Further results on "Reduced order disturbance observer for discrete-time linear systems" ⋆

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Abstract

Reduced order Disturbance OBservers (DOB) have been proposed in Kim et al (2010) and Kim and Rew (2013) for continuous-time and discrete-time linear systems, respectively. The existence condition of the promising algorithm has been established but is not straightforward to check. This work further improves the reduced order DOB design by formulating it as a functional observer design problem. By carefully designing the state functional matrix, a generic DOB is resulted with an easily-checked necessary and sufficient existence condition and an easily-adjusted convergence rate. It is also shown that both the reduced order DOB in Kim and Rew (2013) and the full order DOB in Chang (2006) are special cases of this new DOB.

Key words: Disturbance OBserver (DOB), Functional observer, Reduced-order observer, Unified observer

1 Introduction

Motivated by the prosperous applications in disturbance rejection control and fault diagnosis (see, Wei and Guo (2010); Wei et al (2016); Su and Chen (2017); Yang et al (2018)), a discrete-time Disturbance OBserver (DOB) was proposed in Chang (2006). To relax observer existence conditions and reduce the observer order, a reduced order DOB was designed in Kim et al (2010) to reconstruct disturbances/faults with a minimal observer order for continuous-time linear systems. Recently, this innovative work was further extended to the discrete-time case in Kim and Rew (2013), where the systems under consideration are

$$\begin{cases} x_{k+1} = \Phi x_k + \Gamma u_k + G d_k \\ y_k = C x_k \end{cases}, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $d_k \in \mathbb{R}^q$ and $y_k \in \mathbb{R}^l$ are the states, inputs, disturbances and measurements at k^{th} step. Disturbance distribution matrix G has a full column-rank (i.e., $rank(G) = q$); otherwise, the redundant inputs can be removed (see, Su et al (2015)). The disturbances d_k are assumed to be unknown but slowly time-varying, i.e., the following assumption is assumed

$$d^i_{k+1} = d^i_k + \Delta d^i_{k+1}, \tag{2}$$

with $d_k := [d^1_k, \cdots, d^q_k]^T$, $|\Delta d^i_{k+1}| = |d^i_{k+1} - d^i_k| \le T\mu_i$ where T is the sampling time, μ_i is a small positive value.

Remark 1: The disturbance assumption in (2), in comparison with the ones in Gillijns and De Moor (2007) and Su et al (2015) where no particular disturbance models are assumed, can relax the observer existence condition. However, this is usually at the expense of a degraded performance when the disturbance assumption is violated.

Following the same notations as in Kim and Rew (2013) (see, pp. 970), define $N_c := (I_n - C^+C)$ with C^+ being the Moore-Penrose inverse of C and H_e as

$$H_e := \begin{bmatrix} KN_c \\ K(\Phi - I_n)N_c \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} V^T,$$

with K being a gain matrix to be designed, $H_1 \in \mathbb{R}^{q\times h}$, $H_2 \in \mathbb{R}^{q\times h}$ and $V^T \in \mathbb{R}^{h\times n}$ with $h = rank(H_e)$.

Defining $\eta_k := V^T x_k \in \mathbb{R}^h$, the reduced order DOB constructed in Kim and Rew (2013) is given by

$$\begin{cases} \xi_{k+1} = R\xi_k + Sy_k + W_u u_k + W_d \hat{d}_k \\ \hat{\eta}_k = \xi_k + Qy_k \\ z_{k+1} = z_k + K\{(\Phi - I_n)C^+ y_k + \Gamma u_k\} \\ \qquad + KG\hat{d}_k + H_2\hat{\eta}_k \\ \hat{d}_k = KC^+y_k - z_k + H_1\hat{\eta}_k \end{cases}, \tag{3}$$

where $z_k \in \mathbb{R}^q$ and $\xi_k \in \mathbb{R}^h$, $W_u = (V^T - QC)\Gamma$, $W_d = (V^T - QC)G$ with matrices S, Q, R satisfying

$$(V^T - QC)\Phi - R(V^T - QC) - SC = 0. \tag{4}$$

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Denote disturbance and state function estimation error as $e_k = d_k - \hat{d}_k$ and $\epsilon_k = \eta_k - \hat{\eta}_k$. The composite error dynamic is given by Kim and Rew (2013)

$$\begin{bmatrix} e_{k+1} \\ \epsilon_{k+1} \end{bmatrix} = A_e \begin{bmatrix} e_k \\ \epsilon_k \end{bmatrix} + \begin{bmatrix} \Delta d_{k+1} \\ O_{h\times 1} \end{bmatrix},$$

where the composite error matrix A_e is defined as

$$A_e = \begin{bmatrix} I_q - KG + H_1(V^T - QC)G & H_1R - H_1 - H_2 \\ (V^T - QC)G & R \end{bmatrix}. \quad (5)$$

As pointed by Kim and Rew (2013), the existence of a stable reduced order DOB (3) depends on whether there exists a gain matrix K and other design parameters so that: (i) the Sylvester equation (4) holds; (ii) matrix A_e in (5) is asymptotically stable (i.e., the amplitudes of its eigenvalues are less than 1). Condition (i) is implied by the matrix rank equality

$$rank(\begin{bmatrix} Z_1 \\ V^T\Phi \end{bmatrix}) = rank(Z_1)\,, Z_1 := rank(\begin{bmatrix} C \\ C\Phi \\ V^T \end{bmatrix}). \quad (6)$$

However, condition (ii) is not straightforward to check. Its existence is related to a static output feedback problem. As pointed out in Kim et al (2010) and Kim and Rew (2013), although numerical solutions exist, the general solvability of the static output feedback is not known.

To further develop this promising approach, this paper improves the results in Kim and Rew (2013) by presenting a generic reduced order DOB with an easily-checked existence condition. We transform the reduced order DOB design problem into a State Functional Observer (SFO) design problem (see, Darouach (2000), Fernando et al (2011) for SFO theory). This is achieved by first augmenting the disturbances with the states and then carefully designing the state functional matrix. Consequently, a generic DOB is resulted with its necessary and sufficient existence condition. There are two promising features in the newly developed DOB in comparison with the existing results:

(i) The existence condition is easy to check, where two easily-checked matrix rank equalities are required;
(ii) The observer convergence rate can be easily adjusted via the existing pole assignment techniques.

On this basis, we further investigate the relationship between the proposed DOB with the reduced order DOB in Kim and Rew (2013) and the full order DOB in Chang (2006) in terms of the observer structure and existence conditions. Both of them are shown to be special cases of the newly developed DOB.

2 DOB design using SFO technique

SFO, firstly introduced in Luenberger (1966), received much attention in control engineering (see, Darouach (2000)) owning to its fine properties such as a lower observer order and a relaxed existence condition. Its existence condition has been rigorously established in Darouach (2000). However, little attention has been paid to its applications in disturbance/fault estimation. Consequently, this paper aims to exploit its potential in DOB design, which can provide a framework unifying the existing DOB for discrete-time linear systems.

2.1 Observer design

Combining system (1) and disturbance (2), and defining $\bar{x}_k = [x_k^T, d_k^T]^T$, an augmented system is resulted

$$\begin{cases} \bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{\Gamma}u_k + \Delta\bar{d}_k \\ y_k = \bar{C}\bar{x}_k \end{cases}, \quad (7)$$

where the gain matrices and $\Delta\bar{d}_k$ are given as follows

$$\bar{A} = \begin{bmatrix} \Phi & G \\ O_{q\times n} & I_q \end{bmatrix}, \bar{\Gamma} = \begin{bmatrix} \Gamma \\ O_{q\times m} \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} C & O_{l\times q} \end{bmatrix}, \Delta\bar{d}_k = \begin{bmatrix} O_{n\times 1} \\ \Delta d_k \end{bmatrix}.$$

Remark 2: For the case that measurement outputs are also subject to disturbances, i.e., $y_k = Cx_k + G_2d_k$, this approach is also applicable by choosing $\bar{C} = [C\ G_2]$. Then the remaining design procedures are the same, although the existence condition will be slightly different.

To obtain disturbance estimates, the state functional matrix L (see, Darouach (2000) for its definition) is chosen with a special structure

$$L = \begin{bmatrix} L_0 & O_{\bar{h}\times q} \\ O_{q\times n} & I_q \end{bmatrix}, \quad (8)$$

where the design of sub-matrix $L_0 \in \mathbb{R}^{\bar{h}\times n}$ with full row-rank will be discussed in Section 2.3. Define

$$v_k = L\bar{x}_k, \text{ with } d_k = [O_{q\times n},\ I_q]v_k, \quad (9)$$

which is the state function to be estimated.

Now the problem of DOB design can be transformed into the problem of state functional observer design for the augmented system (7) with state function v_k in (9) to be estimated.

According to SFO theory (see, Darouach (2000) and Fernando et al (2011)), the disturbance observer for $\hat{d}_k$ along with the state functional observer for v_k takes the following form

$$\begin{cases} w_{k+1} = Nw_k + Jy_k + Hu_k, \\ \hat{v}_k = Bw_k + Ey_k, \\ \hat{d}_k = [O_{q\times n}\ I_q]\hat{v}_k. \end{cases} \quad (10)$$

Defining an intermediate error $\chi_k = \bar{P}\bar{x}_k - w_k$ with $\bar{P}$ being an intermediate matrix, its dynamics is given by

$$\begin{aligned} \chi_{k+1} = N\chi_k &+ (\bar{P}\bar{A} - N\bar{P} - J\bar{C})\bar{x}_k \\ &+ (\bar{P}\bar{\Gamma} - H)u_k + \bar{P}\Delta\bar{d}_k. \end{aligned} \quad (11)$$

Moreover, the state function estimation error $e_k = v_k - \hat{v}_k$ can be written as

$$e_k = B\chi_k + (L - E\bar{C} - B\bar{P})\bar{x}_k, \quad (12)$$

from which one can obtain $e_k \to 0$ as $k \to \infty$ for any $\bar{x}_k$ if and only if the following two conditions hold concurrently:
(i) $\chi_k \to 0$ as $k \to \infty$;
(ii) $L - E\bar{C} - B\bar{P} = O$.
For any invertible matrix B, the aforementioned condition (ii) is implied by choosing $\bar{P}$ as

$$\bar{P} = B^{-1}L - B^{-1}E\bar{C}. \quad (13)$$

Ignoring the term $\bar{P}\Delta\bar{d}_k$ as it does not affect the analysis, Eq. (11) implies that $\chi_k \to 0$ as $k \to \infty$ if and only if the following conditions hold simultaneously
(a) $\bar{P}\bar{A} - N\bar{P} - J\bar{C} = O$ (Sylvester equation);
(b) $\bar{P}\bar{\Gamma} - H = O$;
(c) N is asymptotically stable.
Choosing $\bar{P}$ according to (13) with any invertible B and H according to Condition (b), then the existence condition reduces to Condition (c), i.e. N being asymptotically stable under the constraint Sylvester equation Condition (a). The existence condition in the form of easily-checked matrix rank equalities has been established in Darouach (2000) with $B = I$ and later in Fernando et al (2011) with B being any invertible matrix, summarized in Theorem 1.

Theorem 1 *For system (1) under disturbance assumption (2), there exists a stable generic DOB (10) for system (1) if and only if the following conditions hold:*
i) the following matrix rank equality is satisfied:

$$rank(\begin{bmatrix} L_0\Phi \\ C\Phi \\ C \\ L_0 \end{bmatrix}) = rank(\begin{bmatrix} C\Phi \\ C \\ L_0 \end{bmatrix}), \quad (14)$$

ii) the pair (F, M) in Eq. (20) is detectable or equivalently $\forall s \in C$ with $Re(s) \geq 1$,

$$rank(\begin{bmatrix} sL_0 - L_0\Phi & -L_0G \\ O_{q\times n} & sI_q - I_q \\ C\Phi & CG \\ C & O_{l\times q} \end{bmatrix}) = rank(\begin{bmatrix} C\Phi \\ C \\ L_0 \end{bmatrix}) + q. \quad (15)$$

Proof: The existence condition is established by substituting the state function matrix (8) and the definition of the variables to be estimated as in (9) into the existence conditions in Lemmas 1 and 2 of Darouach (2000) and Theorem 3 of Fernando et al (2011). After a number of standard manipulations, Eqs. (14) and (15) are resulted.

2.2 Relationship with the existing results

In this section, we investigate the relationship of the developed DOB using SFO technique with the reduced order DOB in Kim and Rew (2013) and the full order DOB in Chang (2006).

2.2.1 Relationship with reduced order DOB in Kim and Rew (2013)

Inserting $\hat{\eta}_k$ of (3) into $\hat{d}_k$ yields

$$\hat{d}_k = KC^+y_k - z_k + H_1(\xi_k + Qy_k). \quad (16)$$

Combing $\hat{\eta}_k$ of (3) and (16), one can obtain

$$\underbrace{\begin{bmatrix} \hat{\eta}_k \\ \hat{d}_k \end{bmatrix}}_{\hat{v}_k} = \underbrace{\begin{bmatrix} I & O_{h\times q} \\ H_1 & -I \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \xi_k \\ z_k \end{bmatrix}}_{w_k} + \underbrace{\begin{bmatrix} Q \\ KC^+ + H_1Q \end{bmatrix}}_{E} y_k. \quad (17)$$

Substituting the dynamics of $\hat{\eta}_k$ and $\hat{d}_k$ in (17) into that of ξ_k and z_k in (3), a compatible form with SFO based DOB (10) for ξ_k and z_k is given by

$$\underbrace{\begin{bmatrix} \xi_{k+1} \\ z_{k+1} \end{bmatrix}}_{w_{k+1}} = \underbrace{\begin{bmatrix} R + W_dH_1 & -W_d \\ KGH_1 + H_2 & I - KG \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} \xi_k \\ z_k \end{bmatrix}}_{w_k} + \underbrace{\begin{bmatrix} S + W_d(KC^+ + H_1Q) \\ KG(KC^+ + H_1Q) + K(\Phi - I_n)C^+ + H_2Q \end{bmatrix}}_{J} y_k + \underbrace{\begin{bmatrix} W_u \\ K\Gamma \end{bmatrix}}_{H} u_k, \quad (18)$$

which means the reduced order DOB (3) proposed in Kim and Rew (2013) is a special case of the proposed DOB (10) with $L_0 = V^T$ in (8) and B in a special form as in (17). All other corresponding matrices are defined in (17) and (18) (see, the under-braces notations).

It shall be noticed that the existence condition (14) in Theorem 1 is actually the same as condition (i), i.e., Eq. (6) of Kim and Rew (2013) with $L_0 = V^T$. *However, condition (15) with $L_0 = V^T$ in Theorem 1 is a matrix rank equality, which is much easier to check than Condition (ii) of Kim and Rew (2013) as discussed above, i.e., no general solvability for the existence of a static output feedback problem exists.*

2.2.2 Relationship with full order DOB in Chang (2006)

An observer simultaneously estimating full states and disturbances was proposed in Chang (2006) for system (1), given by

$$\begin{cases} \hat{x}_{k+1} = \Phi\hat{x}_k + \Gamma u_k + L_1(y_k - C\hat{x}_k) + G\hat{d}_k \\ \hat{d}_{k+1} = \hat{d}_k + L_2(y_k - C\hat{x}_k) \end{cases}. \quad (19)$$

One can put (19) into an equivalent form to have a compatible structure with the generic DOB (10).

$$\begin{cases} \underbrace{\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix}}_{w_{k+1}} = \underbrace{\begin{bmatrix} \Phi - L_1C & G \\ -L_2C & I \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix}}_{w_k} + \underbrace{\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}}_{J} y_k + \underbrace{\begin{bmatrix} \Gamma \\ O_{q\times m} \end{bmatrix}}_{H} u_k, \\ \hat{v}_k = \underbrace{I_{n+q}}_{B} w_k + \underbrace{O_{(n+q)\times l}}_{E} y_k, \\ \hat{d}_k = [O_{q\times n}, I_q]\hat{v}_k. \end{cases} \quad (20)$$

That means the full order DOB proposed in Chang (2006) is a special case of the proposed DOB with $L_0 = I_n$ and consequently $L = I_{n+q}$.

In addition, with $L_0 = I_n$ the existence condition (14) in Theorem 1 always holds and the condition (15) reduces to $\forall s \in C$ with $Re(s) \geq 1$,

$$rank(\begin{bmatrix} sI_n - \Phi & -G \\ O_{q\times n} & sI_q - I_q \\ C & O_{l\times q} \end{bmatrix}) = n + q,$$

which is equivalent to that of Chang (2006).

Remark 3: Comparing (10) and (20), it can be discovered that the filtered states rather than raw sensor measurements y_k are adopted to derive disturbance estimates in full order DOB. As a result, in comparison with reduced order DOB, full order DOB is less sensitive to sensor noises.

2.3 Design procedure of the generic DOB

The design of generic DOB using SFO technique starts from choosing L_0 to satisfy the existence conditions in Theorem 1. In practice, an observer with a small order may be more desirable. So the selection of L_0 could start with a low order in the complement space of C and then increase the order until the conditions in Theorem 1 are satisfied. This could make sure a disturbance observer with a minimal order is designed. The complete procedures are summarized below:

(1) Form a state functional matrix L according to (8) and choose an invertible matrix B (default, $B = I$).

(2) Define matrices F and M as

$$\begin{cases} F = B^{-1}L\bar{A}L^{+}B - B^{-1}L\bar{A}N_L\Sigma^{+}\begin{bmatrix}\bar{C}\bar{A}L^{+}B \\ \bar{C}L^{+}B\end{bmatrix}, \\ M = [I - \Sigma\Sigma^{+}]\begin{bmatrix}\bar{C}\bar{A}L^{+}B \\ \bar{C}L^{+}B\end{bmatrix}, \end{cases} \quad (20)$$

where $\Sigma = \begin{bmatrix}\bar{C}\bar{A}N_L \\ \bar{C}N_L\end{bmatrix}$, L^{+} is the Moore-Pensrose pseudo-inverse inverse of the matrix L, given by $L^{+} = L^T(L^TL)^{-1}$ due to L being of full-row rank, $N_L = (I - L^{+}L)$.

(3) Calculate matrix N by the existing pole placement techniques for the matrix pair (F, M) as

$$N = F - ZM, \quad (21)$$

where Z is the matrix obtained from the pole placement of the pair (F, M). The observability of the pair (F, M) is guaranteed by condition (15).

(4) Obtain gain matrices J and E based on the following relationship

$$[B^{-1}E \quad J - NB^{-1}E] = B^{-1}L\bar{A}N_L\Sigma^{+} + Z\left[I - \Sigma\Sigma^{+}\right].$$

(5) Obtain matrix H via

$$H = (B^{-1}L - B^{-1}E\bar{C})\bar{\Gamma}.$$

Remark 4: From (11) and (12), one can obtain $e_{k+1} = BNB^{-1}e_k + B\bar{P}\Delta\bar{d}_k$, which means the convergence rate of the DOB is determined by the eigenvalues of N. It follows from Eq. (21) that the convergence rate of the generic DOB can be easily adjusted by using the existing pole assignment techniques for the matrix pair (F, M).

3 Conclusions

In this work, state functional observer technique is applied to reduced-order Disturbance OBserver (DOB) design by augmenting the disturbances as additional states and carefully selecting the state functional matrix. As a result, the existence condition of a fixed order DOB is represented in the form of two easily-checked matrix rank equalities. Besides, the convergence rate of the DOB can be easily adjusted via the existing pole assignment techniques. It is also shown that both the reduced order DOB in Kim and Rew (2013) and full order DOB in Chang (2006) are special cases of the developed generic DOB. Further work can be done to reduce the adverse effects of sensor noises on disturbance estimate performance.

References

Chang, J. L. (2006). Applying discrete-time proportional integral observers for state and disturbance estimations. *IEEE Transactions on Automatic Control*, 51(5), 814–818.

Darouach, M. (2000). Existence and design of functional observers for linear systems. *IEEE Transactions on Automatic Control*, 45(5), 940–943.

Fernando, T. L., Jennings, L. S., & Trinh, H. M. (2011). Generality of functional observer structures. *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 4000–4004.

Gillijns, S., & De Moor, B. (2007). Unbiased minimum-variance input and state estimation for linear discrete-time systems. *Automatica*, 43(1), 111–116.

Kim, K. S., Rew, K. H., & Kim, S. (2010). Disturbance observer for estimating higher order disturbances in time series expansion. *IEEE Transactions on Automatic Control*, 55(8), 1905–1911.

Kim, K. S., & Rew, K. H. (2013). Reduced order disturbance observer for discrete-time linear systems. *Automatica*, 49(4), 968–975.

Luenberger, D. G. (1966). Observers for multivariable systems. *IEEE Transactions on Automatic Control*, 11(2), 190–197.

Su, J., Li, B., & Chen, W. H. (2015). On existence, optimality and asymptotic stability of the Kalman filter with partially observed inputs. *Automatica*, 53, 149–154.

Su, J., & Chen, W. H. (2017). Model based fault diagnosis system verification using reachability analysis. *IEEE Transactions on System, Man and Cybernetics: Systems*, DOI: 10.1109/TSMC.2017.2710132.

Wei, X. J., & Guo, L. (2010). Composite disturbance-observer-based control and H∞ control for complex continuous models. *International Journal of Robust and Nonlinear Control*, 20(1), 106118.

Wei, X. J., Wu, Z. J., & Karimi, H. R. (2016). Disturbance observer-based disturbance attenuation control for a class of stochastic systems. *Automatica*, 63, 21-25.

Yang, J., Cui, H., Li, S., & Zolotas, A. (2018). Optimized active disturbance rejection control for DC-DC buck converters with uncertainties using a reduced-order GPI observer. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 65(2), 832-841.