Level Two of the Quantifier Alternation Hierarchy over Infinite Words

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Abstract The study of various decision problems for logic fragments has a long history in computer science. This paper is on the membership problem for a fragment of first-order logic over infinite words; the membership problem asks for a given language whether it is definable in some fixed fragment. The alphabetic topology was introduced as part of an effective characterization of the fragment Σ_2 over infinite words. Here, Σ_2 consists of the first-order formulas with two blocks of quantifiers, starting with an existential quantifier. Its Boolean closure is $\mathbb{B}\Sigma_2$. Our first main result is an effective characterization of the Boolean closure of the alphabetic topology, that is, given an ω-regular language L, it is decidable whether L is a Boolean combination of open sets in the alphabetic topology. This is then used for transferring Place and Zeitoun's recent decidability result for $\mathbb{B}\Sigma_2$ from finite to infinite words.
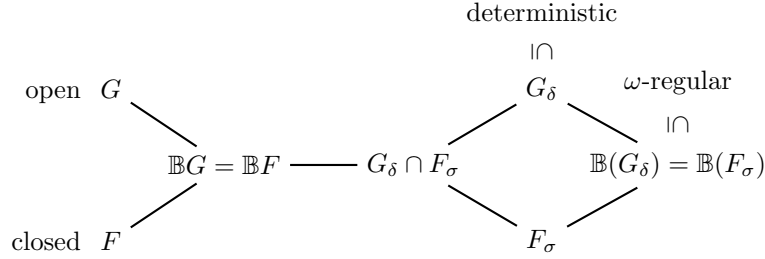
1 Introduction

Over finite words, the connection between finite monoids and regular languages is highly successful for studying logic fragments, see e.g. [2,19]. Over infinite words, the algebraic approach uses infinite repetitions. Not every logic fragment can express whether some definable property P occurs infinitely often. For instance, the usual approach for saying that P occurs infinitely often is as follows: for every position x there is a position $y > x$ satisfying $P(y)$. Similarly, P occurs only finitely often if there is a position x such that all positions $y > x$ satisfy $\neg P(y)$. Each of these formulas requires (at least) one additional change of quantifiers, which not all fragments can provide. It turns out that topology is a very useful tool for restricting the infinite behaviour of the algebraic approach accordingly, see e.g. [3,5,10,22]. In particular, the combination of algebra and topology is convenient for the study of languages in Γ^∞, the set of finite and infinite words over the alphabet Γ. In this paper, an *ω-regular language* is a regular subset of Γ^∞.

Topological ideas have a long history in the study of ω-regular languages. The Cantor topology is the most famous example in this context. We write G

for the Cantor-open sets and F for the closed sets. The open sets in G are the languages of the form $W\Gamma^\infty$ for $W \subseteq \Gamma^*$. If X is a class of languages, then X_δ consists of the countable intersections of languages in X and X_σ are the countable unions; moreover, we write $\mathbb{B}X$ for the Boolean closure of X. Since F contains the complements of languages in G, we have $\mathbb{B}F = \mathbb{B}G$. The Borel hierarchy is defined by iterating the operations $X \mapsto X_\delta$ and $X \mapsto X_\sigma$. The Borel hierarchy over the Cantor topology has many appearances in the context of ω-regular languages. For instance, an ω-regular language is deterministic if and only if it is in G_δ, see [8,21]. By McNaughton's Theorem [9], every ω-regular language is in $\mathbb{B}(G_\delta) = \mathbb{B}(F_\sigma)$. The inclusion $\mathbb{B}G \subset G_\delta \cap F_\sigma$ is strict, but the ω-regular languages in $\mathbb{B}G$ and $G_\delta \cap F_\sigma$ coincide [17].

$$\begin{array}{ccccccc}
 & & & & \text{deterministic} & & \\
 & & & & \shortmid\cap & & \\
\text{open } \; G & & & & G_\delta & \omega\text{-regular} & \\
 & \diagdown & & \diagup & & \shortmid\cap & \\
 & \mathbb{B}G = \mathbb{B}F & \text{------} & G_\delta \cap F_\sigma & & \mathbb{B}(G_\delta) = \mathbb{B}(F_\sigma) & \\
 & \diagup & & \diagdown & & & \\
\text{closed } \; F & & & & F_\sigma & & \\
\end{array}$$

Let FO^k be the fragment of first-order logic which uses (and reuses) at most k variables. By Σ_m we denote the formulas with m quantifier blocks, starting with a block of existential quantifiers. Here, we assume that $x < y$ is the only binary predicate. Let us consider FO^1 as a toy example. With only one variable, we cannot make use of the binary predicate $x < y$. Therefore, in FO^1 we can say nothing but which letters occur, that is, a language is definable in FO^1 if and only if it is a Boolean combination of languages of the form $\Gamma^* a \Gamma^\infty$ for $a \in \Gamma$. Thus $\mathrm{FO}^1 \subseteq \mathbb{B}G$. It is an easy exercise to show that an ω-regular language is in FO^1 if and only if it is in $\mathbb{B}G$ and its syntactic monoid is both idempotent and commutative. The algebraic condition without the topology is too powerful since this would also include the language $\{a,b\}^* a^\omega$, which is not definable in FO^1. For the fragment $\mathbb{B}\Sigma_1$, the same topology $\mathbb{B}G$ with a different algebraic condition works, cf. [10, Theorems VI.3.7, VI.7.4 and VIII.4.5].

In the fragment Σ_2, we can define the language $\{a,b\}^* ab^\infty$ which is not deterministic and hence not in G_δ. Since the next level of the Borel hierarchy already contains all ω-regular languages, another topology is required. For this purpose, Diekert and the first author introduced the *alphabetic topology* [3]: the open sets in this topology are arbitrary unions of languages of the form uA^∞ for $u \in \Gamma^*$ and $A \subseteq \Gamma$. They showed that an ω-regular language is definable in Σ_2 if and only if it satisfies some particular algebraic property and if it is open in the alphabetic topology. Therefore, the canonical ingredient for an effective characterization of $\mathbb{B}\Sigma_2$ is the Boolean closure of the open sets in the alphabetic topology. Our first main result, Theorem 2, shows that, for a given ω-regular language L, it is decidable whether L is a Boolean combination of open sets in the alphabetic topology. As a by-product, we see that every ω-regular language

which is a Boolean combination of arbitrary open sets in the alphabetic topology can be written as a Boolean combination of ω-regular open sets. This resembles a similar result for the Cantor topology [17].

A major breakthrough in the theory of regular languages over finite words is due to Place and Zeitoun [14]. They showed that, for a given regular language $L \subseteq \Gamma^*$, it is decidable whether L is definable in $\mathbb{B}\Sigma_2$. This solved a longstanding open problem, see e.g. [13, Section 8] for an overview. To date, no effective characterization of $\mathbb{B}\Sigma_3$ is known. Our second main result, Theorem 4, is to show that this decidability result transfers to languages in Γ^∞. If $\mathbf{V_2}$ is the algebraic counterpart of $\mathbb{B}\Sigma_2$ over finite words, then we show that $\mathbf{V_2}$ combined with the Boolean closure of the alphabetic topology yields a characterization of $\mathbb{B}\Sigma_2$ over Γ^∞. Combining the decidability of $\mathbf{V_2}$ with our first main result, the latter characterization is effective. The proof that $\mathbb{B}\Sigma_2$ satisfies both the algebraic and the topological restrictions follows a rather straightforward approach. The main difficulty is to show the converse: every language satisfying both the algebraic and the topological conditions is definable in $\mathbb{B}\Sigma_2$.

Missing proofs of some elementary lemmas are given in the appendix.

2 Preliminaries

Words

Let Γ be a finite alphabet. By Γ^* we denote the set of finite words over Γ; we write 1 for the empty word. The set of infinite words is Γ^ω and the set of finite and infinite words is $\Gamma^\infty = \Gamma^* \cup \Gamma^\omega$. By u, v, w we denote finite words and by α, β, γ we denote words in Γ^∞. In this paper a *language* is a subset of Γ^∞. Let $L \subseteq \Gamma^*$ and $K \subseteq \Gamma^\infty$. As usual L^* is the union of powers of L and $LK = \{u\alpha \mid u \in L, \alpha \in K\} \subseteq \Gamma^\infty$ is the concatenation of L and K. By L^ω we denote the set of words which are an infinite concatenation of words in L and the infinite concatenation $uu\cdots$ of the word u is written u^ω. A word $u = a_1 \ldots a_n$ is a *scattered subword* of v if $v \in \Gamma^* a_1 \Gamma^* \ldots a_n \Gamma^*$. The *alphabet* of a word is the set of all letters which appear in the word. The *imaginary alphabet* $\mathrm{im}(\alpha)$ of a word $\alpha \in \Gamma^\infty$ is the set of letters which appear infinitely often in α. Let $A^{\mathrm{im}} = \{\alpha \in \Gamma^\infty \mid \mathrm{im}(\alpha) = A\}$ be the set of words with imaginary alphabet A. In the following, we restrict ourselves to the study of ω-regular languages. A language $L \subseteq \Gamma^*$ is regular if it is recognized by a (deterministic) finite automaton. A language $K \subseteq \Gamma^\omega$ is ω-regular if it is recognized by a Büchi automaton. A language $L \subseteq \Gamma^\infty$ is ω-regular if $L \cap \Gamma^*$ is regular and $L \cap \Gamma^\omega$ is ω-regular.

First-Order logic

We consider first order logic FO over Γ^∞. Variables range over positions of the word. The atomic formulas in this logic are $\top$ for true, $x < y$ to compare two positions x and y and $\lambda(x) = a$ which is true if the word has an a at position x. One may combine those atomic formulas with the boolean connectives $\neg, \wedge$ and

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$\vee$ and quantifiers $\forall$ and $\exists$. A *sentence* φ is an FO formula without free variables. We write $\alpha \models \varphi$ if $\alpha \in \Gamma^\infty$ satisfies the sentence φ. The language defined by φ is $L(\varphi) = \{\alpha \in \Gamma^\infty \mid \alpha \models \varphi\}$. We classify the formulas of FO by counting the number of quantifier alternations, that is the number of alternations of $\exists$ and $\forall$. The fragment Σ_i of FO contains all FO-formulas in prenex normal form with i blocks of quantifiers $\exists$ or $\forall$, starting with a block of existential quantifiers. The fragment $\mathbb{B}\Sigma_i$ contains all Boolean combinations of formulas in Σ_i. We are particularly interested in the fragment Σ_2 and the Boolean combinations of formulas in Σ_2. A language L is definable in a fragment $\mathcal{F}$ (e.g. $\mathcal{F}$ is Σ_2 or $\mathbb{B}\Sigma_2$) if there exists a formula $\varphi \in \mathcal{F}$ such that $L = L(\varphi)$, i.e., if L is definable by some $\varphi \in \mathcal{F}$. The classes of languages defined by Σ_i and $\mathbb{B}\Sigma_i$ form a hierarchy, the quantifier alternation hierarchy. This hierarchy is strict, i.e., $\Sigma_i \subsetneq \mathbb{B}\Sigma_i \subsetneq \Sigma_{i+1}$ holds for all i, cf. [1,20].

Monomials

A *monomial* is a language of the form $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^\infty$ for $n \geq 0$, $a_i \in \Gamma$ and $A_i \subseteq \Gamma$. The number n is called the *degree*. In particular, A_0^∞ is a monomial of degree 0. A monomial is called k-monomial if it has degree at most k. In [3] it is shown that a language $L \subseteq \Gamma^\infty$ is in Σ_2 if and only if it is a finite union of monomials. We are interested in $\mathbb{B}\Sigma_2$ and thus in finite Boolean combination of monomials. For this, let $\equiv_k^\infty$ be the equivalence relation on Γ^∞ such that $\alpha \equiv_k^\infty \beta$ if α and β are contained in exactly the same k-monomials. Thus, $\equiv_k^\infty$-classes are Boolean combinations of monomials and every language in $\mathbb{B}\Sigma_2$ is a union of $\equiv_k^\infty$-classes for some k. Further, since there are only finitely many monomials of degree k, there are only finitely many $\equiv_k^\infty$-classes. The equivalence class of some word α in $\equiv_k^\infty$ is denoted by $[\alpha]_k^\infty$. Note, that such a characterization of $\mathbb{B}\Sigma_2$ in terms of monomials does not yield a decidable characterization.

Our characterization of languages $L \subseteq \Gamma^\infty$ in $\mathbb{B}\Sigma_2$ is based on the characterization of languages in $\mathbb{B}\Sigma_2$ over finite words. For this, we also introduce monomials over Γ^*. A *monomial* over Γ^* is a language of the form $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$ for $n \geq 1$, $a_i \in \Gamma$ and $A_i \subseteq \Gamma$. The degree is defined as above. Let $\equiv_k$ be the congruence on Γ^* which is defined by $u \equiv_k v$ if and only if u and v are contained in the same k-monomials over Γ^*. Again, a language $L \subseteq \Gamma^*$ is in $\mathbb{B}\Sigma_2$ if and only if it is a union of $\equiv_k$-classes for some k.

Algebra

In this paper all monoids are either finite or free. Finite monoids are a common way for defining regular and ω-regular languages. A monoid element e is *idempotent* if $e^2 = e$. An *ordered monoid* $(M, \leq)$ is a monoid equipped with a partial order which is compatible with the monoid multiplication, i.e., $s \leq t$ and $s' \leq t'$ implies $ss' \leq tt'$. Every monoid can be ordered using the identity as partial order. A homomorphism $h : (N, \leq) \to (M, \leq)$ between two ordered monoids must hold $s \leq t \Rightarrow h(s) \leq h(t)$ for $s, t \in N$. A *divisor* is the homomorphic image of a submonoid. A class of monoids which is closed under division and finite direct

products is a *pseudovariety*. Eilenberg showed a correspondence between certain classes of languages (of finite words) and pseudovarieties [4]. A pseudovariety of ordered monoids is defined the same way as with unordered monoids, using homomorphisms of ordered monoids. The Eilenberg correspondence also holds for ordered monoids [12]. Let $\mathbf{V_{3/2}}$ be the pseudovariety of ordered monoids which corresponds to Σ_2 and $\mathbf{V_2}$ be the pseudovariety of monoids which corresponds to languages in $\mathbb{B}\Sigma_2$. Since $\Sigma_2 \subseteq \mathbb{B}\Sigma_2$, we obtain $\mathbf{V_{3/2}} \subseteq \mathbf{V_2}$ when ignoring the order. The connection between monoids and languages is given by the notion of *recognizability*. A language $L \subseteq \Gamma^*$ is *recognized* by an ordered monoid $(M, \leq)$ if there is a monoid homomorphism $h : \Gamma^* \to M$ such that $L = \cup \left\{ h^{-1}(t) \mid s \leq t \text{ for some } s \in h(L) \right\}$. If M is not ordered, then this means that L is an arbitrary union of languages of the form $h^{-1}(t)$.

For ω-languages $L \subseteq \Gamma^\infty$ the notion of recognizability is slightly more technical. For simplicity, we only consider recognition by unordered monoids. Let $h : \Gamma^* \to M$ be a monoid homomorphism. If the homomorphism h is understood, we write $[s]$ for the language $h^{-1}(s)$. We call $(s, e) \in M \times M$ a *linked pair* if $e^2 = e$ and $se = s$. By Ramsey's Theorem [15] for every word $\alpha \in \Gamma^\infty$ there exists a linked pair (s, e) such that $\alpha \in [s][e]^\omega$. A language $L \subseteq \Gamma^\infty$ is *recognized* by h if

$$L = \bigcup \left\{ [s][e]^\omega \mid (s, e) \text{ is a linked pair with } [s][e]^\omega \cap L \neq \emptyset \right\}.$$

Since $1^\omega = 1$, the language $[1]^\omega$ also contains finite words. We thus obtain recognizability of languages of finite words as a special case.

Next, we define syntactic homomorphisms and syntactic monoids; these are the minimal recognizers of an ω-regular language. Let $L \subseteq \Gamma^\infty$ be an ω-regular language. The *syntactic monoid* of L is defined as the quotient $\mathrm{Synt}(L) = \Gamma^*/\approx_L$ where $u \approx_L v$ holds if and only if for all $x, y, z \in \Gamma^*$ we have both $xuyz^\omega \in L \Leftrightarrow xvyz^\omega$ and $x(uy)^\omega \in L \Leftrightarrow x(vy)^\omega \in L$. The syntactic monoid can be ordered by the partial order $\preceq_L$ defined by $u \preceq_L v$ if for all $x, y, z \in \Gamma^*$ we have $xuyz^\omega \in L \Rightarrow xvyz^\omega$ and $x(uy)^\omega \in L \Rightarrow x(vy)^\omega \in L$. The *syntactic homomorphism* $h_L : \Gamma^* \to \mathrm{Synt}(L)$ is given by $h_L(u) = [u]_{\approx_L}$. One can effectively compute the syntactic homomorphism of L. The syntactic monoid $\mathrm{Synt}(L)$ satisfies the property that L is ω-regular if and only if $\mathrm{Synt}(L)$ is finite and the syntactic homomorphism h_L recognizes L, see e.g. [10,21]. Every pseudovariety is generated by its syntactic monoids [4], i.e., every monoid in a given pseudovariety is a divisor of a direct product of syntactic monoids. The importance of the syntactic monoid of a language $L \subseteq \Gamma^\infty$ is that it is the smallest monoid recognizing L:

Lemma 1. *Let $L \subseteq \Gamma^\infty$ be a language which is recognized by a homomorphism $h : \Gamma^* \to (M, \leq)$. Then, $(\mathrm{Synt}(L), \preceq_L)$ is a divisor of $(M, \leq)$.*

3 Alphabetic Topology

As mentioned in the introduction, combining algebraic and topological conditions is a successful approach for characterizations of language classes over Γ^∞.

A topology on a set X is given by a family of subsets of X (called open) which are closed under finite intersections and arbitrary unions. We define the *alphabetic topology* on Γ^∞ by its basis $\{uA^\infty \mid u \in \Gamma^*, A \subseteq \Gamma\}$. Hence, an open set is given by $\bigcup_A W_A A^\infty$ with $W_A \subseteq \Gamma^*$. The alphabetic topology has been introduced in [3], where it is used as a part of the characterization of Σ_2.

Theorem 1 ([3]). *Let $L \subseteq \Gamma^\infty$ be an ω-regular language. Then $L \in \Sigma_2$ if and only if $\mathrm{Synt}(L) \in \mathbf{V_{3/2}}$ and L is open in the alphabetic topology.*

The alphabetic topology has by itself been the subject of further study [16]. We are particularly interested in Boolean combinations of open sets. An effective characterization of a language L being a Boolean combination of open sets in the alphabetic topology is given in the theorem below.

Theorem 2. *Let $L \subseteq \Gamma^\infty$ be an ω-regular language which is recognized by $h : \Gamma^* \to M$. Then the following are equivalent:*

1. *L is a Boolean combination of open sets in the alphabetic topology where each open set is ω-regular.*

2. *L is a Boolean combination of open sets in the alphabetic topology.*

3. *For all linked pairs $(s, e), (t, f)$ it holds that if there exists an alphabet C and words $\hat{e}, \hat{f}$ with $h(\hat{e}) = e, h(\hat{f}) = f$, $\mathrm{alph}(\hat{e}) = \mathrm{alph}(\hat{f}) = C$ and $s \cdot h(C^*) = t \cdot h(C^*)$, then $[s][e]^\omega \subseteq L \Leftrightarrow [t][f]^\omega \subseteq L$.*

Proof. "$1 \Rightarrow 2$": This is immediate.

"$2 \Rightarrow 3$": Let L be a Boolean combination of open sets in the alphabetic topology. Note that for $P, Q \subseteq \Gamma^*$ and $A, B \subseteq \Gamma$ it holds $PA^\infty \cap QB^\infty = (PA^* \cap QB^*)(A \cap B)^\infty$. Therefore, we may assume

$$L = \bigcup_{i=1}^{n} \left((P_i A_i^\infty) \setminus \left(\bigcup_{j=1}^{m_i} Q_{i,j} B_{i,j}^\infty \right) \right)$$

for some $P_i, Q_{i,j} \subseteq \Gamma^*$ and alphabets $A_i, B_{i,j} \subseteq \Gamma$.

Let (s, e) and (t, f) be some linked pairs, $C \subseteq \Gamma$ be an alphabet such that $s \cdot h(C^*) = t \cdot h(C^*)$ holds and there exist words $\hat{e}, \hat{f}$ with $h(\hat{e}) = e, h(\hat{f}) = f$ and $\mathrm{alph}(\hat{e}) = \mathrm{alph}(\hat{f}) = C$. Assume $[s][e]^\omega \subseteq L$, but $[t][f]^\omega \not\subseteq L$. Since h recognizes L, it suffices to show that $[t][f]^\omega \cap L$ is nonempty to obtain a contradiction. Let $u\hat{e}^\omega \in [s][e]^\omega \subseteq L$ for some $u \in [s]$. Since $s \cdot h(C^*) = t \cdot h(C^*)$, we may choose $x, y \in C^*$ such that $s \cdot h(x) = t$ and $t \cdot h(y) = s$.

The idea is to find an increasing sequence of words $u_\ell \in [s]$ and sets $I_\ell \subseteq \{1, \dots, n\}$ such that $u_\ell C^\infty \cap \left(P_i A_i^\infty \setminus \left(\bigcup_{j=1}^{m_i} Q_{i,j} B_{i,j}^\infty \right) \right) = \emptyset$ for all $i \in I_\ell$. We can set $u_0 = u$ and $I_0 = \emptyset$. Consider the word $u_\ell \hat{e}^\omega \in L$. There exists an index $i \in \{1, \dots, n\} \setminus I_\ell$ such that $u_\ell \hat{e}^\omega \in P_i A_i^\infty \setminus \left(\bigcup_{j=1}^{m_i} Q_{i,j} B_{i,j}^\infty \right)$. Choose a number k, such that $u_\ell \hat{e}^k \in P_i A_i^*$. Since $C = \mathrm{alph}(\hat{e}) \subseteq A_i$, we conclude $\beta_\ell = u_\ell \hat{e}^k x \hat{f}^\omega \in P_i A_i^\infty$. By construction we have $\beta_\ell \in [t][f]^\omega$ and therefore, assuming $[t][f]^\omega \cap L = \emptyset$, there exists an index j such that $\beta_\ell \in Q_{i,j} B_{i,j}^\infty$. Analogously, there exists k'
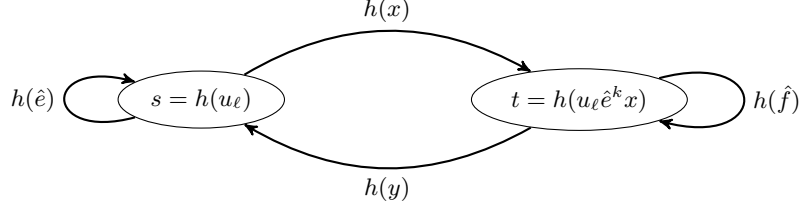
Figure 1. Part of the right Cayley graph of M in the proof of "2 $\Rightarrow$ 3".

such that $u_\ell \hat{e}^k x \hat{f}^{k'} y C^\infty \subseteq Q_{i,j} B_{i,j}^\infty$. Hence we can choose $u_{\ell+1} = u_\ell \hat{e}^k x \hat{f}^{k'} y$ and $I_{\ell+1} = I_\ell \cup \{i\}$. Figure 1 gives an overview of the construction.

Since $u_\ell [e]^\omega \subseteq L \cap u_\ell C^\infty$, this construction has to fail at an index $\ell \leq n$. Therefore, the assumption is not justified and we have $[t][f]^\omega \cap L \neq \emptyset$, proving the claim.

"3 $\Rightarrow$ 1": Let $\alpha \in [s][e]^\omega \subseteq L$ for a linked pair (s, e) and let $C = \text{im}(\alpha)$ denote the imaginary alphabet of α. By $\alpha \in [s][e]^\omega$ and the definition of C, there exists an $\hat{e} \in C^*$ with $\text{alph}(\hat{e}) = C$ and $h(\hat{e}) = e$. Define

$$L(s, C) = [s]C^\infty \setminus \left(\bigcup_{D \subsetneq C} \Gamma^* D^\infty \cup \bigcup_{s \notin t \cdot h(C^*)} [t]C^\infty \right).$$

We have $\alpha \in L(s, C)$ and $L(s, C)$ is a Boolean combination of open sets in the alphabetic topology where each open set is ω-regular. There are only finitely many sets of the type $L(s, C)$. The idea is to saturate L with sets of this type, i.e., it suffices to show $L(s, C) \subseteq L$. For $C = \emptyset$, we have $L(s, C) = [s] \subseteq L$. Thus, we may assume $C \neq \emptyset$. Let $\beta \in L(s, C)$ be an arbitrary element and let (t, f) be a linked pair such that $\beta \in [t][f]^\omega$. Since β is in $L(s, C)$, there exists a prefix u of β such that $\beta \in uC^\omega$ and $u \in [s]$.

By $\beta \in [t][f]^\omega$, one gets $\beta = v\beta'$ with $v \in [t], \beta' \in [f]^\omega$. Using $tf = t$ and $C \neq \emptyset$, we may assume that u is a prefix of v, which implies $\beta' \in C^\omega$. Hence we have $t = h(v) \in h(uC^*) = s \cdot h(C^*)$. By construction $\beta \notin \bigcup_{s \notin t \cdot h(C^*)} [t]C^\infty$ and therefore $s \in t \cdot h(C^*)$. It follows $s \cdot h(C^*) = t \cdot h(C^*)$. Since $\beta \notin \bigcup_{D \subsetneq C} \Gamma^* D^\infty$, there must be a preimage of f of full alphabet C. Therefore, $\beta \in [t][f]^\omega \subseteq L$. $\quad\square$

The alphabetic topology above is a refinement of the well-known Cantor topology. The Cantor topology is given by the basis $u\Gamma^\infty$ for $u \in \Gamma^*$. An ω-regular language L is a Boolean combination of open sets in the Cantor topology if and only if $[s][e]^\omega \subseteq L \Leftrightarrow [t][f]^\omega \subseteq L$ for all linked pairs (s, e) and (t, f) of the syntactic monoid of L with $s \mathrel{\mathcal{R}} t$; cf. [3,10,21]. Here $s \mathrel{\mathcal{R}} t$ denotes one of Green's relations: $s \mathrel{\mathcal{R}} t$ if and only if $s \cdot \text{Synt}(L) = t \cdot \text{Synt}(L)$. Theorem 2 is a similar result, but one had to consider the alphabetic information of the linked pairs. Hence, one does not have $s \mathrel{\mathcal{R}} t$ as condition, but rather $\mathcal{R}$-equivalence within a certain alphabet C.

Remark 1. The *strict alphabetic topology* on Γ^∞, which is introduced in [3], is given by the basis $\left\{ uA^\infty \cap A^{\text{im}} \mid u \in \Gamma^*, A \subseteq \Gamma \right\}$ and the open sets are of

the form $\bigcup_A W_A A^\infty \cap A^{\mathrm{im}}$ with $W_A \subseteq \Gamma^*$. Reusing the proof of Theorem 2 it turns out, that it is equivalent to be a Boolean combination of open sets in the alphabetic topology and in the strictly alphabetic topology. Since $uA^\infty = \bigcup_{B \subseteq A} uA^*B^\infty \cap B^{\mathrm{im}}$, every open set in the alphabetic topology is also open in the strict alphabetic topology. Further, one can adapt the proof of "2 $\Rightarrow$ 3" of Theorem 2 to show that if L is a Boolean combination of open sets in the strict alphabetic topology, then item 3 of Theorem 2 holds.

4 The fragment $\mathbb{B}\Sigma_2$

Place and Zeitoun have shown that $\mathbb{B}\Sigma_2$ is decidable over finite words. In particular, they have shown that, given the syntactic homomorphism of a language $L \subseteq \Gamma^*$, it is decidable if $L \in \mathbb{B}\Sigma_2$. Since every pseudovariety is generated by its syntactic monoids, the result of Place and Zeitoun can be stated as follows:

Theorem 3 ([14]). *The pseudovariety* $\mathbf{V_2}$ *corresponding to the* $\mathbb{B}\Sigma_2$-*definable languages in* Γ^* *is decidable.*

Our second main result charaterizes $\mathbb{B}\Sigma_2$-definable ω-regular languages. We use Theorem 3 as a black-box result.

Theorem 4. *Let* $L \subseteq \Gamma^\infty$ *be* ω-*regular. Then the following are equivalent:*

1. *L is a finite Boolean combination of monomials.*

2. *L is definable in $\mathbb{B}\Sigma_2$.*

3. *The syntactic homomorphism h of L satisfies:*

   (a) $\mathrm{Synt}(L) \in \mathbf{V_2}$ *and*

   (b) *for all linked pairs $(s,e),(t,f)$ it holds that if there exists an alphabet C and words $\hat{e}, \hat{f}$ with $h(\hat{e}) = e, h(\hat{f}) = f$, $\mathrm{alph}(\hat{e}) = \mathrm{alph}(\hat{f}) = C$ and $s \cdot h(C^*) = t \cdot h(C^*)$, then $[s][e]^\omega \subseteq L \Leftrightarrow [t][f]^\omega \subseteq L$.*

Note that item 3 of Theorem 4 is decidable: 3a is decidable by Theorem 3 and 3b is decidable since we can effectively compute the syntactic homomorphism h and $h(C^*)$ for all alphabets C. [1] We start with the difficult direction "3 $\Rightarrow$ 1" in the proof of Theorem 4. This is Proposition 1. The following lemma is an auxiliary result for Proposition 1.

Lemma 2. *For all k there exists a number ℓ such that for every set $\{M_1, \ldots, M_d\}$ of k-monomials over Γ^* and every w with $w \in M_i$ for all $i \in \{1, \ldots, d\}$, there exists an ℓ-monomial N over Γ^* with $w \in N$ and $N \subseteq \bigcap M_i$.*

Proof. Since the number of k-monomials over Γ^* is bounded, this induces a bound on d and one can iterate the statement. Therefore, it suffices to show the

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[1] During the preparation of this submission, we learned that Pierron, Place and Zeitoun [11] independently found another proof for the decidability of $\mathbb{B}\Sigma_2$ over infinite words. For documenting the independency of the two proofs, we also include the technical report of our submission in the list of references [6].

case $d = 2$. Consider two k-monomials $M_1 = A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$ and $M_2 = B_0^* b_1 B_1^* b_2 \cdots B_{m-1}^* b_m B_m^*$. Since $w \in M_1$ and $w \in M_2$, it admits factorizations $w = u_0 a_1 u_1 a_2 \cdots u_{n-1} a_n u_n$ and $w = v_0 b_1 v_1 b_2 \cdots v_{m-1} b_m v_m$ such that $u_i \in A_i^*$ and $v_j \in B_j^*$. The factorizations mark the positions of the a_is and the b_js and pose an alphabetic condition for the factors in between. Thus, there exists a factorization $w = w_0 c_1 w_1 c_2 \cdots w_{\ell-1} c_\ell w_\ell$, such that the positions of c_i are exactly those, that are marked by a_i or b_j, i.e., $c_i = a_j$ or $c_i = b_j$ for some j. The words w_i are over some alphabet C_i such that $C_i = A_j \cap B_k$ for some j and k induced by the factorizations. In the case of consecutive marked positions, one can set $C_i = \emptyset$. Thus, we obtain a monomial $N = C_0^* c_1 C_1^* c_2 \cdots c_{p-1} C_{p-1}^* c_p C_p^*$ with $C_p = A_n \cap B_m$. An illustration of this construction can be found in Figure 2. By construction $N \subseteq M_1$, $N \subseteq M_2$ and $w \in N$ holds. Since there are only finitely many monomials of degree k, the size of the number ℓ is bounded. $\square$
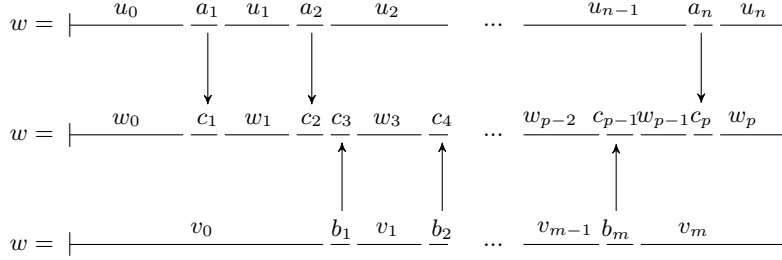

Figure 2. Different factorizations in the proof of Lemma 2. In the situation of the figure it holds $C_0 = A_0 \cap B_0$, $C_1 = A_1 \cap B_0$, $C_2 = \emptyset$, $C_3 = A_2 \cap B_1$, $C_{p-2} = A_{n-1} \cap B_{m-1}$, $C_{p-1} = A_{n-1} \cap B_m$ and $C_p = A_n \cap B_m$.

An analysis of the proof of Lemma 2 yields the bound $\ell \leq n_k \cdot k$, where n_k is the number of distinct k-monomials over Γ^*. Next, we show that a language which is in $\mathbf{V_2}$ and is a Boolean combination of alphabetic open sets is a finite Boolean combination of monomials. One ingredient of the proof is Lemma 2: we are able to compress the information of a set of k-monomials which contain a fixed word into the information of a single ℓ-monomial that contains this fixed word.

Proposition 1. *Let $L \subseteq \Gamma^\infty$ be a Boolean combination of alphabetic open sets such that $\mathrm{Synt}(L) \in \mathbf{V_2}$. Then L is a finite Boolean combination of monomials.*

Proof. Let $h : \Gamma^* \to \mathrm{Synt}(L)$ be the syntactic homomorphism of L and consider the languages $h^{-1}(p)$ for $p \in \mathrm{Synt}(L)$. By Theorem 3 we obtain $h^{-1}(p) \in \mathbb{B}\Sigma_2$. Thus, there exists a number k such that for every $p \in M$ the language $h^{-1}(p)$ is saturated by $\equiv_k$, i.e., $u \equiv_k v \Rightarrow h(u) = h(v)$. By Lemma 2 there exists a number ℓ such that for every set $\{M_1, \ldots, M_n\}$ of k-monomials and every w with $w \in M_i$ for all $i \in \{1, \ldots, n\}$, there exists an ℓ-monomial N with $w \in N \subseteq \cap_{i=1}^n M_i$. Let $\alpha \equiv_\ell^\infty \beta$ and $\alpha \in L$. We show $\beta \in L$ which implies $L = \cup_{\alpha \in L} [\alpha]_\ell^\infty$ and thus that

L is a finite Boolean combination of ℓ-monomials. Using Boolean combinations of monomials of the form $\Gamma^* a A^\infty$, one can test the imaginary alphabet of α and β. Hence we obtain $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$ for the imaginary alphabets. For simplicity, we write $C = \{c_1, \ldots, c_m\}$ for the imaginary alphabet of α and β.

Let $u' \leq \alpha$ and $v' \leq \beta$ be prefixes such that for every ℓ-monomial $N = N' \cdot C^\infty$ with $\alpha, \beta \in N$ we have that some prefix of u', v' is in N'. Further, let $\alpha = u\alpha'$ and $\beta = v\beta'$ such that

- $u' \leq u = u'u''$, $v' \leq v = v'v''$,

- $(c_1 c_2 \cdots c_m)^k$ is a scattered subword of u'' and v'',

- and there exists linked pairs (s, e) and (t, f) such that $s = h(u)$, $t = h(v)$, $\alpha' \in [e]^\omega$ and $\beta' \in [f]^\omega$.

Note that, by the choice of u', v', we have $\alpha', \beta' \in C^\infty$. We show that $s \cdot h(C^*) = t \cdot h(C^*)$, which implies $\beta \in L$ by Theorem 2. By symmetry, it suffices to show $t \in s \cdot h(C^*)$. Consider the set of k-monomials $N_i = N_i' C^\infty$ which hold at u, i.e., such that $u \in N_i'$ and $\alpha' \in C^\infty$. By the choice of ℓ, there exists an ℓ-monomial N' such that $u \in N'$ and $N' \subseteq \bigcap_i N_i'$. Since $u \in N'$, we obtain $\alpha \in N := N'C^\infty$ and by $\alpha \equiv_\ell^\infty \beta$ the membership $\beta \in N$ holds. By construction of v, there exists a word $\hat{v}$ with $\hat{v} \leq v' \leq v$, $\hat{v} \in N'$ and $\hat{\beta} \in C^\infty$ with $\hat{\beta}$ being defined by $\beta = \hat{v}\hat{\beta}$. Let $v = \hat{v}x$, then $x \in C^*$. The situation is depicted in Figure 3. We show that $ux \equiv_k v$ which then implies $t \in sh(C^*)$.
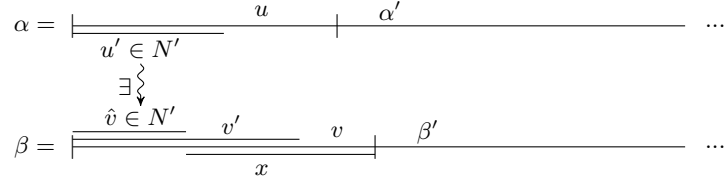

Figure 3. Factorization of α and β in the proof of Proposition 1

Let M be a k-monomial. If $ux \in M$, then there exists a factorization $M = M_1 M_2$ where M_1, M_2 are k-monomials with $u \in M_1$ and $x \in M_2$. Since $u\beta' \in M_1 C^\infty$, we obtain $\hat{v} \in N' \subseteq M_1$ by the definition of N'. We conclude that $v = \hat{v}x \in M_1 M_2 = M$.

If $v = \hat{v}x \in M$, then there exists a factorization of the monomial $M = M_1 M_2$ where M_1, M_2 are k-monomials with $\hat{v} \in M_1$ and $x \in M_2$. Since $(c_1 c_2 \cdots c_m)^k$ is a scattered subword of x, there must be some A_i^* in the monomial M_2 such that $C \subseteq A_i$ by the pigeonhole principle. Thus, there exists a factorisation $M_2 = M_{21} M_{22}$ in k-monomials M_{21}, M_{22} such that $M_{21} \cdot C^* = M_{21}$. Let $x = x'x''$ such that $x' \in M_{21}$ and $x'' \in M_{22}$ and consider $\beta = \hat{v}x\beta' \in M_1 M_{21} \cdot C^\infty$. Since $\alpha \equiv_\ell^\infty \beta$, we obtain $\alpha \in M_1 M_{21} \cdot C^\infty$. By construction, some prefix of u is in $M_1 M_{21}$ and by $M_{21} \cdot C^* = M_{21}$ and $x' \in C^*$, we obtain $ux' \in M_1 M_{21}$. Thus, $ux = ux' \cdot x'' \in M_1 M_{21} \cdot M_{22} = M$ holds. We conclude $ux \equiv_k v$ and thus $t = h(v) = h(ux) \in s \cdot h(C^*)$. □

It is well-known, that the direct product $(g \times h) : \Gamma^* \to M \times N, w \mapsto (g(w), h(w))$ of the homomorphisms $g : \Gamma^* \to M$ and $h : \Gamma^* \to N$ recognizes Boolean combinations:

Lemma 3. *Let L and K be languages such that L is recognized by $g : A^* \to M$ and K is recognized by $h : A^* \to N$. Then, any Boolean combination of L and K is recognized by $(g \times h)$.*

Next, we show that the algebraic characterisation $\mathbf{V_2}$ of $\mathbb{B}\Sigma_2$ over finite words also holds over finite and infinite words simultaneously. The proof of this is based on the fact that the algebraic part of the characterisation of Σ_2 over finite words and finite and infinite words is the same [3] and on the fact that every language of Σ_2 is in $\mathbb{B}\Sigma_2$, i.e., $\mathbf{V_{3/2}} \subseteq \mathbf{V_2}$.

Lemma 4. *If $L \subseteq \Gamma^\infty$ is definable in $\mathbb{B}\Sigma_2$, then $\mathrm{Synt}(L) \in \mathbf{V_2}$.*

Proof. By definition, $L \in \mathbb{B}\Sigma_2$ implies that L is a Boolean combination of languages $L_i \in \Sigma_2$. We have $\mathrm{Synt}(L_i) \in \mathbf{V_{3/2}} \subseteq \mathbf{V_2}$ by [3]. Since L is a Boolean combination of L_i, the direct product of all $\mathrm{Synt}(L_i)$ recognizes L by Lemma 3. In particular, $\mathrm{Synt}(L)$ is a divisor of the direct product of $\mathrm{Synt}(L_i)$ by Lemma 1. Hence, we obtain $\mathrm{Synt}(L) \in \mathbf{V_2}$. □

The proof that monomials are definable in Σ_2 is straightforward which yields:

Lemma 5. *Every monomial $L \subseteq \Gamma^\infty$ is definable in Σ_2.*

Combining our results we are ready to prove Theorem 4.

Proof (Theorem 4). "$1 \Rightarrow 2$": Since $\mathbb{B}\Sigma_2$ is closed under Boolean combinations, it suffices to find a Σ_2-formula for a single monomial. This is provided by Lemma 5.

"$2 \Rightarrow 3$": 3a is proved by Lemma 4. Since $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n$ is a set of finite words, a monomial $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^\infty$ is open in the alphabetic topology. The languages in Σ_2 are unions of such monomials [3] and thus languages in $\mathbb{B}\Sigma_2$ are Boolean combinations of open sets. This implies 3b by Theorem 2.

"$3 \Rightarrow 1$": This is Proposition 1. □

Example 1. In this example we show that $\mathrm{Synt}(L) \in \mathbf{V_2}$ for some language $L \subseteq \Gamma^\infty$ does not imply $L \in \mathbb{B}\Sigma_2$, i.e., the topological property 3b of Theorem 4 is necessary. For this define $L = (\{a, b\}^* aa \{a, b\}^*)^\omega$. We show that $\mathrm{Synt}(L) \in \mathbf{V_2}$, but L is not a Boolean combination of open sets of the alphabetic topology. Computing the syntactic monoid of L yields $\mathrm{Synt}(L) = \{1, a, b, aa, ab, ba\}$. The equations $b^2 = b$, $xaa = aax = aa$ and $bab = b$ hold in $\mathrm{Synt}(L)$. In particular, $(ab)^2 = ab$ and $(aa)^2 = aa$. Thus, $(s, e) = (aa, aa)$ and $(t, f) = (aa, ab)$ are linked pairs. Let h denote the syntactic homomorphism of L. Choosing aab as a preimage for $aa \in \mathrm{Synt}(L)$ yields the alphabetic condition $\mathrm{alph}(aab) = \mathrm{alph}(ab) = C$ on the idempotents. Since $s = t$, we trivially have $s \cdot h(C^*) = t \cdot h(C^*)$. However, $[aa][ab]^\omega \cap L = \emptyset$ but $[aa][aa]^\omega \subseteq L$. Thus, L does not satisfy the topological condition 3b of Theorem 4. It remains to check $\mathrm{Synt}(L) \in \mathbf{V_2}$. It is enough to show that the preimages are in $\mathbb{B}\Sigma_2$.

- $[1] = 1$
- $[a] = (ab^+)^*a$

- $[b] = (b^+a)^*b^+$
- $[ab] = (ab^+)^+$

- $[ba] = (b^+a)^+$
- $[aa] = \{a,b\}^* \, aa \, \{a,b\}^*$

One can find $\mathbb{B}\Sigma_2$ formulas for these languages, e.g., $[ab] = L(\varphi)$ with

$$\varphi \equiv (\exists x \forall y \colon x \leq y \wedge \lambda(x) = a) \wedge (\exists x \forall y \colon x \geq y \wedge \lambda(x) = b) \wedge$$
$$(\forall x \forall y \colon x \geq y \vee (\exists z \colon x < z < y) \vee (\lambda(x) \neq \lambda(y)))$$

and thus $\mathrm{Synt}(L) \in \mathbf{V_2}$. $\diamondsuit$

5 Summary and Open Problems

The alphabetic topology is an essential ingredient in the study of the fragment Σ_2. Thus, in order to study Boolean combinations of Σ_2 formulas, i.e., the fragment $\mathbb{B}\Sigma_2$ over infinite words, we looked closely at properties of Boolean combinations of its open sets. It turns out, that it is decidable whether an ω-regular language is a Boolean combination of open sets. This does not follow immediately from the decidability of the open sets. We used linked pairs of the syntactic homomorphism (which are effectively computable) to get decidability of the topological condition. Combining this result with the decidability of $\mathbf{V_2}$ we obtained an effective characterization of $\mathbb{B}\Sigma_2$ over Γ^∞, the finite and infinite words over the alphabet Γ.

In this paper we dealt with $\mathbb{B}\Sigma_2$, which is the second level of the Straubing-Thérien hierarchy. Another well-known hierarchy is the dot-depth hierarchy. On the level of logic, the difference between the Straubing-Thérien hierarchy and the dot-depth hierarchy is that formulas for the dot-depth hierarchy may also use the successor predicate. A deep result of Straubing is that over finite words each level of the Straubing-Thérien hierarchy is decidable if and only if it is decidable in the dot-depth hierarchy [18]. Thus, the decidability result for $\mathbb{B}\Sigma_2$ by Place and Zeitoun also yields a decidability result of $\mathbb{B}\Sigma_2[<, +1]$. The fragment $\Sigma_2[<, +1]$ is decidable for ω-regular languages [5]. This result also uses topological ideas, namely the factor topology. The open sets in this topology describe which factors of a certain length k may appear in the "infinite part" of the words. The study of Boolean combinations of open sets in the factor topology is an interesting line of future work, and it may yield a decidability result for $\mathbb{B}\Sigma_2[<, +1]$ over infinite words.

Another interesting class of predicates are modular predicates. In [7] the authors have studied $\Sigma_2[<, \mathrm{MOD}]$ over finite words. The results of [7] can be generalised to infinite words by adapting the alphabetic topology to the modular setting. As for successor predicates, we believe that an appropriate effective characterization of this topology might help in deciding $\mathbb{B}\Sigma_2[<, \mathrm{MOD}]$ over infinite words. To the best of our knowledge however, modular predicates have not yet been considered over infinite words.

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A Missing proofs

In this appendix we give the omitted proofs of the main paper.

Lemma 1. *Let $L \subseteq \Gamma^\infty$ be a language which is recognized by a homomorphism $h : \Gamma^* \to (M, \leq)$. Then, $(\mathrm{Synt}(L), \preceq_L)$ is a divisor of $(M, \leq)$.*

Proof. We assume that h is surjective and show that $\mathrm{Synt}(L)$ is a homomorphic image of M. If h is not surjective, we can therefore conclude that $\mathrm{Synt}(L)$ is a divisor of M. We show that $h(u) \leq h(v) \Rightarrow u \preceq_L v$. Let u, v be words with $h(u) \leq h(v)$ and denote $h^{-1}(h(w)) = [h(w)]$ for words w. Assume $xuyz^\omega \in L$, then there exists an index i such that $(h(xuyz^i), h(z)^\omega)$ is a linked pair. Thus, $[h(xuyz^i)][h(z)]^\omega \subseteq L$ and by $h(u) \leq h(v)$ also $[h(xvyz^i)][h(z)]^\omega \subseteq L$. This implies $xvyz^\omega \in L$. The proof that $x(uy)^\omega \in L \Rightarrow x(vy)^\omega \in L$ is similar. Thus, $u \preceq_L v$ holds which shows the claim. $\square$

Lemma 3. *Let L and K be languages such that L is recognized by $g : A^* \to M$ and K is recognized by $h : A^* \to N$. Then, any Boolean combination of L and K is recognized by $(g \times h)$.*

Proof. Since $L \cap [s][e]^\omega \neq \emptyset$ implies $[s][e]^\omega \subseteq L$ for some linked pair (s, e), we obtain $\overline{L} = \cup \left\{ [s][e]^\omega \mid [s][e]^\omega \cap \overline{L} \neq \emptyset \right\}$ for the complement of L. Thus, it suffices to show that $L \cup K$ is recognized by $(g \times h)$. Obviously, L is covered by $[(s,t)][(e,f)]^\omega$, where (s,e) is a linked pair of M with $[s][e]^\omega \subseteq L$ and (t, f) is any linked pair of N. Similiarly one can cover K and thus $M \times N$ recognizes $L \cup K$. $\square$

Lemma 5. *Every monomial $L \subseteq \Gamma^\infty$ is definable in Σ_2.*

Proof. Let $L = A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^\infty$. The Σ_2-formula

$$\exists x_1 \ldots \exists x_n \forall y : \bigwedge_{i=1}^{n} \lambda(x_i) = a_i \wedge \bigwedge_{i=1}^{n-1} x_i < y < x_{i+1} \Rightarrow \lambda(y) \in A_i \wedge$$

$$(y > x_n \Rightarrow \lambda(y) \in A_n) \wedge (y < x_1 \Rightarrow \lambda(y) \in A_0).$$

defines L. $\square$