{(0,0)\}$ and $\tilde{N_b} \cong \{(0,0)\} \times \mathbb{S}^2$.

Also, $Iso_{N_a} = \{1, \gamma^{\hat{\alpha}}, \gamma^{2\hat{\alpha}}, ...\gamma^{(\beta-1)\hat{\alpha}}\}$, $|Iso_{N_a}| = \beta$, $Iso_{N_b} = \{1, \gamma^{\hat{\beta}}, \gamma^{2\hat{\beta}}, ...\gamma^{(\alpha-1)\hat{\beta}}\}$ and $|Iso_{N_b}| = \alpha$.

Now, as in the case of three-dimensional lens spaces, we have for $a = e_1$ and

$r \in \{1, 2, ...(\beta - 1)\}$,

$$A_{\gamma^r \hat{\alpha}}(a) = \begin{pmatrix} \cos \frac{2p_2\pi\alpha r}{\beta} & \sin \frac{2p_2\pi\alpha r}{\beta} \\ \\ -\sin \frac{2p_2\pi\alpha r}{\beta} & \cos \frac{2p_2\pi\alpha r}{\beta} \end{pmatrix},$$

$$I - A_{\gamma^r \hat{\alpha}}(a) = \begin{pmatrix} 1 - \cos \frac{2p_2\pi\alpha r}{\beta} & -\sin \frac{2p_2\pi\alpha r}{\beta} \\ \\ \sin \frac{2p_2\pi\alpha r}{\beta} & 1 - \cos \frac{2p_2\pi\alpha r}{\beta} \end{pmatrix},$$

$$B_{\gamma^r \hat{\alpha}}(a) = (I - A_{\gamma^r \hat{\alpha}}(a))^{-1} = \frac{1}{4\sin^2 \frac{p_2\pi\alpha r}{\beta}} \begin{pmatrix} 1 - \cos \frac{2p_2\pi\alpha r}{\beta} & -\sin \frac{2p_2\pi\alpha r}{\beta} \\ \\ \sin \frac{2p_2\pi\alpha r}{\beta} & 1 - \cos \frac{2p_2\pi\alpha r}{\beta} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -\cot \frac{p_2\pi\alpha r}{\beta} \\ \\ \cot \frac{p_2\pi\alpha r}{\beta} & 1 \end{pmatrix}.$$

So, $|det B_{\gamma^r \hat{\alpha}}(a)| = \frac{1}{4}(1 + \cot^2 \frac{p_2\pi\alpha r}{\beta}) = \frac{1}{4\sin^2 \frac{p_2\pi\alpha r}{\beta}}$.

Similarly we can show that for $b = e_4$ and $r \in \{1, 2, ...(\alpha - 1)\}$,

$$B_{\gamma^r \hat{\beta}}(b) = \frac{1}{2} \begin{pmatrix} 1 & -\cot \frac{p_1\pi\beta r}{\alpha} \\ \\ \cot \frac{p_1\pi\beta r}{\alpha} & 1 \end{pmatrix},$$

and $|det B_{\gamma^r \hat{\beta}}(b)| = \frac{1}{4}(1 + \cot^2 \frac{p_1\pi\beta r}{\alpha}) = \frac{1}{4\sin^2 \frac{p_1\pi\beta r}{\alpha}}$. Note again that for both $B_{\gamma^r \hat{\alpha}}(a)$

and $B_{\gamma^{r\hat{\beta}}}(b)$, $B_{13} = B_{23} = B_{31} = B_{32} = B_{33} = B_{41} = B_{14} = B_{42} = B_{24} = B_{43} = B_{34} = B_{44} = 0$.

Recall that for any k, the functions $b_k(\gamma^{r\hat{\alpha}}, a)$ and $b_k(\gamma^{r\hat{\beta}}, a)$ are universal polynomials in the components of the curvature tensor, its covariant derivatives and the elements of $B_{\gamma^{r\hat{\alpha}}}(a)$ and $B_{\gamma^{r\hat{\beta}}}(b)$ respectively. Since the elements of $B_{\gamma^{r\hat{\alpha}}}(a)$ are $B_{11} = B_{22} = 1/2$, $B_{12} = -\frac{1}{2}\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta}$ and $B_{21} = \frac{1}{2}\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta}$, every $b_k(\gamma^{r\hat{\alpha}}, a)$ will be of the form $\sum_{i=1}^{A_j} C_{ij}^a(R)\cot^{\lambda_i}\frac{p_2\alpha\pi r}{\beta}$ as in case of 3-dimensional lens spaces before, where A_j is the finite number of monomials in the powers of $\cot\frac{p_2\alpha\pi r}{\beta}$, and for each i, $C_{ij}^a(R)$ are constant functions in terms of the curvature tensor and its covariant derivatives. This means that, just as in the case of three-dimensional lens spaces, for each k, we will have,

$$b_k(\tilde{N}_a, a) = \sum_{r=1}^{\beta-1}\sum_{i=1}^{A_k} C_{ik}^a(R)\cot^{\lambda_i}\frac{\pi r}{\beta},$$

and

$$b_k(\tilde{N}_b, b) = \sum_{r=1}^{\alpha-1}\sum_{i=1}^{A_k} C_{ik}^b(R)\cot^{\lambda_i}\frac{\pi r}{\alpha}.$$

This observation gives us the following lemma:

Lemma 6.3.1. *Given two orbifold lens spaces $O_1 = \mathbb{S}^4/G_1$ and $O_2 = \mathbb{S}^4/G_2$, such that $G_1 =<\gamma_1>$ and $G_2 =<\gamma_2>$ where*

$$\gamma_1 = \begin{pmatrix} e^{\frac{2\hat{p_1}\pi i}{q}} & 0 & 0 \\ 0 & e^{\frac{2\hat{p_2}\pi i}{q}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\hat{p_1} \not\equiv \pm\hat{p_2}\,(mod\,q)$, $gcd(\hat{p_1},q) = q_{11}$, $gcd(\hat{p_2},q) = q_{21}$, $\hat{p_1} = p_1q_{11}$, $\hat{p_2} = p_2q_{21}$, $q = \hat{\alpha_1}q_{11} = \hat{\beta_1}q_{21}$, $gcd(\hat{\alpha_1},\hat{\beta_1}) = g_1$, $\hat{\alpha_1} = \alpha_1g_1$, $\hat{\beta_1} = \beta_1g_1$, *and*

$$\gamma_2 = \begin{pmatrix} e^{\frac{2\hat{s_1}\pi i}{q}} & 0 & 0 \\ \\ 0 & e^{\frac{2\hat{s_2}\pi i}{q}} & 0 \\ \\ 0 & 0 & 1 \end{pmatrix},$$

with $\hat{s_1} \not\equiv \pm\hat{s_2}\,(mod\,q)$, $gcd(\hat{s_1},q) = q_{12}$, $gcd(\hat{s_2},q) = q_{22}$, $\hat{s_1} = s_1q_{12}$, $\hat{s_2} = s_2q_{22}$, $q = \hat{\alpha_2}q_{12} = \hat{\beta_2}q_{22}$, $gcd(\hat{\alpha_2},\hat{\beta_2}) = g_2$, $\hat{\alpha_2} = \alpha_2g_2$, $\hat{\beta_2} = \beta_2g_2$.
Then $O_1 = \mathbb{S}^4/G_1$ *and* $O_2 = \mathbb{S}^4/G_2$ *will have the exact same asymptotic expansion of the heat kernel if* $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$.

This lemma gives us a tool to find examples of 4-dimensional orbifold lens spaces that are non-isometric (hence non-isospectral) but have the exact same asymptotic expansion of the heat kernel.

Example 6.3.2. *Suppose* $q = 195$, *and consider the two lens spaces* $O_1 = \tilde{L}_{1+} = L(195 : 3, 5, 0)$ *and* $O_2 = \tilde{L}'_{1+} = L(195 : 6, 35, 0)$ *(using the notation from Proposition 4.4.1). Since there is no integer* l *coprime to* 195 *and no* $e_i \in \{1, -1\}$ *such that* $\{e_1l3, e_2l5\}$ *is a permutation of* $\{6, 35\}(mod\ q)$, O_1 *and* O_2 *are not isometric (and hence non-isospectral). However, in the notation of the lemma above,* $\hat{p_1} = 3$, $\hat{p_2} = 5$, $\hat{s_1} = 6$, $\hat{s_2} = 35$, $gcd(\hat{p_1},q) = 3 = gcd(\hat{s_1},q)$, $gcd(\hat{p_2},q) = 5 = gcd(\hat{s_2},q)$ *and* $q = 195 = 3 \times 65 = 5 \times 39$. *So,* $\hat{\alpha_1} = \hat{\alpha_2} = 65$ *and* $\hat{\beta_1} = \hat{\beta_2} = 39$, *with* $gcd(\hat{\alpha_i},\hat{\beta_i}) = 13$ *(for* $i = 1, 2$*) giving* $\alpha_1 = \alpha_2 = 5$ *and* $\beta_1 = \beta_2 = 3$. *Therefore,* O_1 *and* O_2 *have the exact same asymptotic expansion.*

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