Finite Orbits of the Action of the Pure Braid Group on the Character Variety of the Riemann Sphere with 5 Boundary Components

by

Pierpaolo Calligaris

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Abstract

In this thesis, we classify finite orbits of the action of the pure braid group over a certain large open subset of the $SL_2(\mathbb{C})$ character variety of the Riemann sphere with five boundary components, i.e. Σ_5. This problem arises in the context of classifying algebraic solutions of the Garnier system $\mathcal{G}_2$, that is the two variable analogue of the famous sixth Painlevé equation PVI. The structure of the analytic continuation of these solutions is described in terms of the action of the pure braid group on the fundamental group of Σ_5. To deal with this problem, we introduce a system of co-adjoint coordinates on a big open subset of the $SL_2(\mathbb{C})$ character variety of Σ_5. Our classification method is based on the definition of four restrictions of the action of the pure braid group such that they act on some of the co-adjoint coordinates of Σ_5 as the pure braid group acts on the co-adjoint coordinates of the character variety of the Riemann sphere with four boundary components, i.e. Σ_4, for which the classification of all finite orbits is known. In order to avoid redundant elements in our final list, a group of symmetries G of the large open subset is introduced and the final classification is achieved modulo the action of G. We present a final list of 54 finite orbits.

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Contents

Introduction

The topic of this thesis is the classification of finite orbits of a certain action of the pure braid group on the $SL_2(\mathbb{C})$ character variety of Σ_5, i.e. the Riemann sphere with five boundary components:

$$\mathcal{M}_{\mathcal{G}_2} := \mathrm{Hom}(\pi_1(\Sigma_5), SL_2(\mathbb{C}))/SL_2(\mathbb{C}).$$

After fixing a basis of oriented loops $\gamma_1, \ldots, \gamma_4, \gamma_\infty$ for $\pi_1(\Sigma_5)$ such that $\gamma_\infty^{-1} = \gamma_1 \cdots \gamma_4$, as in Figure 1, an equivalence class of an homomorphism in the character variety $\mathcal{M}_{\mathcal{G}_2}$ can be determined by the five matrices $M_1, \ldots, M_4$, $M_\infty \in SL_2(\mathbb{C})$, that are images of $\gamma_1, \ldots, \gamma_4, \gamma_\infty$. These matrices must satisfy the relation:

$$M_\infty M_4 M_3 M_2 M_1 = \mathbb{1}, \tag{1}$$

up to global conjugation. Assuming M_∞ diagonalizable, then by (1) and global conjugation, M_∞ can be brought to diagonal form:

$$M_\infty = \begin{pmatrix} e^{\pi i \theta_\infty} & 0 \\ 0 & e^{-\pi i \theta_\infty} \end{pmatrix}, \ \theta_\infty \in \mathbb{C}.$$

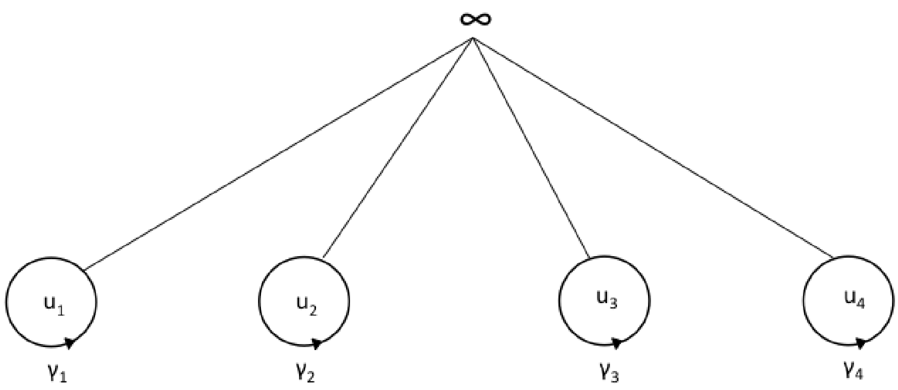

Figure 1: The basis of loops for $\pi_1(\Sigma_5)$.

As a consequence the character variety $\mathcal{M}_{\mathcal{G}_2}$ is identified with the quotient space $\widehat{\mathcal{M}}_{\mathcal{G}_2}$, defined as:

$$\widehat{\mathcal{M}}_{\mathcal{G}_2} := \{(M_1,\ldots,M_4) \in \mathrm{SL}_2(\mathbb{C}) | \, M_\infty M_4 M_3 M_2 M_1 = \mathbb{1}, \\ M_\infty = \mathrm{diag}(e^{\pm i\pi\theta_\infty}) \quad \} / \sim, \tag{2}$$

where $\sim$ is equivalence up to simultaneous conjugation of $M_1,\ldots,M_4$ by a diagonal matrix in $\mathrm{SL}_2(\mathbb{C})$. The action:

$$B_4 \times \widehat{\mathcal{M}}_{\mathcal{G}_2} \longmapsto \widehat{\mathcal{M}}_{\mathcal{G}_2}, \tag{3}$$

of the braid group B_4 on an element in $\widehat{\mathcal{M}}_{\mathcal{G}_2}$ is defined in terms of the following generators:

$$\begin{aligned} \sigma_1 &: (M_1, M_2, M_3, M_4) \mapsto (M_2, M_2 M_1 M_2^{-1}, M_3, M_4), \\ \sigma_2 &: (M_1, M_2, M_3, M_4) \mapsto (M_1, M_3, M_3 M_2 M_3^{-1}, M_4), \\ \sigma_3 &: (M_1, M_2, M_3, M_4) \mapsto (M_1, M_2, M_4, M_4 M_3 M_4^{-1}), \end{aligned} \tag{4}$$

so that M_∞ is preserved and the generators σ_i satisfy the following braid relations:

$$\sigma_1\sigma_3 = \sigma_3\sigma_1, \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3.$$

In this thesis, we classify finite orbits of this action (actually of the action of the pure braid group $P_4 \subset B_4$) on the space $\widehat{\mathcal{M}}_{\mathcal{G}_2}$.

In Chapter 2, we show that this problem arises in the context of classifying algebraic solutions of the 2×2 Schlesinger equations in four variables:

$$\frac{\partial}{\partial u_j}A_i = \frac{[A_i,A_j]}{u_i-u_j}, \quad \frac{\partial}{\partial u_i}A_i = -\sum_{j\neq i}\frac{[A_i,A_j]}{u_i-u_j}, \ i\neq j, \ i,j=1,\dots,4. \tag{5}$$

These equations are the isomonodromic deformations equations of the following Fuchsian system with five singularities $u_1,\dots,u_4,\infty \in \bar{\mathbb{C}}$:

$$\frac{d\Psi}{dz} = \left(\frac{A_1}{z-u_1}+\frac{A_2}{z-u_2}+\frac{A_3}{z-u_3}+\frac{A_4}{z-u_4}\right)\Psi, \ z\in\mathbb{C}\backslash\{u_1,\dots,u_4\}, \tag{6}$$

where the residue matrices A_i, for $i=1,\dots,4$, are traceless and the residue at infinity, i.e. A_∞, defined by:

$$A_\infty := -\left(A_1+A_2+A_3+A_4\right),$$

is assumed to be diagonal:

$$A_\infty = \begin{pmatrix}\theta_\infty & 0\\ 0 & -\theta_\infty\end{pmatrix}, \ \theta_\infty\in\mathbb{C}.$$

Solutions $A_i(u)$, where $u=(u_1,\dots,u_4)$, of (5) locally are (up to Bäcklund transformations) in one to one correspondence with points on $\widehat{\mathcal{M}}_{\mathcal{G}_2}$. The analytic continuation of the solution $A_i(u)$ along a loop on the universal cover of the configuration space of four points, i.e. $\mathbb{C}^4\backslash\{diags\}$, corresponds to another point on $\widehat{\mathcal{M}}_{\mathcal{G}_2}$ that is given by the action (4) of the braid group on $(M_1,\dots,M_4)$, as introduced by Dubrovin-Mazzocco in [DM00] for the Schlesinger equations in three variables. Then, by the generalization due to Cousin [Cou16] of the results of [DM00] and Iwasaki in [Iwa03], algebraic solutions of (5) must correspond to finite orbits of the action (4).

System (5) is equivalent to the Garnier system $\mathcal{G}_2$:

$$\begin{cases} \frac{\partial \nu_j}{\partial u_i} & = \frac{\partial K_i}{\partial \rho_j},\ i,j=1,2, \\ \frac{\partial \rho_j}{\partial u_i} & = -\frac{\partial K_i}{\partial \nu_j},\ i,j=1,2, \end{cases} \tag{7}$$

defined in Chapter 2, that is the two variables analogue of the famous Sixth Painlevé equation, PVI: to be more precise, the Garnier system $\mathcal{G}_2$ is the reduction of the Schlesinger equations (5) to Darboux coordinates on the symplectic leaves. Therefore finite orbits will correspond to algebraic solutions of the Garnier system $\mathcal{G}_2$, see [Cou16]. The simplest example of algebraic solution of $\mathcal{G}_2$ is given by Tsuda in [Tsu06]:

$$(\widetilde{\nu}_i,\widetilde{\rho}_i)=\left(\frac{\theta_i\sqrt{\widetilde{u}_i}}{\theta_\infty},\frac{\theta_\infty}{2\sqrt{\widetilde{u}_i}}\right),\ i=1,2,$$

that is algebraic for $\theta_3=\theta_4=\frac{1}{2}$ and it satisfies (7) after a suitable change of variables $(\nu_i,\rho_i,u_i)\to(\widetilde{\nu}_i,\widetilde{\rho}_i,\widetilde{u}_i)$. In our classification we are going to exclude both cases either when the monodromy group $\langle M_1,M_2,M_3,M_4\rangle$ is reducible or there exists an index $i=1,\dots,4,\infty$ such that $M_i=\pm\mathbb{1}$. Indeed if the monodromy group is reducible the associated solution of $\mathcal{G}_2$ can be reduced to classical solutions in terms of Lauricella hypergeometric functions as proved by Mazzocco in [Maz01a]. Moreover, in case $M_i = \pm\mathbb{1}$ for some index i, again following [Maz01a], the solution of $\mathcal{G}_2$ can be reduced to a solution of PVI. This leads us to define the following big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$:

$$\mathcal{U} = \Big\{(M_1, \ldots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2} | \langle M_1, \ldots, M_4\rangle \text{ irreducible}, \\ M_i \neq \pm 1, \forall i = 1, \ldots, 4, \Big\} / \sim .$$

To explain our classification result, we identify the open subset $\mathcal{U}$ with an affine algebraic variety:

Lemma 1. Let the functions p_i,p_{ij},p_{ijk} be defined as:

$$\begin{aligned} p_i &= \operatorname{Tr} M_i, && i = 1, \ldots, 4, \\ p_{ij} &= \operatorname{Tr} M_i M_j, && i, j = 1, \ldots, 4, \quad i > j, \\ p_{ijk} &= \operatorname{Tr} M_i M_j M_k, && i, j, k = 1, \ldots, 4, \quad i > j > k, \\ p_\infty &= \operatorname{Tr} M_4 M_3 M_2 M_1, \end{aligned} \tag{8}$$

then for every choice of $p_1, \ldots, p_4, p_\infty$, the open subset $\mathcal{U}$ is a four dimensional affine algebraic variety isomorphic to:

$$\mathbb{C}[p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}]/I, \tag{9}$$

where the ideal I is the ideal generated by the polynomials $f_1, \ldots, f_{15}$ defined in (1.53)-(1.67).

Therefore, we think of p_i,p_{ij},p_{ijk} as an overdetermined system of coordinates on a big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and we express the action (3) in terms of p_i,p_{ij},p_{ijk}:

Lemma 2. The transformations $\sigma_i : \widehat{\mathcal{M}}_{\mathcal{G}_2} \longrightarrow \widehat{\mathcal{M}}_{\mathcal{G}_2}$ act on the coordinates p_i,p_{ij},p_{ijk} in the open subset $\mathcal{U}$ as follows:

$$
\begin{aligned}
\sigma_1 :&(p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto\\
&(p_2, p_1, p_3, p_4, p_\infty, p_{21}, p_{32}, p_1p_3 - p_{31} - p_{21}p_{32} + p_2p_{321}, p_{42},\\
&p_1p_4 - p_{41} - p_{21}p_{42} + p_2p_{421}, p_{43}, p_{321}, p_1p_{43} - p_{431} - p_{21}p_{432} + p_2p_\infty,\\
&p_{432}, p_{421}),\\
\sigma_2 :&(p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto\\
&(p_1, p_3, p_2, p_4, p_\infty, p_{31}, p_1p_2 - p_{21} - p_{31}p_{32} + p_3p_{321}, p_{32}, p_{41}, p_{43},\\
&p_2p_4 - p_{42} - p_{32}p_{43} + p_3p_{432}, p_{321}, p_{432}, p_2p_{41} - p_{421} - p_{32}p_{431} + p_3p_\infty,\\
&p_{431}),\\
\sigma_3 :&(p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto\\
&(p_1, p_2, p_4, p_3, p_\infty, p_{21}, p_{41}, p_{42}, p_1p_3 - p_{31} - p_{41}p_{43} + p_4p_{431},\\
&p_2p_3 - p_{32} - p_{42}p_{43} + p_4p_{432}, p_{43}, p_{421}, p_{432}, p_{431},\\
&p_{21}p_3 - p_{321} - p_{421}p_{43} + p_4p_\infty),
\end{aligned} \tag{10}
$$

and they define an action of the braid group B_4.

Hence, our problem can be reformulated as: find all p_i,p_{ij},p_{ijk} in the big open subset $\mathcal{U}$ such that:

- they satisfy the constraints given by I in Lemma 1,
- their orbit under the action of the pure braid group P_4 is finite.

Our approach is based on the observation that given p_i,p_{ij},p_{ijk} such that they generate a finite orbit under the action of the pure braid group P_4, then for any subgroup $H \subset P_4$ the restriction of the action to H produces a finite orbit as well. Such restriction only acts on some of the p_i,p_{ij},p_{ijk} and leaves others invariant. We select subgroups $H \subset P_4$ acting on the set (9) so that the restricted action is isomorphic to the action of the pure braid group P_3 on the $\mathrm{SL}_2(\mathbb{C})$ character variety of Σ_4, i.e. the Riemann sphere with four boundary components, for which all finite orbits are classified in Lisovyy and Tykhyy's work [LT14].

Furthermore, we show that there exist four restrictions $H_1, \dots, H_4$ isomorphic to P_3. Each one of these restrictions allows us to identify some of the p_i,p_{ij},p_{ijk} with coordinates on the $\mathrm{SL}_2(\mathbb{C})$ character variety of Σ_4, as in Table 1: each line shows which p_i,p_{ij},p_{ijk} can be found by imposing that the restriction gives a finite orbit of P_3. We recall the list of all finite orbits of the action of P_3 on the $\mathrm{SL}_2(\mathbb{C})$ character variety of Σ_4 in Chapter 3.

	p_1	p_2	p_3	p_4	p_∞	p_{21}	p_{31}	p_{32}	p_{41}	p_{42}	p_{43}	p_{321}	p_{432}	p_{431}	p_{421}
H_1		•	•	•				•		•	•		•		
H_2	•		•	•			•		•		•			•	
H_3	•	•		•		•			•	•					•
H_4	•	•	•			•	•	•				•			

Table 1: Action on p_i,p_{ij},p_{ijk} defined in (10) of subgroups of P_4 isomorphic to P_3.

In order to avoid redundant solutions to this classification problem, such as for example equivalent solutions obtained by simple *cyclic* relabelling of indices in (8), in Chapter 2, we introduce the symmetry group G of the big open subset $\mathcal{U}$ and factorize our classification modulo the action of G. The symmetry group G can be calculated using known results about Bäcklund transformations of Schlesinger equations (5) and permutations and sign flips on the monodromy matrices.

In Chapter 4, we present a list of 54 finite orbits of action (10) obtained up to the action of the group of symmetries G. Due to the identification of each action of the restriction H_i (determined by the rows in Table 1) with the finite action of P_3 over the $\mathrm{SL}_2(\mathbb{C})$ character variety of Σ_4, we can associate to each restriction an algebraic solution of PVI (see [DM00, Iwa03, Cou16, LT14]). Then in our list each orbit's member has the following properties:

- no more than one restriction (determined by the rows of Table 1) is associated to algebraic solutions of PVI obtained by the *pull-back* of the hypergeometric equation, see Doran [Dor01] and Andreev-Kitaev [AK02],
- no more than one restriction corresponds to the so-called Picard solutions of PVI, see the work of Picard [Pic89] and Mazzocco [Maz01b].

Moreover, we do not allow any orbit's member such that:

- one restriction is associated to algebraic solutions of PVI obtained by the *pull-back* of the hypergeometric equation and another restriction is associated to the so-called Picard solutions of PVI.

Accordingly, our solutions do not include the before mentioned solution obtained by Tsuda in [Tsu06] as a fixed point of a certain birational symmetry of $\mathcal{G}_2$, nor the solutions found by Diarra in [Dia13], who presents all finite orbits that can be obtained using the method of *pull-back* introduced in [Dor01] and [AK02], nor the one found by Girand in [Gir16a, Gir16b], who presents two-parameter families of algebraic solutions of $\mathcal{G}_2$ obtained restricting a logarithmic flat connection defined on the complement of a quintic curve on $\mathbb{P}^2$ on generic lines of the projective plane, these solutions have at least two restrictions obtained by *pull-back* of the hypergeometric equation.

From the monodromy data $M_1, \dots, M_4$, it is possible to recover the explicit formulation of the associated solution of $\mathcal{G}_2$ using the method developed by Lisovyy and Gavrylenko in [GL16] of Fredholm determinant representation for isomonodromic tau functions of Fuchsian systems of the form (6).

The shortest finite orbit classified has length 36, for this reason the associated algebraic solution of $\mathcal{G}_2$ has eventually 36 branches and we doubt that the expression of this solution can have a nice and compact form.

Chapter 1

Action of the braid group B_4 on $\mathcal{M}_{\mathcal{G}_2}$ and restrictions

In this Chapter we are going to describe in details the action:

$$P_4 \times \mathcal{M}_{\mathcal{G}_2} \longmapsto \mathcal{M}_{\mathcal{G}_2}, \tag{1.1}$$

of the pure braid group P_4 on $\mathcal{M}_{\mathcal{G}_2}$, i.e. the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Riemann sphere Σ_5 with five boundary components. In Theorem 3, Lemma 4 and Proposition 5, we will show that there exists a system of co-adjoint coordinates p_i, p_{ij}, p_{ijk} on a big open subset $\mathcal{U}$ of $\mathcal{M}_{\mathcal{G}_2}$. Furthermore, in Theorem 6, the big open subset $\mathcal{U}$ is identified with an affine algebraic variety that is the zero locus of a particular family $\mathcal{F}$ of polynomials.

In Section 1.2, the action (1.1) on p_i, p_{ij}, p_{ijk} is presented explicitly in Lemma 10. In Section 1.3, the problem is reformulated as the classification of finite orbits of the P_4 action over the p_i, p_{ij}, p_{ijk} such that they are in the zero locus of $\mathcal{F}$.

Moreover, in Section 1.4, we discuss the methodology used to achieve this classification problem: indeed, if p_i, p_{ij}, p_{ijk} are known such that they generates a finite P_4 orbit, then for any subgroup $H \subset P_4$, the action of H over p_i, p_{ij}, p_{ijk} still generate a finite orbit. In Theorem 12, we identify four subgroups $H_1, \dots, H_4$ acting as the pure braid group P_3 over the $SL_2(\mathbb{C})$ character variety of the Riemann sphere Σ_4 with four boundary components, that we will denote $\mathcal{M}_{PVI}$: so that we can use the classification result obtained by Lisovyy and Tykhyy in [LT14].

1.1 Co-adjoint coordinates on $\mathcal{M}_{\mathcal{G}_2}$

We identify the character variety $\mathcal{M}_{\mathcal{G}_2}$ with the quotient space:

$$\widehat{\mathcal{M}}_{\mathcal{G}_2} = \{(M_1, M_2, M_3, M_4) \mid M_i \in \mathrm{SL}_2(\mathbb{C}),\ M_\infty M_4 M_3 M_2 M_1 = \mathbb{1}\} / \sim,$$

where $\sim$ is equivalence under global diagonal conjugation. Without loss of generality, the matrix M_∞ can be brought to diagonal form:

$$M_\infty = \begin{pmatrix} e^{\pi i \theta_\infty} & 0 \\ 0 & e^{-\pi i \theta_\infty} \end{pmatrix}, \ \theta_\infty \in \mathbb{C}, \tag{1.2}$$

then, since the trace of M_∞ is a given parameter, $\mathcal{M}_{\mathcal{G}_2}$ is an *eight* dimensional affine algebraic variety: indeed each M_i is an element of $\mathrm{SL}_2(\mathbb{C})$, up to global diagonal conjugation, and $M_1, \dots, M_4$ satisfy the cyclic relation $M_\infty M_4 M_3 M_2 M_1 = \mathbb{1}$ and (1.2).

It is possible to endow the space of functions on $\mathcal{M}_{\mathcal{G}_2}$ with a system of co-adjoint coordinates, this is a generalization of a result proved by Iwasaki for the Sixth Painlevé equation [Iwa03]:

Theorem 3. Let $(M_1, \dots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and define the following complex quantities:

$$(p_1,p_2,p_3,p_4,p_\infty,p_{21},p_{31},p_{32},p_{41},p_{42},p_{43},p_{321},p_{432},p_{431},p_{421})\in\mathbb{C}^{15}, \quad (1.3)$$

to be:

$$\begin{aligned}
p_i &= \mathrm{Tr}\, M_i, & i&=1,\dots,4,\\
p_{ij} &= \mathrm{Tr}\, M_iM_j, & i,j&=1,\dots,4,\ i>j,\\
p_{ijk} &= \mathrm{Tr}\, M_iM_jM_k, & i,j,k&=1,\dots,4,\ i>j>k,\\
p_\infty &= p_{4321} = \mathrm{Tr}\, M_4M_3M_2M_1. & & \quad (1.4)
\end{aligned}$$

Let $g(x,y,z) := x^2+y^2+z^2-xyz-4$, then in the open subset:

$$\mathcal{U}^{(0)}_{jk} := \widehat{\mathcal{M}}_{\mathcal{G}_2}\cap\{(p_{jk}^2-4)g(p_{jk},p_\ell,p_{jk\ell})\neq 0\}, \quad (1.5)$$

the matrices $M_1,\dots,M_4$ can be parametrized as follows:

$$PM_\ell P^{-1} = \begin{pmatrix} \frac{p_{jk\ell}-p_\ell\lambda^-_{jk}}{r_{jk}} & -\frac{g(p_{jk},p_\ell,p_{jk\ell})}{r^2_{jk}} \\ 1 & -\frac{p_{jk\ell}-p_\ell\lambda^+_{jk}}{r_{jk}} \end{pmatrix}, \quad (1.6)$$

$$PM_kP^{-1} = \begin{pmatrix} -\frac{p_j-p_k\lambda^+_{jk}}{r_{jk}} & -\frac{y_{kl}-y_{jl}\lambda^-_{jk}}{r^2_{jk}} \\ \frac{y_{kl}-y_{jl}\lambda^+_{jk}}{g(p_{jk},p_\ell,p_{jk\ell})} & \frac{p_j-p_k\lambda^-_{jk}}{r_{jk}} \end{pmatrix}, \quad (1.7)$$

$$PM_jP^{-1} = \begin{pmatrix} -\frac{p_k-p_j\lambda^+_{jk}}{r_{jk}} & -\frac{y_{jl}-y_{kl}\lambda^+_{jk}}{r^2_{jk}} \\ \frac{y_{jl}-y_{kl}\lambda^-_{jk}}{g(p_{jk},p_\ell,p_{jk\ell})} & \frac{p_k-p_j\lambda^-_{jk}}{r_{jk}} \end{pmatrix}, \quad (1.8)$$

$$PM_iP^{-1} = \begin{pmatrix} \frac{p_{ijk}-p_i\lambda^-_{jk}}{r_{jk}} & -\frac{y_{il}+y_{ijkl}\lambda^+_{jk}}{r^2_{jk}} \\ \frac{y_{il}+y_{ijkl}\lambda^-_{jk}}{g(p_{jk},p_\ell,p_{jk\ell})} & -\frac{p_{ijk}-p_i\lambda^+_{jk}}{r_{jk}} \end{pmatrix}, \quad (1.9)$$

alternatively on the open subset:

$$\mathcal{U}_{jk}^{(1)} := \widehat{\mathcal{M}}_{\mathcal{G}_2} \cap \{(p_{jk}^2 - 4)g(p_j, p_k, p_{jk}) \neq 0\}, \tag{1.10}$$

the matrices $M_1, \ldots, M_4$ can be parametrized as follows:

$$PM_\ell P^{-1} = \begin{pmatrix} \frac{p_{jk\ell}-p_\ell\lambda_{jk}^-}{r_{jk}} & -\frac{y_{kl}-y_{jl}\lambda_{jk}^+}{r_{jk}^2} \\ \frac{y_{kl}-y_{jl}\lambda_{jk}^-}{g(p_{jk},p_j,p_k)} & -\frac{p_{jk\ell}-p_\ell\lambda_{jk}^+}{r_{jk}} \end{pmatrix}, \tag{1.11}$$

$$PM_k P^{-1} = \begin{pmatrix} -\frac{p_j-p_k\lambda_{jk}^+}{r_{jk}} & -\frac{g(p_{jk},p_j,p_k)}{r_{jk}^2} \\ 1 & \frac{p_j-p_k\lambda_{jk}^-}{r_{jk}} \end{pmatrix}, \tag{1.12}$$

$$PM_j P^{-1} = \begin{pmatrix} -\frac{p_k-p_j\lambda_{jk}^+}{r_{jk}} & \frac{g(p_{jk},p_j,p_k)\lambda_{jk}^+}{r_{jk}^2} \\ -\lambda_{jk}^- & \frac{p_k-p_j\lambda_{jk}^-}{r_{jk}} \end{pmatrix}, \tag{1.13}$$

$$PM_i P^{-1} = \begin{pmatrix} \frac{p_{ijk}-p_i\lambda_{jk}^-}{r_{jk}} & -\frac{y_{ik}-y_{ij}\lambda_{jk}^+}{r_{jk}^2} \\ \frac{y_{ik}-y_{ij}\lambda_{jk}^-}{g(p_{jk},p_j,p_k)} & -\frac{p_{ijk}-p_i\lambda_{jk}^+}{r_{jk}} \end{pmatrix}, \tag{1.14}$$

and on the open subset:

$$\mathcal{U}_{jk}^{(2)} := \widehat{\mathcal{M}}_{\mathcal{G}_2} \cap \{(p_{jk}^2 - 4)g(p_{jk}, p_i, p_{ijk}) \neq 0\}, \tag{1.15}$$

the matrices $M_1, \ldots, M_4$ can be parametrized as follows:

$$PM_\ell P^{-1} = \begin{pmatrix} \frac{p_{jk\ell}-p_\ell\lambda_{jk}^-}{r_{jk}} & -\frac{y_{il}+y_{ijkl}\lambda_{jk}^-}{r_{jk}^2} \\ \frac{y_{il}+y_{ijkl}\lambda_{jk}^+}{g(p_{jk},p_i,p_{ijk})} & -\frac{p_{jk\ell}-p_\ell\lambda_{jk}^+}{r_{jk}} \end{pmatrix}, \tag{1.16}$$

$$PM_k P^{-1} = \begin{pmatrix} -\frac{p_j-p_k\lambda_{jk}^+}{r_{jk}} & -\frac{y_{ik}-y_{ij}\lambda_{jk}^-}{r_{jk}^2} \\ \frac{y_{ik}-y_{ij}\lambda_{jk}^+}{g(p_{jk},p_i,p_{ijk})} & \frac{p_j-p_k\lambda_{jk}^-}{r_{jk}} \end{pmatrix}, \tag{1.17}$$

$$PM_jP^{-1} = \begin{pmatrix} -\frac{p_k - p_j\lambda_{jk}^+}{r_{jk}} & -\frac{y_{ij} - y_{ik}\lambda_{jk}^+}{r_{jk}^2} \\ \frac{y_{ij} - y_{ik}\lambda_{jk}^-}{g(p_{jk},p_i,p_{ijk})} & \frac{p_k - p_j\lambda_{jk}^-}{r_{jk}} \end{pmatrix}, \tag{1.18}$$

$$PM_iP^{-1} = \begin{pmatrix} \frac{p_{ijk} - p_i\lambda_{jk}^-}{r_{jk}} & -\frac{g(p_{jk},p_i,p_{ijk})}{r_{jk}^2} \\ 1 & -\frac{p_{ijk} - p_i\lambda_{jk}^+}{r_{jk}} \end{pmatrix}, \tag{1.19}$$

where $P \in \mathrm{GL}_2(\mathbb{C})$ and:

$$\lambda_{jk}^+ := \frac{p_{jk} + r_{jk}}{2}, \ \lambda_{jk}^- = \frac{1}{\lambda_{jk}^+}, \tag{1.20}$$

$$r_{jk} := \sqrt{p_{jk}^2 - 4}, \tag{1.21}$$

$$y_{kl} := 2p_{k\ell} + p_{jk}p_{j\ell} - p_jp_{jk\ell} - p_kp_\ell, \tag{1.22}$$

$$y_{jl} := 2p_{j\ell} + p_{jk}p_{k\ell} - p_kp_{jk\ell} - p_jp_\ell, \tag{1.23}$$

$$y_{ik} := 2p_{ik} + p_{ij}p_{jk} - p_jp_{ijk} - p_ip_k, \tag{1.24}$$

$$y_{ij} := 2p_{ij} + p_{ik}p_{jk} - p_kp_{ijk} - p_ip_j, \tag{1.25}$$

$$y_{il} := 2p_{il} + p_{ijk}p_{jk\ell} - p_{jk}p_{ijk\ell} - p_ip_\ell, \tag{1.26}$$

$$y_{ijkl} := 2p_{ijk\ell} - p_{il}p_{jk} - p_ip_{jk\ell} - p_{ijk}p_\ell + p_ip_{jk}p_\ell. \tag{1.27}$$

Proof. Consider $(M_1, \dots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$. We are going to prove that there exists a parametrization of $M_1, \dots, M_4$ in terms of the invariants p_i, p_{ij}, p_{ijk} in the open subset $\mathcal{U}_{jk}^{(0)} = \widehat{\mathcal{M}}_{\mathcal{G}_2} \cap \{(p_{jk}^2 - 4)g(p_{jk}, p_\ell, p_{jk\ell}) \neq 0\}$. For the parametrizations on the open subsets $\mathcal{U}_{jk}^{(1)}$ and $\mathcal{U}_{jk}^{(2)}$ a similar proof applies.

Under the generic hypothesis that there exist two indices j and k such that $p_{jk} \neq \pm 2$, the product M_jM_k has two distinct eigenvalues $\lambda_{jk}^\pm$, namely:

$$\lambda_{jk}^+ = \frac{p_{jk} + r_{jk}}{2}, \ \lambda_{jk}^- = \frac{1}{\lambda_{jk}^+}, \ r_{jk} = \sqrt{p_{jk}^2 - 4}, \tag{1.28}$$

where the positive branch of the square root is chosen. Consequently, there exists a matrix $P \in \mathrm{GL}_2(\mathbb{C})$ such that the product matrix $M_j M_k$ can be brought into diagonal form:

$$\Lambda_{jk} := P(M_j M_k)P^{-1} = \mathrm{diag}\{\lambda_{jk}^{+}, \lambda_{jk}^{-}\}, \tag{1.29}$$

and we conjugate by P the matrices M_ℓ, M_k, M_j, M_i as follows:

$$P(M_\ell, M_k, M_j, M_i)P^{-1} = (U, V, W, T). \tag{1.30}$$

Since, $W = \Lambda_{jk} V^{-1}$, we proceed with the parametrization of the matrices U, V, T. First, we parametrize the diagonal elements of U, V, T. Indeed, solving the equations $\mathrm{Tr}\, U = p_\ell$ and $\mathrm{Tr}\, \Lambda_{jk} U = p_{jkl}$, we get the diagonal elements of U:

$$\begin{cases} u_{11} &= \frac{p_{jk\ell} - p_\ell \lambda_{jk}^{-}}{r_{jk}}, \\ u_{22} &= -\frac{p_{jk\ell} - p_\ell \lambda_{jk}^{+}}{r_{jk}}. \end{cases} \tag{1.31}$$

Next, solving the equations $\mathrm{Tr}\, V = p_k$ and $\mathrm{Tr}\, \Lambda_{jk} V^{-1} = p_j$, we obtain the diagonal elements of V:

$$\begin{cases} v_{11} &= -\frac{p_j - p_k \lambda_{jk}^{+}}{r_{jk}}, \\ v_{22} &= \frac{p_j - p_k \lambda_{jk}^{-}}{r_{jk}}. \end{cases} \tag{1.32}$$

Finally, equations $\mathrm{Tr}\, T = p_i$ and $\mathrm{Tr}\, TWV = \mathrm{Tr}\, T\Lambda_{jk} = p_{ijk}$, determine the diagonal elements of T:

$$\begin{cases} t_{11} &= \frac{p_{ijk} - p_i \lambda_{jk}^{-}}{r_{jk}}, \\ t_{22} &= -\frac{p_{ijk} - p_i \lambda_{jk}^{+}}{r_{jk}}. \end{cases} \tag{1.33}$$

At this point, we calculate the off-diagonal elements of U, V, T respectively. Consider the matrix U. Once calculated the diagonal elements of U, since $\det U = 1$, then the following identity holds:

$$u_{12}u_{21} = -\frac{g(p_{jk}, p_\ell, p_{jk\ell})}{r_{jk}^2}, \tag{1.34}$$

and we suppose $g(p_{jk}, p_\ell, p_{jk\ell}) \neq 0$. This leads us to define the open subset $\mathcal{U}_{jk}^{(0)}$, as follows:

$$\mathcal{U}_{jk}^{(0)} := \widehat{\mathcal{M}}_{\mathcal{G}_2} \cap \{(p_{jk}^2 - 4)g(p_{jk}, p_\ell, p_{jk\ell}) \neq 0\}. \tag{1.35}$$

Moreover, note that, since P is unique up to left multiplication by a diagonal matrix $D \in \mathrm{GL}_2(\mathbb{C})$, we are allowed to fix $u_{21} = 1$. Then equation (1.34) gives us the element u_{12}. Next, consider the matrix V. The system of equations $\mathrm{Tr}\, VU = p_{k\ell}$ and $\mathrm{Tr}\, \Lambda_{jk} V^{-1} U = p_{j\ell}$ gives us a parametrization for the off-diagonal elements of V:

$$\begin{cases} v_{12} &= -\frac{y_{ik} - y_{ij}\lambda_{jk}^-}{r_{jk}^2}, \\ v_{21} &= \frac{y_{ik} - y_{ij}\lambda_{jk}^+}{g(p_{jk}, p_i, p_{ijk})}, \end{cases} \tag{1.36}$$

where y_{ik} and y_{ij} are defined in (1.24) and (1.25) respectively. Finally we calculate the off-diagonal elements of the matrix T. Consider the system of equations $\mathrm{Tr}\, TU = p_{il}$ and $\mathrm{Tr}\, TWVU = \mathrm{Tr}\, T\Lambda_{jk}U = p_{ijk\ell}$, then we have the following parametrization for t_{12} and t_{21}:

$$\begin{cases} t_{12} &= -\frac{y_{il} + y_{ijkl}\lambda_{jk}^+}{r_{jk}^2}, \\ t_{21} &= \frac{y_{il} + y_{ijkl}\lambda_{jk}^-}{g(p_{jk}, p_\ell, p_{jk\ell})}, \end{cases} \tag{1.37}$$

where y_{il} and y_{ijkl} are defined in (1.26) and (1.27) respectively. This concludes the proof. □

Theorem 3 shows that p, defined in (1.3), parametrizes the following open subset of $\widehat{\mathcal{M}}_{\mathcal{G}_2}$:

$$\bigcup_{j>k} \mathcal{U}_{jk}^{(0)} \cup \mathcal{U}_{jk}^{(1)} \cup \mathcal{U}_{jk}^{(2)} \tag{1.38}$$

We now show that, when the monodromy group is not reducible, and none of the monodromy matrices $M_1, \dots, M_4$ is a multiple of the identity, it is possible to parametrize the monodromy matrices in terms of p, defined in (1.3) and (1.4), also outside of the open subset (1.38).

Lemma 4. Let $(M_1, \dots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and define the complex quantities (1.3) as in (1.4). Assume that: none of the matrices $M_1, \dots, M_4$ is a multiple of the identity, the monodromy group is not reducible, and $p_{jk} \neq \pm 2$ for at least one choice of $j \neq k$, $j, k = 1, \dots, 4$. Moreover, assume that

$$g(p_{jk}, p_\ell, p_{jk\ell}) = g(p_j, p_k, p_{jk}) = g(p_{jk}, p_i, p_{ijk}) = 0, \tag{1.39}$$

then there exists at least an index ℓ for which $p_{lk} \neq \lambda_\ell \lambda_k + \frac{1}{\lambda_\ell \lambda_k}$ and a global conjugation $P \in \mathrm{GL}_2(\mathbb{C})$ such that:

$$PM_kP^{-1} = \begin{pmatrix} \lambda_k & 1 \\ 0 & \frac{1}{\lambda_k} \end{pmatrix}, \tag{1.40}$$

$$PM_jP^{-1} = \begin{pmatrix} \lambda_j & -\lambda_j \lambda_k \\ 0 & \frac{1}{\lambda_j} \end{pmatrix}, \tag{1.41}$$

$$PM_\ell P^{-1} = \begin{pmatrix} \lambda_\ell & 0 \\ p_{lk} - \lambda_\ell \lambda_k - \frac{1}{\lambda_\ell \lambda_k} & \frac{1}{\lambda_\ell} \end{pmatrix}, \tag{1.42}$$

$$PM_iP^{-1} = \begin{cases} \begin{pmatrix} \lambda_i & 0 \\ p_{ik} - \lambda_i\lambda_k - \frac{1}{\lambda_i\lambda_k} & \frac{1}{\lambda_i} \end{pmatrix}, & \text{for } p_{il} = \lambda_i\lambda_\ell + \frac{1}{\lambda_i\lambda_\ell}, \\ \\ \begin{pmatrix} \lambda_i & \frac{p_{il} - \lambda_i\lambda_\ell - \frac{1}{\lambda_i\lambda_\ell}}{p_{lk} - \lambda_\ell\lambda_k - \frac{1}{\lambda_\ell\lambda_k}} \\ 0 & \frac{1}{\lambda_i} \end{pmatrix}, & \text{for } p_{il} \neq \lambda_i\lambda_\ell + \frac{1}{\lambda_i\lambda_\ell}, \end{cases} \tag{1.43}$$

where $\lambda_s + \frac{1}{\lambda_s} = p_s$, $\forall s = 1, \ldots, 4$.

Proof. Proceeding as before, we bring the product matrix M_jM_k into the diagonal form (1.29). Condition (1.39) implies that the following equations must be satisfied:

$$(M_1)_{12}(M_1)_{21} = (M_2)_{12}(M_2)_{21} = (M_3)_{12}(M_3)_{21} = (M_4)_{12}(M_4)_{21} = 0.$$

By global conjugation by a permutation matrix, we can assume that $(M_k)_{12} \neq 0$ and then by global diagonal conjugation we can put M_k in the form (1.40). Then, since $M_j = \Lambda_{jk}M_k^{-1}$ we immediately obtain (1.41).

Now, since the monodromy group must be irreducible, one of the two remaining matrices, call it M_ℓ, must have non zero 21 entry. Then since $\mathrm{Tr}(M_\ell M_k) = p_{lk}$, we obtain $(M_\ell)_{21} = p_{lk} - \lambda_\ell\lambda_k - \frac{1}{\lambda_\ell\lambda_k} \neq 0$, and therefore:

$$M_\ell = \begin{pmatrix} \lambda_\ell & 0 \\ p_{lk} - \lambda_\ell\lambda_k - \frac{1}{\lambda_\ell\lambda_k} & \frac{1}{\lambda_\ell} \end{pmatrix}.$$

Now, if the last matrix is also lower triangular, by imposing $\mathrm{Tr}\, M_iM_k = p_{ik}$, we obtain the first formula in (1.43), and it is immediate to check that $p_{il} = \lambda_i\lambda_\ell + \frac{1}{\lambda_i\lambda_\ell}$. Otherwise, if M_i is upper triangular, by imposing $\mathrm{Tr}\, M_iM_\ell = p_{il}$, we obtain the second formula (1.43), and it is immediate to check that $p_{il} \neq \lambda_i\lambda_\ell + \frac{1}{\lambda_i\lambda_\ell}$. □

Proposition 5. Let $(M_1,\dots,M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and define the complex quantities (1.3) as in (1.4). Assume that: none of the matrices $M_1,\dots,M_4$ is a multiple of the identity, the monodromy group is not reducible, and $p_{jk} = 2\epsilon_{jk}$ for all $j,k = 1,\dots,4$, where $\epsilon_{jk} = \pm 1$. Then, if at least one matrix M_i is diagonalizable, there exists a choice of the ordering of the indices $i,j,k,\ell \in \{1,2,3,4\}$ and a global conjugation $P \in \mathrm{GL}_2(\mathbb{C})$ such that the following parametrization holds true:

$$PM_iP^{-1} = \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix}, \qquad \lambda_i \neq \pm 1, \quad \lambda_i + \frac{1}{\lambda_i} = p_i, \tag{1.44}$$

$$PM_kP^{-1} = \begin{pmatrix} -\frac{p_k - 2\epsilon_{ki}\lambda_i}{\lambda_i^2-1} & -\frac{(p_k\lambda_i - \epsilon_{ki}(\lambda_i^2+1))^2}{(\lambda_i^2-1)^2} \\ 1 & \frac{\lambda_i(p_k\lambda_i - 2\epsilon_{ki})}{\lambda_i^2-1} \end{pmatrix}, \tag{1.45}$$

$$PM_jP^{-1} = \begin{pmatrix} -\frac{p_j-2\epsilon_{ji}\lambda_i}{\lambda_i^2-1} & -\frac{\lambda_i^2(p_k\lambda_i-2\epsilon_{ki})(p_j\lambda_i-2\epsilon_{ji})+\lambda_i(\lambda_i^2-1)(p_{ikj}-2\epsilon_{kj}\lambda_i)}{(\lambda_i^2-1)^2} \\ \frac{(\lambda_i^2-1)(2\epsilon_{kj}-p_{ikj}\lambda_i)+(p_k\lambda_i-2\epsilon_{ki})(p_j\lambda_i-2\epsilon_{ji})}{(p_k\lambda_i-\epsilon_{ki}(\lambda_i^2+1))^2} & \frac{\lambda_i(p_j\lambda_i-2\epsilon_{ji})}{\lambda_i^2-1} \end{pmatrix}, \tag{1.46}$$

$$PM_\ell P^{-1} = \begin{pmatrix} -\frac{p_\ell-2\epsilon_{li}\lambda_i}{\lambda_i^2-1} & -\frac{\lambda_i^2(p_k\lambda_i-2\epsilon_{ki})(p_\ell\lambda_i-2\epsilon_{li})+\lambda_i(\lambda_i^2-1)(p_{ikl}-2\epsilon_{kl}\lambda_i)}{(\lambda_i^2-1)^2} \\ \frac{(\lambda_i^2-1)(2\epsilon_{kl}-p_{ikl}\lambda_i)+(p_k\lambda_i-2\epsilon_{ki})(p_\ell\lambda_i-2\epsilon_{li})}{(p_k\lambda_i-\epsilon_{ki}(\lambda_i^2+1))^2} & \frac{\lambda_i(p_\ell\lambda_i-2\epsilon_{li})}{\lambda_i^2-1} \end{pmatrix}. \tag{1.47}$$

If none of the monodromy matrices is diagonalizable, then there exists a choice of the ordering of the indices $i, j, k, \ell \in \{1, 2, 3, 4\}$ and a global conjugation $P \in \mathrm{GL}_2(\mathbb{C})$ such that the following parametrization holds true:

$$PM_iP^{-1} = \begin{pmatrix} \epsilon_i & 1 \\ 0 & \epsilon_i \end{pmatrix}, \qquad PM_jP^{-1} = \begin{pmatrix} \epsilon_j & 0 \\ 4\epsilon_{ij} & \epsilon_j \end{pmatrix}, \tag{1.48}$$

$$PM_kP^{-1} = \begin{pmatrix} \frac{p_{ijk}-2\epsilon_{ik}\epsilon_j-2\epsilon_{jk}\epsilon_i+2\epsilon_i\epsilon_j\epsilon_k}{4\epsilon_{ij}} & \frac{\epsilon_{jk}-\epsilon_j\epsilon_k}{2\epsilon_{ij}} \\ 2(\epsilon_{ik}-\epsilon_i\epsilon_k) & \frac{2\epsilon_{ik}\epsilon_j+2\epsilon_{jk}\epsilon_i+8\epsilon_{ij}\epsilon_k-2\epsilon_i\epsilon_j\epsilon_k-p_{ijk}}{4\epsilon_{ij}} \end{pmatrix}, \tag{1.49}$$

$$PM_\ell P^{-1} = \begin{pmatrix} \frac{p_{ijl}-2\epsilon_{il}\epsilon_j-2\epsilon_{jl}\epsilon_i+2\epsilon_i\epsilon_j\epsilon_\ell}{4\epsilon_{ij}} & \frac{\epsilon_{jl}-\epsilon_j\epsilon_\ell}{2\epsilon_{ij}} \\ 2(\epsilon_{il}-\epsilon_i\epsilon_\ell) & \frac{2\epsilon_{il}\epsilon_j+2\epsilon_{jl}\epsilon_i+8\epsilon_{ij}\epsilon_\ell-2\epsilon_i\epsilon_j\epsilon_\ell-p_{ijl}}{4\epsilon_{ij}} \end{pmatrix}. \tag{1.50}$$

Proof. First, let us assume that at least one matrix M_i is diagonal and work in the basis in which M_i assumes the form (1.44) with $\lambda_i \neq \pm 1$.

Let $j \neq i$, then we have a set of linear equations in the diagonal elements of M_j:

$$\mathrm{Tr}\,(M_iM_j) = 2\epsilon_{ji}, \qquad \mathrm{Tr}\,M_j = p_j, \quad \epsilon_{ji} = \pm 1,$$

that it is solved by

$$(M_j)_{11} = -\frac{p_j - 2\epsilon_{ji}\lambda_i}{\lambda_i^2 - 1}, \qquad (M_j)_{22} = \frac{\lambda_i(p_j\lambda_i - 2\epsilon_{ji})}{\lambda_i^2 - 1}, \tag{1.51}$$

for $j = 1, \ldots, 4$, $j \neq i$.

Since the monodromy group is not reducible, there is at least one matrix M_k, $k \neq i$ such that in the chosen basis, $(M_k)_{21} \neq 0$, then we use the freedom of global diagonal conjugation to set $(M_k)_{21} = 1$. Since $\det(M_k) = 1$ we obtain the formula (1.45). Observe that:

$$-\frac{(p_k\lambda_i - \epsilon_{ki}(\lambda_i^2+1))^2}{(\lambda_i^2-1)^2} \neq 0$$

otherwise $p_k = \epsilon_{ki} p_i$ and by using $\operatorname{Tr} M_i M_k = 2\epsilon_{ki}$ we would find $\lambda_i = \pm 1$.

We now deal with the other two matrices. We only need to find the off-diagonal elements of these matrices. To this aim we use the following equations for $s = j, \ell$:

$$\operatorname{Tr}(M_s M_k) = 2\epsilon_{sk}, \qquad \operatorname{Tr}(M_i M_k M_s) = p_{iks},$$

which, combined with (1.51) lead to (1.46) and (1.45). This concludes the proof of the first case.

To prove the second case, assume none of the matrices $M_1, \ldots, M_4$ are diagonalizable, then $\operatorname{eigen}(M_i) = \{\epsilon_i, \epsilon_i\}$, $\forall i = 1, \ldots, 4$, where $\epsilon_i = \pm 1$. Let us choose a global conjugation such that one of the matrices M_i is in upper triangular form as in (1.48).

Now, since the monodromy group is not reducible, there exists at least one j such that $(M_j)_{21} \neq 0$. From $\operatorname{Tr} M_i M_j = 2\epsilon_{ij}$ we have $2\epsilon_i\epsilon_j + (M_j)_{21} = 2\epsilon_{ij}$, so that $(M_j)_{21} \neq 0$ implies $\epsilon_i\epsilon_j = -\epsilon_{ij}$. We perform a conjugation by a unipotent upper triangular matrix to impose $(M_j)_{12} = 0$, so that the second equation in (1.48).

For all other matrices we use $\operatorname{Tr} M_i M_s = 2\epsilon_{is}$ and $\operatorname{Tr} M_j M_s = 2\epsilon_{js}$, $s = k, \ell$ to find:

$$(M_s)_{21} = 2(\epsilon_{is} - \epsilon_i\epsilon_s), \qquad (M_s)_{12} = \frac{\epsilon_{js} - \epsilon_j\epsilon_s}{2\epsilon_{ij}},$$

From $\operatorname{Tr} M_s = 2\epsilon_s$ and $\operatorname{Tr}(M_i M_j M_s) = p_{ijs}$ we find the final formula (1.49) for $s = k$ and (1.50) for $s = \ell$ respectively. □

In the following Theorem we show that $\mathcal{M}_{\mathcal{G}_2}$ can be identified with an affine algebraic variety that is the zero locus of a family $\mathcal{F}$ of 15 polynomials in the ring:

$$\mathbb{C}[p_1, p_2, p_3, p_4, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}]. \tag{1.52}$$

Theorem 6. Consider $m := (M_1, \ldots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$.

(i) The co-adjoint coordinates of m defined in (1.3) and (1.4) belong to the zero locus of the following 15 polynomials in the ring (1.52):

$$\begin{aligned} f_1(p) :=& p_{32}p_{31}p_{21} + p_{32}^2 + p_{31}^2 + p_{21}^2 - \\ & (p_1p_{321} + p_2p_3)p_{32} - (p_2p_{321} + p_1p_3)p_{31} - \\ & (p_3p_{321} + p_1p_2)p_{21} + p_3^2 + p_2^2 + p_1^2 + p_{321}^2 + p_3p_2p_1p_{321} - 4, \end{aligned} \tag{1.53}$$

$$\begin{aligned} f_2(p) :=& p_{42}p_{41}p_{21} + p_{42}^2 + p_{41}^2 + p_{21}^2 - \\ & - (p_1p_{421} + p_2p_4)p_{42} - (p_2p_{421} + p_1p_4)p_{41} - \\ & (p_4p_{421} + p_1p_2)p_{21} + p_4^2 + p_2^2 + p_1^2 + p_{421}^2 + p_4p_2p_1p_{421} - 4, \end{aligned} \tag{1.54}$$

$$\begin{aligned} f_3(p) :=& p_{43}p_{41}p_{31} + p_{43}^2 + p_{41}^2 + p_{31}^2 - \\ & (p_1p_{431} + p_3p_4)p_{43} - (p_3p_{431} + p_1p_4)p_{41} - \\ & (p_4p_{431} + p_1p_3)p_{31} + p_4^2 + p_3^2 + p_1^2 + p_{431}^2 + p_4p_3p_1p_{431} - 4, \end{aligned} \tag{1.55}$$

$$\begin{aligned} f_4(p) :=& p_{43}p_{42}p_{32} + p_{43}^2 + p_{42}^2 + p_{32}^2 - \\ & - (p_2p_{432} + p_3p_4)p_{43} - (p_3p_{432} + p_2p_4)p_{42} - \end{aligned}$$

$$(p_4 p_{432} + p_2 p_3) p_{32} + p_4^2 + p_3^2 + p_2^2 + p_{432}^2 + p_4 p_3 p_2 p_{432} - 4, \tag{1.56}$$

$$f_5(p) := -2p_\infty + p_1 p_2 p_3 p_4 + p_1 p_{432} + p_2 p_{431} + p_3 p_{421} + p_{321} p_4 + p_{21} p_{43} + p_{32} p_{41} - p_1 p_2 p_{43} - p_1 p_4 p_{32} - p_2 p_3 p_{41} - p_3 p_4 p_{21} - p_{42} p_{31}, \tag{1.57}$$

$$f_6(p) := p_2 p_3 p_4 - p_{32} p_4 - p_{21} p_3 p_{41} + p_{321} p_{41} - p_3 p_{42} + p_1 p_3 p_{421} - p_{31} p_{421} - p_2 p_{43} + p_{21} p_{431} + 2p_{432} - p_1 p_\infty, \tag{1.58}$$

$$f_7(p) := -p_1 p_4 + 2p_{41} + p_{21} p_{42} - p_2 p_{421} + p_{31} p_{43} + p_{21} p_{32} p_{43} - p_2 p_{321} p_{43} - p_3 p_{431} - p_{21} p_3 p_{432} + p_{321} p_{432} + p_2 p_3 p_\infty - p_{32} p_\infty, \tag{1.59}$$

$$f_8(p) := -p_1 p_2 p_3 + p_{21} p_3 + p_2 p_{31} + p_1 p_{32} - 2p_{321} + p_2 p_{41} p_{43} - p_{421} p_{43} - p_2 p_4 p_{431} + p_{42} p_{431} - p_{41} p_{432} + p_4 p_\infty, \tag{1.60}$$

$$f_9(p) := -p_1 p_2 + 2p_{21} + p_{31} p_{32} - p_3 p_{321} + p_{41} p_{42} - p_4 p_{421} + p_{32} p_{41} p_{43} - p_{32} p_4 p_{431} - p_3 p_{41} p_{432} + p_{431} p_{432} + p_3 p_4 p_\infty - p_{43} p_\infty, \tag{1.61}$$

$$f_{10}(p) := -p_1 p_2 p_4 + p_{21} p_4 + p_2 p_{41} + p_1 p_{42} - 2p_{421} + p_1 p_{32} p_{43} - p_{321} p_{43} - p_{32} p_{431} - p_1 p_3 p_{432} + p_{31} p_{432} + p_3 p_\infty, \tag{1.62}$$

$$f_{11}(p) := p_1p_3p_4 - p_{31}p_4 - p_{21}p_{32}p_4 + p_2p_{321}p_4 - p_3p_{41} - p_{321}p_{42} + p_{32}p_{421} - p_1p_{43} + 2p_{431} + p_{21}p_{432} - p_2p_\infty, \tag{1.63}$$

$$f_{12}(p) := -p_2p_4 + p_{21}p_{41} + 2p_{42} - p_1p_{421} + p_{32}p_{43} - p_{321}p_{431} - p_3p_{432} + p_{31}p_\infty, \tag{1.64}$$

$$f_{13}(p) := p_1p_3 - 2p_{31} - p_{21}p_{32} + p_2p_{321} - p_{41}p_{43} + p_4p_{431} + p_{421}p_{432} - p_{42}p_\infty, \tag{1.65}$$

$$f_{14}(p) := p_2p_3 - p_{21}p_{31} - 2p_{32} + p_1p_{321} - p_{21}p_{41}p_{43} - p_{42}p_{43} + p_1p_{421}p_{43} + p_{21}p_4p_{431} - p_{421}p_{431} + p_4p_{432} - p_1p_4p_\infty + p_{41}p_\infty, \tag{1.66}$$

$$f_{15}(p) := -p_3p_4 + p_{31}p_{41} + p_{21}p_{32}p_{41} - p_2p_{321}p_{41} + p_{32}p_{42} - p_1p_{32}p_{421} + p_{321}p_{421} + 2p_{43} - p_1p_{431} - p_2p_{432} + p_1p_2p_\infty - p_{21}p_\infty. \tag{1.67}$$

(ii) For every given generic $p_1, p_2, p_3, p_4, p_\infty$, the affine algebraic variety:

$$\widehat{\mathcal{M}}_{\mathcal{G}_2} = \mathbb{C}[(p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421})]/I, \tag{1.68}$$

where $I =< f_1, \dots, f_{15} >$, is four dimensional.

Proof. We proceed proving point by point the statement of the Theorem.

(i) We give a detailed proof for the polynomial (1.57), while all the others polynomials are calculated in a similar way and hence we omit their proof. Before proceeding, it is useful to remind the so-called *skein relation*:

$$\operatorname{Tr} AB + \operatorname{Tr} A^{-1}B = \operatorname{Tr} A \operatorname{Tr} B,\ \forall A, B \in \mathrm{SL}_2(\mathbb{C}), \tag{1.69}$$

and the following well-known properties of matrices in $\mathrm{SL}_2(\mathbb{C})$:

$$\operatorname{Tr} A^{-1} = \operatorname{Tr} A,\ \forall A \in \mathrm{SL}_2(\mathbb{C}), \tag{1.70}$$

$$\operatorname{Tr} AB = \operatorname{Tr} BA,\ \forall A, B \in \mathrm{SL}_2(\mathbb{C}), \tag{1.71}$$

$$\operatorname{Tr} ABC = \operatorname{Tr} CAB = \operatorname{Tr} BCA,\ \forall A, B, C \in \mathrm{SL}_2(\mathbb{C}). \tag{1.72}$$

We can now start the proof of (1.57). Firstly rewrite (1) as:

$$M_4 M_3 M_1 = (M_1^{-1} M_2 M_1 M_\infty)^{-1}. \tag{1.73}$$

Then apply the trace operator:

$$\operatorname{Tr} M_4 M_3 M_1 = \operatorname{Tr} M_1^{-1} M_2 M_1 M_\infty, \tag{1.74}$$

and expand the right hand side of (1.74) using rules (1.69) and (1.72):

$$\begin{aligned}
&\operatorname{Tr} M_1^{-1} M_2 M_1 M_\infty = \operatorname{Tr} M_1^{-1} M_2 M_1 \operatorname{Tr} M_\infty - \operatorname{Tr} M_4 M_3 M_2 M_2 M_1 = \\
&\operatorname{Tr} M_2 \operatorname{Tr} M_\infty - \operatorname{Tr} M_2 M_1 \operatorname{Tr} M_4 M_3 M_2 + \operatorname{Tr} M_4 M_3 M_2 M_1^{-1} M_2^{-1} = \\
&\operatorname{Tr} M_2 \operatorname{Tr} M_\infty - \operatorname{Tr} M_2 M_1 \operatorname{Tr} M_4 M_3 M_2 + \operatorname{Tr} M_4 M_2^{-1} \operatorname{Tr} M_3 M_2 M_1^{-1} - \\
&\operatorname{Tr} M_4^{-1} M_2 M_3 M_2 M_1^{-1} = \operatorname{Tr} M_2 \operatorname{Tr} M_\infty - \operatorname{Tr} M_2 M_1 \operatorname{Tr} M_4 M_3 M_2 + \\
&(\operatorname{Tr} M_4 \operatorname{Tr} M_2 - \operatorname{Tr} M_4 M_2)(\operatorname{Tr} M_3 M_2 \operatorname{Tr} M_1 - \operatorname{Tr} M_3 M_2 M_1) - \\
&\operatorname{Tr} M_4^{-1} M_2 M_3 \operatorname{Tr} M_2 M_1^{-1} + \operatorname{Tr} M_2 M_3 M_1 M_2^{-1} M_4^{-1} =
\end{aligned}$$

$$\begin{aligned}
&\operatorname{Tr} M_2 \operatorname{Tr} M_\infty - \operatorname{Tr} M_2M_1 \operatorname{Tr} M_4M_3M_2+\\
&(\operatorname{Tr} M_4 \operatorname{Tr} M_2 - \operatorname{Tr} M_4M_2)(\operatorname{Tr} M_3M_2 \operatorname{Tr} M_1 - \operatorname{Tr} M_3M_2M_1)-\\
&(\operatorname{Tr} M_4 \operatorname{Tr} M_3M_2 - \operatorname{Tr} M_4M_2M_3)(\operatorname{Tr} M_2 \operatorname{Tr} M_1 - \operatorname{Tr} M_2M_1)+\\
&\operatorname{Tr} M_3M_1M_2 \operatorname{Tr} M_4M_2 - \operatorname{Tr} M_2M_3M_1M_4M_2 =\\
&\operatorname{Tr} M_2 \operatorname{Tr} M_\infty - \operatorname{Tr} M_2M_1 \operatorname{Tr} M_4M_3M_2+\\
&(\operatorname{Tr} M_4 \operatorname{Tr} M_2 - \operatorname{Tr} M_4M_2)(\operatorname{Tr} M_3M_2 \operatorname{Tr} M_1 - \operatorname{Tr} M_3M_2M_1)-\\
&(\operatorname{Tr} M_4 \operatorname{Tr} M_3M_2 - \operatorname{Tr} M_4M_2M_3)(\operatorname{Tr} M_2 \operatorname{Tr} M_1 - \operatorname{Tr} M_2M_1)+\\
&\operatorname{Tr} M_3M_1M_2 \operatorname{Tr} M_4M_2 - \operatorname{Tr} M_2 \operatorname{Tr} M_4M_2M_3M_1 + \operatorname{Tr} M_4M_3M_1.
\end{aligned} \tag{1.75}$$

The traces $\operatorname{Tr} M_3M_1M_2$, $\operatorname{Tr} M_4M_2M_3$ and $\operatorname{Tr} M_4M_2M_3M_1$, satisfy the following relations:

$$\operatorname{Tr} M_3M_1M_2 = \operatorname{Tr} M_3 \operatorname{Tr} M_2M_1 + \operatorname{Tr} M_2 \operatorname{Tr} M_3M_1 + \operatorname{Tr} M_1 \operatorname{Tr} M_3M_2- \operatorname{Tr} M_3 \operatorname{Tr} M_2 \operatorname{Tr} M_1 - \operatorname{Tr} M_3M_2M_1, \tag{1.76}$$

$$\operatorname{Tr} M_4M_2M_3 = \operatorname{Tr} M_4 \operatorname{Tr} M_3M_2 + \operatorname{Tr} M_3 \operatorname{Tr} M_4M_2 + \operatorname{Tr} M_2 \operatorname{Tr} M_4M_3- \operatorname{Tr} M_4 \operatorname{Tr} M_3 \operatorname{Tr} M_2 - \operatorname{Tr} M_4M_3M_2, \tag{1.77}$$

$$\begin{aligned}
&\operatorname{Tr} M_4M_2M_3M_1 = \operatorname{Tr} M_4M_2M_1 \operatorname{Tr} M_3 - \operatorname{Tr} M_3M_2^{-1}M_4^{-1}M_1^{-1} =\\
&\operatorname{Tr} M_4M_2M_1 \operatorname{Tr} M_3 - \operatorname{Tr} M_3M_2^{-1} \operatorname{Tr} M_4M_1 + \operatorname{Tr} M_2M_3^{-1}M_4^{-1}M_1^{-1} =\\
&\operatorname{Tr} M_4M_2M_1 \operatorname{Tr} M_3 - (\operatorname{Tr} M_3 \operatorname{Tr} M_2 - \operatorname{Tr} M_3M_2) \operatorname{Tr} M_4M_1+\\
&\operatorname{Tr} M_2 \operatorname{Tr} M_4M_3M_1 - \operatorname{Tr} M_4M_3M_2M_1.
\end{aligned} \tag{1.78}$$

Substitute back in (1.75) the equations (1.76)-(1.78) and apply the definitions given in (1.4), in order to get the following relation:

$$p_2(-p_1p_2p_3p_4 + p_{21}p_3p_4 + p_1p_{32}p_4 - p_{321}p_4 + p_2p_3p_{41} - p_{32}p_{41}+$$

$$p_{31}p_{42} - p_3 p_{421} + p_1 p_2 p_{43} - p_{21}p_{43} - p_2 p_{431} - p_1 p_{432} + 2p_\infty) = 0. \tag{1.79}$$

Since $p \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ is arbitrary, then (1.79) must be true indipendently from the value of p_2, then $f_5(p) = 0$.

(ii) For given $p_1, p_2, p_3, p_4, p_\infty$, we used Macaulay2 [GS], a software for algebraic geometry, in order to compute the dimension of the algebraic variety:

$$\widehat{\mathcal{M}}_{\mathcal{G}_2} = \mathbb{C}[(p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421})]/I. \tag{1.80}$$

The result is that (1.80) has dimension four.

This concludes the proof. □

Corollary 7. The quantities $(p_{21}, \ldots, p_{43}, p_{321}, \ldots, p_{421})$ give a set of overdetermined coordinates on the open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, where:

$$\mathcal{U} = \big\{(M_1, \ldots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2} | \langle M_1, \ldots, M_4\rangle \text{ irreducible}, \tag{1.81}$$
$$M_i \neq \pm 1, \forall i = 1, \ldots, 4, \big\}/\sim,$$

Proof. Thanks to Theorem 3, Lemma 4 and Proposition 5 the quantities p_i, p_{ij}, p_{ijk} parameterize the monodromy matrices up to global conjugation. Thanks to Theorem 6 for every fixed choice of $p_1, p_2, p_3, p_4, p_\infty$ only 4 among the quantities p_{ij}, p_{ijk} for $i, j, k = 1, \ldots, 4$, $i > j > k$, are independent. This concludes the proof. □

1.2 Braid group action on $\mathcal{M}_{\mathcal{G}_2}$

The braid group B_n, $n \in \mathbb{N}$, was firstly introduced by Artin in [Art25]. B_n is defined as the infinite group that can be generated by $n-1$ elementary braids σ_i, for $i = 1, \dots, n-1$, and each σ_i is a collection of n-strands such that the i-th strand pass over the $(i+1)$-th strand.

Definition 8. The so-called Artin's presentation of B_n is given by:

$$B_n = \Big\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \leqslant i \leqslant n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1 \Big\rangle. \tag{1.82}$$

There exists a natural surjective group homomorphism between B_n and the symmetric group S_n . The kernel of this homomorphism is denoted P_n and is called the pure braid group. A complete set of generators for P_n is given by formulae:

$$\beta_{ij} = \sigma_{i-1}^{-1}\sigma_{i-2}^{-1}\cdots\sigma_{j+1}^{-1}\sigma_j^2\sigma_{j+1}\cdots\sigma_{i-2}\sigma_{i-1},\ 1 \leqslant j < i \leqslant n, \tag{1.83}$$

and relations:

$$\beta_{rs}\beta_{ij}\beta_{rs}^{-1} = \begin{cases} \beta_{ij}, & \text{if } j < s < r < i, \\ & \text{or } s < r < j < i, \\ \beta_{rj}^{-1}\beta_{ij}\beta_{rj}, & s < j = r < i, \\ \beta_{rj}^{-1}\beta_{sj}^{-1}\beta_{ij}\beta_{sj}\beta_{rj}, & j = s < r < i, \\ \beta_{rj}^{-1}\beta_{sj}^{-1}\beta_{rj}\beta_{sj}\beta_{ij}\beta_{sj}^{-1}\beta_{rj}^{-1}\beta_{sj}\beta_{rj}, & s < j < r < i. \end{cases} \tag{1.84}$$

We show now that formulae given in (4) express the action of the braid group B_4 over $\widehat{\mathcal{M}}_{\mathcal{G}_2}$:

Lemma 9. Formulae (4) define an action of the braid group B_4 over $\widehat{\mathcal{M}}_{\mathcal{G}_2}$.

Proof. Firstly, we prove that the σ_i for $i = 1, 2, 3$, see (4), define an action over $\widehat{\mathcal{M}}_{\mathcal{G}_2}$, i.e. $\sigma_i(\widehat{\mathcal{M}}_{\mathcal{G}_2}) = \widehat{\mathcal{M}}_{\mathcal{G}_2}$. In order to do this, it is sufficient to prove that the cyclic relation (1) is preserved by the action. If we consider $(M_1', \ldots, M_4') = \sigma_i(M_1, \ldots, M_4)$, then, for every $i = 1, 2, 3$, the M_i' satisfy the cyclic relation and consequently the action is well defined.

Next we prove that the σ_i are generators of the braid group B_4. Suppose $m = (M_1, \ldots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, it is straightforward calculation to check that the σ_i satisfy the so-called "braid relations":

$$\begin{aligned} \sigma_1\sigma_3(m) &= \sigma_3\sigma_1(m), \\ \sigma_1\sigma_2\sigma_1(m) &= \sigma_2\sigma_1\sigma_2(m), \\ \sigma_2\sigma_3\sigma_2(m) &= \sigma_3\sigma_2\sigma_3(m). \end{aligned} \tag{1.85}$$

Then the σ_i generate the full braid group B_4. □

The action of σ_i in (4) can be expressed in terms of co-adjoint coordinates (1.4) on $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$. This is given by:

$$\begin{aligned} \sigma_1 :&(p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto \\ &(p_2, p_1, p_3, p_4, p_\infty, p_{21}, p_{32}, p_1p_3 - p_{31} - p_{21}p_{32} + p_2p_{321}, p_{42}, \\ &p_1p_4 - p_{41} - p_{21}p_{42} + p_2p_{421}, p_{43}, p_{321}, p_1p_{43} - p_{431} - p_{21}p_{432} + p_2p_\infty, \\ &p_{432}, p_{421}), \\ \sigma_2 :&(p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto \\ &(p_1, p_3, p_2, p_4, p_\infty, p_{31}, p_1p_2 - p_{21} - p_{31}p_{32} + p_3p_{321}, p_{32}, p_{41}, p_{43}, \end{aligned}$$

$$
\begin{aligned}
& p_2p_4 - p_{42} - p_{32}p_{43} + p_3p_{432}, p_{321}, p_{432}, p_2p_{41} - p_{421} - p_{32}p_{431} + p_3p_\infty, \\
& p_{431}), \\
\sigma_3 :& (p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto \\
& (p_1, p_2, p_4, p_3, p_\infty, p_{21}, p_{41}, p_{42}, p_1p_3 - p_{31} - p_{41}p_{43} + p_4p_{431}, \\
& p_2p_3 - p_{32} - p_{42}p_{43} + p_4p_{432}, p_{43}, p_{421}, p_{432}, p_{431}, \\
& p_{21}p_3 - p_{321} - p_{421}p_{43} + p_4p_\infty), \qquad (1.86)
\end{aligned}
$$

With the following Lemma we prove that the braids, defined in (1.86), still define the action of B_4 over the co-adjoint coordinates defined in the open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$:

Lemma 10. Formulae (1.86) define the action of the braid group B_4 over the open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$.

Proof. As in the previous proof, we begin proving that action (1.86) is well defined, namely consider the ideal $I =< \mathcal{F} >= \{f_1, \dots, f_{15}\}$, where f_i for $i = 1, \dots, 15$ are given in (1.53)-(1.67), then I is invariant under the B_4 action, i.e. for every $i = 1, 2, 3$:

$$\sigma_i(I) = I.$$

Consider $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, σ_i and $f_j \in \mathcal{F}$. We proceed computing $f_j(\sigma_i(p))$ for $i = 1, \dots, 4$ and $j = 1, \dots, 15$. For σ_1, we obtain:

$$
\begin{aligned}
& f_1(\sigma_1(p)) = f_1(p), \\
& f_2(\sigma_1(p)) = f_2(p), \\
& f_3(\sigma_1(p)) = f_4(p), \\
& f_4(\sigma_1(p)) = f_3(p) + (p_{21}p_{42} - p_2p_{421})f_6(p) + (p_{21}p_{431} - p_2p_\infty)f_{11}(p) +
\end{aligned}
$$

$$
\begin{aligned}
&(p_2p_{321} - p_{21}p_{32})f_{13}(p),\\
f_5(\sigma_1(p)) &= f_1(p) - p_2f_7(p),\\
f_6(\sigma_1(p)) &= f_8(p) - p_{21}f_2(p),\\
f_7(\sigma_1(p)) &= f_3(p) + p_2f_9(p),\\
f_8(\sigma_1(p)) &= f_4(p) - p_{42}f_2(p) - p_{432}f_7(p) + p_{32}f_9(p),\\
f_9(\sigma_1(p)) &= f_5(p) - p_2f_2(p),\\
f_{10}(\sigma_1(p)) &= -f_7(p),\\
f_{11}(\sigma_1(p)) &= f_6(p) - p_{21}f_7(p),\\
f_{12}(\sigma_1(p)) &= -f_2(p),\\
f_{13}(\sigma_1(p)) &= -p_{21}f_9(p) + f_{10}(p),\\
f_{14}(\sigma_1(p)) &= -f_9(p),\\
f_{15}(\sigma_1(p)) &= f_{11}(p) + p_1f_7(p).
\end{aligned}
$$

While, for the inverse σ_1^{-1}, we obtain:

$$
\begin{aligned}
f_1(\sigma_1^{-1}(p)) &= f_1(p),\\
f_2(\sigma_1^{-1}(p)) &= f_2(p),\\
f_3(\sigma_1^{-1}(p)) &= f_4(p) + (p_{21}p_{431} - p_1p_\infty)f_6(p) + (p_{21}p_{41} - p_1p_{421})f_{12}(p) +\\
&\quad (p_1p_{321} - p_{21}p_{31})f_{14}(p),\\
f_4(\sigma_1^{-1}(p)) &= f_3(p),\\
f_5(\sigma_1^{-1}(p)) &= f_5(p) - p_1f_6(p),\\
f_6(\sigma_1^{-1}(p)) &= f_{11}(p) - p_{21}f_6(p),\\
f_7(\sigma_1^{-1}(p)) &= -f_{12}(p),\\
f_8(\sigma_1^{-1}(p)) &= f_8(p) + p_1f_{14}(p),\\
f_9(\sigma_1^{-1}(p)) &= f_9(p) - p_{431}f_6(p) - p_{41}f_{12}(p) + p_{31}f_{14}(p),
\end{aligned}
$$

$$
\begin{aligned}
f_{10}(\sigma_1^{-1}(p)) &= f_{10}(p) - p_1 f_{12}(p),\\
f_{11}(\sigma_1^{-1}(p)) &= -f_6(p),\\
f_{12}(\sigma_1^{-1}(p)) &= f_7(p) - p_{21} f_{12}(p),\\
f_{13}(\sigma_1^{-1}(p)) &= -f_{14}(p),\\
f_{14}(\sigma_1^{-1}(p)) &= -p_{21} f_{14}(p) + f_{13}(p),\\
f_{15}(\sigma_1^{-1}(p)) &= f_{15}(p) + p_2 f_6(p).
\end{aligned}
$$

For σ_2, we obtain:

$$
\begin{aligned}
f_1(\sigma_2(p)) &= f_1(p),\\
f_2(\sigma_2(p)) &= f_3(p),\\
f_3(\sigma_2(p)) &= f_2(p) + (p_{32}p_{31} - p_3 p_{321}) f_8(p) + (p_3 p_\infty - p_{32} p_{431}) f_9(p) +\\
&\quad (p_{32}p_{43} - p_3 p_{432}) f_{12}(p),\\
f_4(\sigma_2(p)) &= f_4(p),\\
f_5(\sigma_2(p)) &= f_1(p) + p_3 f_5(p),\\
f_6(\sigma_2(p)) &= f_2(p) - p_2 f_5(p),\\
f_7(\sigma_2(p)) &= f_3(p) - p_3 f_4(p),\\
f_8(\sigma_2(p)) &= -p_{32} f_4(p) - f_9(p),\\
f_9(\sigma_2(p)) &= -p_{32} f_5(p) - f_7(p),\\
f_{10}(\sigma_2(p)) &= f_6(p) + p_3 f_8(p),\\
f_{11}(\sigma_2(p)) &= f_5(p),\\
f_{12}(\sigma_2(p)) &= -p_{32} f_8(p) + f_{11}(p),\\
f_{13}(\sigma_2(p)) &= f_4(p),\\
f_{14}(\sigma_2(p)) &= f_{10}(p) + p_{31} f_4(p) + p_{43} f_8(p) - p_{431} f_5(p),\\
f_{15}(\sigma_2(p)) &= -f_8(p).
\end{aligned}
$$

While, for the inverse σ_2^{-1}, we obtain:

$$
\begin{aligned}
f_1(\sigma_2^{-1}(p)) &= f_1(p),\\
f_2(\sigma_2^{-1}(p)) &= f_3(p) + (p_{32}p_{421} - p_2p_\infty)f_{11}(p) + (p_2p_{321} - p_{21}p_{32})f_{13}(p) +\\
&\qquad (p_{32}p_{42} - p_2p_{432})f_{15}(p),\\
f_3(\sigma_2^{-1}(p)) &= f_2(p),\\
f_4(\sigma_2^{-1}(p)) &= f_4(p),\\
f_5(\sigma_2^{-1}(p)) &= f_5(p) - p_2f_{11}(p),\\
f_6(\sigma_2^{-1}(p)) &= f_6(p) + p_2f_{15}(p),\\
f_7(\sigma_2^{-1}(p)) &= f_7(p) + p_3f_{11}(p),\\
f_8(\sigma_2^{-1}(p)) &= f_8(p) + p_2f_{13}(p),\\
f_9(\sigma_2^{-1}(p)) &= f_{13}(p),\\
f_{10}(\sigma_2^{-1}(p)) &= f_{11}(p),\\
f_{11}(\sigma_2^{-1}(p)) &= -f_{10}(p) - p_{32}f_{11}(p),\\
f_{12}(\sigma_2^{-1}(p)) &= -f_{15}(p),\\
f_{13}(\sigma_2^{-1}(p)) &= -f_9(p) - p_{32}f_{13}(p),\\
f_{14}(\sigma_2^{-1}(p)) &= f_{14}(p) + p_{421}f_{11}(p) - p_{21}f_{13}(p) + p_{42}f_{15}(p),\\
f_{15}(\sigma_2^{-1}(p)) &= f_{12}(p) - p_{32}f_{15}(p).
\end{aligned}
$$

Finally for σ_3:

$$
\begin{aligned}
f_1(\sigma_3(p)) &= f_2(p),\\
f_2(\sigma_3(p)) &= f_1(p) + (p_4p_\infty - p_{421}p_{43})f_7(p) + (p_4p_{431} - p_{43}p_{41})f_{13}(p) +\\
&\qquad (p_4p_{432} - p_{42}p_{43})f_{14}(p),\\
f_3(\sigma_3(p)) &= f_3(p),
\end{aligned}
$$

$$
\begin{aligned}
f_4(\sigma_3(p)) &= f_4(p),\\
f_5(\sigma_3(p)) &= f_1(p) + p_4 f_3(p),\\
f_6(\sigma_3(p)) &= f_9(p),\\
f_7(\sigma_3(p)) &= f_5(p) - p_{43} f_3(p),\\
f_8(\sigma_3(p)) &= f_4(p) - p_3 f_3(p),\\
f_9(\sigma_3(p)) &= -f_3(p),\\
f_{10}(\sigma_3(p)) &= f_6(p) - p_4 f_{10}(p),\\
f_{11}(\sigma_3(p)) &= f_7(p) - p_4 f_9(p),\\
f_{12}(\sigma_3(p)) &= f_{10}(p),\\
f_{13}(\sigma_3(p)) &= -f_2(p) - p_{43} f_9(p),\\
f_{14}(\sigma_3(p)) &= -p_{43} f_{10}(p) - f_8(p),\\
f_{15}(\sigma_3(p)) &= f_{11}(p) + p_{421} f_3(p) + p_{41} f_9(p) + p_{42} f_{10}(p).
\end{aligned}
$$

While, for the inverse σ_3^{-1}, we obtain:

$$
\begin{aligned}
f_1(\sigma_3^{-1}(p)) &= f_2(p) + (p_{43}p_{31} - p_3 p_{431}) f_7(p) + (p_3 p_\infty - p_{43} p_{321}) f_{10}(p) +\\
&\quad (p_{43}p_{32} - p_3 p_{432}) f_{12}(p),\\
f_2(\sigma_3^{-1}(p)) &= f_1(p),\\
f_3(\sigma_3^{-1}(p)) &= f_3(p),\\
f_4(\sigma_3^{-1}(p)) &= f_4(p),\\
f_5(\sigma_3^{-1}(p)) &= f_5(p) + p_3 f_{10}(p),\\
f_6(\sigma_3^{-1}(p)) &= f_6(p) + p_3 f_{12}(p),\\
f_7(\sigma_3^{-1}(p)) &= -p_{43} f_7(p) - f_{13}(p),\\
f_8(\sigma_3^{-1}(p)) &= -f_{10}(p),\\
f_9(\sigma_3^{-1}(p)) &= f_9(p) + p_4 f_{10}(p),
\end{aligned}
$$

$$\begin{aligned}
f_{10}(\sigma_3^{-1}(p)) &= f_8(p) + p_{43}f_{10}(p),\\
f_{11}(\sigma_3^{-1}(p)) &= f_{11}(p) + p_3f_7(p),\\
f_{12}(\sigma_3^{-1}(p)) &= -f_{14}(p) - p_{43}f_{12}(p),\\
f_{13}(\sigma_3^{-1}(p)) &= f_7(p),\\
f_{14}(\sigma_3^{-1}(p)) &= f_{12}(p),\\
f_{15}(\sigma_3^{-1}(p)) &= f_{15}(p) - p_{31}f_7(p) + p_{321}f_{10}(p) - p_{32}f_{12}(p).
\end{aligned}$$

Then we conclude that the action (1.86) is well defined over the co-adjoint coordinates defined in $\mathcal{U}$.

In order to prove that σ_i for $i = 1, 2, 3$, defined in (1.86), are generators of the braid group B_4, the "braid relations" (1.85) must be satisfied. Consider $p \in \mathcal{U} \subset\in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, as defined in (1.3), then relation:

$$\sigma_1\sigma_3(p) = \sigma_3\sigma_1(p),$$

holds true, and the following relations:

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1(p) &= \sigma_2\sigma_1\sigma_2(p),\\
\sigma_2\sigma_3\sigma_2(p) &= \sigma_3\sigma_2\sigma_3(p),
\end{aligned}$$

follow from the fact that polynomials (1.63),(1.64),(1.65) and (1.66) are zero for every $p \in \mathcal{U}$. □

In order to be consistent, we give explicitly the action of the pure braid group P_4, namely we are going to define the generators of the subgroup $P_4 \subset B_4$. By formulae (1.83), the group P_4 has generators:

$$\beta_{21} = \sigma_1^2,$$

$$\begin{aligned}
\beta_{31} &= \sigma_2^{-1}\sigma_1^2\sigma_2,\\
\beta_{32} &= \sigma_2^2,\\
\beta_{41} &= \sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3,\\
\beta_{42} &= \sigma_3^{-1}\sigma_2^2\sigma_3,\\
\beta_{43} &= \sigma_3^2.
\end{aligned} \tag{1.87}$$

In the last part of this Section, we state a Lemma that is a necessary condition for variables:

$$(p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}),$$

in order to generate a finite orbit under the action of the group P_4:

Lemma 11. Suppose $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ is such that it generates a finite P_4's orbit, then only two possibilities arise:

(i) Or p satisfies:

$$p_{ij} = 2\cos\pi r_{ij},\ \ r_{ij} \in \mathbb{Q},\ \ 0 \leqslant r_{ij} \leqslant 1,\ \ i,j = 1,2,3,4,\ \ i > j. \tag{1.88}$$

(ii) Or there exists a pure braid β_{ij}, for some choice of indices $i, j, k, \ell = 1, 2, 3, 4$ such that $\beta_{ij}(p) = p$. Then p_{ij} is a complex parameter satisfying:

$$p_{\ell i} = \frac{p_{ij}(p_j p_\ell + p_i p_{ij\ell}) - 2(p_\ell p_i + p_j p_{ij\ell})}{p_{ij}^2 - 4}, \tag{1.89}$$

$$p_{\ell j} = \frac{p_{ij}(p_i p_\ell + p_j p_{ij\ell}) - 2(p_\ell p_j + p_i p_{ij\ell})}{p_{ij}^2 - 4}, \tag{1.90}$$

$$p_{ki} = \frac{p_{ij}(p_j p_k + p_i p_{ijk}) - 2(p_k p_i + p_j p_{ijk})}{p_{ij}^2 - 4}, \tag{1.91}$$

$$p_{kj} = \frac{p_{ij}(p_i p_k + p_j p_{ijk}) - 2(p_k p_j + p_i p_{ijk})}{p_{ij}^2 - 4}. \tag{1.92}$$

Proof. We are going to prove the statement for the generator β_{21}, then in a similar way the statement can be proven for all other five generators (1.87).

The braid β_{21} fixes quantities $p_1, p_2, p_3, p_4, p_{321}$ and p_{421} and the resulting action on $(p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43})$ is:

$$\begin{aligned}
\beta_{21}(p_{21}) &= p_{21},\\
\beta_{21}(p_{31}) &= p_1p_3 - 2p_{31} - p_{21}p_{32} + p_2p_{321},\\
\beta_{21}(p_{32}) &= p_2p_3 - 2p_{32} + p_1p_{321} - p_{21}(p_1p_3 - p_{31} - p_{21}p_{32} + p_2p_{321}),\\
\beta_{21}(p_{41}) &= p_1p_4 - 2p_{41} - p_{21}p_{42} + p_2p_{421},\\
\beta_{21}(p_{42}) &= p_2p_4 - 2p_{42} + p_1p_{421} - p_{21}(p_1p_4 - p_{41} - p_{21}p_{42} + p_2p_{421}),\\
\beta_{21}(p_{43}) &= p_{43}.
\end{aligned}$$

Next we show that the pure braid β_{21} acts as linear transformation on variables $(p_{31}, p_{32}, p_{41}, p_{42})$. The cubic relations $f_1(p) = 0$ and $f_2(p) = 0$ are invariant during the action of β_{21}, moreover they are two conics in the variables (p_{31}, p_{32}) and (p_{41}, p_{42}) respectively:

$$\begin{aligned}
&p_{31}^2 + p_{32}^2 + p_{21}(p_{31}p_{32}) - (p_3p_2 + p_1p_{321})p_{32} - (p_3p_1 + p_2p_{321})p_{31} -\\
&\qquad ((p_2p_1 + p_3p_{321})p_{21} - (p_1^2 + p_2^2 + p_3^2 + p_{321}^2 + p_1p_2p_3p_{321}) + 4) = 0,
\end{aligned} \tag{1.93}$$

$$\begin{aligned}
&p_{41}^2 + p_{42}^2 + p_{21}(p_{41}p_{42}) - (p_4p_2 + p_1p_{421})p_{42} - (p_4p_1 + p_2p_{421})p_{41} -\\
&\qquad ((p_2p_1 + p_4p_{421})p_{21} - (p_1^2 + p_2^2 + p_4^2 + p_{421}^2 + p_1p_2p_4p_{421}) + 4) = 0.
\end{aligned} \tag{1.94}$$

If $p_{21} = \pm 2$ then $r_{21} = 0$ or $r_{21} = 1$ and the statement follows. Then, hereafter, we suppose $p_{21} \neq \pm 2$:

(i) The linear action of β_{21} on $(p_{31}, p_{32}, p_{41}, p_{42})$ describes simultaneously a rotation R of (p_{31}, p_{32}) and (p_{41}, p_{42}) on the conics (1.93) and (1.94) respectively. Suppose angle of the rotation R is θ such that $p_{21} = 2\cos\theta$ and if θ is a rational multiple of π then:

$$\exists n \in \mathbb{N} \text{ s.t. } R^n = Id. \tag{1.95}$$

As a consequence the action of β_{21} produces a finite orbit in (p_{31}, p_{32}) and (p_{41}, p_{42}) if and only if $q_{21} = 2\cos\theta$ where θ is a rational multiple of π.

(ii) Suppose p to be a fixed point of the braid β_{21}, i.e. $\beta_{21}(p) = p$, then:

$$p_{32} = \frac{p_{21}(p_1 p_3 + p_2 p_{321}) - 2(p_3 p_2 + p_1 p_{321})}{p_{21}^2 - 4}, \tag{1.96}$$

$$p_{31} = \frac{p_{21}(p_2 p_3 + p_1 p_{321}) - 2(p_3 p_1 + p_2 p_{321})}{p_{21}^2 - 4}, \tag{1.97}$$

$$p_{42} = \frac{p_{21}(p_1 p_4 + p_2 p_{421}) - 2(p_4 p_2 + p_1 p_{421})}{p_{21}^2 - 4}, \tag{1.98}$$

$$p_{41} = \frac{p_{21}(p_2 p_4 + p_1 p_{421}) - 2(p_4 p_1 + p_2 p_{421})}{p_{21}^2 - 4}. \tag{1.99}$$

Then p_{21} plays the role of a complex parameter.

This concludes the proof. □

1.3 The problem in terms of co-adjoint co-ordinates

The aim of this thesis is to classify all finite orbits:

$$\mathcal{O}_{P_4}(p) = \{\beta(p)|\beta \in P_4\},$$

where $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ is the following 15-tuple of complex quantities:

$$p = (p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \in \mathbb{C}^{15},$$

defined in (1.4), and P_4 is the pure braid group defined in (1.87).

Our classification of finite orbits will be presented modulo symmetries Φ, where Φ is an invertible map such that:

$$\Phi : \widehat{\mathcal{M}}_{\mathcal{G}_2} \longrightarrow \widehat{\mathcal{M}}_{\mathcal{G}_2},$$

and given an element $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and its orbit $\mathcal{O}_{P_4}(p)$, the following is true:

$$|\mathcal{O}_{P_4}(\Phi(p))| = |\mathcal{O}_{P_4}(p)|.$$

1.4 Restrictions

Our approach is based on the observation that given p_i,p_{ij},p_{ijk}, defined in (1.4), on the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, defined in (1.38) such that they generate a finite orbit under the action of the pure braid group P_4, then for any subgroup $H \subset P_4$ the action of H over p_i,p_{ij},p_{ijk} produces again a finite orbit. Such restriction H only acts on some of the p_i,p_{ij},p_{ijk} and it leaves others invariant. We select subgroups $H \subset P_4$ acting on p_i,p_{ij},p_{ijk} in the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, such that the restricted action is isomorphic to the action of the pure braid group P_3 on the $SL_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components, i.e. $\mathcal{M}_{PVI}$: indeed in this case all finite orbits of the action of P_3 over the quotient space:

$$\widehat{\mathcal{M}}_{PVI} := \{(N_1, N_2, N_3) \in \mathrm{SL}_2(\mathbb{C}) |\, N_\infty N_3 N_2 N_1 = \mathbb{1}, \\ N_\infty = \mathrm{diag}(e^{\pm i\pi\theta_\infty}),\ \theta_\infty \in \mathbb{C}\,\} / \sim, \tag{1.100}$$

where $\sim$ is the usual equivalence relation up to global diagonal conjugation, are classified in Lisovyy and Tykhyy's work [LT14]. This will be discussed in details in Chapter 3.

There exist four well defined restrictions $H_1, \dots, H_4$ isomorphic to P_3 and each of these restrictions allows us to identify some of the p_i,p_{ij},p_{ijk} with coordinates on $\mathcal{M}_{PVI}$. The four subgroups H_i are defined in the following Theorem:

Theorem 12. There exist four subgroups $H_i \subset P_4$ with $i = 1, \dots, 4$, such that they are generated by:

- $H_1 =< \beta_{32}, \beta_{43}, \beta_{42} >$,
- $H_2 =< \beta_{43}, \beta_{31}, \beta_{41} >$,
- $H_3 =< \beta_{21}, \beta_{42}, \beta_{41} >$,
- $H_4 =< \beta_{21}, \beta_{32}, \beta_{31} >$,

where generators β_{jk}, with $j, k = 1, \dots, 4$ and $j > k$, are defined in (1.87). The subgroups H_i satisfy:

(i) H_i is isomorphic to the pure braid group P_3 for $i = 1, \ldots, 4$.

(ii) Consider $(M_1, M_2, M_3, M_4) \in \mathcal{M}_{\mathcal{G}_2}$ as an ordered 4-tuple of matrices, then each H_i, for $i = 1, \ldots, 4$, acts as pure braid group P_3 on $\widehat{\mathcal{M}}_{PVI}$ leaving matrix M_i out of action, where N_1, N_2, N_3 are given by:

$$H_1 : \ \widehat{N}_1 = M_2, \ \widehat{N}_2 = M_3, \ \widehat{N}_3 = M_4, \ \widehat{N}_\infty = (M_4 M_3 M_2)^{-1}, \quad (1.101)$$

$$H_2 : \ \bar{N}_1 = M_1, \ \bar{N}_2 = M_3, \ \bar{N}_3 = M_4, \ \bar{N}_\infty = (M_4 M_3 M_1)^{-1}, \quad (1.102)$$

$$H_3 : \ \check{N}_1 = M_1, \ \check{N}_2 = M_2, \ \check{N}_3 = M_4, \ \check{N}_\infty = (M_4 M_2 M_1)^{-1}, \quad (1.103)$$

$$H_4 : \ \widetilde{N}_1 = M_1, \ \widetilde{N}_2 = M_2, \ \widetilde{N}_3 = M_3, \ \widetilde{N}_\infty = (M_3 M_2 M_1)^{-1}. \quad (1.104)$$

Proof. We prove explicitly the statements (i) and (ii) for the subgroup:

$$H_1 =< \beta_{32}, \beta_{43}, \beta_{42} >\subset P_4,$$

then for the other subgroups a similar proof applies.

(i) We are going to show that generators of H_1 satisfy the presentation of the pure braid group P_3 given in formulae (1.83)-(1.84), for $n = 3$. In other words generators $\beta_{32}, \beta_{42}, \beta_{43}$ must satisfy:

$$\begin{cases} \beta_{32}\beta_{43}\beta_{32}^{-1} & = \beta_{42}^{-1}\beta_{43}\beta_{42}, \\ \beta_{32}\beta_{42}\beta_{32}^{-1} & = \beta_{42}^{-1}\beta_{43}^{-1}\beta_{42}\beta_{43}\beta_{42}. \end{cases} \quad (1.105)$$

Relations (1.105) are true and they can be checked by direct computations. This implies the isomorphism between H_1 and P_3.

(ii) We prove that $\widehat{n} = (\widehat{N}_1, \widehat{N}_2, \widehat{N}_3)$ is in $\widehat{\mathcal{M}}_{PVI}$. Suppose $m = (M_1,\ M_2,\ M_3,\ M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and consider the identities (1.101): by definition of

$\widehat{\mathcal{M}}_{\mathcal{G}_2}$, matrices $\widehat{N}_i$ for $i = 1, 2, 3, \infty$ are in $SL_2(\mathbb{C})$, moreover, (1.101) transforms the cyclic relation satisfied by m:

$$M_\infty M_4 M_3 M_2 M_1 = \mathbb{1} \Longleftrightarrow M_1 M_\infty M_4 M_3 M_2 = \mathbb{1},$$

to the following cyclic relation:

$$\widehat{N}_\infty \widehat{N}_3 \widehat{N}_2 \widehat{N}_1 = \mathbb{1},$$

This implies $\widehat{n} \in \widehat{\mathcal{M}}_{PVI}$. Now we show that the subgroup H_1 acts as pure braid group P_3 on $\widehat{\mathcal{M}}_{PVI}$ leaving matrix M_i out of action. Since the generators of H_1 are defined in terms of generators σ_2 and σ_3 of the full braid group B_4, see definition (1.87), then, it is enough to prove that σ_2 and σ_3, by (1.101), act as generators of the full braid group B_3. Consider (1.101), then the following relations hold:

$$\begin{aligned} \sigma_2(m) &= (M_1, M_3, M_3 M_2 M_3^{-1}, M_4) \simeq (\widehat{N}_2, \widehat{N}_2 \widehat{N}_1 \widehat{N}_2^{-1}, \widehat{N}_3) = \sigma_1^{(PVI)}(\widehat{n}), \\ \sigma_3(m) &= (M_1, M_2, M_4, M_4 M_3 M_4^{-1}) \simeq (\widehat{N}_1, \widehat{N}_3, \widehat{N}_3 \widehat{N}_2 \widehat{N}_3^{-1}) = \sigma_2^{(PVI)}(\widehat{n}). \end{aligned} \tag{1.106}$$

Furthermore, the generators σ_2 and σ_3 satisfy the "braid relations" (1.82), and then they generate the braid group B_3. Moreover in (1.106) the matrix M_1 remains out of the action as expected.

This completes the proof. □

To avoid extra complications due to the freedom of global conjugation in (1.101),(1.102),(1.103),(1.104), we consider the action of the subgroups H_i for $i = 1, \ldots, 4$ in terms of co-adjoint coordinates. In order to do this, we define:

$$\widehat{q} := (\widehat{q}_1, \widehat{q}_2, \widehat{q}_3, \widehat{q}_\infty, \widehat{q}_{21}, \widehat{q}_{31}, \widehat{q}_{32}) \in \mathbb{C}^7,$$
$$\overline{q} := (\overline{q}_1, \overline{q}_2, \overline{q}_3, \overline{q}_\infty, \overline{q}_{21}, \overline{q}_{31}, \overline{q}_{32}) \in \mathbb{C}^7,$$
$$\breve{q} := (\breve{q}_1, \breve{q}_2, \breve{q}_3, \breve{q}_\infty, \breve{q}_{21}, \breve{q}_{31}, \breve{q}_{32}) \in \mathbb{C}^7,$$
$$\widetilde{q} := (\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3, \widetilde{q}_\infty, \widetilde{q}_{21}, \widetilde{q}_{31}, \widetilde{q}_{32}) \in \mathbb{C}^7, \tag{1.107}$$

where $\widehat{q}_i = \operatorname{Tr} \widehat{N}_i$ for $i = 1, 2, 3, \infty$, $\widehat{q}_{jk} = \operatorname{Tr} \widehat{N}_j \widehat{N}_k$ for $j > k,\ j, k = 1, 2, 3$, etc. As it will be reminded in Chapter 3, the $\widehat{q}, \overline{q}, \breve{q}, \widetilde{q}$ are respectively co-adjoint coordinates on the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components. Then identifications (1.101)-(1.104) imply:

$$\widehat{q}_1 = p_2,\ \widehat{q}_2 = p_3,\ \widehat{q}_3 = p_4,\ \widehat{q}_\infty = p_{432},\ \widehat{q}_{21} = p_{32},\ \widehat{q}_{31} = p_{42},\ \widehat{q}_{32} = p_{43}, \tag{1.108}$$
$$\overline{q}_1 = p_1,\ \overline{q}_2 = p_3,\ \overline{q}_3 = p_4,\ \overline{q}_\infty = p_{431},\ \overline{q}_{21} = p_{31},\ \overline{q}_{31} = p_{41},\ \overline{q}_{32} = p_{43}, \tag{1.109}$$
$$\breve{q}_1 = p_1,\ \breve{q}_2 = p_2,\ \breve{q}_3 = p_4,\ \breve{q}_\infty = p_{421},\ \breve{q}_{21} = p_{21},\ \breve{q}_{31} = p_{41},\ \breve{q}_{32} = p_{42}, \tag{1.110}$$
$$\widetilde{q}_1 = p_1,\ \widetilde{q}_2 = p_2,\ \widetilde{q}_3 = p_3,\ \widetilde{q}_\infty = p_{321},\ \widetilde{q}_{21} = p_{21},\ \widetilde{q}_{31} = p_{31},\ \widetilde{q}_{32} = p_{32}, \tag{1.111}$$

where p_i,p_{ij},p_{ijk} are defined in (1.4). We summarize identities (1.108)-(1.111) in Table 1.1.

In terms of analytic continuation, this means that we are extending our solution on the Riemann sphere with five boundary components in such a way that this continuation doesn't go around one of the singularities $\{0, 1, u_1, u_2, \infty\}$. Moreover, the suborbit of a finite orbit $\mathcal{O}_{P_4}$, generated

	p_1	p_2	p_3	p_4	p_∞	p_{21}	p_{31}	p_{32}	p_{41}	p_{42}	p_{43}	p_{321}	p_{432}	p_{431}	p_{421}
H_1		$\widehat{q}_1$	$\widehat{q}_2$	$\widehat{q}_3$				$\widehat{q}_{21}$		$\widehat{q}_{31}$	$\widehat{q}_{32}$		$\widehat{q}_\infty$		
H_2	$\bar{q}_1$		$\bar{q}_2$	$\bar{q}_3$			$\bar{q}_{21}$		$\bar{q}_{31}$		$\bar{q}_{32}$			$\bar{q}_\infty$	
H_3	$\breve{q}_1$	$\breve{q}_2$		$\breve{q}_3$		$\breve{q}_{21}$			$\breve{q}_{31}$	$\breve{q}_{32}$					$\breve{q}_\infty$
H_4	$\widetilde{q}_1$	$\widetilde{q}_2$	$\widetilde{q}_3$			$\widetilde{q}_{21}$	$\widetilde{q}_{31}$	$\widetilde{q}_{32}$				$\widetilde{q}_\infty$			

Table 1.1: Matching using traces: elements on the same column must be equal.

fixing a conjugacy class of a monodromy matrix M_i must be a finite suborbit describing the analytic continuation of an algebraic solution of PVI: this provides the basis of our method.

Furthermore, by relations (1.101)-(1.104) and (1.108)-(1.111), the following four projections:

$$\widetilde{\pi}, \widehat{\pi}, \breve{\pi}, \bar{\pi} : \widehat{\mathcal{M}}_{\mathcal{G}_2} \mapsto \widehat{\mathcal{M}}_{PVI}, \tag{1.112}$$

can be defined in such a way that, if we know a 4-tuple m of monodromy matrices in $\widehat{\mathcal{M}}_{\mathcal{G}_2}$, then we can project it to four 3-tuples $\widetilde{n}$,$\bar{n}$,$\breve{n}$,$\bar{n} \in \widehat{\mathcal{M}}_{PVI}$ as follows:

$$\begin{aligned}
\widetilde{\pi}(m) &:= (M_1, M_2, M_3) = \widetilde{n}, \\
\widehat{\pi}(m) &:= (M_2, M_3, M_4) = \widehat{n}, \\
\breve{\pi}(m) &:= (M_1, M_3, M_4) = \breve{n}, \\
\bar{\pi}(m) &:= (M_1, M_2, M_4) = \bar{n}.
\end{aligned} \tag{1.113}$$

Equivalently, we can define each projection in terms of co-adjoint coordinates p_i,p_{ij},p_{ijk} on $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and co-adjoint coordinates $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$ on $\widehat{\mathcal{M}}_{PVI}$:

$$\begin{aligned}
\widetilde{\pi}(p) &:= (p_1, p_2, p_3, p_{321}, p_{21}, p_{31}, p_{32}) = \widetilde{q}, \\
\widehat{\pi}(p) &:= (p_2, p_3, p_4, p_{432}, p_{32}, p_{42}, p_{43}) = \widehat{q}, \\
\check{\pi}(p) &:= (p_1, p_2, p_4, p_{421}, p_{21}, p_{41}, p_{42}) = \check{q}, \\
\overline{\pi}(p) &:= (p_1, p_3, p_4, p_{431}, p_{31}, p_{41}, p_{43}) = \overline{q}.
\end{aligned} \tag{1.114}$$

In second instance, given four triples $\widetilde{n}$,$\widehat{n}$,$\check{n}$,$\overline{n} \in \widehat{\mathcal{M}}_{PVI}$ or the associated co-adjoint coordinates $\widetilde{q}$,$\widehat{q}$,$\check{q}$,$\overline{q}$, we can lift them to the respective 4-tuple of matrices $m \in \mathcal{M}_{\mathcal{G}_2}$ or to co-adjoint coordinates p. In general we can not do this for every choice of $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, indeed these four points must satisfy the following relations:

$$\begin{cases}
\widetilde{q}_1 = \check{q}_1 = \overline{q}_1, \\
\widetilde{q}_2 = \widehat{q}_1 = \check{q}_2, \\
\widetilde{q}_3 = \widehat{q}_2 = \overline{q}_2, \\
\widehat{q}_3 = \check{q}_3 = \overline{q}_3, \\
\widetilde{q}_{21} = \check{q}_{21}, \\
\widetilde{q}_{31} = \overline{q}_{21}, \\
\widetilde{q}_{32} = \widehat{q}_{21}, \\
\check{q}_{31} = \overline{q}_{31}, \\
\widehat{q}_{31} = \check{q}_{32}, \\
\widehat{q}_{32} = \overline{q}_{32},
\end{cases} \tag{1.115}$$

derived from (1.108)-(1.111). We give here only the relations between co-adjoint coordinates $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, so that we can forget about the global conjugation.

Definition 13. We call *matching on four points* the procedure of applying the identities (1.115) to four points $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$ in $\widehat{\mathcal{M}}_{PVI}$.

Note that in Table 1.1 the column relative to p_∞ is empty, then p_∞ is not determined by the *matching* procedure but it can be recovered using relation (1.62). Indeed, by (1.108)-(1.111), we can rewrite (1.62) as follows:

$$\begin{aligned} p_\infty = &\frac{1}{2}(\widetilde{q}_1\widetilde{q}_2\widetilde{q}_3\widehat{q}_3 + \widetilde{q}_1\widehat{q}_\infty + \widetilde{q}_2\overline{q}_\infty + \widetilde{q}_3\check{q}_\infty + \widetilde{q}_\infty\widehat{q}_3 + \widetilde{q}_{21}\widehat{q}_{32} + \widetilde{q}_{32}\check{q}_{31} \\ &- \widetilde{q}_1\widetilde{q}_2\widehat{q}_{32} - \widetilde{q}_1\widehat{q}_3\widetilde{q}_{32} - \widetilde{q}_2\widetilde{q}_3\check{q}_{31} - \widetilde{q}_3\widehat{q}_3\widetilde{q}_{21} - \widehat{q}_{31}\widetilde{q}_{31}) \end{aligned} \tag{1.116}$$

where the r.h.s of equations (1.116) depends only on the known coordinates of $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$.

Clearly, before to proceed describing in depth the matching procedure, we need to define the space of all possible $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, therefore in Chapter 3 we will review the main results about PVI.

Chapter 2

Garnier system $\mathcal{G}_2$ and symmetry group G

In this Chapter we recall some known facts about the Garnier system $\mathcal{G}_2$ and we use its birational canonical transformations to define the group G of symmetries acting on the character variety of the Riemann sphere with five boundary components $\mathcal{M}_{\mathcal{G}_2}$.

2.1 Garnier systems $\mathcal{G}_n$

The Garnier system $\mathcal{G}_n$ is a completely integrable Hamiltonian system [Gar12, Gar26, Oka81] in n variables $u_1, \ldots, u_n \in \mathbb{C}$ with $u_i \neq u_j$ when $i \neq j$:

$$\begin{cases} \frac{\partial \nu_j}{\partial u_i} &= \frac{\partial K_i}{\partial \rho_j}, \ i,j = 1,\ldots,n, \\ \frac{\partial \rho_j}{\partial u_i} &= -\frac{\partial K_i}{\partial \nu_j}, \ i,j = 1,\ldots,n. \end{cases} \tag{2.1}$$

The Hamiltonians K_i are defined as:

$$K_i = -\frac{\Lambda(u_i)}{T'(u_i)}\left[\sum_{k=1}^{n}\frac{T(\nu_k)}{(\nu_k-u_i)\Lambda'(\nu_k)}\left\{\rho_k^2-\sum_{m=1}^{n+2}\frac{\theta_m-\delta_{im}}{\nu_k-u_m}\rho_k+\frac{\kappa}{\nu_k(\nu_k-1)}\right\}\right], \tag{2.2}$$

where $\theta_1, \ldots, \theta_{n+2}, \theta_\infty$ are constant parameters and:

$$\Lambda(u) := \Pi_{k=1}^{n}(u-\nu_k),\ T(u) := \Pi_{k=1}^{n+2}(u-u_k), \tag{2.3}$$

$$\kappa = \frac{1}{4}\left\{\left(\sum_{m=1}^{n+2}\theta_m - 1\right)^2 - (\theta_\infty+1)^2\right\}. \tag{2.4}$$

Without loss of generality we fix $u_{n+1} = 0$ and $u_{n+2} = 1$.

When $n = 1$, there is only one complex variable $u = u_1$ and the Hamiltonian $K = K_1$ reads:

$$K = \frac{1}{u(u-1)}\left[\nu(\nu-1)(\nu-u)\rho^2 - \{\theta_2(\nu-1)(\nu-u) + \theta_3\nu(\nu-u) + (\theta_1-1)\nu(\nu-1)\}\rho + \kappa\nu\right], \tag{2.5}$$

where $\kappa = \frac{1}{4}\left[(\theta_1+\theta_2+\theta_3-1)^2 - (\theta_\infty+1)^2\right]$. The system $\mathcal{G}_1$ becomes a system of two first order equations or equivalently a scalar second order ODE that is the famous Painlevé Sixth equation PVI:

$$\nu_{uu} = \frac{1}{2}\left(\frac{1}{\nu}+\frac{1}{\nu-1}+\frac{1}{\nu-u}\right)\nu_u^2 - \left(\frac{1}{u}+\frac{1}{u-1}+\frac{1}{\nu-u}\right)\nu_u + \frac{\nu(\nu-1)(\nu-u)}{u^2(u-1)^2}\left[\alpha+\beta\frac{u}{\nu^2}+\gamma\frac{u-1}{(\nu-1)^2}+\delta\frac{u(u-1)}{(\nu-u)^2}\right], \tag{2.6}$$

with parameters:

$$\alpha = \frac{(\theta_\infty-1)^2}{2}, \quad \beta = -\frac{\theta_1^2}{2}, \quad \gamma = \frac{\theta_2^2}{2}, \quad \delta = \frac{1-\theta_3^2}{2}. \tag{2.7}$$

Hence, the Garnier systems $\mathcal{G}_n$ can be thought as a multivariable generalization of the PVI equation. In this thesis we focus on the first such generalization: $\mathcal{G}_2$.

2.2 Fuchsian system and its monodromy data

As mentioned in the Introduction, the Garnier system $\mathcal{G}_n$ describes isomonodromic deformations of the following Fuchsian system of linear differential equations for a 2×2 matrix valued function $\Psi(z)$ defined over $\overline{\mathbb{C}}$ (see [Gar12, Gar26, Oka81, Iwa91]):

$$\frac{d\Psi}{dz} = A(z)\Psi, \tag{2.8}$$

where $A(z)$ is the following matrix function:

$$A(z) = \sum_{i=1}^{n+2} \frac{A_i}{z - u_i}, \ \ A_i \in \mathfrak{sl}_2(\mathbb{C}).$$

The singularity at ∞ is also Fuchsian so that the residue at ∞ is defined by:

$$A_\infty := -\sum_{i=1}^{n+2} A_i.$$

Without loss of generality, for $i = 1, \ldots, n+2, \infty$, we can choose the matrices A_i traceless and set the eigenvalues of A_i to be $\frac{\pm\theta_i}{2}$ with $\theta_i \in \mathbb{C}$. In addition, by a proper gauge transformation, it is possible to assume the matrix A_∞ to be diagonal:

$$A_\infty = \frac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \ \theta_\infty \in \mathbb{C}^*. \tag{2.9}$$

In order to define the monodromy data of system (2.8), according to the classical results of Wasow [Was65] and Sibuya [Sib90], we choose to fix a fundamental matrix solution Ψ_∞ of (2.8) near ∞. The behaviour of a fundamental solution of (2.8), near the regular singular points, is described in the following Theorem, see [Dub96]:

Theorem 14. Suppose that $\infty \in \overline{\mathbb{C}}$ is a simple pole of (2.8), and the residue matrix A_∞ of the coefficient $A(z)$ near ∞ is in diagonal form (2.9), then system (2.8) has a fundamental solution Ψ_∞ such that:

$$\Psi_\infty(z) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{-J_\infty} z^{-R_\infty}, \ z \to \infty, \tag{2.10}$$

where $J_\infty = A_\infty$ and:

$$(R_\infty)_{12} \neq 0 \text{ if } \theta_\infty \in \mathbb{N}, \ (R_\infty)_{11} = (R_\infty)_{22} = (R_\infty)_{21} = 0,$$
$$(R_\infty)_{21} \neq 0 \text{ if } -\theta_\infty \in \mathbb{N}, \ (R_\infty)_{11} = (R_\infty)_{22} = (R_\infty)_{12} = 0.$$

A branch of the logarithm in the function Ψ_∞ must be chosen.

Near a singular point u_i, system (2.8) has a local fundamental solution Ψ_i such that:

$$\Psi_i(z) = G_i \left(\mathbb{1} + \mathcal{O}\left(z - u_i\right)\right) (z - u_i)^{J_i} (z - u_i)^{R_i}, \ z \to u_i, \tag{2.11}$$

where G_i is an invertible matrix and $J_i = G_i^{-1} A_i G_i$ is the Jordan normal form of the residue matrix A_i of $A(z)$ near u_i, moreover:

$$(R_i)_{12} \neq 0 \text{ if } \theta_i \in \mathbb{N}, \ (R_i)_{11} = (R_i)_{22} = (R_i)_{21} = 0,$$
$$(R_i)_{21} \neq 0 \text{ if } -\theta_i \in \mathbb{N}, \ (R_i)_{11} = (R_i)_{22} = (R_i)_{12} = 0.$$

A branch of the logarithm in the function Ψ_i must be chosen.

We now look at what happens when we continue Ψ_∞ analytically along paths in $\bar{\mathbb{C}}$. A useful notion is:

Definition 15. Two paths γ_1 and γ_2 are homotopic if there exists a continuous deformation of one path to the other, namely there exists a continuous function:

$$T : [0,1] \times [0,1] \mapsto \mathbb{C} \backslash \{u_1, \ldots, u_{n+2}\},$$

such that $T(0,t) = \gamma_1(t)$ and $T(1,t) = \gamma_2(t)$ for all $t \in [0,1]$, and $T(s,0) = \gamma_1(0) = \gamma_2(0)$ and $T(s,1) = \gamma_1(1) = \gamma_2(1)$ for all $s \in [0,1]$.

This is an equivalence relation that permits to identify curves that can be transformed one into the other in a continuous way on Σ_{n+3}, i.e. the Riemann sphere with $n+3$ boundary components. The following Theorem ensures that if we extend analytically our solution along two homotopic paths with the same end points, then we obtain the *same* extension:

Theorem 16. Suppose $\Psi(u)$ be a solution of (2.8) defined in an open set $U \in \mathbb{C} \backslash \{u_1, \ldots, u_{n+2}\}$. Consider $a, b \in U$ and two homotopic paths γ_1 and γ_2 with the same endpoints, then Ψ can be analytically continued along each path. Let $\gamma[\Psi]$ denote the analytic continuation of Ψ along the path γ, then $\gamma_1[\Psi](b) = \gamma_2[\Psi](b)$ and this is again a solution of (2.8).

Thanks to this Theorem, in order to fully characterize the analytic continuation of the fundamental solution Ψ_∞, we fix a basis in the fundamental group:

$$\pi_1\left(\mathbb{C} \backslash \{u_1, \ldots, u_{n+2}\}, \infty\right),$$

and we study the analytic continuation of Ψ_∞ along elements of the basis. We perform branch cuts on the Riemann sphere along the $n+2$ segments

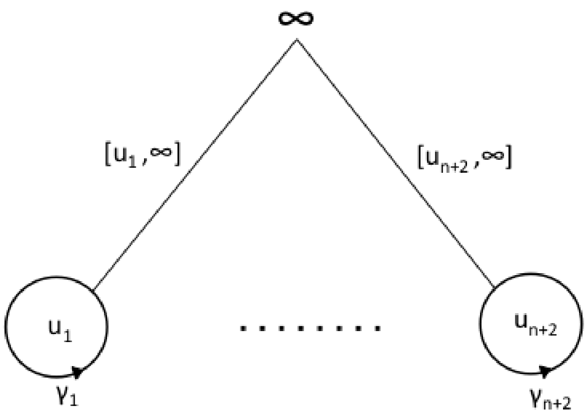

Figure 2.1: The basis of generators for $\pi_1\left(\mathbb{C}\backslash\{u_1,\ldots,u_{n+2}\},\infty\right)$.

$[u_i,\infty]$ and fix a basis of generators for the fundamental group, see Figure 2.1. The segments have all the same direction and they are ordered according to the order of the points $u_1,\ldots,u_{n+2}$. A generator can be thought like a path γ_i starting and ending at ∞, that goes around the singularity u_i in the clockwise direction, leaving the other singular points lying outside.

The product of these $n+2$ loops is equivalent to the loop that encircle the pole at infinity but taken in the counter-clockwise direction:

$$\gamma_1\ldots\gamma_{n+2}=\gamma_\infty^{-1}. \tag{2.12}$$

Fix a fundamental solution Ψ of (2.8). By Theorem 16 and the relation between fundamental solutions, the analytic continuation $\gamma_i[\Psi]$ gives rise to a unique 2×2 invertible matrix M_i, called monodromy matrix, such that:

$$\gamma_i[\Psi]=\Psi M_i. \tag{2.13}$$

It is natural to associate to each analytic continuation of Ψ, along a generator of the fundamental group an element of $\mathrm{GL}_2(\mathbb{C})$:

$$\gamma_i \mapsto M_i, \tag{2.14}$$

this map is a group anti-homomorphism and, moreover, it is a representation of the fundamental group:

Definition 17. The image of $\pi_1\left(\mathbb{C}\backslash\{u_1,\dots,u_{n+2}\},u_\infty\right)$ under the map (2.14) is a subgroup of $\mathrm{GL}_2(\mathbb{C})$ that is called *monodromy group*.

From relation (2.12) we get the following cyclic property for the generators of the monodromy group:

$$M_\infty M_{n+2}\dots M_1 = \mathbb{1}. \tag{2.15}$$

Remark 18. A change of the base point, and the consequent change of basis, leads to a conjugation of the representation by an invertible constant matrix.

Definition 19. We call monodromy data the set:

$$\mathrm{MD} := \left\{M_1,\dots,M_{n+2},R_1,\dots,R_{n+2}\right\}, \tag{2.16}$$

where matrices R_i are defined in Theorem 14.

If we take the fundamental solution Ψ_∞, defined locally near ∞, and another one Ψ_j, defined in a neighbourhood of u_j, then the former can be analytically continued along the path γ_j until a neighbourhood of u_j is reached. Again the two fundamental solutions are related by right multiplication by a constant invertible 2×2 matrix C_j:

$$\Psi_\infty = \Psi_j C_j, \tag{2.17}$$

called *connection matrix*. The connection matrix links the local monodromy at u_j with the global monodromy, in the basis defined by the fundamental matrix Ψ_∞, as follows:

$$M_j = C_j^{-1} e^{2\pi i J_j} e^{R_j} C_j. \tag{2.18}$$

2.3 Isomonodromic deformations

We now deform our system by keeping fixed the monodromy data (2.16). Consider the initial linear system:

$$\frac{d\Psi^0}{dz} = \sum_{k=1}^{n+2} \frac{A_k^0}{z - u_k^0} \Psi^0, \tag{2.19}$$

fix, as above, a basis $\gamma_1, \ldots, \gamma_{n+2}$ for the fundamental group:

$$\pi_1 \left(\mathbb{C} \backslash \{u_1, \ldots, u_{n+2}\}, \infty \right),$$

and a fundamental matrix solution Ψ_∞, near ∞. Isomonodromic deformations are described by the following Theorem, which proof can be found in [Mal91] and [Sib90]:

Theorem 20. There exists an open neighbourhood $U \subset \mathbb{C}^{n+2}$ of the point $u^0 = (u_1^0, \ldots, u_{n+2}^0)$ such that, for any $u = (u_1, \ldots, u_{n+2}) \in U$, there exists a unique $(n+2)$-tuple:

$$(A_1(u), \ldots, A_{n+2}(u)),$$

of analytic matrix valued functions such that:

$$A_j(u^0) = A_j^0,\ i = 1, \dots, n+2, \tag{2.20}$$

and with respect to the same basis of loops $\gamma_1, \dots, \gamma_{n+2}$, the monodromy data (2.16) of the Fuchsian system:

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi = \sum_{k=1}^{n+2} \frac{A_k(u)}{z-u_k}\Psi, \tag{2.21}$$

coincides with the given $M_1, \dots, M_{n+2}$ and $R_1, ..., R_{n+2}$. Furthermore, the monodromy group $\langle M_1, \dots, M_{n+2}\rangle$ is supposed to be irreducible and $M_i \neq \pm\mathbb{1}$, for $i = 1, \dots, n+2, \infty$. The matrices $A_j(u)$ are the solutions of the Cauchy problem with the initial data A_j^0 for the following Schlesinger equations:

$$\frac{\partial}{\partial u_j}A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i}A_i = -\sum_{j\neq i}\frac{[A_i, A_j]}{u_i - u_j}. \tag{2.22}$$

The solution Ψ_∞^0 of (2.19), in the form (2.10), can be uniquely continued, for $z \neq u_i$ $i = 1, \dots, n+2$, to an analytic function $\Psi_\infty(z, u)$ with $u \in U$ with $\Psi_\infty(z, u^0) = \Psi_\infty^0$. This continuation is the local solution of the Cauchy problem with the initial data Ψ_∞^0 for the following system that is compatible to the system (2.21):

$$\frac{\partial}{\partial u_i}\Psi = -\frac{A_i(u)}{z-u_i}\Psi. \tag{2.23}$$

The functions $A_i(u)$ and $\Psi_\infty(z, u)$ can be continued analytically to global meromorphic functions on the universal coverings of:

$$\mathbb{C}^{n+2}\backslash\{diags\} := \left\{(u_1, \dots, u_{n+2}) \in \mathbb{C}^{n+2} \,|\, u_i \neq u_j \,\text{for}\, i \neq j\right\}, \tag{2.24}$$

and:

$$\left\{(z, u_1, \ldots, u_{n+2}) \in \mathbb{C}^{n+3} \,|\, u_i \neq u_j \text{for}\, i \neq j \,\text{and}\, z \neq u_i,\, i = 1, \ldots, n+2\right\}, \tag{2.25}$$

respectively.

We recall here the main result about the solvability of the inverse monodromy problem in dimension two, the following result was proven by Dekkers in [Dek79] and by Bolibruch in [Bol97]:

Theorem 21. Given matrices $M_1, \ldots, M_{n+2} \in \mathrm{SL}_2(\mathbb{C})$ satisfying (2.15), with:

$$M_\infty = \begin{pmatrix} e^{i\pi\theta_\infty} & 0 \\ 0 & e^{-i\pi\theta_\infty} \end{pmatrix}, \ \theta_\infty \in \mathbb{C},$$

and matrices $R_1, \ldots, R_{n+2} \in \mathrm{SL}_2(\mathbb{C})$, then in a neighbourhood U of $u^0 = (u_1^0, \ldots, u_{n+2}^0) \in \mathbb{C}^{n+2} \backslash \{diags\}$, there exists $(u_1, \ldots, u_{n+2}) \in U$ and a Fuchsian system:

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi = \sum_{k=1}^{n+2} \frac{A_k(u)}{z - u_k}\Psi,$$

with $M_1, \ldots, M_{n+2}$ and $R_1, \ldots, R_{n+2}$ as monodromy data and $u_1, \ldots, u_{n+2}$ as poles.

Now, we show how to reduce the Schlesinger equations to the Garnier system $\mathcal{G}_n$. Since Schlesinger equations are invariant under simultaneous diagonal conjugation of matrices A_i for $i = 1, \ldots, n+2$, we introduce $2n$ coordinates on the space of solutions of the Schlesinger equations with respect to the equivalence relation:

$$A_i \sim D^{-1} A_i D,$$

where D is a 2×2 diagonal matrix.

Let $a_{ij}(z,u)$ denote the ij-element of $A(z,u)$, then $a_{12}(z,u)$ has the form:

$$a_{12}(z,u) = \sum_{i=1}^{n+2} \frac{a_{12}^i}{z-u_i}, \tag{2.26}$$

and its denominator is a polynomial of degree n in the variable z. Define $\nu_1, \ldots, \nu_n$ to be the roots of this polynomial, i.e.:

$$a_{12}(\nu_k, u) = 0, \ \ k = 1, \ldots, n,$$

and n quantities ρ_k:

$$\rho_k := \sum_{i=1}^{n+2} \frac{a_{11}^i + \frac{\theta_i}{2}}{\nu_k - u_i}, \ \ k = 1, \ldots, n. \tag{2.27}$$

In this way we introduce $2n$ coordinates $(\nu_1, \ldots, \nu_n, \rho_1, \ldots, \rho_n)$ on the space of the solutions of the above Schlesinger equations, as stated in the following Theorem, due to Iwasaki et al. [Iwa91]:

Theorem 22. If the $n+2$ tuple $(A_1(u), \ldots, A_{n+2}(u))$ of 2×2 matrix is a solution for the Schlesinger system:

$$\frac{\partial}{\partial u_j} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i} A_i = -\sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j}, \tag{2.28}$$

then the functions $(\nu_1(u), \ldots, \nu_n(u), \rho_1(u), \ldots, \rho_n(u))$ with $u = (u_1, \ldots, u_{n+2})$, where $\nu_k(u)$ are the roots of (2.26) and the $\rho_k(u)$ are defined in (2.27), determine $A_1, \ldots, A_n$ uniquely up to global diagonal conjugation and the $\rho_k(u)$ satisfy the Garnier system $\mathcal{G}_n$.

As a consequence of Theorem 22, we can regard $\mathcal{G}_n$ as the system that governs the isomonodromic deformations of the Fuchsian system (2.8), this is why in the following we will refer directly to $\mathcal{G}_n$.

2.4 Braid group B_n and analytic continuation

We now show that the structure of analytic continuation of a solution of $\mathcal{G}_n$ is given by the action of the braid group B_{n+2} (see Definition 8) over the monodromy matrices $(M_1, \ldots, M_{n+2})$, as it was firstly introduced in Dubrovin-Mazzocco in [DM00] for $n = 1$.

Consider a point $u^0 = (u_1^0, \ldots, u_{n+2}^0) \in \mathbb{C}^{n+2}\backslash\{diags\}$ and a solution $(\nu_1(u), \ldots, \nu_n(u), \rho_1(u), \ldots, \rho_n(u))$ of $\mathcal{G}_n$ in a neighbourhood of u^0. We perform $n+2$ cuts on $\mathbb{C}^{n+2}\backslash\{diags\}$, and we choose a basis of loops $\gamma_1, \ldots, \gamma_{n+2}$ for $\pi_1(\mathbb{C}^{n+2}\backslash\{diags\}, u^0)$, as in Figure 2.1. In this way, a branch of our solution is fixed and, by Theorem 22, to this branch we can associate $n+2$ monodromy matrices $M_1, \ldots, M_{n+2}$. Suppose now to continue analytically the solution of $\mathcal{G}_n$ along a loop:

$$\beta \in \pi_1(\mathbb{C}^{n+2}\backslash\{diags\}, u^0),$$

then a new branch is reached with new associated monodromy matrices $M_1', \ldots, M_{n+2}'$. This action of the fundamental group $\pi_1(\mathbb{C}^{n+2}\backslash\{diags\}, u^0)$ over the monodromy matrices extends naturally to the action of the pure braid group P_{n+2}, see (1.83) and (1.84). Indeed, it is well known hat:

$$\pi_1(\mathbb{C}^{n+2}\backslash\{diags\}, u^0) \simeq P_{n+2}.$$

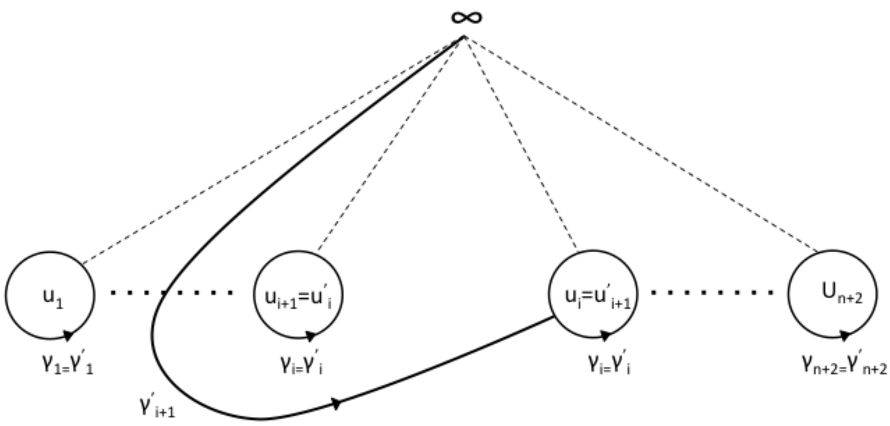

Figure 2.2: The new basis of loops obtained with the action of generator $\sigma_i \in B_3$.

In order to simplify the following computations, the action of the pure braid group P_{n+2} is extended to the action of the full braid group B_{n+2}:

$$\pi_1(\mathbb{C}^{n+2}\backslash\{diags\}/S_{n+2}, u^0) \simeq B_{n+2},$$

where S_{n+2} is the symmetric group over $n+2$ elements.

Consider the i-th generator σ_i of B_{n+2}, then σ_i acts changing position of the poles u_i and u_{i+1}. We show the new basis of loops $\gamma_k^{'}$, for $k = 1, \ldots, n+2$, of $\pi_1(\mathbb{C}^{n+2}\backslash\{diags\}/S_{n+2}, u^0)$ in Figure 2.2.

Since deformations are supposed to be isomonodromic, the new monodromy matrices $M_k^{'}$, in the new basis of loops, are a reordering of the old ones:

$$M_i^{'} = M_{i+1},\ M_{i+1}^{'} = M_i,\ M_k^{'} = M_k,\ k \neq i, i+1.$$

The new basis of loops $\gamma_k^{'}$ and the old basis of loops γ_k for $k = 1, \ldots, n+2$, are related as follows:

$$\gamma_i = \gamma_i^{'}, \quad \gamma_{i+1} = \gamma_i^{'-1}\gamma_{i+1}^{'}\gamma_i^{'}, \quad \gamma_k = \gamma_k^{'},\ k \neq i, i+1.$$

Since the basis of loops must be fixed once for ever, we need to express the new monodromy matrices in the old basis of loops. This leads to the following expression for the i-th generator of B_{n+2}:

$$\sigma_i : (M_1, \dots, M_i, M_{i+1}, \dots M_{n+2}) \mapsto (M_1, \dots, M_{i+1}, M_{i+1}M_iM_{i+1}^{-1}, \dots M_{n+2}). \tag{2.29}$$

Note that the action of σ_i over the monodromy matrices preserves their conjugacy class and the relation $M_\infty M_{n+2} \dots M_1 = \mathbb{1}$.

Once we fix a branch of a solution of $\mathcal{G}_n$, at the same time we fix the monodromy matrices and the matrices $R_1, \dots, R_{n+2}$. The matrices R_i remain invariant under the action of the braid group. Therefore describing the braid action on the monodromy data (2.16) is equivalent to describe the same action on the character variety or equivalently on:

$$\widehat{\mathcal{M}}_{\mathcal{G}_n} := \{(M_1, \dots, M_{n+2}) \mid M_i \in \mathrm{SL}_2(\mathbb{C}),\ M_\infty M_{n+2} \dots M_1 = \mathbb{1}\} / \sim, \tag{2.30}$$

where $\sim$ is equivalence under global diagonal conjugation.

In order to describe all branches of a solution of the Garnier system $\mathcal{G}_n$, we continue this solution analytically along every loop of $\pi_1(\mathbb{C}^{n+2}\backslash\{diags\}, u^0)$. This is done in terms of action of the pure braid group over $\widehat{\mathcal{M}}_{\mathcal{G}_n}$:

$$P_{n+2} \times \widehat{\mathcal{M}}_{\mathcal{G}_n} \longrightarrow \widehat{\mathcal{M}}_{\mathcal{G}_n}.$$

Define the set:

$$\mathcal{O}_{P_{n+2}}(m) := \{\beta(m) | \beta \in P_{n+2}\}, \tag{2.31}$$

being the orbit of an element $m \in \widehat{\mathcal{M}}_{\mathcal{G}_n}$ under the action of the pure braid group P_{n+2}.

In this thesis we are interested in the classification of algebraic solutions of the system $\mathcal{G}_n$:

Definition 23. A $\mathcal{G}_n$'s solution $(\nu_1(u), \ldots, \nu_n(u), \rho_1(u), \ldots, \rho_n(u))$ where $u = (u_1, \ldots, u_n)$ is algebraic if every components ν_i solves an equation:

$$P_i(u_1, \ldots, u_n, \nu_i) = 0,$$

and respectively every component ρ_i solves equation:

$$\tilde{P}_i(u_1, \ldots, u_n, \rho_i) = 0,$$

where P_i and $\tilde{P}_i$ are polynomials in $\mathbb{C}[u_1, \ldots, u_n, \nu_i]$ and $\mathbb{C}[u_1, \ldots, u_n, \rho_i]$ respectively.

Since an algebraic solution has only a finite number of branches, the monodromy associated to this solution under the action of P_{n+2} necessarily generate a finite orbit. We formalize this fact in the following Theorem due to Cousin [Cou16]:

Theorem 24. If a solution of $\mathcal{G}_n$ is algebraic, then the orbit under the P_{n+2} action over the monodromy matrices associated to this solution, up to conjugation by a diagonal matrix, is finite.

This means that the problem of classification of algebraic solutions of $\mathcal{G}_n$ can be seen as the problem of classification of finite orbit of the P_{n+2} action over $\widehat{\mathcal{M}}_{\mathcal{G}_n}$.

2.5 Symmetry group G

In this Section, we study the symmetries acting on the co-adjoint coordinates p_i,p_{ij},p_{ijk} defined on the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$. The definition of a *symmetry* for $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ is the following:

Definition 25. A *symmetry* for $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ is an invertible map $\Phi : \widehat{\mathcal{M}}_{\mathcal{G}_2} \mapsto \widehat{\mathcal{M}}_{\mathcal{G}_2}$ such that given an element $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and its orbit $\mathcal{O}_{P_4}(p)$, the following is true:

$$|\mathcal{O}_{P_4}(\Phi(p))| = |\mathcal{O}_{P_4}(p)|. \tag{2.32}$$

By the above Definition, if $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ generates a finite P_4-orbit of length N, then $\Phi(p)$ generates a finite P_4-orbit of the same length. This leads to define the following equivalence relation between orbits:

Definition 26. The elements p and $p' \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ are said to generate *equivalent orbits* if there exists a symmetry Φ such that $p' = \Phi(p)$.

First, we introduce the group $\widetilde{\mathrm{G}}$ of symmetries obtained by birational transformations of the Garnier system $\mathcal{G}_2$. Subsequently, in order to obtain the group of symmetries G, we extend the group $\widetilde{\mathrm{G}}$ with simpler transformations that arise on the space of monodromy matrices.

2.5.1 Birational transformations of $\mathcal{G}_2$

The birational transformations of the Garnier systems (2.1) were firstly introduced by Kimura et al. [Iwa91], and subsequently studied by Tsuda [Tsu03] and Suzuki [Suz05]. A birational transformation for the Garnier system $\mathcal{G}_2$ is a map of the form:

$$S : (\nu_1, \rho_1, u_1, \nu_2, \rho_2, u_2, \theta_1, \ldots, \theta_4, \theta_\infty) \mapsto (\widetilde{\nu}_1, \widetilde{\rho}_1, \widetilde{u}_1, \widetilde{\nu}_2, \widetilde{\rho}_2, \widetilde{u}_2, \widetilde{\theta}_1, \ldots, \widetilde{\theta}_4, \widetilde{\theta}_\infty), \tag{2.33}$$

where S acts birationally on $(\nu_1, \rho_1, u_1, \nu_2, \rho_2, u_2)$ and by linear affine transformation on the five parameters $(\theta_1, \ldots, \theta_4, \theta_\infty)$.

Since these transformations are birational, they send algebraic solutions to algebraic solutions, preserving the number of branches. In particular, this implies that the action of these transformations on finite orbits of the action of the pure braid group P_4 over $\widehat{\mathcal{M}}_{\mathcal{G}_2}$ are mapped to finite orbits and their length is preserved. If two orbits are related by such transformation, we say that they are *equivalent*. In order to characterize the group of symmetries acting on $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, we define the group:

$$\widetilde{G} = \langle P_{13}, P_{23}, P_{34}, P_{1\infty} \rangle, \tag{2.34}$$

of birational transformations of the Garnier system $\mathcal{G}_2$, and thanks to the work of Dubrovin-Mazzocco [DM07], we are able to explicitly write the action of the generators of $\widetilde{\mathrm{G}}$ over the monodromy matrices M_1, M_2, M_3, M_4 and consequently over the co-adjoint coordinates p_i,p_{ij},p_{ijk}.

We list now all known birational transformations acting on the Garnier system $\mathcal{G}_2$ and compute their effect on the co-adjoint coordinates p_i,p_{ij},p_{ijk} defined on the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$. Firstly we look at the birational transformations s_i for $i = 1, \ldots, 4, \infty$, that act as change of sign on the parameters θ_i for $i = 1, \ldots, 4, \infty$:

$$s_1 : \begin{cases} \widetilde{\nu}_i &= \nu_i,\ i = 1, 2, \\ \widetilde{\rho}_1 &= \rho_1 - \frac{\theta_1}{\nu_1}, \\ \widetilde{\rho}_2 &= \rho_2, \\ \widetilde{u}_i &= u_i,\ i = 1, 2, \\ \widetilde{\theta}_1 &= -\theta_1, \\ \widetilde{\theta}_i &= \theta_i,\ i = 2, 3, 4, \infty, \end{cases} \tag{2.35}$$

$$s_2 : \begin{cases} \widetilde{\nu}_i &= \nu_i,\ i = 1, 2, \\ \widetilde{\rho}_1 &= \rho_1, \\ \widetilde{\rho}_2 &= \rho_2 - \frac{\theta_2}{\nu_2}, \\ \widetilde{u}_i &= u_i,\ i = 1, 2, \\ \widetilde{\theta}_2 &= -\theta_2, \\ \widetilde{\theta}_i &= \theta_i,\ i = 1, 3, 4, \infty, \end{cases} \tag{2.36}$$

$$s_3 : \begin{cases} \widetilde{\nu}_i &= \nu_i,\ i = 1, 2, \\ \widetilde{\rho}_i &= \rho_i - \frac{\theta_3}{u_i(\frac{\nu_1}{u_1} + \frac{\nu_2}{u_2} - 1)},\ i = 1, 2, \\ \widetilde{u}_i &= u_i,\ i = 1, 2, \\ \widetilde{\theta}_3 &= -\theta_3, \\ \widetilde{\theta}_i &= \theta_i,\ i = 1, 2, 4, \infty, \end{cases} \tag{2.37}$$

$$s_4 : \begin{cases} \widetilde{\nu}_i &= \nu_i,\ i = 1,2, \\ \widetilde{\rho}_i &= \rho_i - \frac{\theta_4}{\nu_1+\nu_2-1},\ i = 1,2, \\ \widetilde{u}_i &= u_i,\ i = 1,2, \\ \widetilde{\theta}_4 &= -\theta_4, \\ \widetilde{\theta}_i &= \theta_i,\ i = 1,2,3,\infty, \end{cases} \tag{2.38}$$

$$s_\infty : \begin{cases} \widetilde{\nu}_i &= \nu_i,\ i = 1,2, \\ \widetilde{\rho}_i &= \rho_i,\ i = 1,2, \\ \widetilde{u}_i &= u_i,\ i = 1,2, \\ \widetilde{\theta}_\infty &= -\theta_\infty, \\ \widetilde{\theta}_i &= \theta_i,\ i = 1,2,3,4, \end{cases} \tag{2.39}$$

Lemma 27. Transformations $s_1, \dots, s_4, s_\infty$ act as the identity on $p \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$.

Proof. Since all transformations (2.39) act as the identity on ν_i for $i = 1,2$ and on the independent variables u_i for $i = 1,2$, then the monodromy remains unchanged and each transformation s_i acts on $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ as the identity. □

We consider now transformations acting on both dependent and independent variables permuting the positions of the poles. There are four birational generators P_{ij}:

$$P_{13}:\begin{cases}\widetilde{\nu}_i &= \frac{u_1-\nu_i}{u_1-1},\ i=1,2,\\ \widetilde{\rho}_i &= -(u_1-1)\rho_i,\ i=1,2,\\ \widetilde{u}_1 &= \frac{u_1}{u_1-1},\\ \widetilde{u}_2 &= \frac{u_1-u_2}{u_1-1},\\ \widetilde{\theta}_1 &= \theta_3,\\ \widetilde{\theta}_3 &= \theta_1,\\ \widetilde{\theta}_i &= \theta_i,\ i\neq 1,3.\end{cases} \tag{2.40}$$

$$P_{23}:\begin{cases}\widetilde{\nu}_i &= \frac{u_2-\nu_i}{u_2-1},\ i=1,2,\\ \widetilde{\rho}_i &= -(u_2-1)\rho_i,\ i=1,2,\\ \widetilde{u}_1 &= \frac{u_2-u_1}{u_2-1},\\ \widetilde{u}_2 &= \frac{u_2}{u_2-1},\\ \widetilde{\theta}_2 &= \theta_3,\\ \widetilde{\theta}_3 &= \theta_2,\\ \widetilde{\theta}_i &= \theta_i,\ i\neq 2,3.\end{cases} \tag{2.41}$$

$$P_{34}:\begin{cases}\widetilde{\nu}_i &= 1-\nu_i,\ i=1,2,\\ \widetilde{\rho}_i &= -\rho_i,\ i=1,2,\\ \widetilde{u}_i &= 1-u_i,\ i=1,2,\\ \widetilde{\theta}_3 &= \theta_4,\\ \widetilde{\theta}_4 &= \theta_3,\\ \widetilde{\theta}_i &= \theta_i,\ i\neq 3,4.\end{cases} \tag{2.42}$$

$$P_{1\infty}: \begin{cases} \widetilde{\nu}_i & = \frac{1}{\nu_i - u_1},\ i = 1,2, \\ \widetilde{\rho}_i & = -\rho_i \nu_i^2 - \frac{2\times 2^2 - 1}{2}\nu_i,\ i = 1,2, \\ \widetilde{u}_1 & = \infty, \\ \widetilde{\infty} & = u_1, \\ \widetilde{u}_2 & = \frac{1}{u_2 - u_1}, \\ \widetilde{\theta}_1 & = \theta_\infty - 1, \\ \widetilde{\theta}_\infty & = \theta_1 + 1, \\ \widetilde{\theta}_i & = \theta_i,\ i \neq 1, \infty. \end{cases} \tag{2.43}$$

We restate Theorem 8.1.2 in [Kim90], where a description of the group generated by P_{ij} is given:

Theorem 28. The group $\widetilde{\mathrm{G}}$ generated by:

$$\widetilde{G} := < P_{13}, P_{23}, P_{34}, P_{1\infty} >, \tag{2.44}$$

is a group of symmetries for the Garnier system $\mathcal{G}_2$ and it is isomorphic to the symmetric group with 5 elements, i.e. S_5.

In order to write down the action of the group $\widetilde{\mathrm{G}}$ in terms of co-adjoint coordinates (1.3) defined on the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, we need to understand how transformations P_{ij} act on the monodromy matrices. We calculate this action on a 4-tuple $(M_1, \ldots, M_4)$ of monodromy matrices following Theorem 1.2 in the work of Dubrovin-Mazzocco [DM07] and afterwards the action in terms of co-adjoint coordinates :

Lemma 29. If $(M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ is the monodromy associated to a solution $(\nu_1, \rho_1, u_1, \nu_2, \rho_2, u_2)$ of $\mathcal{G}_2$, then transformations P_{13},P_{23},P_{34} act on the monodromy matrices as follows:

$$\begin{aligned}
P_{13} : (M_1, M_2, M_3, M_4) &\mapsto (M_1^{-1}M_2^{-1}M_3M_2M_1, M_2, M_2M_1M_2^{-1}, M_4),\\
P_{23} : (M_1, M_2, M_3, M_4) &\mapsto ((M_2^{-1}M_3M_2M_1)^{-1}M_1M_2^{-1}M_3M_2M_1,\\
&\quad (M_2M_1)^{-1}M_3M_2M_1, M_2, M_4),\\
P_{34} : (M_1, M_2, M_3, M_4) &\mapsto (M_\infty M_3M_2M_1(M_\infty M_3M_2)^{-1}, M_2,\\
&\quad (M_3M_2M_1M_2^{-1})^{-1}M_4(M_3M_2M_1M_2^{-1}), M_3),
\end{aligned} \tag{2.45}$$

while transformation $P_{1\infty}$ acts on the monodromy matrices as:

$$\begin{aligned}
P_{1\infty} : (M_1, M_2, M_3, M_4) \mapsto &(-C_1M_\infty C_1^{-1}, C_1^{-1}M_2C_1, C_1^{-1}M_3C_1,\\
&C_1^{-1}M_4C_1),
\end{aligned} \tag{2.46}$$

where C_1 is the connection matrix defined in Section 2.2.

Proof. The proof is a consequence of Theorem 1.2 in the work of Dubrovin-Mazzocco [DM07]. □

Finally the action of the group $\widetilde{\mathrm{G}}$ in terms of co-adjoint coordinates (1.3) defined on the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$:

Corollary 30. The group $\widetilde{\mathrm{G}} = \langle P_{13}, P_{23}, P_{34}, P_{1\infty}\rangle$ acts on $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ as follows:

$$P_{13}(p) = \sigma_2\sigma_1^{-1}\sigma_2^{-1}(p), \tag{2.47}$$

$$P_{23}(p) = \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}(p), \tag{2.48}$$

$$P_{34}(p) = \sigma_3\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}(p), \tag{2.49}$$

$$P_{1\infty}(p) = (-p_\infty, p_2, p_3, p_4, -p_1, -p_1p_{43} + p_{431} + p_{21}p_{432} - p_2p_\infty,$$

$$- p_{21}p_4 + p_{421} + p_{321}p_{43} - p_3p_\infty, p_{32}, -p_{321}, p_{42}, p_{43},$$
$$- p_1p_4 + p_{41} + p_{321}p_{432} - p_{32}p_\infty, p_{432}, -p_{21},$$
$$- p_1p_3 + p_{31} + p_{21}p_{32} - p_2p_{321}), \tag{2.50}$$

where σ_i are defined in (1.86).

Proof. Formulae (2.47)-(2.50) can be proven by straightforward computation applying, respectively, to formulae (2.45)-(2.46) the definition of co-adjoint coordinates and the skein relation. □

2.5.2 Symmetries of the monodromy matrices

In this Section, we introduce a set of transformations on the space of monodromy matrices:

(i) Transformations that change signs to matrices M_i, for $i = 1, \ldots, 4$, the so-called Schlesinger transformations introduced by Jimbo-Miwa in [JM81]:

$$(M_1, M_2, M_3, M_4, M_\infty) \mapsto (\epsilon_1 M_1, \epsilon_2 M_2, \epsilon_3 M_3, \epsilon_4 M_4, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 (M_4M_3M_2M_1)^{-1}), \tag{2.51}$$

where $\epsilon_i = \pm 1$ for $i = 1, \ldots, 4$.

(ii) Permutations of the matrices M_i for $i = 1, \ldots, 4$:

$$(M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_{\xi(1)}^{-1}, M_{\xi(2)}^{-1}, M_{\xi(3)}^{-1}, M_{\xi(4)}^{-1}, (M_{\xi(4)}^{-1}, M_{\xi(3)}^{-1} M_{\xi(2)}^{-1} M_{\xi(1)}^{-1})^{-1}), \tag{2.52}$$

where ξ is in a subgroup $H \subset S_4$, of the symmetric group over 4 elements. In (2.52), we consider the inversion to be able to refer to the work of Lisovyy and Tykhyy [LT14].

Given $(M_1, \ldots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, we call *sign flips* the transformations that change sign of the matrices M_i. They are defined as:

$$\text{sign}_{(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (\epsilon_1 M_1, \epsilon_2 M_2, \epsilon_3 M_3, \epsilon_4 M_4, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 (M_4 M_3 M_2 M_1)^{-1}) \quad (2.53)$$

where $\epsilon_i = \pm 1$ for $i = 1, \ldots, 4$. The following four *sign flips* generate all the others:

$$\text{sign}_1 := \text{sign}_{(-1,1,1,1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (-M_1, M_2, M_3, M_4, -M_\infty), \quad (2.54)$$

$$\text{sign}_2 := \text{sign}_{(1,-1,1,1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_1, -M_2, M_3, M_4, -M_\infty), \quad (2.55)$$

$$\text{sign}_3 := \text{sign}_{(1,1,-1,1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_1, M_2, -M_3, M_4, -M_\infty), \quad (2.56)$$

$$\text{sign}_4 := \text{sign}_{(1,1,1,-1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_1, M_2, M_3, -M_4, -M_\infty), \quad (2.57)$$

Sign flips are invertible maps Φ that lead to *equivalent* orbits:

Proposition 31. Given a sign flip sign $\in \langle \text{sign}_1, \ldots, \text{sign}_4 \rangle$ and a braid

$\sigma \in B_4$, then there exists $\text{sign}' \in \langle \text{sign}_1, \ldots, \text{sign}_4 \rangle$ and $\sigma' \in B_4$ s.t.

$$\sigma \text{sign} = \text{sign}' \sigma'. \tag{2.58}$$

Proof. Suppose $m = (M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, we prove that given a generator σ_i for $i = 1, 2, 3$ of the full braid group B_4, then:

$$\sigma_i \text{sign}_j = \text{sign}_{j'} \sigma_{i'}, \tag{2.59}$$

for some choice of the indices $i, i' = 1, 2, 3$ and $j, j' = 1, 2, 3, 4$. We prove it for σ_1 and sign_1. Indeed:

$$\sigma_1 \text{sign}_1(m) = (M_2, -M_2 M_1 M_2^{-1}, M_3, M_4) = \text{sign}_2 \sigma_1(m). \tag{2.60}$$

In a similar way we can prove the following equations:

$$\begin{aligned}
\sigma_1 \text{sign}_2 &= \text{sign}_1 \sigma_1, \\
\sigma_1 \text{sign}_3 &= \text{sign}_3 \sigma_1, \\
\sigma_1 \text{sign}_4 &= \text{sign}_4 \sigma_1, \\
\sigma_2 \text{sign}_1 &= \text{sign}_1 \sigma_2, \\
\sigma_2 \text{sign}_2 &= \text{sign}_3 \sigma_2, \\
\sigma_2 \text{sign}_3 &= \text{sign}_2 \sigma_2, \\
\sigma_2 \text{sign}_4 &= \text{sign}_4 \sigma_2, \\
\sigma_3 \text{sign}_1 &= \text{sign}_1 \sigma_3, \\
\sigma_3 \text{sign}_2 &= \text{sign}_2 \sigma_3, \\
\sigma_3 \text{sign}_3 &= \text{sign}_4 \sigma_3, \\
\sigma_3 \text{sign}_4 &= \text{sign}_3 \sigma_3.
\end{aligned} \tag{2.61}$$

This concludes the proof. □

The following result gives the action of sign flips in terms of co-adjoint coordinates and can be proved by straightforward computations:

Corollary 32. The action of the generators of the group of *sign flips* in terms of co-adjoint coordinates (1.3), on the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, is as follows:

$$\begin{aligned} \text{sign}_1(p) = (&-p_1, p_2, p_3, p_4, -p_\infty, -p_{21}, -p_{31}, p_{32}, -p_{41}, p_{42}, p_{43}, -p_{321}, p_{432}, \\ &-p_{431}, -p_{421}), \end{aligned} \tag{2.62}$$

$$\begin{aligned} \text{sign}_2(p) = (&p_1, -p_2, p_3, p_4, -p_\infty, -p_{21}, p_{31}, -p_{32}, p_{41}, -p_{42}, p_{43}, -p_{321}, -p_{432}, \\ &p_{431}, -p_{421}), \end{aligned} \tag{2.63}$$

$$\begin{aligned} \text{sign}_3(p) = (&p_1, p_2, -p_3, p_4, -p_\infty, p_{21}, -p_{31}, -p_{32}, p_{41}, p_{42}, -p_{43}, -p_{321}, -p_{432}, \\ &-p_{431}, p_{421}), \end{aligned} \tag{2.64}$$

$$\begin{aligned} \text{sign}_4(p) = (&p_1, p_2, p_3, -p_4, -p_\infty, p_{21}, p_{31}, p_{32}, -p_{41}, -p_{42}, -p_{43}, p_{321}, -p_{432}, - \\ &-p_{431}, p_{421}). \end{aligned} \tag{2.65}$$

At this point we introduce permutations on a 4-tuple of monodromy matrices $(M_1, \ldots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$. These permutations are generated by:

$$(12)(34) : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_2^{-1}, M_1^{-1}, M_4^{-1}, M_3^{-1}, M_2 M_1 M_4 M_3), \tag{2.66}$$

$$(1234) : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_4, M_1, M_2, M_3, (M_3 M_2 M_1 M_4)^{-1}). \tag{2.67}$$

As in the case of sign flips also (2.66)-(2.67) can be considered as invertible maps Φ that lead to *equivalent* orbits (see Definitions 37 and 26):

Proposition 33. Given a permutation $\xi \in \langle(12)(34),(1234)\rangle$ and a braid $\sigma \in B_4$, then there exists $\xi' \in \langle(12)(34),(1234)\rangle$ and $\sigma' \in B_4$ such that:

$$\sigma\xi = \xi'\sigma'. \tag{2.68}$$

Proof. Suppose $m = (M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, we prove that given a generator σ_i for $i = 1,2,3$ of the full braid group B_4, then:

$$\xi\sigma_i = \sigma_{i'}\xi',$$

for some choice of the indices $i, i' = 1,2,3$ and $\xi, \xi' \in \langle(12)(34),(1234)\rangle$. We prove in details that the statement is true for the composition of (1234) and σ_1:

$$\begin{aligned}(1234)\sigma_1(m) &= (1234)(M_2, M_2M_1M_2^{-1}, M_3, M_4) = \\ &= (M_4, M_2, M_2M_1M_2^{-1}, M_3),\end{aligned}$$

and:

$$\begin{aligned}\sigma_2(1234)(m) &= \sigma_2(M_4, M_1, M_2, M_3) = \\ &= (M_4, M_2, M_2M_1M_2^{-1}, M_3),\end{aligned}$$

then $\sigma_2(1234) = (1234)\sigma_1$. In a similar way we can prove the following equations:

$$\sigma_1(12)(34) = (12)(34)\sigma_1^{-1}, \tag{2.69}$$

$$\sigma_2(12)(34) = (12)(34)(1234)^3\sigma_2\sigma_3, \tag{2.70}$$

$$\sigma_3(12)(34) = (12)(34)\sigma_3^{-1}, \tag{2.71}$$

$$\sigma_1(1234) = (1234)(1234)\sigma_2^{-1}\sigma_1^{-1}, \tag{2.72}$$

$$\sigma_2(1234) = (1234)\sigma_1, \tag{2.73}$$

$$\sigma_3(1234) = (1234)\sigma_2. \tag{2.74}$$

This concludes the proof. □

The following result gives the action of the permutations in terms of co-adjoint coordinates and can be proved by straightforward computations:

Corollary 34. The action of the generators (12)(34) and (1234) in terms of co-adjoint coordinates, on the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, is:

$$(12)(34)(p) = (p_2, p_1, p_4, p_3, p_\infty, p_{21}, p_{42}, p_{41}, p_{32}, p_{31}, p_{43}, p_{421}, p_{431}, p_{432}, p_{321}), \tag{2.75}$$

$$(1234)(p) = (p_4, p_1, p_2, p_3, p_\infty, p_{41}, p_{42}, p_{21}, p_{43}, p_{31}, p_{32}, p_{421}, p_{321}, p_{432}, p_{431}). \tag{2.76}$$

We resume the results of this Section in the following Theorem:

Theorem 35. The group:

$$G := \langle P_{13}, P_{23}, P_{34}, P_{1\infty}, \mathrm{sign}_1, \dots, \mathrm{sign}_4, (12)(34), (1234)\rangle. \tag{2.77}$$

is a group of symmetries for $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$.

Proof. For the subgroup of transformations $\langle P_{13}, P_{23}, P_{34}, P_{1\infty}\rangle$, the statement follows by construction. For the subgroup of transformations $\langle \mathrm{sign}_1, \dots, \mathrm{sign}_4, (12)(34), (1234)\rangle$, the statement follows by Propositions 31 and 33. □

Chapter 3

Input set of the matching procedure

We briefly recall the main idea underlying our methodology. In order to classify all finite orbits of the action:

$$P_4 \times \mathcal{M}_{\mathcal{G}_2} \longrightarrow \mathcal{M}_{\mathcal{G}_2}, \tag{3.1}$$

in Theorem 12, we introduced four restrictions $H_1, \dots, H_4$ to subgroups of P_4 that are isomorphic to the braid group P_3. These four restrictions are summarized in Table 1.1. In particular each row of Table 1.1 represents a subset $\widetilde{q}, \overline{q}, \check{q}, \overline{q}$ of co-adjoint coordinates p_i, p_{ij}, p_{ijk} that must generate a finite orbit under the "restricted action" of P_3 over the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components, i.e. $\mathcal{M}_{PVI}$.

In Section 3.1, we will remind that the "restricted action" describes the structure of the analytic continuation of algebraic solutions of PVI, the Sixth Painlevé equation, and hence, that each algebraic solution of PVI is associated to a finite orbit of the "restricted action".

In this Chapter we want to produce a list of all possible $\widetilde{q}, \overline{q}, \check{q}, \overline{q}$, as defined in (1.107), such that their orbit, under the action of $H_1, \dots, H_4$ respectively, is finite. In other words we want to define the "input set" for the matching procedure introduced in Section 1.4. It is then fundamental, for our purpouses, to know the classification of all algebraic solutions of PVI. Many authors approached this problem with different methods, see [AK02, Boa06, Boa05, Dub96, DM00, Hit95, Hit03, Kit05, LT14]. In the following the major ideas are inspired from the work of Dubrovin-Mazzocco [DM00] and its natural generalisazion due to Lisovyy and Tykhyy [LT14].

The classification result of Lisovyy and Tykhyy produced a list of $5+45$ distinct finite orbits of the action of P_3 over the character variety $\mathcal{M}_{PVI}$. Lisovyy and Tykhyy's classification is folded up to the action of the group G_{PVI} of symmetries acting on $\mathcal{M}_{PVI}$. The action of the group G_{PVI} will be described in Section 3.2 and we will show that also if G_{PVI} is isomorphic to the affine Weyl group of type F_4 (an infinite group), the action of G_{PVI} over the co-adjoint coordinates of an element in $\mathcal{M}_{PVI}$ is finite, see Lemmata 38, 44, 45 and 50.

In Section 3.3, we give explicitly the Lisovyy and Tykhyy's list of $5+45$ orbits. In this list there exists 5 infinite sublists of finite orbits (corresponding to families of parametric solutions of PVI) and one finite sublist of 45 finite orbits, see Table 3.4. It is precisely the latter finite sublist that will be crucial to our method in order to succeed. Since the action of G_{PVI} over the 45 finite orbits is finite, in Section 3.3.1 we define an "expansion algorithm" that given the "folded list" of 45 orbits, it produces the "unfolded" list of all "equivalent" finite orbits under the action of G_{PVI}.

3.1 PVI: Braid group and analytic continuation

The Sixth Painlevé equation is the isomonodromic deformations equation for the Fuchsian system:

$$\frac{d\Psi}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_u}{z-u}\right)\Psi, \; z \in \mathbb{C}\backslash\{0,u,1\}, \tag{3.2}$$

where the singular points $0, u, 1, \infty$ are simple poles. The matrices A_∞ is defined as:

$$A_\infty := -A_0 - A_1 - A_u = \frac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \; \theta_\infty \in \mathbb{C}^*,$$

and the eigenvalues of A_i are $\frac{\pm\theta_i}{2}$ with $\theta_i \in \mathbb{C}$ for $i = 0, 1, u$. The θ_i are the parameters appearing in the PVI equation (2.6). A solution $\Psi(z)$ of (3.2) is a multivalued analytic function on the Riemann sphere with four punctures: $\mathbb{C}\backslash\{0,u,1\}$. Consider loops γ_0,γ_u,γ_1 and γ_∞ such that they encircle $0, u, 1, \infty$, then we denote the associated monodromy matrices as N_1, N_2, N_3 and N_∞, with N_i in $\mathrm{SL}_2(\mathbb{C})$ and such that:

$$\mathrm{Tr}\, N_i = 2\cos\pi\theta_i, \; i = 1, 2, 3, \infty. \tag{3.3}$$

The $\mathrm{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components is identified with the two dimensional quotient space:

$$\widehat{\mathcal{M}}_{PVI} := \widehat{\mathcal{M}}_{\mathcal{G}_1} = \{(N_1, N_2, N_3) \mid N_i \in \mathrm{SL}_2(\mathbb{C}), \; N_\infty N_3 N_2 N_1 = \mathbb{1}\} / \sim, \tag{3.4}$$

where $\sim$ is simultaneous conjugation of a triple by diagonal matrix.

Every element in $\widehat{\mathcal{M}}_{PVI}$ can be identified with a 7-tuple of complex values:

$$q := (q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32}) \in \mathbb{C}^7, \tag{3.5}$$

where:

$$\begin{aligned} q_i &= \operatorname{Tr} N_i, \ \ i = 1, 2, 3, \infty, \\ q_{ij} &= \operatorname{Tr} N_i N_j, \ \ i, j = 1, 2, 3, \ \ i > j. \end{aligned} \tag{3.6}$$

Moreover, the quantities in (3.5) satisfy the Jimbo-Fricke cubic [Jim82, Boa05]:

$$q_{32}q_{31}q_{21} + q_{32}^2 + q_{31}^2 + q_{21}^2 - \omega_1 q_{32} - \omega_2 q_{31} - \omega_3 q_{21} + \omega_4 - 4 = 0, \tag{3.7}$$

where:

$$\begin{aligned} \omega_1 &:= q_1 q_\infty + q_3 q_2, \\ \omega_2 &:= q_2 q_\infty + q_3 q_1, \\ \omega_3 &:= q_3 q_\infty + q_2 q_1, \\ \omega_4 &:= q_3^2 + q_2^2 + q_1^2 + q_\infty^2 + q_3 q_2 q_1 q_\infty. \end{aligned} \tag{3.8}$$

It was proven by Iwasaki [Iwa03] that, if q_1, q_2, q_3, q_∞ are treated as parameters, then (q_{21}, q_{31}, q_{32}) is a system of coordinates on a big open subset $\mathcal{S} \subset \widehat{\mathcal{M}}_{PVI}$.

By (2.29), explicit formulae for the action of the full braid group B_3 on the monodromy matrices N_i for $i = 1, \dots 4$ are:

$$\sigma_1^{(\mathcal{PVI})} : (N_1, N_2, N_3) \mapsto (N_2, N_2^{-1} N_1 N_2, N_3),$$
$$\sigma_2^{(\mathcal{PVI})} : (N_1, N_2, N_3) \mapsto (N_1, N_3, N_3^{-1} N_2 N_3). \tag{3.9}$$

Moreover action (3.9) can be restated in terms of co-adjoint coordinates (3.6):

$$\sigma_1^{(PVI)} : (q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32}) \mapsto (q_2, q_1, q_3, q_\infty, q_{21}, q_{32}, \omega_2 - q_{31} - q_{32} q_{21}),$$
$$\sigma_2^{(PVI)} : (q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32}) \mapsto (q_1, q_3, q_2, q_\infty, q_{31}, \omega_3 - q_{21} - q_{31} q_{32}, q_{32}). \tag{3.10}$$

By (1.83), we can define the generators of the pure braid group P_3 as follows:

$$P_3 =< \beta_{21}^{(PVI)}, \beta_{31}^{(PVI)}, \beta_{32}^{(PVI)} >, \tag{3.11}$$

where:

$$\beta_{21}^{(PVI)} := (\sigma_1^{(PVI)})^2,$$
$$\beta_{31}^{(PVI)} := (\sigma_2^{(PVI)})^{-1} (\sigma_1^{(PVI)})^2 \sigma_2^{(PVI)},$$
$$\beta_{32}^{(PVI)} := (\sigma_2^{(PVI)})^2. \tag{3.12}$$

Note that sub-indices ij in the generator $\beta_{ij}^{(PVI)}$ determine the q_{ij} that is actually fixed during the action of the generator.

Before proceeding, we reformulate action (3.12) in a slightly different way. Given q satisfying the cubic relation (3.7), we can define $(\underline{q}, \underline{\omega})$ as follows:

$$(\underline{q}, \underline{\omega}) := (q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4), \tag{3.13}$$

where ω_i for $i = 1, \dots 4$ are defined in (3.8). Hence, action (3.12) on q is equivalent to the following action on $(\underline{q}, \underline{\omega})$:

$$\begin{aligned}
\beta_{21}^{(PVI)}(\underline{q}, \underline{\omega}) &:= (q_{21}, \omega_2 - q_{31} - q_{21}q_{32}, \omega_1 - q_{32} - q_{21}(\omega_2 - q_{31} - q_{21}q_{32}), \underline{\omega}),\\
\beta_{31}^{(PVI)}(\underline{q}, \underline{\omega}) &:= (\omega_3 - q_{21} - q_{31}(\omega_1 - q_{21}q_{31} - q_{32}), q_{31}, \omega_1 - q_{21}q_{31} - q_{32}, \underline{\omega}),\\
\beta_{32}^{(PVI)}(\underline{q}, \underline{\omega}) &:= (\omega_3 - q_{21} - q_{31}q_{32}, \omega_2 - q_{31} - q_{32}(\omega_3 - q_{21} - q_{31}q_{32}), q_{32}, \underline{\omega}).
\end{aligned} \tag{3.14}$$

This permits us to identify an orbit with a couple $(\underline{q}, \underline{\omega})$.

The following Lemma, that can be found in [DM00], describes, in a geometric manner, the action of the pure braid group (3.11) and a necessary condition for this action in order to be finite:

Lemma 36. Suppose $q \in \widehat{\mathcal{M}}_{PVI}$ generates a finite P_3-orbit, then only two possibilities arise:

(i) Or q satisfies:

$$q_{ij} = 2\cos \pi r_{ij},\ r_{ij} \in \mathbb{Q},\ 0 \leqslant r_{ij} \leqslant 1,\ i, j = 1, 2, 3,\ i > j. \tag{3.15}$$

(ii) Or there exists a pure braid $\beta_{ij}^{(PVI)}$, for some choice of indices $i, j = 1, 2, 3$, $i > j$, such that $\beta_{ij}^{(PVI)}(q) = q$. Then q_{ij} is a complex parameter satisfying:

$$q_{ki} = \frac{(2\omega_j - \omega_i q_{ij})}{(4 - q_{ij}^2)},\ q_{kj} = \frac{(2\omega_i - \omega_j q_{ij})}{(4 - q_{ij}^2)},\ i, j, k = 1, 2, 3,\ k > i > j. \tag{3.16}$$

Proof. Without loss of generality, we prove the Lemma for $i = 2$ and $j = 1$. The proof in the other cases can be obtained in a similar way.

Consider the generator $\beta_{21}^{(PVI)}$ defined in (3.14), it fixes the coordinate q_{21} and it acts as a linear transformation on the variables (q_{31}, q_{32}). The cubic relation (3.7) is a conic in (q_{31}, q_{32}):

$$q_{31}^2 + q_{32}^2 + q_{21}(q_{31}q_{32}) - \omega_1 q_{32} - \omega_2 q_{31} - (\omega_3 q_{21} - \omega_4 + 4) = 0, \qquad (3.17)$$

that is invariant under the action of $\beta_{21}^{(PVI)}$.

If $q_{21} = \pm 2$, then $r_{21} = 0$ or $r_{21} = 1$ and the statement follows. Hereafter suppose $q_{21} \neq \pm 2$:

(i) The linear action of $\beta_{21}^{(PVI)}$ on (q_{31}, q_{32}) describe a rotation R of (q_{31}, q_{32}) on the conic (3.17). If θ is the angle of the rotation R, then $q_{21} = 2\cos\theta$. Moreover if θ is a rational multiple of π, then:

$$\exists\ n \in \mathbb{N} \text{ s.t. } \mathrm{R}^n = \mathrm{Id}.$$

The action of $\beta_{21}^{(PVI)}$ produce a finite orbit on (q_{31}, q_{32}) if and only if $q_{21} = 2\cos\theta$ where θ is a rational multiple of π.

(ii) Suppose q is a fixed point of the braid $\beta_{21}^{(PVI)}$, i.e. $\beta_{21}^{(PVI)}(q) = q$, then:

$$q_{31} = \frac{(2\omega_2 - \omega_1 q_{21})}{(4 - q_{21}^2)},$$
$$q_{32} = \frac{(2\omega_1 - \omega_2 q_{21})}{(4 - q_{21}^2)}.$$

and q_{21} is a complex parameter.

This concludes the proof. □

As a consequence of the previous Lemma, the classification of all finite orbits of the P_3-action on $\mathcal{M}_{PVI}$ reduces to the classification of triples of rational angles πr_{ij} or fixed points. This classification has been achieved by Lisovyy and Tykhyy in [LT14] modulo action of the symmetry group G_{PVI} that we will describe in the next section.

3.2 PVI: symmetry group G_{PVI}

In this Section, we introduce the group of symmetries G_{PVI} acting on the affine algebraic variety $\widehat{\mathcal{M}}_{PVI}$. An element of G_{PVI} is defined as follows:

Definition 37. A *symmetry* is an invertible map $\Phi : \widehat{\mathcal{M}}_{PVI} \mapsto \widehat{\mathcal{M}}_{PVI}$ such that given an element q and its orbit $\mathcal{O}_{P_3}(q)$, the following is true:

$$|\mathcal{O}_{P_3}(\Phi(q))| = |\mathcal{O}_{P_3}(q)|. \tag{3.18}$$

As a consequence, if q generates a finite orbit of length N, then $\Phi(q)$ generates a finite orbit of the same length. We say that the orbits generated by q and $\Phi(q)$ are equivalent. As for the Garnier system $\mathcal{G}_2$, we firstly study the group of Bäcklund transformations for PVI, the Sixth Painlevé equation, and subsequently we extend this group with more simpler transformations acting on the space of monodromy matrices, in order to obtain the group of symmetries G_{PVI}.

3.2.1 Okamoto transformations of PVI

In this Section we study symmetries of $\widehat{\mathcal{M}}_{PVI}$ that are derived from the so called group of Bäcklund transformations for PVI. Even if this group is isomorphic to the extended affine Weyl group of type F_4, see Okamoto [Oka86], that is an infinite group, we show that the action of the extended affine group F_4 is finite when we express it in terms of co-adjoint coordinates

q.

Bäcklund transformations are birational maps sending a solution $\nu(u)$ of PVI with a fixed set of parameters $(\theta_1, \theta_2, \theta_3, \theta_\infty)$ to a new solution $\nu'(u')$ of PVI with a new set of parameters $(\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\theta}_3, \widetilde{\theta}_\infty)$. Moreover, since these maps are birational, they send algebraic solutions to algebraic solutions, preserving the number of branches. In particular, these transformations act also on the affine algebraic variety $\widehat{\mathcal{M}}_{\mathcal{G}_2}$, sending finite orbits to finite orbits and preserving their length. Two orbits related by such transformations are said to be *equivalent*.

Okamoto [Oka86] showed that the group of birational canonical transformations of the Hamiltonian system associated to the PVI equation, can be identified by an isomorphism with the extended affine Weyl group of type F_4. Usually when the context involves the Sixth Painlevé equation, Bäcklund transformations are referred as Okamoto transformations.

Generators for Bäcklund transformations for PVI are listed in Table 3.1. The first five transformations $s_1, s_2, s_3, s_\infty, s_\delta$ generate a group isomorphic to the affine Weyl group of type D_4, while transformations r_1, r_2, r_3 generate the Klein four-group K_4 and permutations P_{13}, P_{23} generate the symmetric group S_3. Enlarging the set of generators of affine group D_4 by the generators of K_4 and S_3 the extended affine Weyl group of type F_4 is obtained.

We now describe the action of the extended affine Weyl group of type F_4 on the quotient space $\widetilde{\mathcal{M}}_{PVI}$ endowed with the co-adjoint coordinates q introduced in Section 3.1. If the action of the pure braid group P_3 over $(\underline{q}, \underline{\omega})$, defined in (3.13) is finite, then the action of the extended affine Weyl group of type F_4 over $(\underline{q}, \underline{\omega})$ is finite too.

Among all transformations in Table 3.1, we need to be particularly careful on how we treat transformation s_δ, since the parameter δ can be modified

	θ_1	θ_2	θ_3	θ_∞	ν	ρ	u
s_1	$-\theta_1$	θ_2	θ_3	θ_∞	ν	$\rho-\frac{\theta_1}{\nu}$	u
s_2	θ_1	$-\theta_2$	θ_3	θ_∞	ν	$\rho-\frac{\theta_2}{\nu-u}$	u
s_3	θ_1	θ_2	$-\theta_3$	θ_∞	ν	$\rho-\frac{\theta_3}{\nu-1}$	u
s_∞	θ_1	θ_2	θ_3	$-\theta_\infty$	ν	ρ	u
s_δ	$\theta_1-\delta$	$\theta_2-\delta$	$\theta_3-\delta$	$\theta_\infty-\delta$	$\nu+\frac{\delta}{\rho}$	ρ	u
r_1	$\theta_\infty-1$	θ_3	θ_2	θ_1+1	$\frac{u}{\nu}$	$\frac{-\nu(\rho\nu+\rho)}{u}$	u
r_2	θ_3	$\theta_\infty-1$	θ_1	θ_2+1	$\frac{\nu-u}{\nu-1}$	$\frac{(\nu-1)((\nu-1)\rho+\rho)}{u-1}$	u
r_3	θ_2	θ_1	$\theta_\infty-1$	θ_3+1	$\frac{u(\nu-1)}{\nu-u}$	$\frac{-(\nu-u)((\nu-u)\rho+\rho)}{u(u-1)}$	u
P_{13}	θ_3	θ_2	θ_1	θ_∞	$1-\nu$	$-\rho$	$1-u$
P_{23}	θ_1	θ_3	θ_2	θ_∞	$\frac{\nu}{u}$	$u\rho$	$\frac{1}{u}$

Table 3.1: Okamoto bi-rational transformations for Painlevé VI, where: $\delta=\frac{\theta_1+\theta_2+\theta_3+\theta_\infty}{2}$.

by other generators in Table 3.1. Indeed consider a transformation Φ in the extended affine group F_4 then:

$$\Phi:(\theta_1,\dots,\theta_\infty)\longmapsto(\Phi(\theta_1),\dots,\Phi(\theta_\infty))=(\theta_1',\dots,\theta_\infty')$$

then $s_\delta = s_{\delta'}$ where:

$$\delta'=\frac{\theta_1'+\theta_2'+\theta_3'+\theta_\infty'}{2}.$$

We want to develop a strategy in order to deal in an easy way with s_δ. The following Lemma appears in the works of Terajima [Ter03] and Inaba, Iwasaki and Sato [IIS04], in particular we propose it here as in [IIS04]:

Lemma 38. The quantities $(\underline{q},\underline{\omega})$, defined in (3.13), are invariants under the action of the transformations s_i for $i=1,\dots,\infty,\delta$.

As direct consequence of Lemma 38, the affine group D_4 (in particular transformation s_δ) acts trivially on $(\underline{q},\underline{\omega})$. Now we study the action of the remaining generators in Table 3.1 on $(\underline{q},\underline{\omega})$:

Theorem 39. The generators of Bäcklund transformations listed in Table 3.1 act on (3.13) as follows:

$$\begin{aligned}
&s_i(\underline{q},\underline{\omega}) \mapsto (q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4),\ i = 1, 2, 3, \infty, \delta, \\
&r_1(\underline{q},\underline{\omega}) \mapsto (-q_{21}, -q_{31}, q_{32}, \omega_1, -\omega_2, -\omega_3, \omega_4), \\
&r_2(\underline{q},\underline{\omega}) \mapsto (-q_{21}, q_{31}, -q_{32}, -\omega_1, \omega_2, -\omega_3, \omega_4), \\
&r_3(\underline{q},\underline{\omega}) \mapsto (q_{21}, -q_{31}, -q_{32}, -\omega_1, -\omega_2, \omega_3, \omega_4), \\
&P_{13}(\underline{q},\underline{\omega}) \mapsto (q_{32}, \omega_2 - q_{31} - q_{21}q_{32}, q_{21}, \omega_3, \omega_2, \omega_1, \omega_4), \\
&P_{23}(\underline{q},\underline{\omega}) \mapsto (\omega_2 - q_{31} - q_{21}q_{32}, q_{21}, q_{32}, \omega_1, \omega_3, \omega_2, \omega_4)\ .
\end{aligned} \tag{3.19}$$

Proof. The proof for transformations s_1, s_2, s_3, s_∞ and s_δ is a direct consequence of Lemma 38. We proceed with the proof for transformations r_1, r_2, r_3 and P_{13}, P_{23}. In particular we prove in details the statement for r_1 and P_{13}, then for the remaining transformations the proof proceeds in a similar way.

Suppose $\nu(u)$ is a solution to the PVI equation (2.6). Transformation r_1 leaves the independent variable u unchanged but not the dependent variable:

$$\nu(u) \mapsto \tilde{\nu} = \frac{u}{\nu(u)}.$$

By Theorems 2.2 - 2.2' - 2.2" in [DM00], in the sectors $\Sigma_0, \Sigma_1, \Sigma_\infty$ of neighbourhoods of the singular points 0, 1 and ∞ respectively, is as follows:

$$\nu(u) \sim \begin{cases} a_0 u^{1-\sigma_0} + \dots, & \text{for } u \to 0, u \in \Sigma_0, \\ 1 - a_1(1-u)^{1-\sigma_1} + \dots, & \text{for } u \to 1, u \in \Sigma_1, \\ a_\infty u^{\sigma_\infty} + \dots, & \text{for } u \to \infty, u \in \Sigma_\infty, \end{cases} \tag{3.20}$$

The asymptotic behaviour (3.20) can be used in order to determine the action of r_1 over the quantities q_{ij}. Indeed the q_{ij} are related to the exponents of the leading terms in the asymptotic behaviour (3.20) by the identities:

$$q_{21} = 2\cos\pi\sigma_0,\ 0 \leqslant \sigma_0 < 1, \tag{3.21}$$

$$q_{32} = 2\cos\pi\sigma_1,\ 0 \leqslant \sigma_1 < 1, \tag{3.22}$$

$$q_{31} = 2\cos\pi\sigma_\infty,\ 0 \leqslant \sigma_\infty < 1. \tag{3.23}$$

We compute now the asymptotic expansion for $\tilde{\nu}$:

$$\tilde{\nu}(u) = \frac{u}{\nu(u)} \sim \begin{cases} \frac{1}{a_0} u^{\sigma_0} + \dots, & \text{for } u \to 0, u \in \Sigma_0, \\ \frac{u}{1-a_1(1-u)^{(1-\sigma_1)}} + \dots, & \text{for } u \to 1, u \in \Sigma_1, \\ \frac{1}{a_\infty} u^{1-\sigma_\infty} + \dots, & \text{for } u \to \infty, u \in \Sigma_\infty. \end{cases} \tag{3.24}$$

By uniqueness of the asymptotic behaviour and the fact that the independent variable u is invariant under r_1, we can compare the exponents of the leading terms in (3.20) and (3.24), obtaining $\tilde{\sigma_0} = 1 - \sigma_0$, $\tilde{\sigma_1} = \sigma_1$ and $\tilde{\sigma_\infty} = 1 - \sigma_\infty$.

Finally, using equations (3.21)-(3.22), we obtain a change of sign in q_{21} and q_{31}. Moreover the action of r_1 on quantities ω_i for $i = 1, 2, 3, 4$, defined in (3.8), can be directly calculated by the action of r_1 on the θ_i as listed in Table 3.1 and relations $q_i = 2\cos(\pi\theta_i)$ for $i = 1, 2, 3, \infty$.

Consider now transformation P_{13}. It acts not only on $\nu(u)$ but also on u:

$$\nu(u) \mapsto \tilde{\nu}(u) = 1 - \nu(u),$$
$$u \mapsto \tilde{u} = 1 - u.$$

The action of this transformation can no longer be calculated using the asymptotic behaviours of $\nu(u)$ near the singular points, indeed the asymptotics of $\nu(u)$ are defined locally in proper sectors Σ_i for $i = 0, 1, \infty$ but the transformation P_{13} is acting globally by the change of the temporal variable u. Following the approach of Guzzetti in [Guz08], consider the Fuchsian system associated to $\nu(u)$:

$$\frac{d\Psi}{dz} = \left[\frac{A_1}{z - u_1} + \frac{A_2}{z - u_2} + \frac{A_3}{z - u_3} \right] \Psi, \tag{3.25}$$

with $u_1 = 0, u_2 = u, u_3 = 1$ and the Fuchsian system associated to $\tilde{\nu}(\tilde{u})$ obtained applying transformation P_{13}:

$$\frac{d\tilde{\Psi}}{d\tilde{z}} = \left[\frac{\tilde{A}_1}{\tilde{z} - \tilde{u}_1} + \frac{\tilde{A}_2}{\tilde{z} - \tilde{u}_2} + \frac{\tilde{A}_3}{\tilde{z} - \tilde{u}_3} \right] \tilde{\Psi}. \tag{3.26}$$

The two systems are related by a diagonal gauge transformation and the exchange of points $u_1 = 0$ and $u_3 = 1$. This exchange generates a new basis of loops $\tilde{\gamma}_1$,$\tilde{\gamma}_2$,$\tilde{\gamma}_3$ in the fundamental group of the Riemann sphere with 4 boundary components. Since monodromy preserving deformations are considered, the basis $\tilde{\gamma}_1$,$\tilde{\gamma}_2$,$\tilde{\gamma}_3$ can be written in terms of the original basis $\gamma_1, \gamma_2, \gamma_3$. As a consequence, setting (N_1, N_2, N_3) and $(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$ to be respectively the monodromy matrices associated to a fundamental solution of (3.25) and (3.26), then they satisfy:

$$\tilde{N}_1 = N_3, \tag{3.27}$$

$$\tilde{N}_2 = N_3 N_2 N_3^{-1}, \tag{3.28}$$

$$\tilde{N}_3 = N_3 N_2 N_1 N_2^{-1} N_3^{-1}, \tag{3.29}$$

and:

$$\begin{aligned}
\tilde{q}_1 &= \operatorname{Tr}\tilde{N}_1 = \operatorname{Tr} N_3 = q_3,\\
\tilde{q}_2 &= \operatorname{Tr}\tilde{N}_2 = \operatorname{Tr} N_3N_2N_3^{-1} = q_2,\\
\tilde{q}_3 &= \operatorname{Tr}\tilde{N}_3 = \operatorname{Tr} N_3N_2N_1N_2^{-1}N_3^{-1} = q_1,\\
\tilde{q}_\infty &= \operatorname{Tr}\tilde{N}_3\tilde{N}_2\tilde{N}_1 = \operatorname{Tr} N_3N_2N_1 = q_\infty,\\
\tilde{q}_{21} &= \operatorname{Tr}\tilde{N}_2\tilde{N}_1 = \operatorname{Tr} N_3N_2 = q_{32},\\
\tilde{q}_{31} &= \operatorname{Tr}\tilde{N}_3\tilde{N}_1 = \operatorname{Tr} N_3N_2N_1N_2^{-1} = \omega_2 - q_{31} - q_{21}q_{32},\\
\tilde{q}_{32} &= \operatorname{Tr}\tilde{N}_3\tilde{N}_2 = \operatorname{Tr} N_3N_2N_1N_3^{-1} = q_{21}.
\end{aligned}$$

and the action of P_{13} is obtained. □

Corollary 40. The group:

$$\langle s_1, s_2, s_3, s_\infty, s_\delta, r_1, r_2, r_3, P_{13}, P_{23}\rangle, \tag{3.30}$$

is a group of symmetries for $\widehat{\mathcal{M}}_{PVI}$.

Proof. The statement follows by construction. □

We want now to show that given $(\underline{q}, \underline{\omega})$ such that its orbit is finite under the action of P_3, then the set of all possible $(\underline{q}, \underline{\omega})$ obtained by acting on all points in the orbit with the group of transformations generated by $\langle r_1, r_2, r_3, P_{13}, P_{23}\rangle$ is finite too. Before we need few Lemmata, the first one is about the finiteness of group $\langle r_1, r_2, r_3\rangle$:

Lemma 41. Transformations r_i for $i = 1, 2, 3$ generate the Klein four group K_4.

Proof. We prove that $\langle r_1, r_2, r_3\rangle \simeq K_4$. If a, b, c are the generators of the group K_4, then it has presentation:

$$a^2 = b^2 = c^2 = abc = 1 \tag{3.31}$$

Consider $a = r_1$,$b = r_2$,$c = r_3$ then (3.31) is satisfied and the isomorphism follows. □

By Lemma 41, since K_4 is finite, its action on a finite set produces again a finite set. Now we show that P_{13} and P_{23} act as braids:

Lemma 42. Transformations P_{13} and P_{23} are such that:

$$\begin{aligned} P_{13}(\underline{q}, \underline{\omega}) &= \sigma_1^{(PVI)} \sigma_2^{(PVI)} \sigma_1^{(PVI)} (\underline{q}, \underline{\omega}), \\ P_{23}(\underline{q}, \underline{\omega}) &= \sigma_2^{(PVI)\ -1} (\underline{q}, \underline{\omega})\ . \end{aligned} \tag{3.32}$$

Proof. Given the definition (3.10) of the generators $\sigma_i^{(PVI)}$ for $i = 1, 2$ of the full braid group B_3 and the action of P_{13},P_{23} given in (3.19), then identities (3.32) follow. □

Finally, given $(\underline{q}, \underline{\omega})$ generating a finite orbit under the action of the braid group, the next Lemma ensures that $(\underline{q}, \underline{\omega})$ has finite orbit also under the action of the group $\langle r_1, r_2, r_3, P_{13}, P_{23} \rangle$. However, before we proceed with the next Lemma, we need some consideration about the classification of Lisovyy and Tykhyy [LT14]:

Remark 43. Note that in [LT14], if an orbit is finite under the action of the pure braid group P_3, then it is finite under the action of the full braid group B_3. Indeed in [LT14] all orbits are classified under the action of the three generators x, y, z (we keep for the moment the same notations as in [LT14]) of the pure braid group P_3 (see their definition in (7) of [LT14]).

If we consider the transformations (12), (123) (defined in Section 2.2 of [LT14]) and their compositions, we obtain the following identities:

$$x = (12)\beta_x(123)^2, \tag{3.33}$$

$$y = (12)\beta_z, \tag{3.34}$$

$$z = (123)(12)(123)\beta_x. \tag{3.35}$$

We can now state the Lemma:

Lemma 44. If $(\underline{q}, \underline{\omega})$ generates a finite orbit under the action of full braid group B_3 and N is the length of the orbit, then $(\underline{q}, \underline{\omega})$ generates a finite orbit under the action of the group:

$$\mathrm{G}^{(1)}_{PVI} := \langle r_1, r_2, r_3, P_{13}, P_{23}\rangle, \tag{3.36}$$

and the orbit has at most $4N$ elements.

Proof. In order to prove the statement, we firstly prove the following relations:

$$\begin{aligned}
P_{13}r_1 &= r_3P_{13},\\
P_{13}r_2 &= r_2P_{13},\\
P_{13}r_3 &= r_1P_{13},\\
P_{23}r_1 &= r_1P_{23},\\
P_{23}r_2 &= r_3P_{23},\\
P_{23}r_3 &= r_2P_{23}.
\end{aligned} \tag{3.37}$$

Thanks to (3.37), we are allowed to split the action of the whole group $\langle r_1, r_2, r_3, P_{13}, P_{23}\rangle$, into the two separate actions of groups $\langle P_{13}, P_{23}\rangle$ and

$\langle r_1, r_2, r_3\rangle$. Since $< P_{13}, P_{23} >\subset B_3$, then:

$$|\mathcal{O}_{\langle P_{13},P_{23}\rangle}(\underline{q},\underline{\omega})| = N, \ \ N \in \mathbb{N}. \tag{3.38}$$

Moreover, by Lemma 41:

$$|\mathcal{O}_{\langle r_1,r_2,r_3\rangle}(\underline{q},\underline{\omega})| = 4, \tag{3.39}$$

and relations (3.37), if we act on each element in $\mathcal{O}_{\langle P_{13},P_{23}\rangle}(\underline{q},\underline{\omega})$ with $< r_1, r_2, r_3 >$ we obtain:

$$|\mathcal{O}_{\langle r_1,r_2,r_3,P_{13},P_{23}\rangle}(\underline{q},\underline{\omega})| \leqslant 4N. \tag{3.40}$$

This completes the proof. □

By Lemmata 38 and 44 the action of extended affine group F_4 in terms of $(\underline{q},\underline{\omega})$ reduces to the action of group $\langle r_1, r_2, r_3, P_{13}, P_{23}\rangle$ and if $|\mathcal{O}_{B_3}(\underline{q},\underline{\omega})| < \infty$ then also the orbit generated acting on $(\underline{q},\underline{\omega})$ with $\langle r_1, r_2, r_3, P_{13}, P_{23}\rangle$ will be finite.

We focus in the next part on how we can calculate the action of affine group D_4 generated by s_i for $i = 1, \ldots, \delta$ over $(q_1, q_2, q_3, q_\infty)$. As said at the beginning of the Section, in general transformations of affine group D_4 don't act trivially on (3.5) because of the particular nature of s_δ. Anyhow, suppose we know $(\underline{q},\underline{\omega})$, then $(q_1, q_2, q_3, q_\infty)$ must be a solution of (3.8) for the given $\underline{\omega}$. Moreover, due to invariance of $(\underline{q},\underline{\omega})$ under the action of transformations s_i for $i = 1, \ldots, \delta$, we expect that solutions $(q_1, q_2, q_3, q_\infty)$ could be also related by the lift of some transformations in the affine D_4 group in Table 3.1, to co-adjoint coordinates q. This observation is formalized in Proposition 10 in the work of Lisovyy and Tykhyy [LT14] where transformations of the affine group D_4 linking solutions $(q_1, q_2, q_3, q_\infty)$ of (3.8) are explicitly given by the authors. Following Lemma recalls Proposition 10 in [LT14]:

Lemma 45. Suppose $\omega_1, \omega_2, \omega_3, \omega_4$ are given and consider system (3.8) in the variables q_1, q_2, q_3, q_∞, then this system could have at most 24 solutions and any two solutions are related by transformations on the θ_i for $i = 1, \ldots, \infty$ of the affine group D_4. The 24 transformations are:

$$id, \tag{3.41}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2, \tag{3.42}$$

$$s_\delta s_1 s_2 s_\delta s_3 s_\infty, \tag{3.43}$$

$$s_\delta s_1 s_3 s_\delta s_2 s_\infty, \tag{3.44}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_2 s_\delta s_3 s_\infty), \tag{3.45}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_3 s_\delta s_2 s_\infty), \tag{3.46}$$

$$(s_\delta s_1 s_2 s_\delta s_3 s_\infty)(s_\delta s_1 s_3 s_\delta s_2 s_\infty), \tag{3.47}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_2 s_\delta s_3 s_\infty)(s_\delta s_1 s_3 s_\delta s_2 s_\infty), \tag{3.48}$$

$$s_\delta \tag{3.49}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 s_\delta, \tag{3.50}$$

$$s_\delta s_1 s_2 s_\delta s_3 s_\infty s_\delta, \tag{3.51}$$

$$s_\delta s_1 s_3 s_\delta s_2 s_\infty s_\delta, \tag{3.52}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_2 s_\delta s_3 s_\infty) s_\delta, \tag{3.53}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_3 s_\delta s_2 s_\infty) s_\delta, \tag{3.54}$$

$$(s_\delta s_1 s_2 s_\delta s_3 s_\infty)(s_\delta s_1 s_3 s_\delta s_2 s_\infty) s_\delta, \tag{3.55}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_2 s_\delta s_3 s_\infty)(s_\delta s_1 s_3 s_\delta s_2 s_\infty) s_\delta, \tag{3.56}$$

$$s_\delta s_1, \tag{3.57}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 s_\delta s_1, \tag{3.58}$$

$$s_\delta s_1 s_2 s_\delta s_3 s_\infty s_\delta s_1, \tag{3.59}$$

$$s_\delta s_1 s_3 s_\delta s_2 s_\infty s_\delta s_1, \tag{3.60}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_2 s_\delta s_3 s_\infty) s_\delta s_1, \tag{3.61}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_3 s_\delta s_2 s_\infty) s_\delta s_1, \tag{3.62}$$

$$(s_\delta s_1 s_2 s_\delta s_3 s_\infty)(s_\delta s_1 s_3 s_\delta s_2 s_\infty) s_\delta s_1, \tag{3.63}$$

$$(s_1 s_2 s_3 s_\infty s_\delta)^2 (s_\delta s_1 s_2 s_\delta s_3 s_\infty)(s_\delta s_1 s_3 s_\delta s_2 s_\infty) s_\delta s_1. \tag{3.64}$$

3.2.2 Symmetries of monodromy matrices

In the following we remind the reader about other trivial symmetries on the space of monodromy matrices:

(i) Independent sign changes $\epsilon_i = \pm 1$ of the matrices N_i for $i = 1, 2, 3$, due to Schlessinger transformations on the Fuchsian system (3.2) studied by Jimbo and Miwa in [JM81]:

$$(N_1, N_2, N_3, N_\infty) \longmapsto (\epsilon_1 N_1, \epsilon_2 N_2, \epsilon_3 N_3, \epsilon_1 \epsilon_2 \epsilon_3 (N_3 N_2 N_1)^{-1}).$$

(ii) Permutations of the matrices N_i for $i = 1, 2, 3$:

$$(N_1, N_2, N_3, N_\infty) \longmapsto (N_{\xi(1)}^{-1}, N_{\xi(2)}^{-1}, N_{\xi(3)}^{-1}, (N_{\xi(3)}^{-1} N_{\xi(2)}^{-1} N_{\xi(1)}^{-1})^{-1}),$$

where ξ is any permutation in S_3, the symmetric group over 3 elements.

Given $n = (N_1, N_2, N_3) \in \widehat{\mathcal{M}}_{PVI}$, we call the transformations that change sign of the matrices N_i *sign flips* and they are defined as:

$$\mathrm{sign}_{(\epsilon_1,\epsilon_2,\epsilon_3)} : (N_1, N_2, N_3, N_\infty) \mapsto (\epsilon_1 N_1, \epsilon_2 N_2, \epsilon_3 N_3, \epsilon_1\epsilon_2\epsilon_3 N_\infty),$$

where $\epsilon_k = \pm 1$ and we included the action on the monodromy matrix N_∞ as well. The following three sign flips generate all the others:

$$\mathrm{sign}_1 := \mathrm{sign}_{(-1,1,1)} : (N_1, N_2, N_3, N_\infty) \mapsto (-N_1, N_2, N_3, -N_\infty), \tag{3.65}$$

$$\mathrm{sign}_2 := \mathrm{sign}_{(1,-1,1)} : (N_1, N_2, N_3, N_\infty) \mapsto (N_1, -N_2, N_3, -N_\infty), \tag{3.66}$$

$$\mathrm{sign}_3 := \mathrm{sign}_{(1,1,-1)} : (N_1, N_2, N_3, N_\infty) \mapsto (N_1, N_2, -N_3, -N_\infty), \tag{3.67}$$

and they satisfy:

$$\begin{aligned}
&\mathrm{sign}_1^2 = \mathrm{sign}_2^2 = \mathrm{sign}_3^2 = 1,\\
&\mathrm{sign}_1\mathrm{sign}_2 = \mathrm{sign}_2\mathrm{sign}_1,\\
&\mathrm{sign}_1\mathrm{sign}_3 = \mathrm{sign}_3\mathrm{sign}_1,\\
&\mathrm{sign}_2\mathrm{sign}_3 = \mathrm{sign}_3\mathrm{sign}_2,
\end{aligned} \tag{3.68}$$

as a consequence, the group of sign flips is finite and it is isomorphic to the group $C_2 \times C_2 \times C_2$, where C_2 is the cyclic group of order 2.

We need following Lemma in order to prove that sign flips are symmetries of $\widehat{\mathcal{M}}_{PVI}$:

Lemma 46. Given sign $\in \langle \mathrm{sign}_1, \mathrm{sign}_2, \mathrm{sign}_3 \rangle$ and a braid $\sigma \in B_3$, then there exists $\mathrm{sign}' \in \langle \mathrm{sign}_1, \mathrm{sign}_2, \mathrm{sign}_3 \rangle$ and $\sigma' \in B_3$ such that:

$$\sigma \mathrm{sign} = \mathrm{sign}' \sigma'.$$

Proof. Given $n = (N_1, N_2, N_3) \in \widehat{\mathcal{M}}_{PVI}$ we prove the result on the generators σ_i for $i = 1, 2$ of the full braid group B_3, i.e. we show that:

$$\sigma_i \text{sign}_j = \text{sign}_{j'} \sigma_{i'},$$

for some choice of the indices $i, i' = 1, 2$ and $j, j' = 1, 2, 3$. Suppose we consider σ_1 and sign_1, then:

$$\begin{aligned} \sigma_1 \text{sign}_1(n) &= \sigma_1(-N_1, N_2, N_3) = (N_2, -N_2 N_1 N_2^{-1}, N_3) = \\ &= \text{sign}_2(N_2, N_2 N_1 N_2^{-1}, N_3) = \text{sign}_2 \sigma_1(n). \end{aligned} \tag{3.69}$$

In a similar way we can prove all the following equations:

$$\begin{aligned} \sigma_1 \text{sign}_2 &= \text{sign}_1 \sigma_1, \\ \sigma_1 \text{sign}_3 &= \text{sign}_3 \sigma_1, \\ \sigma_2 \text{sign}_1 &= \text{sign}_1 \sigma_2, \\ \sigma_2 \text{sign}_2 &= \text{sign}_3 \sigma_2, \\ \sigma_2 \text{sign}_3 &= \text{sign}_2 \sigma_2. \end{aligned} \tag{3.70}$$

This conclude the proof. □

In Table 3.2 we summarize the action of the sign flips in terms of the co-adjoint coordinates q and the quantities ω_i, as defined in (3.8).

	q_1	q_2	q_3	q_∞	q_{21}	q_{31}	q_{32}	ω_1	ω_2	ω_3	ω_4
sign_1	$-q_1$	q_2	q_3	$-q_\infty$	$-q_{21}$	$-q_{31}$	q_{32}	ω_1	$-\omega_2$	$-\omega_3$	ω_4
sign_2	q_1	$-q_2$	q_3	$-q_\infty$	$-q_{21}$	q_{31}	$-q_{32}$	$-\omega_1$	ω_2	$-\omega_3$	ω_4
sign_3	q_1	q_2	$-q_3$	$-q_\infty$	q_{21}	$-q_{31}$	$-q_{32}$	$-\omega_1$	$-\omega_2$	ω_3	ω_4

Table 3.2: Action of the sign flips in terms of the co-adjoint coordinates q.

Corollary 47. The group $\langle \text{sign}_1, \text{sign}_2, \text{sign}_3 \rangle$ is a group of symmetries for $\widehat{\mathcal{M}}_{PVI}$.

Proof. The statement is a consequence of Lemma 46 and Table 3.2. □

At this point we introduce the permutations on the elements of a triple of monodromy matrices in $\widehat{\mathcal{M}}_{PVI}$. The symmetric group on three elements $S_3 = \{id, (12), (13), (23), (123), (132)\}$ is generated by (123) and (12), i.e. $S_3 = \langle (12), (123) \rangle$. We describe the action of the S_3 on n defining its two generators (123) and (12) as:

$$(123) : (N_1, N_2, N_3, N_\infty) \mapsto (N_3, N_1, N_2, (N_2 N_1 N_3)^{-1}), \tag{3.71}$$

$$(12) : (N_1, N_2, N_3, N_\infty) \mapsto (N_2^{-1}, N_1^{-1}, N_3^{-1}, (N_3 N_1 N_2)^{-1}). \tag{3.72}$$

The action on the monodromy matrices of the entire group S_3 is:

$$(12) : (N_1, N_2, N_3, N_\infty) \mapsto (N_2^{-1}, N_1^{-1}, N_3^{-1}, (N_3^{-1} N_1^{-1} N_2^{-1})^{-1}), \tag{3.73}$$

$$(13) : (N_1, N_2, N_3, N_\infty) \mapsto (N_3^{-1}, N_2^{-1}, N_1^{-1}, (N_1^{-1} N_2^{-1} N_3^{-1})^{-1}), \tag{3.74}$$

$$(23) : (N_1, N_2, N_3, N_\infty) \mapsto (N_1^{-1}, N_3^{-1}, N_2^{-1}, (N_2^{-1} N_3^{-1} N_1^{-1})^{-1}), \tag{3.75}$$

$$(123) : (N_1, N_2, N_3, N_\infty) \mapsto (N_3, N_1, N_2, (N_2 N_1 N_3)^{-1}), \tag{3.76}$$

$$(132) : (N_1, N_2, N_3, N_\infty) \mapsto (N_2, N_3, N_1, (N_1 N_3 N_2)^{-1}). \tag{3.77}$$

As in the case of sign flips, we need following Lemma in order to prove that permutations are symmetries of $\widehat{\mathcal{M}}_{PVI}$:

Lemma 48. Given $\xi \in S_3$ and $\sigma \in B_3$, then there exists $\xi' \in S_3$ and $\sigma' \in B_3$ such that:

$$\sigma\xi = \xi'\sigma'.$$

Proof. Given $n = (N_1, N_2, N_3) \in \widehat{\mathcal{M}}_{PVI}$ we prove the result on the generators $\sigma_i^{(PVI)}$ for $i = 1, 2, 3$ of the full braid group B_3, i.e. we show that:

$$\xi\sigma_i^{(PVI)} = \sigma_{i'}^{(PVI)}\xi', \tag{3.78}$$

for some choice of the indices $i, i' = 1, 2, 3$ and $\xi, \xi' \in \langle(12), (123)\rangle$.

We prove (3.78) for (12) and $\sigma_2^{(PVI)}$:

$$\begin{aligned}
&\sigma_2(12)(N_1, N_2, N_3, N_\infty) = \\
&= \sigma_2(N_2^{-1}, N_1^{-1}, N_3^{-1}, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1}) = \\
&= (N_2^{-1}, N_3^{-1}, N_3^{-1}N_1^{-1}N_3, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1}).
\end{aligned}$$

The triple of monodromy matrices is in $GL(2)^3/GL(2)$ then:

$$\begin{aligned}
&(N_2^{-1}, N_3^{-1}, N_3^{-1}N_1^{-1}N_3, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1}) = \\
&= N_3(N_2^{-1}, N_3^{-1}, N_3^{-1}N_1^{-1}N_3, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1})N_3^{-1} = \\
&= (N_3N_2^{-1}N_3^{-1}, N_3^{-1}, N_1^{-1}, (N_1^{-1}N_2^{-1}N_3^{-1})^{-1}),
\end{aligned}$$

Now, if we consider $(23) = (123)(12)$ and $\sigma_2^{(PVI)}$, then:

$$\begin{aligned}
&(123)(12)\sigma_2(N_1, N_2, N_3, N_\infty) = \\
&= (123)(12)(N_1, N_3, N_3N_2N_3^{-1}, N_\infty) = \\
&= (123)(N_3^{-1}, N_1^{-1}, N_3N_2^{-1}N_3^{-1}, (N_3N_2^{-1}N_3^{-1})^{-1}N_1^{-1}N_3^{-1}) = \\
&= (N_3N_2^{-1}N_3^{-1}, N_3^{-1}, N_1^{-1}, (N_1^{-1}N_2^{-1}N_3^{-1})^{-1}).
\end{aligned}$$

and (3.78) follows.

The following relations can be proven in a similar way:

$$\sigma_2(12) = (123)(12)\sigma_2,$$
$$\sigma_1(123) = (132)\sigma_1^{-1},$$
$$\sigma_1(12) = (12)\sigma_1^{-1},$$
$$\sigma_2(123) = (123)\sigma_1,$$
$$\sigma_2^{-1}(12) = (12)(123)\sigma_1^{-1},$$
$$\sigma_2^{-1}(123) = (123)\sigma_1^{-1},$$
$$\sigma_1^{-1}(123) = \sigma_2,$$
$$\sigma_1^{-1}(12) = (12)\sigma_1.$$

This completes the proof. □

The action of permutations $\langle P_{13}, P_{23}\rangle$ is given, in terms of q and ω_i, in Table 3.3.

	q_1	q_2	q_3	q_∞	q_{21}	q_{31}	q_{32}	ω_1	ω_2	ω_3	ω_4
(12)	q_2	q_1	q_3	q_∞	q_{21}	q_{32}	q_{31}	ω_2	ω_1	ω_3	ω_4
(123)	q_3	q_1	q_2	q_∞	q_{31}	q_{32}	q_{21}	ω_3	ω_1	ω_2	ω_4

Table 3.3: Action of the permutations in terms of the co-adjoint coordinates q.

Corollary 49. The group $\langle (12), (123)\rangle$ is a group of symmetries for $\widehat{\mathcal{M}}_{PVI}$.

Proof. The statement is a consequence of Lemma 48 and Table 3.3. □

Next Lemma ensures us that sign flips (3.65)-(3.67) and permutations (3.71)-(3.72) generate a group and that this is a finite group:

Lemma 50. The group:

$$\mathrm{G}^{(2)}_{PVI} :=< \mathrm{sign}_1, \mathrm{sign}_2, \mathrm{sign}_3, (123), (12) >, \tag{3.79}$$

is a finite group of 48 elements.

Proof. In order to prove that the group $\mathrm{G}^{(2)}_{PVI}$ has 48 elements, we proceed proving the following relations between the generators:

$$\begin{aligned} \mathrm{sign}_1(123) &= (123)\mathrm{sign}_3, \\ \mathrm{sign}_2(123) &= (123)\mathrm{sign}_1, \\ \mathrm{sign}_3(123) &= (123)\mathrm{sign}_2, \\ \mathrm{sign}_1(12) &= (12)\mathrm{sign}_2, \\ \mathrm{sign}_2(12) &= (12)\mathrm{sign}_1, \\ \mathrm{sign}_3(12) &= (12)\mathrm{sign}_3. \end{aligned} \tag{3.80}$$

As a consequence, by relations (3.80) and direct computation, the statement follows. □

3.3 PVI: Classification of finite orbits

In this Section we resume the classification result, about all algebraic solutions of PVI, achieved by Lisovyy and Tykhyy in [LT14]. In particular, for each family of algebraic solutions, we describe the associated finite orbits of the action of P_3 over the Riemann sphere with four boundary components.

Each orbit will be presented as a couple $(\underline{q}, \underline{\omega})$, defined in (3.13), and the action of P_3 is explicitly given in (3.14).

Four families of algebraic solutions of PVI can be distinguished:

- Okamoto solutions.
- three Kitaev-Hitchin solutions.
- Cayley-Picard solutions.
- 45 exceptional solutions.

The list of all finite orbits associate to these families of solutions is:

Okamoto solutions. Each orbit (orbits I in [LT14]) associated to this family of solutions consists of one point $(\underline{q},\omega)$:

$$\{(a,b,c,\underline{\omega})\}\,, \tag{3.81}$$

where a,b,$c \in \mathbb{C}$ are free parameters and $\omega_1,\omega_2,\omega_3,\omega_4$ satisfy:

$$\begin{aligned}
\omega_1 &= 2c+ab,\\
\omega_2 &= 2b+ac,\\
\omega_3 &= 2a+bc,\\
\omega_4 &= 4+2abc+a^2+b^2+c^2.
\end{aligned} \tag{3.82}$$

Definition 51. O is the set of all the *equivalent* orbits that satisfy (3.81) and (3.82).

Hitchin-Kitaev solutions. In this case we distinguish three sub-families of finite orbits:

- K-Type II;
- K-Type III;
- K-Type IV.

Orbits of **K-Type II** (orbits II in [LT14]) consist of two points:

$$\{(0,0,a,\underline{\omega}),(0,0,b,\underline{\omega})\}, \tag{3.83}$$

where $a, b \in \mathbb{C}$, $a \neq b$ is free parameter and $\omega_1, \omega_2, \omega_3, \omega_4$ satisfy:

$$\begin{aligned}\omega_1 &= a + b,\\ \omega_2 &= \omega_3 = 0,\\ \omega_4 &= 4 + ab.\end{aligned} \tag{3.84}$$

Definition 52. K_{II} is the set of all the *equivalent* orbits that satisfy (3.83) and (3.84).

Orbits of **K-Type III** (orbits III in [LT14]) consist of three points:

$$\{(0,0,1,\underline{\omega}),(a,0,1,\underline{\omega}),(0,a,1,\underline{\omega})\}, \tag{3.85}$$

where $a \in \mathbb{C}$, $a \neq 0$ is free parameter and $\omega_1, \omega_2, \omega_3, \omega_4$ satisfy:

$$\begin{aligned}\omega_1 &= 2,\\ \omega_2 &= \omega_3 = a,\\ \omega_4 &= 5.\end{aligned} \tag{3.86}$$

Definition 53. $\mathrm{K}_{\mathrm{III}}$ is the set of all the *equivalent* orbits that satisfy (3.85) and (3.86).

Finally, orbits of **K-Type IV** (orbits IV in [LT14]) consist of four points:

$$\{(1,1,1,\underline{\omega}),(a,1,1,\underline{\omega}),(1,a,1,\underline{\omega}),(1,1,a,\underline{\omega})\}, \tag{3.87}$$

where $a \in \mathbb{C}$, $a \neq 1$ is free parameter and $\omega_1, \omega_2, \omega_3, \omega_4$ satisfy:

$$\begin{aligned} \omega_1 &= \omega_2 = \omega_3 = a + 2, \\ \omega_4 &= 3(a + 2). \end{aligned} \tag{3.88}$$

Definition 54. K_{IV} is the set of all the *equivalent* orbits that satisfy (3.87) and (3.88).

Cayley-Picard solutions. The orbits associated to this family of solutions can be generated from the points:

$$(-2\cos\pi(a+b), 2\cos\pi a, 2\cos\pi b, \underline{\omega}), \ \ a, b \in \mathbb{Q}, \tag{3.89}$$

and $\omega_1, \omega_2, \omega_3, \omega_4$ satisfy:

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0. \tag{3.90}$$

For this particular family the length of orbits varies with the choice of parameters a and b.

Definition 55. CP is the set of all the *equivalent* orbits that satisfy (3.89) and (3.90).

Orbits associated to the 45 exceptional solutions. We summarize the 45 representative of the associated orbits in Table 3.4 (that is exactly Table 5 in [LT14]). The first column identifies the orbit while the second one indicates how many points there are in the orbit. The 4-tuple in the central column gives the values of the parameters (3.8) and in the last column the values n_{ij} are such that:

$$q_{ij} = 2\cos\pi n_{ij}, \ \ i, j = 1, 2, 3, \ \ i > j. \tag{3.91}$$

	size	$(\omega_3, \omega_2, \omega_1, 4-\omega_4)$	(n_{21}, n_{31}, n_{32})
1	5	$(0, 1, 1, 0)$	$(2/3, 1/3, 1/3)$
2	5	$(3, 2, 2, -3)$	$(1/3, 1/3, 1/3)$
3	6	$(1, 0, 0, 2)$	$(1/2, 1/3, 1/3)$
4	6	$(\sqrt{2}, 0, 0, 1)$	$(1/4, 1/3, 3/4)$
5	6	$(3, 2\sqrt{2}, 2\sqrt{2}, -4)$	$(1/2, 1/4, 1/4)$
6	6	$\left(1-\sqrt{5}, \frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, -2+\sqrt{5}\right)$	$(4/5, 1/3, 1/3)$
7	6	$\left(1+\sqrt{5}, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, -2-\sqrt{5}\right)$	$(2/5, 1/3, 1/3)$
8	7	$(1, 1, 1, 0)$	$(1/2, 1/2, 1/2)$
9	8	$(2, 0, 0, 0)$	$(0, 1/3, 2/3)$
10	8	$(1, \sqrt{2}, \sqrt{2}, 0)$	$(1/2, 1/2, 1/2)$
11	8	$\left(\frac{3+\sqrt{5}}{2}, 1, 1, -\frac{\sqrt{5}+1}{2}\right)$	$(1/3, 1/2, 1/2)$
12	8	$\left(\frac{3-\sqrt{5}}{2}, 1, 1, \frac{\sqrt{5}-1}{2}\right)$	$(1/3, 1/2, 1/2)$
13	9	$\left(2-\sqrt{5}, 2-\sqrt{5}, 2-\sqrt{5}, \frac{5\sqrt{5}-7}{2}\right)$	$(4/5, 3/5, 3/5)$
14	9	$\left(2+\sqrt{5}, 2+\sqrt{5}, 2+\sqrt{5}, -\frac{5\sqrt{5}+7}{2}\right)$	$(2/5, 1/5, 1/5)$
15	10	$(1, 0, 0, 1)$	$(1/3, 1/3, 2/3)$
16	10	$\left(3-\sqrt{5}, 3-\sqrt{5}, 3-\sqrt{5}, \frac{7\sqrt{5}-11}{2}\right)$	$(3/5, 3/5, 3/5)$
17	10	$\left(3+\sqrt{5}, 3+\sqrt{5}, 3+\sqrt{5}, -\frac{7\sqrt{5}+11}{2}\right)$	$(1/5, 1/5, 1/5)$
18	10	$\left(-\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, 0\right)$	$(1/2, 1/2, 1/2)$
19	10	$\left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, 0\right)$	$(1/2, 1/2, 1/2)$
20	12	$(0, 0, 0, 3)$	$(2/3, 1/4, 1/4)$
21	12	$(1, 0, 0, 2)$	$(0, 1/4, 3/4)$
22	12	$(2, \sqrt{5}, \sqrt{5}, -2)$	$(1/5, 2/5, 2/5)$
23	12	$\left(\frac{3+\sqrt{5}}{2}, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, -\sqrt{5}\right)$	$(2/5, 2/5, 2/5)$
24	12	$\left(\frac{3-\sqrt{5}}{2}, -\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, \sqrt{5}\right)$	$(4/5, 4/5, 4/5)$
25	12	$\left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}-1}{2}, 1, 0\right)$	$(1/2, 1/2, 1/2)$
26	15	$\left(\frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \sqrt{5}-1\right)$	$(1/2, 3/5, 3/5)$
27	15	$\left(\frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, -\sqrt{5}-1\right)$	$(1/2, 1/5, 1/5)$
28	15	$\left(\frac{5-\sqrt{5}}{2}, 1-\sqrt{5}, 1-\sqrt{5}, \frac{3\sqrt{5}-5}{2}\right)$	$(3/5, 4/5, 4/5)$
29	15	$\left(\frac{5+\sqrt{5}}{2}, 1+\sqrt{5}, 1+\sqrt{5}, -\frac{3\sqrt{5}+5}{2}\right)$	$(1/5, 2/5, 2/5)$
30	16	$(0, 0, 0, 2)$	$(2/3, 2/3, 2/3)$
31	18	$(2, 2, 2, -1)$	$(0, 1/5, 3/5)$
32	18	$(1-2\cos 2\pi/7, 1-2\cos 2\pi/7, 1-2\cos 2\pi/7, 4\cos 2\pi/7)$	$(6/7, 5/7, 5/7)$
33	18	$(1-2\cos 4\pi/7, 1-2\cos 4\pi/7, 1-2\cos 4\pi/7, 4\cos 4\pi/7)$	$(2/7, 3/7, 3/7)$
34	18	$(1-2\cos 6\pi/7, 1-2\cos 6\pi/7, 1-2\cos 6\pi/7, 4\cos 6\pi/7)$	$(4/7, 1/7, 1/7)$
35	20	$\left(\frac{3-\sqrt{5}}{2}, 0, 0, 1+\sqrt{5}\right)$	$(0, 1/3, 2/3)$
36	20	$\left(\frac{3+\sqrt{5}}{2}, 0, 0, 1-\sqrt{5}\right)$	$(0, 1/3, 2/3)$
37	20	$\left(1, -\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$	$(2/3, 3/5, 3/5)$
38	20	$\left(1, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, -\frac{\sqrt{5}-1}{2}\right)$	$(2/3, 1/5, 1/5)$
39	24	$(1, 1, 1, 1)$	$(1/5, 1/2, 1/2)$
40	30	$\left(-\frac{\sqrt{5}+1}{2}, 0, 0, \frac{3-\sqrt{5}}{2}\right)$	$(2/3, 2/3, 2/3)$
41	30	$\left(\frac{\sqrt{5}-1}{2}, 0, 0, \frac{3+\sqrt{5}}{2}\right)$	$(2/3, 2/3, 2/3)$
42	36	$(1, 0, 0, 2)$	$(0, 1/5, 4/5)$
43	40	$\left(0, 0, 0, \frac{5-\sqrt{5}}{2}\right)$	$(2/5, 2/5, 2/5)$
44	40	$\left(0, 0, 0, \frac{5+\sqrt{5}}{2}\right)$	$(4/5, 4/5, 4/5)$
45	72	$(0, 0, 0, 3)$	$(1/2, 1/5, 2/5)$

Table 3.4: 45 Exceptional solutions.

We define the following set:

Definition 56. E_{45} is the set of all the *equivalent* orbits obtained from Table 3.4.

It is important to note that, depending on the family of algebraic solutions of PVI, there is a different number of associated finite orbits. Let us resume this fact:

- The set O of orbits associated to Okamoto solutions is an infinite set.
- The sets K_{II},K_{III},K_{IV} of orbits associated to Hitchin-Kitaev solutions are infinite sets.
- The set CP of orbits associated to Cayley-Picard solutions is an infinite set.
- The set E_{45} of orbits associated to the 45 exceptional solutions in Table 3.4 is a finite set.

Since the sets O,K_{II},K_{III},K_{IV} and CP are infinite, for the moment we focus on the set E_{45}.

3.3.1 Expansion algorithm for Table 3.4

In the last part of this Section we explain how, given an element $(\underline{q}, \underline{\omega})$ in Table 3.4, we can generate the set E_{45} of all equivalent orbits under the groups of symmetries studied in the Section 3.2.

The group G_{PVI} of Okamoto transformations of the Sixth Painlevé equation acts as $K_4 \rtimes S_3$ on $\underline{\omega} = (\omega_1, \dots, \omega_4)$ [LT14]. Extending this action to $\underline{q} = (q_{21}, q_{31}, q_{32})$, we obtain Theorem 39. Moreover, we observe that P_{13} and P_{23} are elements of the braid group B_3, and since we act only on points that have finite orbits under the action of the braid group (see Remark 43), the action of the whole group F_4 produces a finite set of values. All these values will be in the form $(\underline{q}, \underline{\omega})$, in order to extract q_1, q_2, q_3 and q_∞ we use the fact that we can consider the relations (3.8) as a system of equations in q_1, q_2, q_3 and q_∞ and that each q_i has the form:

$$q_i = 2\cos \pi\theta_i, \quad i = 1, 2, 3, \infty.$$

One particular solution of equations (3.8) is listed in [LT14] in terms of θ_1, θ_2, θ_3, θ_∞ for each point in the Table (3.4). We can then compute all other solutions q_1, q_2, q_3 and q_∞ by using Lemma 41.

Consider $(\underline{q}, \underline{\omega})$ in the Lisovyy and Tykhyy's sublist summarized in Table 3.4, then the following *expansion Algorithm* generates all equivalent orbits:

Algorithm 1.

For every line of Table 3.4, consider $(\underline{q}, \underline{\omega})$ and a solution $(q_1, q_2, q_3, q_\infty)$ of system (3.8):

1. Apply to $(q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32}, \underline{\omega})$ all 48 transformations of the group $G^{(2)}_{PVI}$. Save the results in a set E_0.

   For every element $(q_1', q_2', q_3', q_\infty', q_{21}', q_{31}', q_{32}', \underline{\omega}') \in E_0$:

2. Generate the orbit of $(q_1', q_2', q_3', q_\infty', q_{21}', q_{31}', q_{32}', \underline{\omega}')$ under the action of the group B_3. Save the result in a set E_1.

   For every element $(q_1'', q_2'', q_3'', q_\infty'', q_{21}'', q_{31}'', q_{32}'', \underline{\omega}'') \in E_1$:

3. Apply to $(q_1'', q_2'', q_3'', q_\infty'', q_{21}'', q_{31}'', q_{32}'', \underline{\omega}'')$ all the 24 transformations listed in Lemma 45 and save the result in the set E_2.

For every element $(q_1''', q_2''', q_3''', q_\infty''', q_{21}'', q_{31}'', q_{32}'', \underline{\omega}'') \in E_2$:

4. Generate the P_3-orbit of $(q_1''', q_2''', q_3''', q_\infty''', q_{21}'', q_{31}'', q_{32}'', \underline{\omega}'')$ and save the result in the set E_{45}.

Once the Algorithm ends, due to Lemmata 38, 44, 45, 50, the set E_{45} will contain only a finite number of *equivalent* orbits.

Remark 57. Consider $(q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32})$, then in Algorithm 1, the order we apply the transformations of groups $\mathrm{G}^{(1)}_{PVI}$ and $\mathrm{G}^{(2)}_{PVI}$, defined respectively in (3.36) and (3.79), is not relevant. Indeed, the following relations hold true:

$$
\begin{aligned}
P_{13}\mathrm{sign}_1 &= \mathrm{sign}_3 P_{13},\\
P_{13}\mathrm{sign}_2 &= \mathrm{sign}_2 P_{13},\\
P_{13}\mathrm{sign}_3 &= \mathrm{sign}_1 P_{13},\\
P_{23}\mathrm{sign}_1 &= \mathrm{sign}_1 P_{23},\\
P_{23}\mathrm{sign}_2 &= \mathrm{sign}_3 P_{23},\\
P_{23}\mathrm{sign}_3 &= \mathrm{sign}_2 P_{23},\\
P_{13}(123) &= (123)^2 P_{23},\\
P_{23}(123) &= \beta_{32}^{(PVI)} P_{13},\\
P_{13}(12) &= (123)(12) P_{13} P_{23} P_{13},\\
P_{23}(12) &= (12) P_{23} P_{23} P_{13},\\
r_1\mathrm{sign}_1 &= \mathrm{sign}_1 r_1,\\
r_1\mathrm{sign}_2 &= \mathrm{sign}_1\mathrm{sign}_3 r_1,\\
r_1\mathrm{sign}_3 &= \mathrm{sign}_2\mathrm{sign}_1 r_1,\\
r_2\mathrm{sign}_1 &= \mathrm{sign}_2\mathrm{sign}_3 r_2,\\
r_2\mathrm{sign}_2 &= \mathrm{sign}_2 r_2,
\end{aligned}
\tag{3.92}
$$

$$\begin{aligned}
r_2\mathrm{sign}_3 &= \mathrm{sign}_2\mathrm{sign}_1 r_2,\\
r_3\mathrm{sign}_1 &= \mathrm{sign}_2\mathrm{sign}_3 r_3,\\
r_3\mathrm{sign}_2 &= \mathrm{sign}_1\mathrm{sign}_3 r_3,\\
r_3\mathrm{sign}_3 &= \mathrm{sign}_3 r_3,\\
r_1(12) &= (12)r_2,\\
r_1(123) &= (12)r_3,\\
r_2(12) &= (12)r_1,\\
r_2(123) &= (12)r_1,\\
r_3(12) &= (12)r_3,\\
r_3(123) &= (12)r_2.
\end{aligned}$$

In the next Chapter we will describe Algorithms that implement the matching procedure over co-adjoint coordinates $\widehat{q}, \overline{q}, \check{q}, \widetilde{q}$ over $\mathcal{M}_{PVI}$, that eventually leads to the classification of p_i, p_{ij}, p_{ijk} in the open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, that possibly will generate a finite orbit under the action of the pure braid group P_4.

Chapter 4

Matching

In this Chapter, we explain how to implement our methodology. We are classifying finite orbits of action (1.1) in the following way: if $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, then there exist four restrictions $H_1, \ldots, H_4$, see Theorem 12, such that each restriction acts on p as the pure braid group P_3 over $\widehat{\mathcal{M}}_{PVI}$, i.e. the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components. In particular each subgroup acts respectively on projections $\widehat{q}, \check{q}, \bar{q}, \widetilde{q}$, as summarized in Table 1.1.

If p generates a finite orbit under the action (1.1), then the orbits of $\widehat{q}, \check{q}, \bar{q}, \widetilde{q}$ under the restricted action of the respective H_i must be finite too. In the previous Chapter, we described the list of all such $\widehat{q}, \check{q}, \bar{q}, \widetilde{q}$: the list is infinite (this is an issue in order to develop our method), but it contains a finite sublist, called E_{45} (see Definition 56), that will be crucial in the classification presented in this thesis.

In Section 4.1, we propose a procedure that, given three projection points, produces points p that satisfy the necessary conditions to generate a finite orbit under the action of three of the restrictions $H_1, \ldots, H_4$: we call these points *candidate* points.

In Sections 4.2 and 4.3, we introduce algorithms that produce the set $\mathcal{C}$ of *candidate* points p such that:

(C1) Three over four projections are in the set E_{45}.

(C2) Two over four projections are in the set O and one of the remaining projections is in the set E_{45}.

(C3) Two over four projections are in the set E_{45} and one of the remaining two projections is in the set O.

Moreover, all of these algorithms exploit the fact that E_{45} is finite and therefore, they generate a set $\mathcal{C}$ that will be finite too.

Afterwards, since the set $\mathcal{C}$ will contain only a finite number of elements, we extract from $\mathcal{C}$ a list of points such that they produce finite orbits under the action of the pure braid group P_4. Finally we present a list of 54 finite orbits up to the action of the group G, i.e. the group of symmetries of $\mathcal{G}_2$ discussed in Chapter 2. The list of finite orbits is presented in Table 4.2.

4.1 Outline of the procedure

In order to better describe the set $\mathcal{C}$, we introduce the following Definition:

Definition 58. A point p such that its four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, defined in (1.114), generate finite orbits under the action of P_3 is said to be a *candidate* point.

In this Section, we propose a procedure to construct all *candidate* points p in the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$. Note that, to generate a *candidate* point p, it is not necessary to know all four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$. Indeed, looking at Table 1.1, if we know only three projections over four, then only one

	p_1	p_2	p_3	p_4	p_∞	p_{21}	p_{31}	p_{32}	p_{41}	p_{42}	p_{43}	p_{321}	p_{432}	p_{431}	p_{421}
H_1		$\widehat{q}_1$	$\widehat{q}_2$	$\widehat{q}_3$				$\widehat{q}_{21}$		$\widehat{q}_{31}$	$\widehat{q}_{32}$		$\widehat{q}_\infty$		
H_2	$\overline{q}_1$		$\overline{q}_2$	$\overline{q}_3$			$\overline{q}_{21}$		$\overline{q}_{31}$		$\overline{q}_{32}$			$\overline{q}_\infty$	
H_3	$\breve{q}_1$	$\breve{q}_2$		$\breve{q}_3$		$\breve{q}_{21}$			$\breve{q}_{31}$	$\breve{q}_{32}$					$\breve{q}_\infty$

Table 4.1: Matching with three points: elements on the same column must be equal.

p_{ijk} will be missing, but we can calculate it choosing appropriately one of the four relations $f_1, \ldots, f_4$, defined in (1.53)-(1.56). This leads to state a *matching procedure on three points*. For example, we can define the following matching procedure for the three points $\widehat{q}, \breve{q}, \overline{q}$:

Procedure 1.

1. Consider $(\widehat{q}, \breve{q}, \overline{q}) \in \mathcal{FO} \times \mathcal{FO} \times \mathcal{FO}$.

2. Check if $\widehat{q}, \breve{q}, \overline{q}$ satisfy relations given by the columns of Table 4.1, then go to the next Step, otherwise go to Step 1.

3. Calculate the two roots $p_{321}^{(i)}$, for $i = 1, 2$, of the equation (1.53) in which we express p_1,p_2,p_3,p_{21},p_{31},p_{32} in terms of co-adjoint coordinates $\widehat{q}, \breve{q}, \overline{q}$:

$$
\begin{aligned}
&p_{321}^2 + p_{321}(\breve{q}_1\widehat{q}_1\widehat{q}_2 - \breve{q}_{21}\widehat{q}_2 - \widehat{q}_1\overline{q}_{21} - \breve{q}_1\widehat{q}_{21}) + \\
&\breve{q}_1^2 + \widehat{q}_1^2 - \breve{q}_1\widehat{q}_1\breve{q}_{21} + \breve{q}_{21}^2 + \widehat{q}_2^2 - \breve{q}_1\widehat{q}_2\overline{q}_{21} + \overline{q}_{21}^2 - \widehat{q}_1\widehat{q}_2\widehat{q}_{21} + \\
&\breve{q}_{21}\overline{q}_{21}\widehat{q}_{21} + \widehat{q}_{21}^2 - 4 = 0.
\end{aligned} \tag{4.1}
$$

For each $i = 1, 2$:

4. For each root $p^{(i)}_{321}$, determine the value of $p^{(i)}_{\infty}$ using equation (1.116) written in terms of co-adjoint coordinates $\widehat{q}, \check{q}, \overline{q}$ and $p^{(i)}_{321}$, using identities (1.108),(1.109),(1.110) and (1.111) as follows:

$$p^{(i)}_{\infty} = \frac{1}{2}(\check{q}_1\widehat{q}_1\widehat{q}_2\widehat{q}_3 - \check{q}_{21}\widehat{q}_2\widehat{q}_3 - \check{q}_1\widehat{q}_{21}\widehat{q}_3 + p^{(i)}_{321}\widehat{q}_3 - \widehat{q}_1\widehat{q}_2\check{q}_{31} + \widehat{q}_{21}\check{q}_{31} - \overline{q}_{21}\widehat{q}_{31} + \widehat{q}_2\check{q}_{\infty} - \check{q}_1\widehat{q}_1\widehat{q}_{32} + \check{q}_{21}\widehat{q}_{32} + \widehat{q}_1\overline{q}_{\infty} + \check{q}_1\widehat{q}_{\infty}). \tag{4.2}$$

5. Use identities (1.108),(1.109),(1.110) and (1.111) in order to determine the other components of $p^{(i)}$.

6. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise go to Step 1.

7. Save $p^{(i)}$ in the set $\widetilde{\mathcal{C}}$, and go to Step 1.

The procedure ends when all possible choices of three points $\widehat{q}, \check{q}, \overline{q} \in \widehat{\mathcal{M}}_{PVI}$ are exhausted.

Note that, since $\mathcal{FO}$ is not a finite set, this procedure may never end. However, we will adapt it in different cases in such a way to avoid this problem. For the sake of clarity, let us for the moment suppose that $\mathcal{FO}$ is finite.

In order to obtain the big set $\mathcal{C}$ of all *candidate* points, other three procedures similar to Procedure 1 must be developed. We summarize these three matching procedures on three points as follows:

(P1.1) Matching procedure with input triple $(\widehat{q}, \check{q}, \overline{q})$: output set $\widetilde{C}$.

(P1.2) Matching procedure with input triple $(\widetilde{q}, \check{q}, \overline{q})$: output set $\widehat{C}$.

(P1.3) Matching procedure with input triple $(\widetilde{q}, \widehat{q}, \overline{q})$: output set $\check{C}$.

(P1.4) Matching procedure with input triple $(\widehat{q}, \check{q}, \widetilde{q})$: output set $\bar{C}$.

In order to construct the set $\mathcal{C}$, the union of all the above four sets $\widetilde{\mathcal{C}}, \widehat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\mathcal{C}}$ must be taken:

$$\mathcal{C} = \widetilde{\mathcal{C}} \cup \widehat{\mathcal{C}} \cup \check{\mathcal{C}} \cup \bar{\mathcal{C}}. \tag{4.3}$$

As we are going to show in the next Lemma, it is enough to know only one of the sets $\widetilde{\mathcal{C}}, \widehat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\mathcal{C}}$ to generate the whole set $\mathcal{C}$:

Lemma 59. Consider $m \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and the permutation (1234), introduced in Section 2.5:

$$(1234)(M_1, M_2, M_3, M_4, M_\infty) = (M_4, M_1, M_2, M_3, (M_3M_2M_1M_4)^{-1}),$$

that acts on the co-adjoint coordinates of m, in the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, as follows:

$$(1234)(p) = (p_4, p_1, p_2, p_3, p_\infty, p_{41}, p_{42}, p_{21}, p_{43}, p_{31}, p_{32}, p_{421}, p_{321}, p_{431}, p_{432}).$$

If $\widetilde{\mathcal{C}}$,$\widehat{\mathcal{C}}$,$\check{\mathcal{C}}$,$\bar{\mathcal{C}}$ are the sets of *candidate* points p obtained running respectively procedures (P1.1),(P1.2),(P1.3),(P1.4), then:

$$(1234)(\widetilde{\mathcal{C}}) = \check{\mathcal{C}}, \tag{4.4}$$

$$(1234)(\check{\mathcal{C}}) = \bar{\mathcal{C}}, \tag{4.5}$$

$$(1234)(\bar{\mathcal{C}}) = \widehat{\mathcal{C}}, \tag{4.6}$$

$$(1234)(\widehat{\mathcal{C}}) = \widetilde{\mathcal{C}}. \tag{4.7}$$

Proof. We proceed proving the statement of the Theorem for (4.4), then in a similar way the statement can be proved for (4.5)-(4.7).

Consider $\widetilde{n}, \widehat{n}, \check{n}, \bar{n} \in \widehat{\mathcal{M}}_{PVI}$ and $m \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$. Apply to $\widetilde{n}, \widehat{n}, \check{n}, \bar{n}$ the matching procedure stated in (ii) in Theorem 12. If we don't consider the projection $\widetilde{q}$, the matching for procedure (P1.1) is given by the following relations:

$$\widehat{N}_1 = M_2,\ \widehat{N}_2 = M_3,\ \widehat{N}_3 = M_4,\ \widehat{N}_\infty = (M_4M_3M_2)^{-1},$$
$$\bar{N}_1 = M_1,\ \bar{N}_2 = M_3,\ \bar{N}_3 = M_4,\ \bar{N}_\infty = (M_4M_3M_1)^{-1},$$
$$\check{N}_1 = M_1,\ \check{N}_2 = M_2,\ \check{N}_3 = M_4,\ \check{N}_\infty = (M_4M_2M_1)^{-1}.$$

Consider $m' = (1234)(m)$, then the above matching procedure becomes:

$$\widehat{N}'_1 = M_1,\ \widehat{N}'_2 = M_2,\ \widehat{N}'_3 = M_3,\ \widehat{N}'_\infty = (M_3M_2M_1)^{-1},$$
$$\bar{N}'_1 = M_4,\ \bar{N}'_2 = M_2,\ \bar{N}'_3 = M_3,\ \bar{N}'_\infty = (M_3M_2M_4)^{-1},$$
$$\check{N}'_1 = M_4,\ \check{N}'_2 = M_1,\ \check{N}'_3 = M_3,\ \check{N}'_\infty = (M_3M_1M_4)^{-1}.$$

where $\widetilde{n}', \widehat{n}', \check{n}' \in \widehat{\mathcal{M}}_{PVI}$. After the relabelling:

$$\widetilde{N}'_i = \check{N}'_i,\ \check{N}'_i = \bar{N}'_i,\ \widehat{N}'_i = \widetilde{N}'_i,\ i = 1, 2, 3, \infty\ , \tag{4.8}$$

we obtain the relaxed matching procedure for algorithm (P1.2) that produces the set $\widehat{\mathcal{C}}$. □

In the following, when proposing a matching on three points, we will generate the set $\widetilde{C}$, then we will construct the big set $\mathcal{C}$ of all *candidate* points, applying Lemma 59.

Now, we need to determine which points in $\mathcal{C}$ lead to a finite orbit of the P_4-action. As mentioned above, for the moment we suppose the set $\mathcal{FO}$ to be finite (this is not true but we will see how to adapt our procedures), consequently the set $\mathcal{C}$ will be finite and we can develop a way to check if $p \in \mathcal{C}$ may or may not generate a finite orbit, based on the following Lemma:

Lemma 60. Assume $\mathcal{C}$ finite and let $p \in \mathcal{C}$ a *candidate* point, then its orbit is finite if and only if $\beta(p) \in \mathcal{C}$ for every braid $\beta \in P_4$.

Proof. Suppose $\beta(p) \in \mathcal{C}$ for every $\beta \in P_4$, then the orbit is finite since $\mathcal{C}$ is finite too. Vice versa, suppose p has a finite P_4-orbit, then for every β, $\beta(p)$ must have a finite orbit. Hence, $\beta(p)$ must be an element of $\mathcal{C}$. □

We briefly give an explanation on how we are going to operatively use this Lemma. Indeed, in the set $\mathcal{C}$, to select the finite orbits is equivalent to find the subset $\mathcal{C}_0 \subset \mathcal{C}$ such that:

$$\mathcal{C}_0 = \{p \in \mathcal{C} \mid \beta(p) \in \mathcal{C},\ \beta \in P_4\}. \tag{4.9}$$

As mentioned before, the group P_4 is an infinite group and, accordingly to this fact, we can not implement an algorithm able to deal with every pure braid β in P_4. Nevertheless, P_4 is finitely generated:

$$P_4 =< \beta_{21}, \beta_{31}, \beta_{32}, \beta_{41}, \beta_{42}, \beta_{43} >, \tag{4.10}$$

where generators β_{ij} are defined in (1.87). Now, we explain how we can check which element $p \in \mathcal{C}$ generates a finite P_4-orbit. Since every braid $\beta \in P_4$ can be thought as an ordered combination of generators β_{ij} (and their inverses too), we can introduce the so-called *braid word*, namely:

$$\beta = \underbrace{\beta_{i'j'} \dots \beta_{ij}}_{n}, \tag{4.11}$$

where n indicates the length of the *word*. Consider $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and a braid $\beta \in P_4$ such that $\beta(p) \notin \mathcal{C}$. Moreover, consider the following notation:

$$\begin{aligned} p^{(0)} &= p, \\ p^{(1)} &= \beta_{ij}(p^{(0)}), \\ &\vdots \\ p^{(n)} &= \beta(p) = \beta_{i'j'}(p^{(n-1)}) = \underbrace{\beta_{i'j'} \dots \beta_{ij}}_{n}(p^{(0)}). \end{aligned} \tag{4.12}$$

Since we supposed $p^{(n)} \notin \mathcal{C}$, we need to delete $p^{(n)}$ from the set $\mathcal{C}$ and also the element p_{n-1} and so on, till when $p^{(0)} = p$ is deleted from $\mathcal{C}$.

We will then find, in the set $\mathcal{C}_0$, all the points p having finite P_4-orbit and we further factorize by the group of symmetries G introduced in Section 2.5.

In the next Sections we are going to adapt these procedures to different cases to account for the fact that $\mathcal{FO}$ is actually an infinite set.

4.2 Matching with three of the PVI 45 exceptional algebraic solutions

In this Section, we give an algorithm that produces the *finite* set $\mathcal{C}_{\mathrm{E}_{45} \times \mathrm{E}_{45} \times \mathrm{E}_{45}}$ of all *candidate* points p such that three over four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, defined in (1.114), are in the set E_{45}, i.e. the set of all equivalent orbits generated from Table 3.4.

We adapt Procedure 1 in such a way it can process the following triples:

$$(\widehat{q}, \check{q}, \overline{q}) \in \mathrm{E}_{45} \times \mathrm{E}_{45} \times \mathrm{E}_{45}. \tag{4.13}$$

The set E_{45} is generated from Table 3.4, using the expansion Algorithm 1 and it is finite. After generating E_{45}, the following algorithm produces the set of *candidate* points $\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$:

Algorithm 2.

1. Consider $(\widehat{q}, \check{q}, \overline{q}) \in \mathrm{E}_{45} \times \mathrm{E}_{45} \times \mathrm{E}_{45}$.
2. Check if $\widehat{q}, \check{q}, \overline{q}$ satisfy relations given by the columns of Table 4.1, then go to the next Step, otherwise go to Step 1.
3. Determine the values $p^{(i)}_{321}$, for $i = 1, 2$, using equation (4.1).

   For each $i = 1, 2$:
4. Calculate the values of $p^{(i)}_{\infty}$ using equation (1.116).
5. Use identities (1.111),(1.108),(1.110) and (1.109) in order to determine the other components of $p^{(i)}$.
6. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise go to Step 1.
7. Save $p^{(i)}$ in the set $\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$, and go to Step 1.

Since E_{45} is a finite set, the set $\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$ will be finite too. Finally the big set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$ can be generated by Lemma 59 as follows:

$$\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}} = \widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}} \bigcup_{i=1}^{3} (1234)^i(\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}). \tag{4.14}$$

The set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$ contains all *candidate* points $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ such that three projections (1.114) are in the set E_{45}, whereas the remaining projection could be in any other sets: O,K_{II},K_{III},K_{IV},CP and E_{45}, see

Definitions (51) - (56) respectively. There are $3,355,200$ *candidate* points in the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$.

4.3 Matching with Okamoto solutions

By Definition 51, the set O is the set of all orbits related to the infinite family of algebraic solutions of Okamoto type for the PVI equation. The set O is itself an infinite set: it will be crucial to adapt the matching procedure in such a way that the number of required steps is still finite.

We briefly recall which points p are *not relevant* in our classification: we are going to exclude both cases when the monodromy group is reducible or there exists an index $i = 1,\ldots,4,\infty$ such that $M_i = \pm\mathbb{1}$. Indeed, if the monodromy group is reducible the associated solution of $\mathcal{G}_2$ can be reduced to classical solutions in terms of Lauricella hypergeometric functions, as proved by Mazzocco in the [Maz01a], while if $M_i = \pm\mathbb{1}$ for some index i, then, again following [Maz01a], the solution of $\mathcal{G}_2$ can be reduced to solution of PVI. We formalize this fact in the following Definition:

Definition 61. A point p is *not relevant* if the associated monodromy group is reducible or there exists an index $i = 1,\ldots,4,\infty$ such that $M_i = \pm\mathbb{1}$.

The first result of this Section is:

Theorem 62. If a point $p \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ defined in (1.4) is such that any three of its four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, defined in (1.114), are in the set O of all orbits related to the family of Okamoto solutions then the point p is *not relevant.*

Theorem 63. Suppose $m = (M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ with co-adjoint co-ordinates p, defined in (1.4), in the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, defined in

(1.38). Then, $M_\infty = \pm\mathbb{1}$ if and only if p satisfies:

$$p_4 = \pm p_{321},\ p_\infty = \pm 2,\ p_{41} = \pm p_{32},\ p_{42} = \pm(p_{321}p_2 + p_3p_1 - p_{31} - p_{21}p_{32}),$$
$$p_{43} = \pm p_{21},\ p_{432} = \pm p_1,\ p_{431} = \pm p_2,\ p_{421} = \pm p_3. \qquad (4.15)$$

Consequently, all points p satisfying hyphotesis of Theorems 62-63 will be irrelevant to our classification (and then excluded from it), as they are dealt with in [Maz01a].

Before proving Theorem 62, we will enunciate some more results allowing us to further restrict our input of Okamoto points into the matching procedure. All proofs are postponed to Section 4.5. To this aim we need the following two definitions:

Definition 64. The set O_{ID} is the set of all the $q \in \mathrm{O}$ such that the associated triple of monodromy matrices $n \in \widehat{\mathcal{M}}_{PVI}$ admits one matrix equals to $\pm\mathbb{1}$.

Definition 65. The set $\mathrm{O}_{\mathrm{RED}}$ is the set of all the $q \in \mathrm{O}$ such that if we consider the associated triple of monodromy matrices $n \in \widehat{\mathcal{M}}_{PVI}$ then the monodromy group $\langle N_1, N_2, N_3\rangle$ is *reducible*, i.e. it admits a common subspace of dimension one.

Definitions 64 and 65 are given in terms of monodromy matrices: in the following, we will work both with triples $n = (N_1, N_2, N_3) \in \widehat{\mathcal{M}}_{PVI}$ and with 4-tuples of matrices $m = (M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and the associated co-adjoint coordinates q and p, introduced in Sections 3.1 and 1.2 respectively. We are ready to state the results in the following five Lemmata:

Lemma 66. If a point $p \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, defined in (1.4), is such that one of its four projections $\widehat{q}, \check{q}, \bar{q}, \widetilde{q}$, defined in (1.114), is in the set O_{ID} and another one projection is in the set $\mathrm{O}_{\mathrm{RED}}$ then such point p is *not relevant*.

Lemma 67. Let q be the co-adjoint coordinates on $\widehat{\mathcal{M}}_{PVI}$. If q is in the set $\mathrm{O}_{\mathrm{RED}}$, then q satisfies:

$$\begin{cases} q_{ij} &= \frac{1}{2}(q_i q_j - \epsilon_i\epsilon_j s_i s_j),\ i > j,\ i,j = 1,2,3, \\ q_\infty &= \frac{1}{4}(q_1q_2q_3 - \epsilon_1\epsilon_2 s_1 s_2 q_3 - \epsilon_1\epsilon_3 s_1 s_3 q_2 - \epsilon_2\epsilon_3 s_2 s_3 q_1) \end{cases} \tag{4.16}$$

where $s_k = \sqrt{4 - q_k^2}$ and $\epsilon_k = \pm 1$ for $k = 1,2,3$.

Lemma 68. Suppose $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, defined in (1.4), and q being co-adjoint coordinates on $\widehat{\mathcal{M}}_{PVI}$ of one over four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, defined in (1.114). If q is in the set O_{ID}, then q satisfies:

$$\begin{cases} q_{ij} &= \pm q_k, \\ q_\infty &= \pm 2. \end{cases} \tag{4.17}$$

where $i,j,k = 1,2,3$ with $i > j$ and $k \neq i, k \neq j$.

Lemma 69. Suppose $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, defined in (1.4), is such that any two of its four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$, defined in (1.114), are in the set $\mathrm{O}_{\mathrm{RED}}$ and q being co-adjoint coordinates on $\widehat{\mathcal{M}}_{PVI}$ of one of the remaining projections, then there exists a couple of indices (i,j),(i',j') with one index in (i,j) equal to one index in (i',j') such that:

$$\begin{cases} q_{ij}^2 + q_i^2 + q_j^2 - q_{ij}q_iq_j - 4 = 0, & i > j,\ i,j = 1,2,3, \\ q_{i'j'}^2 + q_{i'}^2 + q_{j'}^2 - q_{i'j'}q_{i'}q_{j'} - 4 = 0, & i' > j',\ i',j' = 1,2,3. \end{cases} \tag{4.18}$$

Lemma 70. If a point $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$, defined in (1.4), is such that two of its three projections $\widehat{q}, \check{q}, \overline{q}$, defined in (1.114), are in the set O_{ID}, then, if $\epsilon = \pm 1$, the following cases hold:

(i) If $\widehat{q}, \breve{q} \in \mathrm{O}_{\mathrm{ID}}$, then $\overline{q}$ must satisfy:

$$\overline{q}_2 = \widehat{\epsilon}\,\breve{\epsilon}\,\overline{q}_1,\ \overline{q}_{32} = \widehat{\epsilon}\,\breve{\epsilon}\,\overline{q}_{31}, \tag{4.19}$$

and p is such that:

$$p_1 = \overline{q}_1,\ p_2 = \breve{\epsilon}\,\overline{q}_{31},\ p_3 = \widehat{\epsilon}\,\breve{\epsilon}\,\overline{q}_1,\ p_4 = \overline{q}_3,\ p_{21} = \breve{\epsilon}\,\overline{q}_3,\ p_{31} = \overline{q}_{21},\ p_{32} = \widehat{\epsilon}\,\overline{q}_3,$$
$$p_{41} = \overline{q}_{31},\ p_{42} = \breve{\epsilon}\,\overline{q}_1,\ p_{43} = \widehat{\epsilon}\,\breve{\epsilon}\,\overline{q}_{31},\ p_{432} = \widehat{\epsilon}\,2,\ p_{431} = \overline{q}_\infty,\ p_{421} = \breve{\epsilon}\,2. \tag{4.20}$$

(ii) If $\widehat{q}, \overline{q} \in \mathrm{O}_{\mathrm{ID}}$, then $\breve{q}$ must satisfy:

$$\breve{q}_2 = \widehat{\epsilon}\,\overline{\epsilon}\,\breve{q}_1,\ \breve{q}_{32} = \widehat{\epsilon}\,\overline{\epsilon}\,\breve{q}_{31}, \tag{4.21}$$

and p is such that:

$$p_1 = \breve{q}_1,\ p_2 = \widehat{\epsilon}\,\overline{\epsilon}\,\breve{q}_1, p_3 = \overline{\epsilon}\,\breve{q}_{31},\ p_4 = \breve{q}_3, p_{21} = \breve{q}_{21},\ p_{31} = \overline{\epsilon}\,\breve{q}_3,\ p_{32} = \widehat{\epsilon}\,\breve{q}_3,$$
$$p_{41} = \breve{q}_{31},\ p_{42} = \widehat{\epsilon}\,\overline{\epsilon}\breve{q}_{31},\ p_{43} = \overline{\epsilon}\,\breve{q}_1,\ p_{432} = \widehat{\epsilon}\,2,\ p_{431} = \overline{\epsilon}\,2,\ p_{421} = \breve{q}_\infty. \tag{4.22}$$

(iii) If $\breve{q}, \overline{q} \in \mathrm{O}_{\mathrm{ID}}$, then $\widehat{q}$ must satisfy:

$$\widehat{q}_2 = \breve{\epsilon}\,\overline{\epsilon}\,\widehat{q}_1,\ \widehat{q}_{32} = \breve{\epsilon}\,\overline{\epsilon}\,\widehat{q}_{31}, \tag{4.23}$$

and p is such that:

$$p_1 = \breve{\epsilon}\,\widehat{q}_{31},\ p_2 = \widehat{q}_1,\ p_3 = \breve{\epsilon}\,\overline{\epsilon}\,\widehat{q}_1,\ p_4 = \widehat{q}_3,\ p_{21} = \breve{\epsilon}\,\widehat{q}_3,\ p_{31} = \overline{\epsilon}\,\widehat{q}_3,\ p_{32} = \widehat{q}_{31},$$
$$p_{41} = \breve{\epsilon}\,\widehat{q}_1,\ p_{42} = \widehat{q}_{31},\ p_{43} = \overline{\epsilon}\,\breve{\epsilon}\,\widehat{q}_{31},\ p_{432} = \widehat{q}_\infty,\ p_{431} = \overline{\epsilon}\,2,\ p_{421} = \breve{\epsilon}\,2. \tag{4.24}$$

Lemmata 67,68,70 lead to the development of additional matching algorithms in order to complete our classification for the cases when these points are included.

Suppose that two over three projections are in the set $\mathrm{O}_{\mathrm{RED}}$ and one projection is in the set E_{45}. By Lemma 69 the projection in E_{45} must satisfy conditions (4.18), for an appropriate choice of indices (i, j) and (i', j'). It turned out that actually there are no elements in E_{45} satisfying (4.18). As a consequence of this fact, there are no points $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ with two projections in the set $\mathrm{O}_{\mathrm{RED}}$ and at least one of the remaining two projections in the set E_{45}, so the case in which one projection is in E_{45} and two projections are in O is classified by the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$ of all *candidate* points p such that one over the four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$ is in the set E_{45} and two of the remaining projections are in the set O_{ID}.

To construct this set we proceed as follows: firstly we construct the set $\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$, where one over the three projections $\widehat{q}, \check{q}, \overline{q}$ is in the set E_{45} and two of the remaining projections are in the set O_{ID}, then, applying Lemma 59 we generate the whole set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$.

The set $\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$ is the union of the following three sets of *candidate* points p:

(A3.1) $\overline{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$: *candidate* points p with $\widehat{q}, \check{q} \in \mathrm{O}_{\mathrm{ID}}$, $\overline{q} \in \mathrm{E}_{45}$.

(A3.2) $\check{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$: *candidate* points p with $\widehat{q}, \overline{q} \in \mathrm{O}_{\mathrm{ID}}$, $\check{q} \in \mathrm{E}_{45}$.

(A3.3) $\widehat{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$: *candidate* points p with $\overline{q}, \check{q} \in \mathrm{O}_{\mathrm{ID}}$, $\widehat{q} \in \mathrm{E}_{45}$.

We state the algorithm that generates the subset (A3.1), then in a similar way algorithms for subsets (A3.2) and (A3.3) can be derived:

Algorithm 3.

1. Consider $\overline{q} \in \mathrm{E}_{45}$.

2. Check if $\overline{q}$ satisfies:
$$\begin{cases} \overline{q}_2 = \widehat{\epsilon}\,\check{\epsilon}\,\overline{q}_1, \\ \overline{q}_{32} = \widehat{\epsilon}\,\check{\epsilon}\,\overline{q}_{31}, \end{cases} \tag{4.25}$$
then go to the next Step, otherwise go to Step 1.

3. Determine the components of p involved in identities (4.20).

4. Determine the values $p_{321}^{(i)}$, for $i = 1, 2$, using equation (4.1).

   For each $i = 1, 2$:

5. Calculate the values of $p_{\infty}^{(i)}$ using equation (1.116).

6. Use identities given by the columns of Table 4.1 in order to determine the other components of $p^{(i)}$.

7. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise Step 1.

8. Save $p^{(i)}$ in the set $\overline{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$, and go to Step 1.

When Algorithm 3 and the algorithms for subsets (A3.2) and (A3.3) end, the following set is obtained:

$$\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} = \overline{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} \cup \check{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} \cup \widehat{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}},$$

then, by Lemma 59, we generate the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$ of all *candidate* points p such that one over the four projections $\widehat{q}, \check{q}, \overline{q}, \widetilde{q}$ is in the set E_{45} and two of the remaining projections are in the set O_{ID}:

$$\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} = \widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} \bigcup_{i=1}^{3} (1234)^i (\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}), \tag{4.26}$$

where permutation (1234) is defined in (2.67). There are 6,337 *candidate* points in the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$.

The following algorithm generates the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$ of all *candidate* points p such that one over the four projections $\widehat{q},\check{q},\overline{q},\widetilde{q}$ is in the set $\mathrm{O}_{\mathrm{RED}}$ and two of the remaining projections are in the set E_{45}.

In order to obtain $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$, we construct the set $\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$, where one over three projections $\widehat{q},\check{q},\overline{q}$ is in $\mathrm{O}_{\mathrm{RED}}$ and the remaining two are in E_{45}, afterwards, by Lemma 59, we can construct the big set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$.

Note that the set $\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$ is the union of three subsets of *candidate* points p:

(A4.1) $\overline{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$: *candidate* points p with $\widehat{q},\check{q}\in\mathrm{E}_{45}$, $\overline{q}\in\mathrm{O}_{\mathrm{RED}}$.

(A4.2) $\check{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$: *candidate* points p with $\widehat{q},\overline{q}\in\mathrm{E}_{45}$, $\check{q}\in\mathrm{O}_{\mathrm{RED}}$.

(A4.3) $\widehat{\widetilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$: *candidate* points p with $\overline{q},\check{q}\in\mathrm{E}_{45}$, $\widehat{q}\in\mathrm{O}_{\mathrm{RED}}$.

We describe in detail the algorithm that generates the subset (A4.1), then in a similar way algorithms for subsets (A4.2) and (A4.3) can be derived:

Algorithm 4.

1. Consider $\widehat{q},\check{q}\in\mathrm{E}_{45}\times\mathrm{E}_{45}$.
2. Check if $\widehat{q},\check{q}$ satisfy relations given by the columns of the first two rows of Table 4.1 then go to the next Step, otherwise go to Step 1.
3. Calculate p_{31} and p_{431} using Table 4.1 and conditions (4.16):

$$p_{31}=q_{21}=\frac{1}{2}(q_1q_2-\epsilon_1\epsilon_2s_1s_2)=\frac{1}{2}(p_1p_2-\epsilon_1\epsilon_2s_1s_2),\tag{4.27}$$
$$p_{431}=q_\infty=\frac{1}{4}(p_1p_2p_4-\epsilon_1\epsilon_2s_1s_2p_4-\epsilon_1\epsilon_4s_1s_4p_2-\epsilon_2\epsilon_4s_2s_4p_1)=$$

$$= \frac{1}{4}(q_1q_2q_3 - \epsilon_1\epsilon_2 s_1 s_2 q_3 - \epsilon_1\epsilon_3 s_1 s_3 q_2 - \epsilon_2\epsilon_3 s_2 s_3 q_1). \qquad (4.28)$$

4. Determine the values $p_{321}^{(i)}$, for $i = 1, 2$, using equation (4.1).

   For each $i = 1, 2$:

5. Calculate the values of $p_{\infty}^{(i)}$ using equation (1.116).

6. Use identities given by the columns of Table 4.1 in order to determine the other components of $p^{(i)}$.

7. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise Step 1.

8. Save $p^{(i)}$ in the set $\bar{\bar{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$, and go to Step 1.

When Algorithm 4 and the algorithms for subsets (A4.2) and (A4.3) end, the following set is obtained:

$$\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}} = \bar{\bar{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}} \cup \check{\check{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}} \cup \hat{\hat{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}},$$

then, by Lemma 59, we generate the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$ of all *candidate* points p with one over four projections in the set $\mathrm{O}_{\mathrm{RED}}$ and two over three of the remaining projections are in the set E_{45}:

$$\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}} = \widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}} \bigcup_{i=1}^{3} (1234)^i (\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}), \qquad (4.29)$$

where permutation (1234) is defined in (2.67). There are $342,368$ *candidate* points in the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$.

Last algorithm generates the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}}$ of all *candidate* points p such that one projection is in the set O_{ID} and two of the remaining three projections are in the set E_{45}. Considerations similar to the previous case apply. Indeed in order to obtain $\mathcal{C}_{E_{45}\times E_{45}\times O_{ID}}$, we construct the set $\widetilde{\mathcal{C}}_{E_{45}\times E_{45}\times O_{ID}}$ where one over three projections $\widehat{q}, \breve{q}, \overline{q}$ is in O_{ID} and the remaining two are in E_{45}. Thereafter, by Lemma 59, we construct the whole set $\mathcal{C}_{E_{45}\times E_{45}\times O_{ID}}$. The set $\widetilde{\mathcal{C}}_{E_{45}\times E_{45}\times O_{ID}}$ is the union of three subsets of *candidate* points p:

(A5.1) $\overline{\widetilde{\mathcal{C}}}_{E_{45}\times E_{45}\times O_{ID}}$: *candidate* points p with $\widehat{q}, \breve{q} \in E_{45}$, $\overline{q} \in O_{ID}$.

(A5.2) $\breve{\widetilde{\mathcal{C}}}_{E_{45}\times E_{45}\times O_{ID}}$: *candidate* points p with $\widehat{q}, \overline{q} \in E_{45}$, $\breve{q} \in O_{ID}$.

(A5.3) $\widehat{\widetilde{\mathcal{C}}}_{E_{45}\times E_{45}\times O_{ID}}$: *candidate* points p with $\overline{q}, \breve{q} \in E_{45}$, $\widehat{q} \in O_{ID}$.

We describe in detail algorithm that generates subset (A5.1) and in a similar way algorithms for subsets (A5.2) and (A5.3) can be derived:

Algorithm 5.

1. Consider $\widehat{q}, \breve{q} \in E_{45} \times E_{45}$.

2. Check if $\widehat{q}, \breve{q}$ satisfy relations relations given by the columns of the first two rows of Table 4.1 then go to the next Step, otherwise go to Step 1.

3. Calculate p_{31} and p_{431} using Table 4.1 and conditions (4.16):

$$p_{31} = q_{21} = \pm q_3 = \pm p_4, \tag{4.30}$$

$$p_{431} = q_\infty = \pm 2. \tag{4.31}$$

4. Determine the values $p_{321}^{(i)}$, for $i = 1, 2$, using equation (4.1).

   For each $i = 1, 2$:

5. Calculate the values of $p_\infty^{(i)}$ using equation (1.116).

6. Use identities given by the columns of Table 4.1 in order to determine the other components of $p^{(i)}$.

7. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise go to Step 1.

8. Save $p^{(i)}$ in the set $\bar{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}}$, and go to Step 1.

When Algorithm 5 and algorithms for subsets (A5.2) and (A5.3) end, we obtain:

$$\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}} = \bar{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}} \cup \check{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}} \cup \hat{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}},$$

then, by Lemma 59, we generate the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}}$ of all *candidate* points p with one over four projections in the set O_{ID} and two over three of the remaining projections are in the set E_{45}:

$$\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}} = \widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}} \bigcup_{i=1}^{3} (1234)^i (\widetilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}}) \qquad (4.32)$$

where permutation (1234) is defined in (2.67). There are 245, 760 *candidate* points in the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}}$.

4.4 List of finite orbits

Consider $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and its four projections $\widehat{q}, \check{q}, \bar{q}, \tilde{q}$. We recall that, in this thesis, we construct *candidate* points $p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ such that:

(C1) Three projections are in the set E_{45}.

(C2) Two projections are in the set O and one is in the set E_{45}.

(C3) Two projections are in the set E_{45} and one is in the set O.

By Theorem 62, the case (C4) is not relevant in our classification, since Mazzocco dealt with it in [Maz01a]. Moreover, we exclude the following cases:

- By Theorem 63, we exclude *candidate* points such that p satisfies conditions (4.15).
- By Lemma 66, we exclude *candidate* points such that they have one projection in O_{ID}, see Definition (64), and one projection in O_{RED}, see Definition (65).

We obtained Algorithms such that they generate the following *candidate* points:

(C1) Algorithm 2 produces the set of *candidate* points $\mathcal{C}_{E_{45}\times E_{45}\times E_{45}}$.

(C2) Algorithm 3 produces the set of *candidate* points $\mathcal{C}_{E_{45}\times O_{ID}\times O_{ID}}$.

(C3.1) Algorithm 4 produces the set of *candidate* points $\mathcal{C}_{E_{45}\times E_{45}\times O_{RED}}$.

(C3.2) Algorithm 5 produces the set of *candidate* points $\mathcal{C}_{E_{45}\times E_{45}\times O_{ID}}$.

The finite set $\mathcal{C}$ of all *candidate* points p classified in this thesis is:

$$\mathcal{C} = \mathcal{C}_{E_{45}\times E_{45}\times E_{45}} \cup \mathcal{C}_{E_{45}\times O_{ID}\times O_{RED}} \cup \mathcal{C}_{E_{45}\times E_{45}\times O_{ID}} \cup \mathcal{C}_{E_{45}\times O_{ID}\times O_{ID}}, \quad (4.33)$$

and it contains $3,460,685$ *candidate* points.

Among these points, we need to delete all points in the big open subset $\mathcal{U}$, defined in (1.38), that satisfy relations (4.15) in Theorem 63, since they are not relevant.

Algorithm 6.

1. Consider $p \in \mathcal{C}$.
2. If p satisfies relations (4.15) go to next Step, otherwise save p in $\mathcal{C}'$.
3. If p satisfies at least one of the following conditions:

   (i) $(p_{21}^2 - 4)g(p_{21}, p_3, p_{321})g(p_2, p_1, p_{21})g(p_{21}, p_4, p_{421}) \neq 0.$

   (ii) $(p_{31}^2 - 4)g(p_{31}, p_4, p_{431})g(p_3, p_1, p_{31})g(p_{31}, p_2, p_{321}) \neq 0.$

   (iii) $(p_{32}^2 - 4)g(p_{32}, p_4, p_{432})g(p_3, p_2, p_{32})g(p_{32}, p_1, p_{321}) \neq 0.$

   then the point p is in the open set $\mathcal{U}$, defined in (1.38) and it is not relevant, otherwise save p in $\mathcal{C}'$.

This step permits us to eliminate $173,545$ and the resulting set $\mathcal{C}'$ has $3,287,140$ elements.

Remark 71. During the execution of the previous algorithm we discard also the following point p:

$$p = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2),$$

since the monodromy associated is reducible, as proved in [DM00].

Next, as consequence of Lemma 60, we apply the following Algorithm in order to eliminate all points that don't produce finite orbits:

Algorithm 7.

1. Consider $p \in \mathcal{C}'$.

2. Apply to it all the generators (1.87) of P_4:

$$\beta_{21}(p) = p^{(1)}, \dots, \beta_{32}(p) = p^{(6)}. \tag{4.34}$$

3. If there exists an $i = 1, \dots, 6$ such that $p^{(i)} \notin \mathcal{C}'$ then delete p from the set $\mathcal{C}'$ and go to Step 1, otherwise save p in $\mathcal{C}_0$ and go to Step 1.

This algorithm ends when in the set $\mathcal{C}'$ there are no more elements to delete, and it produces a set $\mathcal{C}_0$ with $1,095,712$ elements that generate finite orbits under the P_4-action. Finally, we can factorize the set $\mathcal{C}_0$ modulo the action of the pure braid group P_4:

$$\mathcal{C}_1 := \mathcal{C}_0/P_4.$$

as follows:

Algorithm 8.

For every $p \in \mathcal{C}_0$:

1. Save $p \in \mathcal{C}_1$.

2. Since p has a finite P_4-orbit by construction. Calculate $|\mathcal{O}_{P_4}(p)|$.

3. Delete $|\mathcal{O}_{P_4}(p)|$ from $\mathcal{C}_0$.

Since the set $\mathcal{C}_0$ is finite, the algorithm ends. This algorithm produces the set $\mathcal{C}_1$, that contains $17,946$ finite orbits of the P_4-action.

At this point, our aim is to factorize the set $\mathcal{C}_1$ by the action of the group of symmetries G, introduced in Section 2.5, where G is an infinite and non commutative group. This obviously poses a problem. However, thanks to the fact that G acts as a finite group on $(p_1, p_2, p_3, p_4, p_\infty)$ and preserves the length of a P_4-orbit, we are able to set up an algorithm to achieve the factorization we are looking for.

First of all, we factorize by the action of the finite subgroup:

$$\langle \text{sign}_1, \ldots, \text{sign}_4, (12)(34), (1234) \rangle \subset \text{G}, \tag{4.35}$$

to obtain the set $\mathcal{C}_2'$. The set $\mathcal{C}_2'$ is finite and it contains 122 points. We do this factorization first as it reduces dramatically from 4,275 to 122 the number of orbits to be processed afterwards. Next, we subdivide the set $\mathcal{C}_2'$ into subsets that contain orbits of the same length and have the same $(p_1, p_2, p_3, p_4, p_\infty)$ modulo change of signs or permutations. Indeed, thanks to the fact that the action of G preserves the length of an orbit and that the $(p_1, p_2, p_3, p_4, p_\infty)$ remain invariant during this action, only points within the same subset can be related by a transformation in G.

Then, in each subset, for all the elements in the subset, we apply a transformation in the subgroup (4.35) extended with the generator $P_{1\infty}$, in such a way that every element p in the subset will have the same ordered $(p_1, p_2, p_3, p_4, p_\infty)$. We do this step by hand, actually explicitly calculating the needed transformation.

In each of the subsets, where every element has the same ordered $(p_1, p_2, p_3, p_4, p_\infty)$, we look for symmetries in G relating the elements. In particular, we relate elements in the same subset with transformations in the subgroup:

$$\langle P_{13}, P_{23}, P_{34} \rangle \subset \text{G}. \tag{4.36}$$

Since elements in the same subset are orbits with the same length and same $(p_1, p_2, p_3, p_4, p_\infty)$, the action of the group of transformations (4.36) reduces to the action of the pure braid group P_4, that in this case is finite by construction. In the following, we state the *factorization algorithm*. Firstly, we factorize with respect to the finite group (4.35):

Algorithm 9.

1. Consider $p \in \mathcal{C}_1$.

2. Remove from $\mathcal{C}_1$ the set $\mathcal{O}_{P_4}(p)$ and save p in the set $\mathcal{C}_2'$.

3. Apply to p all transformations in $\langle \mathrm{sign}_1, \ldots, \mathrm{sign}_4 \rangle$ and save the result in the set A_0.

   For every $p' \in A_0$:

4. Apply to p' all transformations in $\langle (12)(34), (1234) \rangle$ and save the result in the set A_1.

   For every $p'' \in A_1$:

5. If p'' is in $\mathcal{C}_1$, then $\mathcal{O}_{P_4}(p)$ and $\mathcal{O}_{P_4}(p'')$ are equivalent. Remove $\mathcal{O}_{P_4}(p'')$ from $\mathcal{C}_1$. If p'' is not in $\mathcal{C}_1$, apply again the current Step to the next p'' in A_1.

6. If all possible choices of p'' in A_1 are exhausted go to Step 1.

This algorithm ends when all choices of points p in the finite set $\mathcal{C}_1$ are exhausted. The set $\mathcal{C}_2'$, created in Step 2, will contain 122 points.

Now, we are going to further factorize the set $\mathcal{C}_2'$, as anticipated above, firstly subdividing $\mathcal{C}_2'$ in subset which elements are orbits with same length and with the same $(p_1, p_2, p_3, p_4, p_\infty)$ modulo change of signs or permutations.

Algorithm 10.

1. Consider $p \in \mathcal{C}_2'$, with $|\mathcal{O}_{P_4}(p)| = N$, $N \in \mathbb{N}$.

2. Save p in a set A_N.

3. Remove p from $\mathcal{C}_2'$.

   For every $p' \in \mathcal{C}_2'$:

4. If p' is such that:

   - $|\mathcal{O}_{P_4}(p')| = N$.
   - $(p_1, p_2, p_3, p_4, p_\infty)$ and $(p_1', p_2', p_3', p_4', p_\infty')$ differ by change of signs or permutations.

   Save p' in A_N and remove p' from $\mathcal{C}_2'$, otherwise apply again this Step to another $p' \in \mathcal{C}_2'$.

Since the set $\mathcal{C}_2'$ is finite, this algorithm ends when there are no more elements in $\mathcal{C}_2'$. This algorithm generates a finite list of 54 subsets A_N, where N is such that for every $p \in A_N$ we have $|\mathcal{O}_{P_4}(p)| = N$.

Next, in each subset A_N, we apply transformations generated by the subgroup (4.35) extended with the generator $P_{1\infty}$, in such a way that every element in the same subset will have the same ordered $(p_1, p_2, p_3, p_4, p_\infty)$. Afterwards, we quotient each subset with the action of the subgroup of transformations $< P_{13}, P_{23}, P_{34} >$, that inside each subset acts as the pure braid group P_4.

Algorithm 11.

For every subset A_N:

1. Consider $p \in A_N$ and save it in the set $\mathcal{C}_2$.

2. Remove p from A_N.

3. Act with the subgroup:

$$\langle \text{sign}_1, \dots, \text{sign}_4, (12)(34), (1234), P_{1\infty} \rangle \subset \text{G},$$

to each element in the set A_N, producing a new set A'_N in such a way that every element p' in A'_N will have:

$$(p'_1, p'_2, p'_3, p'_4, p'_\infty) = (p_1, p_2, p_3, p_4, p_\infty).$$

For every $p' \in A'_N$:

4. Generate the orbit of p' under the action of the subgroup $\langle P_{13}, P_{23}, P_{34} \rangle$. If p is in this orbit, then $\mathcal{O}_{P_4}(p)$ and $\mathcal{O}_{P_4}(p')$ are equivalent. Apply again this Step to another $p' \in A'_N$, otherwise save p' in $\mathcal{C}_2$ and apply again this Step to another $p' \in A'_N$.

5. When all choices of $p' \in A'_N$ are exhausted, go to Step 1.

Since the number of subsets A_N is 54, and each subset has a finite number of elements, this algorithm ends when there are no more subsets A_N to process. Finally, Algorithm 11 generates a set $\mathcal{C}_2$, that contains 54 elements and hence the classification of all finite orbits with points p

satisfying conditions (C1), (C2), (C3.1), (C3.2). We summarize the content of the set $\mathcal{C}_2$, in Table 4.2.

Remark 72. During the *factorization algorithm*, we apply the generators of G with a specific order. As a consequence, we are factorizing only with respect to a subgroup of the group of symmetries G. However, the set $\mathcal{C}_2$ contains the factorization we were looking for. Indeed, we recall that: under the action of the group P_4 the parameters p_i, for $i = 1, \ldots, 4, \infty$, remain constant (see the definition of the generators of P_4 given in (1.87)), moreover, the group G acts finitely on the parameters p_i, for $i = 1, \ldots, 4, \infty$ and G preserves the length of a finite P_4-orbit. We checked that every two orbits in the set $\mathcal{C}_2$, satisfy:

- If they have same length and parameters $(p_1, p_2, p_3, p_4, p_\infty)$ and $(p'_1$, p'_2, p'_3, p'_4, $p'_\infty)$ respectively, then there does not exist a transformation $\Phi \in G$ such that $\phi(p_i) = p'_i$.
- If two orbits have same parameters p_i, for $i = 1, \ldots, 4, \infty$, then the two orbits have different lengths.

Table 4.2: The 54 finite orbits.

#	sz.	p_1	p_2	p_3	p_4	p_∞	p_{21}	p_{31}	p_{32}	p_{41}	p_{42}	p_{43}
1	36	0	0	-1	0	$\sqrt{2}$	$-\sqrt{2}$	-1	$-\sqrt{2}$	0	0	-1
2	36	0	0	0	1	1	0	2	0	-1	-1	1
3	40	-1	1	$\sqrt{2}$	1	$-\sqrt{2}$	-1	$-\sqrt{2}$	0	1	1	$\sqrt{2}$
4	40	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0
5	40	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	1	1	-1
6	45	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
7	45	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	1	2	1
8	48	$\sqrt{2}$	0	0	0	$\sqrt{2}$	$\sqrt{2}$	-1	$\sqrt{2}$	0	0	1
9	72	0	0	-1	0	0	$\sqrt{2}$	$-\sqrt{2}$	1	-1	0	0
10	72	$-\sqrt{2}$	0	0	-1	$-\sqrt{2}$	0	-1	-1	$\sqrt{2}$	$-\sqrt{2}$	0
11	81	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	-1	$\frac{1-\sqrt{5}}{2}$	-1	0
12	81	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	-1	1
13	96	0	0	0	0	$-\sqrt{2}$	0	$-\sqrt{2}$	-1	$\sqrt{2}$	-1	0
14	96	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	2	1	-1
15	96	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
16	96	0	0	1	0	-1	2	0	0	$-\sqrt{2}$	$\sqrt{2}$	-1
17	105	$-\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	-1
18	105	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	0	0
19	108	$\frac{1+\sqrt{5}}{2}$	1	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-2	0	2
20	108	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{1-\sqrt{5}}{2}$	-2	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0
21	120	1	0	-1	0	-1	0	-1	$\sqrt{2}$	$-\sqrt{2}$	-1	0
22	144	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	-1	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$
23	144	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	$\frac{1+\sqrt{5}}{2}$	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	1	-2	-1	0
24	144	0	1	0	0	$\sqrt{2}$	0	2	0	1	$-\sqrt{2}$	-1
25	192	2	2	-2	-2	-2	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	1
26	192	0	0	0	0	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	-1	$-\sqrt{2}$	-1
27	200	0	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
28	200	$\frac{1+\sqrt{5}}{2}$	0	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$
29	205	-1	1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	-1	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0
30	216	-1	0	0	0	0	0	$\sqrt{2}$	1	$-\sqrt{2}$	0	1
31	220	-1	1	$\frac{-1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	$-\frac{1+\sqrt{5}}{2}$
32	220	$-\frac{1+\sqrt{5}}{2}$	-1	-1	$-\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
33	240	1	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{1-\sqrt{5}}{2}$	0	1	0	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	1	$\frac{-1+\sqrt{5}}{2}$
34	240	$\frac{1-\sqrt{5}}{2}$	0	0	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	-1	-1	-2	$\frac{1-\sqrt{5}}{2}$	0	0
35	240	1	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	-1	$\frac{1-\sqrt{5}}{2}$	0	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$

#	sz.	p_1	p_2	p_3	p_4	p_∞	p_{21}	p_{31}	p_{32}	p_{41}	p_{42}	p_{43}
36	240	0	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	0	$-\frac{1+\sqrt{5}}{2}$	1	0	-1	$\frac{1+\sqrt{5}}{2}$	-1	0
37	300	$\frac{1+\sqrt{5}}{2}$	1	1	1	1	1	0	1	1	$\frac{1+\sqrt{5}}{2}$	1
38	300	1	$\frac{-1+\sqrt{5}}{2}$	1	1	-1	-1	$\frac{1-\sqrt{5}}{2}$	-1	0	0	$\frac{1-\sqrt{5}}{2}$
39	360	0	$\frac{-1+\sqrt{5}}{2}$	0	-1	$\frac{-1+\sqrt{5}}{2}$	-1	-1	-1	1	1	0
40	360	$\frac{1-\sqrt{5}}{2}$	0	0	$-\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	2	1	0	0
41	360	1	0	$-\frac{1+\sqrt{5}}{2}$	0	$-\frac{1+\sqrt{5}}{2}$	-1	0	0	0	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
42	432	1	-1	1	1	-1	-1	0	0	$\frac{1-\sqrt{5}}{2}$	-1	1
43	480	0	0	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	0	1	0	1	-1
44	480	0	0	$\frac{1+\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	0	1	0	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
45	580	$\frac{1-\sqrt{5}}{2}$	0	0	0	$\frac{1+\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	-1	0	-2	-1
46	600	0	-1	0	$\frac{1-\sqrt{5}}{2}$	-1	0	$\frac{1-\sqrt{5}}{2}$	1	$-\frac{1+\sqrt{5}}{2}$	0	-1
47	600	$-\frac{1+\sqrt{5}}{2}$	1	0	0	1	-1	$\frac{-1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	-1	-2
48	900	0	0	0	-1	$\frac{-1+\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	1
49	900	0	0	0	-1	$-\frac{1+\sqrt{5}}{2}$	0	1	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
50	1200	0	0	$\frac{1-\sqrt{5}}{2}$	0	0	$\frac{-1+\sqrt{5}}{2}$	1	1	-1	-1	1
51	1200	0	$\frac{1+\sqrt{5}}{2}$	0	0	0	$\frac{1-\sqrt{5}}{2}$	1	$\frac{-1+\sqrt{5}}{2}$	1	0	1
52	1440	1	0	0	0	-1	0	0	2	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
53	2160	0	0	0	-1	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	-2	0	1	1
54	3072	0	0	0	0	0	$-\frac{1+\sqrt{5}}{2}$	0	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	1

4.5 Proofs of Theorems 62-63 and Lemmata 66-70

In this Section, we give proofs of Theorems 62-63 and Lemmata 66-69. Firstly, we proceed with the proof of Theorems 62-63:

Proof of Theorem 62. In order to prove the statement, we distinguish three cases:

(i) Firstly, we prove that given p with three projections over four in the set O_{ID}, then p is *not relevant*. In particular, it is enough to consider $m \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and the following three projections defined in (1.113):

$$\widetilde{n} = (M_1, M_2, M_3) \in \mathrm{O}_{\mathrm{ID}},$$

$$\widehat{n} = (M_2, M_3, M_4) \in \mathrm{O}_{\mathrm{ID}},$$

$$\breve{n} = (M_1, M_2, M_4) \in \mathrm{O_{ID}},$$

and all other cases follows since they differ from this case only by a permutation of the matrices M_i, see Lemma 59. If any of $M_i = \pm\mathbb{1}$, then we conclude. If not, we are left with the following case:

$$\widetilde{N}_\infty = M_3 M_2 M_1 = \widetilde{\epsilon}\,\mathbb{1}, \tag{4.37}$$

$$\widehat{N}_\infty = M_4 M_3 M_2 = \widehat{\epsilon}\,\mathbb{1}, \tag{4.38}$$

$$\breve{N}_\infty = M_4 M_2 M_1 = \breve{\epsilon}\,\mathbb{1}, \tag{4.39}$$

where $\epsilon = \pm 1$. Then by equations (4.37) and (4.38):

$$M_1 M_3 M_2 = \widetilde{\epsilon}\,\widehat{\epsilon} M_4 M_3 M_2 \Leftrightarrow M_4 = \widetilde{\epsilon}\,\widehat{\epsilon} M_1, \tag{4.40}$$

and by equations (4.37) and (4.39):

$$M_3 M_2 M_1 = \widetilde{\epsilon}\,\breve{\epsilon} M_4 M_2 M_1 \Leftrightarrow M_3 = \widetilde{\epsilon}\,\breve{\epsilon} M_4, \tag{4.41}$$

then $M_3 = \widehat{\epsilon}\,\breve{\epsilon} M_1$. As a consequence, equation (4.39) becomes:

$$M_4 M_2 M_1 = \breve{\epsilon}\,\mathbb{1} \Leftrightarrow \widetilde{\epsilon}\,\widehat{\epsilon} M_1 M_2 M_1 = \breve{\epsilon}\,\mathbb{1} \Leftrightarrow M_2 = \widetilde{\epsilon}\,\widehat{\epsilon}\,\breve{\epsilon} M_1^{-2}, \tag{4.42}$$

and finally:

$$m = (M_1, \widetilde{\epsilon}\,\widehat{\epsilon}\,\breve{\epsilon} M_1^{-2}, \widehat{\epsilon}\,\breve{\epsilon} M_1, \widetilde{\epsilon}\,\widehat{\epsilon} M_1), \tag{4.43}$$

which is reducible. Therefore p is not relevant.

(ii) Suppose p is such that three projections over four are in the set $\mathrm{O_{RED}}$, then p has associated reducible monodromy group. Given $m \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, it is enough to consider the following three projections defined in (1.113):

$$\widetilde{n} = (M_1, M_2, M_3) \in \mathrm{O_{RED}},$$
$$\widehat{n} = (M_2, M_3, M_4) \in \mathrm{O_{RED}},$$
$$\breve{n} = (M_1, M_2, M_4) \in \mathrm{O_{RED}},$$

and all other cases follows since they differ from this case only by a permutation of the matrices M_i, see Lemma 59. Then:

- M_1, M_2, M_3 have $\widetilde{v}$ as common eigenvector.
- M_2, M_3, M_4 have $\widehat{v}$ as common eigenvector.
- M_1, M_2, M_4 have $\breve{v}$ as common eigenvector.

All the matrices M_i for $i = 1, \ldots, 4$ are 2×2 matrices, as a consequence each matrix M_i can have at most two distinct eigenvectors: the matrix M_2 that appear in all the three projections, has $\widetilde{v}, \widehat{v}$ and $\breve{v}$ as eigenvectors then one of the following identities must hold:

$$\widetilde{v} = \widehat{v} \text{ or } \widetilde{v} = \breve{v} \text{ or } \widehat{v} = \breve{v}. \tag{4.44}$$

We can freely chose any of identities (4.44), so that $M_1, \ldots, M_4$ have a common eigenvector, making the monodromy group reducible.

(iii) When there are three projections in O, not all of the same type,we apply Lemma 66. This concludes the proof.

□

Proof of Theorem 63. Suppose $m = (M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ with:

$$M_\infty = (M_4 M_3 M_2 M_1)^{-1} = \pm\mathbb{1}, \tag{4.45}$$

Then, applying the trace operator and the skein relation to (4.45), we obtain relations (4.15). This concludes the first part of the proof.

Suppose $m = (M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$, with co-adjoint coordinates p in the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$. By the definition of the big open subset $\mathcal{U}$, and since p satisfies relations (4.15), then it is straightforward computation to check that the matrix $M_\infty = \pm\mathbb{1}$ in the charts $\mathcal{U}^{(i)}_{jk}$, for $i = 0, 1, 2$, defined in the statement of Theorem 3. This concludes the proof.

□

Next we give the proofs of Lemmata 66-70:

Proof of Lemma 66. Consider $m \in \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and the following two distinct *generic* projections:

$$(M_i, M_j, M_k) \in \mathrm{O}_{\mathrm{ID}},\ i > j > k,\ i, j, k = 1, \dots, 4, \tag{4.46}$$

$$(M_{i'}, M_{j'}, M_{k'}) \in \mathrm{O}_{\mathrm{RED}},\ i' > j' > k',\ i', j', k' = 1, \dots, 4. \tag{4.47}$$

If either M_i, M_j, M_k is equal to $\pm\mathbb{1}$, then we conclude, otherwise suppose:

$$(M_i M_j M_k)^{-1} = \pm\mathbb{1}. \tag{4.48}$$

Moreover, suppose the monodromy group associated to the triple $(M_{i'}, M_{j'}, M_{k'})$ is reducible, then matrices $M_{i'}, M_{j'}, M_{k'}$ have a common eigenvector v. There always exist two indices in (4.46) and in (4.47) that are equal, without loss of generality, suppose $i \neq i'$, $j = j'$ and $k = k'$, then equation (4.48) can be written as:

$$M_i = \pm(M_{j'} M_{k'})^{-1}.$$

The last equation implies that M_i has v as eigenvector, then the monodromy group $< M_i, M_{i'}, M_j, M_k >$ is reducible. This concludes the proof. □

Proof of Lemma 67. Suppose q are the co-adjoint coordinates on $\widehat{\mathcal{M}}_{PVI}$ of the triple $n = (N_1, N_2, N_3)$. Since the monodromy group $< N_1, N_2, N_3 >$ is reducible, we can suppose the three matrices N_1, N_2, N_3 to be upper triangular. Then N_1, N_2, N_3 have the eigenvalues on the diagonal and since $eigenv(N_i) = \exp(\epsilon_l \pi \theta_i)$, where $\epsilon_l = \pm 1$, the following formulae hold:

$$\mathrm{Tr}(N_i N_j) = 2\cos(\pi(\epsilon_i \theta_i + \epsilon_j \theta_j)),\ i, j = 1, 2, 3,\ i > j, \tag{4.49}$$

$$\mathrm{Tr}(N_3 N_2 N_1) = 2\cos(\pi(\epsilon_1 \theta_1 + \epsilon_2 \theta_2 + \epsilon_3 \theta_3)). \tag{4.50}$$

Applying trigonometric identities and being q the co-adjoint coordinates of n in $\widehat{\mathcal{M}}_{PVI}$, we get:

$$q_{ij} = \frac{1}{2}(q_i q_j - \epsilon_i \epsilon_j s_i s_j),\ i, j = 1, 2, 3,\ i > j, \tag{4.51}$$

$$q_\infty = \frac{1}{4}(q_1 q_2 q_3 - \epsilon_1 \epsilon_2 s_1 s_2 q_3 - \epsilon_1 \epsilon_3 s_1 s_3 q_2 - \epsilon_2 \epsilon_3 s_2 s_3 q_1), \tag{4.52}$$

where $s_l := \sqrt{4 - q_l^2}$ for $l = 1, 2, 3$. This concludes the proof. □

Proof of Lemma 68. Suppose q are the co-adjoint coordinates on $\widehat{\mathcal{M}}_{PVI}$ of the projection of p that is supposed to be in the set $\mathrm{O_{ID}}$. Moreover, suppose the triple $n = (N_1, N_2, N_3)$ is associated to q. If any of the N_i is equal to $\pm\mathbb{1}$, by the matching procedure, we end up with a point p that is *not relevant*, therefore, we avoid this case, otherwise, assume $N_\infty = N_3 N_2 N_1 = \pm\mathbb{1}$, then:

$$N_1 = \pm(N_3 N_2)^{-1},\ N_2 = \pm(N_1 N_3)^{-1},\ N_3 = \pm(N_2 N_1)^{-1}. \tag{4.53}$$

Being q the co-adjoint coordinates of n on $\widehat{\mathcal{M}}_{PVI}$, by straightforward computation, we get:

$$\begin{cases} q_{21} &= \pm q_3, \\ q_{31} &= \pm q_2, \\ q_{32} &= \pm q_1, \\ q_\infty &= \pm 2. \end{cases} \tag{4.54}$$

This concludes the proof. □

Proof of Lemma 69. We prove the statement if p has projections $\widehat{q}, \check{q}, \overline{q}$ such that two projections are in the set $\mathrm{O}_{\mathrm{RED}}$. There are three distinct cases: we are going to prove in detail the case when $\widehat{q}, \check{q} \in \mathrm{O}_{\mathrm{RED}}$ then remaining cases can be proven in a similar way.

Given m, the two projections $\widehat{n}, \check{n} \in \mathrm{O}_{\mathrm{RED}}$ are such that:

$$\widehat{n} = (M_2, M_3, M_4), \ \check{n} = (M_1, M_2, M_4). \tag{4.55}$$

Since monodromy groups $< M_2, M_3, M_4 >$ and $< M_1, M_2, M_4 >$ are reducible, then M_2, M_4 are diagonal and M_1,M_3 can be supposed, without loss of generality, upper and lower triangular respectively, and each matrix will have its own eigenvalues on the diagonal. Recall that $eigenv(M_k) = \exp(\epsilon_k \pi \theta_k)$ where $\epsilon_k = \pm 1$. Therefore, by Lemma 67, the following relations hold:

$$\mathrm{Tr}(M_i M_j) = 2\cos(\pi(\epsilon_i \theta_i + \epsilon_j \theta_j)), \ i > j, \ i, j = 2, 3, 4, \tag{4.56}$$

$$\mathrm{Tr}(M_{i'} M_{j'}) = 2\cos(\pi(\epsilon_{i'} \theta_{i'} + \epsilon_{j'} \theta_{j'})), \ i' > j', \ i', j' = 1, 3, 4. \tag{4.57}$$

Consider the remaining projection $\overline{n} = (M_1, M_3, M_4) \in \widehat{\mathcal{M}}_{PVI}$, with associated co-adjoint coordinates $\overline{q}$, then since relations (4.56)-(4.57) hold respectively for $i = 4, j = 1$ and $i' = 4, j' = 3$, using the trigonometric identities and matching (1.115), we get:

$$\bar{q}_{41} = \frac{1}{2}(\bar{q}_4\bar{q}_1 - \epsilon_4\epsilon_1\bar{s}_4\bar{s}_1), \tag{4.58}$$

$$\bar{q}_{43} = \frac{1}{2}(\bar{q}_4\bar{q}_3 - \epsilon_4\epsilon_3\bar{s}_4\bar{s}_3), \tag{4.59}$$

where $\bar{s}_k := \sqrt{4-\bar{q}_k^2}$ for $k = 1,3,4$. Then equations (4.58)-(4.59) can be written as:

$$\bar{q}_{41}^2 + \bar{q}_4^2 + \bar{q}_1^2 - \bar{q}_{41}\bar{q}_4\bar{q}_1 - 4 = 0,$$
$$\bar{q}_{43}^2 + \bar{q}_4^2 + \bar{q}_3^2 - \bar{q}_{43}\bar{q}_4\bar{q}_3 - 4 = 0,$$

and this concludes the proof. □

Proof of Lemma 70. We prove the statement for the case (i), then all the other cases can be proved in a similar way. Suppose $\widehat{q}, \check{q} \in \mathrm{O}_{\mathrm{ID}}$, then the only relevant case for our classification (see the beginning of the previous Section) is the following case:

$$(M_4M_3M_2)^{-1} = \widehat{\epsilon}\,\mathbb{1},\ \ (M_4M_2M_1)^{-1} = \check{\epsilon}\,\mathbb{1}, \tag{4.60}$$

where $\epsilon = \pm 1$. Therefore relations relations (4.19) and (4.20) follow from Lemma 68 and the matching (1.108),(1.109),(1.110).This concludes the proof.

□

Chapter 5

Outlook

In this thesis a list of 54 finite orbits of the action (10) of the pure braid group P_4 on the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with five boundary components is presented in Table 4.2. The list is folded up to the action of the group of symmetries G introduced in Chapter 2. Due to the identification of each action of the restriction H_i (determined by the rows in Table 1.1) with the finite action of P_3 over the $\mathrm{SL}_2(\mathbb{C})$ character variety of Σ_4, we can associate to each restriction an algebraic solution of PVI (see [DM00, Iwa03, Cou16, LT14]). Then in the list of 54 finite orbits each orbit's member has the following properties:

- no more than one restriction (determined by the rows of Table 1) is associated to algebraic solutions of PVI obtained by the *pull-back* of the hypergeometric equation, see Doran [Dor01] and Andreev-Kitaev [AK02],
- no more than one restriction corresponds to the so-called Picard solutions of PVI, see the work of Picard [Pic89] and Mazzocco [Maz01b].

Moreover, we do not allow any orbit's member such that:

- one restriction is associated to algebraic solutions of PVI obtained by the *pull-back* of the hypergeometric equation and another restriction is associated to the so-called Picard solutions of PVI.

Many open questions remain. If we consider the parametrization result given in Theorem (3), Lemma (4) and Proposition (5), we could reconstruct, up to global conjugation, the monodromy matrices associated to a *candidate* point, using the matching procedure (given in Section 4.1) only on two points q.

This means that we could extend the classification result given in this thesis to finite orbits whose members can have up to two projections, defined in (1.114), of Picard or Hitching-Kitaev type. This computation is theoretically possible but it is extremely technical and would require many technical Lemmata in order to cover all sub-cases that we decided not to include them in this thesis.

Another direction of research is to use our method to classify all finite orbits of the action of the pure braid group P_n on the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with $n+1$ boundary components for $n > 4$, or in other words all algebraic solutions of the Garnier system $\mathcal{G}_{n-2}$. We expect that the matching procedure can be adapted in order to work in this case too. For generic $n > 4$, the number of restrictions to the action of the pure braid group P_3 over $\mathcal{M}_{PVI}$ will be $\binom{n}{3}$, consequently many more necessary conditions to be satisfied are introduced in order to produce a *candidate* point.

In our case, for $n = 4$, we relay on a finite extended list E_{45} of $86,768$ points q producing only 54 finite orbits. Since the extended list E_{45} remains the same, and the number of necessary conditions increases with n, we expect that the resulting classification list will contain less and less finite orbits.

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