SRB MEASURES FOR CERTAIN MARKOV PROCESSES

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ABSTRACT. We study Markov processes generated by iterated function systems (IFS). The constituent maps of the IFS are monotonic transformations of the interval. We first obtain an upper bound on the number of SRB (Sinai-Ruelle-Bowen) measures for the IFS. Then, when all the constituent maps have common fixed points at 0 and 1, theorems are given to analyze properties of the ergodic invariant measures δ_0 and δ_1. In particular, sufficient conditions for δ_0 and/or δ_1 to be, or not to be, SRB measures are given. We apply some of our results to asset market games.

1. Introduction

In the 1970's, Sinai, Ruelle and Bowen studied the existence of an important class of invariant measures in the context of deterministic dynamical systems. These invariant measures are nowadays known as SRB (Sinai-Ruelle-Bowen) measures [14]. SRB measures are distinguished among other ergodic invariant measures because of their physical importance. In fact, from ergodic theory point of view, they are the only useful ergodic measures. This is due to the fact that SRB measures are the only ergodic measures for which the Birkhoff Ergodic Theorem holds on a set of positive measure of the phase space. In this note, we study SRB measures in a stochastic setting—Markov processes generated by iterated function systems (IFS).

An IFS[1] is a discrete-time random dynamical system [1, 10] which consists of a finite collection of transformations and a probability vector $\{\tau_s; p_s\}_{s=1}^L$. At each time step, a transformation τ_s is selected with probability $p_s > 0$ and applied to the process. IFS has been a very active topic of research due to its wide applications in fractals and in learning models. The survey articles [5, 13] contain a considerable list of references and results in this area.

The systems which we study in this note do not fall in the category of the IFS[2] considered in [5, 13] and references therein. Moreover, in general, our IFS do not satisfy the classical splitting[3] condition of [7]. In fact, our aim in this note is to depart from the traditional goal of finding sufficient conditions for an IFS to admit a *unique attracting invariant measure* [7, 5, 13]. Instead, we study cases where an IFS may admit more than one invariant measure and aim to identify the *physically relevant* ones; i.e., invariant measures for which the Ergodic Theorem holds on a

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[1]In some of the literature an IFS is called a *random map* or a *random transformation*.

[2]In most articles about IFS, the constituent maps are assumed to be contracting or at least contracting on average. Here we do not impose any assumption of this type. In fact the class of IFS which we study in Section 4 cannot satisfy such assumptions.

[3]In particular, when all the maps have common fixed points at 0 and 1. See Section 4.

set of positive measure of the ambient phase space. We call such invariant measures SRB.

Physical SRB measures for random maps have been studied by Buzzi [3] in the context of random Lasota-Yorke maps. However, Buzzi's definition of a basin of an SRB measure is different from ours. We will clarify this difference in Section 2. A general concept of an SRB measure for general random dynamical systems can be found in the survey article [11]. In this note we study physical SRB measures for IFS whose constituent maps are strictly increasing transformations of the interval. We obtain an upper bound on the number of SRB measures for the IFS. Moreover, when all the constituent maps have common fixed points at 0 and 1, we provide sufficient conditions for δ_0 and/or δ_1 to be, or not to be, SRB measures. To complement our theoretical results, we show at the end of this note that examples of IFS of this type can describe evolutionary models of financial markets [4].

In Section 2 we introduce our notation and main definitions. In particular, Section 2 includes the definition of an SRB measure for an IFS. In Section 3 we identify the structure of the basins of SRB measures and we obtain a sharp upper bound on the number of SRB measures. Section 4 contains sufficient conditions for δ_0 and δ_1, the delta measures concentrated at 0 and 1 respectively, to be SRB. It also contains sufficient conditions for δ_0 and δ_1 not to be SRB measures. Our main results in this section are Theorems 4.3 and Theorem 4.7. In Section 5 we study ergodic properties of δ_0 and δ_1 without having any information about the probability vector of the IFS. In Section 6 we apply our results to asset market games. In particular, we find a generalization of the famous Kelly rule [9] which expresses the principle of "betting your beliefs". The importance of our generalization lies in the fact that it does not require the full knowledge of the probability distribution of the random states of the system. Section 7 contains an auxiliary result which we use in the proof of Theorem 4.7.

2. Preliminaries

2.1. **Notation and assumptions.** Let $([0,1],\mathfrak{B})$ be the measure space where $\mathfrak{B}$ is the Borel σ-algebra on $[0,1]$. Let λ denote Lebesgue measure on $([0,1],\mathfrak{B})$ and δ_r denote the delta measure concentrated at point $r\in[0,1]$. Let $S=\{1,\dots,L\}$ be a finite set and τ_s, $s\in S$, be continuous transformations from the unit interval into itself. We assume:

(A) τ_s are strictly increasing.

Let $\mathbf{p}=(p_s)_{s=1}^L$ be a probability vector on S such that for all $s\in S$, $p_s>0$. The collection

$$F=\{\tau_1,\tau_2,\dots,\tau_L;p_1,p_2,\dots,p_L\}$$

is called an iterated function system (IFS) with probabilities.

We denote the space of sequences $\omega=\{s_1,s_2,\dots\}$, $s_l\in S$, by Ω. The topology on Ω is defined as the product of the discrete topologies on S. Let $\pi_{\mathbf{p}}$ denote the Borel measure on Ω defined as the product measure $\mathbf{p}^{\mathbb{N}}$. Moreover, we write

$$s^t:=(s_1,s_2,\dots,s_t)$$

for the history up to time t, and for any $r_0 \in [0,1]$ we write

$$r_t(s^t) := \tau_{s_t} \circ \tau_{s_{t-1}} \circ \cdots \circ \tau_{s_1}(r_0).$$

Finally, by $E(\cdot)$ we denote the expectation with respect to $\mathbf{p}$, by $E(\cdot|s^t)$ the conditional expectation given the history up to time t and by $\mathrm{var}(\cdot)$ the variance with respect to $\mathbf{p}$.

2.2. **Invariant measures.** F is understood as a Markov process with a transition function

$$\mathbb{P}(r, A) = \sum_{s=1}^{L} p_s \chi_A(\tau_s(r)),$$

where $A \in \mathfrak{B}$ and χ_A is the characteristic function of the set A. The transition function $\mathbb{P}$ induces an operator P on measures on $([0,1], \mathfrak{B})$ defined by

$$\begin{aligned} P\mu(A) &= \int_0^1 \mathbb{P}(r, A) d\mu(r) \\ &= \sum_{s=1}^{L} p_s \mu(\tau_s^{-1} A). \end{aligned} \tag{2.1}$$

Following the standard notion of an invariant measure for a Markov process, we call a probability measure μ on $([0,1], \mathfrak{B})$ F-invariant probability measure if and only if

$$P\mu = \mu.$$

Moreover, it is called ergodic if it cannot be written as a convex combination of other invariant probability measures.

2.3. **SRB measures.** Let μ be an ergodic probability measure for the IFS. Suppose there exists a set of positive Lebesgue measure in $[0,1]$ such that

$$\frac{1}{T} \sum_{t=0}^{T-1} \delta_{r_t(s^t)} \overset{\text{weakly}}{\rightarrow} \mu \qquad \text{with } \pi_{\mathbf{p}}\text{-probability one.} \tag{2.2}$$

Then μ is called an SRB (Sinai-Ruelle-Bowen) measure. The set of points $r_0 \in [0,1]$ for which (2.2) is satisfied will be called the basin[4] of μ and it will be denoted by $B(\mu)$. Obviously, if $\lambda(B(\mu)) = 1$ then μ is the unique SRB measure of F.

3. Number of SRB measures and their basins

The basin of an SRB measure for the systems we are dealing with is described by the following two propositions.

Proposition 3.1. Let μ be an SRB measure and $B(\mu)$ be its basin. Let $r_0, \bar{r}_0 \in B(\mu)$, $r_0 > \bar{r}_0$. Then $[\bar{r}_0, r_0] \subseteq B(\mu)$.

[4]Our definition of a basin is different from Buzzi's definition [3]. In his definition he defines *random* basins $B_\omega(\mu)$ for an SRB measure. In particular, according to Buzzi's definition, for the same SRB measure, basins corresponding to two different ω's may differ on a set of positive lebsegue measure of $[0,1]$. See [3] for more detials.

Proof. When weak convergence is considered on an interval, then $\mu_n \overset{\text{weakly}}{\to} \mu$ if and only if $\mu_n(f) \to \mu(f)$ for any C^1 function[5]. Since every C^1 function is a difference of two continuous increasing functions, this means that $\mu_n \overset{\text{weakly}}{\to} \mu$ if and only if $\mu_n(f) \to \mu(f)$ for any continuous increasing function.

Let $r_0, \bar{r}_0 \in B(\mu)$ and $\bar{r}_0 < r_0' < r_0$. We will show that $r_0' \in B(\mu)$. Let assume that f is continuous and increasing. Let us fix an s^t for which

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(\bar{r}_t(s^t)) = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(r_t(s^t)) = \mu(f).$$

We have $\bar{r}_t(s^t) < r_t'(s^t) < r_t(s^t)$ (since all τ_s are increasing) and

$$\frac{1}{T}\sum_{t=0}^{T-1} f(\bar{r}_t(s^t)) \le \frac{1}{T}\sum_{t=0}^{T-1} f(r_t'(s^t)) \le \frac{1}{T}\sum_{t=0}^{T-1} f(r_t(s^t)).$$

The averages on the left and on the right have common limit $\mu(f)$. Thus,

$$\frac{1}{T}\sum_{t=0}^{T-1} \delta_{r_t'(s^t)}(f) = \frac{1}{T}\sum_{t=0}^{T-1} f(r_t'(s^t)) \to \mu(f).$$

Since the event

$$\{\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(\bar{r}_t(s^t)) = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(r_t(s^t)) = \mu(f)\}$$

occurs with $\pi_{\mathbf{p}}$-probability 1, the event

$$\{\frac{1}{T}\sum_{t=0}^{T-1} f(r_t'(s^t)) \to \mu(f)\}$$

also occurs with $\pi_{\mathbf{p}}$-probability 1. □

Proposition 3.2. Let μ be an SRB measure and $B(\mu) = \langle a, b\rangle$ be its basin, where $\langle a, b\rangle$ denotes an interval closed or open at any of the endpoints. Then,

$$\tau_s(a) \ge a \ , \ \ s = 1, \dots, L, \ \text{ and if } a \neq 0 \ \text{ then } \tau_s(a) = a \ \text{ for at least one } s;$$

$$\tau_s(b) \le b \ , \ \ s = 1, \dots, L, \ \text{ and if } b \neq 1 \ \text{ then } \ \tau_s(b) = b \ \text{ for at least one } s.$$

Proof. We will prove only the second claim with $b \neq 1$. The first claim is proven analogously.

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[5]Here is a sketch of the proof of this claim: Assume

$$\mu_n(f) \to \mu(f)$$

for any $f \in C^1([0,1])$. Let g be a continuous function and let $\{f_k\}_{k\ge1}$ be a sequence of C^1 functions converging to g in C^0 norm. We have

$$\begin{aligned}|\mu_n(g) - \mu(g)| &\le |\mu_n(g) - \mu_n(f_k)| + |\mu_n(f_k) - \mu(f_k)| + |\mu(f_k) - \mu(g)| \\ &\le 2\|f_k - g\|_{C^0} + |\mu_n(f_k) - \mu(f_k)| \ .\end{aligned}$$

Now, for any $\varepsilon > 0$, we can find k_0 such that $2\|f_{k_0} - g\|_{C^0} < \varepsilon/2$ and then we can find n_0 such that for any $n \ge n_0$ we have $|\mu_n(f_{k_0}) - \mu(f_{k_0})| < \varepsilon/2$.

Assume that $\tau_{s_0}(b) > b$, for some $1 \leq s_0 \leq L$. Then, we can find $r_0 \in (a,b)$ such that $\tau_{s_0}(r_0) > b$. For arbitrary continuous function f, for $\omega \in A \subset \Omega$ with $\pi_{\mathbf{p}}(A) = 1$, we have

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(r_t(s^t)) = \mu(f).$$

The set $A_{s_0} = \{(s_1, s_2, \dots) : (s_0, s_1, s_2, \dots) \in A\}$ is also of $\pi_{\mathbf{p}}$-probability 1. Let $r_0' = \tau_{s_0}(r_0)$ and let $(s^t)'$ denote the initial subsequences of length t of $\omega \in A_{s_0}$. Then,

$$\frac{1}{T} \sum_{t=0}^{T-1} f(r_t'((s^t)')) = \frac{1}{T} \sum_{t=0}^{T-1} f(r_t(s^t)) - \frac{1}{T} f(r_0) + \frac{1}{T} f(r_{T-1}'((s^{T-1})')) \underset{T\to+\infty}{\longrightarrow} \mu(f).$$

This shows that $\tau_{s_0}(r_0) \in B_\omega(\mu)$ and contradicts the assumptions.

Now, we assume that $\tau_s(b) < b$, $s = 1, \dots, L$. Then, we can find $r_0 > b$ such that $\tau_s(r_0) \in (a,b)$ for all s. Let

$$A_s = \{\omega : \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(r_t'(s^t)) = \mu(f), \text{ for } r_0' = \tau_s(r_0)\}\ ,\ s = 1, \dots, L\ .$$

We have $\pi_{\mathbf{p}}(A_s) = 1$ for each s. Hence, $\pi_{\mathbf{p}}(A) = 1$, where $A = \cup_{1\leq s\leq L}(s, A_s)$ and $(s, A_s) = \{(s, s_1, s_2, s_3, \dots) : (s_1, s_2, s_3, \dots) \in A_s\}$. For arbitrary continuous function f, for $\omega \in A$, if $\omega_1 = s$ we have

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(r_t(s^t)) \\ &= \lim_{T\to\infty} \left(\frac{1}{T} \sum_{t=0}^{T-1} f(r_t'((s^t)')) + \frac{1}{T} f(r_0) - \frac{1}{T} f(r_{T-1}'((s^{T-1})')) \right) = \mu(f), \end{aligned}$$

where $r_0' = \tau_s(r_0)$ and $(s^t)'$ are the initial subsequences of length t of $\omega \in A_s$. This implies that $r_0 \in B(\mu)$. Since $r_0 > b$, this leads to a contradiction. □

We now state the main result of this section. Firstly, we recall that $\langle\cdot,\cdot\rangle$ denotes an interval which is closed or open at any of the endpoints. Secondly, we define a set **BS** whose elements are intervals of the form $\langle\cdot,\cdot\rangle$ with the following property:

$$\langle a, b\rangle \in \mathbf{BS}$$

if and only if

$$\tau_s(a) \geq a \quad , \quad s = 1, \dots, L \text{ and } \tau_s(a) = a \text{ for at least one } s;$$

and

$$\tau_s(b) \leq b \quad , \quad s = 1, \dots, L \text{ and } \tau_s(b) = b \text{ for at least one } s.$$

Theorem 3.3. *The number of SRB measures of F is bounded above by the cardinality of the set **BS**. In particular, if 0 and 1 are the only fixed points of some τ_{s_0}, $s_0 \in S$, then F admits at most one SRB measure.*

Proof. The fact that number of SRB measures of F is bounded above by the cardinality of the set **BS** is a direct consequence of Proposition 3.2. To elaborate on the second part of the theorem, assume without loss of generality that $\tau_{s_0}(r) > r$ for all $r \in (0,1)$. Obviously, by Proposition 3.2, if all the other maps τ_s, $s \in S \setminus \{s_0\}$ has no fixed points in $(0,1)$, then F admits at most one SRB measure. So let us assume that there exists an $s^* \in S \setminus \{s_0\}$ such that τ_{s^*} has a finite or infinite number of fixed points in $[0,1]$. In the case of finite number of fixed points, denote the fixed points of τ_{s^*} in [0,1] by r_i^*, $i = 1, \dots, q$, such that $0 \le r_1^* < r_2^* < \cdots < r_q^* \le 1$. Since $\tau_s(r_i^*) > r_i^*$ for all $r_i^* \in (0,1)$, the only possible basin for an SRB measure would be either $\langle r_{q-1}^*, 1\rangle$ or $\langle r_q^*, 1\rangle$. In the case of infinite number of fixed points, let

$$\bar{r} = \sup\{r \in (0,1) : \tau_{s^*}(r) = r\}.$$

If $\bar{r} < 1$, then $\tau_{s_0}(\bar{r}) > \bar{r}$. By Proposition 3.2, $\langle \bar{r}, 1\rangle$ is the only possible basin for an SRB measure. If $\bar{r} = 1$, let $\bar{J}$ denote the closure of the set of fixed points of τ_{s^*} and let $\bar{J}_0 \subseteq \bar{J}$ be the minimal closed subset of $\bar{J}$ which contains the point 1. $\bar{J}_0$ is the only possible basin for an SRB measure. Moreover, it cannot be decomposed into basins of different SRB measures. Indeed, let $J_1 \cup J_2 = \bar{J}_0$ such that $J_1 = \langle a, b\rangle$ with $b < 1$. Since $\tau_{s_0}(b) > b$, by Proposition 3.2, J_1 cannot be a basin of an SRB measure. Thus, F admits at most one SRB measure. □

The following example shows that Proposition 3.2 can be used to identify intervals which are not in the basin of an SRB measure. In particular, it shows that the bound obtained on the number of SRB measures in Theorem 3.3 is really sharp.

Example 3.4. Let

$$\tau_1(r) = \begin{cases} 3r^2 & , \text{ for } \ 0 \le r \le 1/3; \\ 1 - \frac{3}{2}(r-1)^2 & , \text{ for } \ 1/3 < r \le 1; \end{cases} ,$$

and

$$\tau_2(r) = \begin{cases} \frac{3}{2}r^2 & , \text{ for } \ 0 \le r \le 2/3; \\ 1 - 3(r-1)^2 & , \text{ for } \ 2/3 < r \le 1. \end{cases}$$

The graphs of the above maps are shown in Figure 1. Using Proposition 3.2, we see that the points of the interval $(1/3, 2/3)$ do not belong to a basin of any SRB measure. Moreover, by Theorem 3.3, F admits at most two SRB measures. Indeed, one can easily check that δ_0 and δ_1 are the only SRB measures with basins $B(\delta_0) = [0, 1/3]$ and $B(\delta_1) = [2/3, 1]$ respectively. For any $r \in [0, 1/3)$ for all ω's the averages $\frac{1}{T}\sum_{t=0}^{T-1} \delta_{r_t(s^t)}$ converge weakly to δ_0. For $r = 1/3$ the only ω for which this does not happen is $\omega = \{1, 1, 1, \dots\}$ so again the averages converge weakly to δ_0 with $\pi_{\mathbf{p}}$-probability 1. Similarly, we can show that $B(\delta_1) = [2/3, 1]$. If $r \in (1/3, 2/3)$, then with positive $\pi_{\mathbf{p}}$-probability the averages converge to δ_0 and with positive $\pi_{\mathbf{p}}$-probability the averages converge to δ_1. Thus, these points do not belong to a basin of any SRB measure and there are only two SRB measures.

4. Properties of δ_0 and δ_1

In addition to condition (A), we assume in this section that for all $s \in S$:

(B) $\tau_s(0) = 0$ and $\tau_s(1) = 1$;

Obviously by Condition (B) the delta measures δ_0 and δ_1 are ergodic probability measures for the IFS. We will be mainly concerned with the following question: When does F have δ_0 and/or δ_1 as SRB measures? We start our analysis by

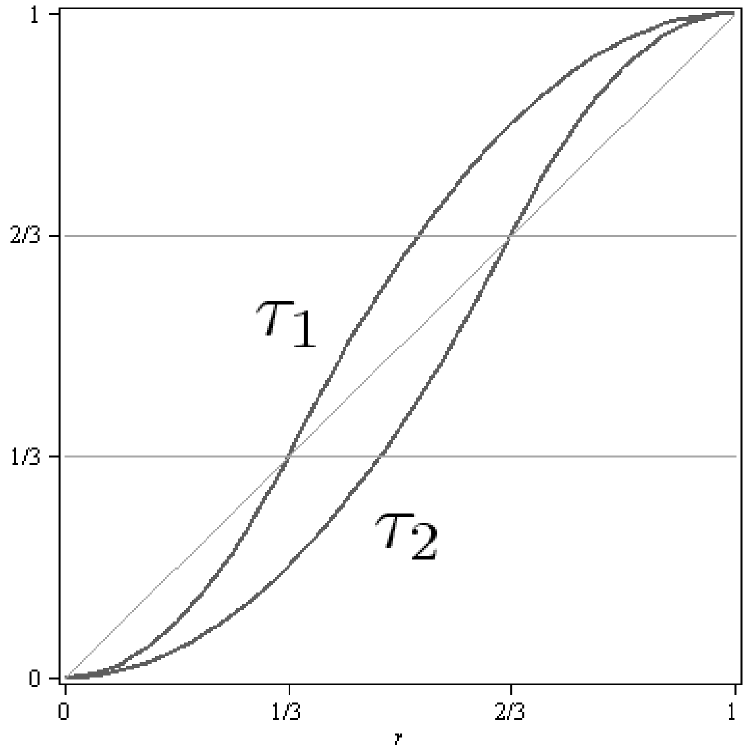

FIGURE 1. Maps τ_1 and τ_2 in Example 3.4

proving a lemma which provides a sufficient condition for δ_x, the point measure concentrated at $x \in [0,1]$, to be an SRB measure.

Lemma 4.1. *Suppose that $\tau_s(x) = x$ for all $s \in \{1, \ldots, L\}$ and that there exists an initial point of a random orbit r_0, $r_0 \neq x$, for which $\lim_{t\to\infty} r_t(s^t) = x$ with probability $\pi_{\mathbf{p}} = 1$. Then δ_x is an SRB measure for F and $B_\omega(\delta_x) \supseteq [x, r_0]$*[6].

Proof. Let f be a continuous function on $[0,1]$. Let $r_0 \neq x$ and fix a history s^t for which $\lim_{t\to\infty} r_t(s^t) = x$. Then

$$\lim_{t\to\infty} f(r_t(s^t)) = f(x).$$

Consequently

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(r_t(s^t)) = f(x).$$

Since the event

$$\{\lim_{t\to\infty} r_t(s^t) = x\}$$

appears with probability one, the event

$$\{\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(r_t(s^t)) = f(x)\}$$

[6]The notation here is for the case when $r_0 > x$.

also appears with probability one. Thus, by Proposition 3.1, δ_x is an SRB measure for F and $B_\omega(\delta_x) \supseteq [x, r_0]$. □

The following lemma, which is easy to prove, is a key observation for our main results in this section.

Lemma 4.2. *Each constituent map of the IFS can be represented as follows:*

$$\tau_s(r) = r^{\beta_s(r)},$$

with $\beta_s(r)$ satisfying:

1. $\beta_s(r) > 0$ *in* $(0,1)$ *;*
2. $(\ln r)\beta_s(r)$ *increasing;*
3. $\lim_{r\to 0}(\ln r)\beta_s(r) = -\infty$*;*
4. $\lim_{r\to 1}(\ln r)\beta_s(r) = 0$.

In the rest of this section, the following notation will be used:

$$\alpha_t \overset{\text{def}}{:=} \beta_s(r_{t-1}) \text{ with probability } p_s,\ t = 1, 2, \dots$$

Theorem 4.3. *Let $F = \{\tau_s; p_s\}_{s\in S}$ be an IFS such that $\tau_s(r) = r^{\beta_s(r)}$. Assume that $0 < b_s \le \beta_s(r) \le B_s < \infty$ for all $r \in [0,1]$.*

1. *If $E(\ln \alpha_t | s^{t-1}) \le 0$ a.s., then $\lim_{t\to\infty} r_t(s^t) \ne 0$ a.s.*
2. *If $\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E(\ln \alpha_t | s^{t-1}) < 0$ a.s., then $\lim_{t\to\infty} r_t(s^t) = 1$ a.s.*
3. *If $\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E(\ln \alpha_t | s^{t-1}) > 0$ a.s., then $\lim_{t\to\infty} r_t(s^t) = 0$ a.s.*

Proof. Let us consider the sequence of random exponents

$$\alpha(t) = \alpha_t\alpha_{t-1}\cdots\alpha_2\alpha_1,$$

where $\alpha_i = \beta_s(r_{i-1})$ with probability p_s, and observe that

$$r_t(s^t) = r^{\alpha(t)}.$$

We have

$$\ln \alpha(t+1) = \ln \alpha_{t+1} + \ln \alpha(t),$$

and, with probability one,

$$E(\ln \alpha(t+1)|s^t) - \ln(\alpha(t)) = E(\ln \alpha_{t+1}|s^t) \le 0.$$

Therefore, $\ln \alpha(t)$ is a supermartingale. Moreover, because $0 < b_s \le \beta_s(r_t) \le B_s < \infty$, $|\ln \alpha(t+1) - \ln \alpha(t)| = |\ln \alpha_{t+1}| < \infty$. Hence $\ln \alpha(t)$ is a supermartingale with bounded increments. Thus, using Theorem 5.1 in Chapter VII of [12], with probability one $\ln \alpha(t)$ does not converge to $+\infty$. Consequently, with probability one, $r_t(s^t) = r^{\alpha(t)}$ does not converge to zero.

We now prove the second statement of the theorem. Again we consider the sequence of random exponents

$$\alpha(t) = \alpha_t\alpha_{t-1}\cdots\alpha_2\alpha_1.$$

Let M_t denote the martingale difference

$$M_t := \ln \alpha_t - E(\ln \alpha_t | s^{t-1}).$$

We have $E(M_t) = 0$ and $\ln \alpha_t$ is uniformly bounded. Therefore, by the strong law of large numbers (see Theorem 2.19 in [8]), with probability one

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} M_t = 0. \tag{4.1}$$

Therefore, with probability one,

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{T}\ln\alpha(T) &= \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\ln\alpha_t\\ &= \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}M_t+\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}E(\ln\alpha_t|s^{t-1})<0.\end{aligned}$$

From this we can conclude that for T large enough there is a positive random variable η such that

$$\alpha(T)\le e^{-T\eta}\text{ a.s.}$$

Thus, since $r\in[0,1]$, for T large enough we obtain

$$r_{T+1}=r^{\alpha(T)}\ge r^{e^{-T\eta}}\text{ a.s.}$$

By taking the limit of T to infinity we obtain

$$\lim_{T\to\infty}r_{T+1}=\lim_{T\to\infty}r^{\alpha(T)}\ge\lim_{T\to\infty}r^{e^{-T\eta}}=1\text{ a.s.}$$

The proof of the third statement is very similar to the proof of the second one with slight changes. In particular, using (4.1), we see that, with probability one,

$$\liminf_{T\to\infty}\frac{1}{T}\ln\alpha(T)>0.$$

From this we can conclude that for T large enough there is a positive random variable η such that

$$\alpha(T)\ge e^{T\eta}\text{ a.s.}$$

Thus, since $r\in[0,1]$, for T large enough we obtain

$$r_{T+1}=r^{\alpha(T)}\le r^{e^{T\eta}}\text{ a.s.}$$

By taking the limit of T to infinity we obtain

$$\lim_{T\to\infty}r_{T+1}=\lim_{T\to\infty}r^{\alpha(T)}\le\lim_{T\to\infty}r^{e^{T\eta}}=0\text{ a.s.}$$

□

Corollary 4.4. Let $F=\{\tau_s;p_s\}_{s\in S}$ be an IFS such that $\tau_s(r)=r^{\beta_s(r)}$. Assume that $0<b_s\le\beta_s(r)\le B_s<\infty$ for all $r\in[0,1]$.

(1) If $\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}E(\ln\alpha_t|s^{t-1})<0$ a.s., then δ_1 is the unique SRB measure of F with $B(\delta_1)=(0,1]$.
(2) If $\liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}E(\ln\alpha_t|s^{t-1})>0$ a.s., then δ_0 is the unique SRB measure of F with $B(\delta_0)=[0,1)$.

Proof. The proof is a consequence of statements (2) and (3) of Theorem 4.3 and Lemma 4.1. □

Remark 4.5. Observe that:

(1) $\sum_s p_s\ln B_s\le 0\implies E(\ln\alpha_t|s^{t-1})\le 0$ a.s.
(2) $\sum_s p_s\ln B_s<0\implies \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}E(\ln\alpha_t|s^{t-1})<0$ a.s.
(3) $\sum_s p_s\ln b_s>0\implies \liminf_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}E(\ln\alpha_t|s^{t-1})>0$ a.s.

Thus, the conditions in the statements of Theorem 4.3 and Corollary 4.4 are very easy to check for certain systems.

Remark 4.6. In the proof of statement (1) of Theorem 4.3, we have with probability $\pi_{\mathbf{p}} = 1$, $\lim_{t\to\infty} \ln \alpha(t) \neq \infty$. In general, it is not clear that this implies that δ_0 is not an SRB measure. However, in the following theorem under additional natural assumption on the variance of $\ln \alpha_t$ we show that δ_0 is indeed not an SRB measure.

Theorem 4.7. *If $E(\ln \alpha_t | s^{t-1}) \leq 0$ and $\mathrm{var}(\ln \alpha_t | s^{t-1}) \geq d > 0$, for all $t \geq 1$, then δ_0 is not an SRB measure of F.*

Proof. Consider the sequence of random exponents

$$\alpha(t) = \alpha_t \alpha_{t-1} \cdots \alpha_2 \alpha_1,$$

where $\alpha_i = \beta_s(r_{t-1})$ with probability p_s, and observe that

$$r_t(s^t) = r^{\alpha(t)}.$$

Observe that

$$\ln \alpha(T) = \sum_{t=1}^{T} \ln \alpha_t.$$

Since

$$E(\ln \alpha(t) | s^{t-1}) - \ln \alpha(t-1) = E(\ln \alpha_t | s^{t-1}) \leq 0,$$

and

$$0 < b_s \leq \beta_s(r_t) \leq B_s < \infty.$$

the sequence $Z_T = \ln \alpha(T)$ forms a supermartingale with bounded increments. Doob's decomposition theorem gives the representation

$$Z_T = W_T + S_T,$$

where $W_T = \sum_{t=1}^{T} E(\ln \alpha_t | s^{t-1})$ is a decreasing predictable sequence and

$$S_T = \sum_{t=1}^{T} [\ln \alpha_t - E(\ln \alpha_t | s^{t-1})],$$

is a 0 mean martingale with bounded increments. By Theorem 5.1 (Ch. VII) of [12] with probability 1 process S_T either converges to finite limit or $\limsup_{T\to\infty} S_T = -\liminf_{T\to\infty} S_T = \infty$. In the first case the process Z_T is bounded from above. We will consider only the second case to show that with positive probability the process Z_T is bounded from above for a set of indices T which has positive density in $\mathbb{N}$, i.e, there exist $M > 0$, $0 < a, b < 1$ such that

$$\pi_{\mathbf{p}}(\limsup_{T\to\infty} \frac{\#\{t \leq T : Z_t \leq M\}}{T} \geq a) > b. \tag{4.2}$$

Let us denote

$$X_t = \ln \alpha_t - E(\ln \alpha_t | s^{t-1}) \ , \quad t \geq 1.$$

This sequence satisfies assumptions of Theorem 7.1, with $\mathcal{A}_t = \sigma(s^t)$. We have

$$\begin{aligned} E(X_t^2 | s^{t-1}) &= E((\ln \alpha_t - E(\ln \alpha_t | s^{t-1}))^2 | s^{t-1}) \\ &= \sum_{s=1}^{L} p_s (\ln \beta_s(r))^2 - \left(\sum_{s=1}^{L} p_s \ln \beta_s(r) \right)^2 = \mathrm{var}(\ln \alpha_t | s^{t-1}) \geq d > 0. \end{aligned}$$

Thus, the sequence X_t satisfies assumptions of Proposition 7.2. In particular, (7.1) holds, i.e., if Pos_T is the number of times $\ln\alpha(t) > 0$ for $t \le T$, then

$$\limsup_{T\to\infty}[\pi_{\mathbf{p}}(\frac{\mathrm{Pos}_T}{T} \le a)] = b > 0,$$

where a, b are some numbers in $(0,1)$. This means that if N_T is the number of times $\ln\alpha(t) \le 0$ for $t \le T$, then

$$\limsup_{T\to\infty}[\pi_{\mathbf{p}}(\frac{N_T}{T} \ge 1-a)] = b > 0.$$

Now, we we show that

$$\pi_{\mathbf{p}}(\limsup_{T\to\infty}\frac{N_T}{T} \ge 1-a) \ge b/2 > 0.$$

For $T > T_0$ we have $\pi_{\mathbf{p}}(\frac{N_T}{T} \ge 1-a) > b/2$. Let $A_T = \{\frac{N_T}{T} \ge 1-a\}$, $T \ge T_0$. The set which contains points from infinitely many A_T is $A = \cap_i \cup_{T>i} A_T$ and since the sequence $(\cup_{T>i}A_T)_i$ is decreasing we have

$$\pi_{\mathbf{p}}(A) = \lim_{i\to\infty}\pi_{\mathbf{p}}(\cup_{T>i}A_T) \ge b/2 \ .$$

Thus, with a positive probability $b/2 > 0$, there exist a sequence $T_n \to \infty$ such that $\frac{N_{T_n}}{T_n} \ge 1-a$ or

$$\pi_{\mathbf{p}}(\limsup_{T\to\infty}\frac{N_T}{T} \ge 1-a) \ge b/2 > 0.$$

Thus, $\ln\alpha(T)$ is negative with positive density, i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\#\{t \le T : \ln\alpha(t) \le 0\} \ge 1-a > 0,$$

with positive probability $b/2$. This implies that $r_T(s_T) \ge \bar{r} > 0$ with positive density $1-a$ and positive probability $b/2$. We can construct a continuous function f which is 0 around 0 and 1 above $\bar{r}$. The averages of this function satisfy

$$\limsup_{T\to\infty}\frac{1}{T}\sum_{T'\le T} f(r_{T'}(s_{T'})) \ge 1-a,$$

with nonzero probability $b/2$ which proves there is no weak convergence to δ_0. □

Remark 4.8. If $E(\ln\alpha_t|s^{t-1}) \ge 0$ and $\mathrm{var}(\ln\alpha_t|s^{t-1}) \ge d > 0$, for all $t \ge 1$, using essentially the proof of Theorem 4.7, we obtain that δ_1 is not an SRB measure of F. In particular, if $E(\ln\alpha_t|s^{t-1}) = 0$ and $\mathrm{var}(\ln\alpha_t|s^{t-1}) \ge d > 0$, for all $t \ge 1$, we obtain that neither δ_0 nor δ_1 is an SRB measure.

5. Properties of δ_0 and δ_1: The case when $\mathbf{p}$ is unknown

In general, one cannot decide whether δ_0 or δ_1 is the unique SRB measure without having information about $\mathbf{p}$. We illustrate this fact in the following example.

Example 5.1. Let $F = \{\tau_1, \tau_2; p_1, p_2\}$ where $\tau_1 = r^2$, $\tau_2 = \sqrt{r}$ and p_1, p_2 are unknown. Observe that the exponents, which are explicit in this case and independent of r, are $\beta_1(r) = 2$ and $\beta_2(r) = 1/2$. Then

$$p_1 \ln B_1(r) + p_2 \ln B_2(r) = (2p_1 - 1)\ln 2.$$

By Corollary 4.4, if $p_1 < 1/2$ the measure δ_1 is the unique SRB measure of F; however, if $p_1 > 1/2$ the measure δ_0 is the unique SRB measure of F. Thus, for this example, without having information about $\mathbf{p}$, no information about the nature of δ_0 or δ_1 can be obtained.

Although Example 5.1 shows that the analysis cannot be definitive in some cases without knowing the probability distribution on S, our aim in this section is to find situations when δ_0 and/or δ_1 are not SRB. Moreover, in addition to studying the properties of δ_0 and δ_1, we are going to study the case when the IFS admit an invariant probability measure whose support is separated from zero and is not necessarily concentrated at one. The definition of such a measure is given below.

Definition 5.2. Let μ be a probability measure on $([0,1],\mathfrak{B})$, where $\mathfrak{B}$ is the Borel σ-algebra. We define the support of μ, denoted by $\text{supp}(\mu)$, as the smallest closed set of full μ measure. We say that $\text{supp}(\mu)$ is separated from zero if there exists an $\eta > 0$ such that $\mu([0,\eta]) = 0$.

In addition to properties (A) and (B), we assume in this section that:

(C) Every τ_s has a finite number of fixed points.

In this section, we use a graph theoretic techniques to analyze ergodic properties of δ_0 and δ_1. This approach is inspired by the concept of a *Markov partition* used in the dynamical systems literature. For instance, in [2], the ergodic properties of a deterministic system which admits a Markov partition is studied via a directed graph and an incidence matrix. In our approach we construct a partition for our random dynamical system akin to that of a Markov partition and use two directed graphs to study ergodic properties of the system.

We now introduce the two graphs, G_d and G_u, which we will use in our analysis.

(1) Both G_d and G_u have the same vertices;
(2) For $s \in \{1,\dots,L\}$, an interval $J_{s,m} = (a_{s,m}, a_{s,m+1})$ is a vertex in G_d and G_u if and only if $\tau_s(a_{s,m}) = a_{s,m}$, $\tau_s(a_{s,m+1}) = a_{s,m+1}$ and $\tau_s(r) \neq r$ for all $r \in (a_{s,m}, a_{s,m+1})$;
(3) Let $J_{s,m}$ and $J_{l,j}$ be two vertices of G_d. There is a directed edge connecting $J_{s,m}$ to $J_{l,j}$ if and only if $\exists$ an $r \in J_{s,m}$, $r > a_{l,j+1}$, and a $t \geq 1$ such that $\tau_s^t(r) \in J_{l,j}$.
(4) Let $J_{s,m}$ and $J_{l,j}$ be two vertices of G_u. There is a directed edge connecting $J_{s,m}$ to $J_{l,j}$ if and only if $\exists$ an $r \in J_{s,m}$, $r < a_{l,j}$, and a $t \geq 1$ such that $\tau_s^t(r) \in J_{l,j}$.
(5) By the *out-degree* of a vertex we mean the number of outgoing directed edges from this vertex in the graph, and by the *in-degree* of a vertex we mean the number of incoming directed edges incident to this vertex in the graph.
(6) A vertex is called a *source* if it is a vertex with in-degree equals to zero. A vertex is called a *sink* if it is a vertex with out-degree equals to zero.

For the above graphs, one can identify two types of vertices: let $(a_{s,m}, a_{s,m+1})$ be a vertex. If $\tau_s(r) > r$ for all $r \in (a_{s,m}, a_{s,m+1})$, then the vertex $(a_{s,m}, a_{s,m+1})$ will be denoted by $\hat{J}_{s,m}$. If $\tau_s(r) < r$ for all $r \in (a_{s,m}, a_{s,m+1})$, then the vertex $(a_{s,m}, a_{s,m+1})$ will be denoted by $\check{J}_{s,m}$. When we prove a statement for a vertex $J_{s,m}$ (without a label), this means that the result holds for both types of vertices. The following lemma contains some properties of G_d and G_u.

Lemma 5.3. *Let G_d and G_u be defined as above.*

1. *If $\hat{J}_{s,m}$ is a vertex in G_d, then $\hat{J}_{s,m}$ is a sink in G_d.*
2. *Let $\check{J}_{s,m}$ and $J_{l,j}$ be two vertices in G_d. There is a directed edge connecting $\check{J}_{s,m}$ to $J_{l,j}$ in G_d if and only if $a_{s,m} < a_{l,j+1} < a_{s,m+1}$. In particular for all $s \in S$ there is no directed edge in G_d connecting $J_{s,m}$ to $J_{s,j}$ for any m and j.*
3. *If $\check{J}_{s,m}$ is a vertex in G_u, then $\check{J}_{s,m}$ is a sink in G_u.*
4. *Let $\hat{J}_{s,m}$ and $J_{l,j}$ be two vertices in G_u. There is a directed edge connecting $\hat{J}_{s,m}$ to $J_{l,j}$ in G_u if and only if $a_{s,m} < a_{l,j} < a_{s,m+1}$. In particular for all $s \in S$ there is no directed edge in G_u connecting $J_{s,m}$ to $J_{s,j}$ for any m and j.*

Proof. The proof of the first statement is straight forward. Indeed, let $J_{l,j}$ be any vertex in G_d and $r \in \hat{J}_{s,m}$ such that $r > a_{l,j+1}$. Then for all $t \geq 1$ $\tau_s^t(r) > \tau_s^{t-1}(r) > \dots \tau_s(r) > r > a_{l,j+1}$. The proof of the second statement follows from the fact that if $r > a_{s,m} \geq a_{l,j+1}$ then for $t \geq 1$ we have $\tau_s^t(r) > a_{s,m} \geq a_{l,j+1}$. If $r > a_{l,j+1} > a_{s,m}$, then there exits a $t \geq 1$ such that $a_{s,m} < \tau_s^t(r) < a_{l,j+1}$. Proofs of the third and fourth statements are similar to the first two. □

For our further analysis we introduce the following notion.

Definition 5.4. We say that a random orbit of F stays above a point c if all the points of the infinite orbit are bigger than or equal to c with probability $\pi_{\mathbf{p}} = 1$.

Lemma 5.5. *Let $J_{l,j}$ be a vertex in G_d such that $a_{l,j+1} \neq 1$. If $J_{l,j}$ is a source in G_d, then the random orbit of F starting from $r > a_{l,j+1}$ stays above $a_{l,j+1}$ with probability $\pi_{\mathbf{p}} = 1$.*

Proof. Suppose $J_{l,j}$ is a source in G_d. Then for all $r > a_{l,j+1}$, we have $\tau_s^t(r) > a_{l,j+1}$ for all $s \in S$ and $t \geq 1$. This means that if $r > a_{l,j+1}$ we have $\tau_{s_1}(r) > a_{l,j+1}$ and $\tau_{s_2} \circ \tau_{s_1}(r) > a_{l,j+1}$ and so on. □

Theorem 5.6. *Let F be an IFS whose transformations satisfy the properties (A), (B) and (C).*

1. *If for all $s \in S$ there is a vertex $\check{J}_{s,m}$ in G_d with $a_{s,m} = 0$, then δ_0 is an SRB measure, $B(\delta_0) \supseteq [0, a)$, where $a = \min_s\{a_{s,m+1}\}$. In particular, for any $r_0 \in [0, a)$, $\lim_t r_t(s^t) = 0$ a.s.*
2. *If for all $s \in S$ there is a vertex $\hat{J}_{s,m}$ in G_d with $a_{s,m+1} = 1$, then δ_1 is an SRB measure. Moreover, $B(\delta_1) \supseteq (b, 1]$, where $b = \max_s\{a_{s,m}\}$. In particular, for any $r_0 \in (b, 1]$, $\lim_t r_t(s^t) = 1$ a.s.*
3. *Let $J_{l,j}$ be a vertex in G_d such that $a_{l,j+1} \neq 1$. If $J_{l,j}$ is a source in G_d[7] then F preserves a probability measure whose support is separated from 0 [8].*
4. *Let $J_{l,j}$ be a vertex in G_u such that $a_{l,j+1} \neq 0$. If $J_{l,j}$ is a source in G_u then F preserves a probability measure whose support is separated from 1.*

[7] In the case where $a_{l,j} = 0$, even if other $\hat{J}_{s,m}$, with $a_{s,m} = 0$, receives a directed edge, the result still holds. Thus, to know the existence of an invariant probability measure whose support is separated from 0, it is enough to check that one vertex $J_{l,j}$ with $a_{l,j} = 0$ which is a source in G_d. Statements of Lemma 5.3 can be useful to visualize cases of this type.

[8] The invariant measure here is not necessarily δ_1.

(5) *Let* $\hat{J}_{s^*,m}$ *be a vertex with* $a_{s^*,m} = 0$ *whose out-degree in* G_u *is at least one. If* $\exists$ *a vertex* $J_{s_0,j}$ *in* G_d, $a_{s_0,j} = 0$ *and* $a_{s_0,j+1} < a_{s^*,m+1}$, *which is a source in* G_d, *then for any* $r_0 \in (0,1]$, $\lim_t r_t(s^t) \neq 0$ *a.s. Moreover,* δ_0 *is not an SRB measure for* F.

(6) *Let* $\check{J}_{s_0,j}$ *be a vertex in* G_d *such that* $a_{s_0,j+1} = 1$ *and whose out-degree in* G_d *is at least one. If* $\exists$ *a* $J_{s^*,m}$ *in* G_u, $a_{s^*,m+1} = 1$ *and* $a_{s^*,m} > a_{s_0,j}$, *which is a source in* G_u, *then for any* $r_0 \in [0,1)$, $\lim_t r_t(s^t) \neq 1$ *a.s. Moreover,* δ_1 *is not an SRB measure for* F.

(7) *If for all* $s \in S$ *the vertices whose* $a_{s,m} = 0$ *are of the form* $\hat{J}_{s,m}$ *and their* $a_{s,m+1} \equiv a$ *are identical, then for any* r_0 *in* $(0,a]$, *with probability one,* $\lim_t r_t(s^t) = a$. *In particular,* δ_a *is an SRB measure with* $B(\delta_a) = (0,a]$ *and* δ_0 *is not an SRB measure.*

(8) *If for all* $s \in S$ *the vertices whose* $a_{s,m+1} = 1$ *are of the form* $\check{J}_{s,m}$ *and their* $a_{s,m} \equiv b$ *are identical, then for any* r_0 *in* $[b,1)$, *with probability one,* $\lim_t r_t(s^t) = b$. *In particular,* δ_b *is an SRB measure with* $B(\delta_b) = [b,1)$ *and* δ_1 *is not an SRB measure.*

Proof. We only prove the odd numbered statements in the theorem. Proofs of the even numbered statements are very similar.
(1) For any $r_0 \in [0,a)$, any random orbit of F will converge to zero. Using Lemma 4.1, this shows that δ_0 is an SRB measure with $B(\delta_0) \supseteq [0,a)$.
(3) Let $r_0 > a_{l,j+1}$. Since $[0,1]$ is a compact metric space and for all $s \in S$ τ_s is continuous, the average $\frac{1}{T}\sum_{t=0}^{T-1} P^t \delta_{r_0}$ of the probability measures converges in the weak* topology to an F invariant probability measure[9]. By Lemma 5.5, this measure is supported on $[a_{l,j+1}, 1]$.
(5) Let $D = \{J_{s,m} \setminus \{0\} : a_{s,m} = 0\}$. For any $r_0 \in D$, there exists a finite $t \geq 1$ such that $\tau^t_{s^*}(r_0) > a_{s_0,j+1}$. Since $J_{s_0,j}$ is a source in G_d, by Lemma 5.5, $\tau^t_{s^*}(r_0)$ stays above $a_{s_0,j+1}$ with probability $\pi_{\mathbf{p}} = 1$. Therefore, for any $r_0 \in D$, with positive probability, the random orbit of r_0 is bounded away from 0. Let us consider now the case of a starting point $r'_0 > a_{s_0,m+1}$. Since all the transformations are homeomorphisms and 0 is a common fixed point, for any $r'_0 > a_{s_0,m+1}$ and any $t \geq 0$, with positive probability, $r_t(s^t) > a_{s_0,m+1}$. Hence, for any $r \in (0,1]$, with strictly positive probability, $\lim_{t\to\infty} r_t(s^t) \geq a_{s_0,m+1}$. Moreover, with strictly positive probability, for any $r \in (0,1]$, there exists a $T-1 > t_0 \geq 1$ such that

$$\frac{1}{T}\sum_{t=0}^{T-1} r_t(s^t) \geq \frac{1}{T}\sum_{t=0}^{t_0-1} r_t(s^t) - \frac{(t_0+1)}{T} a_{s_0,m+1} + a_{s_0,m+1}.$$

Therefore, with strictly positive probability, for any $r \in (0,1]$,

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} r_t(s^t) \geq a_{s_0,m+1}. \tag{5.1}$$

Now, to show that δ_0 is not an SRB measure, it is enough to find a continuous function f on $[0,1]$ such that with positive probability, for any $r \in (0,1]$,

$$\{\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(r_t(s^t)) \neq f(0)\}.$$

[9]This follows from a random version of the Krylov-Bogoliubov Theorem [1].

Indeed, this is the case if we use the function $f(r) = r$ and (5.1). Thus, δ_0 is not an SRB measure..
(7) Obviously, for any $r_0 \in (0, a]$, the random orbit of F starting at r_0 will converge to a. Using Lemma 4.1, this implies that δ_a is an SRB measure with $B(\delta_a) = (0, a]$. Moreover, since all the transformations are homeomorphisms with common fixed point at a, for any $r_0' > a$, the random orbit of F stays above a. Thus, δ_0 is not an *SRB* measure. □

6. Asset Market Games

In this section, we apply our results to *evolutionary models of financial markets*. In particular, we will focus on the model introduced by [4]. First, we recall the model of [4].

6.1. **The Model.** Let S is a finite set and $s_t \in S$, $t = 1, 2, ...,$ be the "state of the world" at date t. Let $\mathbf{p}$ be a probability distribution on S such that for all $s \in S$ $\mathbf{p}(s) > 0$. We also assume that s_t are independent and identically distributed.

In this model there are K *"short-lived" assets* $k = 1, 2, ..., K$ (live one period and are identically reborn every next period). One unit of asset k issued at time t yields payoff $D_k(s_{t+1}) \geq 0$ at time $t + 1$. It is assumed that

$$\sum_{k=1}^{K} D_k(s) > 0 \quad \text{for all} \quad s \in S$$

and

$$ED_k(s_t) > 0$$

for each $k = 1, 2, ..., K$, where E is the expectation with respect to the underlying probability $\mathbf{p}$. The total amount of asset k available in the market is $V^k = 1$.

In this model there are I *investors (traders)* $i = 1, ..., I$. Every investor i at each time $t = 0, 1, 2, ...$ has a *portfolio*

$$x_t^i = (x_{t,1}^i, ..., x_{t,K}^i),$$

where $x_{t,k}^i$ is the number of units of asset k in the portfolio $x_t^i = x_t^i(s^t)$, $s^t = (s_1, ..., s_t)$. We assume that for each moment of time $t \geq 1$ and each random situation s^t, the market for every asset k clears:

$$\sum_{i=1}^{I} x_{t,k}^i(s^t) = 1. \tag{6.1}$$

Each investor is endowed with initial wealth $w_0^i > 0$. Wealth w_{t+1}^i of investor i at time $t + 1$ can be computed as follows:

$$w_{t+1}^i = \sum_{k=1}^{K} D_k(s_{t+1}) x_{t,k}^i. \tag{6.2}$$

Total market wealth at time $t + 1$ is equal to

$$w_{t+1} = \sum_{i=1}^{I} w_{t+1}^i = \sum_{k=1}^{K} D_k(s_{t+1}). \tag{6.3}$$

Investment strategies are characterized in terms of *investment proportions:*

$$\Lambda^i = \{\lambda_0^i, \lambda_1^i, \lambda_2^i, ...\}$$

of K-dimensional vector functions $\lambda_t^i = (\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$, $\lambda_{t,k}^i = \lambda_{t,k}^i(s^t)$ $t \geq 0$, satisfying $\lambda_{t,k}^i > 0$, $\sum_{k=1}^K \lambda_{t,k}^i = 1$. Here, $\lambda_{t,k}^i$ stands for the *share of the budget* w_t^i of investor i that is invested into asset k at time t. In general $\lambda_{t,k}^i$ may depend on $s^t = (s_1, s_2, ..., s_t)$. Given strategies $\Lambda^i = \{\lambda_0^i,\ \lambda_1^i,\ \lambda_2^i,\ ...\}$ of investors $i = 1, ..., I$, the equation

$$p_{t,k} \cdot 1 = \sum_{i=1}^I \lambda_{t,k}^i w_t^i \tag{6.4}$$

determines the market clearing *price* $p_{t,k} = p_{t,k}(s^t)$ of asset k. The number of units of asset k in the portfolio of investor i at time t is equal to

$$x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{p_t^k}. \tag{6.5}$$

Therefore

$$x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^I \lambda_{t,k}^j w_t^j}. \tag{6.6}$$

By using (6.6) and (6.2), we get

$$w_{t+1}^i = \sum_{k=1}^K D_k(s_{t+1}) \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^I \lambda_{t,k}^j w_t^j}. \tag{6.7}$$

Since $w_0^i > 0$, we obtain $w_t^i > 0$ for each t. The main focus of the model is on the analysis of the dynamics of the *market shares* of the investors

$$r_t^i = \frac{w_t^i}{w_t},\ i = 1, 2, ..., I.$$

Using (6.7) and (6.3), we obtain

$$r_{t+1}^i = \sum_{k=1}^K R_k(s_{t+1}) \frac{\lambda_{t,k}^i r_t^i}{\sum_{j=1}^I \lambda_{t,k}^j r_t^j},\ i = 1, 2, ..., I, \tag{6.8}$$

where

$$R_k(s_{t+1}) = \frac{D_k(s_{t+1})}{\sum_{m=1}^K D_m(s_{t+1})}$$

are the *relative* (*normalized*) *payoffs* of the assets $k = 1, 2, ..., K$. We have $R_k(s) \geq 0$ and $\sum_k R_k(s) = 1$.

6.2. **Performance of investment strategies and the Kelly rule.** In the theory of *evolutionary finance* there are three possible grades for investor i (or for the strategy she/he employs):

(i) *extinction*: $\lim r_t^i = 0$ a.s.;
(ii) *survival*: $\limsup r_t^i > 0$ but $\liminf r_t^i < 1$ a.s.;
(iii) *domination*: $\lim r_t^i = 1$ a.s.

Definition 6.1. An investment strategy is called *completely mixed strategy* if it assigns a positive percentage of wealth $\lambda_{t,k}(s^t)$ to every asset $k = 1, \ldots, K$ for all t and s^t; moreover, it is called *simple* if $\lambda_{t,k}(s^t) = \lambda_k > 0$.

In this theory, the following *simple* portfolio rule has been very successful: define

$$\lambda^* = (\lambda_1^*, ..., \lambda_K^*),\ \lambda_k^* = ER_k(s_t),\ k = 1, ..., K,$$

so that $\lambda_1^*, ..., \lambda_K^*$ are the *expected relative payoffs* of assets $k = 1, ..., K$. The portfolio rule λ^* is called the *Kelly rule* which expresses the investment principle of "betting your beliefs" [9]. In [4] under the following two conditions:

E1) There are no *redundant assets*, i.e. the functions $R_1(s), ..., R_K(s)$ of $s \in S$ are linearly independent.

E2) All investors use *simple strategies;*

it was shown that investors who follow the Kelly rule survive and others who use a different simple strategy get extinct. In particular, If only one investor follows the Kelly rule, then this investor dominates the market.

The main challenge in using the Kelly rule lies in the fact that it requires from investors the full knowledge of the probability distribution $\mathbf{p}$. In Subsection 6.4, using an IFS representation of (6.8) and Theorem 4.3, we overcome this difficulty by finding another successful strategy which requires partial knowledge of the probability distribution $\mathbf{p}$.

6.3. **An IFS realization of the model.** In the rest of the paper, we are going to show how the above model can be represented by an IFS. We are going to apply the results of Sections 4 and 5 to study the dynamics of (6.8). As in [4], we assume here that all the investors use simple strategies. Further, we focus on the case[10] when $I = 2$. The market selection process (6.8) reduces to the following one dimensional system:

$$r_{t+1}(s^{t+1}) = \sum_{k=1}^{K} R_k(s_{t+1}) \frac{\lambda_k^1 r_t}{\lambda_k^1 r + \lambda_k^2 (1 - r_t)}, \tag{6.9}$$

where r_t is investor's 1 relative market share at time t and $(\lambda_k^1)_{k=1}^K$ and $(\lambda_k^2)_{k=1}^K$ are the investment strategies of investor 1 and 2 respectively. Then the random dynamical (6.9) of the market selection process can be described by an iterated function system with probabilities:

$$F = \{\tau_1, \tau_2, \ldots, \tau_L; p_1, p_2, \ldots, p_L\},$$

where

$$\tau_s(r) = \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^1 r}{\lambda_k^1 r + \lambda_k^2 (1 - r)}.$$

We first note that the transformations τ_s of the IFS of the market selection process are maps from the unit interval into itself and they satisfy assumptions (A), (B) and (C). In fact, the maps for this model have additional properties. For example, they are differentiable functions.

[10]This is the same as assuming that there are I investors, $I > 2$, where $I - 1$ investors use the same strategy and only one investor deviates from them.

6.4. **Investors with partial information on p and a generalization of the Kelly rule.** We use Theorem 4.3 to provide a rule for investors with partial information on $\mathbf{p}$. The investor who follows this rule cannot be driven out of the market; i.e., she/he either *dominates* or at least *survives*. The importance of this rule lies in the fact that investor 1 does not need to know the Kelly rule exactly[11]. She/he only needs to know a perturbation of the Kelly rule; for example, the Kelly rule plus some error bounds.

Firstly, we show in the following lemma that the logarithms of the exponents $\beta_s(r)$ are uniformly bounded.

Lemma 6.2. *Let*

$$\tau(r) = \sum_{k=1}^{K} R_k \frac{\lambda_k^1 r}{\lambda_k^1 r + \lambda_k^2 (1-r)} \ , \quad r \in [0,1],$$

and

$$\tau(r) = r^{\beta(r)},$$

where, for each $1 \le k \le K$ *we have* $R_k \ge 0$, $\lambda_k^1 > 0$, $\lambda_k^2 > 0$ *and* $\sum_{k=1}^K R_k = \sum_{k=1}^K \lambda_k^1 = \sum_{k=1}^K \lambda_k^2 = 1$. *Then for any* $r \in U$, $U \subseteq [0,1]$, $\ln(\beta(r))$ *is bounded.*

Proof. Without loss of generality, we assume that $U = [0,1]$. We have $\tau(r) = r^{\beta(r)} = \exp(\ln(r)\beta(r))$, so

$$\beta(r) = \frac{\ln(\tau(r))}{\ln(r)}.$$

The minimum and maximum of $\beta(r)$ can be attained at $r = 0$, $r = 1$ or at a point of a local extremum. Using De L'Hospital rule we find

$$\lim_{r \to 0^+} \beta(r) = 1 \qquad \text{and} \qquad \lim_{r \to 1^-} \beta(r) = \sum_{k=1}^{K} R_k \frac{\lambda_k^2}{\lambda_k^1}.$$

A point of local extremum r_* in $(0,1)$ of $\beta(r)$ is found by solving

$$\beta'(r) = \frac{1}{\ln(r)} \left(\frac{\tau'(r)}{\tau(r)} - \frac{B(r)}{r} \right) = 0.$$

Therefore, at the point $r = r_*$ of local extremum

$$\beta(r_*) = \frac{\sum_{k=1}^K R_k \frac{\lambda_k^1 \lambda_k^2}{[\lambda_k^1 r_* + \lambda_k^2 (1-r_*)]^2}}{\sum_{k=1}^K R_k \frac{\lambda_k^1}{\lambda_k^1 r_* + \lambda_k^2 (1-r_*)}}.$$

Observe that the function

$$\frac{\sum_{k=1}^K R_k \frac{\lambda_k^1 \lambda_k^2}{[\lambda_k^1 r + \lambda_k^2 (1-r)]^2}}{\sum_{k=1}^K R_k \frac{\lambda_k^1}{\lambda_k^1 r + \lambda_k^2 (1-r)}}$$

is continuous at $[0,1]$. Thus, it attains its maximum and minimum on $[0,1]$. This completes the proof of the lemma. □

[11] It is often difficult for an investor to know the exact probability distribution of the states of the world.

Corollary 6.3. Let

$$\tau_s(r) = \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^1 r}{\lambda_k^1 r + \lambda_k^2 (1-r)} \ , \quad r \in [0,1].$$

Then for $r \in U$, $U \subseteq [0,1]$,

$$b_s = \min_{r \in \bar{U}} \frac{\sum_{k=1}^{K} R_k \frac{\lambda_k^1 \lambda_k^2}{[\lambda_k^1 r + \lambda_k^2 (1-r)]^2}}{\sum_{k=1}^{K} R_k \frac{\lambda_k^1}{\lambda_k^1 r + \lambda_k^2 (1-r)}} \qquad \text{and} \qquad B_s = \max_{r \in \bar{U}} \frac{\sum_{k=1}^{K} R_k \frac{\lambda_k^1 \lambda_k^2}{[\lambda_k^1 r + \lambda_k^2 (1-r)]^2}}{\sum_{k=1}^{K} R_k \frac{\lambda_k^1}{\lambda_k^1 r + \lambda_k^2 (1-r)}}.$$

Theorem 6.4. *If for each $k \in \{1, \dots, K\}$ λ_k^1 lies between ER_k and λ_k^2, then investor 1 cannot be driven out of the market; i.e., she/he either dominates or at least survives.*

Proof. Let us consider the function

$$G(r) = \sum_{k=1}^{K} v_k \frac{\lambda_k^1}{\Lambda_k(r)} \quad , \quad r \in [0,1],$$

where

$$\Lambda_k(r) = \lambda_k^1 r + \lambda_k^2 (1-r) = (\lambda_k^1 - \lambda_k^2) r + \lambda_k^2,$$

and $V = (v_1, v_2, \dots, v_L)$ is a probability vector. We will find conditions on λ_k^1 which ensure $G(r) \geq 1$, $r \in [0,1]$. It is easy to see that

$$G(1) = \sum_{k=1}^{K} v_k \frac{\lambda_k^1}{\lambda_k^1} = 1. \tag{6.10}$$

We also have

$$G'(r) = \sum_{k=1}^{K} v_k \lambda_k^1 \frac{-(\lambda_k^1 - \lambda_k^2)}{(\Lambda_k(r))^2} \quad , \quad r \in [0,1],$$

and

$$G''(r) = \sum_{k=1}^{K} v_k \lambda_k^1 \frac{2(\lambda_k^1 - \lambda_k^2)^2}{(\Lambda_k(r))^3} > 0 \quad , \quad r \in [0,1].$$

Thus, G is a convex function and its derivative G' is increasing. If $G'(1) \leq 0$ then G is decreasing and because of (6.10) this implies that $G(r) \geq 1$, $r \in [0,1]$. Observe that

$$G'(1) = \sum_{k=1}^{K} v_k \lambda_k^1 \frac{-(\lambda_k^1 - \lambda_k^2)}{(\lambda_k^1)^2} = \sum_{k=1}^{K} \frac{v_k}{\lambda_k^1} (\lambda_k^2 - \lambda_k^1).$$

It is easy to see that a sufficient condition for $G'(1) \leq 0$ is

$$\begin{aligned} v_k &\geq \lambda_k^1 \quad \text{if} \quad \lambda_k^1 \geq \lambda_k^2; \\ v_k &\leq \lambda_k^1 \quad \text{if} \quad \lambda_k^1 \leq \lambda_k^2, \end{aligned} \tag{6.11}$$

or, in short, for each k, $1 \leq k \leq K$, λ_k^1 should be between λ_k^2 and v_k.

Now, let us consider the expression

$$\sum_{s=1}^{L} p_s \ln(\beta_s(r)).$$

We have

(6.12)

$$\begin{aligned}
&\sum_{s=1}^{L} p_s \ln(\beta_s(r)) \le \ln\left(\sum_{s=1}^{L} p_s\beta_s(r)\right) = \ln\left(\sum_{s=1}^{L} p_s \frac{\ln(\tau_s(r))}{\ln r}\right)\\
&\le \ln\left(\frac{1}{\ln r}\ln\left(\sum_{s=1}^{L} p_s\tau_s(r)\right)\right) = \ln\left(\frac{1}{\ln r}\ln\left(\sum_{s=1}^{L} p_s \sum_{k=1}^{K} R_k(s)\frac{\lambda_k^1 r}{\Lambda_k(r)}\right)\right)\\
&= \ln\left(\frac{1}{\ln r}[\ln r + \ln\left(\sum_{k=1}^{K}(\sum_{s=1}^{L} p_s R_k(s))\frac{\lambda_k^1}{\Lambda_k(r)}\right)]\right) = \ln\left(1 + \frac{1}{\ln r}\ln(G(r))\right),
\end{aligned}$$

with v_k being the expected payoff for the k^{th} asset, $v_k = \sum_{s=1}^{L} p_s R_k(s)$, $k = 1, \dots, K$.

A sufficient condition for $\sum_{s=1}^{L} p_s \ln(\beta_s(r)) \le 0$ is for $r \in [0,1]$:

$$\ln(G(r)) \ge 0 \quad \text{or equivalently} \quad G(r) \ge 1.$$

We have shown before that a sufficient condition for this is (6.11) or placing each λ_k^1 between the expected payoff v_k and λ_k^2.

To complete the proof of the theorem, we first use Lemma 6.2 to observe that exponents $\beta_s(r)$ of this system are bounded and then (1) of Theorem 4.3. Indeed, for any fixed partial history s^{t-2}, because the stochastic process s_t is an iid process, we have

$$E(\ln \alpha_t | s^{t-1}) = \sum_{s=1}^{L} p_s \ln(\beta_s(r_{t-2})).$$

□

6.5. **Incorrect beliefs.** Our results in Section 5 are also interesting for studying the dynamics of (6.8). In fact, they can be used to study the dynamics in the situation where both players have 'incorrect beliefs'; i.e., when players do not have the right information or partial information about **p**. Thus, they either use wrong distributions to build their strategies or they arbitrarily choose their strategies. Consequently, their strategies are, in general, different from the Kelly rule and the generalization which we presented in Subsection 6.4. In this case, the results of Section 5 can be used to identify the exact outcome of the game in certain situations. In some situations, as in Example 5.1, one cannot know the outcome of the system without knowing **p**.

7. Appendix

The following general arcsine law has been proved in [6].

Theorem 7.1. [6] *Let $X_1, X_2, \dots$ be a sequence of random variables adapted to the sequence of σ-algebras $\mathcal{A}_1, \mathcal{A}_2, \dots$. Let $S_m = \sum_{i=1}^{m} X_i$, $v_m = \sum_{i=1}^{m} E(X_i^2|\mathcal{A}_{i-1})$ and assume*

$$E(X_{m+1}|\mathcal{A}_m) = 0\ , \qquad EX_m^2 < \infty\ , \quad \text{and} \quad v_m \to \infty \ a.s.$$

Let $T_n = \inf\{m : v_m \ge n\}$ and $L_n = \frac{1}{n}\sum_{i=1}^{T_n} E(X_i^2|\mathcal{A}_{i-1})\chi_{\{S_i>0\}}$. If

$$\frac{1}{n}\sum_{i=1}^{T_n} X_i^2 \chi_{\{X_i^2 > n\varepsilon\}} \xrightarrow[L_1]{} 0 \quad \text{for all } \varepsilon > 0,$$

then the distributions of L_n converge to the arcsine distribution.

We now use Theorem 7.1 to prove a proposition which is used in the proof of Theorem 4.7.

Proposition 7.2. Let $X_1, X_2, \dots$ be a sequence of random variables adapted to the sequence of σ-algebras $\mathcal{A}_1, \mathcal{A}_2, \dots$. Suppose that there exist constants $d > 0$ and $0 < D < \infty$ such that for all $n \geq 1$ we have

$$0 < d \leq E(X_n^2 | \mathcal{A}_{n-1}) \text{ and } X_n^2 \leq D.$$

Then, the sequence satisfies the remaining assumptions of Theorem 7.1. In particular, Theorem 7.1 implies the condition

$$\limsup_{n\to\infty} Pr(\frac{\mathrm{Pos}_n}{n} \leq a) \geq b > 0, \tag{7.1}$$

for some constants $0 < a, b < 1$, where $\mathrm{Pos}_n = \sum_{i=1}^n \chi_{\{S_i > 0\}}$.

Proof. The remaining assumptions of Theorem 7.1 are trivially satisfied. We have $m \cdot d \leq v_m \leq m \cdot D$ for all $m \geq 1$ so $T_n \cdot d \leq n \leq T_n \cdot D$ for all $n \geq 1$. Then,

$$L_n \geq \frac{1}{D}\frac{1}{T_n} d \sum_{i=1}^{T_n} \chi_{\{S_i>0\}}$$

and, for $0 \leq a_1 \leq 1$, we have

$$Pr\left(\frac{d}{D}\frac{1}{T_n}\sum_{i=1}^{T_n} \chi_{\{S_i>0\}} \leq a_1\right) \geq Pr(L_n \leq a_1) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arcsin \sqrt{a_1}.$$

For a_1 small enough we obtain a meaningful estimate

$$Pr\left(\frac{1}{T_n}\sum_{i=1}^{T_n} \chi_{\{S_i>0\}} \leq a\right) \geq \frac{1}{\pi} \arcsin \sqrt{a},$$

for $a = a_1 \frac{D}{d}$ and n large enough. This implies condition (7.1). □

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