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Eigenvalues and eigenvectors of a system of Bernoulli Euler beams connected together in a tree topology

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Abstract

Consideration is given to determining the exact solutions of the eigenproblem posed by a graph with linear tree topology on which the fourth order Sturm Liouville operator is acting. However, this purely mathematical problem in Quantum Graph theory can be solved straightforwardly using a structural mechanics analogy, namely that its solution corresponds precisely to the free vibration problem of a network of beams with identical topology. It is interesting to note that this parallels previous work in which a similar analogy, but with bars rather than beams, was made to establish exact solutions to the simpler problem of the second order Sturm Liouville operator acting on similar tree topologies [1,2]. Such problems remain a continuing source of mathematical interest [3,4].

The exact free vibration of a single, uniform Bernoulli Euler beam can be described by the following fourth order Sturm-Liouville equation

$$\frac{d^2}{dx^2}\left(v_2(x)\frac{d^2y}{dx^2}\right)-\frac{d}{dx}\left(v_1(x)\frac{dy}{dx}\right)+v_0(x)=\lambda wy \qquad (1)$$

where $v_2(x)$ is the flexural rigidity of the beam; $v_1(x)$ is the static axial load; $v_0(x)$ is the distributed foundation stiffness per unit length; w is the mass per unit length of the beam; and λ is ω^2, (ω is the circular frequency). The exact solution of Eq.(1) is most conveniently achieved in the form of a dynamic stiffness matrix, e.g. [5]. This allows any number of tree topologies to be modelled, while use of the Wittrick-Williams algorithm [5] enables any desired eigenvalue to be converged upon to any desired accuracy with the certainty that none have been missed.

The remainder of this paper examines the eigenvalues and corresponding eigenvectors of a series of trees typified by the one drawn to enhance clarity in Fig. 1. However, it should be noted that: every tree must have one or more levels $(n \geq 1)$; within any given tree the branching number, b, must be constant with $b \geq 1$; and that any members (edges) at the same tree level are theoretically collinear in terms of the structural mechanics analogy. Subject to these constraints, the use of theoretical relationships for eigenvalue multiplicity and efficient coding techniques, trees of virtually any complexity can be solved. For brevity in the results presented herein, each member is assumed to be a uniform Bernoulli Euler beam and the effects of static axial load and distributed foundation stiffness are ignored.

The results given in Table 1 are for a tree with clamped boundary conditions at the left hand side (the root of the tree) indicated by the letter ‘A’ in Fig. 1 and at the right hand side indicated by the letters ‘O’ to ‘W’. The table shows the set of fundamental, normalised eigenvalues and their corresponding multiplicities, that completely describe the family of repetitive trees of length n = 1, 2,..., 5. The index, r, defines the subtree length in which a fundamental eigenvalue first occurs, denoted by a ‘1’ in the appropriate column. Any multiplicities are then calculated from Eq.(2). As examples, consider the following: (a) the tenth eigenvalue of the n = 5 tree is given as $\lambda \approx (0.5087\pi)^4$. This fundamental eigenvalue first appears in subtree r = n = 5 and Eq.(2a) gives its multiplicity M_n as 1; (b) the 15th eigenvalue for the same tree lies in a group of multiplicity 18, where $\lambda \approx (0.7314\pi)^4$ and its multiplicity of the fundamental eigenvalue is given by Eq.(2b) with $b = 3$, $n = 3$, $r = 2$, as $M_2 = 2 \times 3^{5-2-1} = 18$; (c) the 16th eigenvalue for an $n = 4$ tree is given as $\lambda \approx (1.192\pi)^4$ whose multiplicity is given by Eq.(2b) as $M_3 = 2 \times 3^{4-3-1} = 2$. The last eigenvalue given in the table corresponds to the clamped clamped eigenvalue of a single beam. For $n = 5$ this eigenvalue has multiplicity 54. For the notation adopted in this paper this multiplicity is defined as M_1=54. The subscript refers to the subtree that is vibrating. The mulitplicites of each eigenvalue M_r are given by

$$M_n = 1;\ M_r = (b-1)b^{n-r-1} \ \text{ for } \ 1 \le r \le n-1 = 1 \tag{2a,b}$$

Table 1. Normalised eigenvalues of a tree with n=5 and b=3. For each eigenvalue the associated subtree length is given by the index r corresponding to a mulitplicity of '1'.

Eigenvalues		Multiplicities				
k	$\sqrt[4]{\lambda_k}/\pi$	$r=$ 5	4	3	2	1
1	0.3234581	1				
2	0.3896334	2	1			
3	0.5040498	6	2	1		
4	0.5087078	1				
5	0.6242067	2	1			
6	0.6965265	1				
7	0.7314065	18	6	2	1	
8	0.8103492	6	2	1		
9	0.8506574	2	1			
10	0.8746724	1				
11	1.127646	1				
12	1.151974	2	1			
13	1.192491	6	2	1		
14	1.272673	18	6	2	1	
15	1.307907	1				
16	1.380471	2	1			
17	1.499681	1				
18	1.499999	6	2	1		
19	1.505618	54	18	6	2	1

For the tree shown in Figure 1 with $n = 4$ the results in the column $r = 5$ can be ignored. The first eigenvalue for this tree has a multiplicity 1 and is the fundamental eigenvalue. Higher harmonics of this fundamental eigenvalue exist higher up the spectrum as shown by the multiple occurrences of the value 1 in the $r = 4$ column. Other multiplicites exist such as 2 and 6. These are different fundamental eigenvalues and are harmonics of subtrees. The first occurence of the multiplicity 2, in this column, is the fundamental eigenvalue for a subtree $r = 3$. Any eigenvalue with multiplicity greater than unity will have unique eigenvector characteristics. Looking at Figure 1, we can see that if the subtree of length 3 emmanating from vertex B to tips OPQ is vibrating with its root clamped then it can be only achieved if the subtree B to RST is vibrating in antiphase or if B to UVW is vibrating in antiphase or a combination of the two subtrees vibrating in antiphase. Hence there exist two independent orthogonal modes and the $n = 4$ tree has an eigenvalue of the subtree of length r=3. This argument can be followed for all subtrees.

The rapid growth in the multiplicities can be seen by looking at eigevectors of the individual member for a clamped clamped eigenvalue. Figure 2a indicates that any pair of beams which emanate to the right of a vertex at level $n-1$ of Figure 1 can vibrate flexurally in antiphase and with their modes having equal amplitude, so that, remembering that the beams are collinear, equilibrium of moment and of transverse force exists at their common vertex. Because there is equilibrium with the force vector results in zero rotation and zero translation which, in effect, is the equivalent of a clamped supprt. Hence the common node is shown as a clamped boundary condition at the left hand side. It is only possible to get equilibirum by examining modes that have members vibrating in antiphase. An eigenvector to a path such as P_tGCBA (t: top) is no longer possible at the same eigenvalue because the associated mode would have to involve zero deflection and rotation at G, C and B and such modes cannot give both moment and transverse force equilibrium at G, C and B (they can only give either moment equilibrium or transverse force equilibrium). Figure 2(b) shows a mode for a subtree of length $r = 2$. The approximate mode and the relative amplitudes shown for the upper four beams give force and moment equilibrium both at their common vertex and, because the lower four beams are in anti-phase, at the left-hand vertex on the figure. Therefore, this is clearly a possible mode for the set of b subtrees emanating to the right from any vertex at level $n-2$. Hence it may be deduced that the mode multiplicities are again the same as those described in equation (1).

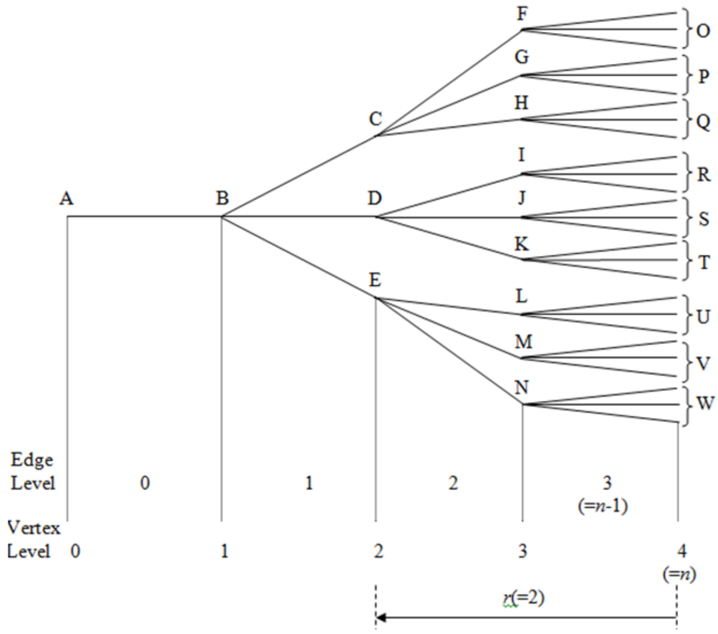
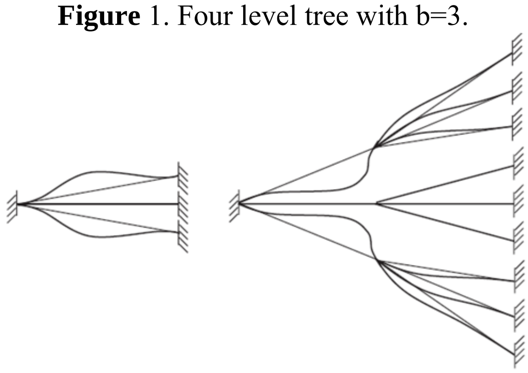


Figure 1. Four level tree with b=3.



(a) (b)

Figure 2. Eigenvectors corresponding to (a) M_1 and (b) M_2 for trees of beams which form the analogous structural mechanics problem of trees of the Sturm-Liouville differential equations.

References

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