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OPTIMAL BOUNDS FOR DISJOINT HAMILTON CYCLES IN STAR GRAPHS

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In interconnection network topologies, the n-dimensional star graph St_n has $n!$ vertices corresponding to permutations $a_{\rho(1)} \dots a_{\rho(n)}$ of n symbols $a_1, \dots, a_n$ and edges which exchange the positions of the first symbol $a_{\rho(1)}$ with any one of the other symbols. The star graph compares favorably with the familiar n-cube on degree, diameter and a number of other parameters. A desirable property which has not been fully evaluated in star graphs is the presence of multiple edge-disjoint Hamilton cycles which are important for fault-tolerance. The only known method for producing multiple edge-disjoint Hamilton cycles in St_n has been to label the edges in a certain way and then take images of a known base 2-labelled Hamilton cycle under different automorphisms that map labels consistently. However, optimal bounds for producing edge-disjoint Hamilton cycles in this way, and whether Hamilton decompositions can be produced, are not known for any St_n other than for the case of St_5 which does provide a Hamilton decomposition. In this paper we show that, for all n, not more than $\varphi(n)/2$, where φ is Euler's totient function, edge-disjoint Hamilton cycles can be produced by such automorphisms. Thus, for non-prime n, a Hamilton decomposition cannot be produced. We show that the $\varphi(n)/2$ upper bound can be achieved for all even n. In particular, if n is a power of 2, St_n has a Hamilton decomposable spanning subgraph comprising more than half of the edges of St_n. Our results produce a better than twofold improvement on the known bounds for any kind of edge-disjoint Hamilton cycles in n-dimensional star graphs for general n.

Keywords: star graphs; Hamilton cycles; automorphisms.

1. Introduction

The n-dimensional star graph St_n [1] has $n!$ vertices corresponding to permutations $a_{\rho(1)} \dots a_{\rho(n)}$ of n symbols $a_1, \dots, a_n$ and edges corresponding to applications of one of the transpositions $(a_{\rho(1)}, a_{\rho(2)}), \dots, (a_{\rho(1)}, a_{\rho(n)})$. It connects $n!$ vertices with degree $n-1$ and diameter $\lfloor 3(n-1)/2 \rfloor$. By comparison the n-cube connects 2^n vertices with degree n and diameter n. The star graph also compares favorably with the n-cube on other properties of symmetry and fault-tolerance. As such, the star graph offers a cheaper alternative to the n-cube, as an interconnection

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topology, requiring less network hardware and incurring less communication delay. Derivatives of the star network such as the incomplete star [8], hierarchical star [10], (n,k)-star [4], arrangement star [3], and starcube [11] have been proposed and their topological properties have been extensively studied and compared. A property of the n-cube that has escaped such studies in all these other topologies has been that of Hamilton decomposability. This property is important for fault tolerance and broadcasting algorithms. Apart from an old result of [7], little was known about Hamilton cycles in star graphs St_n of degree n-1 until fairly recently in [6] where a Hamilton decomposition of St_5 was produced and in [9] where $\varphi(n)/10$ disjoint Hamilton cycles were shown to be present in St_n for all n. Surprisingly, in contrast to the n-cube, the method used in both [6] and [9] generates edge-disjoint Hamilton cycles in a simple and symmetric manner as automorphic images of a single Hamilton cycle. The method defines a labelling for the edges of star graphs and works with automorphisms that map labels consistently. However, so far, no optimal bounds have been given for the numbers of disjoint Hamilton cycles that can be generated by the method, and it is not known whether a Hamilton decomposition can be produced for St_n if n is greater than 5. In this paper we address these two open problems.

This paper is structured as follows. We define the edge labelling for star graphs St_n and corresponding label automorphisms in Section 2. In Section 3, we define 'symmetric' collections of edge-disjoint Hamilton cycles in star graphs to be those collections generated as images of a particular known 2-labelled star graph under label automorphisms. Upper bounds are obtained in Section 4 where we show that St_n cannot have symmetric collections of more than $\varphi(n)/2$ disjoint Hamilton cycles (Theorem 16). From this it follows that St_n is not symmetrically Hamilton decomposable for non-prime n (Corollary 17). Lower bounds are obtained for even n in Section 5 where we show that St_n does have a symmetric collection of $\varphi(n)/2$ Hamilton cycles in Theorem 20.

2. Labelled Star Graphs and Label Automorphisms

Throughout the paper, arithmetic will be modulo n when St_n is the star graph in context. Therefore, $x = y$ will mean $x = y \bmod n$. In arithmetic modulo n, we shall use n instead of 0 so that the set of integers modulo n will be $\{1, \ldots, n\}$.

Definition 1. *The n-dimensional labelled star graph $St_n = (V, E, L)$ is the (n-1)-regular graph of order $|S_n|$, where S_n is the symmetric group of permutations of order n, with a set V of vertices, E of edges and a mapping of edges to integer labels $L : E \mapsto \{1, \ldots, \lfloor n/2 \rfloor\}$, given by:*

$$\begin{aligned}
V(St_n) &= \{a_{\rho(1)} \cdots a_{\rho(n)} \mid \rho \in S_n\}, \\
E(St_n) &= \{e \mid e = \{a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, \\
&\qquad a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}\},\ \rho \in S_n\},
\end{aligned}$$

$$L(\{a_{\rho(1)} \cdots, a_{\rho(i)} \cdots)\} = \delta(a_{\rho(1)}, a_{\rho(i)})$$

$$\begin{tikzcd} v_1 \arrow[r, "\Phi"] \arrow[d, dash, "l_1"'] & \Phi(v_1) \arrow[d, dash, "\phi^l(l_1)"] \\ v_2 \arrow[r, "\Phi"'] & \Phi(v_2) \end{tikzcd}$$

Fig. 1. Label automorphism.

where

$$\delta(a_i, a_j) = \min\{|i-j|, n-|i-j|\} \ \ (1 \leq i, j \leq n)$$

is the distance between a_i and a_j on the cyclic graph whose vertices are $a_1, \ldots, a_n$ in which a_n is adjacent to a_{n-1} and a_1.

The class of automorphisms of St_n used are those which map labels consistently.

Definition 2. *A label map for St_n is a bijection*

$$\phi^l : \{1, \ldots, \lfloor n/2 \rfloor\} \mapsto \{1, \ldots, \lfloor n/2 \rfloor\}$$

of labels. An automorphism

$$\Phi : V(St_n) \mapsto V(St_n)$$

is a label automorphism if there exists a label map ϕ^l such that, for all $\{v_1, v_2\} \in E(St_n)$,

$$L(\{\Phi(v_1), \Phi(v_2)\}) = \phi^l(L\{v_1, v_2\})$$

If G is a graph, H is a subgraph of G, and Φ an automorphism of G, $\Phi(H)$ will refer to the subgraph of G that is the image of the vertices and edges of H under Φ.

Definition 3. *A Hamilton cycle H in a graph G is a subgraph that is a cycle which contains all vertices of G.*

If Φ is an automorphism and H is a Hamilton cycle of G, then clearly $\Phi(H)$ is also a Hamilton cycle of G.The automorphisms used in [9] and [6] are defined 'pointwise' by means of bijections of the elements $\{a_1, \ldots, a_n\}$, which map distances between elements of the cyclic graph $a_1 \to \ldots \to \ldots a_n \to a_1$ consistently in the image.

Lemma 4. *Let $\phi : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ be a bijection. Then $\Phi : V(St_n) \mapsto V(St_n)$, given by*

$$\Phi(a_{\rho(1)} \ldots a_{\rho(n)}) = \phi(a_{\rho(1)}) \ldots \phi(a_{\rho(n)})$$

is an automorphism of the graph St_n.

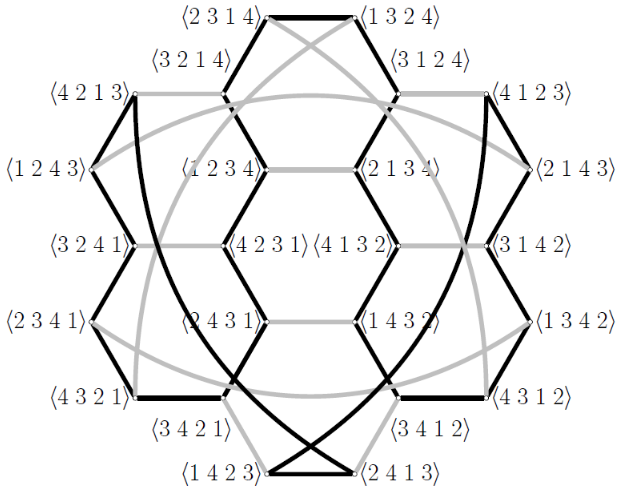

Fig. 2. S_4 with a Hamilton cycle shown in black.

Definition 5. *A pointwise map for St_n is a bijection ϕ as in Lemma 4. The corresponding automorphism is the automorphism Φ as defined in Lemma 4. If ϕ is such that there exists a bijection*

$$\phi^d : \{1, \ldots, \lfloor n/2 \rfloor\} \mapsto \{1, \ldots, \lfloor n/2 \rfloor\}$$

satisfying, for all $a_i, a_j \in \{a_1, \ldots, a_n\}$,

$$\delta(\phi(a_i), \phi(a_j)) = \phi^d(\delta(a_i, a_j)) \tag{1}$$

then Φ is trivially a label automorphism with $\phi^l = \phi^d$ in Definition 2. We shall call ϕ^d the corresponding distance map of Φ.

Distance maps allude to distances in the cyclic graph of the elements $\{a_1, ..., a_n\}$, and not to distances in St_n. Not all pointwise maps yield label automorphisms of St_n. For example, if $n \geq 4$ and ϕ is defined such that $\phi(a_1) = a_2, \phi(a_2) = a_1, \phi(a_i) = a_i$ $(2 < i \leq n)$, then the corrsponding Φ is not a label automorphism as a distance map ϕ^d would have to satisfy (by (1)):

$$1 = \delta(a_1, a_2) = \delta(\phi(a_2), \phi(a_1)) = \phi^d(\delta(a_2, a_1)) =$$

$$\phi^d(\delta(a_2, a_3)) = \delta(\phi(a_2), \phi(a_3)) = \delta(a_1, a_3) = 2$$

The class of label automorphisms generated by a pointwise map and with a distance map as in Definition 5 will be denoted by $\mathcal{A}_n$.

3. Symmetric Collections of Edge-Disjoint Hamilton Cycles

Symmetric collections of edge-disjoint Hamilton cycles are defined with respect to the class of automorphisms $\mathcal{A}_n$ and the Hamilton cycle with edge labels 1 and 2 constructed in [9] as the base Hamilton cycle, so that each Hamilton cycle in a symmetric collection has to be automorphic with this base Hamilton cycle via an automorphism $\Phi \in \mathcal{A}_n$. This is different to the definition of symmetric collections of Hamilton cycles in complete graphs [2]. We will use the following notation.

Definition 6. *A vertex $v \in V(St_n)$ of the form $a_i \dots$ (respectively $\dots a_i$), where $a_i \in \{a_1, \dots a_n\}$, will be denoted by $\overrightarrow{a}_i$ (respectively $\overleftarrow{a}_i$).*

Definition 7. *The base Hamilton cycle $H_{12}(n)$ in St_n is the Hamilton cycle constructed in [9] consisting of alternate paths of $n(n-1)-1$ edges with label 1 and single edges with label 2:*

$$\dots \underbrace{\bullet \overset{1}{—} \bullet \dots\dots\dots \bullet \overset{1}{—} \bullet}_{n(n-1)-1\ edges} \overset{2}{—} \underbrace{\bullet \overset{1}{—} \bullet \dots\dots\dots \bullet \overset{1}{—} \bullet}_{n(n-1)-1\ edges} \overset{2}{—} \bullet \dots$$

The total number of edges with label 1 in $H_{12}(n)$ is $n! - (n-2)!$ which is greater than the number of remaining edges with label 2 $(= n! - (n! - (n-2)!) = (n-2)!)$ in St_n, is such that all vertices v in $H_{12}(n)$ incident with edges with label 2 are of the form $\overleftarrow{a}_n$.
A collection of edge-disjoint Hamilton cycles in St_n are 'symmetric' if any Hamilton cycle in the collection is the image of $H_{12}(n)$ under an automorphism in $\mathcal{A}_n$.

Definition 8. *A collection $\widetilde{H}$ of edge-disjoint Hamilton cycles in St_n is symmetric if $H_{12}(n) \in \widetilde{H}$ and if, for all $H^e, H^f \in \widetilde{H}$, there is a label automorphism $\Phi_{ef} \in \mathcal{A}_n$ such that $\Phi_{ef}(H^e) = H^f$.*

Hamilton cycles in $\widetilde{H}$ all have a similar structure.

Lemma 9. *Let $\Phi \in \mathcal{A}_n$ be a label automorphism with corresponding distance map ϕ^d. Then, $\Phi(H_{12}(n))$ is a Hamilton cycle consisting of alternate paths of $n(n-1)-1$ edges with label $\phi^d(1)$ and single edges with label $\phi^d(2)$:*

$$\dots \underbrace{\bullet \overset{\phi^d(1)}{—} \bullet \dots\dots\dots \bullet \overset{\phi^d(1)}{—} \bullet}_{n(n-1)-1\ edges} \overset{\phi^d(2)}{—} \underbrace{\bullet \overset{\phi^d(1)}{—} \bullet \dots\dots\dots \bullet \overset{\phi^d(1)}{—} \bullet}_{n(n-1)-1\ edges} \overset{\phi^d(2)}{—} \bullet \dots$$

Proof. Follows from Definitions 5 and 7. □

From Lemma 9, we see that a Hamilton cycle which is the image of $H_{12}(n)$ under a label automorphism in $\mathcal{A}_n$, is a succession of edges the majority of which share the same label, and the remaining minority of which share the same second label. This leads to the following definition.

Definition 10. *A Hamilton cycle which is the image of $H_{12}(n)$ under an automorphism as in Lemma 9, will be denoted by $H_{ij}(n)$ (or just H_{ij} if n is clear from the context) where the subscript $i = \phi^d(1)$ is the label for the majority of the edges and the subscript $j = \phi^d(2)$ is the label for the minority of the edges. We shall call these two sets of edges the majority and minority edges of H_{ij} respectively.*

4. Upper Bounds for Symmetric Collections

Not all labels can be majority or minority labels of images of H_{12} under label automorphisms from $\mathcal{A}_n$. The underlying reason for this is the difference in the length of cycles of different labels.

Definition 11. *The spanning subgraph of St_n comprising edges with labels i and j where $i, j \in \{1, \dots, \lfloor n/2 \rfloor\}$ will be denoted by $C_{ij}(n)$ and the spanning subgraph comprising only edges with label i will be denoted $C_i(n)$. Each $C_i(n)$ is a union of disjoint cycles $B_i^x(n)$ of edges with label i*

$$E(C_i(n)) = \bigcup_{x \in X} E(B_i^x(n)) \qquad \textit{(X is some index set)}$$

We shall call a cycle $B_i^x(n)$ an i-ball. Again, we will abbreviate our notation to C_{ij}, C_i and B_i^x when n is clear from the context and will drop the x index in B_i^x when only one i-ball is under consideration. For an i-ball B_i, $|B_i|$ will denote the number of edges in B_i.

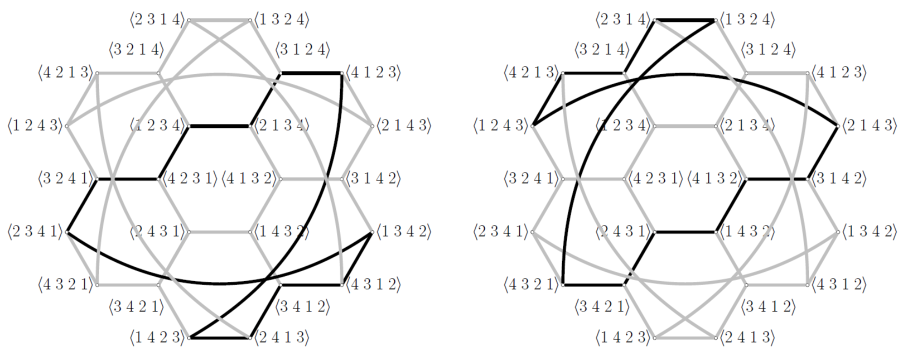

Fig. 3. The two $B_1(4)$ balls comprising $C_1(4)$ shown in black.

Important properties of i-balls are given in the following two lemmata.

Lemma 12. *Let B_i be an i-ball in St_n, where $i \in \{1, \dots, \lfloor n/2 \rfloor\}$. Then,*

(i) $|B_i| = n(n-1)$ *if i is coprime to n, and*
(ii) $|B_i| < n(n-1)$ *if i is not coprime to n.*

Proof. Let $n = dq_1$ and $i = dq_2$ where $d = \gcd(n, i)$ and $\gcd(q_1, q_2) = 1$. Without loss of generality, assume that the vertex

$$a_1 \dots a_n \in B_i$$

Now, the elements

$$a_1, a_{1+i}, \dots, a_{1+(q_1-1)i}$$

are distinct (else, for some r, s such that $0 \leq r < s \leq (q_1 - 1)$ and $k \in \mathbb{N}$, $kn + (1 + ri) = (1 + si)$ and so $kdq_1 = (s - r)dq_2$ and as $\gcd(q_1, q_2) = 1$, q_1 divides $(s - r)$ which is a contradiction as $(s - r) \leq (q_1 - 1)$). The path in B_i of the form:

$$\overrightarrow{a}_1, \overrightarrow{a}_{1+i}, \dots, \overrightarrow{a}_{1+(q_1-1)i},$$

where $\overrightarrow{a}_1 = a_1 \dots a_n$, rotates the elements $a_1, \dots, a_{1+(q_1-1)i}$ within the vertex $a_1 \dots a_n$ in the sequence

$$a_1 \to a_{1+i} \to \dots a_{1+(q_1-1)i} \to a_1,$$

as $q_1 i \bmod n = n$. After $q_1 - 1$ such rotations, the starting vertex $a_1 \dots a_n$ is reached again, i.e. B_i is the cycle of $(q_1 - 1)$ sets of q_1 vertices:

$$\underbrace{\overrightarrow{a}_1, \overrightarrow{a}_{1+i}, \dots \overrightarrow{a}_{1+(q_1-1)i}}_{q_1\ vertices}, \underbrace{\dots\dots\dots}_{q_1\ vertices}, \quad \cdots \quad , \underbrace{\dots\dots\dots}_{q_1\ vertices}, \overrightarrow{a}_1$$

separated by edges with label i, and returning to $\overrightarrow{a}_1$ after $q_1(q_1 - 1)$ edges. If i is coprime to n, $q_1 = n$ and (i) follows. If i is not coprime to n, then $q_1 < n$ and (ii) follows. □

Lemma 13. *Let $\Phi \in \mathcal{A}_n$ and let B_i^x be an i-ball in St_n, where $1 \leq i \leq \lfloor n/2 \rfloor$. Then, there exists an i'-ball $B_{i'}^{x'}$ in St_n, for some i' with $1 \leq i' \leq \lfloor n/2 \rfloor$, such that*

$$\Phi(B_i^x) = B_{i'}^{x'} \text{ and } \gcd(i, n) = 1 \text{ iff } \gcd(i', n) = 1$$

Proof. As Φ is an automorphism, $\Phi(B_i^x)$ is a cycle such that $|\Phi(B_i^x)|$ equals $|B_i^x|$. Also, as Φ is a label automorphism all edges of $\Phi(B_i^x)$ must have the same label and thus $\Phi(B_i^x)$ must be an i'-ball, $B_{i'}^{x'}$ say, for some i' where $1 \leq i' \leq \lfloor n/2 \rfloor$. Then, by Lemma 12,

$$\gcd(i, n) = 1 \text{ iff } |B_i^x| = n(n-1) = |B_{i'}^{x'}| \text{ iff } \gcd(i', n) = 1 \qquad \square$$

As a result of Lemma 13, we are able to give constraints on how automorphisms $\Phi \in \mathcal{A}_n$ map labels. Indeed, we can characterize the pointwise maps ϕ that generate label automorphisms $\Phi \in \mathcal{A}_n$.

Lemma 14. *Let $\Phi \in \mathcal{A}_n$ be a label automorphism with corresponding pointwise and distance maps ϕ and ϕ^d respectively, as in Definition 5. Then:*

(i) for all labels $l \in \{1, \dots, \lfloor n/2 \rfloor\}$,

$$\gcd(l, n) = 1 \quad \text{iff} \quad \gcd(\phi^d(l), n) = 1$$

(ii) there exist $i_0, j \in \{1, \ldots, n\}$, where j is coprime to n, such that

$$\phi(a_i) = a_{i_0+ji} \qquad (1 \le i \le n)$$

Proof. For (i), let B_l^x be a l-ball in St_n. As Φ is a label automorphism with distance map ϕ^d, $\Phi(B_l^x)$ is a $\phi^d(l)$-ball $B_{\phi^d(l)}^{x'}$ in St_n. By Lemma 13, $\gcd(l, n) = 1$ iff $\gcd(\phi^d(l), n) = 1$.

For (ii), let $i_0, i_1 \in \{1, \ldots, n\}$ be such that

$$\phi(a_n) = a_{i_0} \text{ and } \phi(a_1) = a_{i_1}$$

where ϕ is the pointwise map of Φ. Put

$$j_p = \delta(\phi(a_n), \phi(a_1)) = \min\{|i_0 - i_1|, n - |i_0 - i_1|\}$$

As $\delta(a_n, a_1) = 1$ and $\delta(\phi(a_n), \phi(a_1)) = j_p$, it follows by (1) of Definition 5 that

$$\phi^d(1) = j_p \tag{2}$$

Let $a_i \in \{a_1, \ldots, a_n\}$ and consider the $a_g, a_h \in \{a_1, \ldots, a_n\}$ such that

$$\phi(a_i) = a_g \text{ and } \phi(a_{i+1}) = a_h$$

As $\delta(a_i, a_{i+1}) = 1$, by (1) and (2) we have that

$$\delta(a_g, a_h) = j_p$$

Therefore,

$$g - h = j_p \bmod n \quad \text{or} \quad g - h = -j_p \bmod n$$

and so

$$h = g - j_p \bmod n \quad \text{or} \quad h = g + j_p \bmod n$$

As $\Phi(a_n) = a_{i_0}$ and ϕ is injective it is clear that either

$$\Phi(a_n) = a_{i_0}, \Phi(a_1) = a_{i_0 - j_p}, \ldots, \Phi(a_{n-1}) = a_{i_0-(n-1)j_p} \tag{3}$$

or

$$\Phi(a_n) = a_{i_0}, \Phi(a_1) = a_{i_0 + j_p}, \ldots, \Phi(a_{n-1}) = a_{i_0+(n-1)j_p} \tag{4}$$

hold. If (3) is the case put $j = -j_p$ and if (4) is the case put $j = j_p$ and the proof of (ii) is complete. □

Definition 15. *Given a label automorphism $\Phi \in \mathcal{A}_n$ and corresponding pointwise map $\phi(a_i) = a_{i_0+ji}$, i_0 is called the offset and j the generator of ϕ.*

The constraints of label automorphisms in turn impose limits on the number of edge-disjoint Hamilton cycles in symmetric collections.

Theorem 16. *Let $\widetilde{H}$ be a symmetric collection of edge-disjoint Hamilton cycles in St_n. Then $|\widetilde{H}| \le \varphi(n)/2$, where $|\widetilde{H}|$ is the number of Hamilton cycles in $\widetilde{H}$.*

Proof. By Definition 8, as $\widetilde{H}$ is symmetric, any Hamilton cycle in $\widetilde{H}$ is the image of H_{12} under a label automorphism and thus, by Lemma 9 and Definition 10, is of the form H_{ij} with majority edge labels i and minority edge labels j. By Lemma 14 (i) with $l = 1$, $\gcd(i, n) = 1$. Thus, the edge-disjoint Hamilton cycles in $\widetilde{H}$ can be listed as

$$H_{i_1 j_1}, H_{i_2 j_2}, \ldots, H_{i_s j_s}$$

with majority edges with labels $i_1, \ldots, i_s$ respectively and minority edges with labels $j_1, \ldots, j_s$ respectively, and

$$\gcd(i_r, n) = 1 \quad \text{(for all } r \text{ with } 1 \leq r \leq s)$$

Therefore, $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}$ is a set of edge labels coprime to n, and there are at most $\varphi(n)/2$ such integer labels. □

An important corollary to Theorem 16 is that if n is not a prime number, St_n is not symmetrically Hamilton decomposable.

Corollary 17. *If $n \geq 5$ is not a prime number, then there is no symmetric collection of edge-disjoint Hamilton cycles $\widetilde{H}$ such that*

$$E(St_n) = \bigcup_{H \in \widetilde{H}} E(H),$$

where $E(H)$ denotes the set of edges in Hamilton cycle H.

Proof. If the edges $E(St_n)$ of St_n are partitioned into a collection $\widetilde{H}$ of disjoint Hamilton cycles, $\widetilde{H}$ will have $\lfloor n/2 \rfloor$ such cycles if n is odd and $n/2 - 1$ such cycles if n is even. However, if the non-prime n is odd then $\varphi(n) < n - 1$ and if n is even $\varphi(n) \leq n/2$. By Theorem 16, $\widetilde{H}$ cannot be symmetric. □

5. Lower Bounds in Even Dimensions

Although St_n is not symmetrically Hamilton decomposable for any even integer n, we find an optimal symmetric collection of edge-disjoint Hamilton cycles, i.e. a collection with $\varphi(n)/2$ Hamilton cycles, in Theorem 20 below. Constructing a symmetric collection involves finding a collection of label automorphisms which, when applied to H_{12}, generate disjoint Hamilton cycles as the images of H_{12}. Lemma 14 (ii) characterizes the pointwise maps of label automorphisms to be of the form $\phi(a_i) = a_{i_0 + ji}$. In the following Lemma 18 (i) and (ii), the converse is given, i.e. that any pointwise map of the form $\phi(a_i) = a_{i_0+ji}$ consistently defines a distance map of edge labels

$$\phi^d : \{1, ..., \lfloor n/2 \rfloor\} \mapsto \{1, ..., \lfloor n/2 \rfloor\}$$

and therefore a label automorphism.

Lemma 18. *Let n be odd or even and $i_0, j \in \{1, \ldots, n\}$ be such that j is coprime to n. If the bijection $\phi_j : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ is defined by*

$$\phi_j(a_i) = a_{i_0+ji} \quad (1 \leq i \leq n)$$

then the following hold:

(i) for all $a_g, a_h \in \{a_1, ..., a_n\}$,

$$\delta(\phi_j(a_g), \phi_j(a_h)) = \min\{|j(g-h) \bmod n|, n - |j(g-h) \bmod n|\},$$

(ii) there exists a bijection $\phi_j^d : \{1, ..., \lfloor n/2 \rfloor\} \mapsto \{1, ..., \lfloor n/2 \rfloor\}$ such that, for all $a_g, a_h \in \{a_1, ..., a_n\}$,

$$\delta(\phi_j(a_g), \phi_j(a_h)) = \phi_j^d(\delta(a_g, a_h)),$$

(iii) if $i_0 = n$, i.e. $\phi_j(a_i) = a_{ji}$, then for the label automorphism Φ_j corresponding to ϕ_j as in Definition 5, we have that, for all $\overleftarrow{a}_n \in V(St_n)$, there exists $\overleftarrow{a}'_n \in V(St_n)$ such that

$$\Phi_j(\overleftarrow{a}_n) = \overleftarrow{a}'_n,$$

i.e. vertices ending in a_n are mapped to vertices ending in a_n by Φ_j.

Proof. For (i), we have that (arithmetic expressions are evaluated modulo n):

$$\begin{aligned}\delta(\phi_j(a_g), \phi_j(a_h)) &= \min\{|(i_0 + jg) - (i_0 + jh)|, n - |(i_0 + jg) - (i_0 + jh)|\} \\ &= \min\{|j(g-h)|, n - |j(g-h)|\}\end{aligned}$$

To prove (ii), we need to show that if $a_g, a_h, a_{g'}, a_{h'} \in \{a_1, \ldots, a_n\}$, then $\delta(a_g, a_h) = \delta(a_{g'}, a_{h'})$ implies that $\delta(\phi_j(a_g), \phi_j(a_h)) = \delta(\phi_j(a_{g'}), \phi_j(a_{h'}))$. We have that:

$$\begin{aligned}\delta(a_g, a_h) = \delta(a_{g'}, a_{h'}) &\Rightarrow \min\{|g-h|, n - |g-h|\} \\ &= \min\{|g'-h'|, n - |g'-h'|\} \\ &\Rightarrow |g-h| = |g'-h'| \quad \text{or} \quad |g'-h'| = n - |g-h| \\ &\Rightarrow \{|g-h|, n - |g-h|\} = \{|g'-h'|, n - |g'-h'|\} \\ &\Rightarrow \{|j(g-h)|,\ n - |j(g-h)|\} \\ &= \{|j(g'-h')|,\ n - |j(g'-h')|\} \\ &\Rightarrow \delta(\phi_j(a_g), \phi_j(a_h)) = \delta(\phi_j(a_{g'}), \phi_j(a_{h'})) \quad \text{(by (i))}\end{aligned}$$

Condition (iii) follows immediately from the definition of the corresponding label automorphism Φ_j and the fact that $\phi_j(a_n) = a_n$ if $i_0 = n$. □

The offset i_0 in pointwise maps $\phi(a_i) = a_{i_0+ji}$ is important for ensuring that there is no clash of minority edges. Lemma 18 (iii) above shows that, if i_0 is not used, then vertices ending in a_n are mapped to vertices ending in a_n. As, by Definition 7, minority edges have vertices ending in a_n, any collection of Hamilton cycles which use exclusively pointwise maps without i_0, would have all minority edges in the collection with vertices ending in a_n. This would lead to the possibility of the same edges belonging to different Hamilton cycles in the collection, as a clash of edge labels of minority edges is unavoidable for even n. By use of i_0, we can ensure that even though different Hamilton cycles may share the same minority edge labels, they will not share the same edges as their vertices will end in a different $a_i \in \{a_1, \ldots, a_n\}$. The next lemma, Lemma 19, gives the pointwise map ϕ_{+1} which just replaces a_i by a_{i+1}.

Lemma 19. *Let $\phi_{+1} : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ be the pointwise map defined by:*

$$\phi_{+1}(a_i) = a_{i+1} \quad (1 \leq i \leq n)$$

Then:

(i) ϕ_{+1} defines a corresponding distance map

$$\phi^d_{+1} : \{1, \ldots, \lfloor n/2 \rfloor\} \mapsto \{1, \ldots, \lfloor n/2 \rfloor\},$$

such that, for all $l \in \{1, \ldots, \lfloor n/2 \rfloor\}$,

$$\phi^d_{+1}(l) = l$$

(ii) if Φ_{+1} is the label automorphism corresponding to ϕ_{+1} then, for all $\overleftarrow{a}_n \in V(St_n)$, there exists $\overleftarrow{a}_1 \in V(St_n)$ such that

$$\Phi_{+1}(\overleftarrow{a}_n) = \overleftarrow{a}_1$$

i.e. vertices ending in a_n are mapped to vertices ending in a_1 by Φ_{+1}.

Proof. If $a_g, a_h \in \{a_1, ..., a_n\}$ then (with arithmetic being modulo n)

$$\begin{aligned}\delta(\phi_{+1}(a_g), \phi_{+1}(a_h)) &= \min\{|(g+1) - (h+1)|,\ n - |(g+1) - (h+1)|\} \\ &= \min\{|g-h|,\ n - |g-h|\} \\ &= \delta(a_g, a_h)\end{aligned}$$

Thus, ϕ_{+1} defines the identity distance map $\phi^d_{+1} : L \mapsto L$. For (ii), we have that:

$$\begin{aligned}\Phi_{+1}(a_{g_1} \ldots a_{g_{n-1}} a_n) &= \phi_{+1}(a_{g_1}) \ldots \phi_{+1}(a_{g_{n-1}}) \phi_{+1}(a_n) \\ &= a_{g_1+1} \ldots a_{g_{n-1}+1} a_1\end{aligned}$$

□

We now prove that, for all even n, there are $\varphi(n)/2$ symmetric disjoint Hamilton cycles. The Hamilton cycles are generated by the label automorphisms of chosen pointwise maps, and make additional use of the pointwise map ϕ_{+1} of Lemma 19 to resolve any possible clashes of minority edges.

Theorem 20. *For all even n, St_n has a symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles $\widetilde{H}$.*

Proof. Let

$$i_1, \ldots, i_{\varphi(n)/2}$$

be the $\varphi(n)/2$ integers less than $n/2$ which are coprime to n. First of all, for all $j \in \{i_1, \ldots, i_{\varphi(n)/2}\}$ define $\phi_j : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ by

$$\phi_j(a_i) = a_{ji}$$

Then, by Lemma 18 (ii), ϕ_j defines a distance map ϕ_j^d and corresponding label automorphism Φ_j as in Definition 5. Consider the image of H_{12} under Φ_j. From Lemma 18 (i) and as $j < n/2$, we have that:

$$\delta(a_2, a_1) = 1 \text{ and } \delta(\phi_j(a_2), \phi_j(a_1)) = \min\{|j|, n - |j|\} = j$$

and

$$\delta(a_3, a_1) = 2 \text{ and } \delta(\phi_j(a_3), \phi_j(a_1)) = \min\{|2j|, n - |2j|\}$$

Thus, $\phi_j^d(1) = j$ and $\phi_j^d(2) = \pm 2j \bmod n$. Taking the image $\Phi_j(H_{12})$ for each $j \in \{i_1, \ldots, i_{\varphi(n)/2}\}$ we produce a list of Hamilton cycles (with the majority and minority edge labels indicated in the subscripts):

$$H_{i_1 \pm 2i_1}, \ldots, H_{i_{\varphi(n/2)} \pm 2i_{\varphi(n/2)}} \tag{5}$$

as in Definition 10. As $i_1, \ldots, i_{\varphi(n)/2}$ are distinct odd integers coprime to n, each majority edge in any Hamilton cycle in (5) only occurs in that Hamilton cycle as no other Hamilton cycle has the same edge label. However, it is possible that different Hamilton cycles in (5) share the same minority edge labels. We may have, for some distinct $i_r, i_s \in \{i_1, \ldots, i_{\varphi(n)/2}\}$,

$$\min\{|2i_r \bmod n|, n - |2i_r \bmod n|\} = \min\{|2i_s \bmod n|, n - |2i_s \bmod n|\}$$

when $2i_r = -2i_s \bmod n$, i.e.

$$2i_s = n - 2i_r \text{ and so } i_s = n/2 - i_r \tag{6}$$

From (6), it is clear that any minority edge label may be common to at most two Hamilton cycles in (5). To resolve this clash of minority edge labels, we replace one of the Hamilton cycles involved by one with the same labels but different vertices for minority edges. Suppose that the minority edges of $H_{i_r \pm 2i_r}$ and $H_{i_s \pm 2i_s}$ clash, so that $i_s = n/2 - i_r$. Consider the Hamilton cycles:

$$H'_{i_r \pm 2i_r} = \Phi_{i_r}(H_{12}) \text{ and } H'_{i_s \pm 2i_s} = \Phi_{+1}(H_{i_s \pm 2i_s}) = \Phi_{+1}(\Phi_{i_s}(H_{12})) \tag{7}$$

By Definitions 7 and 10, all vertices of minority edges of H_{12} are of the form $\overleftarrow{a}_n$, and so, by Lemma 18 (iii), all vertices of minority edges of $\Phi_{i_r}(H_{12})$ and $\Phi_{i_s}(H_{12})$ are also of the form $\overleftarrow{a}_n$. From the latter it follows, by Lemma 19 (ii), that all vertices of minority edges of $\Phi_{+1}(\Phi_{i_s}(H_{12}))$ are of the form $\overleftarrow{a}_1$. Thus, as the vertices of minority edges of $H_{i_r \pm 2i_r}$ are of the form $\overleftarrow{a}_n$ and those of $H'_{i_s \pm 2i_s}$ are of the form $\overleftarrow{a}_1$, $H_{i_r \pm 2i_r}$ and $H_{i_s \pm 2i_s}$ are edge disjoint despite having the same minority edge labels. By resolving all pairs of clashes in this way in (5) we produce a collection of $\varphi(n)/2$ symmetric and edge-disjoint Hamilton cycles as required. $\square$

6. Conclusions

We leave as an open problem the question of whether the $\varphi(n)/2$ upper bound on the number of symmetric edge-disjoint Hamilton cycles can be achieved for St_n for any odd n other the known (positive) case of St_5 [6]. In the case of even n, the number of Hamilton cycles in a symmetric collection $\widetilde{H}$ is limited to $\varphi(n)/2$ because every majority edge label in $\widetilde{H}$ has to be coprime to n as the majority edge label 1 of the base Hamilton cycle H_{12} is coprime to n. However, in the case of odd n, both the majority and minority edge labels of Hamilton cycles in symmetric collections have to be coprime to n as both the majority and minority edge labels of H_{12}, i.e. 1 and 2, are coprime to n. For this reason, the greatest lower bound for symmetric collections for all odd n may be $\varphi(n)/4$. This bound is nearly achieved by a $2\varphi(n)/9$ bound for odd n in [5]. Along with our $\varphi(n)/2$ bound here for even n, it is clear that the $2\varphi(n)/9$ bound holds comfortably for all n hence achieving a better than twofold improvement of the $\varphi(n)/10$ bound obtained for general n in [9].

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