The Word Problem for Omega-Terms over the Trotter-Weil Hierarchy

(Extended Abstract)

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Abstract. Over finite words, there is a tight connection between the quantifier alternation hierarchy inside two-variable first-order logic FO^2 and a hierarchy of finite monoids: the Trotter-Weil Hierarchy. The various ways of climbing up this hierarchy include Mal'cev products, deterministic and co-deterministic concatenation as well as identities of ω-terms. We show that the word problem for ω-terms over each level of the Trotter-Weil Hierarchy is decidable; this means, for every variety $\mathbf{V}$ of the hierarchy and every identity $u = v$ of ω-terms, one can decide whether all monoids in $\mathbf{V}$ satisfy $u = v$. More precisely, for every fixed variety $\mathbf{V}$, our approach yields nondeterministic logarithmic space (NL) and deterministic polynomial time algorithms, which are more efficient than straightforward translations of the NL-algorithms. From a language perspective, the word problem for ω-terms is the following: for every language variety $\mathcal{V}$ in the Trotter-Weil Hierarchy and every language variety $\mathcal{W}$ given by an identity of ω-terms, one can decide whether $\mathcal{V} \subseteq \mathcal{W}$. This includes the case where $\mathcal{V}$ is some level of the FO^2 quantifier alternation hierarchy. As an application of our results, we show that the separation problems for the so-called corners of the Trotter-Weil Hierarchy are decidable.

1 Introduction

For the study of many regular language classes, it turned out to be fruitful if one finds multiple characterizations for the class. For instance, one can consider the class of languages recognized by *extensive* deterministic finite automata (i. e. automata whose states can be ordered topologically). This is algebraically characterized by the variety $\mathbf{R}$ of $\mathcal{R}$-trivial monoids [3, Chap. 10]. Another example is the class of star-free languages. It is defined as the set of languages which can be defined by a regular expression which may use complementation instead of Kleene's star. Schützenberger's famous theorem [20] yields an algebraic characterization for this class: it coincides with the class of languages which are

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recognized by aperiodic monoids. A monoid M is aperiodic if $x^{|M|!} = x^{|M|!}x$ holds for all $x \in M$. In the case of star-free languages (as in many other cases) this algebraic characterization is particularly useful as it makes it possible to decide whether a given language is star-free: compute the language's syntactic monoid M (which, for a regular language, must be finite) and check whether it is aperiodic. The latter can be achieved by checking the equation $x^{|M|!} = x^{|M|!}x$ for all $x \in M$. Often, this equation is also stated as $x^\omega = x^\omega x$ since this notation is independent of the monoid's size. More formally, we can see the equation as a pair of ω-terms: these are finite words built using letters, which are interpreted as variables, concatenation and an additional formal ω power. In order to check whether the equation $\alpha = \beta$ consisting of the two ω-terms α and β holds in a monoid M one first substitutes the formal ω exponents in α and β by $|M|!$, which results in a finite word in variables. One, then, needs to substitute each variable by all element of M, which is possible if M is finite. These substitutions yield a monoid element belonging to α and one belonging to β. If and only if the respective pairs of monoid elements are equal for all variable substitutions, the equation holds in M.

Often, the question whether an equation holds is not only interesting for a single finite monoid but for a (possibly infinite) set of such monoids. For example one may ask whether all monoids in a certain set are aperiodic. This is trivially decidable if the set is finite. But what if the set is infinite? If the set forms a variety (of finite monoids) – that is a set of finite monoids which is closed under (possibly empty) direct products, submonoids and homomorphic images; sometimes also referred to as pseudo-varieties –, then this problem is called the variety's *word problem for ω-terms*. Usually, the study of a variety's word problem for ω-terms also gives more insight into the variety's structure, which is interesting in its own right. McCammond showed that the word problem for ω-terms of the variety $\mathbf{A}$ of aperiodic finite monoids is decidable [14]. The problem was shown to be decidable in linear time for $\mathbf{J}$ by Almeida [1] and for $\mathbf{R}$ by Almeida and Zeitoun [2]. Later Moura applied their ideas to show decidability in time $\mathcal{O}((nk)^5)$ where k is the maximal nesting depth of the ω-power (which can be linear in n) of the problem for the variety $\mathbf{DA}$ [16]. The variety $\mathbf{DA}$ is the set of finite monoids whose regular $\mathcal{D}$-classes form aperiodic semigroups. This class is interesting because of another characterization of $\mathbf{A}$ and, therefore, star-free languages: a language is star-free if and only if it can be defined by a sentence in first-order logic over words [15]. It is easy to see that any first-order sentence over words is equivalent to one which uses only three variables. Therefore, it is a natural question to ask what happens if one restricts the number of variables to two. This leads to *two-variable* first-order logic (over words). As it turns out, this class of languages is characterized by $\mathbf{DA}$ [24]; see [23] for a survey.

In this paper, we consider the word problems for ω-terms of the varieties of the *Trotter-Weil Hierarchy*. Trotter and Weil [25] used the good understanding of the band varieties (cf. [4]) for studying the lattice of sub-varieties of $\mathbf{DA}$; bands are semigroups satisfying $x^2 = x$. An important aspect of the Trotter-Weil Hierarchy is its connection with the quantifier alternation hierarchy inside two-variable first-order logic. In addition, many characterizations of two-variable first-order logic naturally appear within this hierarchy, see [8]. The Trotter-Weil Hierarchy has a zig-zag shape, see Figure 2. There are non-symmetric varieties, the so-called *corners*; amongst them is the variety $\mathbf{R}$ as well as its symmetric dual $\mathbf{L}$, the variety of $\mathcal{L}$-trivial monoids. Then there are the intersections of corners, the *intersection levels*; and finally there are the joins of the corners, the *join levels*. Two-variable quantifier alternation corresponds to the intersection levels [11]; in particular, the variety $\mathbf{J}$ of $\mathcal{J}$-trivial monoids is one of them. The union of all levels is $\mathbf{DA}$ [10].

In this paper, we present the following results.

- Our main tool for studying a variety $\mathbf{V}$ of the Trotter-Weil Hierarchy is a family of finite index congruences $\equiv_{\mathbf{V},n}$ for $n \in \mathbb{N}$. These congruences have the property that a monoid M is in $\mathbf{V}$ if and only if there exists n for which M divides a quotient by $\equiv_{\mathbf{V},n}$. The congruences are not new but they differ in some minor but crucial details (and these details necessitate new proofs). In the literature, the congruences are usually introduced in terms of rankers [8, 11, 12].
- We lift the combinatorics from finite words to ω-terms using the "linear order approach" introduced by Huschenbett and the first author [6]. They showed that, over varieties of aperiodic monoids, one can use the order $\mathbb{N}+\mathbb{Z}\cdot\mathbb{Q}+(-\mathbb{N})$ for the formal ω-power. In this paper, we use the simpler order $\mathbb{N}+(-\mathbb{N})$. We show that two ω-terms α and β are equal in some variety $\mathbf{V}$ of the Trotter-Weil hierarchy if and only if $[\![\alpha]\!]_{\mathbb{N}+(-\mathbb{N})} \equiv_{\mathbf{V},n} [\![\beta]\!]_{\mathbb{N}+(-\mathbb{N})}$ for all $n \in \mathbb{N}$. Here, $[\![\alpha]\!]_{\mathbb{N}+(-\mathbb{N})}$ denotes the labeled linear order obtained from replacing every ω-power by the linear order $\mathbb{N} + (-\mathbb{N})$. Note that this order is tailor-made for the Trotter-Weil Hierarchy and does not result from simple arguments which work in any variety.
- We show that one can effectively check whether $[\![\alpha]\!]_{\mathbb{N}+(-\mathbb{N})} \equiv_{\mathbf{V},n} [\![\beta]\!]_{\mathbb{N}+(-\mathbb{N})}$ for all $n \in \mathbb{N}$. For some varieties in the Trotter-Weil Hierarchy this is rather straightforward but for the so-called intersection levels it additionally requires some kind of synchronization.
- We further improve the algorithms and show that, for every variety $\mathbf{V}$ of the Trotter-Weil Hierarchy, the word problem for ω-terms over $\mathbf{V}$ is decidable in nondeterministic logarithmic space. The main difficulty is to avoid some blow-up which (naively) is caused by the nesting depth of the ω-power. For the variety $\mathbf{R}$ of $\mathcal{R}$-trivial monoids, this result is incomparable to Almeida and Zeitoun's linear time algorithm [2].
- We also introduce polynomial time algorithms, which are more efficient than the direct translation of these NL algorithms.
- As an application, we show that the separation problem for each corner of the Trotter-Weil Hierarchy is decidability; for $\mathbf{J}$ we adapt the proof of van Rooijen and Zeitoun [26].
- With little additional effort, we also obtain all of the above results for the limit of the Trotter-Weil hierarchy, the variety $\mathbf{DA}$. The decidability of the separation problem re-proves a result of Place, van Rooijen and Zeitoun [19]. The algorithms for the word problem for ω-terms are more efficient than Moura's results [16].

Separability of the join-levels and the intersection-levels is still open. We conjecture that these problems can be solved with similar but more technical reductions.

2 The Trotter-Weil Hierarchy

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, \dots\}$ and $-\mathbb{N} = \{-1, -2, \dots\}$. For the rest of this paper, we fix a finite alphabet Σ. By Σ^*, we denote the set of all finite words over the alphabet Σ, including the empty word ε; Σ^+ denotes that excluding the empty word. Let $w = a_1 a_2 \dots a_n \in \Sigma^*$ be a word of length $n \in \mathbb{N}_0$. The set $\{a_i \mid i = 1, 2, \dots, n\}$ of letters appearing in w shall be denoted by $\mathrm{alph}(w)$. As a finite word $w \in \Sigma^*$ can be seen as a mapping $w : \{1, 2, \dots, n\} \to \Sigma$, we use $\mathrm{dom}(w)$ to denote the set of positions in w.

For a pair $(l, r) \in (\{-\infty\} \uplus \mathrm{dom}(w)) \times (\mathrm{dom}(w) \uplus \{+\infty\})$, define $w_{(l,r)}$ as the restriction of w (seen as a mapping) to the set of positions (strictly) larger than l and (strictly) smaller than r. Note that $w = w_{(-\infty,+\infty)}$ and $w_{(l,r)} = \varepsilon$ for any pair (l, r) with no position between l and r.

Monoids, Divisors, Congruences and Recognition. In this paper, the term *monoid* refers to a finite monoid (except when stated otherwise). it is well known that, for any monoid M, there is a smallest number $n \in \mathbb{N}$ such that m^n is idempotent (i. e. $m^{2n} = m^n$) for every element $m \in M$; this number is called the *exponent* of M and shall be denoted by $M! = n$.[1] A monoid N is a *divisor* of (another) monoid M, written as $N \prec M$, if N is an homomorphic image of a submonoid of M.

A *congruence* (relation) in a (not necessarily finite) monoid M is an equivalence relation $\mathcal{C} \subseteq M \times M$ such that $x_1 \mathrel{\mathcal{C}} x_2$ and $y_1 \mathrel{\mathcal{C}} y_2$ implies $x_1 y_1 \mathrel{\mathcal{C}} x_2 y_2$ for all $x_1, x_2, y_1, y_2 \in M$. If M is a (possibly infinite) monoid and $\mathcal{C} \subseteq M \times M$ is a congruence, then the set of equivalence classes of $\mathcal{C}$, denoted by $M/\mathcal{C}$, is a well-defined monoid (which might still be infinite), whose size is called the *index* of $\mathcal{C}$. For any two congruences $\mathcal{C}_1$ and $\mathcal{C}_2$ over a monoid M, one can define their *join* $\mathcal{C}_1 \vee \mathcal{C}_2$ as the smallest congruence which includes $\mathcal{C}_1$ and $\mathcal{C}_2$; its index is at most as large as the index of $\mathcal{C}_1$ and the index of $\mathcal{C}_2$.

A (possibly infinite) monoid M *recognizes* a language of finite words $L \subseteq \Sigma^*$ if there is a homomorphism $\varphi : \Sigma^* \to M$ with $L = \varphi^{-1}(\varphi(L))$. A language is *regular* if and only if it is recognized by a finite monoid. It is well known that there is a unique smallest monoid which recognizes a given regular language: the *syntactic monoid*.

Varieties, π-Terms and Equations. A *variety* (of finite monoids) – sometimes also referred to as a *pseudo-variety* – is a set of monoids which is closed under submonoids, homomorphic images and – possibly empty – finite direct products. For example, the set $\mathbf{R}$ of $\mathcal{R}$-trivial monoids and the set $\mathbf{L}$ of $\mathcal{L}$-trivial monoids both form a variety, see e.g. [18]. Clearly, if $\mathbf{V}$ and $\mathbf{W}$ are varieties, then so is

[1] Note that all statements remain valid if one assumes that $M!$ is used to denote $|M|!$.

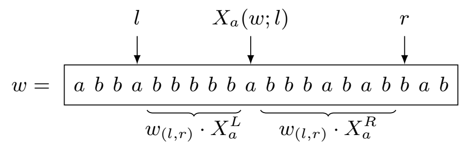

Fig. 1. Application of X_a^L and X_a^R to an example word.

$\mathbf{V} \cap \mathbf{W}$. For example, the set $\mathbf{J} = \mathbf{R} \cap \mathbf{L}$ is a variety; in fact, it is the variety of all $\mathcal{J}$-trivial monoids. For two varieties $\mathbf{V}$ and $\mathbf{W}$, the smallest variety which is a superset of $\mathbf{V} \cup \mathbf{W}$, the so called *join*, is denoted by $\mathbf{V} \vee \mathbf{W}$.

Often, varieties are defined in terms of *equations* (or *identities*). Because it will be useful later, we take a more formal approach towards equations by using *π-terms*[2]. A π-term is a finite word, built using letters, concatenation and an additional formal π-power (and appropriate parentheses), whose π-exponents act as a placeholder for a substitution value.[3]

To state equations using π-terms, one needs to substitute these placeholders by actual values resulting in an ordinary finite word. We define $[\![\gamma]\!]_n$ as the result of substituting the π-exponents in γ by $n \in \mathbb{N}_0$. An equation $\alpha = \beta$ consists of two π-terms α and β over the same alphabet Σ, which, here, can be seen as a set of *variables.* A homomorphism $\sigma : \Sigma^* \to M$ is called an *assignment of variables* in this context. An equation $\alpha = \beta$ *holds* in a monoid M if for every assignment of variables $\sigma\left([\![\alpha]\!]_{M!}\right) = \sigma\left([\![\beta]\!]_{M!}\right)$ is satisfied. If holds in a variety $\mathbf{V}$, if it holds in all monoids in $\mathbf{V}$.

Relations for the Trotter-Weil Hierarchy. In this paper, we approach the Trotter-Weil Hierarchy by using certain congruences. First, however, we give some definitions for factorizations of words at the first or last a-position (i. e. an a-labeled position). For a word w, a position $p \in \mathrm{dom}(w) \uplus \{-\infty\}$ and a letter $a \in \mathrm{alph}(w)$, let $X_a(w;p)$ denote the first a-position (strictly) larger than p (or the first a-position in w if $p = -\infty$). It is undefined if there is no such position. Define $Y_a(w;p)$ symmetrically as the first a-position from the right which is (strictly) smaller than p.

Let w be a word, define

$$w \cdot X_a^L = w_{(-\infty, X_a(w;-\infty))}, \qquad w \cdot X_a^R = w_{(X_a(w;-\infty), +\infty)},$$
$$w \cdot Y_a^L = w_{(-\infty, Y_a(w;+\infty))} \text{ and } \qquad w \cdot Y_a^R = w_{(Y_a(w;+\infty), +\infty)}$$

for all $a \in \mathrm{alph}(w)$. Additionally, define $C_{a,b}$ as a special form of apply X_a^L first and then Y_b^R which is only defined if $X_a(w;-\infty)$ is strictly larger than $Y_b(w;+\infty)$. For an example of X_a^L and X_a^R acting on a word see Figure 1. Note that we have $w = (w \cdot X_a^L)a(w \cdot X_a^R) = (w \cdot Y_a^L)a(w \cdot Y_a^R) = (w \cdot Y_b^L)b(w \cdot C_{a,b})a(w \cdot X_a^R)$ (whenever these factors are defined).

[2] Usually, π-terms are referred to as ω-terms. In this paper, however, we use ω to denote the order type of the natural numbers. Therefore, we follow the approach of Perrin and Pin [17] and use π instead of ω.

[3] See the appendix for an inductive definition.

With these definitions in place, we define the relations[4] $\equiv^X_{m,n}$, $\equiv^Y_{m,n}$ and $\equiv^{\mathrm{WI}}_{m,n}$ of words for $m, n \in \mathbb{N}$. The idea is that these relations hold on two words u and v if both words allow for the same sequence of factorizations at the first or last occurrence of a letter. The parameter m is the remaining number of direction changes (which are caused by an X^L_a or Y^R_a factorizations) in such a sequence and the parameter n is the number of remaining factorization moves (independent of their direction). Thus, if m or n is zero, then all of the three relations shall be satisfied for all words. For m and n larger than zero, our first assertion is that both words have the same alphabet; otherwise, one of them would admit a factorization at a letter while the other would not, as the letter is not in its alphabet. Furthermore, for $u \equiv^X_{m,n} v$ to hold, we require $u \cdot X^L_a \equiv^Y_{m-1,n-1} v \cdot X^L_a$ and $u \cdot X^R_a \equiv^X_{m,n-1} v \cdot X^R_a$ for all a in the common alphabet of u and v. The former states that, after an X_a factorization, the left parts of this factorization in both words have to admit the same factorization sequences where the number of moves as well as the direction changes has decreased by one. We loose one direction change because we factorize at the first a to the right of the words' beginnings but take the factors to the left. On the other hand, if we take the factors to the right, we only lose one move but no change in direction; this is stated in the latter requirement. Additionally, we can also change the starting point of our factorization (which, normally, is the beginning of the words for $\equiv^X_{m,n}$); for this, we loose one move and one change in direction. Therefore, we also require $u \equiv^Y_{m-1,n-1} v$ for $u \equiv^X_{m,n} v$ to hold

Symmetrically, we define $u \equiv^Y_{m,n} v$ if and only if we have $\mathrm{alph}(u) = \mathrm{alph}(v)$, $u \equiv^X_{m-1,n-1} v$ and $u \cdot Y^L_a \equiv^Y_{m,n-1} v \cdot Y^L_a$ as well as $u \cdot Y^R_a \equiv^X_{m-1,n-1} v \cdot Y^R_a$ for all $a \in \mathrm{alph}(u)$. Additionally, we define $\equiv^R_{m,n}$ as the intersection for $\equiv^X_{m,n}$ and $\equiv^Y_{m,n}$ for all $m, n \in \mathbb{N}$.

For $u \equiv^{\mathrm{WI}}_{m,n} v$ with $m, n \in \mathbb{N}$ to hold, we require $\mathrm{alph}(u) = \mathrm{alph}(v)$ and, for all $a \in \mathrm{alph}(u)$, $u \cdot X^L_a \equiv^{\mathrm{WI}}_{m-1,n-1} v \cdot X^L_a$, $u \cdot X^R_a \equiv^{\mathrm{WI}}_{m,n-1} v \cdot X^R_a$, $u \cdot Y^L_a \equiv^{\mathrm{WI}}_{m,n-1} v \cdot Y^L_a$ and $u \cdot Y^R_a \equiv^{\mathrm{WI}}_{m-1,n-1} v \cdot Y^R_a$, as well as that $u \cdot C_{a,b}$ and $v \cdot C_{a,b}$ are either both undefined or both defined and $u \cdot C_{a,b} \equiv^{\mathrm{WI}}_{m-1,n-1} v \cdot C_{a,b}$ holds. All of these requirements except for the last one are analogous to the cases for $\equiv^X_{m,n}$ and $\equiv^Y_{m,n}$. The last assertion states that the first a is to the right of the last b in u if and only if it is so in v and that, in this case, we can continue to factorize in the middle part between b and a with one less move and one less direction change.[5]

By simple inductions, one can see that the relations are congruences of finite index over Σ^*. Also note that $u \equiv^Z_{m,n} v$ implies $u \equiv^Z_{m,k} v$ and, if $m > 0$, also $u \equiv^Z_{m-1,k} v$ for all $k \leq n$ and $Z \in \{X, Y, R, \mathrm{WI}\}$.

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[4] The presented relations could also be defined by (condensed) *rankers* (as it is done in [11] and [12]). Rankers were introduced by Weis and Immerman [27] who reused the *turtle programs* by Schwentick, Thérien and Vollmer [21]. Another concept related to condensed rankers is the *unambiguous interval temporal logic* by Lodaya, Pandya and Shah [13].

[5] The definitions of these congruences in formulas can also be found in the appendix.

The Trotter-Weil Hierarchy. Using these relations, we can define the *Trotter-Weil Hierarchy.* As the name implies, this hierarchy was first studied by Trotter and Weil [25], who obtained it by taking a different approach. For more information on the equivalence of the two definitions see also [12], [7] and [5, Corollary 4.3][6].

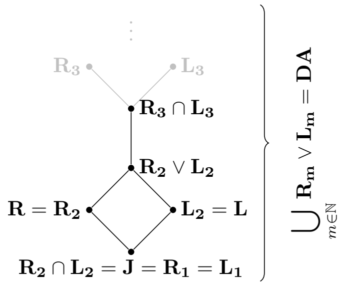

Fig. 2. Trotter-Weil Hierarchy

The Trotter-Weil Hierarchy consists of *corners*, *join levels* and *intersection levels*. The corners of the layer $m \in \mathbb{N}$ are the varieties $\mathbf{R_m}$ and $\mathbf{L_m}$. A monoid M is in $\mathbf{R_m}$ if and only if $M \prec \Sigma^*/\equiv_{m,n}^{X}$ for an $n \in \mathbb{N}_0$ and it is in $\mathbf{L_m}$ if and only if $M \prec \Sigma^*/\equiv_{m,n}^{Y}$ for an $n \in \mathbb{N}_0$. The corresponding join level is $\mathbf{R_m} \vee \mathbf{L_m}$ and the corresponding intersection level is $\mathbf{R_m} \cap \mathbf{L_m}$. A monoid M is in $\mathbf{R_m} \vee \mathbf{L_m}$ if and only if $M \prec \Sigma^*/\equiv_{m,n}^{R}$ for an $n \in \mathbb{N}$ and it is in $\mathbf{R_m} \cap \mathbf{L_m}$ if and only if $M \prec \Sigma^*/\equiv_{m,n}^{\mathrm{WI}}$ for an $n \in \mathbb{N}$.[7]

The term "hierarchy" is justified by the following inclusions: we have $\mathbf{R_m} \cap \mathbf{L_m} \subseteq \mathbf{R_m}, \mathbf{L_m} \subseteq \mathbf{R_m} \vee \mathbf{L_m}$ and $\mathbf{R_m} \vee \mathbf{L_m} \subseteq \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$. The Trotter-Weil Hierarchy contains some well known varieties: we have $\mathbf{R_1} = \mathbf{L_1} = \mathbf{J}$, $\mathbf{R_2} = \mathbf{R}$ and $\mathbf{L_2} = \mathbf{L}$ (for the last two, see [18]).[8]

By taking the union of all varieties in the hierarchy, one gets the variety **DA** [10], which is usually defined as the set of monoids whose regular $\mathcal{D}$-classes form aperiodic semigroups[9]. Though we state this as a fact here, it can also be seen as the definition of **DA** for this paper. These considerations yield the graphic representation given in Figure 2. We also note that the intersection levels corresponds to the quantifier alternation hierarchy of first-order logic with at most two variables.

3 Relations and Equations

Order Types. A linearly ordered set $(P, \leq_P)$ consists of a (possibly infinite) set P and a linear ordering relation $\leq_P$ of P, i. e. a reflexive, anti-symmetric, transitive and total binary relation $\leq_P \subseteq P \times P$. To simplify notation we define two special objects $-\infty$ and $+\infty$. The former is always smaller with regard to $\leq_P$ than any element in P while the latter is always larger. We call two linearly ordered sets $(P, \leq_P)$ and $(Q, \leq_Q)$ *isomorphic* if there is an order-preserving bijection $\varphi : P \to Q$. Isomorphism between linearly order sets is an equivalence relation; its classes are called (linear) *order types.*

[6] as well as the appendix
[7] This follows from the equivalence shown in the appendix.
[8] These assertions are straightforward if one uses the alternative definition given in the appendix.
[9] For finite monoids, $\mathcal{D}$-classes coincide with $\mathcal{J}$-classes; a $\mathcal{D}$-class is called *regular* if it contains an idempotent. A semigroup is called *aperiodic* (or *group-free*) if it has no divisor which is a nontrivial group.

The *sum* of two linearly ordered sets $(P, \leq_P)$ and $(Q, \leq_Q)$ is $(P \uplus Q, \leq_{P+Q})$ where $P \uplus Q$ is the disjoint union of P and Q and $\leq_{P+Q}$ orders all elements of P to be smaller than those of Q while it behaves as $\leq_P$ and $\leq_Q$ on elements from their respective sets. Similarly, the *product* of $(P, \leq_P)$ and $(Q, \leq_Q)$ is $(P \times Q, \leq_{P*Q})$ where $(p, q) \leq_{P*Q} (\tilde{p}, \tilde{q})$ holds if and only if either $q \leq_Q \tilde{q}$ and $q \neq \tilde{q}$ or $q = \tilde{q}$ and $p \leq_P \tilde{p}$ holds. Sum and product of linearly ordered sets are compatible with taking the order type. This allows for writing $\mu + \nu$ and $\mu * \nu$ for order types μ and ν.

We re-use $n \in \mathbb{N}_0$ to denote the order type of $(\{1, 2, \ldots, n\}, \leq)$. One should note that this use of natural numbers to denote order types does not result in contradictions with sums and products: the usual calculation rules apply. Besides finite linear order types, we need ω, the order type of $(\mathbb{N}, \leq)$, and its dual ω^* the order type of $(-\mathbb{N}, \leq)$. Another important order type in the scope of this paper is $\omega + \omega^*$, whose underlying set is $\mathbb{N} \uplus (-\mathbb{N})$. Note that, here, natural numbers and the (strictly) negative numbers are ordered as $1, 2, 3, \ldots, \; \ldots, -3, -2, -1$; therefore, in this order type, we have for example $-1 \geq_{\omega+\omega^*} 1$.

Generalized Words. As already mentioned, any finite word $w = a_1 a_2 \ldots a_n$ of length $n \in \mathbb{N}_0$ with $a_i \in \Sigma$ can be seen as a function which maps a *position* $i \in \mathrm{dom}(w)$ to the corresponding letter a_i (or, possibly, the empty map). By relaxing the requirement of $\mathrm{dom}(w)$ to be finite, one obtains the notion of *generalized words*: a (generalized) word w over the alphabet Σ of order type μ is a function $w : \mathrm{dom}(w) \to \Sigma$, where $\mathrm{dom}(w)$ is a linearly ordered set in μ. For $\mathrm{dom}(w)$, we usually choose $(\mathbb{N}, \leq)$, $(-\mathbb{N}, \leq)$ and $(\mathbb{N} \uplus (-\mathbb{N}), \leq_{\omega+\omega^*})$ as representative of ω, ω^* and $\omega + \omega^*$, respectively. The order type of a finite word of length n is n.

Like finite words, generalized words can be concatenated, i.e. we write u to the left of v and obtain uv. In that case, the order type of uv is the sum of the order types of u and v. Besides, concatenation, we can also take powers of generalized words. Let w be a generalized word of order type μ which belongs to $(P_\mu, \leq_\mu)$ and let ν be an arbitrary order type belonging to $(P_\nu, \leq_\nu)$. Then, w^ν is a generalized word of order type $\mu * \nu$ which determines the ordering of its letters; w maps $(p_1, p_2) \in P_\mu \times P_\nu$ to $w(p_1)$. If $\nu = n$ for some $n \in \mathbb{N}$, then $w^\nu = w^n$ is equal to the n-fold concatenation of w.

In this paper, the term *word* refers to a generalized word. If it is important for a word to be finite, it is referred to explicitly as a *finite word*. One may verify that all previous results still apply if a "word" is considered to be a generalized word instead of a finite word and that previous definitions extend naturally to generalized words. Especially, we can define $\mathrm{alph}(w)$ as the image of w and apply the $\equiv_{m,n}^{Z}$ relations also to generalized words. We also extend the notation $[\![\gamma]\!]_\mu$ to arbitrary order types μ. The result of the π-substitution now, of course, is a generalized word. Only useful for generalized words, however, is the following congruence: for $m \in \mathbb{N}_0$ and $Z \in \{X, Y, R, \mathrm{WI}\}$, define $u \equiv_m^Z v \Leftrightarrow \forall n \in \mathbb{N} : u \equiv_{m,n}^Z v$.

Word Problem for π-terms. The *word problem for π-terms* over a variety $\mathbf{V}$ is the problem to decide whether $\alpha = \beta$ holds in $\mathbf{V}$ for the input π-terms α and β.

In order to solve the word problem for π-terms over the varieties in the Trotter-Weil Hierarchy, one can use the following connection between the relations defined above and equations in these varieties, which is straightforward if one make the transition from finite to infinite words.[10] Besides its use for the word problem for π-terms, this connection is also interesting in its own right as it can be used to prove or disprove equations in any of the varieties. As the set of monoids in which an equation $\alpha = \beta$ holds is a variety, one can see the assertion for the join levels as an implication of the ones for the corners.

Theorem 1. *Let α and β be two π-terms. For every $m \in \mathbb{N}$, we have:*

$$\begin{aligned}
[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^X [\![\beta]\!]_{\omega+\omega^*} &\Leftrightarrow \alpha = \beta \textit{ holds in } \mathbf{R_m} \\
[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^Y [\![\beta]\!]_{\omega+\omega^*} &\Leftrightarrow \alpha = \beta \textit{ holds in } \mathbf{L_m} \\
[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^R [\![\beta]\!]_{\omega+\omega^*} &\Leftrightarrow \alpha = \beta \textit{ holds in } \mathbf{R_m} \vee \mathbf{L_m} \\
[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^{\mathrm{WI}} [\![\beta]\!]_{\omega+\omega^*} &\Leftrightarrow \alpha = \beta \textit{ holds in } \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}
\end{aligned}$$

Corollary 1. $\left(\forall m \in \mathbb{N} : [\![\alpha]\!]_{\omega+\omega^*} \equiv_m^R [\![\beta]\!]_{\omega+\omega^*}\right) \Leftrightarrow \alpha = \beta$ *holds in* $\mathbf{DA}$

4 Decidability

In the previous section, we saw that checking whether $\alpha = \beta$ holds in a variety of the Trotter-Weil Hierarchy boils down to checking $[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^Z [\![\beta]\!]_{\omega+\omega^*}$ (where $\equiv_m^Z$ depends on the variety in question). In this section, we give an introduction on how to do this. The presented approach works uniformly for all varieties in the Trotter-Weil Hierarchy (in particular, it also works for the intersection levels, which tend to be more complicated) and is designed to yield efficient algorithms.

The definition of the relations which need to be tested is inherently recursive. One would factorize $[\![\alpha]\!]_{\omega+\omega^*}$ and $[\![\beta]\!]_{\omega+\omega^*}$ on the first a and/or last b (for $a, b \in \Sigma$) and test the factors recursively. Therefore, the computation is based on working with factors of words of the form $[\![\gamma]\!]_{\omega+\omega^*}$ where γ is a π-term. We have already introduced the notation $w_{(l,r)}$ to denote the factor of a finite w which arises by restricting the domain of w to the open interval (l, r). This notation can easily be extended to the case of generalized words.

What happens if we consecutively factorize at a first/last a is best understood if one considers the structure of $[\![(\alpha)^\pi]\!]_{\omega+\omega^*} = [\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*} = u^{\omega+\omega^*} = w$, which is schematically represented in Figure 3.

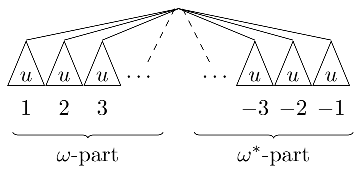

Fig. 3. Representation of $u^{\omega+\omega^*}$

Suppose u only contains a single a and we start with the whole word $w_{(-\infty,+\infty)}$. If we factorize on the first a taking the part to the

[10] The theorem's proof can be found in the appendix.

right, then we end up with the factor $w_{(X_a(w;-\infty),+\infty)}$ with $X_a(w;-\infty) = (p,1)$ where p is the single a-position in u. If we do this again, we obtain $w_{((p,2),+\infty)}$. If we now factorize on the next a but take the part to the left, then we get $w_{((p,2),(p,3))}$. Notice that the difference between 2 and 3 is 1 and that there is no way of getting a (finite) difference larger than one by factorizing on the respective first a. On the other hand, we can reach any number in $\mathbb{N}$ as long as the right position is not in the ω-part.

Notice that there is also no way of reaching $(p,-2)$ as left border without having $(q,-1)$ or $(q,-2)$ as right border for a position $q \in \mathrm{dom}(u)$. These observations (and their symmetrical duals) lead to the notion of *normalizable* pairs of positions.[11]

The choice of words indicates that normalizability of a pair (l,r) can be used to define a normalization. We omit a formal – unfortunately, quite technical – definition of this[12], but give a description of its idea. Let us refers back to the schematic representation of $[\![(\alpha)^\pi]\!]_{\omega+\omega^*} = w$ as given in Figure 3. Basically, there are three different cases for relative positions of the left border l and the right border r which describe the factor $w_{(l,r)}$:

1. l is in the ω-part and r is in the ω^*-part,
2. l and r are either both in the ω-part or both in the ω^*-part and have the same value there, or
3. l and r are either both in the ω-part or both in the ω^*-part but r has a value exactly larger by one than l.

This is ensured by the normalizability of (l,r). Now, in the first case, we can safely move l to value 1 (the first position) and r to value -1 (the last position) without changing the described factor. In the second and third case, we can move l and r to any value – as long as we retain the difference between the values – without changing the described factor. Here, we move them to the left-most values (which are $1,1$ or $1,2$). Afterwards, we go on recursively.

Unfortunately, things get a bit more complicated because l might be $-\infty$ and r might be $+\infty$. In these cases, we normalize to the left-most or right-most value without changing the factor.

For concatenation of π-terms, we have a similar situation: either l and r belong both to the left or to the right factor, in which case we can continue by normalization with respect to that, or l belongs to the left factor and r belongs to the right one. In this case we have to continue the normalization with $(l,+\infty)$ and $(-\infty,r)$ in the respective concatenation parts, as this ensures that the described factor remains unchanged.[13]

One should note that if we normalize a normalizable pair (l,r), then the resulting pair is normalizable itself. Indeed, if we normalize an already normalized pair again, we do not change any values.

[11] for which we give a formal definition in the appendix. Alongside the definition, we also give a proof that any pair (l,r) of positions is normalizable if it describes a factor which arises by multiple first/last a factorizations.

[12] It can be found in the appendix.

[13] A formal proof showing that the factor does not change can be found in the appendix.

Another observation is crucial for the proof of the decidability: after normalizing a pair (l, r) the values belonging to the $\omega + \omega^*$ parts for the two positions are all in $\{1, 2, -2, -1\}$. But: there are only finitely many such positions in any word $w = [\![\gamma]\!]_{\omega+\omega^*}$ for a π-term γ. Because the normalization preserves the described factor, this means that there are only finitely many factors which can result from a sequence of first/last a factorizations.

Plugging all these ideas and observations together yields a proof for the next theorem (note that decidability for **DA** has already been shown by Moura [16]). Here, we only give a sketch of the proof.[14]

Theorem 2. *The word problems for π-terms over $\mathbf{R_m}$, $\mathbf{L_m}$, $\mathbf{R_m} \vee \mathbf{L_m}$ and $\mathbf{R_m} \cap \mathbf{L_m}$ are decidable for any $m \in \mathbb{N}$. Moreover, the word problem for π-terms over* **DA** *is decidable.*

Proof (Sketch). The proof is structurally equivalent for all stated varieties. Though it can also be proved directly, decidability for the join levels can be seen as an implication of the decidability for the corners.

The basic idea is to construct a finite automaton for each input π-term γ. The nodes consist of the normalized position pairs and the edges are labeled by Z_a^D for variables a, $Z \in \{X, Y\}$ and $D \in \{L, R\}$. The node (l, r) has an out-going Z_a^D-edge if $w' = w_{(l,r)} \cdot Z_a^D$ is defined for $w = [\![\gamma]\!]_{\omega+\omega^*}$; its target is obtained by normalizing the pair describing w'. Except for **DA**, we additionally have to keep track of the alternations between X_a and Y_a factorizations; this can be done by taking the intersection of two automata. For the intersection levels, we also need $C_{a,b}$-edges which are defined analogously. If there is a path labeled by $Z_1 Z_2 \ldots Z_k$ in the automaton for α but not in the one for β, we know that $[\![\alpha]\!]_{\omega+\omega^*}$ is not in relation with $[\![\beta]\!]_{\omega+\omega^*}$ under the appropriate relation given by Theorem 1. Therefore, checking $\alpha = \beta$ reduces to checking the automata's symmetric difference for emptiness. $\square$

In the presented algorithm, we have to store and compute normalized pairs of positions in words of the form $[\![\gamma]\!]_{\omega+\omega^*}$ for a π-term γ. To store a single position of such a pair, one could simply store the values for the π-exponents and a position in γ. While this would be sufficient to exactly determine the position, it is impossible to do in logarithmic space. With some additional ideas, however, it is, in fact, possible to solve the problems in nondeterministic logarithmic space, which we state in the following theorem (see the technical report [9] for more details).

Theorem 3. *The word problems for π-term over $\mathbf{R_m}$, $\mathbf{L_m}$, $\mathbf{R_m} \vee \mathbf{L_m}$, $\mathbf{R_m} \cap \mathbf{L_m}$ and* **DA** *can be solved by a nondeterministic Turing machine in logarithmic space (for every $m \in \mathbb{N}$).*

While NL is quite efficient from a complexity class perspective, directly translating the algorithm to polynomial time does not result in a better running time than the algorithm for **DA** given by Moura [16]. However, with some additional

[14] For a complete proof, we refer to the appendix

tweaks, the algorithm's efficiency can be improved, which yields the following theorem [9].

Theorem 4. *The word problems for π-terms over $\mathbf{R_m}$, $\mathbf{L_m}$, $\mathbf{R_m} \vee \mathbf{L_m}$ and $\mathbf{R_m} \cap \mathbf{L_m}$ can be solved by a deterministic algorithm with running time in $\mathcal{O}(n^7 m^2)$ where n is the length of the input π-terms. Moreover, the word problem for π-terms over $\mathbf{DA}$ can be solved by a deterministic algorithms in time $\mathcal{O}(n^7)$.*

5 Separability

Two languages $L_1, L_2 \subseteq \Sigma^*$ are *separable* by a variety $\mathbf{V}$ if there is a language $S \subseteq \Sigma^*$ with $L_1 \subseteq S$ and $L_2 \cap S = \emptyset$ such that S can be recognized by a monoid $M \in \mathbf{V}$. The *separation problem* of a variety $\mathbf{V}$ is the problem to decide whether two regular input languages of finite words are separable by $\mathbf{V}$.

We are going to show the decidability of the separations problems of $\mathbf{R_m}$ for all $m \in \mathbb{N}$ as well as for $\mathbf{DA}$ using the techniques presented in this paper[15]. Note that, by symmetry, this also shows decidability for $\mathbf{L_m}$.

The general idea is as follows. If the input languages are separable, then we can find a separating language S which is recognized by a monoid in the variety in question. This, we can do by recursively enumerating all monoids and all languages in a suitable representation. For the other direction, we show that, if the input languages are inseparable, then there are π-terms α and β which witness their inseparability. Since we can also recursively enumerate these π-terms, we have decidability.

To construct suitable π-terms we need an additional combinatoric property of the $\equiv_{m,n}^X$ relation(s) (which, in a slightly different form, can also be found in [12])[16] Using that, one can prove the following lemma concerning the π-term construction[16] and plug everything together.

Lemma 1. *Let M be a monoid, $\varphi : \Sigma^* \to M$ a homomorphism and $m \in \mathbb{N}_0$. Let $(u_n, v_n)_{n \in \mathbb{N}_0}$ be an infinite sequence of word pairs $(u_n, v_n)_{n \in \mathbb{N}_0}$ with $u_n, v_n \in \Sigma^*$, $u_n \equiv_{m,n}^X v_n$, $\varphi(u_n) = m_u$ and $\varphi(v_n) = m_v$ for fixed monoid elements $m_u, m_v \in M$ and all $n \in \mathbb{N}_0$. Then, the sequence yields π-terms α and β (over Σ) such that $\varphi(\llbracket\alpha\rrbracket_{M!}) = m_u$, $\varphi(\llbracket\beta\rrbracket_{M!}) = m_v$ and $\llbracket\alpha\rrbracket_{\omega+\omega^*} \equiv_m^X \llbracket\beta\rrbracket_{\omega+\omega^*}$ hold.*

Theorem 5. *The separation problem for $\mathbf{R_m}$ and $\mathbf{L_m}$ is decidable for all $m \in \mathbb{N}$.*

Proof (idea). The full proof can be found in the appendix. The idea is to recursively enumerate all separating languages and also all the π-terms which, by the last lemma, witness inseparability. □

Since two languages are separable by $\mathbf{R_m}$ for some $m \in \mathbb{N}$ which depends only on the size of Σ [27] if they are separable by $\mathbf{DA}$, we also get decidability for $\mathbf{DA}$, which has already been shown by Place, van Rooijen and Zeitoun [19].

Corollary 2. *The separation problem for $\mathbf{DA}$ is decidable.*

[15] Decidability for $\mathbf{DA}$ is already known [19]. The proof, however, uses a fix point saturation, which is different from our approach.

[16] See the appendix.

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A More on the Trotter-Weil Hierarchy

Definition of the Congruences for the Trotter-Weil Hierarchy in Formulas.

Definition 1. *Let $m, n \in \mathbb{N}$ and let u and v be words. Define recursively:*

1. $u \equiv_{0,0}^{Z} v$, $u \equiv_{m,0}^{Z} v$ *and* $u \equiv_{0,n}^{Z} v$ *for* $Z \in \{X, Y, \mathrm{WI}\}$ *always hold.*
2. $$\begin{aligned} u \equiv_{m,n}^{X} v \;\Leftrightarrow\;& \mathrm{alph}(u) = \mathrm{alph}(v),\ u \equiv_{m-1,n-1}^{Y} v \text{ and} \\ & \forall a \in \mathrm{alph}\,(u) : u \cdot X_a^L \equiv_{m-1,n-1}^{Y} v \cdot X_a^L \text{ and} \\ & \qquad\qquad\qquad\; u \cdot X_a^R \equiv_{m,n-1}^{X} v \cdot X_a^R \\ u \equiv_{m,n}^{Y} v \;\Leftrightarrow\;& \mathrm{alph}(u) = \mathrm{alph}(v),\ u \equiv_{m-1,n-1}^{X} v \text{ and} \\ & \forall a \in \mathrm{alph}\,(u) : u \cdot Y_a^L \equiv_{m,n-1}^{Y} v \cdot Y_a^L \text{ and} \\ & \qquad\qquad\qquad\; u \cdot Y_a^R \equiv_{m-1,n-1}^{X} v \cdot Y_a^R \\ u \equiv_{m,n}^{\mathrm{WI}} v \;\Leftrightarrow\;& \mathrm{alph}(u) = \mathrm{alph}(v), \\ & \forall a \in \mathrm{alph}\,(u) : u \cdot X_a^L \equiv_{m-1,n-1}^{\mathrm{WI}} v \cdot X_a^L \text{ and} \\ & \qquad\qquad\qquad\; u \cdot X_a^R \equiv_{m,n-1}^{\mathrm{WI}} v \cdot X_a^R, \\ & \forall a \in \mathrm{alph}\,(u) : u \cdot Y_a^L \equiv_{m,n-1}^{\mathrm{WI}} v \cdot Y_a^L \text{ and} \\ & \qquad\qquad\qquad\; u \cdot Y_a^R \equiv_{m-1,n-1}^{\mathrm{WI}} v \cdot Y_a^R \text{ and} \\ & \forall a, b \in \mathrm{alph}\,(u) : u \cdot C_{a,b} \text{ and } v \cdot C_{a,b} \text{ are either both undefined} \\ & \qquad\qquad \text{or both defined and } u \cdot C_{a,b} \equiv_{m-1,n-1}^{\mathrm{WI}} v \cdot C_{a,b} \\ & \qquad\qquad \text{holds.} \end{aligned}$$

Additionally, define $u \equiv_{m,n}^{R} v \;\Leftrightarrow\; u \equiv_{m,n}^{X} v$ *and* $u \equiv_{m,n}^{Y} v$ *for all* $m, n \in \mathbb{N}_0$.

A More Formal Approach to π-Terms. As it is important for inductions over the structure of a π-term, we give an additional formal definition of π-terms: any letter $a \in \Sigma$ is a π-term (over Σ) and if α and β are π-terms (over Σ), then $\alpha\beta$ is also a π-term (over Σ); additionally, if γ is a π-term (over Σ), then so is $(\gamma)^\pi$. As a special case, the empty word ε is also a π-term. To define the substitution of the π-exponents formally, let μ be an arbitrary order type. For a π-term $\gamma \in \Sigma \cup \{\varepsilon\}$, let $[\![\gamma]\!]_\mu = \gamma$. If $\gamma = \alpha\beta$ for two π-terms α and β, let $[\![\alpha\beta]\!]_\mu = [\![\alpha]\!]_\mu [\![\beta]\!]_\mu$, and if $\gamma = (\alpha)^\pi$ for a π-term α, let $[\![(\alpha)^\pi]\!]_\mu = ([\![\alpha]\!]_\mu)^\mu$.

Green's Relations. An important tool for studying monoids are Green's Relations. Let x and y be elements of a monoid M. Define

$$\begin{aligned} x \,\mathcal{R}\, y \;&\Leftrightarrow\; xM = yM, \\ x \,\mathcal{L}\, y \;&\Leftrightarrow\; Mx = My \qquad \text{and} \\ x \,\mathcal{J}\, y \;&\Leftrightarrow\; MxM = MyM \end{aligned}$$

where $xM = \{xm \mid m \in M\}$ is the *right-ideal* of x, $Mx = \{mx \mid m \in M\}$ its *left-ideal* and $MxM = \{m_1 x m_2 \mid m_1, m_2 \in M\}$ its (two-sided) *ideal*.

By simple calculation, one can see that $x \mathrel{\mathcal{R}} y$ holds if and only if there are $z, z' \in M$ such that $xz = y$ and $yz' = x$ and, symmetrically, that $x \mathrel{\mathcal{L}} y$ holds if and only if there are $z, z' \in M$ such that $zx = y$ and $z'y = x$.

Let $\varphi : \Sigma^* \to M$ be a (monoid) homomorphism into a monoid M. The $\mathcal{R}$-*factorization* of a word w is the (unique) factorization $w = w_0 a_1 w_1 a_2 w_2 \dots a_k w_k$ with $w_0, w_1 \dots, w_k \in \Sigma^*$ and $a_1, a_2, \dots, a_k \in \Sigma$ such that on the one hand

$$\varphi(\varepsilon) \mathrel{\mathcal{R}} \varphi(w_0) \text{ and}$$
$$\varphi(w_0 a_1 w_1 a_2 w_2 \dots a_i) \mathrel{\mathcal{R}} \varphi(w_0 a_1 w_1 a_2 w_2 \dots a_i w_i)$$

hold for $i = 1, 2, \dots, k$ and on the other hand

$$\varphi(w_0 a_1 w_1 a_2 w_2 \dots a_i w_i) \mathrel{\not\mathcal{R}} \varphi(w_0 a_1 w_1 a_2 w_2 \dots a_i w_i a_{i+1})$$

holds for $i = 0, 1, \dots, k-1$. Symmetrically, the $\mathcal{L}$-*factorization* of w is the factorization $w = w_0 a_1 w_1 a_2 w_2 \dots a_k w_k$ with $w_0, w_1 \dots, w_k \in \Sigma^*$ and $a_1, a_2, \dots, a_k \in \Sigma$ such that on the one hand

$$\varphi(w_k) \mathrel{\mathcal{L}} \varphi(\varepsilon) \text{ and}$$
$$\varphi(w_{i-1} a_i w_i a_{i+1} w_{i+1} \dots a_k w_k) \mathrel{\mathcal{L}} \varphi(a_i w_i a_{i+1} w_{i+1} \dots a_k w_k)$$

hold for $i = 1, 2, \dots, k$ and on the other hand

$$\varphi(a_i w_i a_{i+1} w_{i+1} a_{i+2} w_{i+2} \dots a_k w_k) \mathrel{\not\mathcal{L}} \varphi(w_i a_{i+1} w_{i+1} a_{i+2} w_{i+2} \dots a_k w_k)$$

holds for $i = 1, 2, \dots, k$.

The Variety **DA** *Revisited.* Remember that we define **DA** as the set of all monoids whose regular $\mathcal{D}$-classes form aperiodic semigroups. But one can also characterize **DA** in terms of an equation.

Fact 1. *Let M be a monoid. Then, we have*

$$M \in \mathbf{DA} \iff (xyz)^\pi y (xyz)^\pi = (xyz)^\pi \textit{ holds in } M.$$

A proof of this fact can be found in [23].

In **DA**, getting into a new $\mathcal{R}$-class is strictly coupled to an element's alphabet, as the following lemma shows[17], where a can be seen as one of the monoids generators (i. e. a letter in its alphabet).

Lemma 2. *Let $M \in \mathbf{DA}$ be a monoid and let $s, t \in M$ such that $s \mathrel{\mathcal{R}} t$. Then*

$$s \mathrel{\mathcal{R}} sa \Rightarrow t \mathrel{\mathcal{R}} ta$$

holds for all $a \in M$.

[17] The curious reader might be interested in the fact that the lemma's assertion also holds for monoids in **DS**, the variety of monoids whose regular $\mathcal{D}$-classes form (arbitrary, but finite) semigroups.

Proof. Since we have $t \mathrel{\mathcal{R}} s \mathrel{\mathcal{R}} sa$, there are $x, y \in M$ with $s = tx$ and $t = say$. We then have

$$t = txay = t(xay)^2 = \cdots = t(xay)^{M!},$$

which yields

$$ta(xay)^{M!} = t(xay)^{M!}a(xay)^{M!} = t(xay)^{M!} = t.$$

using the equation from Fact 1. Thus, we have $ta \mathrel{\mathcal{R}} t$. □

One of the main applications of the previous lemma is the following. If we have a monoid $M \in \mathbf{DA}$, a homomorphism $\varphi : \Sigma^* \to M$ and the $\mathcal{R}$-factorization $w = w_0a_1w_1a_2w_2 \ldots a_kw_k$ of a finite word $w \in \Sigma^*$, then we know that $a_i \notin \mathrm{alph}(w_{i-1})$ for $i = 1, 2, \ldots, k$. If we had $a_i \in \mathrm{alph}(w_{i-1})$, we could factorize $w_{i-1} = ua_iv$ and would have $\varphi(w_0a_1w_1a_2w_2 \ldots a_{i-1}u) \mathrel{\mathcal{R}} \varphi(w_0a_1w_1a_2w_2 \ldots a_{i-1}ua_i)$ and, by the previous lemma, also $\varphi(w_0a_1w_1a_2w_2 \ldots a_{i-1}ua_iv) \mathrel{\mathcal{R}} \varphi(w_0a_1w_1a_2w_2 \ldots a_{i-1}ua_iv\, a_i)$, which results in a contradiction to the definition of $\mathcal{R}$-factorizations. Of course, we can apply a left-right dual of the lemma to get an analogue statement for $\mathcal{L}$-factorizations.

Mal'cev Products. Besides intersection and join, we need one more constructions for varieties: the *Mal'cev product*, which is often defined using *relational morphism*. In this paper, we use a different, yet equivalent, approach based on the congruences $\sim_K$ and $\sim_D$, see [7] or [5, Corollary 4.3]. For their definition, let x and y be elements of a monoid M and define

$$\begin{aligned} x \sim_K y \ &\Leftrightarrow\ \text{if } ex \mathrel{\mathcal{R}} e \text{ or } ey \mathrel{\mathcal{R}} e, \text{ then } ex = ey \\ \text{and } x \sim_D y \ &\Leftrightarrow\ \text{if } xe \mathrel{\mathcal{L}} e \text{ or } ye \mathrel{\mathcal{L}} e, \text{ then } xe = ye. \end{aligned}$$

Obviously, $\sim_K$ and $\sim_D$ are of finite index in any (finite) monoid M. Thus, we have that $M/{\sim_K}$ and $M/{\sim_D}$ are (finite) monoids and can define Mal'cev products of varieties. Let $\mathbf{V}$ be a variety. The varieties $\mathbf{K} Ⓜ \mathbf{V}$ and $\mathbf{D} Ⓜ \mathbf{V}$ are defined by

$$\begin{aligned} M \in \mathbf{K} Ⓜ \mathbf{V} \ &\Leftrightarrow\ M/{\sim_K} \in \mathbf{V} \text{ and} \\ M \in \mathbf{D} Ⓜ \mathbf{V} \ &\Leftrightarrow\ M/{\sim_D} \in \mathbf{V}, \end{aligned}$$

where M is a monoid. Note that, indeed, $\mathbf{K} Ⓜ \mathbf{V}$ and $\mathbf{D} Ⓜ \mathbf{V}$ are varieties for any variety $\mathbf{V}$ and that, furthermore, we have $\mathbf{V} \subseteq \mathbf{K} Ⓜ \mathbf{V}$ and $\mathbf{V} \subseteq \mathbf{D} Ⓜ \mathbf{V}$.

Alternative Definition of the Trotter-Weil-Hierarchy. Using Mal'cev products, we can define the *Trotter-Weil Hierarchy* in a different way. While this approach is yet different to the one originally taken by Trotter and Weil [25], both are equivalent [12].

In this section, we use the following definition of the Trotter-Weil Hierarchy:

$$\begin{aligned} \mathbf{R_1} &= \mathbf{L_1} = \mathbf{J}, \\ \mathbf{R_{m+1}} &= \mathbf{K} Ⓜ \mathbf{L_m} \text{ and} \\ \mathbf{L_{m+1}} &= \mathbf{D} Ⓜ \mathbf{R_m}. \end{aligned}$$

With this definition, we have the inclusions $\mathbf{R_m} \cap \mathbf{L_m} \subseteq \mathbf{R_m}, \mathbf{L_m} \subseteq \mathbf{R_m} \vee \mathbf{L_m}$ trivially. Additionally, one can show $\mathbf{R_m} \vee \mathbf{L_m} \subseteq \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$ by induction.

We repeat some facts about the Trotter-Weil Hierarchy. Firstly, we have that, in addition to $\mathbf{J}$, the varieties $\mathbf{R}$ and $\mathbf{L}$ appear in the hierarchy, as we have $\mathbf{R_2} = \mathbf{R}$ and $\mathbf{L_2} = \mathbf{L}$ [18]. The next fact is that the union of all varieties in the Trotter-Weil Hierarchy is itself a variety. As already mentioned, it is the variety $\mathbf{DA}$ (see e. g. [10]).

Fact 2.
$$\mathbf{DA} = \bigcup_{m \in \mathbb{N}} \mathbf{R_m} \vee \mathbf{L_m} = \bigcup_{m \in \mathbb{N}} \mathbf{R_m} = \bigcup_{m \in \mathbb{N}} \mathbf{L_m}$$

The variety $\mathbf{DA}$ is closely connected to two-variable first-order logic. By $\mathsf{FO}^2[<]$, denote the set of all first-order sentences over words which may only use the $<$ predicate (and equality) and no more than two variables. A language $L \subseteq \Sigma^*$ of finite words is definable by a sentence $\varphi \in \mathsf{FO}^2[<]$ if and only if its syntactic monoid is in $\mathbf{DA}$ [24], which it is if and only if it is in one of the Trotter-Weil Hierarchy's varieties. More precisely, such a language is definable in $\mathsf{FO}^2_m[<]$ (which is the subset of sentences from $\mathsf{FO}^2[<]$ that have at most m blocks of quantifiers on every path in their syntax tree) if and only if its syntactic monoid is in $\mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$ [11].

Beside the definition of the Trotter-Weil Hierarchy using Mal'cev products, one can also characterize its varieties in terms of equations. Here, however, we only need one direction of this characterization.

Lemma 3. *Define the π-terms*

$$U_1 = (sx_1)^\pi s(y_1t)^\pi \quad \text{and} \quad V_1 = (sx_1)^\pi t(y_1t)^\pi$$

over the alphabet $\Sigma_1 = \{s, t, x_1\}$. For $m \in \mathbb{N}$, let x_{m+1} and y_{m+1} be new characters not in the alphabet Σ_m and define the π-terms

$$U_{m+1} = (U_m x_{m+1})^\pi U_m (y_{m+1} U_m)^\pi \quad \text{and} \quad V_{m+1} = (U_m x_{m+1})^\pi V_m (y_{m+1} U_m)^\pi$$

over the alphabet $\Sigma_{m+1} = \Sigma \uplus \{x_{m+1}, y_{m+1}\}$.

Then we have

$$\begin{aligned} M \in \mathbf{R_1} = \mathbf{L_1} &\Leftarrow U_1 = V_1 \text{ holds in } M, \\ M \in \mathbf{R_{m+1}} &\Leftarrow (U_m x_{m+1})^\pi U_m = (U_m x_{m+1})^\pi V_m \text{ holds in } M, \\ M \in \mathbf{L_{m+1}} &\Leftarrow U_m (y_{m+1} U_m)^\pi = V_m (y_{m+1} U_m)^\pi \text{ holds in } M \text{ and} \\ M \in \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}} &\Leftarrow U_m = V_m \text{ holds in } M \end{aligned}$$

for all $m \in \mathbb{N}$.

Proof. For the corners, see [8]. For the intersection levels, suppose that $U_m = V_m$ holds in a monoid M. By the identities for the corners, we directly have $M \in \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$. $\square$

Now, we are prepared to show the equivalence to the definition using divisors of $\Sigma^*/\equiv^Z_{m,n}$. This is done in the following two theorems (see also [12] for the corners and [11] for the intersection levels). We use the notations $X^D_\Sigma = \{X^L_a, X^R_a \mid a \in \Sigma\}$, $Y^D_\Sigma = \{Y^L_a, Y^R_a \mid a \in \Sigma\}$ and some natural variations of it.

Theorem 6. *Let M be a finite monoid, $\varphi : \Sigma^* \to M$ a homomorphism and $m \in \mathbb{N}$. Then:*

- $M \in \mathbf{R_m} \;\Rightarrow\; \big(\exists n \in \mathbb{N} \forall u, v \in \Sigma^* : u \equiv^X_{m,n} v \;\Rightarrow\; \varphi(u) = \varphi(v)\big)$
- $M \in \mathbf{L_m} \;\Rightarrow\; \big(\exists n \in \mathbb{N} \forall u, v \in \Sigma^* : u \equiv^Y_{m,n} v \;\Rightarrow\; \varphi(u) = \varphi(v)\big)$
- $M \in \mathbf{R_m} \vee \mathbf{L_m} \;\Rightarrow\; \big(\exists n \in \mathbb{N} \forall u, v \in \Sigma^* : u \equiv^R_{m,n} v \;\Rightarrow\; \varphi(u) = \varphi(v)\big)$
- $M \in \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}} \;\Rightarrow\; \big(\exists n \in \mathbb{N} \forall u, v \in \Sigma^* : u \equiv^{\mathrm{WI}}_{m,n} v \;\Rightarrow\; \varphi(u) = \varphi(v)\big)$

Proof. We fix a homomorphism $\varphi : \Sigma^* \to M$ and proceed by induction over m. For $m = 1$, we have $\mathbf{R_1} = \mathbf{L_1} = \mathbf{R_1} \vee \mathbf{L_1} = \mathbf{R_2} \cap \mathbf{L_2} = \mathbf{J}$. Let $M \in \mathbf{J}$ and $n = |M|$, which is the number of $\mathcal{J}$-classes in M (and equal to the number of $\mathcal{R}$-classes and the number of $\mathcal{L}$-classes). Assume that $u \equiv^X_{1,n} v$ for two finite words $u, v \in \Sigma^*$ and let $u = u_0 a_1 u_1 a_2 u_2 \dots a_k u_k$ be the $\mathcal{R}$-factorization of u. We have $k + 1 \leq n$ and, because M is $\mathcal{R}$-trivial, $u_0 = u_1 = \dots = u_k = \varepsilon$, which allows for writing $u = a_1 a_2 \dots a_k$. By definition of $\equiv^X_{m,n}$, we have $a_1 \in \mathrm{alph}(v)$ and $u \cdot X^R_{a_1} = a_2 a_3 \dots a_k \equiv^X_{1,n-1} v \cdot X^R_{a_1}$. Therefore, we can find a_2 in $v \cdot X^R_{a_1}$ and have $u \cdot X^R_{a_1} \cdot X^R_{a_2} = a_3 a_4 \dots a_k \equiv^X_{m,n-2} v \cdot X^R_{a_1} \cdot X^R_{a_2}$. Iterating this approach yields that u is a subword of v and, by symmetry, also that v is a subword of u. Thus, u is equal to v and we have $\varphi(u) = \varphi(v)$. The argumentation for $u \equiv^Y_{1,n} v$ is symmetric using the $\mathcal{L}$-factorization, the case for $u \equiv^R_{1,n} v$ follows trivially and the case for $u \equiv^{\mathrm{WI}}_{1,n} v$ uses the same argumentation.

Now, let $M \in \mathbf{R_m}$ for an $m > 1$. This implies $M/{\sim_K} \in \mathbf{L_{m-1}}$ and there is an $n' \in \mathbb{N}$ such that $u' \equiv^Y_{m-1,n'} v' \;\Rightarrow\; \varphi(u') \sim_K \varphi(v')$ holds for all $u', v' \in \Sigma^*$. Let r be the number of $\mathcal{R}$-classes in M and let $n = n' + r$. Consider the $\mathcal{R}$-factorization $u = u_0 a_1 u_1 a_2 u_2 \dots a_k u_k$ of a finite word $u \in \Sigma^*$; note that $k + 1 \leq r$ must hold. We have

$$\begin{aligned} u_i &= u \cdot X^R_{a_1} X^R_{a_2} \dots X^R_{a_i} \text{ for } i = 0, 1, \dots, k-1 \text{ and} \\ u_k &= u \cdot X^R_{a_1} X^R_{a_2} \dots X^R_{a_k}. \end{aligned}$$

For a second finite word $v \in \Sigma^*$ with $u \equiv^X_{m,n} v$, we know that $\mathrm{alph}(u) = \mathrm{alph}(v)$. Thus, we can apply $X^L_{a_1}$ and $X^R_{a_1}$ to v an receive

$$v_0 = v \cdot X^L_{a_1} \quad \text{and} \quad v' = v \cdot X^R_{a_1}.$$

By definition of $\equiv^X_{m,n}$, we have $v_0 \equiv^X_{m-1,n-1} u_0$ and $v' \equiv^X_{m,n-1} u_1 a_2 u_2 a_3 u_3 \dots a_k u_k$. Because of $k \leq r < n$, we can apply the same argument on v' and, by iteration, get

$$\begin{aligned} v_i &= v \cdot X^R_{a_1} X^R_{a_2} \dots X^R_{a_i} \text{ for } i = 0, 1, \dots, k-1 \text{ and} \\ v_k &= v \cdot X^R_{a_1} X^R_{a_2} \dots X^R_{a_k} \end{aligned}$$

with $u_i \equiv^Y_{m-1,n-i-1} v_i$ for $i = 0, 1, \ldots, k-1$ and $u_k \equiv^X_{m,n-k} v_k$. Because of $i \leq k \leq r-1$, we have $n - i - 1 = n' + r - i - 1 \geq n' + r - (r-1) + 1 = n'$ and $u_i \equiv^Y_{m-1,n'} v_i$ for $i = 0, 1, \ldots, k-1$. For u_k and v_k, we have $u_k \equiv^Y_{m-1,n-k-1} v_k$ by the definition of the congruences and, therefore, $u_k \equiv^Y_{m-1,n'} v_k$ because of $n - k - 1 = n' + r - k - 1 \leq n' + r - (r-1) - 1 = n'$. Summing this up, we have $u_i \equiv^Y_{m-1,n'} v_i$ and, thus, $\varphi(u_i) \sim_K \varphi(v_i)$ for all $i = 0, 1, \ldots, k$.

Since we have defined u_i by the $\mathcal{R}$-factorization of u, there is an $s_i \in M$ for any $i \in \{0, 1, \ldots, k\}$ such that $\varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_i u_i) s_i = \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_i)$ holds. For these, we have

$$(\varphi(u_i)s_i)^{M!}\, \varphi(u_i)\ \mathcal{R}\ (\varphi(u_i)s_i)^{M!}$$

because of $(\varphi(u_i)s_i)^{M!}\, \varphi(u_i)s_i\, (\varphi(u_i)s_i)^{M!-1} = (\varphi(u_i)s_i)^{M!}$, which yields

$$(\varphi(u_i)s_i)^{M!}\, \varphi(u_i) = (\varphi(u_i)s_i)^{M!}\, \varphi(v_i)$$

by $\varphi(u_i) \sim_K \varphi(v_i)$. Thus, we have

$$\begin{aligned}
\varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_k u_k) &= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_k u_k)\, (s_k \varphi(u_k))^{M!} \\
&= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_k)\, (\varphi(u_k)s_k)^{M!}\, \varphi(u_k) \\
&= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_k)\, (\varphi(u_k)s_k)^{M!}\, \varphi(v_k) \\
&= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_k) \varphi(v_k) \\
&= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_{k-1} u_{k-1}) \varphi(a_k v_k) \\
&= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_{k-1})\, (\varphi(u_{k-1})s_{k-1})^{M!}\, \varphi(u_{k-1}) \varphi(a_k v_k) \\
&= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_{k-1})\, (\varphi(u_{k-1})s_{k-1})^{M!}\, \varphi(v_{k-1}) \varphi(a_k v_k) \\
&= \varphi(u_0 a_1 u_1 a_2 u_2 \ldots a_{k-2} u_{k-2}) \varphi(a_{k-1} v_{k-1} a_k v_k) \\
&= \ldots \\
&= \varphi(v_0 a_1 v_1 a_2 v_2 \ldots a_k v_k),
\end{aligned}$$

which concludes the proof for $\mathbf{R_m}$.

The proof for $\mathbf{L_m}$ is symmetrical. For $\mathbf{R_m} \vee \mathbf{L_m}$, we observe that a monoid is in the join $\mathbf{V} \vee \mathbf{W}$ of two varieties $\mathbf{V}$ and $\mathbf{W}$ if and only if it is a divisor (i.e. the homomorphic image of a submonoid) of a direct product $M_1 \times M_2$ such that $M_1 \in \mathbf{V}$ and $M_2 \in \mathbf{W}$ [3, Exercise 1.1]. Therefore, if we have a monoid $M \in \mathbf{R_m} \vee \mathbf{L_m}$, there are monoids $M_1 \in \mathbf{R_m}$ and $M_2 \in \mathbf{L_m}$ such that M is a divisor of $M_1 \times M_2$; i.e. there is a submonoid N of $M_1 \times M_2$ and a surjective monoid homomorphism $\psi : N \twoheadrightarrow M$. For every $a \in \Sigma$, we can find elements $m_{a,1} \in M_1$ and $m_{a,2} \in M_2$ such that $\varphi(a) = \psi(\pi(m_{a,1}, m_{a,2}))$, where π is the natural projection from $M_1 \times M_2$ onto N. Indeed, we can define the maps $\varphi_1 : \Sigma \to M_1$ and $\varphi_2 : \Sigma \to M_2$ by setting $\varphi_1(a) := m_{a,1}$ and $\varphi_2(a) := m_{a,2}$. These maps can be lifted into homomorphisms $\varphi_1 : \Sigma^* \to M_1$ and $\varphi_2 : \Sigma^* \to M_2$. By induction, there are n_1 and n_2 such that $u \equiv^X_{m,n_1} v$ implies $\varphi_1(u) = \varphi_2(v)$ and $u \equiv^Y_{m,n_2} v$ implies $\varphi_2(u) = \varphi_2(v)$ for any two finite words $u, v \in \Sigma^*$. By setting $n = \max\{n_1, n_2\}$, we have

$$u \equiv^R_{m,n} v \;\Rightarrow\; \varphi_1(u) = \varphi_1(v) \text{ and } \varphi_2(u) = \varphi_2(v)$$

for all $u, v \in \Sigma^*$. For all $u, v \in \Sigma^*$ with $u \equiv^R_{m,n} v$, this yields

$$\begin{aligned}
\varphi(a_1 a_2 \dots a_k) &= \varphi(a_1)\varphi(a_2)\dots\varphi(a_k) \\
&= \psi(\pi(m_{a_1,1}, m_{a_1,2}))\psi(\pi(m_{a_2,1}, m_{a_2,2}))\dots\psi(\pi(m_{a_k,1}, m_{a_k,2})) \\
&= \psi(\pi(m_{a_1,1}, m_{a_1,2})\pi(m_{a_2,1}, m_{a_2,2})\dots\pi(m_{a_k,1}, m_{a_k,2})) \\
&= \psi(\pi((m_{a_1,1}, m_{a_1,2})(m_{a_2,1}, m_{a_2,2})\dots(m_{a_k,1}, m_{a_k,2}))) \\
&= \psi\left(\pi\left(\varphi_1(u), \varphi_2(u)\right)\right) \\
&= \psi\left(\pi\left(\varphi_1(v), \varphi_2(v)\right)\right) \\
&= \varphi(b_1 b_2 \dots b_l)
\end{aligned}$$

where $u = a_1 a_2 \dots a_k$, $v = b_1 b_2 \dots b_l$ and $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l \in \Sigma$.

Finally, let $M \in \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$ with $m > 1$. Denote by 2^Σ the monoid of subsets of Σ whose binary operation is the union of sets. It is easy to see that 2^Σ is $\mathcal{J}$-trivial. Therefore, we have $M \times 2^\Sigma \in \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$. Next, we lift $\varphi : \Sigma^* \to M$ into a homomorphism $\hat{\varphi} : \Sigma^* \to M \times 2^\Sigma$ by taking the word's alphabet as the entry in the second component. If we show $u \equiv_{m,n} v \;\Rightarrow\; \hat{\varphi}(u) = \hat{\varphi}(v)$ for a suitable $n \in \mathbb{N}$, we especially have $u \equiv_{m,n} v \;\Rightarrow\; \varphi(u) = \varphi(v)$. The advantage of this approach is that we have $\hat{\varphi}(u) = \hat{\varphi}(v) \;\Rightarrow\; \mathrm{alph}(u) = \mathrm{alph}(v)$ for all $u, v \in \Sigma^*$ by the construction of $\hat{\varphi}$. Instead of continuing to write $\hat{\varphi}$, we simply substitute M by $M \times 2^\Sigma$ and φ by $\hat{\varphi}$.

We have $M/{\sim_K} \in \mathbf{L_m}$ and $M/{\sim_D} \in \mathbf{R_m}$. By $\approx$, denote the join of $\sim_K$ and $\sim_D$. Since it is a homomorphic image of both, $M/{\sim_K}$ and $M/{\sim_D}$, the monoid $M/{\approx}$ is in $\mathbf{R_m} \cap \mathbf{L_m}$ and we can apply induction, which yields an $n' \in \mathbb{N}$ such that $u \equiv^{\mathrm{WI}}_{m-1,n'} v$ implies $\varphi(u) \approx \varphi(v)$ for all finite words $u, v \in \Sigma^*$. Let c be the sum of the number of $\mathcal{R}$-classes and the number of $\mathcal{L}$-classes in M and set $n = n' + c$. Suppose we have $u \equiv^{\mathrm{WI}}_{m,n} v$ for two finite words $u, v \in \Sigma^*$. Consider the $\mathcal{R}$-factorization $u = u'_0 a_1 u'_1 a_2 u'_2 \dots a_r u'_r$ of u and the $\mathcal{L}$-factorization $v = v'_0 b_1 v'_1 b_2 v'_2 \dots b_l v'_l$ of v. Clearly, we have $r + 1 + l + 1 \leq c$. Define the positions $p^w_0 = -\infty$, $p^w_{r+1} = +\infty$ and $p^w_i = X_{a_i}(w; p_{i-1})$ for $i = 1, 2, \dots, r$ and $w = u, v$. By Lemma 2, we know that p^u_i denotes the position of a_i in the $\mathcal{R}$-factorization for $i = 1, 2, \dots, r$. Symmetrically, we can define $q^w_{l+1} = +\infty$, $q^w_0 = -\infty$ and $q_j = Y_{a_j}(w; q_{j+1})$ for $j = l, l-1, \dots, 1$ and $w = u, v$. Again, we know that q^v_j is the position of b_j in the $\mathcal{L}$-factorization of v for $j = 1, 2, \dots, l$. Additionally, we have

$$\begin{aligned}
&p^w_0 < p^w_1 < \dots < p^w_r < p^w_{r+1} \text{ and} \\
&q^w_0 < q^w_1 < \dots < q^w_l < q^w_{l+1}
\end{aligned}$$

for $w = u$ and $w = v$ by their definition. We are going to show that we have $p^u_i \,\triangledown\, q^u_j \;\Leftrightarrow\; p^v_i \,\triangledown\, q^v_j$ for $\triangledown \in \{<, =, >\}$ and all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, l$. Together, these results yield that the sequence which is obtained by ordering the

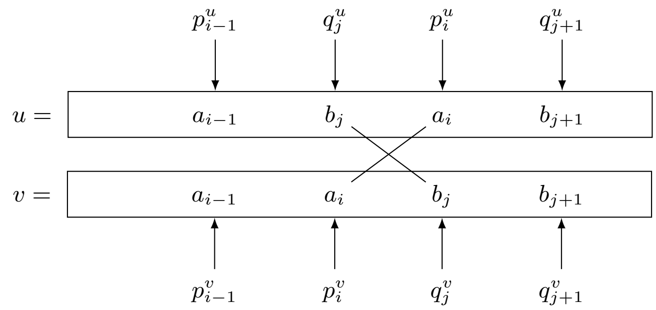

Fig. 4. Contradiction: p_i is to the right of q_j in u but to its left in v.

p_i and q_j positions in u is equal to the corresponding sequence in q_j. To prove this assertion, assume that we have $q^u_j \le p^u_i$ but $q^v_j > p^v_i$ for an $i \in \{1, 2, \dots, r\}$ and a $j \in \{1, 2, \dots, l\}$ (all other cases are symmetric or analogous). Without loss of generality, we may assume that $p^u_{i-1} < q^u_j \le p^u_i$ holds since, otherwise, we can substitute i by a smaller i for which the former holds. Note that this substitution does not violate the condition $q^v_j > p^v_i$ as p^v_i gets strictly smaller if i decreases. Equally without loss of generality, we may assume $q^u_j \le p^u_i < q^u_{j+1}$ by a dual argumentation. The situation is presented in Figure 4. We have

$$u_{(p^u_{i-1}, q^u_{j+1})} = u \cdot X^R_{a_1} X^R_{a_2} \dots X^R_{a_{i-1}} Y^L_{b_l} Y^L_{b_{l-1}} \dots Y^L_{b_{j+1}} \text{ and}$$
$$v_{(p^v_{i-1}, q^v_{j+1})} = v \cdot X^R_{a_1} X^R_{a_2} \dots X^R_{a_{i-1}} Y^L_{b_l} Y^L_{b_{l-1}} \dots Y^L_{b_{j+1}}$$

and $u_{(p^u_{i-1}, q^u_{j+1})} \equiv^{\mathrm{WI}}_{m, n-(i-1)-(l-(j+1)+1)} v_{(p^v_{i-1}, q^v_{j+1})}$, which yields $u_{(p^u_{i-1}, q^u_{j+1})} \equiv^{\mathrm{WI}}_{m,2} v_{(p^v_{i-1}, q^v_{j+1})}$ because of

$$\begin{aligned} n - (i-1) - (l - (j+1) + 1) &= n' + c - i + 1 - l + j + 1 - 1 \\ &= n' + c - i - l + j + 1 \\ &\ge n' + c - (r + l) + 1 \\ &\ge n' + c - (c - 2) + 1 = n' + 3 \\ &> 2. \end{aligned}$$

If $q^u_j = p^u_i$, we have a contradiction since $u_{(p^u_{i-1}, q^u_{j+1})} \cdot Y^L_{b_j}$ contains no a_i while $v_{(p^v_{i-1}, q^v_{j+1})} \cdot Y^L_{b_j}$ does. For $q^u_j < p^u_i$, we can apply C_{a_i, b_j} to $u_{(p^u_{i-1}, q^u_{j+1})}$ while we cannot apply it to $v_{(p^v_{i-1}, q^v_{j+1})}$ by its definition. Both situations constitute a contradiction.

We have proved that if we order the set $\{p^u_i, q^u_j \mid i = 1, 2, \dots, r, j = 1, 2, \dots, l\} = \{P^u_1, P^u_2, \dots, P^u_t\}$ (with $t \in \mathbb{N}_0$) such that

$$P^u_1 < P^u_2 < \cdots < P^u_t$$

holds, then we can set

$$P_s^v = \begin{cases} p_i^v & P_s^u = p_i^u \text{ for some } i \in \{1, 2, \dots, r\} \\ q_j^v & P_s^u = q_j^u \text{ for some } j \in \{1, 2, \dots, l\} \end{cases}$$

for $s = 1, 2, \dots, t$ and get

$$P_1^v < P_2^v < \dots < P_t^v.$$

These positions yield factorizations $u = u_0 c_1 u_1 c_2 u_2 \dots c_t u_t$ and $v = v_0 c_1 v_1 c_2 v_2 \dots c_t v_t$ such that $c_s \in \{a_i, b_j \mid i = 1, 2, \dots, r, j = 1, 2, \dots, l\}$ and P_s^w denotes the position of c_s in $w \in \{u, v\}$ for $s = 1, 2, \dots, t$. To apply induction, we are going to show $u_s \equiv_{m-1,n'}^{\mathrm{WI}} v_s$ for all $s = 1, 2, \dots, t$ next.

To simplify notation, we say "P_s is an $\mathcal{R}$-position" for any $s \in \{1, 2, \dots, t\}$ if $P_s^u = p_i^u$ for some $i \in \{1, 2, \dots, r\}$ (or, equivalently, if $P_s^v = p_i^v$ for some i) and we say "P_s is an $\mathcal{L}$-position" if $P_s^u = q_j^u$ for some $j \in \{1, 2, \dots, l\}$ (or, equivalently again, if $P_s^v = q_j^v$ for some j). Note that this definition is *not* exclusive, i. e. there can be a position which is both, an $\mathcal{R}$-position and an $\mathcal{L}$-position.

Next, we consider the corner cases of u_0/v_0 and u_t/v_t. If P_1 is an $\mathcal{R}$ position, we have $c_1 = a_1$ and

$$u_0 = u \cdot X_{a_1}^L \quad \text{as well as} \quad v_0 = v \cdot X_{a_1}^L,$$

which yields $u_0 \equiv_{m-1,n'}^{\mathrm{WI}} v_0$ by definition of $\equiv_{m,n}^{\mathrm{WI}}$ and because of $c > 0$. If P_1 is an $\mathcal{L}$-position, we have $c_1 = b_1$ and

$$\begin{aligned} u_0 &= u \cdot Y_{b_l}^L Y_{b_{l-1}}^L \dots Y_{b_1}^L \text{ as well as} \\ v_0 &= v \cdot Y_{b_l}^L Y_{b_{l-1}}^L \dots Y_{b_1}^L. \end{aligned}$$

Because $l < c$, $u_0 \equiv_{m-1,n'}^{\mathrm{WI}} v_0$ holds also in this case. For u_t and v_t, we can apply a symmetric argumentation.

Finally, we distinguish four cases for a fixed $s \in \{1, 2, \dots, t-1\}$. If P_s and P_{s+1} are both $\mathcal{R}$-positions, then we have $c_s = a_i$ and $c_{s+1} = a_{i+1}$ for some $i \in \{1, 2, \dots, r\}$ and also

$$\begin{aligned} u_s &= u \cdot X_{a_1}^R X_{a_2}^R \dots X_{a_i}^R X_{a_{i+1}}^L \text{ as well as} \\ v_s &= v \cdot X_{a_1}^R X_{a_2}^R \dots X_{a_i}^R X_{a_{i+1}}^L. \end{aligned}$$

By definition of $\equiv_{m,n}^{\mathrm{WI}}$, because of $i + 1 \leq c$, we thus have $u_s \equiv_{m-1,n'}^{\mathrm{WI}} v_s$. A symmetric argument applies if both, P_s and P_{s+1}, are $\mathcal{L}$-positions. If P_s is an $\mathcal{R}$-position but P_{s+1} is an $\mathcal{L}$-position, then $c_s = a_i$ for some $i \in \{1, 2, \dots, r\}$ and $c_{s+1} = b_j$ for some $j \in \{1, 2, \dots, l\}$, which yields

$$\begin{aligned} u_s &= u \cdot X_{a_1}^R X_{a_2}^R \dots X_{a_i}^R Y_{b_j}^L Y_{b_{l-1}}^L \dots X_{b_j}^L \text{ as well as} \\ v_s &= v \cdot X_{a_1}^R X_{a_2}^R \dots X_{a_i}^R Y_{b_j}^L Y_{b_{l-1}}^L \dots X_{b_j}^L. \end{aligned}$$

Therefore, we have $u_s \equiv^{\mathrm{WI}}_{m-1,n'} v_s$ because of the definition of $\equiv^{\mathrm{WI}}_{m,n}$ and $n - i - (l - j + 1) = n' + c - (i + 1 + l) + j \geq n' + c - c + 0 = n'$. The fourth case is the most interesting: if P_s is an $\mathcal{L}$-but not an $\mathcal{R}$-position while P_{s+1} is an $\mathcal{R}$ but not an $\mathcal{L}$-position, then $c_s = b_j$ for some $j \in \{1, 2, \ldots, l\}$ and $c_{s+1} = a_i$ for some $i \in \{1, 2, \ldots, r\}$. Additionally, we have $p^w_{i-1} < P^w_s = q^w_j < P^w_{s+1} = p^w_i < q_{j+1}$ for $w = u$ and for $w = v$. We define

$$\tilde{u} = u \cdot X^R_{a_1} X^R_{a_2} \ldots X^R_{a_{i-1}} Y^L_{b_l} Y^L_{b_{l-1}} \ldots Y^L_{b_{j+1}} \text{ as well as}$$
$$\tilde{v} = v \cdot X^R_{a_1} X^R_{a_2} \ldots X^R_{a_{i-1}} Y^L_{b_l} Y^L_{b_{l-1}} \ldots Y^L_{b_{j+1}}$$

(we consider the X-blocks as empty – meaning that we do not factorize – if $i = 1$ and the Y-blocks as empty if $j = l$). We have $\tilde{u} \equiv^{\mathrm{WI}}_{m,n-(i-1)-(l-j)} \tilde{v}$. Because of $n - (i - 1) - (l - j) = n' + c - (i + l) + j + 1 \geq n' + c - (r + l) + 1 \geq n' + 1$, $u_s = \tilde{u} \cdot C_{a_i,b_j}$, $v_s = \tilde{v} \cdot C_{a_i,b_j}$ and the definition of $\equiv^{\mathrm{WI}}_{m,n}$, we have $u_s \equiv^{\mathrm{WI}}_{m-1,n'} v_s$.

We have shown $u_s \equiv_{m-1,n'} v_s$ for all $s = 1, 2, \ldots, t$ and, by induction, therefore, know that $\varphi(u_s) \approx \varphi(v_s)$, i.e. for a fixed $s \in \{1, 2, \ldots, t\}$, there are $w_1, w_2, \ldots, w_k \in \Sigma^*$ such that

$$\varphi(u_s) = \varphi(w_1) \sim_K \varphi(w_2) \sim_D \cdots \sim_K \varphi(w_{k-1}) \sim_D \varphi(w_k) = \varphi(v_s)$$

holds.

Remember that we substituted M by $M \times 2^\Sigma$ so that we can assume $\varphi(u) = \varphi(v) \Rightarrow \mathrm{alph}(u) = \mathrm{alph}(v)$ for all $u, v \in \Sigma^*$. We can extend this implication: if we have $\varphi(u) \sim_K \varphi(v)$ for two $u, v \in \Sigma^*$, then we trivially have $\varphi(u)^{M!}\varphi(u) \; \mathcal{R} \; \varphi(u)^{M!}$ and, by definition of $\sim_K$ also $\varphi(u)^{M!}\varphi(u) = \varphi(u)^{M!}\varphi(v)$. Therefore, we have $\mathrm{alph}(u) = \mathrm{alph}(u) \cup \mathrm{alph}(v)$ by the implication stated above. By symmetry, we, thus, have $\mathrm{alph}(u) = \mathrm{alph}(v)$. Since we can apply a similar argumentation for $\sim_D$, we have $\varphi(u) \sim_K \varphi(v)$ or $\varphi(u) \sim_D \varphi(v) \Rightarrow \mathrm{alph}(u) = \mathrm{alph}(v)$ for all $u, v \in \Sigma^*$. This yields $\mathrm{alph}(u_s) = \mathrm{alph}(w_1) = \mathrm{alph}(w_2) = \cdots = \mathrm{alph}(w_k) = \mathrm{alph}(v_s)$.

Since the factorizations $u = u_0 c_1 u_1 c_2 u_2 \ldots c_t u_t$ and $v = v_0 c_1 v_1 c_2 v_2 \ldots c_t v_t$ are subfactorizations from the $\mathcal{R}$-factorization of u and the $\mathcal{L}$-factorization of v, there are $x_s, y_s \in M$ with

$$\varphi(u_0 c_1 u_1 c_2 u_2 \ldots c_s) = \varphi(u_0 c_1 u_1 c_2 u_2 \ldots c_s u_s) x_s \text{ and}$$
$$\varphi(c_{s+1} v_{s+1} c_{s+2} v_{s+2} \ldots c_t v_t) = y_s \varphi(v_s c_{s+1} v_{s+1} c_{s+2} v_{s+2} \ldots c_t v_t).$$

Because of $\mathrm{alph}(u_s) = \mathrm{alph}(w_i)$ for all $i \in \{1, 2, \ldots, k\}$ and by Lemma 2,

$$(\varphi(u_s)x_s)^{M!} \; \mathcal{R} \; (\varphi(u_s)x_s)^{M!} \varphi(u_s) \text{ implies } (\varphi(u_s)x_s)^{M!} \; \mathcal{R} \; (\varphi(u_s)x_s)^{M!} \varphi(w_i).$$

Similarly, we have

$$(y_s \varphi(v_s))^{M!} \; \mathcal{L} \; \varphi(w_i) \, (y_s \varphi(v_s))^{M!}$$

for all $i \in \{1, 2, \ldots, k\}$. For $w_i \sim_K w_{i+1}$, this implies

$$(\varphi(u_s)x_s)^{M!} \varphi(w_i) = (\varphi(u_s)x_s)^{M!} \varphi(w_{i+1})$$

and

$$\varphi(w_i)\left(y_s\varphi(v_s)\right)^{M!} = \varphi(w_{i+1})\left(y_s\varphi(v_s)\right)^{M!}$$

for $w_i \sim_D w_{i+1}$. In either case, we have

$$\left(\varphi(u_s)x_s\right)^{M!}\varphi(w_i)\left(y_s\varphi(v_s)\right)^{M!} = \left(\varphi(u_s)x_s\right)^{M!}\varphi(w_{i+1})\left(y_s\varphi(v_s)\right)^{M!},$$

which yields for any $i \in \{1, 2, \ldots, k-1\}$:

$$\begin{aligned}
&\varphi(u_0c_1u_1c_2u_2 \ldots c_sw_ic_{s+1}v_{s+1}c_{s+2}v_{s+2} \ldots c_tv_t) \\
&= \varphi(u_0c_1u_1c_2u_2 \ldots c_s)\left(\varphi(u_s)x_s\right)^{M!}\varphi(w_i)\left(y_s\varphi(v_s)\right)^{M!}\varphi(c_{s+1}v_{s+1}c_{s+2}v_{s+2} \ldots c_tv_t) \\
&= \varphi(u_0c_1u_1c_2u_2 \ldots c_s)\left(\varphi(u_s)x_s\right)^{M!}\varphi(w_{i+1})\left(y_s\varphi(v_s)\right)^{M!}\varphi(c_{s+1}v_{s+1}c_{s+2}v_{s+2} \ldots c_tv_t) \\
&= \varphi(u_0c_1u_1c_2u_2 \ldots c_sw_{i+1}c_{s+1}v_{s+1}c_{s+2}v_{s+2} \ldots c_tv_t)
\end{aligned}$$

So, we can substitute w_i by w_{i+1} and, therefore, also u_i by v_i, i. e. we have

$$\begin{aligned}
&\varphi(u_0c_1u_1c_2u_2 \ldots c_su_sc_{s+1}v_{s+1}c_{s+2}v_{s+2} \ldots c_tu_t) \\
&= \varphi(u_0c_1u_1c_2u_2 \ldots c_sv_sc_{s+1}v_{s+1}c_{s+2}v_{s+2} \ldots c_tv_t).
\end{aligned}$$

Consecutively applying the former equation for $s = t$, then for $s = t - 1$ and so on yields

$$\begin{aligned}
\varphi(u) &= \varphi(u_0c_1u_1c_2u_2 \ldots c_{t-1}u_{t-1}c_tu_t) \\
&= \varphi(u_0c_1u_1c_2u_2 \ldots c_{t-1}u_{t-1}c_tv_t) \\
&= \varphi(u_0c_1u_1c_2u_2 \ldots c_{t-1}v_{t-1}c_tv_t) \\
&\ \vdots \\
&= \varphi(v_0c_1v_1c_2v_2 \ldots c_{t-1}v_{t-1}c_tv_t) \\
&= \varphi(v),
\end{aligned}$$

which concludes the proof. □

It remains to show the other direction, stated in the next theorem.

Theorem 7. *Let $m, n \in \mathbb{N}$. Then:*

- $\Sigma^*/{\equiv^X_{m,n}} \in \mathbf{R_m}$
- $\Sigma^*/{\equiv^Y_{m,n}} \in \mathbf{L_m}$
- $\Sigma^*/{\equiv^R_{m,n}} \in \mathbf{R_m} \vee \mathbf{L_m}$
- $\Sigma^*/{\equiv^{\mathrm{WI}}_{m,n}} \in \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$

Proof (Sketch of Proof). To prove the theorem's assertion one needs to show that the equations from Lemma 3 hold in the respective monoid. To do this, it is worthwhile to make an observation: choose $m, n \in \mathbb{N}$ and $Z \in \{X, Y, R, \mathrm{WI}\}$ arbitrarily and let $M = \Sigma^*/{\equiv^Z_{m,n}}$. The observation is that an equation $\alpha = \beta$ holds in M if and only if $\sigma\left([\![\alpha]\!]_{n \cdot M!}\right) \equiv^Z_{m,n} \sigma\left([\![\beta]\!]_{n \cdot M!}\right)$ holds for all assignments $\sigma : \Gamma \to \Sigma^*$ where Γ is the alphabet of α and β (i. e. the set of variables appearing in α and β).

Proving the equations from Lemma 3 now boils down to an outer induction over m and an inner induction over n. We will only give a sketch of this induction.

For $\mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$, it suffices to show that $\sigma\left(\llbracket U_m \rrbracket_{n \cdot M!}\right) \equiv^{\mathrm{WI}}_{m,n} \sigma\left(\llbracket V_m \rrbracket_{n \cdot M!}\right)$ holds for all assignments $\sigma : \Sigma_m \to \Sigma^*$. Indeed, we can show this assertion for $\equiv^{Z}_{m,n}$ with $Z \in \{X, Y, R, \mathrm{WI}\}$ arbitrarily. For $m = 1$, we have

$$U_1 = (sx_1)^\pi s (y_1 t)^\pi \quad \text{and} \quad V_1 = (sx_1)^\pi t (y_1 t)^\pi.$$

Let $u = \sigma\left(\llbracket U_1 \rrbracket_{n \cdot M!}\right)$ and $v = \sigma\left(\llbracket V_1 \rrbracket_{n \cdot M!}\right)$. First, assume $Z = X$. We are only interested in at most n consecutive simultaneous X^R_Σ factorizations of u and v because, as soon as we apply at least one X^L_Σ factorization, we know that $\equiv^X_{0,n}$ holds. As long as we apply only factorizations X^R_a with $a \in \mathrm{alph}(\sigma(sx_1))$, the factorization position stays in the $(sx_1)^\pi$ part of u and v. Since the number of remaining factorizations decreases, the right parts will eventually be in relation under $\equiv^X_{m,0}$. If there is at least one X^R_a factorization in the sequence where a is in $\mathrm{alph}(\sigma(y_1 t)) \setminus \mathrm{alph}(\sigma(sx_1))$, the right side of u belongs to the $(y_1 t)^\pi$ part and the right side of v belongs to the $t(y_1 t)^\pi$ part; but in both words there are still at least $n - 1$ instance of $\sigma(y_1 t)$, which implies that the right sides are equal under $\equiv^X_{m,n-1}$. For $Z = Y$, the argumentation is symmetric, which also handles the $Z = R$ case. The additional $C_{a,b}$ of $Z = \mathrm{WI}$ needs no special handling since it decreases the first index of $\equiv^{\mathrm{WI}}_{m,n}$ to $m - 1 = 0$ anyway.

To conclude the induction, we show $\sigma\left(\llbracket U_{m+1} \rrbracket_{n \cdot M!}\right) \equiv^Z_{m+1,n} \sigma\left(\llbracket V_{m+1} \rrbracket_{n \cdot M!}\right)$ next. Since, by induction, $\sigma\left(\llbracket U_m \rrbracket_{n \cdot M!}\right) \equiv^{Z'}_{m,n} \sigma\left(\llbracket V_m \rrbracket_{n \cdot M!}\right)$ holds for all $Z' \in \{X, Y, R, \mathrm{WI}\}$ and since $\equiv^{Z'}_{m,n}$ is a congruence, we have $\sigma\left(\llbracket U_{m+1} \rrbracket_{n \cdot M!}\right) \equiv^{Z'}_{m,n} \sigma\left(\llbracket V_{m+1} \rrbracket_{n \cdot M!}\right)$ for all $Z' \in \{X, Y, R, \mathrm{WI}\}$. Therefore, we do not need to consider factorizations of the form X^L_a, Y^R_a or $C_{a,b}$ any further. Neither do we need to consider Y^L_a factorizations due to symmetry. If we apply a sequence of X^R_Σ factorization, then two situations can emerge: first, all of the factorization positions can belong to the $(U_m x_{m+1})^\pi$ part. In that case, we are done since that part is identical in U_{m+1} and in V_{m+1} and since in the end the number of remaining possible factorizations is 0. In the second case, there is a factorization position which belongs to the $(y_{m+1} U_m)^\pi$ part. In this case, we are done as well, as the remaining right side of the factorization is identical in U_{m+1} and in V_{m+1}. Note, that no factorization position can belong to the U_m or V_m part in the middle: this is the case because $x_{m+1}, y_{m+1} \notin \mathrm{alph}\left(\sigma\left(\llbracket U_m \rrbracket_{n \cdot M!}\right)\right)$ and because any other letter from Σ_m appears at least n times in $\left(\llbracket (U_m x_{m+1})^\pi \rrbracket_{n \cdot M!}\right)$ and in $\left(\llbracket (y_{m+1} U_m)^\pi \rrbracket_{n \cdot M!}\right)$. This establishes $\sigma\left(\llbracket U_m \rrbracket_{n \cdot M!}\right) \equiv^Z_{m,n} \sigma\left(\llbracket V_m \rrbracket_{n \cdot M!}\right)$ for all $m, n \in \mathbb{N}$ and $Z \in \{X, Y, R, \mathrm{WI}\}$ and, thus, $M/{\equiv^{\mathrm{WI}}_{m,n}} \in \mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$.

Showing

$$\sigma\left(\llbracket (U_m x_{m+1})^\pi \, U_m \rrbracket_{n \cdot M!}\right) \equiv^X_{m+1,n} \sigma\left(\llbracket (U_m x_{m+1})^\pi \, V_m \rrbracket_{n \cdot M!}\right)$$

and

$$\sigma\left(\llbracket U_m \, (U_m y_{m+1})^\pi \rrbracket_{n \cdot M!}\right) \equiv^Y_{m+1,n} \sigma\left(V_m \llbracket (U_m y_{m+1})^\pi \rrbracket_{n \cdot M!}\right)$$

can be done using similar argumentation, which proves $M/{\equiv^X_{m,n}} \in \mathbf{R_m}$ and $M/{\equiv^Y_{m,n}} \in \mathbf{L_m}$.

To prove that $M/{\equiv^R_{m,n}}$ is in $\mathbf{R_m} \vee \mathbf{L_m}$, one can recycle an observation from the proof of Theorem 6: a monoid is in the join $\mathbf{V} \vee \mathbf{W}$ of two varieties $\mathbf{V}$ and $\mathbf{W}$ if and only if it is a divisor of a direct product $M_1 \times M_2$ such that $M_1 \in \mathbf{V}$ and $M_2 \in \mathbf{W}$. Indeed, for any two congruences $\mathcal{C}_1$ and $\mathcal{C}_2$ over a monoid N, $N/\left(\mathcal{C}_1 \cap \mathcal{C}_2\right)$ is a divisor of $N/\mathcal{C}_1 \times N/\mathcal{C}_2$ (as can be shown easily). Therefore, $M/{\equiv^R_{m,n}}$ is a divisor of the direct product of $M/{\equiv^X_{m,n}} \in \mathbf{R_m}$ and $M/{\equiv^Y_{m,n}} \in \mathbf{L_m}$. $\square$

Proof for Theorem 1

To prove Theorem 1, we need the following technical lemmas.

Lemma 4. *Let $m \in \mathbb{N}_0$, $Z \in \{X, Y, R, \mathrm{WI}\}$ and let u and v be words over Σ. Then:*

$$u \equiv^Z_{m,n} v \;\Rightarrow\; \forall 0 \le k \le n : u^k \equiv^Z_{m,k} v^{\omega+\omega^*}$$

Proof. The case $m = 0$ is trivial. Therefore, let $m > 0$ and continue by induction over k. Again, the case $k = 0$ is trivial. To complete the induction, it remains to show that $uu^k \equiv^Z_{m,k+1} v^{\omega+\omega^*}$ holds for $k < n$. Obviously, $\mathrm{alph}(u^{k+1}) = \mathrm{alph}\left(v^{\omega+\omega^*}\right)$ is satisfied by assumption. Now assume $Z = X$. The assumption $u \equiv^X_{m,n} v$ implies $u \equiv^Y_{m-1,n-1} v$. By induction on m, this yields $u^k \equiv^Y_{m-1,k} v^{\omega+\omega^*}$ and $u \equiv^Y_{m-1,k} v$. Because $\equiv^Y_{m-1,k}$ is a congruence, this shows $uu^k \equiv^Y_{m-1,k} v^{\omega+\omega^*}$. Let $a \in \mathrm{alph}(u) = \mathrm{alph}(v)$. If factorization on the first a in u and v yields $u = u_0 a u_1$ and $v = v_0 a v_1$, then such a factorization on uu^k and $v^{\omega+\omega^*}$ yields $uu^k = u_0 a u_1 u^k$ and $v^{\omega+\omega^*} = vv^{\omega+\omega^*} = v_0 a v_1 v^{\omega+\omega^*}$. The assumption $u \equiv^X_{m,n} v$ implies $u_0 \equiv^Y_{m-1,k} v_0$ and $u_1 \equiv^X_{m,k} v_1$. The latter yields $u_1 u^k \equiv^X_{m,k} v_1 v^{\omega+\omega^*}$ because $\equiv^X_{m,k}$ is a congruence and $u^k \equiv^X_{m,k} v^{\omega+\omega^*}$ holds by induction on k.

The case for $Z = Y$ is symmetric and the case for $Z = R$ follows directly. Finally, for $Z = \mathrm{WI}$ the argumentation is analogous because there clearly are no letters $a, b \in \Sigma$ which yield a factorization $v^{\omega+\omega^*} = v_0 b v_1 a v_2$ with $a \notin \mathrm{alph}(v_0 b v_1)$ and $b \notin \mathrm{alph}(v_1 a v_2)$. $\square$

Lemma 5. *Let $m, n \in \mathbb{N}_0$, $Z \in \{X, Y, R, \mathrm{WI}\}$ and let γ be a π-term. Then*

$$[\![\gamma]\!]_k \equiv^Z_{m,n} [\![\gamma]\!]_{\omega+\omega^*}$$

holds for all $k \in \mathbb{N}_0$ with $k \ge n$.

Proof. The cases for $m = 0$ or $n = 0$ are trivial. Thus, assume $m > 0$ and $n > 0$. If $\gamma = \varepsilon$ or $\gamma = a$ for an $a \in \Sigma$, then $[\![\gamma]\!]_k = \gamma = [\![\gamma]\!]_{\omega+\omega^*}$. If $\gamma = \alpha\beta$ for two π-terms α and β, then by induction $[\![\alpha]\!]_k \equiv^Z_{m,n} [\![\alpha]\!]_{\omega+\omega^*}$ and $[\![\beta]\!]_k \equiv^Z_{m,n} [\![\beta]\!]_{\omega+\omega^*}$ hold. As $\equiv^Z_{m,n}$ is a congruence, this implies $[\![\gamma]\!]_k \equiv^Z_{m,n} [\![\gamma]\!]_{\omega+\omega^*}$.

Finally, let $\gamma = (\alpha)^\pi$ for a π-term α. It remains to show that $[\![\alpha]\!]_k^k \equiv^Z_{m,n} [\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*}$. Clearly, the alphabetic condition is satisfied and, by induction, $[\![\alpha]\!]_k \equiv^Z_{m,n} [\![\alpha]\!]_{\omega+\omega^*}$ holds. For $Z = X$, let $a \in \mathrm{alph}(\alpha)$. Then there are factorizations $[\![\alpha]\!]_k = u_0 a u_1$ and $[\![\alpha]\!]_{\omega+\omega^*} = v_0 a v_1$ with $a \notin \mathrm{alph}(u_0) \cup \mathrm{alph}(v_0)$. This yields the factorizations $[\![\alpha]\!]_k^k = u_0 a u_1 [\![\alpha]\!]_k^{k-1}$ and $[\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*} = v_0 a v_1 [\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*}$. By induction, we have $u_0 a u_1 = [\![\alpha]\!]_k \equiv_{m,n}^X [\![\alpha]\!]_{\omega+\omega^*} = v_0 a v_1$. This yields $u_0 \equiv_{m-1,n-1}^Y v_0$ and $u_1 \equiv_{m,n-1}^X v_1$. Therefore, if we show $[\![\alpha]\!]_k^{k-1} \equiv_{m,n-1}^X [\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*}$, then we are done with this case. For that, write $k-1 = k' + n - 1$ for a $k' \in \mathbb{N}_0$ and then

$$[\![\alpha]\!]_k^{k-1} = [\![\alpha]\!]_k^{k'} [\![\alpha]\!]_k^{n-1} \text{ and}$$
$$[\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*} = [\![\alpha]\!]_{\omega+\omega^*}^{k'} [\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*}.$$

Because $\equiv_{m,n-1}^X$ is a congruence and by Lemma 4, this concludes the proof for $Z = X$. The case for $Z = Y$ is symmetric, which also shows the case for $Z = R$. For $Z = \mathrm{WI}$, the only remaining case is that in which there are $a, b \in \Sigma$ which yield factorizations $[\![\alpha]\!]_k^k = u_0 b u_1 a u_2$ and $[\![\alpha]\!]_{\omega+\omega^*}^{\omega+\omega^*} = v_0 b v_1 a v_2$ with $a \notin \mathrm{alph}(u_0 b u_1) \cup \mathrm{alph}(v_0 b v_1)$ and $b \notin \mathrm{alph}(u_1 a u_2) \cup \mathrm{alph}(v_1 a v_2)$. Clearly, this can only happen for $k = 1 \geq n > 0$, which is equivalent to $n = 1$. Therefore, one may apply Lemma 4. $\square$

Proof (for Theorem 1). The proof is structurally identical for all stated varieties. Therefore, we limit our discussion to $\mathbf{R_m}$.

First, let $[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^X [\![\beta]\!]_{\omega+\omega^*}$. Choose a monoid $M \in \mathbf{R_m}$ and an assignment for variables $\sigma : \Sigma^* \to M$. By Theorem 6, there is an $n \in \mathbb{N}$ such that $u \equiv_{m,n}^X v$ implies $\sigma(u) = \sigma(v)$ for any two words $u, v \in \Sigma^*$. Now, choose $c \in \mathbb{N}$ with $M! \cdot c \geq n$. Then by assumption and Lemma 5, we have

$$\Sigma^* \ni [\![\alpha]\!]_{M! \cdot c} \equiv_{m,n}^X [\![\alpha]\!]_{\omega+\omega^*} \equiv_{m,n}^X [\![\beta]\!]_{\omega+\omega^*} \equiv_{m,n}^X [\![\beta]\!]_{M! \cdot c} \in \Sigma^*$$

and, therefore, $\sigma([\![\alpha]\!]_{M!}) = \sigma([\![\alpha]\!]_{M! \cdot c}) = \sigma([\![\beta]\!]_{M! \cdot c}) = \sigma([\![\beta]\!]_{M!})$, which is equivalent to $\alpha = \beta$ holding in M.

Now, let $[\![\alpha]\!]_{\omega+\omega^*} \not\equiv_m^X [\![\beta]\!]_{\omega+\omega^*}$, which implies that there is an $n \in \mathbb{N}$ such that $[\![\alpha]\!]_{\omega+\omega^*} \not\equiv_{m,n}^X [\![\beta]\!]_{\omega+\omega^*}$. Define $M := \Sigma^* / \equiv_{m,n}^X$, which is in $\mathbf{R_m}$ according to Theorem 7, and choose $c \in \mathbb{N}$ such that $M! \cdot c \geq n$. Then, by assumption and Lemma 5, we have

$$\Sigma^* \ni [\![\alpha]\!]_{M! \cdot c} \equiv_{m,n}^X [\![\alpha]\!]_{\omega+\omega^*} \not\equiv_{m,n}^X [\![\beta]\!]_{\omega+\omega^*} \equiv_{m,n}^X [\![\beta]\!]_{M! \cdot c} \in \Sigma^*.$$

As assignment of variables $\sigma : \Sigma^* \to M$ choose the canonical projection. This yields $\sigma([\![\alpha]\!]_{M!}) = \sigma([\![\alpha]\!]_{M! \cdot c}) \neq \sigma([\![\beta]\!]_{M! \cdot c}) = \sigma([\![\beta]\!]_{M!})$, which means that $\alpha = \beta$ does *not* hold in M. $\square$

More on Decidability

We start by giving a formal definition of normalizable pairs.

Definition 2. *Let γ be a π-term and let $w = [\![\gamma]\!]_{\omega+\omega^*}$. A pair (l, r) of positions in w such that l is strictly smaller than r is called* normalizable *(with respect to γ) based on the following rules:*

- *Any pair is normalizable with respect to $\gamma = \varepsilon$ or $\gamma = a$ for an $a \in \Sigma$.*

- $(-\infty, +\infty)$ *is normalizable with respect to any* π*-term.*
- *If* $\gamma = \alpha\beta$ *for* π*-terms* α *and* β, $l \in \mathrm{dom}(\llbracket\alpha\rrbracket_{\omega+\omega^*}) \uplus \{-\infty\}$ *and* $r \in \mathrm{dom}(\llbracket\beta\rrbracket_{\omega+\omega^*}) \uplus \{+\infty\}$*, then* (l, r) *is normalizable with respect to* γ *if* $(l, +\infty)$ *is with respect to* α *and* $(-\infty, r)$ *is with respect to* β*.*
- *If* $\gamma = \alpha\beta$ *for* π*-terms* α *and* β *and* $l \in \mathrm{dom}(\llbracket\alpha\rrbracket_{\omega+\omega^*}) \uplus \{-\infty\}$ *as well as* $r \in \mathrm{dom}(\llbracket\alpha\rrbracket_{\omega+\omega^*})$ *(or* $l \in \mathrm{dom}(\llbracket\beta\rrbracket_{\omega+\omega^*})$ *as well as* $r \in \mathrm{dom}(\llbracket\beta\rrbracket_{\omega+\omega^*}) \uplus \{+\infty\}$*), then* (l, r) *is normalizable with respect to* γ *if it is with respect to* α *(or* β*, respectively).*
- *If* $\gamma = (\alpha)^\pi$ *for a* π*-term* α, $l = (l', n)$ *for* $l' \in \mathrm{dom}(\llbracket\alpha\rrbracket_{\omega+\omega^*})$ *and* $n \in \mathbb{N} \uplus -\mathbb{N}$ *and* $r = +\infty$*, then* (l, r) *is normalizable with respect to* γ *if* $(l', +\infty)$ *is with respect to* α *and* n *is in* $\mathbb{N} \uplus \{-1\}$*.*
- *If* $\gamma = (\alpha)^\pi$ *for a* π*-term* α, $l = -\infty$*, and* $r = (r', m)$ *for* $r' \in \mathrm{dom}(\llbracket\alpha\rrbracket_{\omega+\omega^*})$ *and* $m \in \mathbb{N} \uplus -\mathbb{N}$*, then* (l, r) *is normalizable with respect to* γ *if* $(-\infty, r')$ *is with respect to* α *and* m *is in* $\{1\} \uplus -\mathbb{N}$*.*
- *If* $\gamma = (\alpha)^\pi$ *for a* π*-term* α, $l = (l', n)$ *for* $l' \in \mathrm{dom}(\llbracket\alpha\rrbracket_{\omega+\omega^*})$ *and* $n \in \mathbb{N} \uplus -\mathbb{N}$ *and* $r = (r', m)$ *for* $r' \in \mathrm{dom}(\llbracket\alpha\rrbracket_{\omega+\omega^*})$ *and* $m \in \mathbb{N} \uplus -\mathbb{N}$*, then* (l, r) *is normalizable with respect to* γ *if*
  - $n \in \mathbb{N}$, $m \in -\mathbb{N}$ *and* $(l', +\infty)$ *and* $(-\infty, r')$ *are normalizable with respect to* α*,*
  - $n, m \in \mathbb{N}$ *or* $n, m \in -\mathbb{N}$ *and in both cases* $m = n$ *and* (l', r') *is normalizable with respect to* α*, or*
  - $n, m \in \mathbb{N}$ *or* $n, m \in -\mathbb{N}$ *and in both cases* $m = n + 1$ *and* $(l', +\infty)$ *and* $(-\infty, r')$ *are normalizable with respect to* α*.*

Next, we formally prove that, in fact, our observations from above hold for all positions which can be reached by consecutive factorization at the first/last a, i. e. that all these pairs are normalizable. We extend our notation: we write $(l, r) \cdot Z_a^D$ for the pair of positions $(l, r) \in (\{-\infty\} \uplus \mathrm{dom}(w)) \times (\mathrm{dom}(w) \uplus \{+\infty\})$ in w and mean the pair of positions (l', r') such that $w_{(l',r')} = w_{(l,r)} \cdot Z_a^D$ (for $Z \in \{X, Y\}$ and $D \in \{L, R\}$). We also use this notation with $C_{a,b}$.

Lemma 6. *Let* γ *be a* π*-term and let* $w = \llbracket\gamma\rrbracket_{\omega+\omega^*}$*. Additionally, let* (l, r) *be a normalizable pair of positions in* w*. Then the pairs*

$$(l, r) \cdot X_a^L, (l, r) \cdot X_a^R, (l, r) \cdot Y_a^L \text{ and } (l, r) \cdot Y_a^R$$

are normalizable with respect to γ *for any* $a \in \mathrm{alph}(w_{(l,r)})$*.*

Therefore, $(-\infty, +\infty) \cdot F_1 F_2 \ldots F_n$ *is normalizable with respect to* γ *for any* $F_1, F_2, \ldots, F_n \in \{X_a^L, X_a^R, Y_a^L, Y_a^R, C_{a,b} \mid a, b \in \Sigma\}$ *(if it is defined).*

Proof. As the cases for Y_a^L and Y_a^R are symmetrical, we only show those for X_a^L and X_a^R. Let $p = X_a(w; l)$ for an $a \in \mathrm{alph}(w_{(l,r)})$. Clearly, we have $l <_\mu p <_\mu r$, where μ is the order type of w, and we need to show that (l, p) and (p, r) are normalizable. For this, we proceed by induction on the structure of γ. The base case $\gamma = \varepsilon$ or $\gamma \in \Sigma$ is trivial.

Case 1. $\gamma = \alpha\beta$ Define $u = \llbracket\alpha\rrbracket_{\omega+\omega^*}$ and $v = \llbracket\beta\rrbracket_{\omega+\omega^*}$. For $l \in \mathrm{dom}(u) \uplus \{-\infty\}$ and $r \in \mathrm{dom}(u)$ we have $p \in \mathrm{dom}(u)$ as well. Additionally, (l, r) needs to be normalizable with respect to α by the definition of normalizability and we can apply induction. The same argument, but on β, works for $l \in \mathrm{dom}(v)$ and $r \in \mathrm{dom}(v) \uplus \{+\infty\}$. For $l \in \mathrm{dom}(u) \uplus \{-\infty\}$ and $r \in \mathrm{dom}(v) \uplus \{+\infty\}$ we know that $(l, +\infty)$ is normalizable with respect to α and $(-\infty, r)$ is with respect to β by the definition of normalizablity. If $p \in \mathrm{dom}(u)$, then $(p, +\infty) = (l, +\infty) \cdot X_a^R$ and $(l, p) = (l, +\infty) \cdot X_a^L$. Induction yields normalizability with respect to α for both and, by the definition of normalizability, we have that (p, r) and (l, p) are normalizable with respect to γ. For $p \in \mathrm{dom}(v)$, we can apply a similar argument, as then $(-\infty, p) = (-\infty, +\infty) \cdot X_a^L$ and $(p, r) = (-\infty, r) \cdot X_a^R$ are normalizable with respect to β.

Case 2. $\gamma = (\alpha)^\pi$ Define $u = [\![\alpha]\!]_{\omega+\omega^*}$ and let $p = (p', k)$. If $l = (l', n)$ for an $n \in \mathbb{N} \uplus -\mathbb{N}$ and $r = +\infty$, then, by the definition of normalizability, we have that $(l', +\infty)$ is normalizable with respect to α and $n \in \mathbb{N} \uplus \{-1\}$. There are two cases: for $k = n \in \mathbb{N} \uplus \{-1\}$ we know that $p' = X_a(u; l')$ and, by induction, that $(l', p'), (p', +\infty)$ are normalizable with respect to α. This yields the normalizability with respect to γ of (l, p) and $(p, +\infty)$. For $k = n + 1$ we know that $n \neq -1$ and, therefore, that $n, k \in \mathbb{N}$. We also have $p' = X_a(u; -\infty)$ and, thus, that $(-\infty, p')$ and $(p', +\infty)$ are normalizable with respect to α by induction. By definition, $(p, +\infty)$ and (l, p) are normalizable with respect to γ then. Note that k cannot have any other value than n or $n + 1$ since otherwise it could not be the smallest a-position to the right of l.

If $l = -\infty$ and $r = (r', m)$, then $k = 1$, and $p' = X_a(u; -\infty)$, which yields $(-\infty, p') = (-\infty, +\infty) \cdot X_a^L$ and $(p', +\infty) = (-\infty, +\infty) \cdot X_a^R$. By induction, both of these pairs are normalizable with respect to α and, by definition of the normalizability, $(-\infty, p)$ is normalizable with respect to γ. Furthermore in this case, we know that $(-\infty, r')$ is normalizable with respect to α ynd that m is in $\{1\} \uplus -\mathbb{N}$. For $m \in -\mathbb{N}$, this shows the normalizability with respect to γ of (p, r). For $m = 1$, we have $(p', r') = (-\infty, r') \cdot X_a^R$ and, by induction, its normalizability with respect to α. This yields that (p, r) is normalizable with respect to γ.

If $l = (l', n)$ and $r = (r', m)$ for $n \in \mathbb{N}$ and $m \in -\mathbb{N}$, we know that $(l', +\infty)$ and $(-\infty, r')$ are normalizable with respect to α. For $k = n \in \mathbb{N}$, we also know that $p' = X_a(u; l')$ and, therefore, that $(l', p') = (l', +\infty) \cdot X_a^L$ and $(p', \infty) = (l', +\infty) \cdot X_a^R$ are normalizable with respect to α by induction. Then, by definition, (l, p) and (p, r) are normalizable with respect to γ. For $k = n + 1 \in \mathbb{N}$ we have that $p' = X_a(u; -\infty)$ and, therefore, the normalizability with respect to α of $(-\infty, p') = (-\infty, +\infty) \cdot X_a^L$ and $(p', +\infty) = (-\infty, +\infty) \cdot X_a^R$ by induction. This yields the normalizability with respect to γ of (l, p) and (p, r).

Finally, if $l = (l', n)$ and $r = (r', m)$ for $n, m \in \mathbb{N}$ or $n, m \in -\mathbb{N}$, we know that $0 \leq m - n \leq 1$. Because p must be in between l and r, $n = m$ also implies $n = m = k$ and that p' is in between l' and r' as well as $p' = X_a(u; l')$. In that case, we have that (l', r') and, by induction, also $(l', p') = (l', r') \cdot X_a^L$ and $(p', r') = (l', r') \cdot X_a^R$ are normalizable with respect to α. This yields the normalizability with respect to γ of (l, p) and (p, r). For $m = n + 1$, we know that $(l', +\infty)$ and $(-\infty, r')$ are normalizable with respect to α. Moreover, there are only two cases: $k = n$ and $k = m$. In the former case, we have $p' = X_a(u; l')$ and the normalizability with respect to α of $(l',p') = (l',-\infty) \cdot X_a^L$ and $(p',+\infty) = (l',+\infty) \cdot X_a^R$ by induction, which yields the normalizability of (l,p) and (p,r) with respect to γ. In the latter case, we have $X_a(u;-\infty)$ and the normalizability with respect to α of $(-\infty,p') = (-\infty,+\infty) \cdot X_a^L$ and $(p',r') = (-\infty,r') \cdot X_a^R$, which yields the normalizability with respect to γ of (l,p) and (p,r).

□

The formal definition of the normalization is as follows.

Definition 3. *Let γ be a π-term, $w = [\![\gamma]\!]_{\omega+\omega^*}$ and (l,r) a normalizable pair of positions in w. The normalized pair $\overline{(l,r)}^{\gamma} = (\bar{l},\bar{r})$ with respect to γ is defined recursively:*

- *For $\gamma = \varepsilon$ or $\gamma = a \in \Sigma$ define $\bar{l} = l$ and $\bar{r} = r$.*
- *If $\gamma = \alpha\beta$ for π-terms α and β, $l \in \mathrm{dom}([\![\alpha]\!]_{\omega+\omega^*}) \uplus \{-\infty\}$ and $r \in \mathrm{dom}([\![\beta]\!]_{\omega+\omega^*}) \uplus \{+\infty\}$, then define $\bar{l}$ as the first component of $\overline{(l,+\infty)}^{\alpha}$ and $\bar{r}$ as the second component of $\overline{(-\infty,r)}^{\beta}$.*
- *If $\gamma = \alpha\beta$ for π-terms α and β and $l \in \mathrm{dom}([\![\alpha]\!]_{\omega+\omega^*}) \uplus \{-\infty\}$ as well as $r \in \mathrm{dom}([\![\alpha]\!]_{\omega+\omega^*})$ (or $l \in \mathrm{dom}([\![\beta]\!]_{\omega+\omega^*})$ as well as $r \in \mathrm{dom}([\![\beta]\!]_{\omega+\omega^*}) \uplus \{+\infty\}$), then define $(\bar{l},\bar{r}) = \overline{(l,r)}^{\alpha}$ (or $(\bar{l},\bar{r}) = \overline{(l,r)}^{\beta}$, respectively).*
- *If $\gamma = (\alpha)^\pi$ for a π-term α, then:*
  - *if $l = -\infty$, define $\bar{l} = -\infty$,*
  - *if $r = +\infty$, define $\bar{r} = +\infty$,*
  - *if $l = (l',n)$ and $r = +\infty$, define $\bar{l} = (\bar{l'},\bar{n})$ with $\bar{l'}$ given by the first component of $\overline{(l',+\infty)}^{\alpha}$ and $\bar{n}$ given by*
  $$\bar{n} = \begin{cases} 1 & \text{if } n \in \mathbb{N} \\ -1 & \text{if } n = -1, \end{cases}$$
  - *if $l = -\infty$ and $r = (r',m)$, define $\bar{r} = (\bar{r'},\bar{m})$ with $\bar{r'}$ given by the second component of $\overline{(-\infty,r')}^{\alpha}$ and $\bar{m}$ given by*
  $$\bar{m} = \begin{cases} 1 & \text{if } n = 1 \\ -1 & \text{if } n \in -\mathbb{N}, \end{cases}$$
  - *if $l = (l',n)$ and $r = (r',m)$ with $n \in \mathbb{N}$ and $m \in -\mathbb{N}$, define $\bar{l} = (\bar{l'},1)$ with $\bar{l'}$ being by the first component of $\overline{(l',+\infty)}^{\alpha}$ and define $\bar{r} = (\bar{r'},-1)$ with $\bar{r'}$ given by the second component of $\overline{(-\infty,r')}^{\alpha}$,*
  - *if $l = (l',n)$ and $r = (r',m)$ with $n = m$, define $\bar{l} = (\bar{l'},\bar{n})$ and $\bar{r} = (\bar{r'},\bar{m})$ with $(\bar{l'},\bar{r'}) = \overline{(l',r')}^{\alpha}$ and $\bar{n} = \bar{m} = 1$, and*
  - *if $l = (l',n)$ and $r = (r',m)$ with $m = n+1$, define $\bar{l} = (\bar{l'},\bar{n})$ and $\bar{r} = (\bar{r'},\bar{m})$ with $\bar{l'}$ given by the first component of $\overline{(l',+\infty)}^{\alpha}$, $\bar{r'}$ given by the second component of $\overline{(-\infty,r')}^{\alpha}$, $\bar{n} = 1$ and $\bar{m} = \bar{n} + 1 = 2$.*

We proceed by a formal proof of the normalization not changing the described factor:

Lemma 7. *Let γ be a π-term, $w = [\![\gamma]\!]_{\omega+\omega^*}$ and $(l,r) \in \bar{P}(\gamma)$. Then*

$$w_{(l,r)} = w_{\overline{(l,r)}^{\gamma}}$$

holds.

Proof. Define $\overline{(l,r)}^{\gamma} = (\bar{l}, \bar{r})$ and proceed by induction on the structure of γ. The base cases for $\gamma = \varepsilon$ and $\gamma \in \Sigma$ are trivial.

If $\gamma = \alpha\beta$ for π-terms α and β, then define $u = [\![\alpha]\!]_{\omega+\omega^*}$ and $v = [\![\beta]\!]_{\omega+\omega^*}$. If $l \in \mathrm{dom}(u) \uplus \{-\infty\}$ and $r \in \mathrm{dom}(v) \uplus \{+\infty\}$, then

$$w_{(l,r)} = u_{(l,+\infty)} v_{(-\infty,r)} = u_{\overline{(l,+\infty)}^{\alpha}} v_{\overline{(-\infty,r)}^{\beta}} = w_{\overline{(l,r)}^{\gamma}}.$$

If $l \in \mathrm{dom}(u) \uplus \{-\infty\}$ and $r \in \mathrm{dom}(u)$, then

$$w_{(l,r)} = u_{(l,r)} = u_{\overline{(l,r)}^{\alpha}} = w_{\overline{(l,r)}^{\gamma}}.$$

The case $l \in \mathrm{dom}(v)$ and $r \in \mathrm{dom}(v) \uplus \{+\infty\}$ is symmetrical.

If $\gamma = (\alpha)^{\pi}$ for a π-term α, then define $u = [\![\alpha]\!]_{\omega+\omega^*}$. The case $l = -\infty$ and $r = +\infty$ is trivial. If $l = (l', n)$ for an $n \in \mathbb{N} \uplus -\mathbb{N}$ and $r = +\infty$, define $\bar{l}'$ by $\overline{(l', +\infty)}^{\alpha} = (\bar{l}', +\infty)$. For $n \in \mathbb{N}$ we then have

$$w_{(l,r)} = w_{((l',n),+\infty)} = \left(u^{\omega+\omega^*}\right)_{((l',n),+\infty)} = \left(u^{\omega+\omega^*}\right)_{((l',1),+\infty)}$$

because of $u^{\omega+\omega^*} = uu^{\omega+\omega^*}$ and further

$$\begin{aligned} w_{(l,r)} &= u_{(l',+\infty)} u^{\omega+\omega^*} = u_{\overline{(l',+\infty)}^{\alpha}} u^{\omega+\omega^*} = u_{(\bar{l}',+\infty)} u^{\omega+\omega^*} \\ &= \left(u^{\omega+\omega^*}\right)_{((\bar{l}',1),+\infty)} = w_{\overline{((l',n),+\infty)}^{\gamma}} = w_{\overline{(l,r)}^{\gamma}} \end{aligned}$$

and for $n = -1$ – the only remaining case – we have

$$w_{(l,r)} = w_{((l',-1),+\infty)} = u_{(l',+\infty)} = u_{\overline{(l',+\infty)}^{\alpha}} = u_{(\bar{l}',+\infty)} = w_{((\bar{l}',-1),+\infty)} = w_{\overline{(l,r)}^{\gamma}}.$$

The case for $l = -\infty$ and $r = (r', m)$ is symmetrical.

Therefore, we can assume $l = (l', n)$ and $r = (r', m)$. The case $n \in \mathbb{N}$ and $m \in -\mathbb{N}$ is proved by a calculation similar to the one given above. For $n = m$ we have

$$w_{(l,r)} = w_{((l',n),(r',n))} = u_{(l',r')} = u_{\overline{(l',r')}^{\alpha}} = w_{\overline{(l,r)}^{\gamma}}$$

and for $m = n + 1$ we have

$$w_{(l,r)} = u_{(l',+\infty)} u_{(-\infty,r')} = u_{\overline{(l',\infty)}^{\alpha}} u_{\overline{(-\infty,r')}^{\alpha}} = w_{\overline{(l,r)}^{\gamma}}.$$

□

These results allow us to prove Theorem 2 formally. Rather than giving more details on the construction of the automata described in the main paper's proof sketch, we describe a direct algorithm.

Theorem 2. *The word problems for π-terms over $\mathbf{R_m}$, $\mathbf{L_m}$, $\mathbf{R_m} \vee \mathbf{L_m}$ and $\mathbf{R_m} \cap \mathbf{L_m}$ are decidable for any $m \in \mathbb{N}$. Moreover, the word problem for π-terms over* **DA** *is decidable.*

Proof. We only describe the decision algorithm for $\mathbf{R_{m+1}} \cap \mathbf{L_{m+1}}$, as the others are similar. By Theorem 1, we need to test whether

$$u := [\![\alpha]\!]_{\omega+\omega^*} \equiv_m^{\mathrm{WI}} [\![\beta]\!]_{\omega+\omega^*} =: v$$

holds for the input π-terms α and β. For this, we have to perform consecutive factorizations at the first or last a in the current factors of u and v while we keep track of the remaining possible values of m. If at some point the factorization is only possible in one of the two words, then we know that $u \not\equiv_m^{\mathrm{WI}} v$.

In more detail, we have the variables `core` and `fringe` which contain subsets of $\bar{P}(\alpha) \times \bar{P}(\beta) \times \{1, 2, \ldots, m\}$, where $\bar{P}(\alpha)$ (or $\bar{P}(\beta)$) is the set of all normalized pairs of positions in u (or in v, respectively). Initially, `core` is empty and `fringe` contains only $((-\infty, +\infty), (-\infty, +\infty), m)$; then we execute the following algorithm:

while `fringe` $\neq \emptyset$ **do**
  Remove $((l_\alpha, r_\alpha), (l_\beta, r_\beta), k)$ from `fringe`
  if $k > 0$ **then**
    for all $a \in \Sigma$ **do**
      if either $(l_\alpha, r_\alpha) \cdot X_a^L$ or $(l_\beta, r_\beta) \cdot X_a^L$ is defined (but not both) **then**
        return $u \not\equiv_m^{\mathrm{WI}} v$ ▷ Alphabets of $u_{(l_\alpha, r_\alpha)}$ and $v_{(l_\beta, r_\beta)}$ differ
      else
        Add $(\overline{(l_\alpha, r_\alpha) \cdot X_a^L}^{\alpha}, \overline{(l_\beta, r_\beta) \cdot X_a^L}^{\beta}, k-1)$ to `finge` unless it is in `core` ▷ We must have $u_{(l_\alpha, r_\alpha) \cdot X_a^L} \equiv_{k-1}^{\mathrm{WI}} v_{(l_\beta, r_\beta) \cdot X_a^L}$
      end if
      if either $(l_\alpha, r_\alpha) \cdot X_a^R$ or $(l_\beta, r_\beta) \cdot X_a^R$ is defined (but not both) **then**
        return $u \not\equiv_m^{\mathrm{WI}} v$
      else
        Add $(\overline{(l_\alpha, r_\alpha) \cdot X_a^R}^{\alpha}, \overline{(l_\beta, r_\beta) \cdot X_a^R}^{\beta}, k)$ to `finge` unless it is in `core` ▷ We must have $u_{(l_\alpha, r_\alpha) \cdot X_a^R} \equiv_{k}^{\mathrm{WI}} v_{(l_\beta, r_\beta) \cdot X_a^R}$
      end if
      Handle Y_a^L and Y_a^R analogously
      for all $b \in \Sigma$ **do**
        if either $(l_\alpha, r_\alpha) \cdot C_{a,b}$ or $(l_\beta, r_\beta) \cdot C_{a,b}$ is defined (not both) **then**
          return $u \not\equiv_m^{\mathrm{WI}} v$
        else
          Add $(\overline{(l_\alpha, r_\alpha) \cdot C_{a,b}}^{\alpha}, \overline{(l_\beta, r_\beta) \cdot C_{a,b}}^{\beta}, k-1)$ to `finge` unless it is in `core`
          ▷ We must have $u_{(l_\alpha, r_\alpha) \cdot C_{a,b}} \equiv_{k-1}^{\mathrm{WI}} v_{(l_\beta, r_\beta) \cdot C_{a,b}}$
        end if
      end for
    end for
  end if
  Add $((l_\alpha, r_\alpha), (l_\beta, r_\beta), k)$ to `core`

end while
return $u \equiv_m^{\mathrm{WI}} v$

What the algorithm does is trying to guess a sequence of factorizations at the first or last a such that the factorization can be applied to u but not to v (or vice versa). Because normalization does not change the factor of the word by Lemma 7, we can normalize the pair which describes the factor at any time. Here, we perform a normalization before we add the pairs to `fringe`. For $\equiv_m^{\mathrm{WI}}$, we are in the special situation that we can perform a factorization at the first a and the last b in one step, which means that we only have to go to $m-1$ instead of $m-2$; this case has to be handled specially in the above algorithm.

By the definition of $\equiv_m^{\mathrm{WI}}$, we can find a sequence of factorizations which can be applied to u but not to v (or vice versa) if $u \not\equiv_m^{\mathrm{WI}} v$. The definition yields also the other way: if $u \equiv_m^{\mathrm{WI}} v$, then we can apply any sequence of factorizations at the first/last a (which respects the value of m) to u if and only if we can apply it to v. Which shows the correctness of the algorithm.

Termination is guaranteed because there are only finitely many pairs in $\bar{P}(\alpha)$ and in $\bar{P}(\beta)$.

The algorithm can be adapted for $\mathbf{R_m}$, $\mathbf{L_m}$ and $\mathbf{R_m} \vee \mathbf{L_m}$ by changing the way how we compute k for the next tuple accordingly. All we have to do for this is keep track of whether the last factorization operation was in $\{X_a^L, X_a^R \mid a \in \Sigma\}$ or in $\{Y_a^L, Y_a^R \mid a \in \Sigma\}$ using an additional position in the tuple.

An algorithm for **DA** can be obtained by omitting the counting of k in the algorithm for $\mathbf{R_m} \vee \mathbf{L_m}$ by Fact 2. □

B More on Separability

First, we state the mentioned and proof the combinatoric property of $\equiv_{m,n}^X$.

Lemma 8. *Let $n, m \in \mathbb{N}$ with $m \geq 2$ and let $u \equiv_{m,n}^X v$ for two words u and v. Then, $u \cdot X_a^L \equiv_{m,n-1}^X v \cdot X_a^L$ holds for all $a \in \mathrm{alph}(u) = \mathrm{alph}(v)$.*

Proof. We prove the lemma by induction over n. For $n = 1$, the assertion is satisfied by definition. Therefore, assume we have $u \equiv_{m,n+1}^X v$ and we want to show $u_0 := u \cdot X_a^L \equiv_{m,n}^X v \cdot X_a^L =: v_0$. We already have $u_0 \equiv_{m-1,n}^Y v_0$ by definition of $\equiv_{m,n+1}^X$. This especially implies $\mathrm{alph}(u_0) = \mathrm{alph}(v_0)$ since we have $m \geq 2$ and $n \geq 1$ as well as $u_0 \equiv_{m-1,n-1}^Y v_0$. Additionally, we have

$$u_0 \cdot X_b^L = u \cdot X_b^L \equiv_{m-1,n}^Y v \cdot X_b^L = v_0 \cdot X_b^L$$

for all $b \in \mathrm{alph}(u_0) = \mathrm{alph}(v_0)$, which implies $u_0 \cdot X_b^L \equiv_{m-1,n-1}^Y v_0 \cdot X_b^L$. All that remains to be shown is that $u_0 \cdot X_b^R \equiv_{m,n-1}^X v_0 \cdot X_b^R$ holds for all $b \in \mathrm{alph}(u_0) = \mathrm{alph}(v_0)$. Applying induction on $u \cdot X_b^R \equiv_{m,n}^X v \cdot X_b^R$ (for the same a) yields $u \cdot X_b^R X_a^L \equiv_{m,n-1}^X v \cdot X_b^R X_a^L$. Since we have $u_0 \cdot X_b^R = u \cdot X_a^L X_b^R = u \cdot X_b^R X_a^L$ and $v_0 \cdot X_b^R = v \cdot X_a^L X_b^R = v \cdot X_b^R X_a^L$, we are done. □

Next, we present the omitted proof for decidability of the separation problem for the variety **J** of $\mathcal{J}$-trivial monoids ($m = 1$). It is basically an adaption of the ideas from the proof showing decidability given by van Rooijen and Zeitoun [26] to our setting.

Lemma 9. *Let M be a monoid and $\varphi : \Sigma^* \to M$ a homomorphism. An infinite sequence of word pairs $(u_n, v_n)_{n \in \mathbb{N}_0}$ with*

- $u_n, v_n \in \Sigma^*$,
- $u_n \equiv_{1,n}^X v_n$,
- $\varphi(u_n) = m_u$ *and*
- $\varphi(v_n) = m_v$

for fixed monoid elements $m_u, m_v \in M$ and all $n \in \mathbb{N}_0$ yields π-terms α and β (over Σ) such that $\varphi(\llbracket\alpha\rrbracket_{M!}) = m_u$, $\varphi(\llbracket\beta\rrbracket_{M!}) = m_v$ and $\llbracket\alpha\rrbracket_{\omega+\omega^*} \equiv_1^X \llbracket\beta\rrbracket_{\omega+\omega^*}$ hold.

Proof. This proof is based on Simon's Factorization Forest Theorem [22]. For a finite word $w \in \Sigma^*$, a factorization tree is a rooted, finite, unranked, labeled ordered tree such that

- the tree's root is labeled with w,
- the leaves are baled by letters (from Σ) and
- any internal node has at least two children and, if its children are labeled with $w_1, w_2, \dots, w_k \in \Sigma^*$, then the node is labeled with $w_1 w_2 \dots w_k$.

For every homomorphism $\psi : \Sigma^* \to N$ into a monoid N, Simon's Factorization Forest Theorem yields a factorization tree for every finite word $w \in \Sigma^*$ such that ψ maps the labels of a node's children to the same idempotent in N if the node has at least three children. Furthermore, the tree's height[18] is finite and limited by some constant that solely depends on $|N|$ (and, especially, not on w).

Before we begin with the actual proof, we note that, if we remove pairs from the sequence $(u_n, v_n)_{n \in \mathbb{N}_0}$ and still have an infinite sequence, then the resulting sequence still satisfies all conditions states above, especially $u_n \equiv_{1,n}^X v_n$.

We extend φ into a homomorphism $\Sigma^* \to M \times 2^\Sigma$ which maps a word w to its alphabet $\mathrm{alph}(w)$ for the second component.[19] Then, we observe that there has to be an infinite subsequence such that all first components as well as all second components have the same alphabet[20]; we remove all other words from the sequence. To the remaining words u_n and v_n, we apply Simon's Factorization Forest Theorem, which yields a sequence of factorization tree pairs $(T_{u,n}, T_{v,n})$. We first construct α from $(T_{u,n})_{n \in \mathbb{N}_0}$ such that we have $\varphi(\llbracket\alpha\rrbracket_{M!}) = m_u$ and the following conditions:

- If $w \in \Sigma^*$ is a subword[21] of u_n for an $n \in \mathbb{N}_0$, then w is a subword of $\llbracket\alpha\rrbracket_{\omega+\omega^*}$.

[18] A single node has height 0.
[19] 2^Σ is the monoid of all subsets of Σ with taking union as the monoid's operation.
[20] Indeed, these two alphabets have to coincide by the definition of $\equiv_{1,n}^X$
[21] A finite word $u = a_1 a_2 \dots a_n$ with $a_i \in \Sigma$ is a subword of a (not necessarily finite) word v if we can write $v = v_0 a_1 v_1 a_2 v_2 \dots a_n v_n$ for some words $v_0, v_1, \dots, v_n$.

– If $w \in \Sigma^*$ is a subword of $[\![\alpha]\!]_{\omega+\omega^*}$, then it is a subword of all u_n with $n \geq n_0$ for an $n_0 \in \mathbb{N}_0$.

Afterwards, we proceed with $(T_{v,n})_{n\in\mathbb{N}_0}$ to construct β in the same manner.

We may assume that all trees $T_{u,n}$ have the same height H as the height is bounded by a constant and we can remove all words u_n from the underlying sequence which yield a tree not of height H. If H is zero, all trees consist of a single leaf and all words u_n consist of a single letter. Among these, one letter $a \in \Sigma$ has to appear infinitely often; we remove all other words from the sequence and choose $\alpha = a$. Clearly, all conditions for α are satisfied.

For $H > 0$, we consider the situation at the root of each $T_{u,n}$. Let $u_{n,1}, u_{n,2}, \dots, u_{n,K_n}$ be the labels of the root's children in $T_{u,n}$. If the sequence $(K_n)_{n\in\mathbb{N}_0}$ is bounded, there is an infinite subsequence such that K_n is equal to a specific $K \geq 2$ for all $n \in \mathbb{N}_0$; we remove all words not belonging to this subsequence. Additionally, there is an infinite subsequence such that, for each sequence $(u_{n,k})_{n\in\mathbb{N}_0}$ with $1 \leq k \leq K$, all $u_{n,k}$ get mapped to the same monoid element by φ; we remove all other words. As each child of the root yields a subtree, taking these subtrees gives K infinite sequences of factorization trees of height $H-1$. Applying induction on H, yields $\alpha_1, \alpha_2, \dots, \alpha_K$. We define $\alpha := \alpha_1\alpha_2 \dots \alpha_K$. Because $\alpha_1, \alpha_2, \dots, \alpha_K$ satisfy the conditions stated above for their respective subtree sequence, so does α for $(T_{u,n})_{n\in\mathbb{N}_0}$.

If the sequence $(K_n)_{n\in\mathbb{N}_0}$ is unbounded, we can, without loss of generality, assume $K_n \geq 3$ for all $n \in \mathbb{N}_0$ – again taking the appropriate infinite subsequence. Also, we can assume that all $u_{n,1}, u_{n,2}, \dots, u_{n,K_n}$ get mapped to the same idempotent $e \in M \times 2^\Sigma$. Choose $w \in \varphi^{-1}(e)$ arbitrarily and define $\alpha := (w)^\pi$. Note that we now have $\mathrm{alph}(u_{n,1}) = \mathrm{alph}(u_{n,2}) = \cdots = \mathrm{alph}(u_{n,K_n}) = \mathrm{alph}(u_n) = \mathrm{alph}(w)$ for all $n \in \mathbb{N}_0$. Therefore, α satisfies the conditions above.

All which remains to be shown is that we now have $[\![\alpha]\!]_{\omega+\omega^*} \equiv_1^X [\![\beta]\!]_{\omega+\omega^*}$. The important observation here is that $w_1 \equiv_{1,n}^X w_2$ with $n \in \mathbb{N}_0$ holds if and only if w_1 and w_2 have the same subwords of length $\leq n$. This means we have to show that $[\![\alpha]\!]_{\omega+\omega^*}$ and $[\![\beta]\!]_{\omega+\omega^*}$ have the same subwords (of arbitrary length). To show the subword equality, assume w is a subword of $[\![\alpha]\!]_{\omega+\omega^*}$ (without loss of generality). By the conditions above, w is a subwords of all u_n with $n \geq n_0$ for an $n_0 \in \mathbb{N}_0$. Let $\tilde{n} = \max\{n_0, |w|\}$. Since we have $u_{\tilde{n}} \equiv_{1,\tilde{n}}^X v_{\tilde{n}}$ and by applying our observation regarding subwords and $\equiv_{1,\tilde{n}}^X$, w is a subword of $v_{\tilde{n}}$ and, thus, a subword of $[\![\beta]\!]_{\omega+\omega^*}$. $\square$

Now, we prove the general case for $m > 1$.

Lemma 1. *Let M be a monoid, $\varphi : \Sigma^* \to M$ a homomorphism and $m \in \mathbb{N}_0$. Let $(u_n, v_n)_{n\in\mathbb{N}_0}$ be an infinite sequence of word pairs $(u_n, v_n)_{n\in\mathbb{N}_0}$ with $u_n, v_n \in \Sigma^*$, $u_n \equiv_{m,n}^X v_n$, $\varphi(u_n) = m_u$ and $\varphi(v_n) = m_v$ for fixed monoid elements $m_u, m_v \in M$ and all $n \in \mathbb{N}_0$. Then, the sequence yields π-terms α and β (over Σ) such that $\varphi([\![\alpha]\!]_{M!}) = m_u$, $\varphi([\![\beta]\!]_{M!}) = m_v$ and $[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^X [\![\beta]\!]_{\omega+\omega^*}$ hold.*

Proof. The assertion is trivial for $m = 0$. The case $m = 1$ is covered by the previous lemma. For $m \geq 1$, we proceed by induction over $|\Sigma|$. For $\Sigma = \emptyset$, we set $\alpha = \beta = \varepsilon = u_n = v_n$. For $|\Sigma| > 0$, we start by making an observation: if we take an infinite subsequence $(u'_n, v'_n)_{n\in\mathbb{N}_0}$ of $(u_n, v_n)_{n\in\mathbb{N}_0}$, this sequence still satisfies all conditions of the lemma. In particular, we would still have $u'_n \equiv^X_{m,n} v'_n$ for all $n \in \mathbb{N}_0$.

Now, we factorize $u_n = w_{n,0}a_{n,0}w_{n,1}a_{n,1} \dots w_{n,K_n}a_{n,K_n}w_{n,K_n+1}$ for all $n \in \mathbb{N}_0$ such that $\mathrm{alph}(w_{n,k}) = \mathrm{alph}(u_n) \setminus \{a_{n,k}\}$ for all $k \in \{0, 1, \dots, K_n\}$ and $\mathrm{alph}(w_{n,K_n}) \subsetneq \mathrm{alph}(u_n)$. If the sequence $(K_n)_{n\in\mathbb{N}_0}$ is bounded, let K be one of the numbers which appear infinitely often in it. If $(K_n)_{n\in\mathbb{N}_0}$ is unbounded, let $K = |M|^2 + 1$. In either case, restrict all further considerations to the subsequence of word pairs corresponding to K. Because Σ is of finite size, a single letter $a_k \in \Sigma$ has to appear infinitely often in the sequence $(a_{n,k})_{n\in\mathbb{N}_0}$ for all $k \in \{0, 1, \dots, K\}$. We restrict our consideration to the appropriate subsequence. Then, we define $x_{n,k} = u_n \cdot X^R_{a_0}X^R_{a_1} \dots X^R_{a_{k-1}}X^L_{a_k}$ and $y_{n,k} = v_n \cdot X^R_{a_0}X^R_{a_1} \dots X^R_{a_{k-1}}X^L_{a_k}$ for $k \in \{0, 1, \dots, K\}$ as well as $x_{n,K+1} = u_n \cdot X^R_{a_0}X^R_{a_1} \dots X^R_{a_K}$ and $y_{n,K+1} = v_n \cdot X^R_{a_0}X^R_{a_1} \dots X^R_{a_K}$. We, thus, have $u_n = x_{n,0}a_0x_{n,1}a_1 \dots x_{n,K} a_Kx_{n,K+1}$ and $v_n = y_{n,0}a_0y_{n,1}a_1 \dots y_{n,K}a_Ky_{n,K+1}$ for all $n \in \mathbb{N}_0$. Because K is constant, we can safely assume that φ maps all elements of the sequence $(x_{n,k})_{n\in\mathbb{N}_0}$ (for every $k \in \{0, 1, \dots, K\}$) to the same element $s_k \in M$: one element has to appear infinitely often and we take the appropriate subsequence. In the same way, we can ensure that φ maps all element of $(y_{n,k})_{n\mathbb{N}_0}$ to the same element $t_k \in M$ (again, for all $k \in \{0, 1, \dots, K\}$). By removing the first $K + 2$ pairs of words, we can also ensure $u_n \equiv^X_{m,n+K+2} v_n$ for all $n \in \mathbb{N}_0$. This implies $x_{n,k} \equiv^X_{m,n+K+2-k-2} y_{n,k}$ for all $n \in \mathbb{N}_0$ and all $k \in \{0, 1, \dots, K\}$ by Lemma 8. Directly by the definition of $\equiv^X_{m,n}$, we already have $x_{n,K+1} \equiv^X_{m,n+K+2-K-1} y_{n,K+1}$ and, therefore, $x_{n,k} \equiv^X_{m,n} y_{n,k}$ for all $k \in \{0, 1, \dots, K + 1\}$. We can apply induction to $(x_{n,k}, y_{n,k})_{n\in\mathbb{N}_0}$ for $k \in \{0, 1, \dots, K\}$ since we have $a_k \notin \mathrm{alph}(x_{n,k})$ by construction. This yields π-terms $\alpha_0, \alpha_1, \dots, \alpha_K, \beta_0, \beta_1, \dots, \beta_K$. If $(K_n)_{n\in\mathbb{N}_0}$ was bounded, then $\mathrm{alph}(x_{n,K+1}) = \mathrm{alph}(y_{n,K+1}) \subsetneq \mathrm{alph}(u_n) = \mathrm{alph}(v_n)$ holds and we can apply induction as well, which yields π-terms α_{K+1} and β_{K+1}. Setting $\alpha = \alpha_0a_0\alpha_1a_1 \dots \alpha_Ka_K\alpha_{K+1}$ and $\beta = \beta_0a_0\beta_1a_1 \dots \beta_Ka_K\beta_{K+1}$ satisfies $[\![\alpha]\!]_{\omega+\omega^*} \equiv^X_m [\![\beta]\!]_{\omega+\omega^*}$ since $\equiv^X_m$ is a congruence. If $(K_n)_{n\in\mathbb{N}_0}$ was unbounded, we set $K = |M|^2 + 1$ and, by the pigeon hole principle, there are $i, j \in \{0, 1, \dots, K\}$ with $i < j$ and

$$s_0\varphi(a_0)s_1\varphi(a_1) \dots s_i\varphi(a_i) = s_0\varphi(a_0)s_1\varphi(a_1) \dots s_j\varphi(a_j) \text{ and}$$
$$t_0\varphi(a_0)t_1\varphi(a_1) \dots t_i\varphi(a_i) = t_0\varphi(a_0)t_1\varphi(a_1) \dots t_j\varphi(a_j).$$

We define

$$\alpha = \alpha_0a_0\alpha_1a_1 \dots \alpha_ia_i \left(\alpha_{i+1}a_{i+1}\alpha_{i+2}a_{i+2} \dots \alpha_ja_j\right)^\pi \alpha_{K+1} \text{ and}$$
$$\beta = \beta_0a_0\beta_1a_1 \dots \beta_ia_i \left(\beta_{i+1}a_{i+1}\beta_{i+2}a_{i+2} \dots \beta_ja_j\right)^\pi \beta_{K+1}$$

where α_K and β_K are obtained by using induction on m (and symmetry) for the sequences $(x_{n,K+1})_{n\in\mathbb{N}_0}$ and $(y_{n,K+1})_{n\in\mathbb{N}_0}$. Thus, they map to the same monoid element as the elements in their respective sequence. Therefore, we have $\varphi\left([\![\alpha]\!]_{M!}\right) = m_u$ and $\varphi\left([\![\beta]\!]_{M!}\right) = m_v$ by construction. We also have $[\![\alpha]\!]_{\omega+\omega^*} \equiv^X_m$

$[\![\beta]\!]_{\omega+\omega^*}$: for the part left to and including the $(\cdot)^\pi$, we have equivalence by induction and because $\equiv_m^X$ is a congruence; the right part, we cannot reach by arbitrarily many X_a factorizations since any letter appears infinitely often in the $(\cdot)^\pi$ part and, if we reach it by using at least one Y_a factorization, we are done since m decreases. □

This allows us to prove Theorem 5.

Theorem 5. *The separation problem for* $\mathbf{R_m}$ *and* $\mathbf{L_m}$ *is decidable for all* $m \in \mathbb{N}$.

Proof. We only consider $\mathbf{R_m}$ as the case for $\mathbf{L_m}$ is symmetric. If the input languages are separable, we can find a separating language by enumerating all candidates. If the languages are inseparable, we have to apply the previous lemma. As regular languages, the input languages $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Sigma^*$ can be recognized by monoids M_1 and M_2 via the homomorphisms φ_1 and φ_2 and the homomorphism can be computed. Therefore, they are also be recognized by $M := M_1 \times M_2$ via the homomorphism φ which maps a word to a pair whose first component is determined by φ_1 and whose second component is determined by φ_2. Let $n \in \mathbb{N}_0$ be arbitrary. Since we have $\Sigma^*/\equiv_{m,n}^X \in \mathbf{R_m}$ and since L_1 and L_2 cannot be separated by $\mathbf{R_m}$, there have to be finite words $u_n, v_n \in \Sigma^*$ with $u_n \in L_1$, $v_n \in L_2$ and $u_n \equiv_{m,n}^X v_n$; otherwise, we could construct a separating language. The homomorphism φ has to map infinitely many element of the sequence $(u_n, v_n)_{n \in \mathbb{N}_0}$ to the same element in M since M is finite. If we remove all other elements, we still have an infinite sequence $(u_n, v_n)_{n \in \mathbb{N}_0}$ with $u_n \equiv_{m,n}^X v_n$ for all $n \in \mathbb{N}_0$ which also satisfies all conditions of Lemma 1. Therefore, there are π-terms α and β with $[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^X [\![\beta]\!]_{\omega+\omega^*}$, $\varphi([\![\alpha]\!]_{M!}) \in \varphi(L_1)$ and $\varphi([\![\beta]\!]_{M!}) \in \varphi(L_2)$. Since we can test whether $[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^X [\![\beta]\!]_{\omega+\omega^*}$ holds for any two π-terms α and β using the algorithm described in Section 4, we can also recursively enumerate all possible π-term pairs and check whether the conditions above are met. We know that we can find such a pair if L_1 and L_2 are inseparable. On the other hand, suppose L_1 and L_2 can be separated by $S \subseteq \Sigma^*$ which is recognized by the monoid $N \in \mathbf{R_m}$ via a homomorphism $\psi : \Sigma^* \to N$ and we have found a pair α and β with $[\![\alpha]\!]_{\omega+\omega^*} \equiv_m^X [\![\beta]\!]_{\omega+\omega^*}$, $\varphi([\![\alpha]\!]_{M!}) \in \varphi(L_1)$ and $\varphi([\![\beta]\!]_{M!}) \in \varphi(L_2)$. Then, we have $\varphi([\![\alpha]\!]_{N! \cdot M!}) = \varphi([\![\alpha]\!]_{M!}) \in \varphi(L_1)$ and, thus, $[\![\alpha]\!]_{N! \cdot M!} \in L_1$ as well as $[\![\beta]\!]_{N! \cdot M!} \in L_2$ (by a similar argumentation). Also, $\alpha = \beta$ holds in $\mathbf{R_m}$ by Theorem 1, which implies $n := \psi([\![\alpha]\!]_{N! \cdot M!}) = \psi([\![\alpha]\!]_{N!}) = \psi([\![\beta]\!]_{N!}) = \psi([\![\beta]\!]_{N! \cdot M!})$. If we have $n \in \psi(S)$, then we have $[\![\beta]\!]_{N! \cdot M!} \in S \cap L_2$; otherwise, we have $[\![\alpha]\!]_{N! \cdot M!} \in L_1$ but $[\![\alpha]\!]_{N! \cdot M!} \notin S$ and, thus, a contradiction in either case. □