Nonparametric Production Technologies with Weakly Disposable Inputs

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Abstract

In models of production theory and efficiency analysis, the inputs and outputs are assumed to satisfy some form of disposability. In this paper, we consider the assumption of weak input disposability. It states that any activity remains feasible if its inputs are simultaneously scaled up in the same proportion. As suggested in the literature, the Shephard technology incorporating weak input disposability could be used to evaluate the effect of input congestion. We show that the Shephard technology is not convex and therefore introduces bias in evaluation of congestion. To address this, we develop an alternative convex technology whose use in the evaluation of congestion removes the noted bias. We undertake a further axiomatic investigation and obtain a range of production technologies, all of which exhibit weak input disposability but are based on different, progressively relaxed, convexity assumptions. Apart from the evaluation of input congestion, such technologies should also be useful in applications in which some inputs are closely related or are overlapping, and therefore satisfy only the weak input disposability assumption incorporated in the new models.

Keywords: Data envelopment analysis; Production technology; Weak input disposability; Congestion

1. Introduction

Assumptions of disposability of inputs and outputs are central to production theory and to various models of efficiency analysis based on them (Shephard, 1974; Färe et al., 1985; Ray, 2004). The most common variant of such assumptions is the strong (free) disposability of inputs and outputs. It means that, if an activity (or decision making unit in the DEA terminology) produces a vector of outputs from a vector of inputs, then it is also possible to produce the same or a smaller (component-wise) vector of outputs from the same or a larger (component-wise) vector of inputs. The assumption of strong disposability is incorporated in the constant (CRS) and variable (VRS) returns-to-scale technologies of data envelopment analysis (DEA) of Charnes et al. (1978) and Banker et al. (1984), and also in the free disposal hull (FDH) technology of Deprins et al. (1984).

In some types of application, the assumption of strong disposability is deemed to be unrealistic, so a more conservative assumption of weak disposability is used (Shephard,

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1974; Färe et al., 1985; Färe et al., 1994). The traditional non-parametric specifications of technologies that replace strong disposability of inputs or outputs by its weak analogue, are often referred to as the Shephard technologies.

An established example of this is the assumption of weak *output* disposability. It is used in the Shephard technology that incorporates undesirable (bad) outputs (Färe and Grosskopf, 2003), and in its convex analogue (Kuosmanen, 2005).[1] In these models, the assumption of weak output disposability means that the whole vector of good and bad outputs can be reduced in an arbitrary proportion, which does not allow free disposal of bad outputs (although the free disposal of good outputs is allowed by an additional axiom).

Similarly, the assumption of weak *input* disposability states that, if a vector of outputs is produced from a vector of inputs, then the same vector of outputs can be produced if we scale up all components of the input vector in the same proportion.

Although the assumptions of weak input and output disposability are conceptually clear, their incorporation in the mathematical statement of production technology is less straightforward. Kuosmanen and Podinovski (2009) show that the Shephard technology, while correctly incorporating the assumption of weak output disposability, at the same time invalidates another basic assumption stating that the production technology is a convex set. In this context, Kuosmanen and Podinovski (2009) use the axiomatic approach to formally derive the convex technology that exhibits weak output disposability. The resulting convex technology is the technology of Kuosmanen (2005), which is the convex hull of the Shephard technology.

The focus of our paper is on the assumption of weak input disposability. The Shephard technology based on this assumption is used in the methodology for measuring input congestion developed by Färe and Grosskopf (1983), and is further discussed, e.g., by Färe et al. (1985), Grosskopf (1986) and Byrnes et al. (1988).

In our paper, we show that the Shephard technology that incorporates weak input disposability is generally not a convex technology, and that, consequently, its use for measuring input congestion is problematic. As shown by Färe and Grosskopf (1983), one can obtain the congestion measure as the ratio of the technical efficiency of an activity in the standard VRS technology to its efficiency in the same technology in which strong input disposability is replaced by weak input disposability. It is assumed that the latter technology should be the Shephard technology.[2] However, it would be incorrect to attribute the observed effect entirely to input congestion, because the (unintended) loss of convexity in the Shephard technology would introduce bias in this evaluation. This bias is subsequently transferred to various aggregate efficiency indices based on the congestion measure (Färe and Karagiannis, 2017) and the corresponding decomposition approaches (Byrnes et al., 1988; Dervaux et al., 1998; Färe and Grosskopf, 2000).

It is clear that, in order to remove the noted bias in the measurement of congestion, we need to construct a technology that exhibits weak input disposability and, at the same time, preserves convexity. Comparing the technical efficiency of an activity in the VRS technology

[1] The technology of Kuosmanen (2005) is closely related to a special case of the hybrid returns-to-scale technology of Podinovski (2004). For a discussion and graphical illustration of this fact, see Kuosmanen and Podinovski (2009).

[2] Tone and Sahoo (2004) introduce an alternative measure of congestion which does not assume any input disposability at all and utilizes a smaller reference technology than the Shephard technology. It is clear that the approach of Färe and Grosskopf (1983) measures congestion arising from the distortion of the relative input proportions, while the model of Sahoo and Tone (2004) measures congestion arising from any increased levels of individual inputs. In our paper we are considering the former approach. The latter approach is given further consideration in Mehdiloozad et al. (2017).

with its efficiency in this new convex technology would allow us to attribute any difference in efficiency purely to the difference between the input disposability assumptions, i.e., to the input congestion.

Motivated by the above problem, in our paper we address a broader spectrum of related issues. We undertake a full axiomatic investigation of a range of production technologies, all of which exhibit only weak input disposability, but differ in the type of convexity assumed. Specifically, we define and obtain operational statements of such technologies for the following three cases: 1) when the technology is assumed convex; 2) when only the input sets are assumed convex; and 3) when no convexity assumption is made.

This progression of models is similar to the direction of study undertaken by Podinovski and Kuosmanen (2011) in the case of weak output disposability, but our models and results are of course different. In the above case 1, we obtain the *convex* analogue of the VRS technology that allows only weak input disposability. In case 2, we obtain a weak input disposable analogue of the technology with convex input sets of Petersen (1990) and Bogetoft (1996). Case 3 results in an analogue of the FDH technology.[3]

We also obtain a full axiomatic characterization of the Shephard technology exhibiting weak input disposability and show that it is positioned between the technologies exhibiting full convexity and convexity only with respect to its input sets, developed in cases 1 and 2 above. We prove that the convex technology exhibiting weak input disposability (developed in case 1) is the closure of the convex hull of the Shephard technology.

We finally consider an important situation in which only a subset of inputs is weakly disposable, while the remaining inputs are strongly disposable, and show how this assumption could be incorporated in the model of technology. From a practical perspective, this scenario is applicable when there is a subset of closely related inputs (e.g., overlapping, embedded one in another or highly statistically correlated). Assuming strong disposability of such related inputs implies allowing arbitrary, including meaningless, proportions between them. Instead, assuming a more conservative weak disposability means that all such inputs act as a whole. In relation to the other inputs, this group of inputs is jointly strongly disposable, but within the group, the inputs are only weakly disposable.

Evaluating technical efficiency of activities in the suggested technologies is straightforward. In particular, for the convex technology with weak input disposability, which could be viewed as a convex replacement of the Shephard technology, calculating the input or output radial efficiency of any activity requires solving a linear program.[4] In contrast, in models based on the Shephard technology, only the output orientation results in a linear program, while the input orientation leads to a nonlinear problem.

Despite the seemingly close relationship of our development to the results of Podinovski and Kuosmanen (2011), it is worth emphasizing that the two cases are not fully symmetrical and are different in several respects. Thus, the range of models of Podinovski and Kuosmanen (2011) are specifically developed for the simultaneous treatment of good and bad outputs in the same technology. These models are based on the assumption that the good outputs are both strongly and weakly disposable, the latter in conjunction with the bad outputs. This dual assumption about the good outputs makes the models of Podinovski and Kuosmanen

[3] There are many other approaches to modelling nonconvex production technologies developed in the literature, including those of Bogetoft et al. (2000), Kuosmanen (2001), Briec et al. (2004), Agrell et al. (2005), Podinovski (2005) and Mehdiloozad et al. (2014), to name a few. In our paper, we limit our investigation only to the three cases identified above.

[4] Using a more general approach based on directional distance functions (Chambers et al., 1998) also results in a linear program.

(2011) unsuitable for the measurement of output congestion, which would be symmetrical to the input congestion considered in the current paper.

Because the axioms underlying the cases of weak input and output disposability are different, the formal inference of technologies based on them is also different and independent of each other. Interestingly, Podinovski and Kuosmanen (2011) do not explicitly assume that the technology is closed, because this property follows from the other assumed axioms. In contrast, the technology generated by the assumption of weak input disposability is not generally closed, which requires a different mathematical development.

From a practical perspective, our results identify a bias arising from the use of the nonconvex Shephard technology for the measurement of congestion, and suggest a corrected fully convex model suitable for the same purpose.

We proceed as follows. In Sections 2 and 3, we introduce basic definitions, notation and axioms used in this paper. In Sections 4 and 5, we obtain a closed convex technology which exhibits weak input disposability. In further development, this technology is proved to be the closure of the convex hull of the Shephard technology. In Sections 6 and 7, we obtain two further technologies exhibiting weak input disposability. Both of these technologies are not convex. One of them has convex input sets and the other does not assume any form of convexity. In Section 8, we give an axiomatic characterization of the Shephard technology. We show that it is based on the rather unusual axiom of convexity first stated by Podinovski and Kuosmanen (2011). This axiom allows convex combinations of observed activities, but not of all activities in the technology. In Section 9, we discuss extensions of our models to the practically important case in which only a subset of inputs is weakly disposable. The concluding Section 10 presents a summary and outlines avenues for future research. All proofs of mathematical statements are given in Appendix A.

2. Notation and preliminaries

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, and let $\mathbb{R}^d_+$ denote its nonnegative orthant. We denote sets by uppercase calligraphic letters, and vectors by lowercase boldface letters. By convention, all vectors are considered to be column vectors. The transposition of a vector is indicated by superscript T. Vectors $\mathbf{0}_d$ and $\mathbf{1}_d$ are the d-dimensional vectors, all components of which are equal to 0 and 1, respectively. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, the inequality $\mathbf{a} \geq \mathbf{b}$ ($\mathbf{a} > \mathbf{b}$) means that $a_i \geq b_i$ ($a_i > b_i$), for all $i = 1, ..., d$.

Throughout this paper, a production activity is represented by the pair $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (x_1, \dots x_m)^T \in \mathbb{R}^m_+$ is the non-zero input vector and $\mathbf{y} = (y_1, \dots y_s)^T \in \mathbb{R}^s_+$ is the non-zero output vector. We assume that we have n observed activities $(\mathbf{x}_j, \mathbf{y}_j)$, $j \in \mathcal{J} = \{1, ..., n\}$.

A production technology $\mathcal{T}$ transforms input vectors $\mathbf{x} \in \mathbb{R}^m_+$ into output vectors $\mathbf{y} \in \mathbb{R}^s_+$, and is interpreted as follows:

$$\mathcal{T} = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \mathbf{x} \text{ can produce } \mathbf{y} \right\}.$$

Equivalently, the technology can be represented using its input sets defined as follows:

$$\mathcal{Q}_{\mathcal{T}}(\mathbf{y}) = \left\{ \mathbf{x} \in \mathbb{R}^m_+ \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{T} \right\}, \ \mathbf{y} \in \mathbb{R}^s_+.$$

Because in empirical applications the technology is usually unknown, it has to be approximated based on observed data and a set of assumed properties of production technology, stated as production axioms (Shephard, 1974; Banker et al., 1984; Färe et al., 1985). The following are the main axioms that we use in the subsequent sections.

Axiom IO (*Inclusion of Observations*) $(\mathbf{x}_j, \mathbf{y}_j) \in \mathcal{T}$, for all $j \in \mathcal{J}$.

Axiom SOD (*Strong Output Disposability*) If $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}$, then $(\mathbf{x}, \hat{\mathbf{y}}) \in \mathcal{T}$, for all $\mathbf{0}_s \leq \hat{\mathbf{y}} \leq \mathbf{y}$.

Axiom WID (*Weak Input Disposability*) If $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}$, then $(\delta\mathbf{x}, \mathbf{y}) \in \mathcal{T}$, for all $\delta \geq 1$.

According to Axiom WID, if an output vector $\mathbf{y}$ can be produced by an input vector $\mathbf{x}$, then $\mathbf{y}$ can also be produced by the input vector $\delta\mathbf{x}$ which represents a proportional increase of all components of the original input vector $\mathbf{x}$ by the scaling factor $\delta \geq 1$. It is clear that Axiom WID is weaker than strong input disposability, in the sense that the latter implies the former, but the converse does not hold true, unless there is a single input, i.e., $m = 1$.

Axiom C (*Closedness*) $\mathcal{T}$ is closed.

Axiom CT (*Convexity of Technology*) $\mathcal{T}$ is convex.

Axiom CIS (*Convexity of Input Sets*) The input set $\mathcal{Q}_{\mathcal{T}}(\mathbf{y})$ is convex, for any $\mathbf{y} \in \mathbb{R}^s_+$.

This last axiom implies that, if two input vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ produce an output vector $\mathbf{y}$, then any convex combination of these vectors can also produce $\mathbf{y}$. It is clear that Axiom CIS does not imply Axiom CT, i.e., it does not imply that technology $\mathcal{T}$ is convex.

Finally, we remark that all our developments below are based on the assumption of VRS, unless stated otherwise.

3. The input and output disposal hulls

In this section, we introduce several definitions used throughout this paper.

Definition 3.1. *Let $\mathcal{S}$ be a subset of $\mathbb{R}^m_+ \times \mathbb{R}^s_+$. Then the strong output disposal hull (SODH) of $\mathcal{S}$ is defined as*

$$\mathcal{S}^{\mathrm{S}} = \{(\mathbf{x}, \mathbf{z}) \,|\, (\mathbf{x}, \mathbf{y}) \in \mathcal{S},\, \mathbf{0}_s \leq \mathbf{z} \leq \mathbf{y}\}.$$

Definition 3.2. *Let $\mathcal{S}$ be a subset of $\mathbb{R}^m_+ \times \mathbb{R}^s_+$. Then the weak input disposal hull (WIDH) of $\mathcal{S}$ is defined as*

$$\mathcal{S}^{\mathrm{W}} = \{(\delta\mathbf{x}, \mathbf{y}) \,|\, (\mathbf{x}, \mathbf{y}) \in \mathcal{S},\, \delta \geq 1\}.$$

Based on the above definitions, let $\mathcal{D}^{\mathrm{S}}_j$ be the SODH of the jth observed activity:

$$\mathcal{D}^{\mathrm{S}}_j = \{(\mathbf{x}_j, \mathbf{y}_j)\}^{\mathrm{S}} = \{(\mathbf{x}_j, \mathbf{z}_j) \,|\, \mathbf{0}_s \leq \mathbf{z}_j \leq \mathbf{y}_j\},\ j \in \mathcal{J}.$$

We also define the *disposal hull* of the jth observed activity to be the WIDH of $\mathcal{D}^{\mathrm{S}}_j$, and denote it by $\mathcal{D}^{\mathrm{SW}}_j$, i.e.,

$$\mathcal{D}^{\mathrm{SW}}_j = \left(\mathcal{D}^{\mathrm{S}}_j\right)^{\mathrm{W}} = \{(\delta_j\mathbf{x}_j, \mathbf{z}_j) \,|\, \mathbf{0}_s \leq \mathbf{z}_j \leq \mathbf{y}_j,\, \delta_j \geq 1\},\ j \in \mathcal{J}. \quad (1)$$

As an illustration, consider activities A and B defined in Table 1 and shown in Fig. 1. In this example, x_1 and x_2 are inputs, and y is output.

Table 1: Input-output data for activities A and B

Activity	x_1	x_2	y
A	2	1	1
B	1	3	5

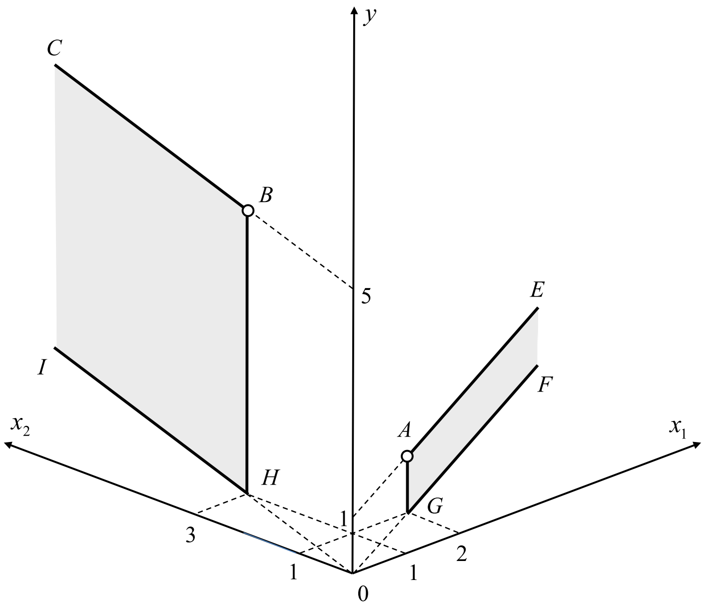

Figure 1: Disposal hulls generated by activities A and B.

By definition, the SODHs $\mathcal{D}_A^{\mathrm{S}}$ and $\mathcal{D}_B^{\mathrm{S}}$ are constructed by reducing the outputs of activities A and B along the output axis. Hence, $\mathcal{D}_A^{\mathrm{S}}$ and $\mathcal{D}_B^{\mathrm{S}}$ are, respectively, the vertical line segments AG and BH in Fig. 1. The WIDHs $\mathcal{D}_A^{\mathrm{W}}$ and $\mathcal{D}_B^{\mathrm{W}}$ are, respectively, the rays AE and BC. These are obtained by the radial expansion of the input vectors of activities A and B. It is now clear that the flat surfaces $EAGF$ and $CBHI$ are, respectively, the disposal hulls $\mathcal{D}_A^{\mathrm{SW}}$ and $\mathcal{D}_B^{\mathrm{SW}}$. It is straightforward to verify that both disposal hulls $\mathcal{D}_A^{\mathrm{SW}}$ and $\mathcal{D}_B^{\mathrm{SW}}$ are closed convex sets and that both satisfy Axioms IO, SOD and WID.

4. An intermediate not closed convex WID technology

As shown by Banker et al. (1984), the VRS technology is defined as the smallest technology (in the sense of the minimum extrapolation principle) that satisfies Axioms IO, CT and strong output and input disposability. Our goal is to replace the latter with weak input disposability (WID). This leads to the following definition.

Definition 4.1. *The WID technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is the intersection of all technologies $\mathcal{T} \subset \mathbb{R}_+^m \times \mathbb{R}_+^s$ that satisfy Axioms IO, SOD, WID and CT.*

It is straightforward to verify that technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ itself satisfies all Axioms IO, SOD, WID and CT. Therefore, $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is the smallest (or minimal) technology that satisfies the specified axioms. Importantly, this technology includes only those activities that are needed to satisfy the assumed axioms, and does not include any arbitrary activities. (Without further mention, a similar observation applies to all the other technologies defined below.)

The following result shows that technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ coincides with the convex hull of the union of disposal hulls of observed activities $\mathcal{D}_j^{\mathrm{SW}}$, $j \in \mathcal{J}$.

Theorem 4.1. *Technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is equivalently stated as follows:*

$$\tilde{\mathcal{T}}_0^{\mathrm{WID}} = \operatorname{conv}\Big(\bigcup_{j\in\mathcal{J}} \mathcal{D}_j^{\mathrm{SW}}\Big). \tag{2}$$

The proofs of Theorem 4.1 and the other statements are given in Appendix A.

Fig. 2 illustrates Theorem 4.1. It shows technology $\tilde{\mathcal{T}}_0^{\rm WID}$ generated by observed activities A and B. To construct this technology, we first identify disposal hulls $EAGF$ and $CBHI$, as in Fig. 1. Technology $\tilde{\mathcal{T}}_0^{\rm WID}$ is obtained as their convex hull. Note that the relative interior points on the side $DBAE$ are obtained as convex combinations of activities B and E when the latter tends to infinity. The ray BD cannot be obtained as such convex combination and is not included in technology $\tilde{\mathcal{T}}_0^{\rm WID}$. Furthermore, the relative interior points of the surface CBD are not included in $\tilde{\mathcal{T}}_0^{\rm WID}$, although the ray BC is included. This example shows that technology $\tilde{\mathcal{T}}_0^{\rm WID}$ is generally not a closed set and therefore does not satisfy Axiom C.

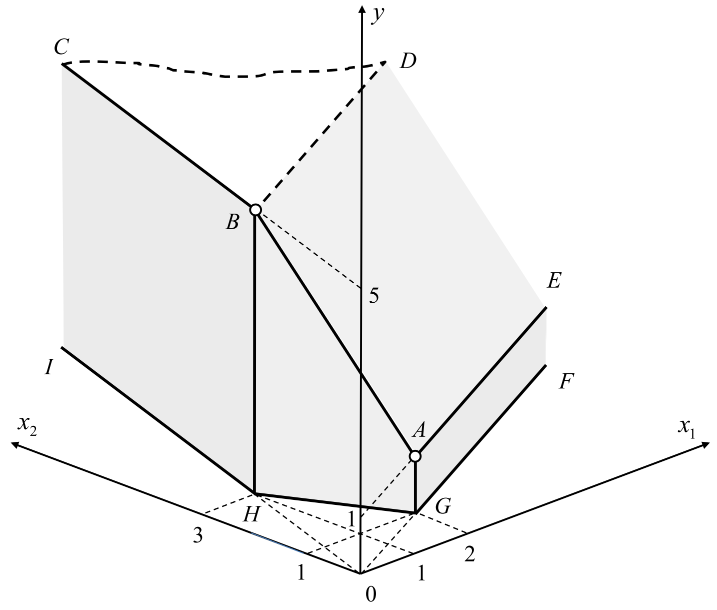

Figure 2: The convex but not closed technology $\tilde{\mathcal{T}}_0^{\rm WID}$ generated by activities A and B.

The following result gives an equivalent constructive definition of technology $\tilde{\mathcal{T}}_0^{\rm WID}$.

Theorem 4.2. *Technology* $\tilde{\mathcal{T}}_0^{\rm WID}$ *is equivalently stated as follows:*

$$\tilde{\mathcal{T}}_0^{\rm WID} = \Big\{(\mathbf{x},\mathbf{y}) \,\Big|\, \sum_{j\in\mathcal{J}} \lambda_j \delta_j \mathbf{x}_j = \mathbf{x},\ \sum_{j\in\mathcal{J}} \lambda_j \mathbf{y}_j \geq \mathbf{y} \geq \mathbf{0}_s,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geq \mathbf{0}_n,\ \boldsymbol{\delta} \geq \mathbf{1}_n \Big\}. \quad (3)$$

Note that the equality for inputs included in (3) disallows strong input disposability. However, it allows weak input disposability by incorporating the scaling factors δ_j that are generally different for each $j \in \mathcal{J}$.

Remark 1. It is easy to extend our development to the case of constant returns to scale (CRS). In this case we require an additional axiom of "ray unboundness" (Banker et al., 1984). This axiom states that any activity remains feasible in the technology if its input and output vectors are proportionally scaled up or down by a nonnegative factor. If this axiom is assumed, then Axiom WID becomes redundant, as it follows from ray unboundness and Axiom SOD.

Define the CRS technology $\tilde{\mathcal{T}}_0^{\rm CRS-WID}$ with weakly disposable inputs as the intersection of all technologies $\mathcal{T} \subset \mathbb{R}_+^m \times \mathbb{R}_+^s$ that satisfy Axioms IO, SOD, CT and the axiom of ray unboundness. It is straightforward to prove that $\tilde{\mathcal{T}}_0^{\rm CRS-WID}$ is obtained from (3) by omitting the constraint $\mathbf{1}_n^T \boldsymbol{\lambda} = 1$ and setting all scaling factors δ_j equal to 1. It follows that technology $\tilde{\mathcal{T}}_0^{\mathrm{CRS-WID}}$ is the conventional CRS technology in which the input inequalities are replaced by equalities. Unlike the VRS technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$, technology $\tilde{\mathcal{T}}_0^{\mathrm{CRS-WID}}$ is a polyhedral, and therefore, a closed set, i.e., it satisfies Axiom C.[5]

5. The closed convex WID technology

In the previous section we showed that technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ defined by Axioms IO, SOD, WID and CT, is generally not closed. This fact was highlighted in our discussion of Fig. 2. Requiring that, in addition to the above axioms, the technology is a closed set leads to the following definition.[6]

Definition 5.1. *The closed convex WID technology $\mathcal{T}_0^{\mathrm{WID}}$ is the intersection of all technologies $\mathcal{T} \subset \mathbb{R}_+^m \times \mathbb{R}_+^s$ that satisfy Axioms IO, SOD, WID, CT and C.*

The following result shows that technology $\mathcal{T}_0^{\mathrm{WID}}$ is the closure of technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$.

Theorem 5.1. *The following equality is true:*

$$\mathcal{T}_0^{\mathrm{WID}} = \mathrm{cl}\left(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\right). \tag{4}$$

The next result gives a constructive statement of technology $\mathcal{T}_0^{\mathrm{WID}}$. It is obtained from the statement (3) of technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ by the following substitution:

$$\mu_j = \lambda_j (\delta_j - 1), \quad j \in \mathcal{J}, \tag{5}$$

which implies $\lambda_j \delta_j = \lambda_j + \mu_j$, for all $j \in \mathcal{J}$. It turns out that the above substitution expands technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ "just enough" to add all its missing limit points and make it a closed set.

Theorem 5.2. *Technology $\mathcal{T}_0^{\mathrm{WID}}$ is equivalently stated as follows:*

$$\mathcal{T}_0^{\mathrm{WID}} = \Big\{ (\mathbf{x}, \mathbf{y}) \,\Big|\, \sum_{j \in \mathcal{J}} (\lambda_j + \mu_j)\, \mathbf{x}_j = \mathbf{x},\ \sum_{j \in \mathcal{J}} \lambda_j \mathbf{y}_j \geq \mathbf{y} \geq \mathbf{0}_s,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda}, \boldsymbol{\mu} \geq \mathbf{0}_n \Big\}. \tag{6}$$

Fig. 3 shows technology $\mathcal{T}_0^{\mathrm{WID}}$ generated by activities A and B. This technology is the closure of technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ depicted in Fig. 2. It includes the ray BD and the relative interior of the facet CBD, which consist of the limit points of technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$. For example, the ray BD is obtained as the set of of limit points for the line segments BE when the point E tends to infinity along the ray AE. The facet CBD is the convex hull of rays BC and BD.

It is straightforward to show that technology $\mathcal{T}_0^{\mathrm{WID}}$ is a polyhedral set.[7] As known from convex analysis, any polyhedral set can be decomposed as the Minkowski sum of a bounded polyhedral set (polytope) and a polyhedral cone (see, e.g., Rockafellar 1970, Theorem 19.1). For technology $\mathcal{T}_0^{\mathrm{WID}}$, such decomposition follows directly from Theorem 5.2. In the statement below, the operator "cone" denotes the conical hull.

Corollary 5.1. *(Decomposition theorem) Technology $\mathcal{T}_0^{\mathrm{WID}}$ can be represented as*

$$\mathcal{T}_0^{\mathrm{WID}} = \mathrm{conv}\Big(\bigcup_{j \in \mathcal{J}} \mathcal{D}_j^s\Big) + \mathrm{cone}\left(\{(\mathbf{x}_1, \mathbf{0}_s), \ldots, (\mathbf{x}_n, \mathbf{0}_s)\}\right).$$

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[5] A polyhedral set in $\mathbb{R}^n$ is defined as the intersection of a finite number of closed half-spaces (Rockafellar, 1970). The fact that technology $\tilde{\mathcal{T}}_0^{\mathrm{CRS-WID}}$ is a polyhedral set follows from the more general Proposition 1 proved in Podinovski et al. (2016).

[6] Note that the VRS technology of Banker et al. (1984) also satisfies Axiom C. However, this axiom does not need to be stated explicitly because it follows from the other axioms that define the VRS technology. Therefore, requiring that technology $\mathcal{T}_0^{\mathrm{WID}}$ satisfy Axiom C does not impose an additional condition over what is already incorporated in the VRS technology.

[7] The fact that technology $\mathcal{T}_0^{\mathrm{WID}}$ is a polyhedral set is formally stated by Lemma 4 in Appendix A.

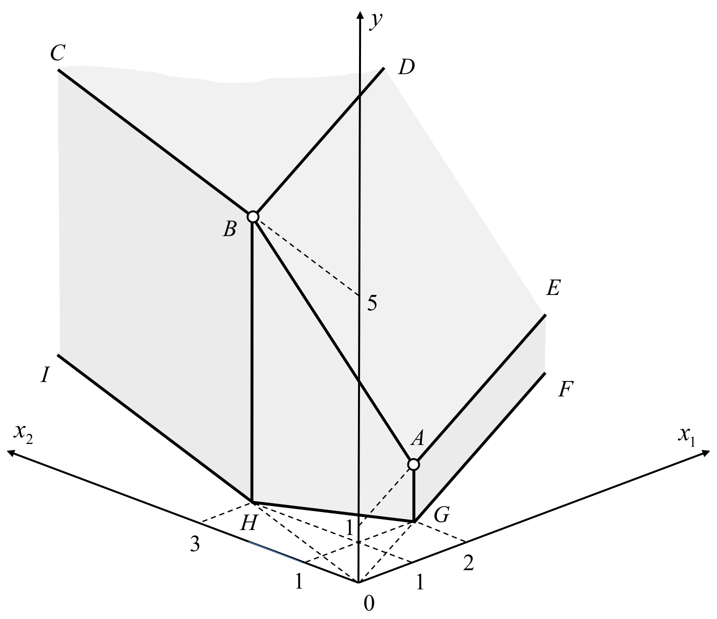

Figure 3: The closed convex technology $\mathcal{T}_0^{\mathrm{WID}}$ generated by activities A and B.

6. The WID technology with convex input sets

In this section we relax the assumption that technology should be a convex set. Instead, we require that only the input sets be convex, i.e., we assume Axiom CIS instead of Axiom CT. The resulting technology can be regarded as a weakly disposable (with respect to inputs) analogue of the technologies considered by Petersen (1990) and Bogetoft (1996).

This technology is also useful in our further discussion of the Shephard technology in Section 8. We show that the Shephard technology is generally not convex but has convex input sets. A natural question arises if the Shephard technology is the smallest technology that has convex input sets and satisfies Axiom WID. The answer to this question is negative. The smallest technology that satisfies these properties is developed below, in this section.

Definition 6.1. *The WID technology with convex input sets $\mathcal{T}_1^{\mathrm{WID}}$ is the intersection of all technologies $\mathcal{T} \subset \mathbb{R}_+^m \times \mathbb{R}_+^s$ that satisfy Axioms IO, SOD, WID and CIS.*

To give a constructive definition of technology $\mathcal{T}_1^{\mathrm{WID}}$, let us introduce the following definitions. For a given vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that $\mathbf{1}_n^T \boldsymbol{\lambda} = 1$, let $\mathcal{J}^+(\boldsymbol{\lambda}) = \{j \in \mathcal{J} \mid \lambda_j > 0\}$ be the index set of strictly positive components of vector $\boldsymbol{\lambda}$. Define the component-wise minimum of the observed output vectors $\mathbf{y}_j = (y_{j1}, \ldots, y_{js})^T \in \mathbb{R}_+^s$, $j \in \mathcal{J}$, as follows:

$$\min_{\boldsymbol{\lambda}} (\mathbf{y}_1, \ldots, \mathbf{y}_n) = \Big(\min_{j \in \mathcal{J}^+(\boldsymbol{\lambda})} \{y_{j1}\}, \ldots, \min_{j \in \mathcal{J}^+(\boldsymbol{\lambda})} \{y_{js}\} \Big)^T.$$

Using the above notation, we obtain the following statement of technology $\mathcal{T}_1^{\mathrm{WID}}$.

Theorem 6.1. *Technology $\mathcal{T}_1^{\mathrm{WID}}$ is equivalently stated as follows:*

$$\mathcal{T}_1^{\mathrm{WID}} = \Big\{ (\mathbf{x}, \mathbf{y}) \,\Big|\, \sum_{j \in \mathcal{J}} \delta_j \lambda_j \mathbf{x}_j = \mathbf{x},\ \min_{\boldsymbol{\lambda}} (\mathbf{y}_1, \ldots, \mathbf{y}_n) \geq \mathbf{y} \geq \mathbf{0}_s,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geq \mathbf{0}_n,\ \boldsymbol{\delta} \geq \mathbf{1}_n \Big\}. \tag{7}$$

Theorem 6.2. *Technology $\mathcal{T}_1^{\mathrm{WID}}$ satisfies Axiom C.*

Technology $\mathcal{T}_1^{\mathrm{WID}}$ also allows an alternative statement based on the notion of *selective convexity* (Podinovski, 2005). This notion applies to technologies in which only a subset of inputs I^C and a subset of outputs O^C can form convex combinations. The remaining inputs and outputs from the sets I^{NC} and O^{NC} are taken at their most conservative levels (minimum for outputs and maximum for inputs), across all combined activities.

For our purposes, we assume that $\mathrm{I}^C = \{1,\ldots,m\}$ and $\mathrm{O}^{NC} = \{1,\ldots,s\}$. Consider any technology $\mathcal{T} \subset \mathbb{R}_+^m \times \mathbb{R}_+^s$. The following axiom is a special case of the axiom of selective convexity (A3′) given by Podinovski (2005).[8]

Axiom SCI (*Selective Convexity of Inputs*) For any $(\mathbf{x}^1,\mathbf{y}^1),(\mathbf{x}^2,\mathbf{y}^2) \in \mathcal{T}$, and any λ_1, $\lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$, the selective convex (*s-convex*) combination $(\lambda_1\mathbf{x}^1 + \lambda_2\mathbf{x}^2, \min_{\boldsymbol{\lambda}}(\mathbf{y}^1,\mathbf{y}^2))$ belongs to $\mathcal{T}$.

If technology $\mathcal{T}$ satisfies Axiom SCI, we refer to it as an *s-convex technology*. In technology $\mathcal{T}$ is s-convex, then any convex combination of a pair of input vectors taken with positive weights, can produce the minimum of each of the component outputs that they can produce individually.

It is clear that Axiom SCI implies Axiom CIS. The converse is also true, provided technology $\mathcal{T}$ satisfies Axiom SOD.

Theorem 6.3. *Let technology $\mathcal{T}$ satisfy Axiom SOD. Then $\mathcal{T}$ satisfies Axiom SCI if and only if it satisfies Axiom CIS.*

This last result, together with Definition 6.1, implies that technology $\mathcal{T}_1^{\mathrm{WID}}$ is equivalently defined as the smallest technology that satisfies Axioms IO, SOD, WID and SCI.

Definition 6.2. *Let $\mathcal{S}$ be a non-empty set in $\mathbb{R}_+^m \times \mathbb{R}_+^s$. We define the s-convex hull of the set $\mathcal{S}$ as*

$$\begin{aligned}\operatorname{sconv}(\mathcal{S}) = \Big\{ \Big(\sum_{k=1}^{p} \lambda_k \mathbf{x}^k, \min_{\boldsymbol{\lambda}} \left(\mathbf{y}^1,\ldots,\mathbf{y}^p\right)\Big) \Big|\ \left(\mathbf{x}^k,\mathbf{y}^k\right) \in \mathcal{S},\ k = 1,\ldots,p,\\ \mathbf{1}_p^T\boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geq \mathbf{0}_p \Big\}.\end{aligned} \tag{8}$$

Definition 6.2 states that the s-convex hull of the set $\mathcal{S}$ consists of all (finite) s-convex combinations of elements of $\mathcal{S}$. It can be proved that $\operatorname{sconv}(\mathcal{S})$ is equivalently defined as the intersection of all s-convex sets that contain $\mathcal{S}$. Therefore, $\operatorname{sconv}(\mathcal{S})$ is the smallest s-convex set containing $\mathcal{S}$.

The following theorem shows that technology $\mathcal{T}_1^{\mathrm{WID}}$ is the s-convex hull of the union of the disposal hulls generated by all observed activities.

Theorem 6.4. *Technology $\mathcal{T}_1^{\mathrm{WID}}$ is equivalently stated as follows:*

$$\mathcal{T}_1^{\mathrm{WID}} = \operatorname{sconv}\Big(\bigcup_{j\in\mathcal{J}} \mathcal{D}_j^{\mathrm{SW}} \Big). \tag{9}$$

[8] Olesen et al. (2015, 2017) use the notion of selective convexity to define technologies in which some inputs and outputs are represented by ratio data, such as percentages and averages. This reflects the fact that ratio measures cannot be combined in convex combinations in the same way as volume measures. In the context of technologies with ratio measures, Olesen et al. (2017) refer to s-convex combinations of activities (decision making units) as their ratio-convex (R-convex) combinations.

Fig. 4 shows technology $\mathcal{T}_1^{\text{WID}}$ generated by activities A and B. Note that, for every level of output y, the corresponding input set $\mathcal{Q}(y)$ is convex. For example, the output of activity A is equal to 1, and $\mathcal{Q}(1)$ is the unbounded polyhedron $MLAE$. Similarly, the output of activity B is equal to 5, and the set $\mathcal{Q}(5)$ is the ray BC. Furthermore, according to the statement (9), $\mathcal{T}_1^{\text{WID}}$ includes all s-convex combinations of its activities. For example, combining activities A and B with the weights $\lambda_A > 0$ and $\lambda_B = 1 - \lambda_A > 0$, we obtain the convex combinations only for their input components, while keeping the resulting output equal to 1, which is the minimum of the outputs of activities A and B. This generates the open line segment LA, which is included in the technology.

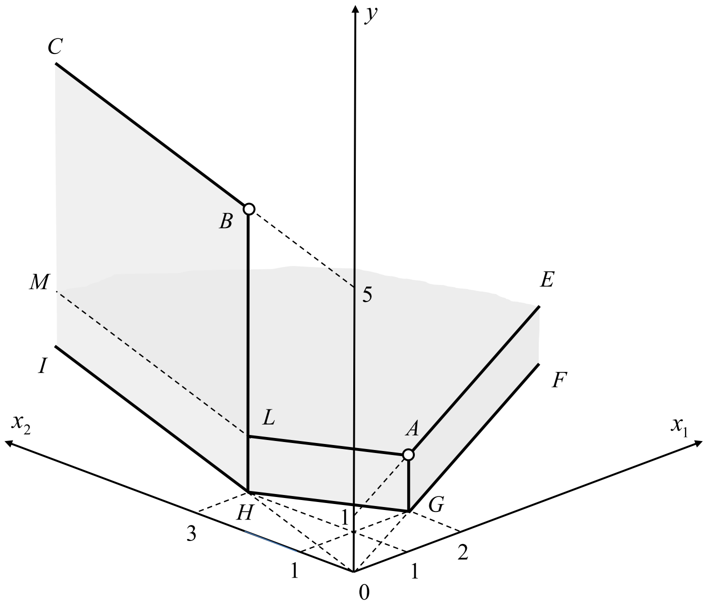

Figure 4: Technology $\mathcal{T}_1^{\text{WID}}$ generated by activities A and B.

Remark 2. The statement (7) of technology $\mathcal{T}_1^{\text{WID}}$ is nonlinear. Using substitution (5) and implementing the linearization technique employed by Kuosmanen (2001), it is straightforward to show that $\mathcal{T}_1^{\text{WID}}$ can be restated in the following equivalent form:

$$\mathcal{T}_1^{\text{WID}} = \left\{ (\mathbf{x}, \mathbf{y}) \,\middle|\, \sum_{j \in \mathcal{J}} (\lambda_j + \mu_j)\, \mathbf{x}_j = \mathbf{x},\ \lambda_j y_{rj} \geq \lambda_j y_r, y_r \geq 0,\ \forall j, r,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda}, \boldsymbol{\mu} \geq \mathbf{0}_n \right\}. \tag{10}$$

The above statement of technology $\mathcal{T}_1^{\text{WID}}$ is useful for computational purposes. For example, the evaluation of the input radial efficiency of activity $(\mathbf{x}_o, \mathbf{y}_o)$ in technology $\mathcal{T}_1^{\text{WID}}$ based on its statement (10) requires solving a linear program. As shown by Podinovski (2005) and Podinovski and Kuosmanen (2011) for similar programs, computations of the output radial efficiency and, more generally, the efficiency using the directional distance function approach of Chambers et al. (1998), require solving a mixed integer linear program.

7. The WID technology without convexity assumptions

In this section we obtain a technology that exhibits weak input disposability and strong output disposability, without imposing any convexity assumption. Such technology may be regarded as an adaptation of the FDH technology to the situation in which the inputs are closely related and cannot be freely disposed of. We show that the resulting technology is generally significantly smaller than the conventional FDH technology.

Definition 7.1. *Technology $\mathcal{T}_2^{\mathrm{WID}}$ is the intersection of all technologies $\mathcal{T} \subset \mathbb{R}_+^m \times \mathbb{R}_+^s$ that satisfy Axioms IO, SOD and WID.*

The next result shows that technology $\mathcal{T}_2^{\mathrm{WID}}$ has a very simple structure.

Theorem 7.1. *Technology $\mathcal{T}_2^{\mathrm{WID}}$ is equivalently stated as follows:*

$$\mathcal{T}_2^{\mathrm{WID}} = \bigcup_{j \in \mathcal{J}} \mathcal{D}_j^{\mathrm{SW}}. \tag{11}$$

Corollary 7.1. *Technology $\mathcal{T}_2^{\mathrm{WID}}$ satisfies Axiom C.*

As an illustration, in Fig. 1, technology $\mathcal{T}_2^{\mathrm{WID}}$ is the union of disposal hulls $\mathcal{D}_A^{\mathrm{SW}}$ and $\mathcal{D}_B^{\mathrm{SW}}$ of activities A and B, i.e., the union of the surfaces $EAGF$ and $CBHI$.

Based on Theorem 7.1, we have the following constructive statement of technology $\mathcal{T}_2^{\mathrm{WID}}$.

Theorem 7.2. *Technology $\mathcal{T}_2^{\mathrm{WID}}$ is equivalently stated as follows:*

$$\mathcal{T}_2^{\mathrm{WID}} = \Big\{ (\mathbf{x}, \mathbf{y}) \,\Big|\, \sum_{j \in \mathcal{J}} \lambda_j \delta_j \mathbf{x}_j = \mathbf{x},\ \sum_{j \in \mathcal{J}} \lambda_j \mathbf{y}_j \geq \mathbf{y} \geq \mathbf{0}_s,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \in \{0,1\}^n,\ \boldsymbol{\delta} \geq \mathbf{1}_n \Big\}. \tag{12}$$

Comparing the statement (3) of technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ with the statement (12) of technology $\mathcal{T}_2^{\mathrm{WID}}$, we observe that, in the former case, the components of vector $\boldsymbol{\lambda}$ take on real values, while in the latter case, they are binary. Therefore, we always have the embedding $\mathcal{T}_2^{\mathrm{WID}} \subseteq \tilde{\mathcal{T}}_0^{\mathrm{WID}}$. In fact, a stronger result is available showing that $\mathcal{T}_2^{\mathrm{WID}}$ is always a subset of technology $\mathcal{T}_1^{\mathrm{WID}}$. This follows directly from the definitions of the two technologies.

Corollary 7.2. *The following embedding is true: $\mathcal{T}_2^{\mathrm{WID}} \subseteq \mathcal{T}_1^{\mathrm{WID}}$.*

Remark 3. It is clear that, in practical applications, the disposal hulls $\mathcal{D}_j^{\mathrm{SW}}$ generated by observed activities $(\mathbf{x}_j, \mathbf{y}_j)$, $j \in \mathcal{J}$, are likely to be mutually disjoint sets. In this case, each observed activity belongs only to its own disposal hull, and is efficient in technology $\mathcal{T}_2^{\mathrm{WID}}$ as a result. The only exception of this is the case in which the input vector of one activity is obtained by proportional scaling of the input vector of the other.

The evaluation of the input or output radial efficiency of activity $(\mathbf{x}_o, \mathbf{y}_o)$ in technology $\mathcal{T}_2^{\mathrm{WID}}$ stated in the form (12) would require solving a mixed integer nonlinear program. A more straightforward approach is to use a simple enumeration algorithm at each step of which the efficiency of activity $(\mathbf{x}_o, \mathbf{y}_o)$ is evaluated in the single disposal hull $\mathcal{D}_j^{\mathrm{SW}}$, $j \in \mathcal{J}$, which is an elementary task. The input or output radial efficiency of activity $(\mathbf{x}_o, \mathbf{y}_o)$ is then obtained as the minimum of its efficiencies calculated in the individual sets $\mathcal{D}_j^{\mathrm{SW}}$, for which $(\mathbf{x}_o, \mathbf{y}_o) \in \mathcal{D}_j^{\mathrm{SW}}$, $j \in \mathcal{J}$.[9]

Alternatively, using Theorem 7.2, we can state technology $\mathcal{T}_2^{\mathrm{WID}}$ in the following equivalent form:

$$\mathcal{T}_2^{\mathrm{WID}} = \Big\{ (\mathbf{x}, \mathbf{y}) \,\Big|\, \sum_{j \in \mathcal{J}} \delta_j \mathbf{x}_j = \mathbf{x},\ \sum_{j \in \mathcal{J}} \lambda_j \mathbf{y}_j \geq \mathbf{y} \geq \mathbf{0}_s,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \in \{0,1\}^n,\ \boldsymbol{\lambda} \leq \boldsymbol{\delta} \leq M\boldsymbol{\lambda} \Big\}, \tag{13}$$

where M is a sufficiently large constant representing an upper bound on the feasible values of components of vector $\boldsymbol{\delta}$. This upper bound can be assessed using the input equality in (13). Assessing the input or output radial efficiency of activity $(\mathbf{x}_o, \mathbf{y}_o)$ in technology $\mathcal{T}_2^{\mathrm{WID}}$ using its statement (13) requires solving a mixed integer linear program.

[9] Obviously, if $(\mathbf{x}_o, \mathbf{y}_o) \in \mathcal{T}_2^{\mathrm{WID}}$, then there exists at least one $j' \in \mathcal{J}$ such that $(\mathbf{x}_o, \mathbf{y}_o) \in \mathcal{D}_{j'}^{\mathrm{SW}}$.

8. The Shephard WID technology

The Shephard technology is often used in applications that involve undesirable (bad) outputs (Färe and Grosskopf, 2003). In this technology, the good outputs are assumed to be strongly disposable, while the bad and good outputs are jointly only weakly disposable. The assumption of weak disposability of outputs is modeled by allowing the outputs of convex combinations of observed activities to be scaled down by a single (abatement) factor δ.

Kuosmanen and Podinovski (2009) show that the Shephard technology with bad outputs is not convex, and that this is because it effectively associates the same abatement factor δ with each observed activity. Furthermore, Podinovski and Kuosmanen (2011) give an axiomatic characterization of the Shephard technology. They show that an axiomatic reason why the Shephard technology is not convex is that it is based on a rater unusual axiom of convexity that allows only observed activities to form convex combinations. This axiom is stated as follows.

Axiom CO (*Convexity of Observations*) $\mathcal{T}$ contains all convex combinations of the observed activities.

Below we explore the Shephard technology stated under the assumption of weak disposability of inputs (Färe and Grosskopf, 1983; Grosskopf, 1986). Similar to its output analogue, this technology attaches a single scaling factor δ to all inputs. As a result, and as shown below, it is not a convex technology.

To maintain consistency with the previous sections, we start by the axiomatic definition of the Shephard technology and explore its operational statements and theoretical properties afterwards.

Definition 8.1. *Technology $\mathcal{T}_S^{\text{WID}}$ is the intersection of all technologies $\mathcal{T} \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+$ that satisfy Axioms IO, SOD, WID and CO.*

The following theorem reveals the structure of the Shephard technology $\mathcal{T}_S^{\text{WID}}$ in terms of SODHs of observed activities.

Theorem 8.1. *Technology $\mathcal{T}_S^{\text{WID}}$ is equivalently stated as follows:*

$$\mathcal{T}_S^{\text{WID}} = \Big(\text{conv}\Big(\bigcup_{j\in\mathcal{J}} \mathcal{D}_j^{\text{S}}\Big)\Big)^{\text{W}}. \tag{14}$$

Let $\mathcal{C}$ be the convex hull of the set of all observed activities $(\mathbf{x}_j, \mathbf{y}_j)$, $j \in \mathcal{J}$, and let $\mathcal{C}^{\text{SW}} = \big(\mathcal{C}^{\text{S}}\big)^{\text{W}}$ be the disposal hull of the set $\mathcal{C}$, as defined in Section 3.

Corollary 8.1. $\mathcal{T}_S^{\text{WID}} = \mathcal{C}^{\text{SW}} = \big(\mathcal{C}^{\text{S}}\big)^{\text{W}}$.

The following theorem gives a constructive statement of technology $\mathcal{T}_S^{\text{WID}}$. It is equivalent to the statement given by Grosskopf (1986).

Theorem 8.2. *Technology $\mathcal{T}_S^{\text{WID}}$ is equivalently stated as follows:*

$$\mathcal{T}_S^{\text{WID}} = \Big\{(\mathbf{x},\mathbf{y}) \,\Big|\, \delta \sum_{j\in\mathcal{J}} \lambda_j \mathbf{x}_j = \mathbf{x},\ \sum_{j\in\mathcal{J}} \lambda_j \mathbf{y}_j \geq \mathbf{y} \geq \mathbf{0}_s,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geq \mathbf{0}_n,\ \delta \geq 1\Big\}. \tag{15}$$

Theorem 8.3. *Technology $\mathcal{T}_S^{\text{WID}}$ satisfies Axiom C.*

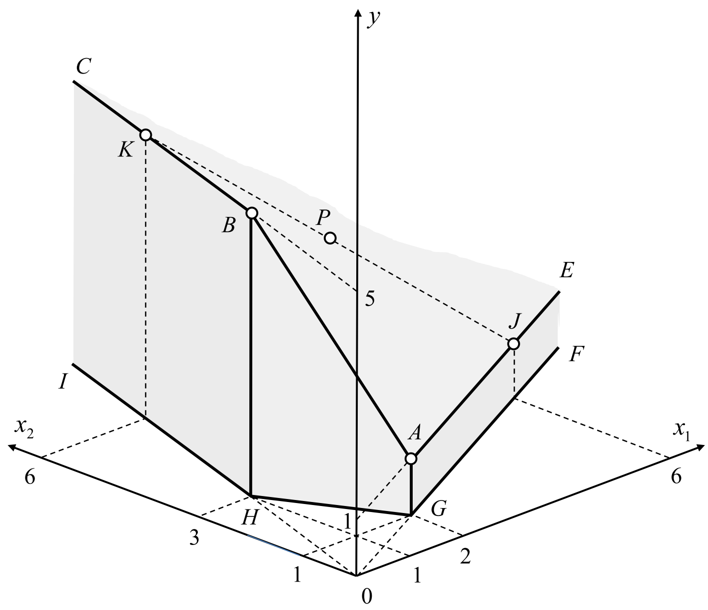

Figure 5: The non-convex Shephard technology $\mathcal{T}_S^{\mathrm{WID}}$ generated by activities A and B.

Table 2: Input-output data for activities J and K

Activity	x_1	x_2	y
J	6	3	1
K	2	6	5

Fig. 5 shows technology $\mathcal{T}_S^{\mathrm{WID}}$ generated by observed activities A and B. To construct this technology, we first identify the line segments AG and BH, representing the SODHs $\mathcal{D}_A^{\mathrm{S}}$ and $\mathcal{D}_B^{\mathrm{S}}$, respectively. The trapezium $ABHG$ is the convex hull of these SODHs. Technology $\mathcal{T}_S^{\mathrm{WID}}$ is obtained as the WIDH of trapezium $ABHG$, i.e., by adding the horizontal rays emerging from all activities in $ABHG$ in the directions away from the output axis.

Note that the Shephard technology $\mathcal{T}_S^{\mathrm{WID}}$ in Fig. 5 is not convex. For example, consider activities J and K that are defined in Table 2 and are in technology $\mathcal{T}_S^{\mathrm{WID}}$. (Activity J is obtained from activity A by increasing its inputs by a factor 3. Similarly, K is obtained from B by increasing its inputs by a factor 2.) Note that the relative interior of the line segment joining activities J and K defined in Table 2 is outside this technology, which implies that $\mathcal{T}_S^{\mathrm{WID}}$ is not a convex set.[10]

The reason why the Shephard technology $\mathcal{T}_S^{\mathrm{WID}}$ is not convex becomes clear by comparing its statement (14) with the statement (2) of the convex technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$. The latter can be restated as

$$\tilde{\mathcal{T}}_0^{\mathrm{WID}} = \mathrm{conv}\Big(\Big(\bigcup_{j\in\mathcal{J}} \mathcal{D}_j^{\mathrm{S}}\Big)^{\mathrm{W}}\Big). \tag{16}$$

[10]For example, consider the midpoint P of the line segment JK. Its two inputs and single output are equal to 4, 9/2 and 3, respectively. Substitute this into (15), taking into account that the two observed activities are A and B defined by Table 1. The resulting two input equations and the equation $\mathbf{1}_n^T\boldsymbol{\lambda} = 1$ have the unique solution $\lambda_1 = 3/5$, $\lambda_2 = 2/5$ and $\delta = 5/2$. However, for this solution, the output inequality in (15) is not satisfied. Therefore, the point P does not belong to $\mathcal{T}_S^{\mathrm{WID}}$.

The only difference between the statements (14) and (16) is in the order in which the operations of taking the convex hull and constructing the WIDH of the union of the sets $\mathcal{D}_j^{\mathrm{S}}$, $j \in \mathcal{J}$, are performed. Taking the convex hull before the WIDH is constructed, as in formula (14), results in the nonconvex technology $\mathcal{T}_S^{\mathrm{WID}}$.

It is clear that, because Axiom CT implies Axiom CO, we have $\mathcal{T}_S^{\mathrm{WID}} \subseteq \tilde{\mathcal{T}}_0^{\mathrm{WID}}$. The following result is a more precise statement of this relationship.

Theorem 8.4. *Technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is the convex hull of technology $\mathcal{T}_S^{\mathrm{WID}}$:*

$$\tilde{\mathcal{T}}_0^{\mathrm{WID}} = \mathrm{conv}\left(\mathcal{T}_S^{\mathrm{WID}}\right).$$

Taking into account (4), we immediately obtain the following statement:

Corollary 8.2. *Technology $\mathcal{T}_0^{\mathrm{WID}}$ is the closure of the convex hull of technology $\mathcal{T}_S^{\mathrm{WID}}$:*

$$\mathcal{T}_0^{\mathrm{WID}} = \mathrm{cl}\left(\mathrm{conv}\left(\mathcal{T}_S^{\mathrm{WID}}\right)\right).$$

Although, as demonstrated, technology $\mathcal{T}_S^{\mathrm{WID}}$ is not convex, the following result shows that all of its input sets are convex.

Theorem 8.5. *For any $\mathbf{y} \in \mathbb{R}_+^s$, the input set $\mathcal{Q}_S^{\mathrm{WID}}(\mathbf{y})$ is convex.*

The above theorem shows that technology $\mathcal{T}_S^{\mathrm{WID}}$ satisfies Axiom CIS. However, as already noted, $\mathcal{T}_S^{\mathrm{WID}}$ is not the smallest technology that satisfies this property and exhibits weak input disposability. Indeed, taking into account the definition of technology $\mathcal{T}_1^{\mathrm{WID}}$, we obtain the following result.

Corollary 8.3. *The following is true*

$$\mathcal{T}_1^{\mathrm{WID}} \subseteq \mathcal{T}_S^{\mathrm{WID}}.$$

Finally, we consider a special case in which all observed activities produce the same output vector.

Theorem 8.6. *Let all observed activities have the same output vector $\tilde{\mathbf{y}}$, i.e., $\mathbf{y}_j = \tilde{\mathbf{y}}$, for all $j \in \mathcal{J}$. Then $\mathcal{T}_S^{\mathrm{WID}} = \tilde{\mathcal{T}}_0^{\mathrm{WID}} = \mathcal{T}_0^{\mathrm{WID}} = \mathcal{T}_1^{\mathrm{WID}}$.*

The above result identifies a special case in which the generally nonconvex technologies $\mathcal{T}_1^{\mathrm{WID}}$ and $\mathcal{T}_S^{\mathrm{WID}}$ are convex, and the generally not closed technology $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is closed.

9. Convex technology with a subset of inputs exhibiting WID

It is straightforward to extend our previous results to the case in which only a subset of inputs is weakly disposable, while the rest of the inputs are strongly disposable. Let us point out two application areas for such an extension. First, as suggested by Byrnes et al. (1988), this can be used to identify a subset of inputs that cause congestion. This requires comparing the technical efficiency of an activity in the technology in which a subset of inputs exhibits weak disposability with its efficiency in the technology in which all inputs exhibit weak disposability.

Second, in many applications there is often a subset of highly correlated or overlapping inputs. For example, in the public sector applications such as schools or hospitals, the inputs representing the teaching or doctor and nurse provision may be strongly linked to administration costs. In applications to banking, operational expenses may be linked to the value of deposits. In various applications, the input representing staff holding high level of qualification is a subset of all staff.

In all such cases it may be deemed inappropriate to assume strong disposability of related inputs, as this would imply allowing unrestricted proportions between them. Instead, it might be reasonable to assume only weak disposability of such inputs relative to each other, which will not distort proportions between them, while allowing strong disposability for the remaining inputs. In this model, the group of related inputs acts as a whole: it is effectively strongly disposable with respect to the remaining inputs, while *within the group*, the inputs are mutually only weakly disposable.

Obtaining the required technologies is similar to our previous development. To avoid repetition, below we outline only the case of the fully convex technology which is likely to be of the most practical interest.

Let us represent each input vector as $\mathbf{x} = \left(\mathbf{x}^{\mathrm{W}}, \mathbf{x}^{\mathrm{S}}\right)$, where $\mathbf{x}^{\mathrm{W}}$ and $\mathbf{x}^{\mathrm{S}}$ are the subvectors of weakly and strongly disposable inputs. The difference between the two subvectors is reflected in the following axioms.

Axiom SDSI (*Strong Disposability of a Subset of Inputs*) If $\left(\mathbf{x}^{\mathrm{W}}, \mathbf{x}^{\mathrm{S}}, \mathbf{y}\right) \in \mathcal{T}$, then $\left(\mathbf{x}^{\mathrm{W}}, \hat{\mathbf{x}}^{\mathrm{S}}, \mathbf{y}\right) \in \mathcal{T}$, for all $\hat{\mathbf{x}}^{\mathrm{S}} \geq \mathbf{x}^{\mathrm{S}}$.

Axiom WDSI (*Weak Disposability of a Subset of Inputs*) If $\left(\mathbf{x}^{\mathrm{W}}, \mathbf{x}^{\mathrm{S}}, \mathbf{y}\right) \in \mathcal{T}$, then $\left(\delta\mathbf{x}^{\mathrm{W}}, \mathbf{x}^{\mathrm{S}}, \mathbf{y}\right) \in \mathcal{T}$, for all $\delta \geq 1$.

Using the minimum extrapolation principle, define the closed convex technology $\mathcal{T}_0^{\mathrm{WDSI}}$ that satisfies the above two axioms and the axiom of strong output disposability, as follows.

Definition 9.1. *Technology $\mathcal{T}_0^{\mathrm{WDSI}}$ is the intersection of all technologies $\mathcal{T} \subset \mathbb{R}_+^m \times \mathbb{R}_+^s$ that satisfy Axioms IO, SOD, SDSI, WDSI, CT and C.*

The following result is an analogue of Theorem 5.2.

Theorem 9.1. *Technology $\mathcal{T}_0^{\mathrm{WDSI}}$ is equivalently stated as follows:*

$$\mathcal{T}_0^{\mathrm{WDSI}} = \Big\{ \left(\mathbf{x}^{\mathrm{W}}, \mathbf{x}^{\mathrm{S}}, \mathbf{y}\right) \Big| \sum_{j \in \mathcal{J}} (\lambda_j + \mu_j)\, \mathbf{x}_j^{\mathrm{W}} = \mathbf{x}^{\mathrm{W}},\ \sum_{j \in \mathcal{J}} \lambda_j \mathbf{x}_j^{\mathrm{S}} \leq \mathbf{x}^{\mathrm{S}}, \\ \sum_{j \in \mathcal{J}} \lambda_j \mathbf{y}_j \geq \mathbf{y} \geq \mathbf{0}_s,\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda}, \boldsymbol{\mu} \geq \mathbf{0}_n \Big\}.$$

The proof of Theorem 9.1 is similar to that of Theorem 5.2, and is omitted.

10. Concluding remarks

In this paper, we consider a range of production technologies that incorporate the assumption of weak input disposability. According to this assumption, if an output vector $\mathbf{y}$ can be produced from an input vector $\mathbf{x}$, then $\mathbf{y}$ can also be produced if all inputs are increased in the same proportion $\delta \geq 1$, i.e., from the input vector $\delta\mathbf{x}$.

The assumption of weak input disposability is more conservative than the assumption of strong input disposability incorporated in the conventional VRS and CRS technologies of data envelopment analysis, and also in the FDH technology. The latter assumption allows all inputs to be increased independently of each other. As pointed in our paper, this strong assumption may be inappropriate if some or all inputs are deemed to be closely related to each other. This may be because there is a strong statistical or perceptual relation between them, or because they represent overlapping measures, or because one input is embedded in another. In these cases, using a model that allows strong disposability of all inputs would imply allowing arbitrary, including meaningless, subvectors of the related inputs.

The assumption of weak input disposability is incorporated in the known Shephard technology. Färe and Grosskopf (1983) suggest using this technology to measure the level of input congestion. The idea of this method is to calculate the technical efficiency of an activity in the strongly disposable VRS technology and also in the Shephard technology. The ratio of the two efficiencies is interpretable as the congestion measure.

The mathematical statement of the Shephard technology is obtained from the conventional statement of the VRS technology, by assigning a single scaling factor δ to the input vectors of all observed activities, and replacing the input inequalities by equalities. In our paper, we show that such a modification, while correctly representing weak input disposability, simultaneously makes the resulting Shephard technology nonconvex. We argue that this unintended side effect appears problematic, for example, in the calculation of congestion measure. Ideally, congestion should be measured relative to a technology which has the same axiomatic properties as the VRS technology, except for replacing the strong input disposability by weak input disposability. Using the Shephard technology for this purpose is problematic because not only the type of input disposability is changed, but also the convexity assumption is disposed of. In other words, the Shephard technology is too small for this purpose, and its use creates a bias in the evaluation of congestion measure.

Our development is motivated by the two factors outlined above. First, it is a practical need to have suitable technologies that incorporate weak input disposability, to be used in situations with closely related inputs. Second, it is a need to correct the bias arising from the use of the Shephard technology in the evaluation of congestion measure.

The range of technologies introduced in our paper exhibit weak input disposability, strong output disposability, and are based on progressively relaxed convexity assumptions. Technology $\mathcal{T}_0^{\mathrm{WID}}$ is a convex production technology, technology $\mathcal{T}_1^{\mathrm{WID}}$ is not convex but has convex input sets, and technology $\mathcal{T}_2^{\mathrm{WID}}$ does not assume any type of convexity.

Additionally, we prove that the Shephard technology, denoted $\mathcal{T}_S^{\mathrm{WID}}$, fits in the above progression by satisfying the rather unusual axiom of convexity introduced by Podinovski and Kuosmanen (2011). Namely, the Shephard technology allows convex combinations of only observed activities. This makes it smaller than the fully convex technology $\mathcal{T}_0^{\mathrm{WID}}$ but larger than the technology $\mathcal{T}_1^{\mathrm{WID}}$ with convex input sets.

Overall, combining several results stated in the main sections of this paper, we have the following embedding:

$$\mathcal{T}_2^{\mathrm{WID}} \subseteq \mathcal{T}_1^{\mathrm{WID}} \subseteq \mathcal{T}_S^{\mathrm{WID}} \subseteq \mathcal{T}_0^{\mathrm{WID}}.$$

The above technologies treat all inputs as being weakly disposable. In the final section of our paper we consider a situation in which only a subset of inputs is weakly disposable, while the remaining inputs are strongly disposable. We show that the above models are easy to adjust to this scenario. In practice, the resulting models should be useful in two cases. First, they are more precise in representing situations in which only some inputs are closely related and require modelling using weak input disposability. Second, as shown by Byrnes et al. (1988), such models could be used in identifying which inputs are congested.

From a computational perspective, assessing the technical efficiency of activities in the output and input orientations (i.e., assessing the input and output radial efficiency) in technology $\mathcal{T}_0^{\mathrm{WID}}$ requires solving only linear programs, and the same is true for using the directional distance function approach of Chambers et al. (1998). This contrasts favourably with the Shephard technology whose use in evaluation of technical efficiency leads to a linear program only in the output orientation. In technologies $\mathcal{T}_1^{\mathrm{WID}}$ and $\mathcal{T}_2^{\mathrm{WID}}$ similar computations require solving mixed integer linear programs.

The new models developed in our paper open up a number of further research avenues. First, it is hoped that the assumption of weak input disposability in the convex technology $\mathcal{T}_0^{\text{WID}}$, especially applied to a subset of inputs, would be attractive in practical applications with a group of closely related inputs, as an alternative to the standard VRS technology. Second, it should be interesting to extend the treatment of weak input disposability in the nonconvex technology $\mathcal{T}_1^{\text{WID}}$ to a related and mathematically more general model incorporating ratio inputs and outputs of Olesen et al. (2015, 2017). From a practical perspective, it should also be worthwhile to explore the dual representation of technology $\mathcal{T}_0^{\text{WID}}$ and its use for evaluation and interpretation of the scale elasticity and returns to scale on the efficient frontier.

Appendix A. Proofs

Lemma 1. *Let $\mathcal{K}$ be the set on the right-hand side of* (2)*. Then*

$$\begin{aligned}\mathcal{K} = \Big\{(\mathbf{x},\mathbf{y}) \,\Big|\, &\sum_{j\in\mathcal{J}} \lambda_j \delta_j \mathbf{x}_j = \mathbf{x},\ \sum_{j\in\mathcal{J}} \lambda_j \mathbf{z}_j = \mathbf{y}, \\ &\mathbf{0}_s \le \mathbf{z}_j \le \mathbf{y}_j, j \in \mathcal{J},\ \mathbf{1}_n^T \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \ge \mathbf{0}_n,\ \boldsymbol{\delta} \ge \mathbf{1}_n\Big\}.\end{aligned} \tag{A.1}$$

Proof of Lemma 1. Denote $\mathcal{W}$ the set on the right-hand side of (A.1). We need to prove that $\mathcal{K} = \mathcal{W}$. Let $(\mathbf{x},\mathbf{y}) \in \mathcal{K}$. By Carathéodory's theorem, activity $(\mathbf{x},\mathbf{y})$ is a convex combination of a finite number of activities from the sets $\mathcal{D}_j^{\mathrm{SW}}$, $j \in \mathcal{J}$. Because each set $\mathcal{D}_j^{\mathrm{SW}}$ is convex, without loss of generality, we assume that all such activities are taken from different sets $\mathcal{D}_j^{\mathrm{SW}}$. By definition of the sets $\mathcal{D}_j^{\mathrm{SW}}$, each of these activities is stated in the form $(\delta_j \mathbf{x}_j, \mathbf{z}_j)$, where $\delta_j \ge 1$ and $\mathbf{0}_s \le \mathbf{z}_j \le \mathbf{y}_j$. Therefore, $(\mathbf{x},\mathbf{y}) \in \mathcal{W}$, and $\mathcal{K} \subseteq \mathcal{W}$.

Conversely, let $(\mathbf{x},\mathbf{y}) \in \mathcal{W}$. Then each activity $(\delta_j \mathbf{x}_j, \mathbf{z}_j)$ defined by (A.1) is in the corresponding set $\mathcal{D}_j^{\mathrm{SW}}$, $j \in \mathcal{J}$. Because $(\mathbf{x},\mathbf{y})$ is their convex combination, $(\mathbf{x},\mathbf{y}) \in \mathcal{K}$. Therefore, $\mathcal{W} \subseteq \mathcal{K}$. ■

Lemma 2. *Let $\mathcal{K}$ be the set on the right-hand side of* (2)*. Then $\mathcal{K}$ satisfies Axioms IO, SOD, WID and CT.*

Proof of Lemma 2. Clearly, $\mathcal{K}$ satisfies Axiom IO and CT. To prove that $\mathcal{K}$ satisfies Axiom SOD, let $(\mathbf{x},\mathbf{y}) \in \mathcal{K}$, and let $\mathbf{0}_s \le \hat{\mathbf{y}} \le \mathbf{y}$. We need to prove that $(\mathbf{x},\hat{\mathbf{y}}) \in \mathcal{K}$.

By Lemma 1, $(\mathbf{x},\mathbf{y})$ can be stated as in (A.1). By definition of vector $\hat{\mathbf{y}}$, for each $r = 1,...,s$, we have $0 \le \hat{y}_r \le y_r = \sum_{j\in\mathcal{J}} \lambda_j z_{jr}$. Then there exists a $\sigma_r \in [0,1]$ such that $\hat{y}_r = \sum_{j\in\mathcal{J}} \lambda_j (\sigma_r z_{jr})$. Denote $\hat{z}_{jr} = \sigma_r z_{jr}$. Then

$$\hat{\mathbf{y}} = \sum_{j\in\mathcal{J}} \lambda_j \hat{\mathbf{z}}_j,\quad \mathbf{0}_s \le \hat{\mathbf{z}}_j \le \mathbf{y}_j,\quad j \in \mathcal{J}.$$

Therefore, $(\mathbf{x},\hat{\mathbf{y}}) \in \mathcal{K}$, and $\mathcal{K}$ satisfies Axiom SOD. To prove that $\mathcal{K}$ satisfies Axiom WID, let $\gamma \ge 1$. Then $(\gamma\mathbf{x},\mathbf{y})$ satisfies all conditions in (A.1) with the vector $\boldsymbol{\delta}$ replaced by $\gamma\boldsymbol{\delta}$. Therefore, $(\gamma\mathbf{x},\mathbf{y}) \in \mathcal{K}$. ■

Proof of Theorem 4.1. Let $\mathcal{K}$ be the set on the right-hand side of (2). By Lemma 2, $\mathcal{K}$ satisfies Axioms IO, SOD, WID and CT. Because $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is the smallest set that satisfies these axioms, we have $\tilde{\mathcal{T}}_0^{\mathrm{WID}} \subseteq \mathcal{K}$.

Conversely, because the set $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is convex, it suffices to show that $\mathcal{D}_j^{\mathrm{SW}} \subseteq \tilde{\mathcal{T}}_0^{\mathrm{WID}}$, for all $j \in \mathcal{J}$. For any $j \in \mathcal{J}$, let $(\delta_j \mathbf{x}_j, \mathbf{z}_j) \in \mathcal{D}_j^{\mathrm{SW}}$, where $\mathbf{0}_s \le \mathbf{z}_j \le \mathbf{y}_j$ and $\delta_j \ge 1$. Because $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ satisfies Axioms IO and WID, we have $(\delta_j \mathbf{x}_j, \mathbf{z}_j) \in \tilde{\mathcal{T}}_0^{\mathrm{WID}}$, and the proof follows. ■

Proof of Theorem 4.2. By Theorem 4.1 and Lemma 1, $\tilde{\mathcal{T}}_0^{\mathrm{WID}} = \mathcal{K}$, where the set $\mathcal{K}$ is stated in the form (A.1). Let $\mathcal{M}$ be the set on the right-hand side of (3). Clearly, $\tilde{\mathcal{T}}_0^{\mathrm{WID}} \subseteq \mathcal{M}$. Conversely, let $(\bar{\mathbf{x}},\bar{\mathbf{y}}) \in \mathcal{M}$. Then $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ satisfies all conditions of (3) with some vectors $\bar{\boldsymbol{\lambda}} \ge \mathbf{0}_n$ and $\bar{\boldsymbol{\delta}} \ge \mathbf{1}_n$. Similar to the proof of Lemma 2, we define vectors $\bar{\mathbf{z}}_j$, $j \in \mathcal{J}$, such that $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ satisfies (A.1) with the vectors $\bar{\boldsymbol{\lambda}}$, $\bar{\boldsymbol{\delta}}$ and $\bar{\mathbf{z}}_j$, $j \in \mathcal{J}$. Therefore, $(\bar{\mathbf{x}},\bar{\mathbf{y}}) \in \tilde{\mathcal{T}}_0^{\mathrm{WID}}$. ■

Lemma 3. *The following embedding is true: $\tilde{\mathcal{T}}_0^{\mathrm{WID}} \subseteq \mathcal{T}_0^{\mathrm{WID}}$.*

Proof of Lemma 3. The set $\mathcal{T}_0^{\mathrm{WID}}$ satisfies axioms IO, SOD, WID and CT. The proof is completed by noting that $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is the smallest set that satisfies these axioms. ■

Lemma 4. *Denote $\mathcal{P}$ the set on the right-hand side of equality* (6)*. Then $\mathcal{P}$ is a polyhedral set, which implies that it satisfies Axioms CT and C. Furthermore, $\mathcal{P}$ satisfies Axioms IO, SOD, and WID.*

Proof of Lemma 4. The fact that $\mathcal{P}$ is a polyhedral set follows from the projection lemma (see, e.g., the proof of Proposition 1 in Podinovski et al. (2016)). Therefore, $\mathcal{P}$ satisfies Axioms CT and C. Clearly, $\mathcal{P}$ also satisfies Axiom IO. Let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{P}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (6) with some vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\mu}}$.

Consider Axiom SOD. Let $\mathbf{0}_s \leq \hat{\mathbf{y}} \leq \bar{\mathbf{y}}$. Then $(\bar{\mathbf{x}}, \hat{\mathbf{y}})$ satisfies (6) with the same $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\mu}}$. Therefore, $\mathcal{P}$ satisfies Axiom SOD. Finally, consider any $\gamma \geq 1$. Then $(\gamma\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (6) with $\hat{\boldsymbol{\lambda}} = \bar{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\mu}}$ found from the equation $\bar{\boldsymbol{\lambda}} + \hat{\boldsymbol{\mu}} = \gamma\left(\bar{\boldsymbol{\lambda}} + \bar{\boldsymbol{\mu}}\right)$, i.e., with $\hat{\boldsymbol{\mu}} = \gamma\left(\bar{\boldsymbol{\lambda}} + \bar{\boldsymbol{\mu}}\right) - \bar{\boldsymbol{\lambda}}$. Therefore, $\mathcal{P}$ satisfies Axiom WID. ■

Lemma 5. *Let $\mathcal{P}$ be the set on the right-hand side of equality* (6)*. Then $\mathcal{P} = \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big)$.*

Proof of Lemma 5. If $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \tilde{\mathcal{T}}_0^{\mathrm{WID}}$, then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (3) with some vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\delta}}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (6) with the same $\bar{\boldsymbol{\lambda}}$ and the vector $\bar{\boldsymbol{\mu}}$ defined by (5). Therefore $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{P}$, and $\tilde{\mathcal{T}}_0^{\mathrm{WID}} \subseteq \mathcal{P}$. By Lemma 4, $\mathcal{P}$ is a closed set. Therefore, $\mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big) \subseteq \mathcal{P}$.

Conversely, let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{P}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (6) with some vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\mu}}$. We need to prove that $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big)$. Two cases arise.

Case 1. Suppose that $\bar{\boldsymbol{\lambda}} > \mathbf{0}_n$. Using (5), define vector $\bar{\boldsymbol{\delta}}$ as follows. For all $j \in \mathcal{J}$, let $\bar{\delta}_j = \left(\bar{\lambda}_j + \bar{\mu}_j\right)/\bar{\lambda}_j$, which implies $\bar{\lambda}_j + \bar{\mu}_j = \bar{\lambda}_j\bar{\delta}_j$. Note that $\bar{\boldsymbol{\delta}} \geq \mathbf{1}_n$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (3) with vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\delta}}$. Therefore, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \tilde{\mathcal{T}}_0^{\mathrm{WID}} \subseteq \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big)$.

Case 2. Suppose that the inequality $\bar{\boldsymbol{\lambda}} > \mathbf{0}_n$ is not true. Let us show that, in this case, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is the limit of some sequence $\{(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t)\}$, $t = 1, 2\ldots$, each element of which is in $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$. Because $\mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big)$ is a closed set, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big)$.

To define the required sequence, first let $(\mathbf{x}^*, \mathbf{y}^*)$ be the simple average of all observed activities $(\mathbf{x}_j, \mathbf{y}_j)$, $j \in \mathcal{J}$. Note that $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies (6) with vectors $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^* = \mathbf{0}_n$, where $\lambda_j^* = 1/n$, for all $j \in \mathcal{J}$. Define the elements of the above sequence as follows:

$$(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t) = (1 - 1/t)\,(\bar{\mathbf{x}}, \bar{\mathbf{y}}) + (1/t)\,(\mathbf{x}^*, \mathbf{y}^*), \quad t = 1, 2, \ldots \tag{A.2}$$

Each $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t)$ defined by (A.2) satisfies (6) with vectors $\boldsymbol{\lambda}_t = (1 - 1/t)\,\bar{\boldsymbol{\lambda}} + (1/t)\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}_t = (1 - 1/t)\,\bar{\boldsymbol{\mu}} + (1/t)\boldsymbol{\mu}^* = (1 - 1/t)\,\bar{\boldsymbol{\mu}}$. Because $\boldsymbol{\lambda}_t > \mathbf{0}_n$, as proved above, $(\bar{\mathbf{x}}_t, \bar{\mathbf{y}}_t) \in \tilde{\mathcal{T}}_0^{\mathrm{WID}}$. Then the limit of this sequence $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big)$.

The two considered cases imply that $\mathcal{P} \subseteq \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big)$, which completes the proof. ■

Proof of Theorem 5.1. By Lemma 4, $\mathcal{P}$ satisfies Axioms IO, SOD, WID, CT and C. By definition of $\mathcal{T}_0^{\mathrm{WID}}$, we have $\mathcal{T}_0^{\mathrm{WID}} \subseteq \mathcal{P}$. Conversely, by Lemma 3, we have $\tilde{\mathcal{T}}_0^{\mathrm{WID}} \subseteq \mathcal{T}_0^{\mathrm{WID}}$. Then, taking into account Lemma 5, we have $\mathcal{P} = \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big) \subseteq \mathrm{cl}\big(\mathcal{T}_0^{\mathrm{WID}}\big) = \mathcal{T}_0^{\mathrm{WID}}$. ■

Proof of Theorem 5.2. By Theorem 5.1 and Lemma 5, we have $\mathcal{T}_0^{\mathrm{WID}} = \mathrm{cl}\big(\tilde{\mathcal{T}}_0^{\mathrm{WID}}\big) = \mathcal{P}$, where $\mathcal{P}$ be the set on the right-hand side of equality (6). ■

Lemma 6. *Let $\mathcal{H}$ be the set on the right-hand side of equality* (7)*. Then $\mathcal{H}$ satisfies Axioms IO, SOD, WID and CIS.*

Proof of Lemma 6. Clearly, $\mathcal{H}$ satisfies Axiom IO. Let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{H}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies the conditions in (7) with some vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\delta}}$. Let $\hat{\mathbf{y}} \leq \bar{\mathbf{y}}$. Then $(\bar{\mathbf{x}}, \hat{\mathbf{y}})$ satisfies (7) with the vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\delta}}$. Therefore, $\mathcal{H}$ satisfies Axiom SOD. Let $\gamma \geq 1$. Then $(\delta\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (7) with the vectors $\bar{\boldsymbol{\lambda}}$ and $\boldsymbol{\delta} = \gamma\bar{\boldsymbol{\delta}}$. Therefore, $\mathcal{H}$ satisfies Axiom WID.

To prove that $\mathcal{H}$ satisfies Axiom CIS, consider $(\bar{\mathbf{x}}^1, \bar{\mathbf{y}})$, $(\bar{\mathbf{x}}^2, \bar{\mathbf{y}}) \in \mathcal{H}$ and any $\sigma \in (0,1)$. We need to prove that $(\mathbf{x}^\sigma, \mathbf{y}^\sigma) = \sigma(\bar{\mathbf{x}}^1, \bar{\mathbf{y}}) + (1-\sigma)(\bar{\mathbf{x}}^2, \bar{\mathbf{y}}) \in \mathcal{H}$. Let $(\bar{\mathbf{x}}^1, \bar{\mathbf{y}})$ and $(\bar{\mathbf{x}}^2, \bar{\mathbf{y}})$ satisfy the conditions in (7) with vectors $\boldsymbol{\lambda}^1$ and $\boldsymbol{\delta}^1$, and $\boldsymbol{\lambda}^2$ and $\boldsymbol{\delta}^2$, respectively. It is straightforward to verify that $(\mathbf{x}^\sigma, \mathbf{y}^\sigma)$ satisfies the conditions in (7) with the vectors $\boldsymbol{\lambda}^\sigma$ and $\boldsymbol{\delta}^\sigma$, where

$$\lambda_j^\sigma = \sigma\lambda_j^1 + (1-\sigma)\lambda_j^2, \qquad \delta_j^\sigma = \frac{\sigma\delta_j^1\lambda_j^1 + (1-\sigma)\delta_j^2\lambda_j^2}{\lambda_j^\sigma}, \qquad j \in \mathcal{J}.$$

Note that $\delta_j^\sigma \geq 1$, for all $j \in \mathcal{J}$. Therefore, $(\mathbf{x}^\sigma, \mathbf{y}^\sigma) \in \mathcal{H}$. ■

Proof of Theorem 6.1. By Lemma 6, the set $\mathcal{H}$ on the right-hand side of (7) satisfies Axioms IO, SOD, WID and CIS. Because $\mathcal{T}_1^{\text{WID}}$ is the smallest technology that satisfies these axioms, we have $\mathcal{T}_1^{\text{WID}} \subseteq \mathcal{H}$. Conversely, let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{H}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies all conditions in (7) with some vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\delta}}$. Because $\mathcal{T}_1^{\text{WID}}$ satisfies Axioms IO, SOD and WID, $\left(\bar{\delta}_j\mathbf{x}_j, \bar{\mathbf{y}}\right) \in \mathcal{T}_1^{\text{WID}}$, for all $j \in \mathcal{J}$ such that $\bar{\lambda}_j > 0$. Because $\mathcal{T}_1^{\text{WID}}$ satisfies Axiom CIS, we have $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{T}_1^{\text{WID}}$. Therefore, $\mathcal{H} \subseteq \mathcal{T}_1^{\text{WID}}$. ■

Proof of Theorem 6.2. Consider the statement of technology $\mathcal{T}_1^{\text{WID}}$ by (7). Let some sequence $\left\{\left(\mathbf{x}^k, \mathbf{y}^k\right) \mid k = 1, 2, \ldots\right\} \subset \mathcal{T}_1^{\text{WID}}$ converge to $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. For any k, $\left(\mathbf{x}^k, \mathbf{y}^k\right)$ satisfies all conditions in (7) with some vectors $\boldsymbol{\lambda}^k$ and $\boldsymbol{\delta}^k$. Because the sequence $\left\{\boldsymbol{\lambda}^k\right\}$ belongs to a compact set, without loss of generality, let $\left\{\boldsymbol{\lambda}^k\right\}$ converge to some vector $\hat{\boldsymbol{\lambda}}$. A more careful treatment is required for the sequence $\left\{\boldsymbol{\delta}^k\right\}$.

It holds that $\mathbf{1}_n^T\hat{\boldsymbol{\lambda}} = 1$. Consider any $j^* \in \mathcal{J}$ such that $\hat{\lambda}_{j^*} > 0$. Based on the assumption about the observed activities made in Section 2, $\mathbf{x}_{j^*} \neq \mathbf{0}_m$. Therefore, there exists an input $i^* \in \{1, \ldots, m\}$ such that $x_{j^*i^*} > 0$. Furthermore, there exists a sufficiently large $\bar{k}$ such that, for all $k \geq \bar{k}$, we have $\lambda_{j^*}^k \geq \hat{\lambda}_{j^*}/2$ and $x_{i^*}^k \leq \hat{x}_{i^*} + 1$. Then, based on (7), for input i^* we have:

$$\hat{x}_{i^*} + 1 \geq x_{i^*}^k = \sum_{j\in\mathcal{J}} \delta_j^k\lambda_j^k x_{ji^*} \geq \delta_{j^*}^k\lambda_{j^*}^k x_{j^*i^*} \geq \delta_{j^*}^k\frac{\hat{\lambda}_{j^*}}{2}x_{j^*i^*}.$$

Therefore, for all $k \geq \bar{k}$, we have $\delta_{j^*}^k \in [1, D]$, where $D = 2\left(\hat{x}_{i^*} + 1\right)/\left(\hat{\lambda}_{j^*}x_{j^*i^*}\right)$. Then, without loss of generality, we can assume that the sequence $\left\{\delta_{j^*}^k\right\}$ converges to some $\hat{\delta}_{j^*}$. For any $j \in \mathcal{J}$ such that $\hat{\lambda}_j = 0$, we arbitrarily define $\hat{\delta}_j = 1$. Taking the limits by k of both sides of the input equality in (7) stated for $\mathbf{x}^k$, we have $\sum_{j\in\mathcal{J}}\hat{\delta}_j\hat{\lambda}_j\mathbf{x}_j = \hat{\mathbf{x}}$. Finally, to prove that the output inequality in (7) is true for $\hat{\mathbf{y}}$ and $\hat{\boldsymbol{\lambda}}$, note that, if $\hat{\lambda}_j > 0$, then, for all $k \geq \bar{k}$, we have $\lambda_j^k > 0$. Therefore, $\mathbf{y}_j \geq \mathbf{y}^k$. Then at the limit, we have $\mathbf{y}_j \geq \hat{\mathbf{y}}$. We have proved that $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathcal{T}_1^{\text{WID}}$. Therefore, $\mathcal{T}_1^{\text{WID}}$ is a closed set. ■

Proof of Theorem 6.3. The proof follows follows from a more general Lemma 2 proved in Podinovski (2005). ■

Lemma 7. *Let $\mathcal{B}$ be the set on the right-hand side of* (9). *Then $\mathcal{B}$ satisfies Axioms IO, SOD, WID and CIS.*

Proof of Lemma 7. Clearly, $\mathcal{B}$ satisfies Axioms IO and CIS. Suppose that $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{B}$. Then there exist a finite number of pairs $\left(\bar{\mathbf{x}}^k, \bar{\mathbf{y}}^k\right) \in \bigcup_{j\in\mathcal{J}}\mathcal{D}_j^{\text{SW}}$, $k = 1, \ldots, p$, and a vector $\bar{\boldsymbol{\lambda}} \geq \mathbf{0}_p$, such that $\mathbf{1}_p^T\bar{\boldsymbol{\lambda}} = 1$ and

$$\bar{\mathbf{x}} = \sum_{k=1}^{p}\bar{\lambda}_k\bar{\mathbf{x}}^k, \quad \bar{\mathbf{y}} = \min_{\bar{\boldsymbol{\lambda}}}\left(\bar{\mathbf{y}}^1, \ldots, \bar{\mathbf{y}}^p\right). \tag{A.3}$$

For each $k=1,\dots,p$, there exists a $j(k)\in\mathcal{J}$ such that $\left(\bar{\mathbf{x}}^k,\bar{\mathbf{y}}^k\right)\in\mathcal{D}^{\mathrm{SW}}_{j(k)}$. By (1), we have $\bar{\mathbf{x}}^k=\bar{\delta}_k\mathbf{x}_{j(k)}$, where $\bar{\delta}_k\geq 1$, and $\bar{\mathbf{y}}^k\leq\mathbf{y}_{j(k)}$. Restate the first equality in (A.3) as

$$\bar{\mathbf{x}}=\sum_{k=1}^{p}\bar{\lambda}_k\bar{\delta}_k\mathbf{x}_{j(k)}. \tag{A.4}$$

To prove that $\mathcal{B}$ satisfies Axiom SOD, let $\hat{\mathbf{y}}\leq\bar{\mathbf{y}}$. Consider any $k=1,\dots,p$, such that $\bar{\lambda}_k>0$. By (A.3), $\hat{\mathbf{y}}\leq\mathbf{y}_{j(k)}$. Because the set $\mathcal{D}^{\mathrm{SW}}_j$ satisfies Axiom SOD, we have $\left(\bar{\delta}_k\mathbf{x}_{j(k)},\hat{\mathbf{y}}\right)\in\mathcal{D}^{\mathrm{SW}}_{j(k)}$. By (8), $(\bar{\mathbf{x}},\hat{\mathbf{y}})\in\mathcal{B}$. Therefore, $\mathcal{B}$ satisfies Axiom SOD.

Consider any $\gamma\geq 1$. Then $(\gamma\bar{\mathbf{x}},\bar{\mathbf{y}})$ satisfies conditions (A.4) in which we replace $\bar{\boldsymbol{\delta}}$ by $\gamma\bar{\boldsymbol{\delta}}$. Therefore, $(\gamma\bar{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{B}$, and $\mathcal{B}$ satisfies Axiom WID. ■

Proof of Theorem 6.4. By Lemma 7, the set $\mathcal{B}$ satisfies Axioms IO, SOD, WID and CIS. By Definition 6.1, we have $\mathcal{T}^{\mathrm{WID}}_1\subseteq\mathcal{B}$. Conversely, by Definition 6.1 and Theorem 6.3, $\mathcal{T}^{\mathrm{WID}}_1$ is s-convex. Therefore, it suffices to show that $\mathcal{D}^{\mathrm{SW}}_j\subseteq\mathcal{T}^{\mathrm{WID}}_1$, for all $j\in\mathcal{J}$. For any $j\in\mathcal{J}$, let $(\delta_j\mathbf{x}_j,\mathbf{z}_j)\in\mathcal{D}^{\mathrm{SW}}_j$, where $\mathbf{0}_s\leq\mathbf{z}_j\leq\mathbf{y}_j$ and $\delta_j\geq 1$. Because $\mathcal{T}^{\mathrm{WID}}_1$ satisfies Axioms IO and WID, we have $(\delta_j\mathbf{x}_j,\mathbf{z}_j)\in\mathcal{T}^{\mathrm{WID}}_1$, and the proof follows. ■

Proof of Theorem 7.1. Let $\mathcal{F}$ denote the set on the right-hand side of (11). We need to prove that $\mathcal{T}^{\mathrm{WID}}_2=\mathcal{F}$. Clearly, $\mathcal{F}$ satisfies Axiom IO. Furthermore, let $(\bar{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{F}$. Then there exists a $j'\in\mathcal{J}$ such that $(\bar{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{D}^{\mathrm{SW}}_{j'}$. Because $\mathcal{D}^{\mathrm{SW}}_{j'}$ satisfies Axioms SOD and WID, $\mathcal{F}$ also satisfies them. Because $\mathcal{T}^{\mathrm{WID}}_2$ is the smallest set that satisfy Axioms IO, SOD and WID, we have $\mathcal{T}^{\mathrm{WID}}_2\subseteq\mathcal{F}$. Conversely, because $\mathcal{T}^{\mathrm{WID}}_2$ satisfies Axioms IO, SOD and WID, we have $\mathcal{D}^{\mathrm{SW}}_j\subseteq\mathcal{T}^{\mathrm{WID}}_2$, for all $j\in\mathcal{J}$. Therefore, $\mathcal{F}\subseteq\mathcal{T}^{\mathrm{WID}}_2$. ■

Proof of Corollary 7.1. Technology $\mathcal{T}^{\mathrm{WID}}_2$ is closed as the union of a finite number of closed sets $\mathcal{D}^{\mathrm{SW}}_j$, $j\in\mathcal{J}$. ■

Proof of Theorem 7.2. Denote $\mathcal{G}$ the set on the right-hand side of (12). Let $(\bar{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{G}$. Then $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ satisfies all conditions of (12) with some vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\delta}}$. There exists a single $j'\in\mathcal{J}$ such that $\bar{\lambda}_{j'}=1$, and $\bar{\lambda}_j=0$, for all $j\neq j'$, $j\in\mathcal{J}$. Then (12) implies that $(\bar{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{D}^{\mathrm{SW}}_{j'}$. By (11), $(\bar{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{T}^{\mathrm{WID}}_2$. Therefore, $\mathcal{G}\subseteq\mathcal{T}^{\mathrm{WID}}_2$.

Conversely, the set $\mathcal{G}$ clearly satisfies Axioms IO, SOD and WID. Because $\mathcal{T}^{\mathrm{WID}}_2$ is the smallest set that satisfies these axioms, we have $\mathcal{T}^{\mathrm{WID}}_2\subseteq\mathcal{G}$, and the proof follows. ■

Lemma 8. *Let $\mathcal{V}$ be the set on the right-hand side of* (14). *Then $\mathcal{V}$ satisfies Axioms IO, SOD, WID and CO.*

Proof of Lemma 8. Clearly, $\mathcal{V}$ satisfies Axiom IO and CO. Denote

$$\mathcal{V}_0=\operatorname{conv}\Bigl(\bigcup_{j\in\mathcal{J}}\mathcal{D}^{\mathrm{S}}_j\Bigr). \tag{A.5}$$

Then

$$\mathcal{V}=\mathcal{V}^{\mathrm{W}}_0. \tag{A.6}$$

Consider any $(\bar{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{V}$. By (A.6), there exist a $(\tilde{\mathbf{x}},\bar{\mathbf{y}})\in\mathcal{V}_0$ and a $\bar{\delta}\geq 1$ such that $(\bar{\mathbf{x}},\bar{\mathbf{y}})=\left(\bar{\delta}\tilde{\mathbf{x}},\bar{\mathbf{y}}\right)$. To prove that $\mathcal{V}$ satisfies Axiom SOD, let $\mathbf{0}_s\leq\hat{\mathbf{y}}\leq\bar{\mathbf{y}}$. We need to prove that $(\bar{\mathbf{x}},\hat{\mathbf{y}})\in\mathcal{V}$. As in the proof of Lemma 2, it is straightforward to show that $\mathcal{V}_0$ satisfies Axiom SOD. Therefore, $(\tilde{\mathbf{x}},\hat{\mathbf{y}})\in\mathcal{V}_0$. Then $\left(\bar{\delta}\tilde{\mathbf{x}},\hat{\mathbf{y}}\right)\in\mathcal{V}^{\mathrm{W}}_0$, i.e., $(\bar{\mathbf{x}},\hat{\mathbf{y}})\in\mathcal{V}$.

To prove that $\mathcal{V}$ satisfies Axiom WID, let $\gamma\geq 1$. Then $(\gamma\bar{\mathbf{x}},\bar{\mathbf{y}})=\left((\gamma\bar{\delta})\tilde{\mathbf{x}},\bar{\mathbf{y}}\right)\in\mathcal{V}$. ■

Lemma 9. *Let $\mathcal{C}$ be the convex hull of the set of all observed activities $(\mathbf{x}_j, \mathbf{y}_j)$, $j \in \mathcal{J}$, and let $\mathcal{C}^{\mathrm{S}}$ be the SODH of the set $\mathcal{C}$. Then $\mathcal{C}^{\mathrm{S}} = \mathcal{V}_0$, where the set $\mathcal{V}_0$ is defined by* (A.5).

Proof of Lemma 9. To prove that $\mathcal{V}_0 \subseteq \mathcal{C}^{\mathrm{S}}$, consider any $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{V}_0$. By Carathéodory's theorem, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a convex combination of a finite number of activities from the union of sets $\mathcal{D}_j^{\mathrm{S}}$, $j \in \mathcal{J}$. Because each set $\mathcal{D}_j^{\mathrm{S}}$ is convex, without loss of generality, we assume that all such activities are taken from different sets $\mathcal{D}_j^{\mathrm{S}}$. Therefore, there exist activities $(\mathbf{x}_j, \bar{\mathbf{z}}_j) \in \mathcal{D}_j^{\mathrm{S}}$, such that $\mathbf{0}_s \leq \bar{\mathbf{z}}_j \leq \mathbf{y}_j$, for all $j \in \mathcal{J}$, and a vector $\bar{\boldsymbol{\lambda}} \geq \mathbf{0}_n$, such that $\mathbf{1}_n^T \bar{\boldsymbol{\lambda}} = 1$ and

$$\bar{\mathbf{x}} = \sum_{j \in \mathcal{J}} \bar{\lambda}_j \mathbf{x}_j, \qquad \bar{\mathbf{y}} = \sum_{j \in \mathcal{J}} \bar{\lambda}_j \bar{\mathbf{z}}_j.$$

Let $\hat{\mathbf{y}} = \sum_{j \in \mathcal{J}} \bar{\lambda}_j \mathbf{y}_j$. Then $(\bar{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathcal{C}$. Because $\bar{\mathbf{y}} \leq \hat{\mathbf{y}}$, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{C}^{\mathrm{S}}$. Therefore, $\mathcal{V}_0 \subseteq \mathcal{C}^{\mathrm{S}}$.

Conversely, note that $\mathcal{C} \subseteq \mathcal{V}_0$. As in the proof of Lemma 2, it is straightforward to prove that $\mathcal{V}_0$ satisfies Axiom SOD. Therefore, $\mathcal{C}^{\mathrm{S}} \subseteq \mathcal{V}_0$. ■

Proof of Theorem 8.1. In this proof we use notation introduced in Lemmas 8 and 9. By Lemma 8, the set $\mathcal{V}$ satisfies Axioms IO, SOD, WID and CO. Because $\mathcal{T}_S^{\mathrm{WID}}$ is the smallest set that satisfies these axioms, we have $\mathcal{T}_S^{\mathrm{WID}} \subseteq \mathcal{V}$.

Conversely, because $\mathcal{T}_S^{\mathrm{WID}}$ satisfies Axioms IO and CO, we have $\mathcal{C} \subseteq \mathcal{T}_S^{\mathrm{WID}}$. Because $\mathcal{T}_S^{\mathrm{WID}}$ satisfies Axiom SOD, we have $\mathcal{C}^{\mathrm{S}} \subseteq \mathcal{T}_S^{\mathrm{WID}}$. By Lemma 9, $\mathcal{V}_0 \subseteq \mathcal{T}_S^{\mathrm{WID}}$, where $\mathcal{V}_0$ is defined by (A.5). Finally, because $\mathcal{T}_S^{\mathrm{WID}}$ satisfies Axiom WID, we have $\mathcal{V}_0^{\mathrm{W}} \subseteq \mathcal{T}_S^{\mathrm{WID}}$. Noting (A.6) completes the proof. ■

Proof of Corollary 8.1. By Theorem 8.1 and (A.6), we have $\mathcal{T}_S^{\mathrm{WID}} = \mathcal{V} = \mathcal{V}_0^{\mathrm{W}}$. By Lemma 9, $\mathcal{V}_0^{\mathrm{W}} = \left(\mathcal{C}^{\mathrm{S}}\right)^{\mathrm{W}} = \mathcal{C}^{\mathrm{SW}}$. ■

Proof of Theorem 8.2. Let $\mathcal{U}$ denote the set on the right-hand side of (15). By Corollary 8.1, it suffices to prove that $\mathcal{U} = \left(\mathcal{C}^{\mathrm{S}}\right)^{\mathrm{W}}$. Let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{U}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies all conditions of (15) with some vector $\bar{\boldsymbol{\lambda}}$ and scalar $\bar{\delta}$. By definition of the set $\mathcal{C}$, we have $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathcal{C}$, where

$$\hat{\mathbf{x}} = \sum_{j \in \mathcal{J}} \bar{\lambda}_j \mathbf{x}_j, \qquad \hat{\mathbf{y}} = \sum_{j \in \mathcal{J}} \bar{\lambda}_j \mathbf{y}_j. \tag{A.7}$$

Then $(\hat{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{C}^{\mathrm{S}}$, and $\left(\bar{\delta}\hat{\mathbf{x}}, \bar{\mathbf{y}}\right) = (\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \left(\mathcal{C}^{\mathrm{S}}\right)^{\mathrm{W}}$. Therefore, $\mathcal{U} \subseteq \left(\mathcal{C}^{\mathrm{S}}\right)^{\mathrm{W}}$. Conversely, let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \left(\mathcal{C}^{\mathrm{S}}\right)^{\mathrm{W}}$. Then there exists a $\bar{\delta} \geq 1$ such that $\bar{\mathbf{x}} = \bar{\delta}\hat{\mathbf{x}}$ and $(\hat{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{C}^{\mathrm{S}}$. Furthermore, by definition of $\mathcal{C}^{\mathrm{S}}$, there exists a vector $\hat{\mathbf{y}}$ such that $\hat{\mathbf{y}} \geq \bar{\mathbf{y}}$ and $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathcal{C}$. By definition of $\mathcal{C}$, there exists a vector $\bar{\boldsymbol{\lambda}}$ such that equalities (A.7) are true. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies all conditions of (15) with the above $\bar{\boldsymbol{\lambda}}$ and $\bar{\delta}$. Therefore, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathcal{U}$, and $\left(\mathcal{C}^{\mathrm{S}}\right)^{\mathrm{W}} \subseteq \mathcal{U}$. ■

Proof of Theorem 8.3. The proof is similar to the proof of Theorem 6.2, so we only outline its idea. Let some sequence $\left\{\left(\mathbf{x}^k, \mathbf{y}^k\right) \mid k = 1, 2, \dots\right\} \subset \mathcal{T}_S^{\mathrm{WID}}$ converge to $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. We need to prove that $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathcal{T}_S^{\mathrm{WID}}$. For any k, $\left(\mathbf{x}^k, \mathbf{y}^k\right)$ satisfies all conditions in (15) with some vector $\boldsymbol{\lambda}^k$ and scalar δ^k. Because the sequence $\left\{\boldsymbol{\lambda}^k\right\}$ belongs to a compact set, without loss of generality, we assume that it converges to some vector $\hat{\boldsymbol{\lambda}}$. Furthermore, we have $\hat{\lambda}_j > 0$ for some $j \in \mathcal{J}$. This and the fact that $\mathbf{x}_j \neq \mathbf{0}_m$ imply that there exists a constant D such that $\delta^k \in [1, D]$. Therefore, without loss of generality, the sequence $\{\delta^k\}$ converges to some $\hat{\delta}$. Taking the limit of all conditions in (15), we prove that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ satisfies (15). ■

Proof of Theorem 8.4. Because $\mathcal{T}_S^{\mathrm{WID}} \subseteq \tilde{\mathcal{T}}_0^{\mathrm{WID}}$ and $\tilde{\mathcal{T}}_0^{\mathrm{WID}}$ is a convex technology, we have $\operatorname{conv}\left(\mathcal{T}_S^{\mathrm{WID}}\right) \subseteq \tilde{\mathcal{T}}_0^{\mathrm{WID}}$. Conversely, let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \tilde{\mathcal{T}}_0^{\mathrm{WID}}$. By (2) and by Carathéodory's theorem, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a convex combination of a finite number of activities from the sets $\mathcal{D}_j^{\mathrm{SW}}$,

$j \in \mathcal{J}$. Because $\mathcal{T}_S^{\text{WID}}$ satisfies Axioms WID and SOD, we have $\mathcal{D}_j^{\text{SW}} \subseteq \mathcal{T}_S^{\text{WID}}$, for all $j \in \mathcal{J}$. Therefore, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \text{conv}\left(\mathcal{T}_S^{\text{WID}}\right)$. ■

Proof of Theorem 8.5. Let $\mathbf{y} \in \mathbb{R}_+^s$ and let $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{Q}_{\text{S}}^{\text{WID}}(\mathbf{y})$. Then $(\mathbf{x}^1, \mathbf{y})$ and $(\mathbf{x}^2, \mathbf{y})$ satisfy all conditions of (15) with some scalars δ^1, δ^2 and vectors $\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2$. For any $\sigma \in [0, 1]$, define $\mathbf{x}^\sigma = \sigma \mathbf{x}^1 + (1 - \sigma)\mathbf{x}^2$. Then $(\mathbf{x}^\sigma, \mathbf{y})$ satisfies all conditions of (15) with

$$\delta' = \sigma\delta^1 + (1 - \sigma)\delta^2, \qquad \boldsymbol{\lambda}' = \frac{\sigma\delta^1}{\delta'}\boldsymbol{\lambda}^1 + \frac{(1 - \sigma)\delta^2}{\delta'}\boldsymbol{\lambda}^2.$$

Therefore, $\mathbf{x}^\sigma \in \mathcal{Q}_{\text{S}}^{\text{WID}}(\mathbf{y})$, which completes the proof. ■

Lemma 10. *Let all observed activities produce the same output vector* $\tilde{\mathbf{y}}$. *Then* $\tilde{\mathcal{T}}_0^{\text{WID}} = \mathcal{T}_S^{\text{WID}}$.

Proof of Lemma 10. Any $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathcal{T}_S^{\text{WID}}$ satisfies (15) with some vector $\hat{\boldsymbol{\lambda}}$ and scalar $\hat{\delta}$. Then $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ satisfies (3) with the same vector $\hat{\boldsymbol{\lambda}}$ and $\delta_j = \hat{\delta}$, for all $j \in \mathcal{J}$. Therefore, $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \tilde{\mathcal{T}}_0^{\text{WID}}$, and $\mathcal{T}_S^{\text{WID}} \subseteq \tilde{\mathcal{T}}_0^{\text{WID}}$.

Conversely, let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \tilde{\mathcal{T}}_0^{\text{WID}}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (3) with some vectors $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{\delta}}$. Because all observed activities have the same output vector $\tilde{\mathbf{y}}$, from (3) we have:

$$\bar{\mathbf{x}} = \sum_{j \in \mathcal{J}} \bar{\lambda}_j \bar{\delta}_j \mathbf{x}_j, \qquad \bar{\mathbf{y}} \leq \sum_{j \in \mathcal{J}} \bar{\lambda}_j \bar{\mathbf{y}}_j = \tilde{\mathbf{y}}. \tag{A.8}$$

Define $\tilde{\delta} = \sum_{j \in \mathcal{J}} \bar{\lambda}_j \bar{\delta}_j$. Restating (A.8), we obtain

$$\bar{\mathbf{x}} = \tilde{\delta} \sum_{j \in \mathcal{J}} \frac{\bar{\lambda}_j \bar{\delta}_j}{\tilde{\delta}} \mathbf{x}_j, \qquad \bar{\mathbf{y}} \leq \sum_{j \in \mathcal{J}} \frac{\bar{\lambda}_j \bar{\delta}_j}{\tilde{\delta}} \tilde{\mathbf{y}}.$$

Define $\tilde{\lambda}_j = \bar{\lambda}_j \bar{\delta}_j / \tilde{\delta}$, for all $j \in \mathcal{J}$. Then $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies (15) with the vector $\tilde{\boldsymbol{\lambda}}$ and scalar $\tilde{\delta}$. Therefore, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \tilde{\mathcal{T}}_S^{\text{WID}}$, and $\tilde{\mathcal{T}}_0^{\text{WID}} \subseteq \mathcal{T}_S^{\text{WID}}$. ■

Lemma 11. *Let all observed activities produce the same output vector* $\tilde{\mathbf{y}}$. *Then* $\tilde{\mathcal{T}}_0^{\text{WID}} = \mathcal{T}_1^{\text{WID}}$.

Proof of Lemma 11. Under the condition of this lemma, the output inequalities of statement (3) of technology $\tilde{\mathcal{T}}_0^{\text{WID}}$ and statement (7) of technology $\mathcal{T}_1^{\text{WID}}$ take on the form $\tilde{\mathbf{y}} \geq \mathbf{y}$. The two statements are identical and, therefore, $\tilde{\mathcal{T}}_0^{\text{WID}} = \mathcal{T}_1^{\text{WID}}$. ■

Proof of Theorem 8.6. By Lemma 10, we have $\mathcal{T}_S^{\text{WID}} = \tilde{\mathcal{T}}_0^{\text{WID}}$. Taking the closure of both sides and noting that, by Theorem 8.3, $\mathcal{T}_S^{\text{WID}}$ is a closed set, we have $\mathcal{T}_S^{\text{WID}} = \text{cl}\left(\tilde{\mathcal{T}}_0^{\text{WID}}\right) = \mathcal{T}_0^{\text{WID}}$, where the last equality follows from (4). Noting Lemma 11 completes the proof. ■

References

Agrell, P. J., Bogetoft, P., Brock, M., & Tind, J. (2005). Efficiency evaluation with convex pairs. *Advanced Modeling and Optimization*, *7*(2), 211–237.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale efficiencies in data envelopment analysis. *Management Science*, *30*(9), 1078–1092.

Bogetoft, P. (1996). DEA models on relaxed convexity assumptions. *Management Science*, *42*(3), 457–465.

Bogetoft, P., Tama, J. M., & Tind, J. (2000). Convex input and output projections of nonconvex production possibility sets. *Management Science*, *46*(6), 858–869.

Briec, W., Kerstens, K., & Vanden Eeckaut, P. (2004). Non-convex technologies and cost functions: Definitions, duality and nonparametric tests of convexity. *Journal of Economics*, *81*(2), 155–192.

Byrnes, P., Färe, R., Grosskopf, S., & Lovell, C. A. K. (1988). The effect of unions on productivity: US surface mining of coal. *Management Science*, *34*(9), 1037–1053.

Chambers, R. G., Chung, Y., & Färe, R. (1998). Profit, directional distance functions, and Nerlovian efficiency. *Journal of Optimization Theory and Applications*, *98*(2), 351–364.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, *2*(6), 429–444.

Deprins, D., Simar, L., & Tulkens, H. (1984). Measuring labor-efficiency in post offices. In M. Marchand, P. Pestieau, H. Tulkens (Eds.), *The performance of public enterprises: concepts and measurements* (pp. 243–267). Amsterdam: North-Holland.

Dervaux, B., Kerstens, K., & Vanden Eeckaut, P. (1998). Radial and nonradial static efficiency decompositions: A focus on congestion measurement. *Transportation Research. Part B: Methodological*, *32*(5), 299–312.

Färe, R., & Grosskopf, S. (1983). Measuring congestion in production. *Zeitschrift für Nationalökonomie / Journal of Economics*, *43*(3), 257–271.

Färe, R., & Grosskopf, S. (2000). Research note. Decomposing technical efficiency with care. *Management Science*, *46*(1), 167–168.

Färe, R., & Grosskopf, S. (2003). Non-parametric productivity analysis with undesirable outputs: Comment. *American Journal of Agricultural Economics*, *85*(4), 1070–1074.

Färe, R., Grosskopf, S., & Lovell, C. A. K. (1985). *The measurement of efficiency of production*. Boston: Kluwer Academic Publishers.

Färe, R., Grosskopf, S., & Lovell, C. A. K. (1994). *Production frontiers*. Cambridge: Cambridge University Press.

Färe, R., & Karagiannis, G. (2017). The denominator rule for share-weighting aggregation. *European Journal of Operational Research*, 260(3), 1175–1180.

Grosskopf, S. (1986). The role of the reference technology in measuring productive efficiency. *The Economic Journal*, *96*(382), 499–513.

Kuosmanen, T. (2001). DEA with efficiency classification preserving conditional convexity. *European Journal of Operational Research*, *132*(2), 326–342.

Kuosmanen, T. (2005). Weak disposability in nonparametric productivity analysis with undesirable outputs. *American Journal of Agricultural Economics*, *87*(4), 1077–1082.

Kuosmanen, T., & Podinovski, V. V. (2009). Weak disposability in nonparametric production analysis: Reply to Färe and Grosskopf. *American Journal of Agricultural Economics*, *91*(2), 539–545.

Mehdiloozad, M., Sahoo, B. K., & Roshdi, I. (2014). A generalized multiplicative directional distance function for efficiency measurement in DEA. *European Journal of Operational Research*, *232*(3), 679–688.

Mehdiloozad, M., Zhu, J., & Sahoo, B. K. (2017). Identification of congestion in data envelopment analysis under the occurrence of multiple projections: A reliable method capable of dealing with negative data. *European Journal of Operational Research*, doi.org/10.1016/j.ejor.2017.07.065.

Olesen, O. B., Petersen, N. C., & Podinovski, V. V. (2015). Efficiency analysis with ratio measures. *European Journal of Operational Research*, *245*(2), 446–462.

Olesen, O. B., Petersen, N. C., & Podinovski, V. V. (2017). Efficiency measures and computational approaches for data envelopment analysis models with ratio inputs and outputs. *European Journal of Operational Research*, *261*(2), 640–655.

Petersen, N. C. (1990). Data envelopment analysis on a relaxed set of assumptions. *Management Scoience*, *36*(3), 305–314.

Podinovski, V. V. (2004). Bridging the gap between the constant and variable returns-to-scale models: Selective proportionality in data envelopment analysis. *Journal of the Operational Research Society*, *55*(3), 265–276.

Podinovski, V. V. (2005). Selective convexity in DEA models. *European Journal of Operational Research*, *161*(2), 552–563.
Podinovski, V. V., Chambers, R. G., Atici, K. B., & Deineko, I. D. (2016). Marginal values and returns to scale for nonparametric production frontiers. *Operations Research*, *64*(1), 236–250.
Podinovski, V. V., & Kuosmanen, T. (2011). Modelling weak disposability in data envelopment analysis under relaxed convexity assumptions. *European Journal of Operational Research*, *211*(3), 577–585.
Ray, S. C. (2004). *Data envelopment analysis. Theory and techniques for economics and operations research.* Cambridge: Cambridge University Press.
Rockafellar, R. T. (1970). *Convex analysis.* Princeton, NJ: Princeton University Press.
Shephard, R. W. (1974). Semi-homogeneous production functions and scaling of production. In W. Eichhorn, R. Henn, O. Opitz, R.W. Shephard (Eds.), *Production theory* (pp. 253-285). New York: Springer-Verlag.
Tone, K., & Sahoo, B. K. (2004). Degree of scale economies and congestion: A unified DEA approach. *European Journal of Operational Research*, *158*(3), 755–772.