SPECTRAL PROPERTIES OF THE MASSLESS RELATIVISTIC QUARTIC OSCILLATOR

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ABSTRACT. An explicit solution of the spectral problem of the non-local Schrödinger operator obtained as the sum of the square root of the Laplacian and a quartic potential in one dimension is presented. The eigenvalues are obtained as zeroes of special functions related to the fourth order Airy function, and closed formulae for the Fourier transform of the eigenfunctions are derived. These representations allow to derive further spectral properties such as estimates of spectral gaps, heat trace and the asymptotic distribution of eigenvalues, as well as a detailed analysis of the eigenfunctions. A subtle spectral effect is observed which manifests in an exponentially tight approximation of the spectrum by the zeroes of the dominating term in the Fourier representation of the eigenfunctions and its derivative.

1. INTRODUCTION

In this paper our aim is to obtain explicit formulae on the spectrum and eigenfunctions of the specific non-local Schrödinger operator

$$(1.1) \qquad H = \sqrt{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}} + x^4,$$

defined as the sum in an appropriate sense of the one-dimensional fractional Laplacian of index $\frac{1}{2}$ and the potential $V(x) = x^4$ acting as a multiplication operator. While for the classical Schrödinger operator featuring the Laplacian and the same potential there are no closed-form solutions of the eigenvalue problem, this can be obtained for the relativistic Schrödinger operator above.

Non-local Schrödinger operators of the type

$$(1.2) \qquad H^{\Psi} = \Psi(-\Delta) + V$$

currently receive an increasing attention displaying interesting features from the perspective of functional analysis, stochastic processes and mathematical physics, with a rich interplay between these aspects [2, 8, 6, 10, 11, 12, 9]. Here Ψ is a so called Bernstein function, which can be represented by the Lévy-Khintchine formula for subordinators, and V is a Borel-measurable function acting as a multiplication operator, called potential. For a specific choice of Ψ one obtains fractional Schrödinger operators of the form $(-\Delta + m^{2/\alpha})^{\alpha/2} - m + V$, $0 < \alpha < 2$, with or without rest mass m. A basic motivation of the study of models with

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fractional Laplacians comes from models of relativistic quantum mechanics and anomalous transport theory. Another fact that makes these operators particularly interesting is that they are Markov generators of jump processes.

One approach to investigating the spectral properties of H^Ψ exploits the fact that (1.2) is a perturbation of a pseudo-differential operator, and uses direct methods of analysis to derive various spectral and regularity properties. Another approach, see [6, 7, 10, 11, 12], uses a probabilistic language starting from the fact that non-local operators of the type $\Psi(-\Delta)$ generate Brownian motion sampled at random times, i.e., subordinate Brownian motions $(B_{S_t^\Psi})_{t\geq 0}$, where the subordinator $(S_t^\Psi)_{t\geq 0}$ is uniquely determined by the Bernstein function Ψ, see [21]. Subordinate Brownian motion is a jump Lévy process, and when V is added to the operator, it has the effect of killing or reinforcing paths according to its sign and magnitude. In particular, fractional Schrödinger operators with $m = 0$ or $m > 0$ generate rotationally symmetric α-stable and relativistic α-stable processes, respectively. The two cases differ essentially by the concentration properties of the Lévy measure of the underlying process, which has a significant impact on some properties of eigenfunctions of the related non-local Schrödinger operators.

In this paper we take the analytic approach, which complements our previous investigations of non-local Schrödinger operators using a stochastic description. Although the literature in this field is rapidly increasing, there is a shortage of a detailed understanding of examples which can stand as benchmark cases to the general theory. Our aim in this paper is to contribute to filling this gap and by choosing $d = 1$, $\alpha = 1$ and $V(x) = x^4$, in the remainder of this paper we consider the operator (1.1) in detail. There is special interest in this operator also from a physics viewpoint in that it describes a semi-relativistic quantum particle with no rest mass, moving in the force field of a quartic potential. Probabilistically, H is the generator of a one-dimensional Cauchy process (isotropic 1-stable process) with a space-dependent killing by the potential, which can be derived by the Feynman-Kac formula and subordination.

Using a standard argument [17, Ch. 3] it follows that the spectrum of H is purely discrete consisting of eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \to \infty$, each having finite multiplicity. The corresponding eigenfunctions $\psi_1, \psi_2, \ldots$ are bounded continuous functions and form an orthonormal basis in $L^2(\mathbb{R})$. The first eigenfunction (ground state) ψ_1, corresponding to the bottom of the spectrum λ_1, can be shown to be unique and have a strictly positive version. Making use of recent general results, it follows by [11, Cor. 2.2] that for large enough $|x|$ the ground state has a polynomial decay at infinity like

$$(1.3) \qquad \psi_1(x) \asymp \frac{1}{V(x)|x|^{d+\alpha}} = \frac{1}{|x|^6}.$$

Furthermore, by [11, Cor. 2.1] we have for the other eigenfunctions that

$$(1.4) \qquad |\psi_n(x)| \leqslant C_n \psi_1(x), \quad x \in \mathbb{R}, \ n \in \mathbb{N},$$

with a suitable constant C_n. (We note that such a pointwise ground state domination fails to hold in general for usual Schrödinger operators, for instance, in the case of $-\Delta + x^2$.) An implication of this is that the heat kernel $u(t, x, y) = e^{-tH}(x, y)$ asymptotically factorizes and

takes the shape of the ground state exponentially quickly in the sense that

$$\left| \frac{e^{\lambda_1 t} u(t,x,y)}{\psi_1(x)\psi_1(y)} - 1 \right| \leqslant C e^{-(\lambda_2 - \lambda_1)t},$$

for sufficiently large t and uniformly in the space variables, at a rate given by the spectral gap $\lambda_2 - \lambda_1$. In particular, we obtain that $u(t,x,y) \asymp \frac{e^{-\lambda_1 t}}{(1+x^6)(1+y^6)}$ for large enough $t > 0$ and all $x, y \in \mathbb{R}$.

Our goal here is to go beyond these results and derive explicit formulae on the spectrum and eigenfunctions of H. A previous work aiming to solve explicitly the eigenvalue problem for a specific fractional Schrödinger operator has been carried out in [18] where the operator $(- \mathrm{d}^2/\mathrm{d}x^2)^{1/2} + x^2$, i.e., the case of a quadratic potential has been considered. In this case the spectrum was found to be the alternating sequence given by the zeroes of the Airy function Ai and its derivative Ai'. Also, we have obtained closed formulae for the Fourier transform of the eigenfunctions in terms of Airy functions. These results allowed to upgrade the general estimates on the fall-off of eigenfunctions obtained in [11] to tighter bounds and a full asymptotic expansion, and derive detailed properties of the ground state, heat kernel, heat trace, and spectral gaps.

The features that emerge in the case of the quartic potential differ from the quadratic case on several counts and we observe a new phenomenon. In the quartic case we can express the Fourier transform of the eigenfunctions in terms of special functions derived from the fourth-order Airy function of the first kind, denoted below by Ai_4. Unlike in the quadratic case, where the Fourier transforms are expressed in a single term involving the Airy function, in the quartic case we have two. One is a highly oscillatory integral given in terms of Ai_4, while the second, which we will denote by $\widetilde{\mathrm{Ai}}_4(y)$, is comparable to $y^{-3/8} e^{-\frac{2\sqrt{2}}{5} y^{5/4}}$ for $y > 0$ and to $|y|^{-3/8} e^{\frac{4}{5}|y|^{5/4}}$ for $y < 0$. A combination of these functions in the expression of the Fourier transform of ψ_n produces a function dominated by Ai_4 even for low values of n, with a subtle contribution from $\widetilde{\mathrm{Ai}}_4$. The net effect is that the spectrum λ_n of H is located exponentially close with increasing n to the negative zeroes μ_n of Ai_4 and Ai_4' in an alternating order, and ψ_n is exponentially well approximated by the inverse Fourier transform of Ai_4 at argument shifted by λ_n.

We note that numerical and partially rigorous evidence supports our conjecture that this small effect persists for the higher order anharmonic non-local oscillators $(- \mathrm{d}^2/\mathrm{d}x^2)^{1/2} + x^{2k}$, $k = 3, 4, \ldots$ While even for the $k = 3$ case the expressions rapidly become more complex, the case formally obtained in the limit $k \to \infty$ can be put in a new light through this approach. Note that in this limit V becomes an infinitely deep potential well with boundaries at ± 1, and the problem becomes equivalent with the non-local Dirichlet problem for $(- \mathrm{d}^2/\mathrm{d}x^2)^{1/2}$ in the interval $(-1, 1)$, i.e., the Cauchy process killed outside this interval. It can be expected that studying the small spectral (non-uniform) shift observed in the present paper will lead to further understanding of the infinite well problem for which thus far only approximate solutions are around [16]. Our results in this direction will be further discussed elsewhere.

The remainder of this paper is organized as follows. In Section 2 we first state and then by Fourier transform reformulate the eigenvalue problem. The so obtained ODE can then be reduced to a fourth-order Airy equation with specific boundary conditions, and we use suitable modifications of Ai_4 to express the solutions of this problem (formulae (2.10)-(2.11)).

In Section 3 we identify the spectrum as the zeroes of higher transcendental functions derived from Ai_4 (Theorem 3.1) and find an expression of these functions in terms of generalized Fresnel integrals (Theorem 3.2). Using asymptotic expansions of Ai_4, we derive formulae showing more explicitly the dependence of λ_n on n. Next we analyze the small spectral effect discussed above and establish exponential bounds on the differences between the spectrum and the zeroes of Ai_4 and Ai_4' (Theorem 3.5). We also obtain bounds on the full sequence of spectral gaps $\lambda_{n+1} - \lambda_n$. In a last subsection we derive the behaviour of the heat trace around the origin, and obtain a Weyl-type law on the asymptotic behaviour of the counting measure of the eigenvalues (Theorem 3.7, Corollary 3.8). In Section 4 we turn to discussing the eigenfunctions. Since fourth-order Airy functions appear to be little explored in the literature, we start by a brief presentation of their properties particularizing the results obtained in [4]. To keep the length of this paper reasonable while offering detail, we only discuss the material which will be strictly needed here and refer for more to the given source. Next we obtain expressions of the Fourier transforms of the eigenfunctions in terms of these special functions (Theorem 4.5), and also prove analyticity of the eigenfunctions (Theorem 4.7). In Theorem 4.8 we analyze the small effect on the level of eigenfunctions and show that the ψ_n differ by exponentially small errors from the inverse Fourier transform of the L^2-normalized Ai_4 at values shifted by λ_n. Proofs will be presented in Section 5.

2. Eigenvalue problem for the quartic oscillator

Let $H_0 = (-\,\mathrm{d}^2/\,\mathrm{d}x^2)^{1/2}$ be the square root of the Laplace operator in one dimension, defined by

$$(\mathcal{F}(-\,\mathrm{d}^2/\,\mathrm{d}x^2)^{1/2} f)(y) = |y|(\mathcal{F}f)(y)$$

with domain $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : |y|\mathcal{F}f \in L^2(\mathbb{R})\}$, where $\mathcal{F}$ denotes Fourier transform. H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ and $\mathrm{Spec}\, H_0 = \mathrm{Spec}_{\mathrm{ess}}\, H_0 = [0, \infty)$.

In this paper we consider the fractional Schrödinger operator formally written as $H = H_0 + x^4$. One way to define this operator as a self-adjoint operator with dense domain in $L^2(\mathbb{R})$ is by using the Feynman-Kac formula

$$(f, e^{-tH}g)_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(x)\mathbb{E}^x[e^{-\int_0^t X_s^4 ds}g(X_t)]dx, \quad t \geq 0,\ f, g \in L^2(\mathbb{R}),$$

where $(X_t)_{t \geq 0}$ is a one-dimensional Cauchy process, and the expectation at the right hand side is taken with respect to the measure of this process starting from $x \in \mathbb{R}^d$. The right hand side can be proven to be a strongly continuous semigroup, and thus it follows by the Hille-Yoshida theorem that H is self-adjoint and equals the form sum $H_0 \dot{+} x^4$. Since our framework here is other than using functional representation, we refer for further details to [6, Th. 4.8]. As the potential is positive and growing to infinity, we have as a consequence of Rellich's theorem [17, Th. 3.20] that H has compact resolvent, in particular, $\mathrm{Spec}\, H \subset [0, \infty)$ is discrete, consisting of isolated eigenvalues, each of finite multiplicity. Our concern in what follows is to determine and analyze this spectrum and the corresponding eigenfunctions.

Consider the eigenvalue equation

$$(2.1) \qquad \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} \psi_n + x^4 \psi_n = \lambda_n \psi_n, \quad \psi_n \in H^1(\mathbb{R}),\ n \in \mathbb{N}.$$

From relations (1.3)-(1.4) it directly follows that

$$(2.2) \qquad\qquad x^4 \psi_n \in L^1(\mathbb{R}), \quad n \in \mathbb{N}.$$

This implies, in particular, $\psi_n \in L^1(\mathbb{R})$ for every $n \in \mathbb{N}$. We write $\phi_n(y) := (\mathcal{F}\psi_n)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} \psi_n(x)\,\mathrm{d}x$, $y \in \mathbb{R}$, for the Fourier transform of ψ_n. Since

$$\left(\mathcal{F} \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} \mathcal{F}^{-1} \right)(y) = |y| \quad \text{and} \quad (\mathcal{F}x^4 \mathcal{F}^{-1})(y) = \frac{\mathrm{d}^4}{\mathrm{d}y^4},$$

equation (2.1) transforms into

$$(2.3) \qquad\qquad \frac{\mathrm{d}^4}{\mathrm{d}y^4} \phi_n(y) + (|y| - \lambda_n)\phi_n(y) = 0, \quad y \in \mathbb{R}.$$

By (2.2) we have

$$\left| \frac{\mathrm{d}^4}{\mathrm{d}y^4} \phi_n(y) \right| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} x^4 e^{-ixy} \psi_n(x)\,\mathrm{d}x \right| \leqslant \frac{1}{\sqrt{2\pi}} \|x^4 \psi_n\|_1 < \infty, \quad y \in \mathbb{R}.$$

Furthermore, by multiplying and dividing, and using the Schwarz inequality

$$\int_{-\infty}^{\infty} |\phi_n(y)|dy \leqslant \left(\int_{-\infty}^{\infty} \frac{dy}{1+y^2} \int_{-\infty}^{\infty} (1+y^2)|\phi_n(y)|^2 dy \right)^{1/2} = \sqrt{\pi} \|\psi_n\|_{H^1(\mathbb{R})}.$$

From these two observations we have that $\phi_n \in C^4(\mathbb{R}) \cap L^1(\mathbb{R})$, for all $n \in \mathbb{N}$.

Note, moreover, that if $\phi_n(y)$ is a solution of (2.3), then so is $\phi_n(-y)$. Thus it suffices to consider (2.3) only for $y > 0$, and construct odd and even solutions on the whole of $\mathbb{R}$. They respectively satisfy

$$(2.4) \qquad\qquad \phi_n(0+) = \phi_n''(0+) = 0, \quad n = 2, 4, 6, \ldots$$

and

$$(2.5) \qquad\qquad \phi_n'(0+) = \phi_n'''(0+) = 0, \quad n = 1, 3, 5, \ldots$$

Our goal translates then to studying the spectrum and the eigenfunctions of H given by (2.1) through the L^1 solutions of equation (2.3) satisfying the boundary conditions (2.4)-(2.5).

Let $y > 0$. By the change of variable $y - \lambda_n \mapsto y$ and using the notation $\varphi_n(y) \equiv \phi_n(y + \lambda_n)$, we obtain

$$(2.6) \qquad\qquad \frac{\mathrm{d}^4}{\mathrm{d}y^4} \varphi_n(y) + y\varphi_n(y) = 0.$$

Equation (2.6) can be thought of being a higher order Airy equation. Following the terminology in [13] we call the particular solution

$$(2.7) \qquad\qquad \mathrm{Ai}_4(y) := \frac{1}{\pi} \int_0^{\infty} \cos \left(\frac{t^5}{5} + yt \right) \mathrm{d}t$$

a *fourth-order Airy function*. These functions seem to be relatively little known in the literature and we will use Section 4.1 to presenting some of their basic properties used in this paper, see [4] for details and a more general study.

Notice that equation (2.6) is invariant under the rotations $y \mapsto ye^{2l\pi i/5}$, $l \in \mathbb{Z}$, thus we have four other solutions by rotation of the argument of $\mathrm{Ai}_4(y)$, i.e.,

$$(2.8) \qquad \mathrm{Ai}_4(ye^{-2\pi i/5}), \quad \mathrm{Ai}_4(ye^{-4\pi i/5}), \quad \mathrm{Ai}_4(ye^{4\pi i/5}), \quad \mathrm{Ai}_4(ye^{2\pi i/5}).$$

Any four of the above five solutions can be shown to have a non-vanishing Wronskian. Consider

$$A_1(y) := e^{-2\pi i/5} \operatorname{Ai}_4(ye^{-2\pi i/5}), \quad A_2(y) := e^{-4\pi i/5} \operatorname{Ai}_4(ye^{-4\pi i/5})$$
$$A_3(y) := e^{4\pi i/5} \operatorname{Ai}_4(ye^{4\pi i/5}), \quad A_4(y) := e^{2\pi i/5} \operatorname{Ai}_4(ye^{2\pi i/5}).$$

Cauchy's theorem on contour integration then gives $\operatorname{Ai}_4(y) = -\sum_{r=1}^4 A_r(y)$. By taking $\operatorname{Ai}_4(y)$ and suitable linear combinations of $A_r(y)$ for $r = 1, 2, 3, 4$, a complete set of independent solutions to the differential equation (2.6) is given by $\widetilde{\operatorname{Ai}}_4(y) = i \sum_{r=1}^4 (-1)^{r+1} A_r(y)$ with integral representation

$$(2.9) \qquad \widetilde{\operatorname{Ai}}_4(y) = \frac{1}{\pi} \int_0^\infty \left[e^{-yt - \frac{t^5}{5}} - \sin\left(\frac{t^5}{5} + yt \right) \right] \mathrm{d}t,$$

and $G_3(y) = A_1(y) + A_4(y) - (A_2(y) + A_3(y))$, $G_4(y) = i\,(A_4(y) - A_1(y)) + i\,(A_3(y) - A_2(y))$. Since G_3, G_4 diverge as $y \to \infty$, these solutions are ruled out by the L^1 condition. Hence finally we obtain that the even and odd solutions of equation (2.3) satisfying (2.4)–(2.5) are

$$(2.10) \qquad \phi_{2j-1}(y) = c_{1,2j-1} \operatorname{Ai}_4(|y| - \lambda_{2j-1}) + c_{2,2j-1}\widetilde{\operatorname{Ai}}_4(|y| - \lambda_{2j-1})$$

$$(2.11) \qquad \phi_{2j}(y) = c_{1,2j}\operatorname{sgn}(y) \operatorname{Ai}_4(|y| - \lambda_{2j}) + c_{2,2j}\operatorname{sgn}(y)\widetilde{\operatorname{Ai}}_4(|y| - \lambda_{2j}),$$

for all $y \in \mathbb{R}$, $j \in \mathbb{N}$, with constant prefactors to be determined below. The above calculations are simple and we leave the details to the reader; alternatively see [4].

3. The spectrum of H

3.1. **Identification of the spectrum.** The boundary conditions (2.4)–(2.5) applied to (2.10) and (2.11) yield for $n = 2, 4, 6, \ldots$ and $n = 1, 3, 5, \ldots$, respectively, that

$$(3.1) \quad \det \begin{pmatrix} \operatorname{Ai}(-\lambda_n) & \widetilde{\operatorname{Ai}}_4(-\lambda_n) \\ \operatorname{Ai}''(-\lambda_n) & \widetilde{\operatorname{Ai}}_4''(-\lambda_n) \end{pmatrix} = 0 \quad \text{and} \quad \det \begin{pmatrix} \operatorname{Ai}'(-\lambda_n) & \widetilde{\operatorname{Ai}}_4'(-\lambda_n) \\ \operatorname{Ai}'''(-\lambda_n) & \widetilde{\operatorname{Ai}}_4'''(-\lambda_n) \end{pmatrix} = 0.$$

Define

$$(3.2) \quad \Phi_1(y) := \operatorname{Ai}(y)\widetilde{\operatorname{Ai}}_4''(y) - \operatorname{Ai}''(y)\widetilde{\operatorname{Ai}}_4(y) \quad \text{and} \quad \Phi_2(y) := \operatorname{Ai}'(y)\widetilde{\operatorname{Ai}}_4'''(y) - \operatorname{Ai}'''(y)\widetilde{\operatorname{Ai}}_4'(y).$$

Then we have the following main result of this section.

Theorem 3.1.

$$(3.3) \qquad \operatorname{Spec} H = \{\lambda_n,\, n \in \mathbb{N} : \lambda_{2j-1} = -a_{2,j} \text{ and } \lambda_{2j} = -a_{1,j},\, j \in \mathbb{N}\},$$

where $a_{1,j}$ and $a_{2,j}$ are the negative real zeroes of the higher transcendental functions Φ_1 and Φ_2, respectively, arranged in increasing order.

The following result allows to identify the eigenvalues as the zeroes of more familiar special functions. Recall the generalised Fresnel sine and cosine integrals $\operatorname{si}(a, z)$ and $\operatorname{ci}(a, z)$ (see also (5.5) below). The proof of this and the forthcoming results will be presented in Section 5.

Proposition 3.2. *For all $\lambda > 0$, we have the expressions*

$$(3.4) \qquad \Phi_1(-\lambda) = -\frac{1}{2\pi^2} \int_0^\infty \left[h_1(v) \cos\left(\frac{v^5}{20} + \lambda v \right) - h_2(v) \sin\left(\frac{v^5}{20} + \lambda v \right) \right] \mathrm{d}v$$

and

$$(3.5) \qquad \Phi_2(-\lambda) = -\frac{1}{2\pi^2} \int_0^\infty \left[h_3(v) \cos\left(\frac{v^5}{20} + \lambda v\right) - h_4(v) \sin\left(\frac{v^5}{20} + \lambda v\right) \right] dv,$$

where

$$h_1(v) := v^{1/2} \mathrm{si}\left(\frac{1}{2}, \frac{v^5}{16}\right), \quad h_2(v) := v^{1/2} \mathrm{ci}\left(\frac{1}{2}, \frac{v^5}{16}\right)$$

$$h_3(v) := 2\sin\frac{v^5}{16} - v^{5/2} \mathrm{ci}\left(\frac{1}{2}, \frac{v^5}{16}\right), \quad h_4(v) := 2\cos\frac{v^5}{16} + v^{5/2} \mathrm{si}\left(\frac{1}{2}, \frac{v^5}{16}\right).$$

3.2. **Approximations of the spectrum.** To have a more specific idea of the dependence of the eigenvalues on n, we can use Proposition 3.2 to derive asymptotic relations for $\Phi_1(-\lambda)$ and $\Phi_2(-\lambda)$.

Corollary 3.3. *There exists $0 < \lambda_0 \leqslant 1/2$ such that for every $\lambda > \lambda_0$, we have*

$$(3.6) \qquad \Phi_1(-\lambda) = \frac{1}{\pi} \lambda^{-1/4} e^{\frac{4}{5}\lambda^{5/4}} \sin\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) + O\left(\frac{e^{\frac{4}{5}\lambda^{5/4}}}{\lambda^{7/8}}\right)$$

and

$$(3.7) \qquad \Phi_2(-\lambda) = \frac{1}{\pi} \lambda^{1/4} e^{\frac{4}{5}\lambda^{5/4}} \cos\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) + O\left(\frac{e^{\frac{4}{5}\lambda^{5/4}}}{\lambda^{3/8}}\right).$$

Solving the equations $\Phi_1(-\lambda_n) = 0$ for $n = 2j$ and $\Phi_2(-\lambda_n) = 0$ for $n = 2j-1$, $j \in \mathbb{N}$, the above asymptotic relations give some more explicit expressions for the eigenvalues.

Corollary 3.4. *For each $n \in \mathbb{N}$,*

$$(3.8) \qquad \lambda_n = \left(\frac{5(2n-1)\pi}{16}\right)^{4/5} \left[1 + O\left(\frac{1}{n^{3/2}}\right)\right].$$

The next result describes the small spectral effect discussed in the Introduction, i.e., that the eigenvalues λ_n differ from the zeroes of Ai_4 and Ai_4' by exponentially small terms. Define

$$(3.9) \qquad \mu_n := \begin{cases} -\alpha_j' & n = 2j-1 \\ -\alpha_j & n = 2j \end{cases}$$

for $j \in \mathbb{N}$, where α_j and α_j' are the negative real zeroes of Ai_4 and Ai_4', respectively, see (4.4)-(4.6) below.

Theorem 3.5. *For every $n \in \mathbb{N}$ we have that*

$$(3.10) \qquad |\lambda_n - \mu_n| \leqslant \frac{2^{5/4}\sqrt{\Gamma\left(\frac{4}{5}\right)}}{5^{1/10}\sqrt{\pi}} \gamma_n$$

where $\gamma_n := \sqrt{c_{1,n} c_{2,n}}$ and $c_{1,n}, c_{2,n}$ are the normalization constants given in Theorem 4.5 below. Moreover,

$$\gamma_n \asymp \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} e^{-\frac{2}{5}\lambda_n^{5/4}} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} e^{-\frac{2}{5}n},$$

for all $n \in \mathbb{N}$.

We note that throughout this paper the standard Landau notations are used, and $f(x) \sim g(x)$ means $f(x) = g(x)[1 + o(1)]$, while $f(x) \asymp g(x)$ means that there exist real numbers $C_2 \geqslant C_1 > 0$ such that $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$.

This gives the following bounds on the sequence $\lambda_{n+1} - \lambda_n$ of spectral gaps.

Corollary 3.6. *For every $n \in \mathbb{N}$ we have that*

$$(3.11) \qquad \frac{\pi}{2}\left(\frac{8}{15\pi}\right)^{1/5}\left(n - \frac{1}{2}\right)^{-1/5} \leqslant \lambda_{n+1} - \lambda_n \leqslant \frac{\pi}{2}\left(\frac{8}{5\pi}\right)^{1/5}\left(n - \frac{1}{2}\right)^{-1/5}.$$

3.3. **Heat trace and Weyl-type theorem.** Using the asymptotic expression in Corollary 3.4, we are able to derive an expression for the behaviour of the trace of the semigroup

$$Z(t) := \operatorname{Tr} e^{-tH} = \sum_{n=1}^{\infty} e^{-\lambda_n t}, \quad t > 0,$$

in a neighbourhood of the origin.

Theorem 3.7. *We have*

$$(3.12) \qquad \lim_{t \downarrow 0} t^{5/4} Z(t) = \frac{2\Gamma\left(\frac{5}{4}\right)}{\pi}.$$

A consequence of this is the following Weyl-type asymptotic formula on the distribution of eigenvalues. Denote by

$$N(\lambda) := |\{n \in \mathbb{N} : \lambda_n \leq \lambda\}|$$

the counting measure of the number of eigenvalues λ_n of H up to level $\lambda > 0$.

Corollary 3.8. *We have that*

$$(3.13) \qquad \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{5/4}} = \frac{8}{5\pi}.$$

4. EXPRESSIONS AND PROPERTIES OF EIGENFUNCTIONS

4.1. **Fourth-order Airy function.** The fourth-order Airy function of the first kind Ai_4 is a particular case of the more general class Ai_k, $k = 2, 4, 6, \ldots$, formally defined by the integrals

$$\mathrm{Ai}_k(y) = \frac{1}{2\pi i}\int_{\mathcal{G}} e^{-\gamma_k y s - \frac{s^{k+1}}{k+1}}\, \mathrm{d}s, \qquad y \in \mathbb{R}, \ \ \gamma_k := (-1)^{k/2},$$

where $\mathcal{G}$ is any infinite contour in the complex s-plane that starts at infinity in the sector $-\frac{k\pi}{k+1} - \delta < \arg(s) < \delta - \frac{k\pi}{k+1}$ and ends at infinity in the sector $\frac{k\pi}{k+1} - \delta < \arg(s) < \frac{k\pi}{k+1} + \delta$ for $0 \leqslant \delta < \frac{(k-1)\pi}{2(k+1)}$, cutting through the negative real semi-axis. The integral is convergent since $\Re e(s^{k+1}) \to \infty$ as $|s| \to \infty$ within $\left|\arg(s) \pm \frac{k\pi}{k+1}\right| < \delta$. It is also possible to continuously deform the contour $\mathcal{G}$ to align with the imaginary axis without altering the value of the integral, and hence by the change of variable $s = it$ and for $k = 4$, we obtain (2.7). The function Ai_k is smooth and bounded, and extends to the complex plane as an entire function, and hence it follows directly that Ai_k satisfies the higher order Airy differential equation

$$\frac{\mathrm{d}^k}{\mathrm{d}y^k}\varphi(y) + \gamma_k y \varphi(y) = 0.$$

Since proofs involve lengthy calculations, for more details we refer to [4]; here we present some properties of Ai_4 taken from here for the specific case $k = 4$ which are of particular importance in this paper.

Proposition 4.1. *The following expansion holds:*

$$(4.1) \qquad \mathrm{Ai}_4(y) = \sum_{p=0}^{3} \frac{\mathrm{Ai}_4^{(p)}(0)}{p!} y^p \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \prod_{\substack{j=0 \\ j \neq p}}^{3} \frac{\Gamma\left(\frac{p-j}{5} + 1\right)}{\Gamma\left(\frac{p-j}{5} + l + 1\right)} \frac{y^{5l}}{5^{4l}}, \qquad y \in \mathbb{R}.$$

Moreover, for the derivatives we have

$$(4.2) \qquad \mathrm{Ai}_4^{(p)}(0) = \frac{\cos\left(\frac{\pi}{2}\frac{p+1}{5} + \frac{p\pi}{2}\right)}{5^{\frac{4-p}{5}} \Gamma\left(\frac{4-p}{5}\right) \sin\left(\frac{p+1}{5}\pi\right)}, \qquad \text{for each } p = 0, 1, 2, 3.$$

Formula (4.1) can alternatively be expressed as the sum

$$(4.3) \qquad \mathrm{Ai}_4(y) = \sum_{p=0}^{3} \mathrm{Ai}_4^{(p)}(0) \frac{y^p}{p!} \mathrm{U}_p(y),$$

in terms of the generalized hypergeometric functions $_0F_3(\alpha, \beta, \gamma; z)$, where we denote $\mathrm{U}_0(y) :=$ $_0F_3\left(\frac{2}{5}, \frac{3}{5}, \frac{4}{5}; -\frac{y^5}{5^4}\right)$, $\mathrm{U}_1(y) := {}_0F_3\left(\frac{3}{5}, \frac{4}{5}, \frac{6}{5}; -\frac{y^5}{5^4}\right)$, $\mathrm{U}_2(y) := {}_0F_3\left(\frac{4}{5}, \frac{6}{5}, \frac{7}{5}; -\frac{y^5}{5^4}\right)$, and $\mathrm{U}_3(y) :=$ $_0F_3\left(\frac{6}{5}, \frac{7}{5}, \frac{8}{5}; -\frac{y^5}{5^4}\right)$.

Proposition 4.2. *There exist constants $C_1, C_2 > 0$ such that*

$$|\mathrm{Ai}_4(y)| \leqslant \min\left\{C_1 e^{-2y}, C_2 |y|^{-\frac{5}{8}}\right\}, \qquad y \in \mathbb{R}.$$

Moreover, we have that $\mathrm{Ai}_4\rceil_{\mathbb{R}^+} \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$.

The next result gives the behaviour of Ai_4 and Ai_4' on the negative semi-axis from which the asymptotic expansions of the negative real zeroes of $\mathrm{Ai}_4(y)$ and $\mathrm{Ai}_4'(y)$ are derived.

Proposition 4.3. *We have that*

$$\mathrm{Ai}_4(-y) = \frac{1}{\pi^{1/2} y^{3/8}} \left(\cos\left(\xi - \frac{\pi}{4}\right) P(\xi) + \sin\left(\xi - \frac{\pi}{4}\right) Q(\xi)\right)$$

and

$$\mathrm{Ai}_4'(-y) = \frac{1}{\pi^{1/2} y^{1/8}} \left(\sin\left(\xi - \frac{\pi}{4}\right) R(\xi) - \cos\left(\xi - \frac{\pi}{4}\right) S(\xi)\right),$$

where

$$P(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{C_{2r}}{\xi^{2r}} \quad \text{and} \quad Q(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{C_{2r+1}}{\xi^{2r+1}} \quad \text{as } \xi \to +\infty$$

and

$$R(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{C_{2r}'}{\xi^{2r}} \quad \text{and} \quad S(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{C_{2r+1}'}{\xi^{2r+1}} \quad \text{as } \xi \to +\infty,$$

with $\xi = \frac{4}{5} y^{5/4}$, and coefficients C_l, C_l' expressed in terms of the Bell polynomials.

Finally, we give some asymptotic expansions of the negative real zeroes of Ai_4 and Ai_4' denoted by α_n and α_n', respectively.

Proposition 4.4. *For every $n \in \mathbb{N}$ we have*

$$(4.4) \qquad \alpha_n = -f\left(\frac{5(4n-1)\pi}{16}\right) \quad and \quad \alpha_n' = -g\left(\frac{5(4n-3)\pi}{16}\right),$$

where

$$(4.5) \qquad f(\tau) \sim \tau^{4/5}\left(1 + \frac{0.1586783204}{\tau^2} - \frac{0.03595263992}{\tau^4} + \frac{0.01511323043}{\tau^6} - \cdots\right),$$

$$(4.6) \qquad g(\tau) \sim \tau^{4/5}\left(1 - \frac{0.05289277344}{\tau^2} - \frac{0.003797437462}{\tau^4} - \frac{0.0005042991615}{\tau^6} - \cdots\right),$$

as $\tau \to \infty$.

4.2. **Eigenfunctions.** The Fourier transforms of the eigenfunctions of H can be expressed in terms of fourth-order extended Airy functions. Define

$$(4.7) \qquad \Lambda_1(\xi) := \widetilde{\mathrm{Ai}}_4(\xi)\,\mathrm{Ai}_4'(\xi) - \mathrm{Ai}_4(\xi)\widetilde{\mathrm{Ai}}_4'(\xi)$$

$$(4.8) \qquad \Lambda_2(\xi) := \widetilde{\mathrm{Ai}}_4'(\xi)\,\mathrm{Ai}_4''(\xi) - \mathrm{Ai}_4'(\xi)\widetilde{\mathrm{Ai}}_4''(\xi)$$

$$(4.9) \qquad \Lambda_3(\xi) := \widetilde{\mathrm{Ai}}_4''(\xi)\,\mathrm{Ai}_4'''(\xi) - \mathrm{Ai}_4''(\xi)\widetilde{\mathrm{Ai}}_4'''(\xi).$$

Theorem 4.5. *The Fourier transform of the L^2-normalized eigenfunctions of H is given by*

$$(4.10) \qquad (\mathcal{F}\psi_n)(y) = \begin{cases} c_{1,n}\,\mathrm{Ai}_4(|y|-\lambda_n) + c_{2,n}\widetilde{\mathrm{Ai}}_4(|y|-\lambda_n), & n = 2j-1 \\[2mm] c_{1,n}\mathrm{sgn}(y)\,\mathrm{Ai}_4(|y|-\lambda_n) + c_{2,n}\mathrm{sgn}(y)\widetilde{\mathrm{Ai}}_4(|y|-\lambda_n), & n = 2j \end{cases}$$

for all $y \in \mathbb{R}$ and $j \in \mathbb{N}$, where

$$(4.11) \qquad c_{1,n} := \begin{cases} -\dfrac{1}{\sqrt{2}}\dfrac{\widetilde{\mathrm{Ai}}_4'(-\lambda_n)}{\sqrt{\lambda_n\Lambda_1^2(-\lambda_n)+\Lambda_2^2(-\lambda_n)}} & n = 2j-1 \\[4mm] \dfrac{1}{2}\dfrac{\widetilde{\mathrm{Ai}}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} & n = 2j \end{cases}$$

and

$$(4.12) \qquad c_{2,n} := \begin{cases} \dfrac{1}{\sqrt{2}}\dfrac{\mathrm{Ai}_4'(-\lambda_n)}{\sqrt{\lambda_n\Lambda_1^2(-\lambda_n)+\Lambda_2^2(-\lambda_n)}} & n = 2j-1 \\[4mm] -\dfrac{1}{2}\dfrac{\mathrm{Ai}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} & n = 2j \end{cases}$$

We now proceed with further properties of the eigenfunctions ψ_n. First we obtain a full asymptotic expansion.

Theorem 4.6. *For every $j \in \mathbb{N}$ and $N = 2, 3, \dots$*

$$(4.13) \qquad \psi_{2j-1}(x) = \sum_{l=1}^{N-1}(-1)^{l+1}\frac{\mathcal{P}(\lambda_{2j-1})}{x^{4+2l}} + O\left(\frac{1}{x^{2N+4}}\right)$$

$$(4.14) \qquad \psi_{2j}(x) = \sum_{l=1}^{N-1}(-1)^{l+1}\frac{\mathcal{Q}(\lambda_{2j})}{x^{5+2l}} + O\left(\frac{1}{x^{2N+5}}\right)$$

holds as $x \to \infty$, where we have that $\mathcal{P}(-\lambda_1) = \sqrt{\frac{2}{\pi}} \left(c_{1,1} \operatorname{Ai}_4(-\lambda_1) + c_{2,1} \widetilde{\operatorname{Ai}}_4(-\lambda_1) \right)$, $\mathcal{Q}(-\lambda_2) = 2\sqrt{\frac{2}{\pi}} \left(c_{1,2} \operatorname{Ai'}_4(-\lambda_2) + c_{2,2} \widetilde{\operatorname{Ai}}_4'(-\lambda_2) \right)$, *and for all* $j = 2, 3, \dots$

$$\mathcal{P}(\lambda_{2j-1}) = \sqrt{\frac{2}{\pi}} \left(c_{1,2j-1} \operatorname{Ai}_4^{(3+2l)}(-\lambda_{2j-1}) + c_{2,2j-1} \widetilde{\operatorname{Ai}}_4^{(3+2l)}(-\lambda_{2j-1}) \right)$$

$$\mathcal{Q}(\lambda_{2j}) = \sqrt{\frac{2}{\pi}} \left(c_{1,2j-1} \operatorname{Ai}_4^{(4+2l)}(-\lambda_{2j}) + c_{2,2j} \widetilde{\operatorname{Ai}}_4^{(4+2l)}(-\lambda_{2j}) \right).$$

From general results on the smoothing property of the evolution semigroup e^{-tH} we know that the eigenfunctions are bounded and continuous functions [11]. Using the extra information for the specific case here, we obtain stronger regularity properties.

Theorem 4.7. *For every $n \in \mathbb{N}$ the eigenfunction ψ_n is analytic on $\mathbb{R}$ and has the expansion*

$$(4.15) \qquad \psi_n(x) = \begin{cases} \sum_{r=0}^{\infty} (-1)^r a_{2r}(\lambda_n) x^{2r}, & n = 2j-1 \\[2ex] \sum_{r=0}^{\infty} (-1)^r a_{2r+1}(\lambda_n) x^{2r+1}, & n = 2j \end{cases}$$

for all $j \in \mathbb{N}$ and $x \in \mathbb{R}$, where

$$a_p(u) = \frac{1}{p!} \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^p \left(c_{1,n} \operatorname{Ai}_4(y-u) + c_{2,n} \widetilde{\operatorname{Ai}}_4(y-u) \right) \mathrm{d}y, \quad p \in \mathbb{N}.$$

The following result shows another aspect of the small effect discussed before on the level of the spectrum. Its occurrence on the level of eigenfunctions consists in the fact that the eigenfunctions ψ_n are exponentially well approximated by the inverse Fourier transform of the dominating term in (2.10)-(2.11). Write $\chi_n(y) := c_{1,n} \operatorname{Ai}_4(|y| - \mu_n)$, for which we have $\|\chi_n\|_2 = 1$ for all $n \in \mathbb{N}$. Using that $|\psi_n - \mathcal{F}^{-1} \chi_n| \leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\phi_n(y) - \chi_n(y)| \, \mathrm{d}y$, we derive a direct estimate on the sup-norm of the difference, and another using the local cancellations between successive eigenvalues.

Theorem 4.8. *For every $n \in \mathbb{N}$ we have*

$$(4.16) \qquad \left\| \psi_n - \mathcal{F}^{-1} \chi_n \right\|_{\infty} \leqslant \frac{2^{9/4}}{\sqrt{5\pi}} c_{2,n},$$

where $c_{2,n} \asymp \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} e^{-\frac{4}{5} \lambda_n^{5/4}} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} e^{-\frac{4}{5} n}$. *Moreover, for every $n, p \in \mathbb{N}$ it follows that*

$$(4.17) \qquad \left(\int_{-\infty}^{\lambda_1} + \sum_{p \geqslant 1} \int_{\lambda_p}^{\lambda_{p+1}} \right) \left| \phi_n(y) - \chi_n(y) \right| \mathrm{d}y \leqslant \frac{2^{3/4}}{\sqrt{5}} J(n)$$

where

$$J(n) = 3c_{2,n} + B_n + \frac{8C_n}{5^{5/4} 3^{1/4}} + \left(\frac{1}{e^{\frac{2\pi}{5^{5/4} 3^{1/4}}} - 1} + \frac{e^{-\frac{\pi}{15^{1/4} 2}}}{e^{\frac{\pi}{15^{1/4} 2}} - 1} \right) D_n - \frac{e^{\frac{\pi}{15^{1/4} 2}} E_n}{\left(e^{\frac{\pi}{15^{1/4} 2}} - 1 \right)}$$

with

$$B_n = e^{-\frac{2^{7/4}\pi n}{5^{5/4}3^{1/4}}} c_{2,n} \sim \sqrt{\frac{\pi}{2}}\, n^{-1/10} e^{-1.8739\,n} \quad and \quad C_n = n^{-1/4} c_{2,n} \sim \sqrt{\frac{\pi}{2}}\, n^{-7/20} e^{-\frac{4}{5}n}$$

$$D_n = e^{\frac{\pi n}{5^{5/4}3^{1/42}}} c_{2,n} \sim \sqrt{\frac{\pi}{2}}\, n^{-1/10} e^{-0.00183\,n} \quad and \quad E_n = e^{-\frac{2\pi n}{5^{5/4}3^{1/4}}} c_{2,n} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} e^{-1.4385\,n}.$$

From (4.17) it is seen that the ψ_n stay near $\mathcal{F}^{-1}\chi_n$ also piecewise, i.e., inside each interval given by the successive eigenvalues.

Finally, write $\chi_n(y) := C_n \operatorname{Ai}_4(|y| - \lambda_n)$, where $C_n := 1/\|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2$ for each $n \in \mathbb{N}$. Then we have

Theorem 4.9. *For all $n \in \mathbb{N}$*

$$\|\psi_n - \mathcal{F}^{-1}\chi_n\|_2 \leqslant \frac{2^{33/20}\, 3^{1/5}\, 5^{1/10}}{\pi^{13/10}} \sqrt{\Gamma\left(\frac{4}{5}\right)}\, \beta_n,$$

where $\beta_n := n^{1/5} C_n \sqrt{c_{2,n}/c_{1,n}} \asymp \frac{\pi}{2}\, n^{1/5}\lambda_n^{-1/8} e^{-\frac{2}{5}\lambda_n^{5/4}} \sim \frac{\pi}{2}\, n^{1/10} e^{-\frac{2}{5}n}.$

5. Proofs

5.1. **Proof of Proposition 3.2.** We only show the proof for (3.4), by a similar argument (3.5) also follows. Let $\lambda > 0$ and consider the split-up $\operatorname{Ai}_4(-\lambda)\widetilde{\operatorname{Ai}}_4''(-\lambda) = I(\lambda) + J(\lambda)$, where

$$I(\lambda) := \frac{1}{2\pi^2 i}\left(\int_{-i\infty}^{i\infty} e^{\lambda t - \frac{t^5}{5}}\, dt\right)\left(\int_0^\infty u^2 e^{\lambda u - \frac{u^5}{5}}\, du\right)$$

and

$$J(\lambda) := \frac{1}{\pi^2}\left(\int_0^\infty \cos\left(\frac{t^5}{5} - \lambda t\right) dt\right)\left(\int_0^\infty u^2 \sin\left(\frac{u^5}{5} - \lambda u\right) du\right).$$

Rewriting $I(\lambda)$ as a double integral and applying the change of variables $X = t+u$, $Y = t-u$, we readily obtain that $I(\lambda) = -\frac{1}{16\pi^2 i}\int_{-i\infty}^{i\infty} e^{\lambda X + \frac{X^5}{20}}\left(\int_0^\infty (X+Y)^2 e^{-\frac{X}{16}(Y^2+X^2)^2}\, dY\right) dX$. For the second integral a similar straightforward computation gives $J(\lambda) = 0$. Hence

$$(5.1) \qquad \operatorname{Ai}_4(-\lambda)\widetilde{\operatorname{Ai}}_4''(-\lambda) = -\frac{1}{16\pi^2 i}\int_{-i\infty}^{i\infty} e^{\lambda X + \frac{X^5}{20}}\left(\int_0^\infty (X+Y)^2 e^{-\frac{X}{16}(Y^2+X^2)^2}\, dY\right) dX.$$

Next consider similarly $\operatorname{Ai}_4''(-\lambda)\widetilde{\operatorname{Ai}}_4(-\lambda) = \tilde{I}(\lambda) + \tilde{J}(\lambda)$, where

$$\tilde{I}(\lambda) := \frac{1}{2\pi^2 i}\left(\int_{-i\infty}^{i\infty} t^2 e^{\lambda t - \frac{t^5}{5}}\, dt\right)\left(\int_0^\infty e^{\lambda u - \frac{u^5}{5}}\, du\right)$$

and

$$\tilde{J}(\lambda) := \frac{1}{\pi^2}\left(\int_0^\infty t^2 \cos\left(\frac{t^5}{5} - \lambda t\right) dt\right)\left(\int_0^\infty \sin\left(\frac{u^5}{5} - \lambda u\right) du\right).$$

We obtain $\tilde{I}(\lambda) = -\frac{1}{16\pi^2 i}\int_{-i\infty}^{i\infty} e^{\lambda X + \frac{X^5}{20}}\left(\int_0^\infty (X-Y)^2 e^{-\frac{X}{16}(Y^2+X^2)^2}\, dY\right) dX$ and $\tilde{J}(\lambda) = 0$, which give

$$(5.2) \qquad \operatorname{Ai}_4''(-\lambda)\widetilde{\operatorname{Ai}}_4(-\lambda) = -\frac{1}{16\pi^2 i}\int_{-i\infty}^{i\infty} e^{\lambda X + \frac{X^5}{20}}\left(\int_0^\infty (X-Y)^2 e^{-\frac{X}{16}(Y^2+X^2)^2}\, dY\right) dX.$$

A combination of (5.1)-(5.2) gives furthermore

$$(5.3) \qquad \Phi_1(-\lambda) = -\frac{1}{4\pi^2 i} \int_{-i\infty}^{i\infty} X b(X) e^{\lambda X + \frac{X^5}{20}} \, \mathrm{d}X$$

with $b(X) = \int_0^\infty Y e^{-\frac{X}{16}(Y^2 + X^2)^2} \, \mathrm{d}Y$. Computing the integral, we have

$$\int_0^\infty Y e^{-\frac{X}{16}(Y^2 + X^2)^2} \, \mathrm{d}Y = \frac{1}{2} \int_0^\infty e^{-\frac{X}{16}(Y + X^2)^2} \, \mathrm{d}Y = X^{-1/2} \Gamma\left(\frac{1}{2}, \frac{X^5}{16}\right).$$

Thus $\Phi_1(-\lambda) = -\frac{1}{4\pi^2 i} \int_{-i\infty}^{i\infty} X^{1/2} \Gamma\left(\frac{1}{2}, \frac{X^5}{16}\right) e^{\lambda X + \frac{X^5}{20}} \, \mathrm{d}X$, and hence

$$(5.4) \qquad \Phi_1(-\lambda) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} |X|^{1/2} \Gamma\left(\frac{1}{2}, \frac{iX^5}{16}\right) e^{i\left(\frac{X^5}{20} + \lambda X + \frac{\pi}{4}\mathrm{sgn}(X)\right)} \, \mathrm{d}X$$

$$= -\frac{1}{2\pi^2} \Re e \int_0^\infty X^{1/2} \Gamma\left(\frac{1}{2}, \frac{iX^5}{16}\right) e^{i\left(\frac{X^5}{20} + \lambda X + \frac{\pi}{4}\right)} \, \mathrm{d}X.$$

Finally, we use the identity $\Gamma\left(\frac{1}{2}, \frac{iX^5}{16}\right) = e^{\frac{\pi i}{4}} \left[\mathrm{ci}\left(\frac{1}{2}, \frac{X^5}{16}\right) - \mathrm{si}\left(\frac{1}{2}, \frac{X^5}{16}\right)\right]$, see [20], to complete the proof for $\Phi_1(-\lambda)$ in (3.4), where
$$(5.5)$$
$$\mathrm{si}(a, z) = \int_z^\infty t^{a-1} \sin t \, \mathrm{d}t \quad \text{and} \quad \mathrm{ci}(a, z) = \int_z^\infty t^{a-1} \cos t \, \mathrm{d}t \quad (\Re e(a) < 1 \text{ and } \Re e(z) > 0)$$

are the generalised Fresnel sine and cosine integral functions, respectively.

5.2. Proof of Corollary 3.3.
Using a steepest descent-argument we get

$$(5.6) \qquad \mathrm{Ai}_4(-\lambda) = \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} \sin\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) \left(1 + O(\lambda^{-5/8})\right)$$

and

$$(5.7) \qquad \widetilde{\mathrm{Ai}}_4(-\lambda) = \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} e^{\frac{4}{5}\lambda^{5/4}} \left(1 + O(\lambda^{-5/8})\right)$$

for every $\lambda > \lambda_0$ for some $0 < \lambda_0 \leqslant 1/2$. Since $\mathrm{Ai}_4(y)$ and $\widetilde{\mathrm{Ai}}_4(y)$ are analytic functions, (3.6) and (3.7) follow by differentiation in (5.6)-(5.7). Since these calculations are lengthy but do not use special methods, we leave the details to the reader; alternatively see [4].

5.3. Proof of Theorem 3.5.
Before proceeding with the proof of Theorem 3.5 we will need some auxiliary results. Denote $\mathcal{L} := \frac{\mathrm{d}^4}{\mathrm{d}y^4} + |y|$, so that $(\mathcal{L} - \lambda_n) \phi_n(y) = 0$.

Lemma 5.1. *For every $n \in \mathbb{N}$ we have*

$$(5.8) \qquad \|(\mathcal{L} - \mu_n) \mathrm{Ai}_4(|\cdot| - \lambda_n)\|_2^2 \leqslant \frac{4\sqrt{2}\,\Gamma\left(\frac{4}{5}\right)}{5^{1/5}\pi} \frac{c_{2,n}}{c_{1,n}}.$$

Proof. Since $\mu_n \geqslant \lambda_n$ for $n \in \mathbb{N}$, we have

$$(5.9) \qquad |\left(\mathcal{L} - \mu_n\right) \mathrm{Ai}_4(|y| - \lambda_n)| = \frac{1}{c_{1,n}} \left| \left(\mathcal{L} - \mu_n\right) \phi_n(y) - c_{2,n} \left(\mathcal{L} - \mu_n\right) \widetilde{\mathrm{Ai}}_4(|y| - \lambda_n) \right|$$

$$\leqslant \frac{1}{c_{1,n}} \left| \left(\mathcal{L} - \lambda_n\right) \phi_n(y) - c_{2,n} \left(\mathcal{L} - \mu_n\right) \widetilde{\mathrm{Ai}}_4(|y| - \lambda_n) \right|$$

$$\leqslant \frac{c_{2,n}}{c_{1,n}} \left| \left(\mathcal{L} - \lambda_n\right) \widetilde{\mathrm{Ai}}_4(|y| - \lambda_n) \right|.$$

Hence,

$$(5.10) \quad \int_{-\infty}^{\infty} |\left(\mathcal{L} - \mu_n\right) \mathrm{Ai}_4(|y| - \lambda_n)|^2 \, \mathrm{d}y \leqslant \frac{2 c_{2,n}}{c_{1,n}} \int_0^{\infty} \left| \left(\frac{\mathrm{d}^4}{\mathrm{d}y^4} + y - \lambda_n \right) \widetilde{\mathrm{Ai}}_4(y - \lambda_n) \right|^2 \, \mathrm{d}y.$$

Using that $\widetilde{\mathrm{Ai}}_4(z) \asymp \frac{1}{\pi} \int_0^{\infty} e^{-zt - \frac{t^5}{5}} \, \mathrm{d}t$ for every $z \in \mathbb{R}$, it follows by (5.10) that

$$(5.11) \quad \| \left(\mathcal{L} - \mu_n\right) \mathrm{Ai}_4(|\cdot| - \lambda_n)\|_2^2 \leqslant \frac{2 c_{2,n}}{\pi^2 c_{1,n}} \int_0^{\infty} \left| \int_0^{\infty} \left(t^4 + y - \lambda_n \right) e^{-(y - \lambda_n)t - \frac{t^5}{5}} \, \mathrm{d}t \right|^2 \mathrm{d}y$$

$$\leqslant \frac{4 c_{2,n}}{\pi^2 c_{1,n}} \int_0^{\infty} \left(\left| \int_0^{\infty} t^4 e^{-(y - \lambda_n)t - \frac{t^5}{5}} \, \mathrm{d}t \right|^2 + \left| \int_0^{\infty} (y - \lambda_n) e^{-(y - \lambda_n)t - \frac{t^5}{5}} \, \mathrm{d}t \right|^2 \right) \mathrm{d}y.$$

To estimate the inner integral in (5.11) we write $t = (\mu_n - y)^{1/4} u$ and $h(u) = \frac{u^5}{5} - u$, and apply a standard calculation to obtain

$$(5.12) \qquad \left| \int_0^{\infty} t^4 e^{-(y - \mu_n)t - \frac{t^5}{5}} \, \mathrm{d}t \right| \leqslant \sqrt{\frac{\pi}{2}} |y - \mu_n|^{5/8} e^{-\frac{2\sqrt{2}}{5} |y - \mu_n|^{5/4}}.$$

Similarly, it follows that

$$(5.13) \qquad \left| \int_0^{\infty} (y - \lambda_n) e^{-(y - \lambda_n)t - \frac{t^5}{5}} \, \mathrm{d}t \right| \leqslant \sqrt{\frac{\pi}{2}} |y - \lambda_n|^{5/8} e^{-\frac{2\sqrt{2}}{5} |y - \lambda_n|^{5/4}}.$$

A combination of (5.12)-(5.13) and an application of (5.11) gives that

$$(5.14) \qquad \| \left(\mathcal{L} - \mu_n\right) \mathrm{Ai}_4(|\cdot| - \mu_n)\|_2^2 \leqslant \frac{4}{\pi} \frac{c_{2,n}}{c_{1,n}} \int_0^{\infty} |y - \mu_n|^{5/4} e^{-\frac{4\sqrt{2}}{5} |y - \mu_n|^{5/4}} \, \mathrm{d}y$$

$$= \frac{4}{\pi} \frac{c_{2,n}}{c_{1,n}} [I_1(\mu_n) + I_2(\mu_n)].$$

Here

$$I_1(\lambda_n) := \int_0^{\lambda_n} (\lambda_n - y)^{5/4} e^{-\frac{4\sqrt{2}}{5}(\lambda_n - y)^{5/4}} \, \mathrm{d}y = \frac{1}{\sqrt{2}} \left[-\lambda_n e^{-\frac{4\sqrt{2}}{5} \lambda_n^{5/4}} + \int_0^{\lambda_n} e^{-\frac{4\sqrt{2}}{5}(\lambda_n - y)^{5/4}} \, \mathrm{d}y \right]$$

$$\leqslant \frac{1}{\sqrt{2}} \left[-\lambda_n e^{-\frac{4\sqrt{2}}{5} \lambda_n^{5/4}} + \int_0^{\infty} e^{-\frac{4\sqrt{2}}{5} z^{5/4}} \, \mathrm{d}z \right] = \frac{1}{\sqrt{2}} \left[-\lambda_n e^{-\frac{4\sqrt{2}}{5} \lambda_n^{5/4}} + \frac{1}{5^{1/5}} \Gamma\left(\frac{4}{5}\right) \right],$$

and similarly

$$I_2(\lambda_n) := \int_{\lambda_n}^{\infty} (y - \lambda_n)^{5/4} e^{-\frac{4\sqrt{2}}{5}(y - \lambda_n)^{5/4}} \, \mathrm{d}y = \frac{1}{\sqrt{2}} \int_{\lambda_n}^{\infty} e^{-\frac{4\sqrt{2}}{5}(y - \lambda_n)^{5/4}} \, \mathrm{d}y \leqslant \frac{1}{\sqrt{2}} \frac{1}{5^{1/5}} \Gamma\left(\frac{4}{5}\right).$$

Thus from (5.14) we obtain

$$\| \left(\mathcal{L} - \mu_n\right) \mathrm{Ai}_4(|\cdot| - \lambda_n)\|_2^2 \leqslant \frac{4}{\pi\sqrt{2}} \frac{c_{2,n}}{c_{1,n}} \left[\frac{2}{5^{1/5}} \Gamma\left(\frac{4}{5}\right) - \lambda_n e^{-\frac{4\sqrt{2}}{5} \lambda_n^{5/4}} \right].$$

Since the second term in the bracket is positive, the claim follows. □

Next we derive formulae for the normalization constants explicitly in terms of n.

Lemma 5.2. *For all $n \in \mathbb{N}$, we have that*

$$(5.15) \quad c_{1,n} \asymp \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} \sim \sqrt{\frac{\pi}{2}} \, n^{-1/10} \quad and \quad c_{2,n} \asymp \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} e^{-\frac{4}{5}\lambda_n^{5/4}} \sim \sqrt{\frac{\pi}{2}} \, n^{-1/10} e^{-\frac{4}{5}n}.$$

Proof. Recall (4.11)-(4.12) and (4.7)-(4.9) for $\lambda > 0$. From (5.6)-(5.7) we obtain

$$\mathrm{Ai}_4(-\lambda) = \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} \sin\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) \left(1 + O(\lambda^{-5/8})\right);$$

$$\widetilde{\mathrm{Ai}}_4(-\lambda) = \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} e^{\frac{4}{5}\lambda^{5/4}} \left(1 + O(\lambda^{-5/8})\right)$$

for every $\lambda > \lambda_0$ with some $0 < \lambda_0 \leqslant 1/2$. Since $\mathrm{Ai}_4(y)$ and $\widetilde{\mathrm{Ai}}_4(y)$ are analytic functions, by differentiating and some straightforward calculations we obtain

$$\Lambda_1(-\lambda) \asymp \frac{1}{2\pi} \left[\sin\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) - \cos\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right)\right] \lambda^{-1/2} e^{\frac{4}{5}\lambda^{5/4}}$$

$$\Lambda_2(-\lambda) \asymp \frac{1}{2\pi} \left[\sin\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) + \cos\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right)\right] e^{\frac{4}{5}\lambda^{5/4}}$$

$$\Lambda_3(-\lambda) \asymp \frac{1}{2\pi} \left[\cos\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) - \sin\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right)\right] \lambda^{1/2} e^{\frac{4}{5}\lambda^{5/4}}.$$

Using these relations, we have $\lambda \Lambda_1^2(-\lambda) + \Lambda_2^2(-\lambda) \asymp \frac{1}{2\pi^2} e^{\frac{8}{5}\lambda^{5/4}}$. In particular,

$$-\frac{1}{\sqrt{2}} \frac{\widetilde{\mathrm{Ai}}_4'(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \asymp \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} \quad \text{for every } n = 1, 3, 5, \ldots$$

Similarly, we have

$$\frac{1}{\sqrt{2}} \frac{\mathrm{Ai}_4'(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \asymp -\sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} \cos\left(\frac{4}{5}\lambda_n^{5/4} + \frac{\pi}{4}\right) e^{-\frac{4}{5}\lambda_n^{5/4}} \quad \text{for every } n = 1, 3, 5, \ldots$$

We also have that

$$\Lambda_2(-\lambda)\Lambda_3(-\lambda) \asymp \frac{1}{4\pi^2} \left[\sin^2\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right) - \cos^2\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right)\right] \lambda^{1/2} e^{\frac{8}{5}\lambda^{5/4}}$$

$$= \frac{1}{4\pi^2} \sin\left(\frac{8}{5}\lambda^{5/4}\right) \lambda^{1/2} e^{\frac{8}{5}\lambda^{5/4}}.$$

In particular, setting $\lambda = \left[\frac{5}{16}(2n-1)\pi\right]^{4/5}$, we obtain for all $n = 2, 4, 6, \ldots$, that

$$\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n) \asymp \frac{1}{4\pi^2} \lambda_n^{1/2} e^{\frac{8}{5}\lambda_n^{5/4}}$$

and thus

$$\frac{1}{2} \frac{\widetilde{\mathrm{Ai}}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} \asymp \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8}$$

and

$$-\frac{1}{2} \frac{\mathrm{Ai}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} \asymp \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} \sin\left(\frac{4}{5}\lambda_n^{5/4} + \frac{\pi}{4}\right) e^{-\frac{4}{5}\lambda_n^{5/4}}.$$

The fact that $\lambda_n \sim n^{4/5}$ proves the claim.   □

Let

$$C_n := 1/\|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2. \tag{5.16}$$

Using the identity $\left(z \operatorname{Ai}_4^2(z) + 2\operatorname{Ai}_4'(z)\operatorname{Ai}_4'''(z) - [\operatorname{Ai}_4''(z)]^2\right)' = \operatorname{Ai}_4^2(z)$, we have that

$$\|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2 = \left(\int_{-\infty}^{\infty} \operatorname{Ai}_4(|y| - \lambda_n)\,\mathrm{d}y\right)^{1/2} = \left(2\int_{-\lambda_n}^{\infty} \operatorname{Ai}_4(z)\,\mathrm{d}z\right)^{1/2}$$

$$= \sqrt{2}\left(\lambda_n \operatorname{Ai}_4^2(-\lambda_n) + [\operatorname{Ai}_4''(-\lambda_n)]^2 - 2\operatorname{Ai}_4'(-\lambda_n)\operatorname{Ai}_4'''(-\lambda_n)\right)^{1/2}. \tag{5.17}$$

Recall that $\mu_n = -\alpha_n'$ for $n = 1, 3, 5, \ldots$ and $\mu_n = -\alpha_n$ for $n = 2, 4, 6, \ldots$, where α_n and α_n' are the negative real zeroes of Ai_4 and Ai_4', respectively. Using (5.16)-(5.17), we have that

$$C_n \asymp \frac{1}{\sqrt{2}} \times \begin{cases} \left(\lambda_n \operatorname{Ai}_4^2(-\lambda_n) + [\operatorname{Ai}_4''(-\lambda_n)]^2\right)^{-1/2} & n = 2j-1 \\ \left([\operatorname{Ai}_4''(-\lambda_n)]^2 - 2\operatorname{Ai}_4'(-\lambda_n)\operatorname{Ai}_4'''(-\lambda_n)\right)^{-1/2} & n = 2j \end{cases} \tag{5.18}$$

for every $j \in \mathbb{N}$. We also recall from (5.6) that

$$\operatorname{Ai}_4(-\lambda) = \frac{1}{\sqrt{2\pi}}\lambda^{-3/8}\sin\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right)\left[1 + O\left(\lambda^{-5/8}\right)\right] \tag{5.19}$$

for every $\lambda > \lambda_0$ given that $0 < \lambda_0 \leqslant 1/2$. Using that Ai_4 is analytic, by differentiation in (5.19) we obtain

$$\lambda[\operatorname{Ai}_4(-\lambda)]^2 + [\operatorname{Ai}_4''(-\lambda)]^2 \asymp \frac{1}{\pi}\lambda^{1/4}\sin^2\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right)$$

and

$$[\operatorname{Ai}_4''(-\lambda)]^2 - 2\operatorname{Ai}_4'(-\lambda)\operatorname{Ai}_4'''(-\lambda) \asymp \frac{1}{2\pi}\lambda^{1/4}\left[1 + \cos^2\left(\frac{4}{5}\lambda^{5/4} + \frac{\pi}{4}\right)\right].$$

Setting again $\lambda = [5(2n-1)\pi/16]^{4/5}$ and using (5.18), we obtain

$$C_n \asymp \sqrt{\frac{\pi}{2}}\lambda_n^{-1/8} \sim \sqrt{\frac{\pi}{2}}\,n^{-1/10}, \quad n \in \mathbb{N}. \tag{5.20}$$

A comparison of (5.15) and (5.20) implies $c_{1,n} \equiv C_n := 1/\|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2$.

Proof of Theorem 3.5. Since $(\phi_l)_{l \in \mathbb{N}}$ forms a complete orthornormal set in $L^2(\mathbb{R})$, and $\operatorname{Ai}_4(|\cdot| - \lambda_n) \in L^2(\mathbb{R})$, we make the expansion $\operatorname{Ai}_4(|y| - \lambda_n) = \sum_{l \geqslant 1} b_l \phi_l(y)$, $n \in \mathbb{N}$, where $b_l := (\operatorname{Ai}_4(|\cdot| - \lambda_n), \phi_l)_{L^2}$, and the Parseval identity

$$\sum_{l \geqslant 1} b_l^2 = \|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2^2 \tag{5.21}$$

holds. We then have that

$$\|(\mathcal{L} - \mu_n)\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2^2 = \sum_{l \geqslant 1} b_l^2(\lambda_l - \mu_n)^2. \tag{5.22}$$

Let $\lambda_{q(n)}$ denote an eigenvalue closest to μ_n. It follows from (5.22) and an application of (5.21) that

$$\sum_{l \geqslant 1} b_l^2(\lambda_l - \mu_n)^2 \geqslant (\lambda_{q(n)} - \mu_n)^2 \sum_{l \geqslant 1} b_l^2 = (\lambda_{q(n)} - \mu_n)^2\|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2^2. \tag{5.23}$$

Using (5.22)-(5.23) and (5.8), we obtain

$$|\lambda_{q(n)} - \mu_n| \leqslant 2\sqrt{\frac{\sqrt{2}\,\Gamma\left(\frac{4}{5}\right)}{\pi 5^{1/5}}} \frac{\sqrt{c_{2,n}/c_{1,n}}}{\|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2}.$$

Since by the above $\|\operatorname{Ai}_4(|\cdot| - \lambda_n)\|_2^2 = 2\sum_{r=0}^{2}(-1)^r \binom{2}{r} \operatorname{Ai}_4^{(r)}(-\lambda_n) \operatorname{Ai}_4^{(4-r)}(-\lambda_n) \equiv 1/c_{1,n}^2$, we have

$$|\lambda_{q(n)} - \mu_n| \leqslant \frac{2^{5/4}}{5^{1/10}\sqrt{\pi}} \sqrt{\Gamma\left(\frac{4}{5}\right)} \gamma_n, \tag{5.24}$$

with $\gamma_n := \sqrt{c_{1,n}\,c_{2,n}}$, and using the results of Lemma 5.2, we have $\gamma_n \asymp \sqrt{\frac{\pi}{2}}\,\lambda_n^{-1/8} e^{-\frac{2}{5}\lambda_n^{5/4}} \sim \sqrt{\frac{\pi}{2}}\,n^{-1/10}e^{-\frac{2}{5}n}$. Take $\delta = \frac{5\pi}{4}\left(\frac{\pi}{17}\right)^{5/4} \approx 0.475813$ and let $n_0 = \lceil 2\ln\left(C/\delta\right)\rceil + 1$, where $C = \frac{2^{5/4}}{5^{1/10}}\sqrt{\frac{\Gamma(4/5)}{\pi}}\sup_{n\in\mathbb{N}}\gamma_n n^{1/10}e^{2n/5} \approx 0.667134$. We claim that

$$\lambda_{q(n)} \in \left(\left(\mu_n^{5/4} - \delta\right)^{4/5}, \left(\mu_n^{5/4} + \delta\right)^{4/5}\right) \quad \text{for } n \geqslant n_0. \tag{5.25}$$

By (5.24) we have $|\lambda_{q(n)} - \mu_n| \leqslant C\,n^{-1/10}e^{-2n/5} \leqslant \delta$ for $n \geqslant n_0$. On the other hand,

$$\frac{(2n-1)\pi}{4} + \max\left\{0, \frac{(n-3)\pi}{8}\right\} < \mu_n^{5/4} - \delta \leqslant \mu_n^{5/4} + \delta \leqslant \frac{5n\pi}{8} \quad \text{for } n \geqslant n_0. \tag{5.26}$$

Using the inequality $\sigma(\xi - \eta)\xi^{\sigma-1} \leqslant \xi^\sigma - \eta^\sigma \leqslant \sigma(\xi - \eta)\eta^{\sigma-1}$ for $0 < \sigma \leqslant 1$ and $\xi \geqslant \eta \geqslant 0$, see [5, (2.15.2)]), and setting

$$A = \frac{\pi}{C}\left(\frac{\pi}{17}\right)^{5/4}\left(\frac{8}{5\pi}\right)^{1/5}\inf_{n\geqslant n_0} n^{-1/10}e^{2n/5} \geqslant 1,$$

we have that

$$\left|\left(\mu_n^{5/4} \pm \delta\right)^{4/5} - \mu_n\right| > \frac{4\delta}{5}\frac{1}{(\mu_n^{5/4} + \delta)^{1/5}} \geqslant \frac{4\delta}{5}\left(\frac{8}{5\pi}\right)^{1/5}n^{-1/5} = \pi\left(\frac{\pi}{17}\right)^{5/4}\left(\frac{8}{5\pi}\right)^{1/5}n^{-1/5}$$

$$\geqslant A\,C\,n^{-1/10}e^{-2n/5} \geqslant A|\lambda_{q(n)} - \mu_n| \quad \text{for } n \geqslant n_0.$$

This shows (5.25). Next by showing that $\mu_{n+1}^{5/4} - \mu_n^{5/4} > 2\delta$, we conclude that the intervals $\left((\mu_n^{5/4} - \delta)^{4/5}, (\mu_n^{5/4} + \delta)^{4/5}\right)$ are mutually disjoint, and hence $\lambda_{q(n)}$ for all $n \geqslant n_0$ are distinct. Using now the inequality $\sigma(\xi - \eta)\xi^{\sigma-1} \geqslant \xi^\sigma - \eta^\sigma \geqslant \sigma(\xi - \eta)\eta^{\sigma-1}$ for $\sigma \geqslant 1$ and $\xi \geqslant \eta \geqslant 0$, we have

$$\mu_{n+1}^{5/4} - \mu_n^{5/4} > \frac{5}{4}(\mu_{n+1} - \mu_n)\,\mu_n^{1/4}. \tag{5.27}$$

Take $B = \inf_{n\geqslant n_0}(n + 1/2)^{-1/5}\mu_n^{1/4} \geqslant 1$ and note that (see (5.34) below)

$$\mu_{n+1} - \mu_n \geqslant \frac{\pi}{2}\left(\frac{8}{15\pi}\right)^{1/5}\left(n + \frac{1}{2}\right)^{-1/5}, \quad n \in \mathbb{N}. \tag{5.28}$$

A combination of (5.28) and (5.27) gives

$$\mu_{n+1}^{5/4} - \mu_n^{5/4} > \frac{5\pi}{8}\left(\frac{8}{15\pi}\right)^{1/5}\left(n + \frac{1}{2}\right)^{-1/5}\mu_n^{1/4} \geqslant \frac{5\pi}{8}\left(\frac{8}{15\pi}\right)^{1/5}B > 2\delta, \quad \text{for all } n \geqslant n_0.$$

Furthermore, since $\lambda_{q(n)}$ is the closest eigenvalue to μ_n so that $\lambda_n \leqslant \lambda_{q(n)} \leqslant \mu_n$ for $n \geqslant n_0$, and given that $\lambda_{q(n)}^{5/4} < \mu_n^{5/4} + \delta$ for $n \geqslant n_0$, we conclude that $\lambda_n < \left(\mu_n^{5/4} + \delta\right)^{4/5}$ for $n \geqslant n_0$. Also, we have that

$$\frac{5n\pi}{8} < \lambda_{n+1}^{5/4} \leqslant \frac{5(n+1)\pi}{8}, \quad n \in \mathbb{N},$$

which together with (5.26) implies that $\left(\mu_n^{5/4} + \delta\right)^{4/5} < \lambda_{n+1}$ for $n \geqslant n_0$. Thus

$$(5.29) \qquad \lambda_n < \left(\mu_n^{5/4} + \delta\right)^{4/5} < \lambda_{n+1} \quad \text{for } n \geqslant n_0.$$

To complete, we make use of an idea in [15, Th. 1]. We claim that there are at most $n_0 - 1$ eigenvalues not included in the above class. We use from Section 5.5 below that $\sum_{n \geqslant 1} e^{-\lambda_n t} = \int_0^\infty e^{-\lambda t} \, dN(\lambda)$, $t > 0$, where $N(\lambda)$ is the spectral counting function associated with H. Let

$$L := \{l \in \mathbb{N} : l \neq q(n) \text{ for all } n \geqslant n_0\}.$$

Note that L is the set of all mismatches $l \neq q(n)$ for $n \geqslant n_0$, and hence its complement in $\mathbb{N}$ is exactly the set for which $q(n)$ matches $l \in \mathbb{N}$ for $n \geqslant n_0$. Using below $\lambda_{q(n)} < (\mu_n + \delta)^{4/5}$ and the monotonicity of $e^{-\lambda t}$, we have

$$\sum_{l \in L} e^{-\lambda_l t} = \sum_{l \geqslant 1} e^{-\lambda_l t} - \sum_{n \geqslant n_0} e^{-\lambda_{q(n)} t} \leqslant \int_0^\infty e^{-\lambda t} \, dN(\lambda) - \sum_{n \geqslant n_0} e^{-\left(\mu_n^{5/4} + \delta\right)^{4/5} t}.$$

Since $\int_{\left(\mu_{n_0}^{5/4} + \delta\right)^{4/5}}^\infty e^{-\lambda t} \, dN(\lambda) \leqslant \sum_{n \geqslant n_0} e^{-\left(\mu_n^{5/4} + \delta\right)^{4/5} t}$, we get

$$(5.30) \qquad \sum_{l \in L} e^{-\lambda_l t} \leq e^{-\left(\mu_{n_0}^{5/4} + \delta\right)^{4/5} t} N\left((\mu_{n_0}^{5/4} + \delta)^{4/5}\right) + t \int_0^{\left(\mu_{n_0}^{5/4} + \delta\right)^{4/5}} e^{-\lambda t} N(\lambda) \, d\lambda.$$

As $N(\lambda)$ is increasing on $(0, (\mu_{n_0}^{5/4} + \delta)^{4/5})$, we have

$$(5.31) \qquad \int_0^{\left(\mu_{n_0}^{5/4} + \delta\right)^{4/5}} e^{-\lambda t} N(\lambda) \, d\lambda \leqslant \frac{N\left((\mu_{n_0}^{5/4} + \delta)^{4/5}\right)}{t} \left(1 - e^{-\left(\mu_{n_0}^{5/4} + \delta\right)^{4/5} t}\right).$$

Inserting (5.31) into (5.30), we obtain $\sum_{l \in L} e^{-\lambda_l t} < N\left((\mu_{n_0}^{5/4} + \delta)^{4/5}\right)$. As $t \searrow 0$, the left-hand side converges to $|L|$, and it follows that $|L| < N\left((\mu_{n_0}^{5/4} + \delta)^{4/5}\right)$. Using (5.29), it follows that the right hand side above equals n for $n \geqslant n_0$. Therefore, we have that $|L| < n_0$, which proves the claim.

Given that $\lambda_n \leqslant (5n\pi/8)^{4/5}$ for all $n \in \mathbb{N}$, we have in particular for $l < n_0$ that $\lambda_l \leqslant (5l\pi/8)^{4/5} \leqslant (5(n_0 - 1)\pi/8)^{4/5}$. On the other hand, for $n \geqslant n_0$, we have that $\lambda_{q(n)} > \left(\mu_{n_0}^{5/4} - \delta\right)^{4/5} > (5(n-1)\pi/8)^{4/5} \geqslant (5(n_0 - 1)\pi/8)^{4/5}$. Thus, $L \supseteq \{1, 2, \ldots, n_0 - 1\}$. However, $|L| \leqslant n_0 - 1$, which implies that $L \subseteq \{1, 2, \ldots, n_0 - 1\}$. Hence they are actually equal, and therefore we conclude that $q(n) = n$ for $n \geqslant n_0$.

By direct calculation, we have that $n_0 = 2$ implying that λ_1 is the only eigenvalue excluded from the above set of eigenvalues. However, $\lambda_1 \leqslant (5\pi/16)^{4/5}$ and since $\mu_1 \geqslant (\pi/5)^{4/5}$, we obtain

$$\lambda_1 - \mu_1 \leqslant \left(\frac{5\pi}{16}\right)^{4/5} - \left(\frac{\pi}{5}\right)^{4/5} < \frac{9\pi}{100}\left(\frac{5}{\pi}\right)^{1/5} \approx 0.310282 < \frac{2^{5/4}}{5^{1/10}}\sqrt{\frac{\Gamma(4/5)}{\pi}}\gamma_1 \approx 0.447194.$$

Hence we conclude that the result (3.10) holds for all $n \geqslant 1$. $\qquad\square$

Proof of Theorem 3.6. To derive the upper bound recall from (3.8) that

$$(5.32) \qquad \lambda_{n+1} - \lambda_n \asymp \left(\frac{5(2n+1)\pi}{16}\right)^{4/5} - \left(\frac{5(2n-1)\pi}{16}\right)^{4/5}.$$

Applying the same inequality to (5.32) as in the previous proof, we obtain

$$\left(\frac{5(2n+1)\pi}{16}\right)^{4/5} - \left(\frac{5(2n-1)\pi}{16}\right)^{4/5} \leqslant \frac{\pi}{2}\left(\frac{8}{5\pi}\right)^{1/5}\left(n - \frac{1}{2}\right)^{-1/5}.$$

For the lower bound recall that $\mu_{2j-1} = -\alpha'_{4,j} \leqslant \left(\frac{5(4j-3)\pi}{16}\right)^{4/5}$, $\mu_{2j} = -\alpha_{4,j} \geqslant \left(\frac{5(4j-1)\pi}{16}\right)^{4/5}$, for each $j \in \mathbb{N}$. The used inequality gives then again

$$(5.33) \quad \mu_{2j} - \mu_{2j-1} \geqslant \left(\frac{5(4j-1)\pi}{16}\right)^{4/5} - \left(\frac{5(4j-3)\pi}{16}\right)^{4/5} \geqslant \frac{\pi}{2}\left(\frac{4}{5\pi}\right)^{1/5}\left(j - \frac{1}{4}\right)^{-1/5}.$$

Setting $n = 2j - 1$ in (5.33), we obtain

$$\mu_{n+1} - \mu_n \geqslant \frac{\pi}{2}\left(\frac{8}{5\pi}\right)^{1/5}\left(n + \frac{1}{2}\right)^{-1/5},$$

and $n = 2j$ gives

$$\mu_n - \mu_{n-1} \geqslant \frac{\pi}{2}\left(\frac{8}{5\pi}\right)^{1/5}\left(n - \frac{1}{2}\right)^{-1/5} \geqslant \frac{\pi}{2}\left(\frac{8}{5\pi}\right)^{1/5}\left(n + \frac{1}{2}\right)^{-1/5}.$$

Thus we conclude that

$$(5.34) \qquad \mu_{n+1} - \mu_n \geqslant \frac{\pi}{2}\left(\frac{8}{5\pi}\right)^{1/5}\left(n + \frac{1}{2}\right)^{-1/5} \qquad \text{for all } n \in \mathbb{N}.$$

Since $\lambda_n \asymp \mu_n$, using (5.34) we can optimize over $C > 0$ in

$$\frac{\pi}{2}\left(\frac{8}{5\pi}\right)^{1/5}\left(n + \frac{1}{2}\right)^{-1/5} \geqslant C\left(n - \frac{1}{2}\right)^{-1/5}.$$

Using $1 \leqslant \frac{n+1/2}{n-1/2} \leqslant 3$ for $n \geq 2$, gives $C = \frac{\pi}{2}\left(\frac{8}{15\pi}\right)^{1/5}$, and this completes the proof. $\qquad\square$

5.4. Proof of Theorem 3.7.
Consider the decomposition $\sum_{n=1}^{\infty} e^{-\lambda_n t} = F(t) + G(t)$, where

$$F(t) := \sum_{j=1}^{\infty} e^{-\lambda_{2j-1}t} \quad \text{and} \quad G(t) := \sum_{j=1}^{\infty} e^{-\lambda_{2j}t}.$$

For every $0 \leqslant t \leqslant 1$ it follows from (3.8) that there is a constant $0 < C < 1$ such that

$$e^{-Ct}\sum_{j=1}^{\infty} e^{-\left(\frac{5(4j-3)\pi}{16}\right)^{4/5}t} \leqslant F(t) \leqslant e^{Ct}\sum_{j=1}^{\infty} e^{-\left(\frac{5(4j-3)\pi}{16}\right)^{4/5}t}.$$

Passing from summation to integration gives

$$e^{-Ct} \int_1^\infty e^{-\left(\frac{5(4x-3)\pi}{16}\right)^{4/5} t} \, \mathrm{d}x \leqslant F(t) \leqslant e^{Ct} \int_1^\infty e^{-\left(\frac{5(4x-3)\pi}{16}\right)^{4/5} t} \, \mathrm{d}x.$$

By the change of variable $y = \left(\frac{5(4x-3)\pi}{16}\right)^{4/5} t$ it furthermore follows that

$$\int_1^\infty e^{-\left(\frac{5(4x-3)\pi}{16}\right)^{4/5} t} \, \mathrm{d}x = \frac{1}{\pi t^{5/4}} \int_{\left(\frac{5\pi}{16}\right)^{4/5} t}^\infty y^{\frac{1}{4}} e^{-y} \, \mathrm{d}y,$$

which implies that $\lim_{t \downarrow 0} t^{5/4} F(t) = \frac{1}{\pi} \Gamma\left(\frac{5}{4}\right)$. Similarly, we obtain $\lim_{t \downarrow 0} t^{5/4} G(t) = \frac{1}{\pi} \Gamma\left(\frac{5}{4}\right)$. By combining these results (3.12) follows.

5.5. **Proof of Corollary 3.8.** Let m_l count the multiplicity of λ_l for each $l \in \mathbb{N}$. With $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \leqslant R$, for every $R > 0$ and finite, we have

$$N(\lambda) = \begin{cases} m_1 + m_2 + \ldots + m_l & \lambda_l \leqslant \lambda < \lambda_{l+1}; \ l = 1, 2, \ldots, n \\ 0 & 0 < \lambda < \lambda_1. \end{cases}$$

Since eigenvalues come with multiplicities, $m_l = 1$ for all $l \in \mathbb{N}$, and we have

$$N(\lambda) = \sum_{l=1}^n l \mathbf{1}_{[\lambda_l, \lambda_{l+1})}(\lambda) \quad \text{for } 0 < \lambda \leqslant R,$$

with $N(\lambda) = 0$ for $\lambda \leqslant 0$. Note that $N(\lambda)$ is right-continuous at each $\lambda = \lambda_n$, $n \in \mathbb{N}$, and jumps by $d_n := N(\lambda_n^+) - N(\lambda_n^-) = 1$, $n \in \mathbb{N}$. Also, $N(\lambda)$ is of bounded variation in $(0, R)$, and the integral $\int_0^R e^{-\lambda t} \, \mathrm{d}N(\lambda)$, $t > 0$, exists. Furthermore, taking the limit as $R \to \infty$, we have that $\int_0^\infty e^{-\lambda t} \, \mathrm{d}N(\lambda) = \lim_{R \to \infty} \int_0^R e^{-\lambda t} \, \mathrm{d}N(\lambda)$. This limit exists for every $t \in \mathbb{R}^+$ since by integration-by-parts, $\lim_{R \to \infty} \int_0^R e^{-\lambda t} \, \mathrm{d}N(\lambda) = t \int_0^\infty e^{-\lambda t} N(\lambda) \, \mathrm{d}\lambda < \infty$. Since

$$t \int_0^\infty e^{-\lambda t} N(\lambda) \, \mathrm{d}\lambda = t \left(\int_0^{\lambda_1} + \sum_{n \geqslant 1} \int_{\lambda_n}^{\lambda_{n+1}} \right) e^{-\lambda t} N(\lambda) \, \mathrm{d}\lambda$$

$$= t \sum_{n \geqslant 1} n \int_{\lambda_n}^{\lambda_{n+1}} e^{-\lambda t} \, \mathrm{d}\lambda = \sum_{n \geqslant 1} n \left(e^{-\lambda_n t} - e^{-\lambda_{n+1} t} \right).$$

By telescopic summation we then obtain $\int_0^\infty e^{-\lambda t} \, \mathrm{d}N(\lambda) = \sum_{n \geqslant 1} e^{-\lambda_n t}$. By definition, $Z(t) = \sum_{n=1}^\infty e^{-\lambda_n t}$, and from Theorem 3.7 we have

$$(5.35) \qquad \int_0^\infty e^{-\lambda t} \, \mathrm{d}N(\lambda) = Z(t) \sim \frac{2\Gamma\left(\frac{5}{4}\right)}{\pi} t^{-5/4} \quad \text{as } t \downarrow 0.$$

A Karamata-Tauberian theorem [14, Th. 15.3] applied to (5.35) gives (3.13), which completes the proof.

5.6. **Proof of Theorem 4.5.** We are only required to derive expressions for $c_{1,n}$ and $c_{2,n}$. Take $\phi_n(y) = \varphi(y - \lambda_n)$ for $y > 0$ and recall from (2.6) that

$$(5.36) \qquad\qquad\qquad \frac{\mathrm{d}^4}{\mathrm{d}z^4} \varphi(z) = -z\varphi(z).$$

Consider

$$\int_{-\infty}^{\infty} \phi_n^2(y)\,\mathrm{d}y = 2\int_0^{\infty} \varphi^2(y-\lambda_n)\,\mathrm{d}y = 2\int_{-\lambda_n}^{\infty} \varphi^2(z)\,\mathrm{d}z.$$

Using (5.36), we have the identity

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z\varphi^2(z) + 2\varphi'(z)\varphi'''(z) - [\varphi''(z)]^2\right) = \varphi^2(z).$$

Thus we obtain

$$(5.37) \qquad \int_{-\infty}^{\infty} \phi_n^2(y)\,\mathrm{d}y = 2\left(\lambda_n\varphi^2(-\lambda_n) + [\varphi''(-\lambda_n)]^2 - 2\varphi'(-\lambda_n)\varphi'''(-\lambda_n)\right).$$

From the boundary conditions (2.4), we have for each $n = 1, 3, 5, \ldots$ that

$$(5.38) \qquad \varphi'(-\lambda_n) = \varphi'''(-\lambda_n) = 0$$

implying that

$$(5.39) \qquad c_2 = -c_1\frac{\mathrm{Ai}_4'(-\lambda_n)}{\widetilde{\mathrm{Ai}}_4'(-\lambda_n)} \equiv -c_1\frac{\mathrm{Ai}_4'''(-\lambda_n)}{\widetilde{\mathrm{Ai}}_4'''(-\lambda_n)}$$

Plancherel's theorem and a combination of (5.37)-(5.38) yield

$$(5.40) \quad 1 = \int_{-\infty}^{\infty} \psi_n^2(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \phi_n^2(y)\,\mathrm{d}y = 2\left(\lambda_n\varphi^2(-\lambda_n) + \varphi''(-\lambda_n)^2\right) \qquad n = 1, 3, 5, \ldots$$

With (5.39), we have that

$$\varphi(-\lambda_n) = c_1\,\mathrm{Ai}_4(-\lambda_n) + c_2\widetilde{\mathrm{Ai}}_4(-\lambda_n) = -\frac{c_1}{\widetilde{\mathrm{Ai}}_4'(-\lambda_n)}\Lambda_1(-\lambda_n)$$

$$\varphi''(-\lambda_n) = c_1\,\mathrm{Ai}_4''(-\lambda_n) + c_2\widetilde{\mathrm{Ai}}_4''(-\lambda_n) = \frac{c_1}{\widetilde{\mathrm{Ai}}_4'(-\lambda_n)}\Lambda_2(-\lambda_n).$$

From here we deduce

$$c_{1,n} \equiv c_1 = -\frac{1}{\sqrt{2}}\frac{\widetilde{\mathrm{Ai}}_4'(-\lambda_n)}{\sqrt{\lambda_n\Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}}$$

$$c_{2,n} \equiv c_2 = \frac{1}{\sqrt{2}}\frac{\mathrm{Ai}_4'(-\lambda_n)}{\sqrt{\lambda_n\Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}}$$

for each $n = 1, 3, 5, \ldots$. To obtain the results for $n = 2, 4, 6, \ldots$, we proceed similarly by using now the boundary conditions (2.5) giving

$$(5.41) \qquad \varphi(-\lambda_n) = \varphi''(-\lambda_n) = 0.$$

Proceedings as above, we finally obtain

$$c_{1,n} \equiv c_1 = \frac{1}{2}\frac{\widetilde{\mathrm{Ai}}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} \quad \text{and} \quad c_{2,n} \equiv c_2 = -\frac{1}{2}\frac{\mathrm{Ai}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}},$$

for each $n = 2, 4, 6, \ldots$

5.7. **Proof of Theorem 4.6.** We recall (4.10) for each $n = 1, 3, 5, \ldots$, and by taking inverse cosine transform, we have that

$$(5.42) \qquad \psi_{2j-1}(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xy)\phi_{2j-1}(y)\,\mathrm{d}y \quad \text{for each } j \in \mathbb{N},$$

where

$$\phi_{2j-1}(y) = c_{1,2j-1}\,\mathrm{Ai}_4(y - \lambda_{2j-1}) + c_{2,2j-1}\widetilde{\mathrm{Ai}}_4(y - \lambda_{2j-1}), \quad \text{for } y > 0.$$

On integrating by parts $(2N+4)$-times in (5.42), together with $\lim_{y\to\infty} \sin\left(xy + \frac{r\pi}{2}\right)\phi_{2j-1}^{(l)}(y) = 0$ for all $r, l \in \mathbb{N}_0$, we obtain

$$(5.43) \qquad \int_0^\infty \cos(xy)\phi_{2j-1}(y)\,\mathrm{d}y = \sum_{r=0}^N (-1)^{r+1}\frac{\phi_{2j-1}^{(2r+1)}(0)}{x^{2r+2}} + E_{1,N}(x).$$

Here

$$E_{1,N}(x) := \frac{(-1)^{N+1}}{x^{2N+2}}\int_0^\infty \cos(xy)\phi_{2j-1}^{(2N+2)}(y)\,\mathrm{d}y$$

$$= \frac{(-1)^N}{x^{2N+4}}\left(\phi_{2j-1}^{(2N+3)}(0) + \int_0^\infty \cos(xy)\phi_{2j-1}^{(2N+4)}(y)\,\mathrm{d}y\right),$$

for each $N = 2, 3, 4, \ldots$. For sufficiently large $M_1, M_2 > 0$, dependent only on N, we have that

$$|\phi_{2j-1}^{(2N+3)}(0)| \leqslant c_{1,2j-1}|\,\mathrm{Ai}_4^{(2N+3)}(-\lambda_{2j-1})| + c_{2,2j-1}|\widetilde{\mathrm{Ai}}_4^{(2N+3)}(-\lambda_{2j-1})| \leqslant M_1$$

and

$$\int_0^\infty |\phi_{2j-1}^{(2N+4)}(y)|\,\mathrm{d}y$$

$$\leqslant c_{1,2j-1}\int_0^\infty |\,\mathrm{Ai}_4^{(2N+4)}(y - \lambda_{2j-1})|\,\mathrm{d}y + c_{2,2j-1}\int_0^\infty |\widetilde{\mathrm{Ai}}_4^{(2N+4)}(y - \lambda_{2j-1})|\,\mathrm{d}y \leqslant M_2.$$

Hence $|E_{1,N}(x)| \leqslant \frac{C_{1,j,N}}{|x|^{2N+4}}$ with some $C_{1,j,N} > 0$. Using the conditions $\phi'_{\lambda_{2j-1}}(0) = \phi'''_{\lambda_{2j-1}}(0) = 0$, $j = 1, 2, \ldots$, and differentiating in (2.3), we have $\phi_{2j-1}^{(5)}(y) = -\phi_{2j-1}(y) - (y - \lambda_{2j-1})\phi'_{2j-1}(y)$, which reduces to $\phi_{2j-1}^{(5)}(0) = -\phi_{2j-1}(0)$ with $\phi_{2j-1}(0) > 0$ for all $j \in \mathbb{N}$. Then (4.13) follows from (5.43) combined with (5.42). The proof of (4.14) follows completely similarly.

5.8. **Proof of Theorem 4.7.** We only show the proof for $n = 2j - 1$ case, the result is obtained similarly for $n = 2j$. For each $n = 2j - 1$, $j \in \mathbb{N}$ we have that

$$(5.44) \qquad \psi_n(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xy)\left[c_{1,n}\,\mathrm{Ai}_4(y - \lambda_n) + c_{2,n}\widetilde{\mathrm{Ai}}_4(y - \lambda_n)\right]\mathrm{d}y$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sum_{r=0}^\infty (-1)^r\frac{(xy)^{2r}}{(2r)!}\left[c_{1,n}\,\mathrm{Ai}_4(y - \lambda_n) + c_{2,n}\widetilde{\mathrm{Ai}}_4(y - \lambda_n)\right]\mathrm{d}y.$$

Define

$$a_p(\lambda_n) := \frac{1}{p!}\sqrt{\frac{2}{\pi}} \int_0^\infty y^p\left[c_{1,n}\,\mathrm{Ai}_4(y - \lambda_n) + c_{2,n}\widetilde{\mathrm{Ai}}_4(y - \lambda_n)\right]\mathrm{d}y, \quad p = 2r, \ r \in \mathbb{N}.$$

We show that for each $n = 2j - 1$, $j \in \mathbb{N}$,

$$(5.45) \qquad \sum_{r=0}^{\infty} a_{2r}(\lambda_n)|x|^{2r} < \infty$$

for almost all $x \in \mathbb{R}$; then we can conclude using the monotone and dominated convergence theorems that the result follows from (5.44) for each $n = 2j - 1$, $j \in \mathbb{N}$, by interchanging summation with integration. Let $R := \lim_{r \to \infty} \left| \frac{a_{2r}(\lambda_n)}{a_{2r+2}(\lambda_n)} \right|$. Since $|\widetilde{\mathrm{Ai}}_4(y)| \leqslant |\mathrm{Ai}_4(y)|$ for $y > 0$, we have

$$\mathrm{Ai}_4(y) \asymp \frac{1}{2\pi^{1/2}} y^{-\frac{3}{8}} e^{-\frac{4}{5} y^{\frac{5}{4}}}, \quad y > 0,$$

which implies that there exists a constant $C_n > 0$ such that

$$|c_{1,n} \mathrm{Ai}_4(y - \lambda_n) + c_{2,n}\widetilde{\mathrm{Ai}}_4(y - \lambda_n)| < C_n e^{-\frac{4}{5} y^{\frac{5}{4}}}, \quad y > 0.$$

Thus we obtain

$$a_{2r}(\lambda_n) \leqslant \frac{C_n}{(2r)!} \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{2r} e^{-\frac{4}{5} y^{\frac{5}{4}}} \, \mathrm{d}y.$$

By the change of variable $z = \frac{4}{5} y^{5/4}$ we have

$$\tilde{a}_{2r} := \frac{1}{(2r)!} \sqrt{\frac{2}{\pi}} \left(\frac{5}{4} \right)^{\frac{8r-1}{5}} \int_0^{\infty} z^{\frac{8r-1}{5}} e^{-z} \, \mathrm{d}z = \frac{1}{(2r)!} \sqrt{\frac{2}{\pi}} \left(\frac{5}{4} \right)^{\frac{8r-1}{5}} \Gamma\left(\frac{4(2r+1)}{5} \right).$$

Applying Stirling's formula to $\Gamma(\alpha)$, we furthermore have

$$\frac{\tilde{a}_{2r}}{\tilde{a}_{2r+2}} = (2r+2)(2r+1) \left(\frac{4}{5} \right)^{\frac{8}{5}} \frac{\Gamma\left(\frac{4(2r+1)}{5} \right)}{\Gamma\left(\frac{4(2r+3)}{5} \right)}$$

$$\asymp (2r+2)(2r+1) \left(\frac{4}{5} \right)^{\frac{8}{5}} e^{\frac{8}{5}} \left(\frac{2r+1}{2r+3} \right)^{\frac{4(2r+1)}{5} - \frac{1}{2}} \left(\frac{5}{4(2r+3)} \right)^{\frac{8}{5}},$$

which implies that $R \to \infty$ as $r \to \infty$.

5.9. **Proof of Theorem 4.8.** With $\phi_n(y) = c_{1,n} \mathrm{Ai}_4(|y| - \lambda_n) + c_{2,n}\widetilde{\mathrm{Ai}}_4(|y| - \lambda_n)$ and $\chi_n(y) := c_{1,n} \mathrm{Ai}_4(|y| - \lambda_n)$, we have that

$$\left| \psi_n(x) - \left(\mathscr{F}^{-1}\chi_n \right)(x) \right| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{ixy} \left[\phi_n(y) - \chi_n(y) \right] \, \mathrm{d}y \right|$$

$$(5.46) \qquad = \frac{c_{2,n}}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{ixy} \widetilde{\mathrm{Ai}}_4(|y| - \lambda_n) \, \mathrm{d}y \right| \leqslant c_{2,n} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left| \widetilde{\mathrm{Ai}}_4(y - \lambda_n) \right| \, \mathrm{d}y.$$

By a similar procedure leading to (5.12), we can also show that

$$\left| \widetilde{\mathrm{Ai}}_4(y - \lambda_n) \right| \leqslant \frac{1}{\sqrt{2\pi}} |y - \lambda_n|^{-3/8} e^{-\frac{2\sqrt{2}}{5} |y - \lambda_n|^{5/4}}.$$

Using this estimate in (5.46), we have that

$$
\int_0^\infty \left| \widetilde{\mathrm{Ai}}_4(y - \lambda_n) \right| \mathrm{d}y \leqslant \frac{1}{\sqrt{2\pi}} \int_0^\infty |y - \lambda_n|^{-3/8} e^{-\frac{2\sqrt{2}}{5}|y-\lambda_n|^{5/4}} \, \mathrm{d}y
$$

$$
= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \left[\int_0^{\lambda_n - \epsilon} \frac{e^{-\frac{2\sqrt{2}}{5}(\lambda_n - y)^{5/4}}}{(\lambda_n - y)^{3/8}} \, \mathrm{d}y + \int_{\lambda_n + \epsilon}^\infty \frac{e^{-\frac{2\sqrt{2}}{5}(y - \lambda_n)^{5/4}}}{(y - \lambda_n)^{3/8}} \, \mathrm{d}y \right]
$$

$$
(5.47) \qquad \leqslant \frac{2}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \int_\epsilon^\infty z^{-3/8} e^{-\frac{2\sqrt{2}}{5} z^{5/4}} \, \mathrm{d}z = \frac{2^{7/4}}{\sqrt{5}}.
$$

The result (4.16) then follows by a combination of (5.46)-(5.47).

To prove (4.17), we proceed in a number of steps. Let $p \neq n - 1$ and $p \neq n$. We have that

$$
\int_{\lambda_p}^{\lambda_{p+1}} \left| \phi_n(y) - \chi_n(y) \right| \mathrm{d}y = c_{2,n} \int_{\lambda_p}^{\lambda_{p+1}} \left| \widetilde{\mathrm{Ai}}_4(y - \lambda_n) \right| \mathrm{d}y
$$

$$
\leqslant \frac{c_{2,n}}{\sqrt{2\pi}} \int_{\lambda_p}^{\lambda_{p+1}} |y - \lambda_n|^{-3/8} e^{-\frac{2\sqrt{2}}{5}|y-\lambda_n|^{5/4}} \, \mathrm{d}y
$$

$$
(5.48) \qquad = \frac{c_{2,n}}{\sqrt{2\pi}} \left[\int_0^{|\lambda_{p+1} - \lambda_n|} - \int_0^{|\lambda_p - \lambda_n|} \right] z^{-3/8} e^{-\frac{2\sqrt{2}}{5} z^{5/4}} \, \mathrm{d}z =: c_{2,n} I(p, n).
$$

The change of variables $u = \left(\frac{2\sqrt{2}}{5} \right)^{1/2} z^{5/8}$ gives

$$
I(p, n) = \frac{8}{5\sqrt{2\pi}} \left(\frac{5}{2\sqrt{2}} \right)^{1/2} \left[\int_0^{\left(\frac{2\sqrt{2}}{5} \right)^{1/2} |\lambda_{p+1} - \lambda_n|^{5/8}} - \int_0^{\left(\frac{2\sqrt{2}}{5} \right)^{1/2} |\lambda_p - \lambda_n|^{5/8}} \right] e^{-u^2} \, \mathrm{d}u.
$$

Using the inequality (see [3, (1)-(3)])

$$
(5.49) \qquad \frac{\sqrt{\pi}}{2} \left(1 - e^{-\frac{4}{\pi} \xi^2} \right)^{1/2} \leqslant \int_0^\xi e^{-u^2} \, \mathrm{d}u \leqslant \frac{\sqrt{\pi}}{2} \left(1 - e^{-\frac{4}{\pi} \xi^2} \right)^{1/2}, \qquad \xi \geqslant 0,
$$

we get

$$
I(p, n) \leqslant \frac{2^{3/4}}{\sqrt{5}} \left[\left(1 - e^{-\frac{8\sqrt{2}}{5\pi} |\lambda_{p+1} - \lambda_n|^{5/4}} \right)^{1/2} - \left(1 - e^{-\frac{2\sqrt{2}}{5} |\lambda_p - \lambda_n|^{5/4}} \right)^{1/2} \right]
$$

$$
(5.50) \qquad \leqslant \frac{2^{3/4}}{\sqrt{5}} \left(e^{-\frac{2\sqrt{2}}{5} |\lambda_p - \lambda_n|^{5/4}} - \frac{1}{2} e^{-\frac{8\sqrt{2}}{5\pi} |\lambda_p - \lambda_n|^{5/4}} \right) \leqslant \frac{2^{3/4}}{\sqrt{5}} e^{-\frac{2\sqrt{2}}{5} |\lambda_p - \lambda_n|^{5/4}}.
$$

Given $n \in \mathbb{N}$, for $p > n$ we have by (3.11) that

$$
\lambda_{n+q} - \lambda_{n+q-1} \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi} \right)^{1/5} (n + q - 1)^{-1/5}, \quad q \in \mathbb{N}
$$

so that

$$
\lambda_{n+q} - \lambda_n \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi} \right)^{1/5} q(n + q - 1)^{-1/5}.
$$

For $n \in \mathbb{N}$, write $p = n + q$ for every $q \in \mathbb{N}$ giving

$$
\lambda_p - \lambda_n \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi} \right)^{1/5} (p - n)(p - 1)^{-1/5}.
$$

Using $(p-n)^{5/4}(p-1)^{-1/4} \geqslant p(1-n/p)^{5/4} \geqslant p(1-\frac{5n}{4p}) = p - \frac{5}{4}n$, it follows by (5.50) that

$$(5.51) \qquad I(p,n) \leqslant \frac{2^{3/4}}{\sqrt{5}} e^{-\frac{2\sqrt{2}}{5}(\frac{\pi}{2})^{5/4}(\frac{8}{15\pi})^{1/4}(p-\frac{5}{4}n)} \quad \text{for each } p \geqslant n+1.$$

Next for $n \in \mathbb{N}$ and $p < n - 1$, again we have by (3.11) that

$$\lambda_{n+1-j} - \lambda_{n-j} \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi}\right)^{1/5} (n-j)^{-1/5}, \quad 1 \leqslant j \leqslant n-1$$

giving

$$\lambda_n - \lambda_{n-q} \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi}\right)^{1/5} q(n-1)^{-1/5}.$$

Setting $p = n - q$ for every $2 \leqslant q \leqslant n - 1$, we get

$$\lambda_n - \lambda_p \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi}\right)^{1/5} (n-p)(n-1)^{-1/5}.$$

Using $(n-p)^{5/4}(n-1)^{-1/4} \geqslant n(1-p/n)^{5/4} \geqslant n(1-\frac{5p}{4n}) = n - \frac{5}{4}p$, it follows from (5.50) that

$$(5.52) \qquad I(p,n) \leqslant \frac{2^{3/4}}{\sqrt{5}} e^{-\frac{2\sqrt{2}}{5}(\frac{\pi}{2})^{5/4}(\frac{8}{15\pi})^{1/4}(n-\frac{5}{4}p)}, \quad 1 \leqslant p \leqslant n-2.$$

When $p = n$, it directly follows from (5.48) that

$$\int_{\lambda_n}^{\lambda_{n+1}} \left|\phi_n(y) - \chi_n(y)\right| \mathrm{d}y = c_{2,n} \int_{\lambda_n}^{\lambda_{n+1}} \left|\widetilde{\mathrm{Ai}}_4(y - \lambda_n)\right| \mathrm{d}y$$

$$\leqslant \frac{c_{2,n}}{\sqrt{2\pi}} \int_{\lambda_n}^{\lambda_{n+1}} |y - \lambda_n|^{-3/8} e^{-\frac{2\sqrt{2}}{5}|y-\lambda_n|^{5/4}} \mathrm{d}y$$

$$= \frac{c_{2,n}}{\sqrt{2\pi}} \int_0^{\lambda_{n+1}-\lambda_n} z^{-3/8} e^{-\frac{2\sqrt{2}}{5}z^{5/4}} \mathrm{d}z =: c_{2,n} I(n).$$

By a similar change of variables $u = \left(\frac{2\sqrt{2}}{5}\right)^{1/2} z^{5/8}$ and a use of (5.49), we have

$$I(n) = \frac{2^{3/4}}{\sqrt{5}} \int_0^{\left(\frac{2\sqrt{2}}{5}\right)^{1/2}(\lambda_{n+1}-\lambda_n)^{5/8}} e^{-u^2} \mathrm{d}u$$

$$\leqslant \frac{2^{3/4}}{\sqrt{5}} \left(1 - e^{-\frac{8\sqrt{2}}{5\pi}(\lambda_{n+1}-\lambda_n)^{5/4}}\right)^{1/2} \leqslant \frac{2^{3/4}}{\sqrt{5}} \left(1 - \frac{1}{2} e^{-\frac{8\sqrt{2}}{5\pi}(\lambda_{n+1}-\lambda_n)^{5/4}}\right).$$

From (3.11) we immediately have that $\lambda_{n+1} - \lambda_n \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi}\right)^{1/5} n^{-1/5}$, and hence

$$(5.53) \qquad \int_{\lambda_n}^{\lambda_{n+1}} \left|\phi_n(y) - \chi_n(y)\right| \mathrm{d}y \leqslant \frac{2^{3/4}}{\sqrt{5}} \left(1 - \frac{1}{2} e^{-\frac{8}{5^{5/4}3^{1/4}}n^{-1/4}}\right) c_{2,n}, \quad n \in \mathbb{N}.$$

Similarly, if $p = n - 1$, we have

$$\int_{\lambda_{n-1}}^{\lambda_n} \left|\phi_{n-1}(y) - \chi_n(y)\right| \mathrm{d}y \leqslant \frac{c_{2,n}}{\sqrt{2\pi}} \int_0^{\lambda_n-\lambda_{n-1}} z^{-3/8} e^{-\frac{2\sqrt{2}}{5}z^{5/4}} \mathrm{d}z =: c_{2,n} I(n-1).$$

We then have that

$$I(n-1) = \frac{2^{3/4}}{\sqrt{5}} \int_0^{\left(\frac{2\sqrt{2}}{5}\right)^{1/2}(\lambda_n-\lambda_{n-1})^{5/8}} e^{-u^2}\, \mathrm{d}u \leqslant \frac{2^{3/4}}{\sqrt{5}}\left(1 - \frac{1}{2}e^{-\frac{8\sqrt{2}}{5\pi}(\lambda_n-\lambda_{n-1})^{5/4}}\right),$$

and hence

$$(5.54) \qquad \int_{\lambda_{n-1}}^{\lambda_n} \left|\phi_{n-1}(y) - \chi_n(y)\right| \mathrm{d}y \leqslant \frac{2^{3/4}}{\sqrt{5}}\left(1 - \frac{1}{2}e^{-\frac{8}{5^{5/4}3^{1/4}}n^{-1/4}}\right)c_{2,n}, \quad n \in \mathbb{N}.$$

Combining (5.48), (5.51)-(5.54), and using $1 - e^{-\varrho n^{-1/4}} \leqslant \varrho n^{-1/4}$ for $0 < \varrho \leqslant 1$ and $n \geqslant 1$, we obtain

(5.55)

$$\sum_{p\geqslant 1} \int_{\lambda_p}^{\lambda_{p+1}} \left|\phi_n(y) - \chi_n(y)\right| \mathrm{d}y$$

$$\leqslant \frac{2^{3/4}}{\sqrt{5}}\left(1 + \frac{8n^{-1/4}}{5^{5/4}3^{1/4}} + \left(\frac{1}{e^{\frac{2\pi}{5^{5/4}3^{1/4}}}-1} + \frac{e^{-\frac{\pi}{15^{1/4}2}}}{e^{\frac{\pi}{15^{1/4}2}}-1}\right)e^{\frac{\pi}{15^{1/4}2}n} - \frac{e^{\frac{\pi}{15^{1/4}2}}e^{\frac{2\pi}{5^{5/4}3^{1/4}}n}}{\left(e^{\frac{\pi}{15^{1/4}2}}-1\right)}\right)c_{2,n}.$$

Finally, consider

$$\int_{-\infty}^{\lambda_1} \left|\phi_n(y)-\chi_n(y)\right|\mathrm{d}y = c_{2,n}\int_{-\infty}^{\lambda_1}\left|\widetilde{\mathrm{Ai}}_4(|y|-\lambda_n)\right|\mathrm{d}y = c_{2,n}\left(\int_0^\infty + \int_0^{\lambda_1}\right)\left|\widetilde{\mathrm{Ai}}_4(y-\lambda_n)\right|\mathrm{d}y$$

$$\leqslant \frac{c_{2,n}}{\sqrt{2\pi}}\left(\int_0^\infty + \int_0^{\lambda_1}\right)|y-\lambda_n|^{-3/8}e^{-\frac{2\sqrt{2}}{5}|y-\lambda_n|^{5/4}}\,\mathrm{d}y$$

$$= \frac{c_{2,n}}{\sqrt{2\pi}}\left(\int_0^\infty + \int_0^{\lambda_n} + \int_{\lambda_n-\lambda_1}^{\lambda_n}\right)z^{-3/8}e^{-\frac{2\sqrt{2}}{5}z^{5/4}}\,\mathrm{d}z$$

$$\leqslant \frac{2^{3/4}}{\sqrt{5}}\left[1 + 2\left(1 - e^{-\frac{2^{7/2}}{5\pi}\lambda_n^{5/4}}\right)^{1/2} - \left(1 - e^{-\frac{2^{3/2}}{5}(\lambda_n-\lambda_1)^{5/4}}\right)^{1/2}\right]c_{2,n}$$

$$\leqslant \frac{2^{3/4}}{\sqrt{5}}\left[2 + e^{-\frac{2^{3/2}}{5}(\lambda_n-\lambda_1)^{5/4}} - e^{-\frac{2^{7/2}}{5\pi}\lambda_n^{5/4}}\right]c_{2,n}$$

$$(5.56) \qquad \leqslant \frac{2^{3/4}}{\sqrt{5}}\left(2 + e^{-\frac{2^{3/2}}{5}(\lambda_n-\lambda_1)^{5/4}}\right)c_{2,n}.$$

Since $\lambda_n - \lambda_1 \geqslant \frac{\pi}{2}\left(\frac{8}{15\pi}\right)^{1/5}n(n-1)^{-1/5} \geqslant \frac{\pi}{2}\left(\frac{8}{15\pi}\right)^{1/5}n^{4/5}$, it follows from (5.56) that

$$(5.57) \qquad \int_{-\infty}^{\lambda_1}\left|\phi_n(y)-\chi_n(y)\right|\mathrm{d}y \leqslant \frac{2^{3/4}}{\sqrt{5}}\left(2 + e^{-\frac{2\pi}{5^{5/4}3^{1/4}}n}\right)c_{2,n}.$$

Hence (4.17) follows by a combination of (5.55) and (5.57), which completes the proof.

5.10. **Proof of Theorem 4.9.** Since $\chi_n \in L^2(\mathbb{R})$, we have

$$(5.58) \qquad\qquad\qquad\qquad \chi_n(y) = \sum_{l\geqslant 1} a_l \phi_l(y)$$

with $a_l := \langle \chi_n, \phi_l \rangle$ and $\sum_{l \geqslant 1} a_l^2 = \|\varphi_n\|_2^2 = 1$. We now consider

$$(5.59) \qquad \|(\mathcal{L} - \lambda_n)\chi_n\|_2^2 = \sum_{l \geqslant 1} a_l^2 (\lambda_l - \lambda_n)^2 = \sum_{l \neq n} a_l^2 (\lambda_l - \lambda_n)^2.$$

For $l \neq n$, we have

$$|\lambda_l - \lambda_n| \geqslant \lambda_{n+1} - \lambda_n \geqslant \frac{\pi}{2} \left(\frac{8}{15\pi} \right)^{1/5} n^{-1/5}.$$

Therefore, from (5.59), we simply have that

$$(5.60) \qquad \sum_{l \neq n} a_l^2 \leqslant \left(\frac{2}{\pi} \right)^2 \left(\frac{15\pi}{8} \right)^{2/5} n^{2/5} \|(\mathcal{L} - \lambda_n)\chi_n\|_2^2.$$

Furthermore, from (5.58), we have $\chi_n(y) - a_n \phi_n(y) = \sum_{l \neq n} a_l \phi_l(y)$ so that using (5.60), we have

$$\|\chi_n - a_n \phi_n\|_2^2 = \sum_{l \neq n} a_l^2 \leqslant \frac{4}{\pi^2} \left(\frac{15\pi}{8} \right)^{2/5} n^{2/5} \|(\mathcal{L} - \lambda_n)\chi_n\|_2^2$$

$$(5.61) \qquad = \frac{4}{\pi^2} \left(\frac{15\pi}{8} \right)^{2/5} C_n^2 n^{2/5} \|(\mathcal{L} - \lambda_n) \mathrm{Ai}_4(\cdot - \lambda_n)\|_2^2.$$

By similar procedure adopted in the proof of (5.8), we can also easily show that

$$\|(\mathcal{L} - \lambda_n) \mathrm{Ai}_4(\cdot - \lambda_n)\|_2^2 \leqslant \frac{4\sqrt{2}\, \Gamma \left(\frac{4}{5} \right)}{\pi 5^{1/5}} \frac{c_{2,n}}{c_{1,n}}.$$

Hence, it follows that

$$\|\chi_n - a_n \phi_n\|_2 \leqslant \frac{2}{\pi} \left(\frac{15\pi}{8} \right)^{1/5} \sqrt{\frac{4\sqrt{2}\, \Gamma \left(\frac{4}{5} \right)}{\pi 5^{1/5}}} \, n^{1/5} C_n \sqrt{c_{2,n}/c_{1,n}}.$$

Finally, we observe that $\sum_{n \geqslant 1} a_n^2 = 1$ implying that $a_n \leqslant 1$ for each $n \in \mathbb{N}$. Therefore, we have that $\|\chi_n - \phi_n\|_2 \leqslant \|\chi_n - a_n \phi_n\|_2$, and hence, we complete the proof by recalling Plancherel's theorem.

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