Structural Optimisation using Analytical Equations

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Abstract: Optimisation of structures requires the minimisation of an objective function subject to a set of constraints. Typically the objective function is mass for mass sensitive structures or cost for heavy engineering projects. However environmental sensitivity can be incorporated without any difficulty. For example in an energy scarce environment energy inputs can be considered as the objective function which can take into account all energy inputs such as material manufacture and structural element forming. Mass and energy are in fact more important considerations than cost. The barrier to this has been shaped by the political world where cost considerations are considered to be of paramount importance.

Keywords: Structural Optimisation, Analytical Solutions.

Introduction

Optimization requires the minimization of an objective function subject to a set of constraints. In structural engineering the objective function typically uses mass minimization as the objective function for example in aeronautical and automotive vehicles. A globally optimized structure will typically result in a structure with the lowest mass that also satisfies all the constraints. The constraint set may involve a large set of variables. In this paper we will examine mass as the prime variable for minimization. Alternatively embodied energy or and cost can be minimized. The structures considered have of a small set of basic structures that have stiffness, buckling and stress constraints.

Designers need to consider the longevity of a materials resource base and need to have an appreciation of the exponential expiry time of a materials resource base. Whole lifecycle awareness of a structure also prevents poor decision making in structural design. Low cost structures for example may have high maintenance costs so whole life cycles costs will in fact be higher than an initially high cost structure with low maintenance costs. An often overlooked part of the structural design process is embodied energy, that is the energy used in the manufacture of the structure. In an energy scare environment energy minimisation will become an increasingly more important objective function. Energy is closely linked to the environment so environmental sensitivity can be incorporated using materials with low energy embodiment values.

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Detailed structural design can be complex and this paper seeks only to encourage a more resilient manner in the design of structures.

Beam Design subject to Stiffness Constraints

The analysis of a simply supported beam subject to a central point load i.e. three point bending is achieved using Eq. (1). This equation gives the central deflection, δ, of the beam when subject to a central point load, P. The beam has length L and is made of an isotropic material with a Youngs modulus E. The constant cross section has a second moment of area, I.

$$\delta = \frac{PL^3}{48EI} \qquad (1)$$

Initially we will assume that the cross section of the beam is square and assume the height (and breadth) of the beam is h. The density of the material can be assumed to be ρ.

There is a linear relationship between load and displacement in Eq.(1) but it also shows that for a given beam there is a constant value of stiffness that is the value of P/δ. An aircraft wing can be considered as a cantilever beam. A simplified analysis of the wing will show that a minimum second moment of area is required to ensure that the wing tip does not touch the ground. So a stiffness requirement with a known load and deflection can be calculated for our initial beam. This is stated in Eq.(2) assuming the square cross section.

$$\frac{P}{\delta} = \frac{4Eh^4}{L^3} \qquad (2)$$

The mass of the beam is given by

$$M = \rho h^2 L \qquad (3)$$

where h^2 is the cross sectional area. Rearranging Eq.(3) gives an expression for mass, M, in terms of the density and length. Which can be substituted into Eq.(2) to give

$$\frac{P}{\delta} = \frac{4E}{L^3} \times \left(\frac{M}{L\rho}\right)^2 \qquad (4)$$

Finally we can then establish that the mass M of the beam is given as:

$$M = \left(\frac{P}{\delta}\right)^{0.5} \left(\frac{L^5}{4}\right)^{0.5} \left(\frac{\rho^2}{E}\right)^{0.5} \qquad (5)$$

Eq.(5) states that the mass of the beam is function of the defined stiffness P/δ, length L and the material properties of the beam namely density and Youngs modulus. If the assumption is made that the beam is constant has a required stiffness and fixed length then the lowest mass beam is achieved by searching for a material that has the lowest mass index given by the third bracketed term in Eq.(5). This equation is the analytical solution to the minimum mass design for a square cross section beam that is simply supported and subjected to a central point load.

Column Design subject to Buckling Constraints

The critical buckling load, P_{CR}, of a simply supported column can be calculated from Eq.(6). The beam has length L and is made of an isotropic material with a Youngs modulus E. The constant cross section has a second moment of area, I.

$$P_{CR} = \frac{\pi^2 E I}{L^2} \qquad (6)$$

Initially we again will assume that the cross section of the beam is square and assume the width of the column b. The density of the material can be assumed to be ρ. It is assumed that the column is transmitting an axial load of a fixed value defined as R. We will assume that the column is on the point of buckling so the load will equal the critical load defined in Eq.(6) and the aim is to achieve the minimum mass for the column.

Proceeding as for the first example we can define the load R as

$$R = \frac{\pi^2 E b^4}{12L^2} \qquad (7)$$

The mass of the column is given by

$$M = \rho b^2 L \qquad (8)$$

Finally we can then establish that the mass M of the beam is given as:

$$M = \left(\frac{12R}{\pi^2}\right)^{0.5} \left(L^2\right) \left(\frac{\rho^2}{E}\right)^{0.5} \qquad (9)$$

For the buckling of an isotropic fixed length column transmitting a defined axial load the lowest mass can be achieved by searching for a material that has the lowest mass index given by the third bracketed term in Eq.(9).

Solid circular shaft subject to torsional stiffness constraints

The torsion behavior of the circular shaft can be achieved using Eq.(10). We will assume we want to achieve the lowest mass of a shaft with a circular cross section. There is a linear relationship between torque, T, and angular displacement, θ. The elastic shear modulus is G and shaft has a length L. The torsion constant is J and the radius is R.

$$\frac{T}{J} = \frac{G\theta}{L} \qquad (10)$$

A torsional stiffness requirement can be defined i.e. for a given torque there will be a defined angular displacement. Rearranging Eq.(10) and knowing that the cross section is circular we obtain.

$$\frac{T}{\theta} = \frac{G\pi R^4}{2L} \qquad (11)$$

The mass of the shaft is given by

$$M = \rho \pi R^2 L \qquad (12)$$

Hence we can obtain mass in by substituting in for the radius R from Eq.(12) into Eq.(11) to obtain the following:

$$M = \left(\frac{T}{\theta}\right)^{0.5} \left(2L^3\pi\right)^{0.5} \left(\frac{\rho^2}{G}\right)^{0.5} \qquad (5)$$

Materials Selection

For the beam in three point bending to achieve the lowest mass from the materials given in Table 1 then the material that should be selected should be CFRP. Note because it was assumed that the material is isotropic it would be necessary to ensure that the lay-up of the composite beam was quasi-isotropic. The heaviest beam would be that composed of Steel. The range of materials in Table 1 is very limited but interesting does show that wood, which is a naturally occurring material, performs very well. The complete list of materials in the order ash shown is Steel, Aluminium, Titanium, Glass Fibre Reinforced Plastic, Wood and Carbon Fibre Reinforced Plastic.

Table 1:   Mass index for range of materials.

Material	Density ρ, kgm^{-3}	Young's Modulus, E, Nm^{-2}	Mass index (x10^6)
Steel	7800	200x10^9	304
Al	2700	69x10^9	106
Ti	4500	120x10^9	169
GFRP	2000	40x10^9	100
Wood	600	12x10^9	30
CFRP	1500	200x10^9	11.3

For the column buckling problem the Table 1 can be used again to establish the material producing the lowest because the mass index for this problem is identical to the beam bending problem. For the torsion problem the elastic modulus is the shear modulus not the Youngs modulus. However the

mass index has the same form as the first two problems and so Table 1 can be used again to establish the lowest mass index to produce the lowest mass shaft.

Cost and Energy Index

As an alternative to mass it may be that cost or energy is the objective function. If cost is the objective function. For lowest mass it was established that CFRP was the ideal choice however Table 2 shows that this would be the highest cost structure. In fact wood turns out to be the cheapest material. In addition to cost energy can be considered. The energy index is a direct correlation to mass and is based on producing the material from its source. It does not take into account the energy required to form the structure.

Table 2:   Mass, Cost and Energy indices.

Material	Mass index $(\times 10^6)$	Cost Index	Energy index
Steel	304	55	44
Al	106	128	64
Ti	169	1150	287
GFRP	100	310	155
Wood	30	11.9	0.68
CFRP	11.3	2200	55

Table 2 shows that wood has the lowest energy index. As a designer wood is in fact the best choice if mass, cost and energy are the primary considerations.

Conclusions

Optimization requires the minimization of an objective function subject to a set of constraints. In structural engineering the objective function typically uses mass minimization as the objective function for example in aeronautical and automotive vehicles. Although wood is shown to be a material that performs well the size of the structural element was not shown. The structural design process presented was very simplified but puts a process that should enable designers to articulate individual problems in a more analytical manner enabling improved choices to be made.

Sustainability is an increasingly more important consideration and so energy is becoming an increasingly significant consideration. Energy use is closely linked to the environment so environmental sensitivity can be incorporated using materials with low energy embodiment values.

Detailed structural design can be complex and this paper seeks only to encourage a more resilient manner in the design of structures.

The structural problems presented in this paper are based on those presented by Ashby and Jones, (2005).

References

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