Appl. Math. J. Chinese Univ.
2017, 32(2): 237-252

Competitive analysis of online inventory problem with interrelated prices

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Abstract. This paper investigates the online inventory problem with interrelated prices in which a decision of when and how much to replenish must be made in an online fashion even without concrete knowledge of future prices. Four new online models with different price correlations are proposed in this paper, which are the linear-decrease model, the log-decrease model, the logarithmic model and the exponential model. For the first two models, the online algorithms are developed, and as the performance measure of online algorithm, the upper and lower bounds of competitive ratios of the algorithms are derived respectively. For the exponential and logarithmic models, the online algorithms are proposed by the solution of linear programming and the corresponding competitive ratios are analyzed, respectively. Additionally, the algorithm designed for the exponential model is optimal, and the algorithm for the logarithmic model is optimal only under some certain conditions. Moreover, some numerical examples illustrate that the algorithms based on the dprice-conservative strategy are more suitable when the purchase price fluctuates relatively flat.

§1 Introduction

The development of inventory problem depends much on the economic order quantity (EOQ) model. The research of inventory problem has ample academic achievements and has formed a system of study [13] and [6]. Prices are generally assumed to be a probability distribution or constant in the classical inventory problem. Serel [12] studied the optimal ordering and pricing problem based on the interrelated demand and price in the rapid response system which has twice orders. Banerjee and Sharma [2] studied the inventory model with seasonal demand in two potentially replaceable markets. Sana [11] generalized the EOQ model to the

Received: 2015-11-14. Revised: 2017-03-28.
MR Subject Classification: 11Y16, 90B35, 68Q25.
Keywords: online, inventory problem, interrelated prices, competitive analysis.
Digital Object Identifier(DOI): 10.1007/s11766-017-3360-4.
Supported by the National Natural Science Foundation of China (11571013, 11471286).
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case of perishable products with sensitive demand to price. Lin and Ho [8] studied the optimal ordering and pricing problem of the joint inventory model with sensitive demand for price based on the quantity discount. Studies mentioned above generally assumed that the parameters are determined. In fact, some parameters possess uncertainty and the research for the uncertain parameters is mainly divided into two categories, fuzzy and random. Webster and Weng [15] studied the ordering and pricing problem of the fashion product's supply chain which consisted of producer and seller, and random demand is sensitive to price in the supply chain. Tajbakhsh, Lee and Zolfaghari [14] studied the inventory model in which the time points of price discount are from the Poisson distribution and the discount price is a discrete random variable. Abouee-Mehrizi et al. [1] studied inventory problem of joint production with demand which is from the Poisson distribution in two levels of inventory system.

In the classical inventory problem, the optimal solution changes as the parameter distribution changes. But in fact, there is no clear law of change or distribution in the prices, with only part of the price information to be obtained. In the international commodity markets, the precise distributions of the prices of oil, iron ore and gold have not been well defined. Within the volatility of the oil price, a country has to make a decision of when and how to complete the oil reserve, which is of particularly strategic importance. Therefore, it is essential to develop some new methods to solve such online inventory problem [7], which does not depend on any specific parameter distribution.

The online inventory problem with price is challenging where the decision maker, the retailer, must make a decision of when and how much to purchase without future prices. The online inventory problem with price considered also has some applications in the stock fund market. In the establishing stage of fund, the fund manager's task is to determine what stocks to buy and how much to buy, based on stock price volatility in a certain period. And the goal of the fund manager is to purchase a certain number of shares at the least cost.

The online inventory problem with price can be seen as an extension of the time series search problem and the financial one-way trading problem [5], [4], [3] and [17]. Some papers considered the online inventory problem. Larsen and Wøhlk [7] considered a real-time version of the inventory problem with continuous deterministic demand and involved the fixed order cost, the inventory cost per item unit per time unit but obtained algorithmic upper and lower bounds of the competitive ratio whereas the gap grows with the complexity of the modes. The inventory problem considered in [9] is a demand online inventory problem where the decision maker only knows the upper bound and lower bound of the daily demand and decides how many products should be prepared everyday. Ma and Pan [10] considered the online inventory problem with the assumption that the decision maker has the knowledge of the same upper and lower bounds of all prices.

This paper focuses on the price and analyzes the impact of price and the price-related patterns to the algorithms and results. And the bounds considered in the extant reference are difficult to be obtained if not impossible. In China stock market, the stock prices of the day vary in the interval from 90% to 110% of the previous day's closing price. We assume that the variation range of each price is interrelated with its preceding price. The problems considered in this paper become more practical and quite different from the problems in [7] and [10].

Four kinds of price interrelation: linear-decrease, log-decrease, exponential and logarithmic interrelation are considered in this paper, the first two interrelations and the third interrelation originate from [17], respectively, and the fourth interrelation is presented based on inspirations of the second model. The time series search problem and the financial one-way trading problem are searching for the maximum price and convert more currency in the large price to maximize the profits. And the price online inventory problem considered possible purchases more items in the low price to minimize the cost. So the methods and the conclusions of [5], [4], [3] and [17] can not be directly applied to the online inventory problem with price, which should be modified according the new models.

This paper is organized as follows. We define the problem and present four models in Section 2. In Section 3, we present an online strategy and its upper and lower bounds for the linear-decrease model. In Section 4, we provide the competitive analysis of the log-decrease model and compare the competitive ratios for the linear-decrease and log-decrease models. In Section 5, we present an optimal online algorithm for the exponential model. In Section 6, we discuss the competitive analysis of the logarithmic model. In Section 7 we demonstrate some numerical examples for our consequences. Section 8 summarizes our conclusions and indicates some directions for future work.

§2 Problem statement

This paper considers an online inventory problem that the decision maker, the retailer, should decide when and how much to purchase every day without knowing future prices during the purchasing process. The storage capacity, U (without loss of generality, we assume the storage capacity U is 1), must be reached when the game is over. Additionally, the initial inventory level is zero. The objective of the decision maker is to minimize the total cost. In order to be more general, we introduce some different variation ranges of price. That is, the price has its own variation range and the range is variable. Let n denote the number of purchasing days. Denoted by p and p_i the current purchase price and the price of the ith day, respectively.

The online inventory problem with price is unlikely to find the optimal offline solution, because it does not have enough price information to determine. Thus, the competitive ratio is introduced to evaluate the performance of algorithm which solves the online inventory problem with price. An arbitrary online algorithm, ALG, is referred to as c-competitive, if for an arbitrary input price instance I has $ALG(I) \leq c \cdot OPT(I)$, where $ALG(I)$ denotes the cost of the online algorithm ALG, and $OPT(I)$ is the cost of the optimal offline algorithm OPT. The competitive ratio of algorithm ALG is defined as the minimum c, which satisfies the inequality. The competitive ratio, c^*, of the optimal online algorithm is the lower bound of the price online inventory problem, defined as

$$c^* = \inf_{ALG} c_{ALG},$$

where c_{ALG} is the competitive ratio of algorithm ALG.

Let θ, θ_1 and θ_2 denote the known parameters, where $\theta > 0$ and $0 < \theta_1 \le \theta_2$. The four price interrelation models are depicted below.

- The linear-decrease model: $p_i \in [p_{i-1} - \theta, p_{i-1} + \theta],\ 2 \le i \le n$.
- The log-decrease model: $p_1 \in [p - \theta, p + \theta], p_i \in [p - \theta \ln i, p + \theta \ln i],\ 2 \le i \le n$.
- The exponential model: $p_{i+1} \in [\theta_1 p_i, \theta_2 p_i],\ 1 \le i \le n-1$.
- The logarithmic model: $p_i \in [\theta_1 p_1 \ln i, \theta_2 p_1 \ln i],\ 2 \le i \le n$.

§3 Competitive analysis of the linear-decrease model

For this model, we present a dprice-conservative (DPC) strategy [7] which is described as follows. Let c be competitive ratio that can be achieved by an algorithm.

- DPC-l: At the end of the game, ensure that the inventory capacity is reached.
- DPC-2: Purchase items only at the lowest price of the day.
- DPC-3: When the price reaches a new lowest, purchase enough to ensure that the competitive ratio c can be obtained even if purchase the remaining items at the highest price of the day.

Lemma 3.1. *If ALG is a c-competitive DPC strategy, then the daily purchasing quantity is*

$$s_1 = \frac{p+\theta}{2\theta} - \frac{c(p-\theta)}{2\theta}, \tag{1}$$

$$s_i = \frac{c}{2i} + \frac{D_{i-1}}{2i},\ (i = 2, 3, \cdots, n). \tag{2}$$

Proof. When $i = 1$, by DPC-3, we must have

$$\frac{(p-\theta)s_1 + (1-s_1)(p+\theta)}{p-\theta} = c.$$

Here the numerator represents the cost of the DPC strategy and the denominator is the cost of the optimal offline algorithm for such an price sequence. By simplification, we obtain Equation 1. For $i \ne 1$, we can obtain

$$\frac{V_i + D_i(p+i\theta)}{p-i\theta} = c, \tag{3}$$

where V_i denotes the total cost after the ith day and D_i denotes the number of items that still need to purchase after the ith day. It is easy to get that

$$V_i = \sum_{j=1}^{i} p_j s_j \quad \text{and} \quad D_i = D_{i-1} - s_i.$$

Substituting $V_i = V_{i-1} + (p - i\theta)s_i$ into Equation 3, we obtain

$$\frac{V_{i-1} + s_i(p-i\theta) + D_{i-1}(p+i\theta) - s_i(p+i\theta)}{p-i\theta} = c. \tag{4}$$

From Equation 3 with $i-1$, we have

$$V_{i-1} + D_{i-1}[p + (i-1)\theta] = c[p - (i-1)\theta]. \tag{5}$$

Combining Equations 4 and 5, we obtain

$$c[p - (i-1)\theta] + s_i(p - i\theta) - s_i(p + i\theta) + D_{i-1}\theta = c(p - i\theta),$$

which implies that Equation 2 holds. $\square$

Theorem 3.1. *In the linear-decrease model, the upper and lower bounds of competitive ratio of the DPC strategy are*

$$c^{\max} = 1 + \frac{ln(n+1)}{\frac{p-\theta}{2\theta} - \frac{ln(n+1)}{2}}$$

and

$$c^{\min} = 1 + \frac{\frac{ln(n+1)}{2} - \frac{1}{2}}{\frac{p-\theta}{2\theta} - \frac{ln(n+1)}{2} + \frac{1}{2}}.$$

Proof. Because the storage capacity is reached after the last purchase, we have $\sum\limits_{i=1}^{n} s_i = 1$. By Equations 1 and 2, we obtain

$$\frac{p+\theta}{2\theta} - \frac{c(p-\theta)}{2\theta} + \sum_{i=2}^{n} \frac{c}{2i} + \sum_{i=2}^{n} \frac{D_{i-1}}{2i} = 1.$$

It follows that

$$c = \frac{\frac{p-\theta}{2\theta} + \sum_{i=2}^{n} \frac{D_{i-1}}{2i}}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n} \frac{1}{2i}}.$$

By the definition of D_i, we have $0 \le D_{i-1} \le 1$, and thus

$$\frac{\frac{p-\theta}{2\theta}}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n} \frac{1}{2i}} \le c \le \frac{\frac{p-\theta}{2\theta} + \sum_{i=2}^{n} \frac{1}{2i}}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n} \frac{1}{2i}}.$$

That is,

$$c^{\min}(n) = 1 + \frac{\sum_{i=2}^{n} \frac{1}{2i}}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n} \frac{1}{2i}} \le c \le c^{\max}(n) = 1 + \frac{2\sum_{i=2}^{n} \frac{1}{2i}}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n} \frac{1}{2i}}. \tag{6}$$

Because $\ln(n+1) - \ln n = \frac{1}{\xi}(n < \xi < n+1)$, we have $\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$. Thus,

$$\ln(n+1) - 1 < \sum_{i=2}^{n} \frac{1}{i} < \ln(n+1) - \frac{1}{n+1}. \tag{7}$$

Combining Inequalities 6 and 7, we obtain

$$c^{\max}(n) < 1 + \frac{\ln(n+1) - \frac{1}{n+1}}{\frac{p-\theta}{2\theta} - \frac{\ln(n+1)}{2} + \frac{1}{2(n+1)}} < 1 + \frac{\ln(n+1)}{\frac{p-\theta}{2\theta} - \frac{\ln(n+1)}{2}} = c^{\max}, \tag{8}$$

$$c^{\min}(n) > 1 + \frac{\frac{\ln(n+1)}{2} - \frac{1}{2}}{\frac{p-\theta}{2\theta} - \frac{\ln(n+1)}{2} + \frac{1}{2}} = c^{\min}. \tag{9}$$

$\square$

Note that Inequality 8 holds only if $\frac{p-\theta}{2\theta} > \frac{\ln(n+1)}{2}$. Therefore, the price volatility, 2θ, must be small enough to satisfy this inequality for the lager n. Moreover, the rationality of DPC strategy highly depends on the choice of the price volatility and the initial price.

§4 Competitive analysis of the log-decrease model

For this model, we still use the dprice-conservative (DPC) strategy.

Lemma 4.1. *If ALG is a c-competitive DPC strategy, then the daily purchasing quantity is*

$$s_1 = \frac{p+\theta}{2\theta} - \frac{c(p-\theta)}{2\theta}, \tag{10}$$

$$s_i = \frac{c\ln\frac{i}{i-1}}{2\ln i} + \frac{D_{i-1}\ln\frac{i}{i-1}}{2\ln i},\ (i=2,3,\cdots,n). \tag{11}$$

Proof. Similarly, we can obtain Equation 10. When $i \neq 1$, by DPC-3, we must have

$$\frac{V_i + D_i(p+\theta\ln i)}{p-\theta\ln i} = c, \tag{12}$$

where the numerator represents the cost of DPC strategy and the denominator is the cost of the optimal offline algorithm for such an price sequence. Because $V_i = V_{i-1} + (p-\theta\ln i)s_i$, substituting it into Equation 12, we obtain

$$c = \frac{V_{i-1} + s_i(p-\theta\ln i) + D_{i-1}(p+\theta\ln i) - s_i(p+\theta\ln i)}{p-\theta\ln i}. \tag{13}$$

Substituting $i-1$ into Equation 12, we obtain

$$V_{i-1} + D_{i-1}[p+\theta\ln(i-1)] = c[p-\theta\ln(i-1)]. \tag{14}$$

Combining Equations 13 and 14 we obtain

$$c[p-\theta\ln(i-1)] + s_i(p-\theta\ln i) - s_i(p+\theta\ln i) + D_{i-1}\theta\ln\frac{i}{i-1} = c(p-\theta\ln i).$$

After simplification, we obtain Equation 11. □

Theorem 4.1. *In the log-decrease model, the upper and lower bounds of competitive ratio of DPC strategy are*

$$c^{\max} = 1 + \frac{2\sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}$$

and

$$c^{\min} = 1 + \frac{\sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]},$$

respectively.

Proof. Because the storage capacity is reached after the last purchase, we have $\sum\limits_{i=1}^{n} s_i = 1$. From Equations 10 and 11, we have

$$\frac{p+\theta}{2\theta} - \frac{c(p-\theta)}{2\theta} + \sum_{i=2}^{n}\frac{c\ln\frac{i}{i-1}}{2\ln i} + \sum_{i=2}^{n}\frac{D_{i-1}\ln\frac{i}{i-1}}{2\ln i} = 1.$$

After simplification, we obtain the competitive ratio which is

$$c = \frac{\frac{p-\theta}{2\theta} + \sum_{i=2}^{n}\left[\left(D_{i-1}\ln\frac{i}{i-1}\right)/2\ln i\right]}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}.$$

Note that $0 \le D_{i-1} \le 1$, we obtain the upper and lower bounds of c

$$\frac{\frac{p-\theta}{2\theta}}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]} \le c \le \frac{\frac{p-\theta}{2\theta} + \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}.$$

That is,

$$c^{\min}(n) = 1 + \frac{\sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]},$$

$$c^{\max}(n) = 1 + \frac{2\sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}{\frac{p-\theta}{2\theta} - \sum_{i=2}^{n}\left[\left(\ln\frac{i}{i-1}\right)/2\ln i\right]}.$$

□

§5 Competitive analysis of the exponential model

For $1 \le i \le n-1$, if $\theta_1 \ge 1$, then $p_{i+1} \ge \theta_1 p_i \ge p_i$. That is, the price sequence is monotonically increasing. So the minimum cost can be obtained by purchasing all on the first day. In addition, if $0 \le \theta_2 \le 1$, then $p_{i+1} \le \theta_2 p_i \le p_i$. That is, the price sequence is monotonically decreasing. So the minimum cost can be obtained by purchasing all on the last day. Thus, we only need to consider the case where $0 < \theta_1 < 1 < \theta_2$.

Firstly, we investigate a linear programming problem with variables $\{r, s_1, s_2, \ldots, s_n\}$ as follows.

$$\begin{aligned} \textbf{minimize} \quad & r & \text{(LP1)} \\ \textbf{such that} \quad & F_i(s_1, s_2, \ldots, s_n) \le r,\ i = 1, 2, \ldots, n \\ & s_1 + s_2 + \cdots + s_n = 1 \\ & s_i \ge 0,\ i = 1, 2, \ldots, n \end{aligned}$$

where $F_i(s_1, s_2, \ldots, s_n) = \frac{s_1}{\theta_1^{i-1}} + \frac{s_2}{\theta_1^{i-2}} + \cdots + \frac{s_{i-1}}{\theta_1} + s_i + s_{i+1}\theta_2 + \cdots + s_n\theta_2^{n-i}$.

Lemma 5.1. *The solution to the linear programming problem LP1 exists.*

Proof. We only need to prove that there exists $\{r', s'_1, s'_2, \cdots, s'_n\}$ such that

$$F_i(s'_1, s'_2, \ldots, s'_n) \le r', \quad i = 1, 2, \ldots, n \tag{15}$$

$$s'_1 + s'_2 + \cdots + s'_n = 1, \tag{16}$$

$$s'_i \ge 0, \qquad i = 1, 2, \ldots, n. \tag{17}$$

We construct them as following. Let $s'_1 = s'_2 = \cdots = s'_{n-1} = 0$, $s'_n = 1$. It is obvious that $s'_i \ge 0$ for $i = 1, 2, \ldots, n$ and $s'_1 + s'_2 + \cdots + s'_n = 0 + \cdots + 0 + 1 = 1$. That is, $\{s'_1, \cdots, s'_{n-1}, s'_n\} = \{0, \cdots, 0, 1\}$ satisfies Equation 16 and Inequality 17. In addition, we obtain

$$F_i(s'_1, s'_2, \ldots, s'_n) = F_i(0, 0, \ldots, 1) = \theta_2^{n-i}, \qquad i = 1, 2, \ldots, n.$$

Because $\theta_2 > 1$, $F_i(s'_1, s'_2, \ldots, s'_n) = F_i(0, 0, \ldots, 1)$ is monotonically decreasing with respect to i. That is, $\max\limits_{1 \le i \le n} F_i(0, 0, \ldots, 1) = \theta_2^{n-1}$. Let $r' = \theta_2^{n-1}$. It is clear that $F_i(0, 0, \ldots, 1) \le r'$ for all $1 \le i \le n$.

From the above results, we know that there exists $\{r', s'_1, s'_2, \cdots, s'_n\} = \{\theta_2^{n-1}, 0, 0, \cdots, 1\}$ which satisfies Equation 16 and Inequalities 15 and 17. Thus, the solution to the above linear programming problem LP1 exists. □

From Lemma 5.1, we know that the optimal solution to the linear programming problem LP1 can be obtained in polynomial time. Since the online algorithm purchases items according to the solution to the linear programming problem LP1, we denote the online algorithm by SLP_1.

Algorithm 1 :SLP$_1$

1: Solving the linear programming problem LP1, and let $\{r^*, s_1^*, s_2^*, \cdots, s_n^*\}$ be the solution.
2: Purchasing $s_i^*(1 \le i \le n)$ units at period i.

Theorem 5.1. *The competitive ratio of SLP_1 is r^*.*

Proof. Let $\sigma = p_1, p_2, \ldots, p_n$ be an arbitrary price sequence. Without loss of generality, we assume that the lowest price in σ is p_i. Apparently, $OPT(\sigma) = p_i$ while $SLP_1(\sigma) = \sum\limits_{j=1}^{n} s_j^* p_j$.

Because $p_{j+1} \in [\theta_1 p_j, \theta_2 p_j]$ when $1 \le j \le n-1$, we have

$$p_j \le \frac{p_i}{\theta_1^{i-j}} \qquad j = 1, 2, \ldots, i,$$

$$p_j \le \theta_2^{j-i} p_i \quad j = i+1, i+2, \ldots, n.$$

Because $\frac{SLP_1(\sigma)}{OPT(\sigma)} = \frac{\sum_{j=1}^{n} s_j^* p_j}{p_i}$, we obtain

$$\frac{SLP_1(\sigma)}{OPT(\sigma)} \le \frac{\frac{s_1^*}{\theta_1^{i-1}} p_i + \frac{s_2^*}{\theta_1^{i-2}} p_i + \cdots + \frac{s_{i-1}^*}{\theta_1} p_i + s_i^* p_i + s_{i+1}^* \theta_2 p_i + \cdots + s_n^* \theta_2^{n-i} p_i}{p_i}.$$

Thus,

$$\frac{SLP_1(\sigma)}{OPT(\sigma)} \le \frac{s_1^*}{\theta_1^{i-1}} + \frac{s_2^*}{\theta_1^{i-2}} + \cdots + \frac{s_{i-1}^*}{\theta_1} + s_i^* + s_{i+1}^* \theta_2 + \cdots + s_n^* \theta_2^{n-i} = F_i(s_1^*, s_2^*, \cdots, s_n^*).$$

Combining the optimal solution to the linear programming problem LP1, we rewrite the above inequality in the following.

$$\frac{SLP_1(\sigma)}{OPT(\sigma)} \le F_i(s_1^*, s_2^*, \ldots, s_n^*) \le r^*, \qquad i = 1, 2, \ldots, n,$$

where r^* is the minimum one that satisfies the above inequality.

Hence, r^* is the competitive ratio of SLP_1. □

We show that SLP_1 algorithm is optimal for the exponential model in the following.

Theorem 5.2. *The competitive ratio of any online algorithm for the exponential model is not less than r^*.*

Proof. Let ALG be an arbitrary online algorithm for the exponential model. We construct n special sequences $\hat{\sigma_1}, \hat{\sigma_2}, \ldots, \hat{\sigma_n}$, where

$$\hat{\sigma_i} = \frac{1}{\theta_1^{i-1}}, \frac{1}{\theta_1^{i-2}}, \ldots, \frac{1}{\theta_1}, 1, \theta_2, \ldots, \theta_2^{n-i}.$$

Without loss of generality, we assume that the algorithm ALG purchases s'_i (where $1 \le i \le n$ and $\sum_{j=1}^{n} s'_j = 1$) units at period i.

To obtain the desired result, we only need to prove that there exists at least one sequence $\hat{\sigma_j}$ such that $\frac{ALG(\hat{\sigma_j})}{OPT(\hat{\sigma_j})} \ge r^*$. We show it by contradiction.

Otherwise, assume that $\frac{ALG(\hat{\sigma_j})}{OPT(\hat{\sigma_j})} < r^*$ for all j $(1 \le j \le n)$, we obtain a contradiction in the following. Because $0 < \theta_1 < 1 < \theta_2$, for every j $(1 \le j \le n)$, we know that $OPT(\hat{\sigma_j}) = 1$,

$$ALG(\hat{\sigma_j}) = \frac{s'_1}{\theta_1^{j-1}} + \frac{s'_2}{\theta_1^{j-2}} + \cdots + \frac{s'_{j-1}}{\theta_1} + s'_j + s'_{j+1}\theta_2 + \cdots + s'_n\theta_2^{n-j} = F_j(s'_1, s'_2, \ldots, s'_n).$$

Hence,

$$\frac{ALG(\hat{\sigma_j})}{OPT(\hat{\sigma_j})} = F_j(s'_1, s'_2, \ldots, s'_n).$$

By $\frac{ALG(\hat{\sigma_j})}{OPT(\hat{\sigma_j})} < r^*$ for all j, we obtain

$$F_j(s'_1, s'_2, \ldots, s'_n) < r^*, \qquad j = 1, 2, \ldots, n.$$

It indicates that there exists a $r' < r^*$ such that $F_j(s'_1, s'_2, \ldots, s'_n) \le r'$ for all $j (1 \le j \le n)$, which contradicts the minimality of r^*. $\square$

§6 Competitive analysis of the logarithmic model

In this model, if $\theta_1 = \theta_2$, then $p_i \in [\theta_1 p_1 \ln i, \theta_1 p_1 \ln i] = \theta_1 p_1 \ln i$, $2 \le i \le n$. That is, $p_i = \theta_1 p_1 \ln i \, (2 \le i \le n)$, which is monotonically increasing with respect to i. It is clear that $p_n > p_{n-1} > \cdots > p_2$. Then we only need to purchase all the items on the day with the price of $min p_1, p_2$. We focus on the case where $0 < \theta_1 < \theta_2$. Let

$$G_1(s_1, s_2, \ldots, s_n) = s_1 + s_2\theta_2 \ln 2 + \cdots + s_n\theta_2 \ln n \tag{18}$$

$$G_i(s_1, s_2, \ldots, s_n) = \frac{s_1 + s_2\theta_2 \ln 2 + \cdots + s_n\theta_2 \ln n}{\theta_1 \ln i}, \quad i = 2, 3, \ldots, n. \tag{19}$$

Before giving the competitive ratio, we consider a linear programming problem with variables $\{r, s_1, s_2, \ldots, s_n\}$.

$$\begin{aligned} &\textbf{minimize} && r && \text{(LP2)} \\ &\textbf{such that} && G_i(s_1, s_2, \ldots, s_n) \le r, \ i = 1, 2, \ldots, n \\ &&& s_1 + s_2 + \cdots + s_n = 1 \\ &&& s_i \ge 0, \ i = 1, 2, \ldots, n \end{aligned}$$

Lemma 6.1. *The solution to the linear programming problem LP2 exists.*

Proof. Similar to the proof of Lemma 5.1, we only need to prove that there exists $\{r', s'_1, s'_2, \cdots, s'_n\}$ such that

$$G_i(s'_1, s'_2, \ldots, s'_n) \le r', \quad i = 1, 2, \ldots, n. \tag{20}$$

$$s'_1 + s'_2 + \cdots + s'_n = 1 \tag{21}$$

$$s'_i \ge 0, \qquad i = 1, 2, \ldots, n. \tag{22}$$

We construct them as following. Let $s'_2 = \cdots = s'_n = 0$, $s'_1 = 1$. It is obvious that $s'_i \ge 0$ for $i = 1, 2, \ldots, n$ and $s'_1 + s'_2 + \cdots + s'_n = 0 + \cdots + 0 + 1 = 1$. That is $\{s'_1, s'_2, \cdots, s'_n\} = \{1, 0, \cdots, 0\}$ satisfies Equation 21 and Inequality 22. In addition, we obtain $G_1(s'_1, s'_2, \ldots, s'_n) = G_1(1, 0, \ldots, 0) = 1$,

$$G_i(s'_1, s'_2, \ldots, s'_n) = G_i(1, 0, \ldots, 0) = \frac{1}{\theta_1 \ln i}, \qquad i = 2, 3, \ldots, n.$$

It is clear that

$$G_2(1, 0, \ldots, 0) > G_3(1, 0, \ldots, 0) > \cdots > G_n(1, 0, \ldots, 0). \tag{23}$$

Let $\alpha = \max\{1, \frac{1}{\theta_1 \ln 2}\}$, and from Inequality 23, we obtain $\max\limits_{1 \le i \le n} G_i(1, 0, \ldots, 0) = \alpha$. Let $r' = \alpha$, we have $G_i(1, 0, \ldots, 0) \le r'$ for all $1 \le i \le n$.

From the above results, we know that there exists $\{r', s'_1, s'_2, \cdots, s'_n\} = \{\alpha, 1, 0, \cdots, 0\}$ which satisfies Equation 21 and Inequalities 20 and 22. Thus, the solution to the above linear programming problem LP2 exists. □

From Lemma 6.1, we know that the optimal solution to the linear programming problem LP2 can be obtained in polynomial time. Since the online algorithm purchases items according to the solution to the linear programming problem LP2, we denote the online algorithm by SLP_2.

Algorithm 2 :SLP_2

1: Solving the linear programming problem LP2, and let $\{\bar{r}, \bar{s_1}, \bar{s_2}, \cdots, \bar{s_n}\}$ be the solution.
2: Purchasing $\bar{s_i} (1 \le i \le n)$ units at period i.

Theorem 6.1. *The competitive ratio of SLP_2 is $\bar{r}$.*

Proof. Let $\sigma = p_1, p_2, \ldots, p_n$ denote an arbitrary price sequence. Without loss of generality, we assume that the lowest price in σ is p_i.

For $i = 1$, we have $OPT(\sigma) = p_1$ while $SLP_2(\sigma) = \sum\limits_{j=1}^{n} \bar{s_j} p_j$. Because $p_j \in [\theta_1 p_1 \ln j, \theta_2 p_1 \ln j]$ for $2 \le j \le n$, we obtain $p_j \le \theta_2 p_1 \ln j$, $j = 2, 3, \ldots, n$. Thus,

$$\begin{aligned}
\frac{SLP_2(\sigma)}{OPT(\sigma)} = \frac{\sum\limits_{j=1}^{n} \bar{s_j} p_j}{p_1} &\le \frac{\bar{s_1} p_1 + \bar{s_2} \theta_2 \ln 2 p_1 + \cdots + \bar{s_n} \theta_2 \ln n p_1}{p_1} \\
&= \bar{s_1} + \bar{s_2} \theta_2 \ln 2 + \cdots + \bar{s_n} \theta_2 \ln n \\
&= G_1(\bar{s_1}, \bar{s_2}, \cdots, \bar{s_n}).
\end{aligned}$$

For $2 \le i \le n$, we have $OPT(\sigma) = p_i$ while $SLP_2(\sigma) = \sum_{j=1}^{n} \bar{s_j} p_j$. By the assumption of this model, we know that $p_i \ge \theta_1 p_1 \ln i$, $p_j \le \theta_2 p_1 \ln j$, $j = 2, 3, \ldots, n$. Thus,

$$\begin{aligned}\frac{SLP_2(\sigma)}{OPT(\sigma)} = \frac{\sum_{j=1}^{n} \bar{s_j} p_j}{p_i} &\le \frac{\bar{s_1} p_1 + \bar{s_2}\theta_2 \ln 2 p_1 + \cdots + \bar{s_n}\theta_2 \ln n p_1}{\theta_1 p_1 \ln i} \\ &= \frac{\bar{s_1} + \bar{s_2}\theta_2 \ln 2 + \cdots + \bar{s_n}\theta_2 \ln n}{\theta_1 \ln i} \\ &= G_i(\bar{s_1}, \bar{s_2}, \cdots, \bar{s_n}).\end{aligned}$$

Combining the cases of $i = 1$ and $2 \le i \le n$, we obtain

$$\frac{SLP_2(\sigma)}{OPT(\sigma)} \le G_i(\bar{s_1}, \bar{s_2}, \cdots, \bar{s_n}) \le \bar{r}, \qquad i = 1, 2, \ldots, n,$$

where $\bar{r}$ is the minimum one that satisfies the above inequality. Hence, $\bar{r}$ is the competitive ratio of the algorithm SLP_2. □

Next, we prove that the algorithm SLP_2 is optimal for the logarithmic model under certain conditions.

Theorem 6.2. *The competitive ratio of any online algorithm for the logarithmic model is not less than $\bar{r}$ when $\theta_1 \ln 2 \ge 1$.*

Proof. By Equation 19 we obtain

$$G_2(s_1, s_2, \ldots, s_n) > G_3(s_1, s_2, \ldots, s_n) > \cdots > G_n(s_1, s_2, \ldots, s_n).$$

When $\theta_1 \ln 2 \ge 1$, combining Equations 18 and 19, we have $G_1(s_1, s_2, \ldots, s_n) \ge G_2(s_1, s_2, \ldots, s_n)$. Thus,

$$G_1(s_1, s_2, \ldots, s_n) \ge G_2(s_1, s_2, \ldots, s_n) > \cdots > G_n(s_1, s_2, \ldots, s_n). \tag{24}$$

Let ALG be an arbitrary online algorithm for the logarithmic model. We construct a special sequences $\hat{\sigma}$, where $\hat{\sigma} = \{1, \theta_2 \ln 2, \ldots, \theta_2 \ln n\}$. Without loss of generality, we assume that ALG algorithm purchases s'_i (where $1 \le i \le n$ and $\sum_{j=1}^{n} s'_j = 1$) units at period i.

Now we prove that $\frac{ALG(\hat{\sigma})}{OPT(\hat{\sigma})} \ge \bar{r}$ by contradiction.

Otherwise, assume that $\frac{ALG(\hat{\sigma})}{OPT(\hat{\sigma})} < \bar{r}$. For $\ln i$ is monotonically increasing and $0 < \theta_1 < \theta_2$, we obtain

$$\theta_1 \ln 2 < \theta_2 \ln 2 < \theta_2 \ln 3 < \cdots < \theta_2 \ln n.$$

When $\theta_1 \ln 2 \ge 1$, we have $\theta_2 \ln 2 > 1$. Hence,

$$1 < \theta_2 \ln 2 < \cdots < \theta_2 \ln n.$$

So the lowest price in $\hat{\sigma}$ is 1. That is,

$$OPT(\hat{\sigma}) = 1 \quad \text{and} \quad ALG(\hat{\sigma}) = s'_1 + s'_2\theta_2 \ln 2 + \cdots + s'_n\theta_2 \ln n,$$

we can implying

$$\frac{ALG(\hat{\sigma})}{OPT(\hat{\sigma})} = s'_1 + s'_2\theta_2 \ln 2 + \cdots + s'_n\theta_2 \ln n = G_1(s'_1, s'_2, \cdots, s'_n).$$

Because $\frac{ALG(\hat{\sigma})}{OPT(\hat{\sigma})} < \bar{r}$, we have $G_1(s'_1, s'_2, \cdots, s'_n) < \bar{r}$. From Inequality 24, we obtain $G_i(s'_1, s'_2, \cdots, s'_n) < \bar{r}$, $i = 1, 2, \ldots, n$. It indicates that there exists a $r' < \bar{r}$ such that $G_i(s'_1, s'_2, \ldots, s'_n) \leq r'$ for all $i\,(1 \leq i \leq n)$, which contradicts the minimality of $\bar{r}$. That is, $\frac{ALG(\hat{\sigma})}{OPT(\hat{\sigma})} \geq \bar{r}$. □

§7 Numerical examples

In this section, we provide some numerical examples to illustrate our results.

Table 1: linear-decrease model, $n = 100, \theta = 0.5$

p	$r^{\max}$	$r^{\min}$	Δ
8	1.8888	1.3175	0.5713
10	1.6417	1.2350	0.4067
12	1.5021	1.1865	0.3156
14	1.4123	1.1546	0.2578
16	1.3498	1.1320	0.2178

Table 2: log-decrease model, $n = 100, \theta = 0.5$

p	$r^{\max}$	$r^{\min}$	Δ
8	1.4485	1.2243	0.2243
10	1.3381	1.1691	0.1691
12	1.2713	1.1357	0.1357
14	1.2266	1.1133	0.1133
16	1.1945	1.0973	0.0973

Here, $\Delta = c^{\max} - c^{\min}$ denotes the gap between the upper and lower bounds of competitive ratio. From Table 1 and Table 2, it is clear that the increase of p will lead to the decrease of $c^{\max}$, $c^{\min}$ and Δ in the linear-decrease and log-decrease models when n and θ are fixed. The online decision maker prefers to adopt the DPC strategy using the competitive analysis for a larger p when n and θ are fixed.

Table 3: linear-decrease model, $n = 100, p = 10$

θ	$r^{\max}$	$r^{\min}$	Δ
0.3	1.3330	1.1259	0.2071
0.4	1.4762	1.1773	0.2988
0.5	1.6417	1.2350	0.4067
0.6	1.8352	1.3000	0.5352
0.7	2.0645	1.3738	0.6907

Table 4: log-decrease model, $n = 100, p = 10$

θ	$r^{\max}$	$r^{\min}$	Δ
0.3	1.1857	1.0929	0.0929
0.4	1.2586	1.1293	0.1293
0.5	1.3381	1.1691	0.1691
0.6	1.4254	1.2127	0.2127
0.7	1.5215	1.2607	0.2607

From Table 3 and Table 4, we can find that when n and p are constants, $c^{\max}, c^{\min}$ and Δ are the increasing function of θ. The online decision maker prefers to adopt DPC strategy using the competitive analysis for a smaller θ when n and p are fixed. Clearly, whether the online decision maker decides to adopt DPC strategy depends on the obtained information(p, θ, n) and calculates the upper bound of competitive ratio.

We find that the speed of log-decrease purchase price is less than that of linearly-decrease purchase price, which means that the log-decrease model is more suitable for the DPC strategy. Next, we provide some numerical examples about the competitive ratios of linear-decrease model and log-decrease model when p and θ are fixed.

Table 5: linear-decrease model and log-decrease model, $p = 10, \theta = 0.5$

Duration	Linear		Log	
n	$r^{\max}(n)$	$r^{\min}(n)$	$r^{\max}(n)$	$r^{\min}(n)$
10	1.2260	1.1130	1.2432	1.1216
11	1.2379	1.1190	1.2485	1.1242
12	1.2489	1.1245	1.2531	1.1266
13	1.2592	1.1296	1.2573	1.1287
14	1.2689	1.1344	1.2611	1.1305
15	1.2779	1.1390	1.2645	1.1323
16	1.2865	1.1433	1.2677	1.1338
17	1.2946	1.1473	1.2706	1.1353
18	1.3023	1.1512	1.2732	1.1366
19	1.3097	1.1549	1.2757	1.1379
20	1.3169	1.1584	1.2781	1.1390

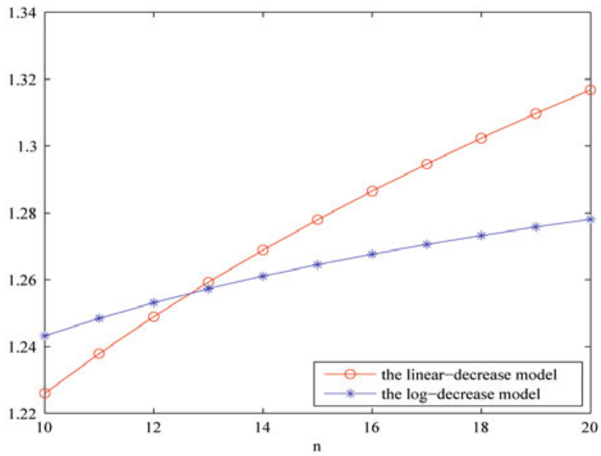

Figure 1: Comparison of the competitive ratios of two models

From Table 5 and Figure 1, we can get $n_0 = 13$ when $p = 10$, and $\theta = 0.5$, which means that the competitive ratios of log-decrease model are slightly larger than that of linearly-decrease model when $n < n_0$ and the competitive ratios of log-decrease model are much less than that of linearly-decrease model when $n > n_0$. Therefore, the performance of DPC strategy is totally different for different purchase price ranges model.

From the above results, we know whether the online decision maker decides to adopt the DPC strategy not only depends on the obtained information(p, θ, n) but also the purchase price ranges model. And the log-decrease model is more suitable for the DPC strategy when $n > n_0$. Moreover, DPC strategy can be used as one of alternative strategies to the online decision maker. We can estimate the competitive ratio and the daily purchase quantity when n is small so as to guide the actual purchasing action.

Table 6: exponential model, $n = 20, \theta_2 = 1.05$

θ_1	r
0.8	2.1425
0.82	2.1064
0.84	2.0645
0.86	2.0163
0.88	1.9572
0.9	1.8845
0.92	1.7947
0.94	1.6798
0.96	1.5258
0.98	1.312

Table 7: exponential model, $n = 20, \theta_1 = 0.95$

θ_2	r
1.02	1.312
1.04	1.5257
1.06	1.6794
1.08	1.7955
1.1	1.8854
1.12	1.9564
1.14	2.0161
1.16	2.0641
1.18	2.1066
1.2	2.1409

Table 8: exponential model, $\theta_1 = 0.95, \theta_2 = 1.05$

n	r
12	1.317
14	1.3845
16	1.4555
18	1.5301
20	1.6086
22	1.691
24	1.7777
26	1.8688
28	1.9646
30	2.0653

Table 9: logarithmic model, $n = 20, \theta_2 = 2$

θ_1	r
0.8	1.8034
0.9	1.603
1	1.4427
1.1	1.3115
1.2	1.2022
1.3	1.1098
1.4	1.0305
1.5	1
1.6	1
1.7	1

Table 10: logarithmic model, $n = 20, \theta_1 = 0.8$

θ_2	r
0.9	1.125
1	1.25
1.1	1.375
1.2	1.5
1.3	1.625
1.4	1.75
1.5	1.8034
1.6	1.8034
1.7	1.8034
1.8	1.8034

Table 11: logarithmic model, $\theta_1 = 0.95, \theta_2 = 1.05$

n	r
12	1.1053
14	1.1053
16	1.1053
18	1.1053
20	1.1053
22	1.1053
24	1.1053
26	1.1053
28	1.1053
30	1.1053

From Table 6 and Table 9, we can see that the increase of θ_1 will lead to the decrease of r both in the exponential and logarithmic models when n and θ_2 are fixed. In addition, r is always equal to 1 in the logarithmic model when n and θ_2 are fixed and θ_1 is greater than a certain value. By numerical experiment, we find that the value is in the vicinity of 1.443. The online decision maker prefers to adopt the algorithms SLP_1 and SLP_2 using the competitive analysis for a lager θ_1 when n and θ_2 are fixed.

From Table 7 and Table 10, we can find that r is proportional to θ_2 both in the exponential and logarithmic models when n and θ_1 are fixed. In addition, r is a constant in the logarithmic model when n and θ_1 are fixed and θ_2 is greater than a certain value. By numerical experiment, we find that the value is in the vicinity of 1.443. The online decision maker prefers to adopt the algorithms SLP_1 and SLP_2 using the competitive analysis for a small θ_2 when n and θ_1 are fixed.

From Table 8, we know that the increase of n will lead to the increase of r in the exponential model when θ_1 and θ_2 are fixed. From Table 11, we know that the increase of n will lead to the same r in the logarithmic model when θ_1 and θ_2 are fixed. The online decision maker prefers to adopt the algorithm SLP_1 using the competitive analysis for a small n and adopt the SLP_2 algorithm for arbitrary n when θ_1 and θ_2 are fixed.

§8 Conclusions

We investigate four models for the online inventory problem with interrelated prices. We describe the algorithms based on the DPC strategy and derive upper and lower bounds of competitive ratio for the linear-decrease model and the log-decrease model respectively. In addition, we derive the competitive ratio of the SLP_1 algorithm for the exponential model and the SLP_2 algorithm for the logarithmic model respectively, and prove that the SLP_1 algorithm is the optimal online algorithm for the exponential model and the SLP_2 algorithm is the optimal online algorithm for the logarithmic model under certain conditions. Moreover, we demonstrate some numerical examples for the four models. And by using some numerical examples of competitive ratio attained by linear-decrease model and log-decrease model, we find that the DPC strategy is more suitable to purchase when the price fluctuates smoothly.

In the future, it is interesting to consider one online inventory problem where the price information is updated, which means that we can further minimize the cost by modifying the initial online decision according to the updated price information.

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