# Tilting the Classroom

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This article describes and illustrates 12 simple ways to make large mathematics lectures more engaging. These include a variety of short-and-snappy activities, framed by organisational practices that support concentration and maintain a positive atmosphere. These practices can be implemented individually or in combination, with no need for a wholesale classroom restructure.

## Introduction

I borrowed this article's title from Calvin Smith, who told me that his classroom, while not flipped, is *tilted*. This perfectly captures my own approach to lecturing. My lectures are in a sense traditional: students sit in rows, listen to me, and take notes. But they also engage with a variety of conceptual reasoning tasks. I do not claim that this approach is perfect, and I do not intend to be prescriptive — I have opinions, based on research in undergraduate mathematics education, but I think that good teaching is partly about authenticity and there is no single way to do it right.

What I do think important is that lecturers are free to try out new ideas on a small scale and without pressure for radical innovation. Radical innovation is currently fashionable: teaching development schemes often require it, and lecturers are encouraged to flip their classrooms, experiment with new technologies, and so on. But I find this troubling. I am all for trying new things, but innovative teaching is time-consuming and can easily fail. Radical changes are risky by nature, and traditional teaching can excel.

With that in mind, this article describes 12 practices that I use in lectures, each of which requires minimal effort to implement. I have applied these practices most recently in a real analysis course for 200 first- and second-year students. Like any such course, this is difficult. Its fundamental definitions are logically complex — no-one deals with triply quantified statements in everyday life or in earlier mathematics — and it is completely different from procedure-based learning. I can't work miracles, and I do not know how to make it intelligible to every student. But I can help many to engage with the complex ideas and to recognise their own development. In this article, I frame the twelve practices with three background principles and some thoughts about influencing students toward effective study habits.

## Principles

The first principle is that there is no point in the lecturer covering the material if the students don't. That is simplistic, of course: I teach to the curriculum, and I only partially control what is learned — students need to work after class on the more difficult ideas. But I also offer numerous opportunities to engage and re-engage during lectures.

The second principle is that students are not inherently lazy or bad people. This can be hard to remember — I certainly have moments at which it is not uppermost in my thoughts. But my experience is that the vast majority of students, the vast majority of the time, have good intentions. They also have moments of weakness, and they respond poorly to sensations of failure. But that is not because they are students, it is because they are *people*.

The third principle is that learning results from student activity more than from lecturer activity. As I gain experience, I think less about what I will say, and more about what students will do both in lectures and in independent study.

## Practices: Organisation

I want all of the students' intellectual energy available for mathematics. And I want all of their emotional energy available for maintaining resilience in the face of struggle. So the first four practices are about setting up the environment so that everyone feels secure and can invest their energies wisely.

### *1. Announcements*

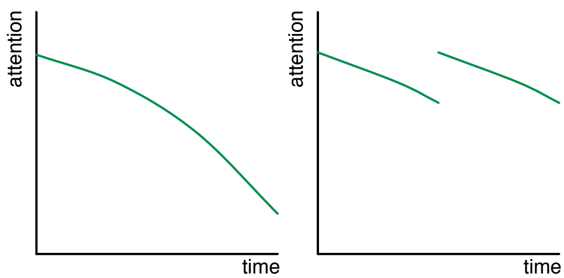
In the changeover before each lecture, I put handwritten announcements on the visualiser. These announcements say boring things like this.

> Good morning.
>
> Please pick up a set of notes.
>
> Turn to page 54. What is your answer to this morning's question?

This helps students to feel confident about practicalities, which is particularly important for first years. It helps them to help one another — the half who read the announcements can answer questions when the other half ask. Consequently, it dramatically cuts the amount of time I spend repeating myself.

#### *2. Break*

Around middle of each lecture, I use a natural break in the content to give a two-minute breather. The time is a bit different each day because I don't want anyone clock-watching. And I don't care what students do in the break. Some use it to review what we have just covered, others get out their phones. I think the only question to consider about breaks is: which graph of attention against time do you want?




#### *3. Notes*

I use gappy notes (or skeleton or partially populated notes), distributed weekly. Students have copies, and I have one that I use at the visualiser. I cover about four pages per lecture, and the amount of pre-printed material varies considerably. Each week's notes have a problem sheet attached to the back, so I don't have to distribute these separately. And page numbers for the whole course are contiguous, so anyone who mixes up their paperwork can reorder it easily.

#### *4. Routine*

My lectures are currently on Monday (11am and 5pm) and Wednesday (11am). On Monday morning, students pick up notes on the way in. Between Monday and Wednesday they are expected to read a few pre-printed pages, where I make clear that this should take less than an hour and that I advise setting a regular reading time. Wednesday's lecture starts with ten true/false questions. After that lecture, completed notes go on the virtual learning environment (ours is based on moodle), followed on Friday by problem solutions. The actual routine doesn't matter, of course, it just matters that there is one. Like all of these organisational practices, this helps students to know where everything is and what they are supposed to be doing, so that they can focus their energies on learning.

### Practices: Study guidance

There is substantial evidence that students — and people in general — hold erroneous and unhelpful beliefs about learning [3]. First years, certainly, know little about what is expected in undergraduate study. Some have been micromanaged by earlier teachers, and have not developed good planning skills or self-discipline. Some have found earlier mathematics fairly easy, and do not know how to handle themselves in the face of a challenge. The next three practices offer practical advice and encouragement.

#### *5. Clarifying expectations*

The week 2 reading begins with information on what real analysis is like. Here is a short sample.

> Here is what happens when I teach Analysis. In week 1, everyone is in a good mood because they're starting something new. In weeks 2 and 3, there is a buildup of increasingly challenging material. In week 4, the mood in the lecture theatre is dreadful. The whole class has realized that this is difficult stuff and that it isn't going to get any easier. Everyone hates Analysis and, by extension, quite a few people hate me. I am not fazed by this, though, because I have taught Analysis about twenty times now and I know what will happen next. . . (from [1]).

Someone needs to say this, because new students who experience difficulty will believe that they are failing, and some will respond with avoidance rather than redoubled effort. Reading tasks are good for such content because labouring it in lectures takes time and can seem patronising. This reading goes on to discuss strategies for keeping up, how much time I expect students to spend studying notes and trying problems, and what to try and where and how to seek help when stuck.

### *6. Self-explanation training*

The week 3 reading is a research-based booklet providing self-explanation training adapted for mathematics students (see setmath.lboro.ac.uk). Self-explanation training teaches students to read effectively, and has been used across a range of academic subjects and mathematical levels [6]. This self-explanation training states that when reading mathematics, students should explicitly relate each line to earlier material and to their existing knowledge, questioning their own understanding. It teaches them to differentiate self-explanation from monitoring ('Yeah, yeah, I get that') and from paraphrasing. Experimental and eye-movement studies have established that it leads to better proof comprehension and more expert-like reading behaviour [2, 5].

### *7. Early feedback opportunity*

After nearly 20 years of lecturing, I finally do what training courses say that you should: at the end of the first main topic, I give out big sticky notes and ask students to write down something they like about the course, and something that they don't like or are concerned about or didn't understand. The positive responses are straightforward and predictable. The negative things are more varied — everyone is unhappy in their own way — and include things like:

- Analysis is difficult.
- Pace is too fast.
- Worried about constructing proofs.
- A few don't like interactive discussions.
- Would like lecture capture used.
- Would like more worked examples.

Each year I put a full list on the visualiser and take ten minutes to discuss it. How many elements of the course do you think I say I will change? That's right: none. I know a lot more than undergraduates do about teaching and learning mathematics. But the value of such feedback is not in finding things to change, it's in arranging an opportunity to explain why things are as they are. It helps students to see that not everyone wants the same things, and that some requests are mutually exclusive — you can't have both more examples and slower delivery. And I stress that the concerns are reasonable, which helps the students to feel understood.

## Practices: Activities

In-lecture mathematical activities can provide students with opportunities to be wrong, opportunities to be right, and opportunities to feel unsure. I believe that all of these are important for engagement and a sense of progress. And, handled well, a large class is ideal for generating emotional investment. Instead of an unmemorable 'Yeah okay, yeah okay', I want students to experience a more memorable 'Oh I know that... Oh no wait, maybe I don't... Gosh that is harder than it looks... Oh I get it now!'.

Gappy notes are great for this. They allow me to pre-print information that I want to record but not write. They facilitate variety and short-and-snappy tasks, which is important because momentum is easily lost. I don't have students do routine calculations; these take too long, and if there is one thing that new undergraduates can do on their own, it's routine calculations. I use lecture time instead to develop conceptual understanding by having students articulate their thinking to one another. Here are some things that I ask them to do and discuss.

### *8. Filling things in*

If something can be filled in by students without my assistance, I think it should be. This works for routine extensions, applications to examples, and conceptual thought about mathematical claims. For instance, students can complete this theorem.

$$\lim_{n\to\infty} x^n = \begin{cases} \infty & \text{if} \\ 1 & \text{if} \\ 0 & \text{if} \end{cases}$$

Everyone can get this right, and it requires thought about the roles of $x$ and $n$, which are less likely to seem important if I print or write the full theorem.

Similarly, if provided with definitions of *bounded above*, *upper bound*, and *supremum*, students can complete definitions of *bounded below*, *lower bound*, and *infimum*. And, of course, they can fill in tables that provide examples related to these concepts.

My favourite filling-in task is about the axioms for the real numbers. I didn't want to write these out — that list is long. But I knew that printing them would not prompt much thought. I toyed with the idea of printing the axioms and writing in the names ('commutativity of addition'). Then I had a brainwave: I now print the list of axioms and the list of names, and have students match them up. This is a few weeks into the term so, after some initial hesitation, most people can get them all right. More importantly, they have to think about the meanings of commutativity, distributivity, and so on. And that's key for all of these activities. They are quick and doable, but they require thought about meaning.

***9. Deciding***

Another type of activity is deciding. My lectures often include several decision tasks, which start with 30 seconds or a minute or three minutes for thinking and discussion. I then ask for a vote, using the old-school technology of raised hands. The hand-raising works because I raise my own hand, right up in the air, for both answers ('Votes for true [raise hand... lower hand]... Votes for false [raise hand]'). Before every vote, I say 'I don't care who is right or wrong, I just care that you're thinking and that you're willing to change your mind if someone gives you good reason to'. If there are not enough votes, I say 'That is not enough votes', and give the students a minute to think some more and vote again.

My favourite decision questions are those that I know will split the class 50:50. I draw attention to these by asking everyone to vote again and look around the room. Then I say 'Whatever you think, half of the class disagrees. *Do you want to change your mind?*'. The room then comes alive: everyone knows that their peers are not stupid, yet apparently half of them are wrong. This dramatically increases everyone's motivation to work out whether they might have overlooked a crucial idea.

A useful type of decision question is: *What symbol goes in the gap in this theorem? ⇒, ⇐, or ⇔?* Here are some theorems for which that works.

- $(a_n) \to a \qquad (|a_n|) \to |a|$.
- $(a_n)$ is convergent $\qquad (a_n)$ is bounded.
- $\sum_{n=1}^{\infty} a_n$ is convergent $\qquad (a_n) \to 0$.

The last takes two or three rounds of voting because, even when we have just studied the series $\sum 1/n$, about 75% answer incorrectly — the intuition that a series converges if its terms tend to zero is tough to dislodge. But that's the point. My drawing attention to a counterintuitive result is not enough. Being wrong a couple of times is more memorable.

Another useful question type involves a true/false decision, which can set up what is coming next. Here are some of those, with the set-up that $A \subseteq \mathbb{R}$ has a supremum $\sup A$.

- $\sup A \in A$.
- If we define $-A = \{-a \mid a \in A\}$, then $\sup(-A) = -\sup A$.
- $\forall \varepsilon > 0$, $\exists a \in A$ such that $\sup A - \varepsilon < a \leq \sup A$.

After considering these, students are more ready to hear my comments. And this is true whether or not they have made much progress. Those who struggle to interpret a quantified statement learn as much about interpretation as they do about the result.

***10. Reading and explaining***

A third type of activity is reading something and explaining it to your neighbour. This, in my view, is worth doing: independent reading is an important skill, and if something is important then it merits lecture time. And explanation tasks can be short. I often ask students to read a definition, theorem or proof and to use gestures, diagrams or examples to explain what it means (and why it is true or valid). This, again, requires thought about meaning. Of course, some definitions, theorems and proofs are difficult, so I adjust for this. For the definition of sequence convergence, for instance, I first give students enough time to try to understand it and realise that they don't. I then offer an extended explanation, building up a diagram and an informal verbal expression. I then ask them to explain to one another what I just said. When they realise that they can't quite do that either, I say that I'll run through it once more and give them another go. Attention, by that point, is high.

*11. True/false questions*

Wednesday's lecture starts with ten true/false questions, printed on one side of paper with space for each response. Here are a few examples.

- The number $\sqrt{47}$ is irrational.
- The number $47/225$ has a non-repeating decimal expansion.
- The set of even numbers is countably infinite.
- For all $x \in \mathbb{N}$, $4|x^3 \Rightarrow 4|x$.
- If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ then $x + y \notin \mathbb{Q}$.

The instruction is to state whether each statement is true or false and, if it is false, to give a counterexample or a brief reason. I give about seven minutes for individual, silent attempts, about three for students to discuss their answers, and about two for them to consult their notes. Then I run through the answers.

I originally intended these questions to encourage students to do the reading — those who haven't done it spend a few minutes feeling uncomfortable. But their real value is in providing retrieval practice, which is important because repeated retrieval is known to strengthen memory [3]. And, ironically, they provide individualised feedback — many students comment that the true/false questions highlight what they need to review.

*12. Tests*

Three times during the term, the true/false quiz is replaced by a 20-minute for-credit test. This contains ten true/false questions with the usual instructions, and two or three more challenging questions. The challenging questions are published a week in advance so that students can prepare. They can work together and look up whatever they want, but they are not allowed to ask tutors or staff in our mathematics learning support centres. This allows me to ask questions that go beyond what has been covered in lectures, while holding everyone accountable for producing their own answers; those who want to cheat have to remember what their clever friend said, not just copy it out.

## Influencing students

My overall aim is to be a positive influence on student behaviour, and in this I've been guided by the book *Influencer* [4]. Its authors argue that there are six sources of influence, sorted into a three-by-two grid. The columns are motivation (do I want to do it?) and ability (can I do it?), and the rows capture individual, social and structural influences.

Individual motivation sounds straightforward. My students, after all, have chosen to study mathematics. But every lecturer knows that desire to obtain a degree is not directly linked to desire to engage with difficult ideas in the day's sixth lecture. Fortunately, I think there are two sources of individual motivation, one of which is often overlooked. Some students are interested in real analysis. Some are not. But *everyone* is interested in their own intellectual development. Everyone likes to be right, and most are pretty happy to be wrong and then right, having gained an insight. Activities can engage that.

Individual ability, counterintuitively, is easier to manipulate. Some abilities can be improved: study planning and mathematical reading can be addressed directly [1, 2, 5]. And perhaps more important is *perception* of ability. Students at this level often can fill in definitions, explain theorems and proofs, and get most of our true/false questions right. That provides a sense of progress and developing capability, which makes the difficult things more palatable.

Social motivation is a strange one. Many undergraduates tell one another that they don't need to study if the first year doesn't count for credit, and I can't generate a comprehensive culture shift. But I can create an environment in which it is clear that the vast majority are, in fact, keen to do well, and willing to work hard and support one another.

Social influences on ability can be direct: my students are encouraged to help one another, both to understand the mathematics and to keep going when it gets tough. Or they can be indirect: students who struggle in isolation can have skewed ideas about what it means to do well, whereas students who see regular evidence that no-one else knows all the answers either tend to have a better calibrated sense of their own performance, and to suffer less worry.

Structural influences on motivation are tricky. The *Influencer* authors stress that carrots and sticks are not effective replacements for individual and social motivation. In academia, for instance, tests can make people study, but I do not believe that they make them *want* to study. So I am leery of set-ups involving frequent assessed work: I want students to develop deep understanding of a body of mathematics, not to chase after bits of credit. Because of this, I am content that my true/false questions are formative only, and that tests form a minor part of my strategy.

Structural supports for ability are easier. They often involve simply removing rocks from the path, and my organisational practices are designed for this. Gappy notes enable students to keep up while still thinking. A consistent routine minimises time-wasting confusion about what is happening when. And tired, tense and irritable students find it difficult to concentrate; a friendly atmosphere of mutual support can loosen the tension and help them engage.

**That all sounds nice, but...**

The material in this article raises consistent concerns among mathematicians, so I conclude by addressing some common questions[1].

Some concerns are about time. Many mathematicians can't imagine having time to include activities in lectures. But time problems are caused by writing out longhand everything that you want to say. Of course, writing is valuable for students, for practice and because it is hard to pay attention when there is nothing physical to do. But I am selective about what merits this treatment. Do I want students to spend two minutes copying a definition? Maybe, if I want to draw attention to an aspect of its formulation. But maybe I'd rather they spend two minutes explaining it to their neighbour.

Other concerns are about participation. Some worry that not all students will engage with in-lecture tasks. That is certainly true: some will talk about something else or check social media feeds. But it is worrying only if you think that they won't do that otherwise. Some worry that students will provide one another with incorrect mathematical explanations. That, also, is true, but less so than you might think. And again it is worrying only if they would fully understand a lecturer — talking is an imperfect way to learn, but so is listening. Some worry about regaining attention after a task. But if you can do that at the beginning of a lecture, you can do it again later. Finally, some mathematicians ask why I don't use clickers to gather responses — these, after all, permit full anonymity. The answer used to be laziness: the set-up takes some work and I never got around to it. But then I realised that one crucial thing for undergraduates to learn is that mathematics requires persistence, and that struggle is normal. Students in my class often get things wrong or have to admit that they don't know. Familiarity and ease with that is exactly what I want to encourage.

### FURTHER READING

[1] L. Alcock, How to think about analysis, Oxford University Press, 2014.
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[1]This article is based on a talk given at Manchester, Reading, Leeds, the sigma conference, and the HEA/HoDoMs New Lecturers' Course. My thanks to those audiences for useful feedback.