QUASI-INVARIANT MEASURES, ESCAPE RATES AND THE EFFECT OF THE HOLE

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ABSTRACT. Let T be a piecewise expanding interval map and T_H be an abstract perturbation of T into an interval map with a hole. Given a number ℓ, $0 < \ell < 1$, we compute an upper-bound on the size of a hole needed for the existence of an absolutely continuous conditionally invariant measure (accim) with escape rate not greater than $-\ln(1 - \ell)$. The two main ingredients of our approach are Ulam's method and an abstract perturbation result of Keller and Liverani.

1. Introduction

Open dynamical systems have recently been a very active topic of research in ergodic theory and dynamical systems. Such dynamical systems are used in studying nonequilibrium statistical mechanics [19] and metastable chaos [22].

A dynamical system is called open if there is a subset in the phase space such that whenever an orbit lands in it, the dynamics of this obit is terminated; i.e, the orbit dies or disappears. The subset through which orbits escape is called a hole, denoted H. The escape rate through H can be measured if the system admits an absolutely continuous conditionally invariant measure (accim). The first result in this direction is due to Pianigiani and Yorke [18]. The survey article [5] contains a considerable list of references on the existence of accim and its relation to other measures. One of the most intuitive existence results is found in Section 7 of [16]. It is mainly concerned with systems having small holes and its idea is based on the perturbation result of [10]. It roughly says that if a mixing interval map is perturbed by introducing a 'sufficiently small' hole, then the resulting open dynamical system admits an accim. Our main goal in this paper is to show how the condition 'sufficiently small' can be computationally verified in some of these results, in particular, results from Section 7 of [16].

More precisely, for a given Lasota-Yorke map T, we use Ulam's method on the closed dynamical system T to give a computable size of the hole H for which the open dynamical system T_H must admit an accim.

Historically, Ulam approximations have been used to provide rigorous estimates of invariant densities of closed systems (see [15] and references cited there) or to

approximate other dynamical invariants (see [8]). The use of Ulam's method in the study of open systems is natural. In [1], Ulam approximations were used to rigorously estimate the escape rate for certain open systems. In that computation the Ulam matrix was derived from the Perron-Frobenius operator associated to the open system (i.e., a sub-stochastic matrix), whereas here, we approximate the closed system. The method in [1] also demanded existence of an accim as a basic assumption.

We remark that, as a consequence of the spectral methods discussed here, upper bounds on the escape rate can be obtained from analysis of the closed system. However this does not generally replace the computation in [1] where the Ulam approximation of the open system yields both upper and lower bounds on the escape rate (but under the additional assumption of existence of an accim). Hence, there is potential to apply a two step method – the current algorithm would be used to guarantee an accim and to provide rough (upper) bounds on the escape rate, followed by the method of [1], once the size of the hole is fixed, to more accurately estimate the latter.

Our paper is organized as follows. In Section 2 we present a version of Keller and Liverani's abstract perturbation theorem. The constants which are involved in this theorem are essential in all our computations and thus, we need to state all the details of this theorem explicitly. Section 3 contains a precise setting of the problem. Section 4 contains technical lemmas, mostly well-known and stated without proof. Section 5 presents the algorithm (Algorithm 5.1) whose outputs $(\delta_{com}, \varepsilon_{com})$ are the parameters used to solve our problem. Theorem 5.3, takes these parameters and computes a maximum hole size leading to an accim. In Section 6 we provide, in detail, rigorous computations of the size of a hole for two examples as well as a discussion of computational overhead and some techniques for reducing computation time. In Section 7 we discuss how our methods can be implemented in a smooth setting where there are interesting results concerning the effect of the position of a hole on the escape rate. This is in connection with the recent results of [4] and [11]. The examples of Section 6 and the discussion of Section 7 highlight the new results that Algorithm 5.1 brings to open dynamical systems.

2. The First Keller-Liverani Perturbation Result

Let $(I, \mathfrak{B}, \lambda)$ be the measure space where $I = [0, 1]$, $\mathfrak{B}$ is the Borel σ-algebra and λ is Lebesgue measure. Let $L^1 = L^1(I, \mathfrak{B}, \lambda)$. For $f \in L^1$, we define

$$V f = \inf_{\overline{f}} \{ \operatorname{var} \overline{f} : f = \overline{f} \text{ a.e.} \},$$

where

$$\operatorname{var} \overline{f} = \sup \{ \sum_{i=0}^{l-1} |\overline{f}(x_{i+1}) - \overline{f}(x_i)| : 0 = x_0 < x_1 < \cdots < x_l = 1 \}.$$

We denote by BV the space of functions of bounded variation on I equipped with the norm $\| \cdot \|_{BV} = V(\cdot) + \| \cdot \|_1$ [7]. Let $P_i : BV(I) \to BV(I)$ be two bounded linear operators, $i = 1, 2$. We assume that: For $f \in L^1$

(2.1) $$\|P_i f\|_1 \leq \|f\|_1,$$

and $\exists\, \alpha \in (0,1)$, $A > 0$ and $B \geq 0$ such that

(2.2) $\qquad \|P_i^n f\|_{BV} \leq A\alpha^n \|f\|_{BV} + B\|f\|_1 \quad \forall n \in \mathbb{N}\ \forall f \in BV(I),\ i = 1, 2.$

Further, we introduce the mixed operator norm:

$$|||P_i||| = \sup_{\|f\|_{BV} \leq 1} \|P_i f\|_1.$$

For any bounded linear operator $P : BV \to BV$ with spectrum $\sigma(P)$, consider the set

$$V_{\delta,r}(P) = \{z \in \mathbb{C} : |z| \leq r \text{ or } \operatorname{dist}(z, \sigma(P)) \leq \delta\}.$$

Since the complement of $V_{\delta,r}(P)$ belongs to the resolvent of P, it follows that ([7] Lemma 11, VII.6.10)

$$H_{\delta,r}(P) = \sup_z \{\|(z - P)^{-1}\|_{BV} : z \in \mathbb{C} \setminus V_{\delta,r}\} < \infty.$$

Remark 2.1. α in (2.2) is an upper bound on the essential spectral radius of P_i [2].

Theorem 2.2. [10, 15] *Consider two operators $P_i : BV(I) \to BV(I)$ which satisfy (2.1) and (2.2). For $r \in (\alpha, 1)$, let*

$$n_1 = \lceil \frac{\ln 2A}{\ln r/\alpha} \rceil$$

$$C = r^{-n_1}; \qquad D = A(A + B + 2)$$

$$n_2 = \lceil \frac{\ln 8BDCH_{\delta,r}(P_1)}{\ln r/\alpha} \rceil.$$

If

$$|||P_1 - P_2||| \leq \frac{r^{n_1 + n_2}}{8B(H_{\delta,r}(P_1)B + (1 - r)^{-1})} \stackrel{def}{=} \varepsilon_1(P_1, r, \delta)$$

then for each $z \in \mathbb{C} \setminus V_{\delta,r}(P_1)$, we have

$$\|(z - P_2)^{-1} f\|_{BV} \leq \frac{4(A + B)}{1 - r} r^{-n_1} \|f\|_{BV} + \frac{1}{2\varepsilon_1} \|f\|_1.$$

Set

$$\gamma = \frac{\ln(r/\alpha)}{\ln(1/\alpha)},$$

$$a = \frac{8[2A(A + B) + (1 - r)^{-1}](A + B)^2 r^{-n_1} + 1}{1 - r}$$

and

$$b = 2[(4(A + B)^2(D + B) + B)(1 - r)^{-1} r^{-n_1} + B].$$

If

(2.3)

$$|||P_1 - P_2||| \leq \min\{\varepsilon_1(P_1, r, \delta), \left[\frac{r^{n_1}}{4B\left(H_{\delta,r}(P_1)(D + B) + 2A(A + B) + (1 - r)^{-1}\right)} \right]^{\gamma}\}$$

$$\stackrel{def}{=} \varepsilon_0(P_1, r, \delta)$$

then for each $z \in \mathbb{C} \setminus V_{\delta,r}(P_1)$, we have

(2.4)

$$|||(z - P_2)^{-1} - (z - P_1)^{-1}||| \leq |||P_1 - P_2|||^{\gamma}(a\|(z - P_1)^{-1}\|_{BV} + b\|(z - P_1)^{-1}\|_{BV}^2).$$

Corollary 2.3. [10, 15] *If* $|||P_1 - P_2||| \leq \varepsilon_1(P_1, r, \delta)$ *then* $\sigma(P_2) \subset V_{\delta,r}(P_1)$. *In addition, if* $|||P_1 - P_2||| \leq \varepsilon_0(P_1, r, \delta)$, *then in each connected component of* $V_{\delta,r}(P_1)$ *that does not contain* 0 *both* $\sigma(P_1)$ *and* $\sigma(P_2)$ *have the same multiplicity; i.e., the associated spectral projections have the same rank.*

3. Expanding Interval Maps, Perturbations and Holes

Let T be a non-singular interval map and denote by P the Perron-Frobenius operator associated with T [2]. Typically, and this will be the case for our examples, T will be a Lasota-Yorke map[1]. We assume:

(A1) $\exists \ \alpha_0 \in (0, 1)$, and $B_0 \geq 0$ such that $\forall f \in BV(I)$
$$VPf \leq \alpha_0 Vf + B_0 \|f\|_1.$$

Remark 3.1. Condition (A1) implies that $\rho = 1$ is an eigenvalue of P. In particular, T admits an absolutely continuous invariant measure [2, 3]. Moreover, for any $r \in (\alpha_0, 1)$ there exists a $\bar{\delta}_0 > 0$, $\bar{\delta}_0$ depends on r, such that for any $\delta_0 \in (0, \bar{\delta}_0]$ and any eigenvalue ρ_i of P, with $|\rho_i| > r$, we have:

  (1) $B(\rho_i, \delta_0) \cap B(0, r) = \emptyset$;
  (2) $B(\rho_i, \delta_0) \cap B(\rho_j, \delta_0) = \emptyset$, $i \neq j$.

The inequality of assumption (A1) is known as a Lasota-Yorke inequality[2]. For a given Lasota-Yorke map with $\beta > 2$, the constant $\alpha_0 = 2/\beta$ in inequality (A1) and B_0 may be found in terms of bounds on the second derivative of the map T and the minimum of $x_{i+1} - x_i$.

3.1. **Ulam's approximation of** P. Let η be a finite partition of I into intervals. Let mesh(η) be the mesh size of η; i.e, the maximum length of an interval in η, and let $\mathfrak{B}_\eta$ be the finite σ-algebra associated with η. For $f \in L^1$, let
$$\Pi_\eta f = \mathbb{E}(f|\mathfrak{B}_\eta),$$
where $\mathbb{E}(\cdot|\mathfrak{B}_\eta)$ denotes the conditional expectation with respect to $\mathfrak{B}_\eta$. Specifically, if $x \in I_\eta \in \eta$
$$(\Pi_\eta f)(x) = \frac{1}{\lambda(I_\eta)} \int_{I_\eta} f d\lambda.$$
Define
$$P_\eta = \Pi_\eta \circ P \circ \Pi_\eta.$$
P_η is called Ulam's approximation of P. Using the basis $\{\frac{1}{\lambda(I_\eta)}\chi_{I_\eta}\}$ in L^1, P_η can be represented by a (row) stochastic matrix acting on vectors from $\mathbb{R}^{d(\eta)}$ by right multiplication: $x \to xP_\eta$. The entries of Ulam's matrix are given by:
$$P_{I_\eta J_\eta} = \frac{\lambda(I_\eta \cap T^{-1}J_\eta)}{\lambda(I_\eta)}.$$

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[1]A map T acting from an interval $[a, b]$ to itself is a *Lasota-Yorke map* if it is piecewise C^2 with respect to a finite partition $a = x_0 < x_1 < \cdots < x_n = b$, has well-defined left and right limits of derivatives up to second order at each x_i and is expanding: $\beta := \inf_{x \neq x_i} |T'(x)| > 1$.

[2]In fact, (A1) is slightly stronger than the original Lasota-Yorke inequality. In particular, when T is a general piecewise expanding C^2 map the Lasota-Yorke inequality is given by $VPf \leq 2\beta^{-1}Vf + B_0\|f\|_1$. See [3] for details and for generalizations of the original result of [14]. In certain situations, in particular, when T is piecewise expanding and piecewise onto or when $\inf_x |T'(x)| > 2$, the original Lasota-Yorke inequality reduces to (A1). In principle, when dealing with Ulam's scheme for Lasota-Yorke maps, as we do in this paper, (A1) cannot be relaxed. See [17] for details.

Since P_η can be represented as stochastic matrix, it has a dominant eigenvalue $\rho_\eta = 1$ [13]. Any associated left eigenvectors represent invariant functions $f_\eta \in BV(I)$ for the operator P_η.

3.2. **Interval maps with holes.** Let $H \subset I$ be an open interval. Denote by $T_H \overset{\text{def}}{:=} T_{|X_0}$, where $X_0 = I \setminus H$. We call T_H an interval map with a hole; H being the hole. Its Perron-Frobenius operator, which we denote by P_H, is defined as follows: for $f \in L^1$ and $n \geq 1$

$$P_H^n f = P^n(f\chi_{X_{n-1}}),$$

where $X_{n-1} = \cap_{i=0}^{n-1} T^{-i} X_0$, the set of points whose orbits do not meet the hole H in the first $n-1$ steps.

Definition 3.2. A probability measure μ on $[0, 1]$, $d\mu = f_H^* d\lambda$, is said to be an absolutely continuous conditionally invariant measure (accim) if there exists $0 < e_H < 1$ such that $P_H f_H^* = e_H f_H^*$. In this case $-\ln e_H$ is the escape rate associated to μ.

4. Some Technical Lemmas

We state (for the most part, without proof) some well-known technical results to be used in our computations later in the paper.

4.1. **Lasota-Yorke inequalities and estimates on the difference of operators in the mixed norm.** Lasota-Yorke inequalities for P and P_η are standard results from the literature (see [20] for an original source). The inequality for P_H is not so well-known, but it is straightforward and we derive it below for completeness.

Lemma 4.1. *The operators P and P_η satisfy a common Lasota-Yorke inequality as follows:* $\forall n \in \mathbb{N}$ $\forall f \in BV(I)$

$$\|P^n f\|_{BV} \leq \alpha_0^n \|f\|_{BV} + \hat{B}\|f\|_1$$

$$\|P_\eta^n f\|_{BV} \leq \alpha_0^n \|f\|_{BV} + \hat{B}\|f\|_1$$

with $\hat{B} = 1 + \frac{B_0}{1-\alpha_0}$, independent of η. For T_H, the map with a hole, and under the stronger assumption $\alpha_0 < 1/3$[3] we have: $\forall n \in \mathbb{N}$ $\forall f \in BV(I)$

$$\|P_H^n f\|_{BV} \leq \alpha^n \|f\|_{BV} + B\|f\|_1,$$

where $\alpha = 3\alpha_0 < 1$ and $B = \frac{2\alpha_0 + B_0}{1-\alpha}$ with constants independent of H.

Proof. For the operator P_H, by assumption (A1), for $f \in BV(I)$ we have,

$$
\begin{aligned}
(4.1) \qquad V P_H f = V P(f\chi_{X_0}) &\leq \alpha_0 V(f\chi_{X_0}) + B_0\|f\|_1 \\
&\leq \alpha_0(Vf + 2\sup_{x \in I} f(x)) + B_0\|f\|_1 \\
&\leq \alpha_0(Vf + 2Vf + 2\|f\|_1) + B_0\|f\|_1 \\
&= \alpha Vf + (2\alpha_0 + B_0)\|f\|_1.
\end{aligned}
$$

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[3]The assumption that $\alpha_0 < 1/3$ can be relaxed simply to $\alpha_0 < 1$. This relaxation can still produce a common Lasota-Yorke inequality for P, P_η and P_H with constants A and B independent of H. A common Lasota-Yorke inequality for P and P_H with $\alpha_0 < 1$ can be found in section 7 of [16]. Hence the results of this paper are still valid for $1/3 < \alpha_0 < 1$ using the appropriate Lasota-Yorke inequaility. For the purpose of the examples which we want to present in Section 6, it is more sensible to use the constants produced by Lemma 4.1.

Therefore,

$$VP_H^n f \leq \alpha^n Vf + (\sum_{k=1}^{n} \alpha^{k-1})(2\alpha_0 + B_0)\|f\|_1,$$

and consequently, for all $n \geq 1$

$$\|P_H^n f\|_{BV} \leq \alpha^n \|f\|_{BV} + B\|f\|_1.$$

$\square$

Remark 4.2. Since $\alpha_0 < \alpha$ and $\hat{B} < B$ in the above lemma, we can obtain a common Lasota-Yorke inequality, independent of H and η using coefficients α and B:

$$\|P_*^n f\|_{BV} \leq \alpha^n \|f\|_{BV} + B\|f\|_1$$

where P_* represents any of the three operators under discussion. For ease of exposition, we will use this common inequality in what follows. (However, see Example 6.2 in Section 6 for a discussion indicating how using the strongest possible inequality can significantly reduce computational overhead.)

Lemma 4.3. *Let* $\Gamma = \max\{\alpha_0 + 1, B_0\}$ *and* $\varepsilon = mesh(\eta)$.
  (1) $|||P_\eta - P||| \leq \Gamma \varepsilon$.
  (2) *If* $\lambda(H) \leq \Gamma \varepsilon$ *then* $|||P_\eta - P_H||| \leq 2\Gamma \varepsilon$.

Proof. The first statement is standard. For the proof of the second statement, let $f \in BV(I)$ and observe that

$$\|(P_\eta - P_H)f\|_1 \leq \|(P_\eta - P)f\|_1 + \|(P - P_H)f\|_1$$
$$\leq \varepsilon\Gamma\|f\|_{BV} + \lambda(H)\|f\|_{BV} \leq 2\varepsilon\Gamma\|f\|_{BV}.$$

$\square$

4.2. **Computer-assisted estimates on the spectrum of P.** All the constants arising in Theorem 2.2 are (in principle) computable for the finite-dimensional operator P_η[4]. Thus, as proposed in [15], we are going to apply Theorem 2.2 with P_η as P_1 and P as the perturbation P_2. This entails some *a priori* estimates.

Lemma 4.4. *Given* P, $\delta > 0$ *and* $r \in (\alpha, 1)$, *there exists* $\varepsilon_2 > 0$ *such that for each* η *with* $0 < mesh(\eta) \leq \varepsilon_2$, *we have*

$$(4.2) \qquad\qquad mesh(\eta) \leq (2\Gamma)^{-1}\varepsilon_0(P_\eta, r, \delta),$$

and

$$(4.3) \qquad\qquad |||P_\eta - P||| \leq \frac{1}{2}\varepsilon_0(P_\eta, r, \delta).$$

Proof. See Lemma 4.2 of [15]. $\square$

The computation of a lower bound on $\varepsilon_0(P_\eta, \delta, r)$ follows the argument in Lemma 3.10 of [1] the only difference arising from the fact that here, P is associated to T and not T_H as in [1]. The key idea is to estimate the BV-norm of the resolvent of P_η (difficult to compute) by the $\|\cdot\|_1-$norm. Hence, following Lemma 3.10 of [1] we define

$$H_{\delta,r}^*(P_\eta) = \sup\{(\frac{B_0}{r - \alpha_0}+1)\|(z-P_\eta)^{-1}v\|_1+\frac{1}{r-\alpha_0}+\frac{2}{r} : \|v\|_1 = 1, z \in \mathbb{C}\backslash V_{\delta,r}(P_\eta)\},$$

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[4]In fact, more precisely, we make our computations on the matrix representation of P_η acting on the basis $\{\frac{1}{\lambda(I_\eta)}\chi_{I_\eta}\}$ as in Section 3.

$$\varepsilon_0^*(P_\eta, \delta, r) \stackrel{\text{def}}{=} \min\{ \frac{r^{n_1 + \lceil \frac{\ln 8BDCH_{\delta,r}^*(P_\eta)}{\ln r/\alpha} \rceil}}{8B(H_{\delta,r}^*(P_\eta)B + (1-r)^{-1})},$$

$$\left[\frac{r^{n_1}}{4B\left(H_{\delta,r}^*(P_\eta)(D+B) + 2(1+B) + (1-r)^{-1}\right)} \right]^\gamma \}.$$

Lemma 4.5.

(1) $\varepsilon_0^*(P_\eta, \delta, r)$ *is uniformly bounded below;*
(2) $\varepsilon_0^*(P_\eta, \delta, r) \le \varepsilon_0(P_\eta, \delta, r)$;
(3) $mesh(\eta) \le (2\Gamma)^{-1}\varepsilon_0^*(P_\eta, \delta, r)$ *implies* $mesh(\eta)$ *satisfies (4.2).*

Proof. Follow the proof of Lemma 3.12 [1] verbatum. $\square$

5. MAIN RESULT

Now we have our tools ready to use the computer and rigorously solve the following problem: Given a map T satisfying (A1) and given a number $\ell > 0$ compute a number $\varepsilon > 0$ such that if $\lambda(H) < \Gamma\varepsilon$ then the map T_H has an accim with escape rate $-\ln e_H < -\ln(1 - \ell)$.

The critical step is to obtain control on the separation of the point spectrum of P outside the essential spectral radius α. Naturally, from a computational viewpoint we can only really do this for P_η after which we use Theorem 2.2 to transfer the picture to the spectrum of P.

More precisely, the following algorithm will, given the number ℓ with $0 < \ell < 1 - \alpha$, compute a number $\delta = \delta_{com}$ with $0 < \delta < \ell$ and $\varepsilon = \varepsilon_{com} > 0$ such that with $r = 1 - \ell$, and any η with $mesh(\eta) < \varepsilon$

(1) $mesh(\eta) \le (2\Gamma)^{-1}\varepsilon_0(P_\eta, r, \delta)$;
(2) $B(1, \delta) \cap B(\rho_i, \delta) = \emptyset$, whenever ρ_i is an eigenvalue of P_η, $\rho_i \notin B(1, \delta)$ and $|\rho_i| > r$.

Thus we obtain the required spectral separation (near the eigenvalue 1) for P_η as well as the conditions necessary to apply Theorem 2.2.

Algorithm 5.1. *T and ℓ given as above, then*

(1) *Set $r = 1 - \ell$.*
(2) *Pick $\delta = \frac{1}{k} < \ell$, $k \in \mathbb{N}$.*
(3) *Feed in a partition of I into intervals. Call it η.*
(4) *Compute ε the mesh size of η.*
(5) *Find $P_\eta = (P_{I_\eta J_\eta})$ where*

$$P_{I_\eta J_\eta} = \frac{\lambda(I_\eta \cap T^{-1}J_\eta)}{\lambda(I_\eta)}.$$

(6) *Compute the following: $H_{\delta,r}^*(P_\eta)$, $n_1 = \lceil \frac{\ln 2}{\ln r/\alpha} \rceil$, $C = r^{-n_1}$, $D = 3 + B$,*
$n_2 = \lceil \frac{\ln 8BDCH_{\delta,r}^*(P_\eta)}{\ln r/\alpha} \rceil$, $\gamma = \frac{\ln(r/\alpha)}{\ln(1/\alpha)}$, $B = \frac{1 - \alpha_0 + B_0}{1 - \alpha}$, $\Gamma = \max\{1 + \alpha_0, B_0\}$.
(7) *Check if $\varepsilon \le (2\Gamma)^{-1}\varepsilon_0^*(P_\eta, \delta, r)$.*
   If (7) is not satisfied, feed in a new η with a smaller mesh size and repeat (3)-(7); otherwise, continue.

(8) *List the eigenvalues of P_η whose modulus is bigger than r: $\rho_{\eta,i}$, $i = 1, \dots, d$.*

(9) *Define:*

$$CL = \{ \text{ all the eigenvalues from the list which are in } B(1,\delta)\}.$$

(10) *Check that if $\rho_{\eta,i} \notin CL$, then $\overline{B}(\rho_{\eta,i},\delta) \cap \overline{B}(1,\delta) = \emptyset$.*

(11) *If (10) is satisfied, report $\delta_{com} := \delta$ and $\varepsilon_{com} := \varepsilon$; otherwise, multiply k by 2 and repeat steps (2)-(11) starting with the last η that satisfied (7).*

Proposition 5.2. *Algorithm 5.1 stops after finitely many steps.*

Proof. By Lemma 4.5, for each $\delta > 0$ and $r \in (\alpha,1)$ $\exists \varepsilon = \mathrm{mesh}(\eta) > 0$ such that

$$\varepsilon < (2\Gamma)^{-1}\varepsilon_0^*(P_\eta, r, \delta).$$

Therefore, the internal loop of algorithm 5.1 (2)-(7) stops after finitely many steps. To prove that the outer loop of stops after finitely many steps, observe that there exist a $K \in \mathbb{N}$, $K < +\infty$, such that $\delta = \frac{1}{K} < \min\{\ell, \overline{\delta}_0\}$, $r = 1 - \ell$ and η with $\varepsilon = \mathrm{mesh}(\eta) > 0$ such that

$$\varepsilon < \min\{(2\Gamma)^{-1}\varepsilon_0(P_\eta, r, \delta), (2\Gamma)^{-1}\varepsilon_0(P, r, \delta)\}.$$

This implies $\sigma(P_\eta) \subset V_{\delta,r}(P) \subset V_{\overline{\delta}_0,r}(P)$. Thus, any P_η eigenvalue which is not in CL is contained in $B(0,r)$ or it is at distance of at least δ from $B(1,\delta)$. By Remark 3.1, (11) of Algorithm 5.1 is satisfied for this K.   □

5.1. **A computer assisted bound on a hole size ensuring the existence of ACCIM.** Given the output $\varepsilon_{\mathrm{com}}$ and δ_{com} from Algorithm 5.1 it is now straightforward to prove the existence of an accim for T_H. As a byproduct of the computation, the spectral information obtained from the algorithm shows that the associated escape rate is at most $-\ln(1-\ell)$.

Theorem 5.3. *Let T_H be a perturbation of T into an interval map with a hole. If $\lambda(H) \leq \Gamma\varepsilon_{com}$ then:*

(1) *P_H has dominant eigenvalue $e_H > 0$ whose associated eigenfunction $f_H^* \geq 0$ is the density of a T_H-accim;*

(2) *$1 - e_H < \delta_{com}$;*

(3) *$1 - e_H \leq (1 + \frac{2\alpha_0 + B_0}{1 - \ell - \alpha})\lambda(H)$.*

Proof. Let $\lambda(H) \leq \Gamma\varepsilon_{\mathrm{com}}$ and set $\mathrm{mesh}(\eta) = \varepsilon_{\mathrm{com}}$. Then by (2) of Lemma 4.3 we have

$$|||P_\eta - P_H||| \leq 2\Gamma\varepsilon_{\mathrm{com}} \leq \varepsilon_0(P_\eta, 1 - \ell, \delta_{\mathrm{com}}).$$

Using Corollary 2.3 with $P_\eta = P_1$ and $P_H = P_2$ we obtain that $\sigma(P_H) \subset V_{\delta_{\mathrm{com}},r}(P_\eta)$. Now, from Algorithm 5.1, recall that $B(1,\delta_{\mathrm{com}}) \cap B(0,r) = \emptyset$ and if $|\rho_i| > r$, ρ_i is an eigenvalue of P_η, is not in CL, then $B(1,\delta_{\mathrm{com}}) \cap B(\rho_i, \delta_{\mathrm{com}}) = \emptyset$. Then Corollary 2.3 implies that the spectral projections of P_η and P_H on $B(1,\delta_{\mathrm{com}})$ have the same rank. Hence, P_H must have at least one isolated eigenvalue in $B(1,\delta_{\mathrm{com}})$. Let e_H denote the spectral radius of P_H. Since P_H is a positive linear operator, $e_H \in \sigma(P_H)$. Moreover, P_H has isolated eigenvalues in $B(1,\delta_{\mathrm{com}})$. Thus, e_H is an eigenvalue of P_H and it must be in $B(1,\delta_{\mathrm{com}})$. This ends the proof of the first two statements of the theorem. To prove (3) of the theorem, we first find a uniform[5] upper bound on the BV-norm of f_H^*. By Lemma 4.1 we have

$$V(e_H f_H^*) = V P_H f_H^* \leq \alpha V f_H^* + (2\alpha_0 + B_0)\|f_H^*\|_1.$$

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[5]By uniform we mean here an upper bound which is independent of H. Hence it holds for all T_H with $\lambda(H) \leq \Gamma\varepsilon_{\mathrm{com}}$.

Therefore,

$$V f_H^* \leq \frac{2\alpha_0 + B_0}{e_H - \alpha} \leq \frac{2\alpha_0 + B_0}{1 - \ell - \alpha};$$

and hence we obtain

$$\|f_H^*\|_{BV} \leq 1 + \frac{2\alpha_0 + B_0}{1 - \ell - \alpha}.$$

Using the fact that P, the Perron-Frobenius operator associated with T, preserves integrals, we obtain

(5.1)
$$
\begin{aligned}
1 - e_H &= |\int_0^1 P f_H^* d\lambda - e_H \int_0^1 f_H^* d\lambda| \\
&= |\int_0^1 P f_H^* d\lambda - \int_0^1 P_H f_H^* d\lambda| \\
&\leq \||P - P_H\|| \cdot \|f_H^*\|_{BV} \leq (1 + \frac{2\alpha_0 + B_0}{1 - \ell - \alpha})\lambda(H).
\end{aligned}
$$

$\square$

6. EXAMPLES

In this section we implement Algorithm 5.1 and Theorem 5.3 of the previous section on two sample computations. Our aim is to show the feasibility of the computation, while at the same time, to discuss some analytic techniques that can be used to reduce the weight of computations for some of the larger matrices P_η that may arise during application of Algorithm 5.1. Large matrices should be expected when α is close to 1 or alternatively, when the escape rate tolerance ℓ is small. We will take advantage of the second mechanism; in both examples we use the same map T:

$$T(x) = \begin{cases} \frac{9x}{1-x} & \text{for } 0 \leq x \leq \frac{1}{10} \\ 10x - i & \text{for } \frac{i}{10} < x \leq \frac{i+1}{10} \end{cases},$$

where $i = 1, 2, \ldots, 9$. However, in the first example $\ell = 1/25$ and in the second example $\ell = 1/40$; i.e., in second example we will be looking for the size of a hole which guarantees the smaller escape rate. We remark none of our computations are particulary time consuming[6] except for the computation of an upper bound on $H_{\delta,r}^*(P_\eta)$[7]. We now turn to the computations.

The Lasota-Yorke inequality for P is given by:

$$VPf \leq 1/9 Vf + 2/9\|f\|_1.$$

Therefore P satisfies (A1) with $\alpha_0 = 1/9$ and $B_0 = 2/9$ and consequetly, $\Gamma = 10/9$, $\alpha = 1/3$, $B = 5/3$ and $D = A(A + B + 2) = 14/3$.

Example 6.1. Given $\ell = 1/25$, using Algorithm 5.1, we show that if $\lambda(H) \in (0, \frac{20}{9} \times 10^{-4}]$, T_H has an accim with escape rate $-\ln e_H < -\ln(24/25)$. The values of the variables [8] involved in the computation are summarized in Table 1.

---

[6] In particular, creating an Ulam matrix of size 5000×5000, or even much bigger, is not really time demanding. Once a computer code is developed for this purpose, which only requires the formula of the map and the number of bins of the Ulam partition as an input, it will excute the nonzero entries in few minutes if not less.

[7] In our computations we found a rigorous upper bound on $H_{\delta,r}^*(P_\eta)$ for an Ulam matrix of size 5000×5000. This computation took few hours using MATLAB on a desktop computer.

[8] All these variables depend on r and δ. $H_{\delta,r}^*$ and n_2 also depend on $\varepsilon = \text{mesh}(\eta)$.

We present here the method which we have followed to rigorously compute an upper bound on $H^*_{\delta,r}(P_\eta)$, for $\mathrm{mesh}(\eta) = 2 \times 10^{-4}$. Using MATLAB we found the dominant eigenvalue 1 of P_η is simple and that there are no other peripheral eigenvalues. Moreover, the modulus of any non-peripheral eigenvalues is smaller than $\alpha_0 = 1/9$. Therefore we have the following estimate (see [9])

$$(6.1) \qquad \|(z - P_\eta)^{-1}\|_1 \le \delta^{-1}\|\Pi_1\|_1 + \|R(z)\|_1,$$

where $\|\Pi_1\|$ is the projection associated with the eigenvalue 1 of the operator P_η, and $R(z)$ is the resolvent of the operator $P_\eta(\mathbf{1} - \Pi_1)$. Since $|z| > r = 1 - \ell > \alpha_0$, $R(z)$ can be represented by a convergent Neumann series. Indeed, we have

$$
\begin{aligned}
\|R(z)\|_1 &= \| \sum_{n=0}^{\infty} \frac{(P_\eta(\mathbf{1} - \Pi_1))^n}{z^{n+1}} \|_1 \\
&\le \frac{1}{r} \left(\sum_{n=0}^{5} \frac{\| (P_\eta(\mathbf{1} - \Pi_1))^n \|_1}{r^n} + \sum_{n=6}^{\infty} \frac{\| (P_\eta(\mathbf{1} - \Pi_1))^n \|_1}{r^n} \right) \\
&\le \frac{1}{r} \left[\sum_{n=0}^{5} \frac{\| (P_\eta(\mathbf{1} - \Pi_1))^n \|_1}{r^n} \left(1 + \sum_{m=1}^{\infty} \left(\frac{\| (P_\eta(\mathbf{1} - \Pi_1))^6 \|_1}{r^6} \right)^m \right) \right] \\
&= 7.444310493.
\end{aligned}
$$

(6.2)

The computation of the estimate in (6.2) is the most time consuming step in the algorithm[9]. Using the definition of $H^*_{\delta,r}$ and inequality (6.1), we obtain that

$$H^*_{\delta,r} \le 45.46070939.$$

r	24/25
δ	1/26
ε	2×10^{-4}
$H^*_{\delta,r}$	45.46070939
n_1	1
C	25/24
n_2	8
$(2\Gamma)^{-1}\varepsilon^*_0$	0.0002319492040
Loop I	Pass
Loop II	Pass
Output I	$\varepsilon_{\mathrm{com}} = 2 \times 10^{-4}, \delta_{\mathrm{com}} = 1/26$
Output II	$\lambda(H) \in (0, \frac{20}{9} \times 10^{-4}] \implies T_H$ admits an accim μ with escape rate $-\ln e_H < -\ln(24/25)$

TABLE 1. The output of Algorithm 5.1 for $l = 1/25$

Example 6.2. Given $\ell = 1/40$, using Algorithm 5.1, we show that if $\lambda(H) \in (0, \frac{10}{9} \times 10^{-5}]$, T_H has an accim with escape rate $-\ln e_H < -\ln(39/40)$. The values of the variables involved in the computations are summarized in Table 2.

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[9]Precisely, the work is in the computation of the powers $(P_\eta(\mathbf{1} - \Pi_1))^n$, $n = 1, \ldots, 6$. Once these powers are known the computation of the norm is very fast. However, we will see in the next example how we can benefit from these numbers and avoid time consuming computations when dealing with a higher order Ulam approximation in the case of a smaller hole.

Here, we explain how we have obtained some of the values which appear in Table 2. In particular, we will explain how we have avoided time demanding computation of $H^*_{\delta,r}(P_{\eta'})$ and rigorously estimated

$$H_{\delta,r}(P_{\eta'}) \leq 1036.693385,$$

where $\operatorname{mesh}(\eta') \leq \operatorname{mesh}(\eta) = 2 \times 10^{-4}$. In the first pass through the Algorithm 5.1, we start with $\varepsilon = \operatorname{mesh}(\eta) = 2 \times 10^{-4}$. This is the same ε which closed Algorithm 5.1 in Example 6.1. So the numbers $\|(P_\eta(\mathbf{1}-\Pi_1))^n\|_1$, $n = 1, \ldots, 6$, can be obtained from the computation of Example 6.1 as they do not depend on r and δ. Thus, the rigorous estimate

$$H^*_{\delta,r}(P_\eta) \leq 63.73181657$$

easily follows. However, the inner loop of Algorithm 5.1 will fail because

$$2 \times 10^{-4} = \varepsilon = \operatorname{mesh}(\eta) > (2\Gamma)^{-1}\varepsilon^*_0 = 0.0001763820641.$$

Next, Algorithm 5.1 asks us to feed another Ulam partition η' with $\operatorname{mesh}(\eta') < 2 \times 10^{-4}$ and to repeat the inner loop of Algorithm 5.1. Here, we have used a 3-step trick to avoid time demanding estimate of the new value $H^*_{\delta,r}(P_{\eta'})$:

(1) Let us suppose for a moment that we are only concerned with a rigorous approximation of the spectrum of P, the operator associated with T. Then P, P_η and $P_{\eta'}$ satisfy a common Lasota-Yorke inequality which does not involve α, but rather α_0 (see Remark 4.2). Now, re-checking the computations which were obtained in the first run of Algorithm 5.1 and this time with $\alpha \equiv \alpha_0 = 1/9$ and the modfications of B to $\hat{B} = 1 + \frac{B_0}{1-\alpha_0}$ and D to $\hat{D} = 3 + \hat{B}$. Then the value of $(2\Gamma)^{-1}\varepsilon^*_0$ changes to

$$(2\Gamma)^{-1})\varepsilon^*_0 = 0.0002425063815 > 2 \times 10^{-4} = \operatorname{mesh}(\eta).$$

Consequently, for any η' with $\operatorname{mesh}(\eta') \leq \operatorname{mesh}(\eta)$, we have

$$|||P_\eta - P_{\eta'}||| \leq |||P_\eta - P||| + |||P - P_{\eta'}|||$$
$$\leq \Gamma\operatorname{mesh}(\eta) + \Gamma\operatorname{mesh}(\eta')$$
$$\leq 2\Gamma\operatorname{mesh}(\eta) < \varepsilon^*_0(P_\eta, r, \delta).$$

Therefore, we can use part one of Theorem 2.2 with $P_1 = P_\eta$ and $P_2 = P_{\eta'}$.

(2) In particular, for any $z \in \mathbb{C} \setminus V_{\delta,r}(P_{\eta'})$ we have

$$\|(z - P_{\eta'})^{-1}\|_{BV} \leq \frac{4(1 + \hat{B})}{1 - r}r^{-n_1} + \frac{1}{2\varepsilon_1} \leq 1036.693385.$$

Recall that $\varepsilon_1 = \frac{r^{n_1+n_2}}{8\hat{B}(H_{\delta,r}(P_\eta)+\frac{1}{1-r})}$.

(3) Now we go back to the problem of finding the size of a hole which guarantees the existence of a T_H-accim with the desired escape rate. Here $\alpha = 3\alpha_0$ and all we have to do is to feed the estimate on $H_{\delta,r}(P_\eta)$ obtained in Step 2, together with the new n_2, in the formula of ε_0 to obtain that

$$(2\Gamma)^{-1}\varepsilon_0 \geq 0.00001216687545.$$

Hence, we can deduce that $\operatorname{mesh}(\eta') = 10^{-5}$ will do the job; i.e., $\lambda(H) \in (0, \frac{10}{9} \times 10^{-5}]$ $\implies$ T_H admits an accim μ with escape rate $-\ln e_H < -\ln(39/40)$.

r	$39/40$	$39/40$
δ	$1/41$	$1/41$
ε	2×10^{-4}	10^{-5}
Upper bound on $H_{\delta,r}$	63.73181657	1036.693385
n_1	1	1
C	$40/39$	$40/39$
n_2	8	11
Lower bound on $(2\Gamma)^{-1}\varepsilon_0$	0.0001763820641	0.00001216687545
Loop I	Fail: reduce ε	Pass
Loop II		Pass
Output I		$\varepsilon_{\text{com}} = 10^{-5}, \delta_{\text{com}} = 1/41$
Output II		$\lambda(H) \in (0, \frac{10}{9} \times 10^{-5}] \implies T_H$ admits an accim μ with escape rate $-\ln e_H < -\ln(39/40)$

TABLE 2. The output of Algorithm 5.1 for $l = 1/40$

7. THE EFFECT OF THE POSITION OF A HOLE

The results of the previous section give upper bounds on the escape rate that are uniform for a given size of hole, independent of the position of the hole. However, it has been observed already in [4] that the position of the hole can affect the escape rate; i.e., given a map T and two holes H_1, H_2, with $\lambda(H_1) = \lambda(H_2)$, it may happen that the escape through H_1, say, may be bigger than the escape rate through H_2. For example, define the sets

$$\text{Per}(H_i) = \{p : p \in \mathbb{N} \text{ s.t. for some } x \in H_i, T^p(x) = x, \text{ and } T^{p-1}(x) \neq x\}; i = 1, 2,$$

For certain maps, if

$$\text{Minimum}\{p \in \text{Per}(H_1)\} \leq \text{Minimum}\{p \in \text{Per}(H_2)\}$$

then the escape rate through H_1 will be smaller than the escape rate through H_2 .

In [11] Keller and Liverani obtained precise asymptotic information about the effect of the location of the hole. Roughly speaking, for a system of holes shrinking to a single point, the rate of decay of escape rate depends on two things: the value of the invariant density of the map T at the point the holes shrink to, and whether or not this point is periodic. We now state a version of this result and will discuss in the next subsection, in a smooth setting, how a combination of Algorithm 5.1, with the proper modification of P_η, can be used with this theorem when the formula of the invariant density of T cannot be found explicitly. When smoothness is not assumed, as in this paper, obtaining asymptotics for the escape rate relative to the size of the hole appears to be an open problem.

Theorem 7.1. [11] *Let T be piecewise C^2 on a finite partition of $[0, 1]$ and assume it is piecewise expanding and mixing. Let $\{H_\kappa\}$ be a sequence of holes such that $H_\kappa \supset H_{\kappa'}$, for $0 \leq \kappa' < \kappa$, with $H_0 = \{y\}$ for some point $y \in [0, 1]$ which is a point of continuity of both T and f^*, f^* is the invariant density of T. Let T_{H_κ} be a perturbation of T into a map with a hole. Assume that $\inf f^*_{|H_\kappa} > 0$. For $\lambda(H_\kappa)$ sufficiently small[10] we have:*

---

[10]We can of course quantify what we mean by sufficiently small using Algorithm 5.1.

(1) *If y is non-periodic then $\lim_{\kappa \to 0} \frac{1 - e_{H_\kappa}}{\lambda(H_\kappa)} = f^*(y)$.*

(2) *If y is periodic with period p then $\lim_{\kappa \to 0} \frac{1 - e_{H_\kappa}}{\lambda(H_\kappa)} = f^*(y) \left(1 - \frac{1}{|(T^p)'(y)|}\right)$.*

7.1. C^3 circle maps.

Theorem 7.1 requires the knowledge of the value of the invariant density f^*; in particular, its value at the point y. Unfortunately, the approximate invariant density which is obtained by Ulam's method in Algorithm 5.1 does not provide a pointwise approximation of f^*. However, in a smooth setting, one can modify Ulam's scheme, and the function spaces where P and P_η act, to obtain rigorous approximation of $\|f^* - f_\eta\|_\infty$, where f_η denotes the fixed function for the modified finite rank operator P_η. Of course our main theoretical tool, Theorem 2.2 will need to be modified. In [10] an abstract version of Theorem 2.2 is proved. The result requires bounded linear operators P_1 and P_2 acting on two abstract Banach spaces whose norms $\|\cdot\|$ and $|\cdot|$ satisfy $|\cdot| \le \|\cdot\|$, and the unit ball of $\|\cdot\|$ is $|\cdot|$-compact. Thus, Theorem 2.2 is a particular application of the general result of [10]. When T is a C^3 circle map, it is well known that its invariant density f^* is C^2. Thus, instead of L^1 and BV, one can study the action of P and a smooth version[11] of P_η on the spaces $W^{1,1}$ and $W^{1,2}$ with norms

$$\|f\|_{W^{1,1}} = \|f\|_1 + \|f'\|_1$$

$$\|f\|_{W^{1,2}} = \|f\|_1 + \|f'\|_1 + \|f''\|_1$$

respectively. A common Lasota-Yorke inequality of P and P_η in this setting is given by: for $f \in W^{1,2}$ and $n \in \mathbb{N}$ we have

$$\|Pf\|_{W^{1,2}} = \alpha_0^{2n}\|f\|_{W^{1,2}} + \bar{B}\|f\|_{W^{1,1}},$$

$$\|P_\eta f\|_{W^{1,2}} = \alpha_0^{2n}\|f\|_{W^{1,2}} + \bar{B}\|f\|_{W^{1,1}},$$

where $\bar{B} \ge 0$ which depends on T only. Using this setting, one obtains the estimate

$$\|f^* - f_\eta\|_{W^{1,1}} \le \bar{C} \cdot \mathrm{mesh}(\eta).$$

For more details and for a proof of the above Lasota-Yorke inequality we refer to Section 10.2 of [15].

In a setting like this, one can then repeat Algorithm 5.1 with the smooth version of P_η and obtain the following reformulation of Theorem 7.1 to a setting where the invariant density f^* is a priori unknown. Note that for $f \in W^{1,2}$, $\|f\|_\infty \le \|f\|_{W^{1,1}}$.

Theorem 7.2. *Let T be a C^3 circle map. Let $\{H_\kappa\}$ be a sequence of holes such that $H_\kappa \supset H_{\kappa'}$, for $0 \le \kappa' < \kappa$, with $H_0 = \{y\}$ for some point $y \in [0,1]$. Let f^* be the invariant density of T, and T_{H_κ} be a perturbation of T into a map with a hole[12]. Let $\varepsilon = mesh(\eta)$. $\exists$ a constant $\bar{C} = \bar{C}(P_\eta)$ such that for $\lambda(H_\kappa) \in (0, \Gamma\varepsilon_{com}]$, we have:*

(1) *If y is non-periodic then*

$$f_\eta(y) - \bar{C} \cdot \varepsilon \le \lim_{\kappa \to 0} \frac{1 - e_{H_\kappa}}{\lambda(H_\kappa)} \le f_\eta(y) + \bar{C} \cdot \varepsilon.$$

---

[11]For instance one can use a piecewise linear approximation method [6].

[12]$\inf_{x \in [0,1]} f^* > 0$ for C^2 circle maps. See [12] or [17]. Thus, the assumption $\inf f^*_{|H_\kappa} > 0$ is automatically satisfied for such maps.

(2) *If y is periodic with period p then*

$$\left(f_\eta(y) - \bar{C} \cdot \varepsilon\right)\left(1 - \frac{1}{|(T^p)'(y)|}\right) \leq \lim_{\kappa \to 0} \frac{1 - e_{H_\kappa}}{\lambda(H_\kappa)} \leq$$

$$\left(f_\eta(y) + \bar{C} \cdot \varepsilon\right)\left(1 - \frac{1}{|(T^p)'(y)|}\right).$$

Proof. We only give a sketch of the proof. Suppose that we have used Algorithm 5.1 with the proper modification of P_η and the function spaces. Then the invariant density, which is a byproduct of the algorithm, would provide the following estimate:

$$\|f_\eta - f^*\|_\infty \leq \bar{C} \cdot \varepsilon.$$

Consequently, for any $y \in [0, 1]$, we have

(7.1) $$|f_\eta(y) - f^*(y)| \leq \bar{C} \cdot \varepsilon.$$

$\square$

Thus, the proof follows by using (7.1) and Theorem 7.1.

Remark 7.3. All the constants which are hiding in the computation of $\bar{C} = \bar{C}(P_\eta)$ can be rigorously computed using Theorem 2.2 with the spaces $W^{1,1}$ and $W^{1,2}$. It should be pointed out that these constants cannot be computed if one attempts to do this a approximation in the L^1, BV framework. This is because the estimates will depend on $(f^*)''$ which is a priori unknown.

Remark 7.4. For C^2 Lasota-Yorke maps, a result similar to Theorem 7.2 is not obvious at all. The problem for C^2 Lasota-Yorke maps involves two issues:

(1) The invariant density f^* is C^1. This means that the framework of $W^{1,1}$, $W^{1,2}$ cannot be used.
(2) If one uses the function spaces BV and L^1, then to the best of our knowledge, only the original Ulam method will fit in this setting. The problem with Ulam's method is that it provides only good estimates in the L^1 norm $\|f^* - f_\eta\|_1 = \bar{C} \cdot \varepsilon \ln 1/\varepsilon$. However, typically, $\|f^* - f_\eta\|_{BV} \not\to 0$.

Our last comment on this is that providing a scheme for C^2 Lasota-Yorke maps to obtain a result similar to that of Theorem 7.2 would be an interesting problem.

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