A second-order theory for wave energy converters with curved geometry

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Introduction

We present a second-order nonlinear theory for a gate-type Wave Energy Converter (WEC) in a semi-infinite bi-dimensional channel. The gate model is similar to that shown in [1], except for a generalized weak displacement of the gate wetted surface about the vertical. Both free surface elevation and gate vertical displacement are assumed small if compared to the channel depth. Hence, the boundary conditions on the gate can be conveniently Taylor-expanded about the vertical. Perturbation-harmonic expansion allows us to decompose the nonlinear governing equations in a sequence of linear boundary-value problems of order n and harmonic m. Then, we show that the effect of the gate shape forces the first-harmonic component at the second order only. We also show that hydrodynamic interactions between the curved gate and short waves can have either constructive or disruptive effects in terms of power absorption.

Mathematical model

With reference to Figure 1(a), consider a semi-infinite channel of constant depth h' and width w'. Define a Cartesian reference system (x', z') with the x' axis lying on the undisturbed free surface level and the z' axis pointing upward. Primes indicate physical variables. At $x' = 0$ rests a gate-type WEC of mass M', allowed to oscillate horizontally along the channel without friction, under the action of incident harmonic waves. The WEC is connected with a wall by a spring-damper system in parallel. The spring has elastic constant C', while the linear damper, which simulates a power take-off (PTO) system, has PTO coefficient ν'_{pto}. Hence the PTO exerts a force proportional to the gate velocity. The fluid is assumed inviscid and incompressible and the flow irrotational. As a consequence, the velocity field satisfies the Laplace equation in the fluid domain $\Omega(x', z')$ for the velocity potential $\Phi'(x', z', t')$. Let us assume the following equation for the wetted gate surface

$$f'(x', z', t') = x' - \delta'(z') - X'(t') = 0, \tag{1}$$

where δ' denotes the deviation of the gate surface about $x' = 0$, while $X'(t')$ indicates the displacement depending on time t', positive rightward. Suppose that the mean of δ' has zero value. Let A' be the amplitude of the incident waves, ω' the incident wave frequency and g' the acceleration due to gravity. Then introduce the following non-dimensional quantities (see also [2]):

$$(x, z) = \frac{(x', z')}{h'}, \ \Phi = \frac{\Phi'}{A'\omega'h'}, \ \zeta = \frac{\zeta'}{A'}, \ t = t'\omega', \ \delta = \frac{\delta'}{\delta'_g}, \ X = \frac{X'}{A'}, \ G = \frac{g'}{\omega'^2 h'}, \tag{2}$$

where ζ' is the free surface elevation, δ'_g the length scale for δ' and G the non-dimensional frequency. Moreover we introduce the following two length ratios: $\epsilon = A'/h' \ll 1$, $\mu = \delta'_g/h' \ll 1$. Let us assume the following perturbation expansion for the non-dimensional unknowns up to second order $O(\epsilon)$:

$$\{\Phi, \zeta, X\} = \sum_{n=1}^{2} \epsilon^{n-1} \{\Phi_n, \zeta_n, X_n\}. \tag{3}$$

Using the latter expansion, together with the quantities defined by (2), yields a non-dimensional form of the Laplace governing equation and a no-flux boundary condition at the bottom:

$$\nabla^2 \Phi_n = 0, \ (x, z) \in \Omega; \quad \Phi_{n_z} = 0, \ z = -h, \tag{4}$$

where the x, z, t subscripts denote differentiation with respect to the relevant variable. Taylor expansion of the boundary conditions about $z = 0$ gives the free-surface dynamic condition:

$$-G\zeta_n = \mathcal{B}_n; \quad \mathcal{B}_1 = \Phi_{1_t}, \ \mathcal{B}_2 = \Phi_{2_t} + \zeta_1 \Phi_{1_{tz}}, \quad z = 0, \tag{5}$$

and the free surface mixed condition:

$$\Phi_{n_{tt}} + G\Phi_{n_z} = \mathcal{F}_n; \quad \mathcal{F}_1 = 0, \ \mathcal{F}_2 = -\left[\zeta_1\left(\Phi_{1_{ttz}} + G\Phi_{1_{zz}}\right) + |\nabla\Phi_1|_t^2\right], \quad z = 0. \tag{6}$$

Similarly, the kinematic boundary condition on the gate surface can be Taylor-expanded about $x = 0$:

$$\Phi_{n_x} = X_{n_t} + \mathcal{G}_n; \quad \mathcal{G}_1 = 0, \ \mathcal{G}_2 = -\Phi_{1_{xx}}\left[\frac{\mu}{\epsilon}\delta + X_1\right] + \Phi_{1_z}\frac{\mu}{\epsilon}\delta_z, \quad x = 0. \tag{7}$$

Finally, the equation of motion of the gate can be written as:

$$MX_{n_{tt}} + GCX_n + \nu_{pto}X_{n_t} = \int_{-1}^{0} dz\Phi_{n_t} + \mathcal{D}_n, \tag{8}$$

$$\mathcal{D}_1 = 0, \ \mathcal{D}_2 = \int_{-1}^{0} dz\left\{\Phi_{1_{tx}}\left[\frac{\mu}{\epsilon}\delta + X_1\right] + \frac{1}{2}|\nabla\Phi_1|^2\right\} + \frac{G}{2}\zeta_1^2 + \Phi_{1_t}\zeta_1. \tag{9}$$

in which $M = M'/(\rho'h'^2w')$ is the non-dimensional mass, $C = C'/(\rho'g'h'w')$ the non-dimensional elastic constant, $\nu_{pto} = \nu'_{pto}/(\rho'\omega'h'^2w')$ the non-dimensional PTO-coefficient and ρ' the water density. Because of harmonic motion, higher-order solutions imply higher harmonics. Hence, return in physical variables, omit the primes for convenience and assume the following harmonic expansion [2]:

$$\{\Phi_n, \zeta_n, X_n\} = \sum_{m=0}^{n} \{\phi_{nm}, \eta_{nm}, X_{nm}\} e^{-im\omega t} + *, \tag{10}$$

where the symbol $*$ indicates the complex conjugate. Substitution of the latter expansion into the governing equation and boundary conditions gives a sequence of linear boundary problems of order n and harmonic m.

Leading order problem $O(1)$

Performing harmonic expansion (10) of the governing equations, we obtain that the zeroth harmonic problem at the leading order is unforced with homogeneous Neumann boundary conditions, hence $\phi_{10} = 0$. The incident wave field is assumed to be at the leading order $O(1)$. As a result, this is a well-known diffraction problem forced by the incident wave field. The resulting velocity potential reads:

$$\phi_{11} = -\frac{igA}{\omega}\frac{\cosh k_0(h+z)}{\cosh k_0 h}\cos k_0 x - \sum_{n=0}^{\infty} \frac{\omega X_{11}D_n}{k_n C_n}\cosh k_n(h+z)\, e^{ik_n x}, \tag{11}$$

where the first term on the right represents the diffraction (incident + reflected) potential, while the second term the radiation potential due to body motion. In (11), the k_n's denote the roots of the dispersion relation

$$\omega^2 = gk_0\tanh k_0 h, \ \omega^2 = -g\overline{k}_n\tan\overline{k}_n h, \quad k_n = i\overline{k}_n, \quad n = 1, 2, \ldots \tag{12}$$

while the coefficients C_n, D_n correspond to:

$$C_n = \int_{-h}^{0} dz\,\cosh^2 k_n(h+z) = \frac{1}{2}\left(h + \frac{g}{\omega^2}\sinh^2 k_n h\right), \ D_n = \int_{-h}^{0} dz\,\cosh k_n(h+z) = \frac{\sinh k_n h}{k_n}. \tag{13}$$

The gate response is finally given by

$$X_{11} = -\frac{A\rho g w D_0/\cosh k_0 h}{-\omega^2 M + C - i\omega\nu_{pto} - i\omega^2\rho w \sum_{n=0}^{\infty}\frac{D_n^2}{k_n C_n}}. \tag{14}$$

Second order problem $O(\epsilon)$

At this order we obtain a zeroth-harmonic drift, plus two harmonics ω and 2ω forced by quadratic products of the leading order solution $O(1)$. The bound wave and the static displacement have expressions:

$$\eta_{20} = \frac{1}{g\epsilon}\left(2\omega^2|\eta_{11}|^2 - |\nabla\phi_{11}|^2\right), \ X_{20} = -\frac{\rho w}{\epsilon C}\left(\omega^2 h|X_{11}|^2 + g|\eta_{11}|^2\right). \tag{15}$$

The related velocity potential ϕ_{20} differs from zero and is forced both on the free surface and on the gate. However, it does not affect energy extraction and will be evaluated in future work. Effects of the gate shape on the total wave field influence the first harmonic solution only. In particular, we have:

$$\phi_{21} = \sum_{n=0}^{\infty} \left(-\frac{X_{21}D_n\omega}{C_n k_n} + \frac{\Psi_n}{\epsilon} \right) e^{ik_n x} \cosh k_n (z+h), \ \ \Psi_n = \frac{i}{k_n C_n} \int_{-h}^{0} dz \cosh k_n (z+h) \{\phi_{11_{xx}}\delta - \phi_{11_z}\delta_z\}.$$
(16)

The corresponding displacement is given by:

$$X_{21} = -\frac{i\omega\rho w \left[\sum_{n=0}^{\infty} \Psi_n D_n + \int_{-h}^{0} dz\, \phi_{11_x}\delta \right]}{\epsilon \left(-M\omega^2 + C - i\omega\nu_{pto} - i\omega^2\rho w \sum_{n=0}^{\infty} \frac{D_n^2}{C_n k_n} \right)}.$$
(17)

The numerator of (17) is the complex exciting torque, and represents the effect of the vertical profile of the gate. Note that, for a flat gate with $\delta = \delta_z = 0$, the displacement X_{21} becomes null. The denominator of expression (17) is made by terms that do not depend on the gate shape. Hence the displacement X_{21} is maximum if the exciting torque function is maximized accordingly.

The second harmonic boundary-value problem presents the following inhomogeneous forcing terms, respectively, on the free surface and on the gate surface:

$$\phi_{22_z} = \frac{4\omega^2\phi_{22}}{g} - \frac{i\omega}{g\epsilon}\left[\frac{\phi_{11}}{g}\left(-\omega^2\phi_{11_z} + g\phi_{11_{zz}}\right) - 2\left|\nabla\phi_{11}\right|^2 \right], \ \ \phi_{22_x} = -2i\omega X_{22} + \frac{1}{\epsilon}\phi_{11_{xx}}X_{11}.$$
(18)

Linearity allows decomposition of the velocity potential, i.e. $\phi_{22} = \phi_{22}^S + \phi_{22}^F$, in which ϕ_{22}^S represents the solution with homogenized condition on the gate, while ϕ_{22}^F is the velocity potential solution with homogenized condition on the free surface. Solution can be found by the eigenfunction expansion method (see also [2]). After some lengthy but straightforward algebra we obtain:

$$\phi_{22}^S = \sum_{l=0}^{\infty} \cosh\kappa_l(z+h)\frac{\cosh\kappa_l h}{g\kappa_l C_l}\left\{ \Gamma_1\left(-\frac{1}{2\kappa_l} - \frac{\kappa_l\cos 2k_0 x}{-8k_0^2 + 2\kappa_l^2}\right) + \Gamma_2\frac{e^{i\kappa_l x}(k_m + k_n) - e^{i(k_m+k_n)x}\kappa_l}{\kappa_l^2 - (k_m+k_n)^2} \right.$$

$$+ \Gamma_3\frac{e^{ik_n x}\kappa_l(k_0^2 - \kappa_l^2 + k_n^2)\cos k_0 x + k_n\left[e^{i\kappa_l x}(k_0^2 + \kappa_l^2 - k_n^2) - 2ie^{ik_n x}k_0\kappa_l\sin k_0 x\right]}{\left[k_0^2 - (\kappa_l + k_n)^2\right]\left[k_0^2 - (\kappa_l - k_n)^2\right]} + \Gamma_4\left(-\frac{1}{2\kappa_l} + \frac{\kappa_l\cos 2k_0 x}{-8k_0^2 + 2\kappa_l^2}\right)$$

$$\left. + \Gamma_5\frac{e^{ik_n x}\kappa_l(k_0^2 - \kappa_l^2 + k_n^2)\sin k_0 x + k_0\left[-ie^{i\kappa_l x}(-k_0^2 + \kappa_l^2 + k_n^2) + 2ie^{ik_n x}k_n\kappa_l\cos k_0 x\right]}{\left[k_0^2 - (\kappa_l + k_n)^2\right]\left[k_0^2 - (\kappa_l - k_n)^2\right]} \right\},$$
(19)

where κ_l are the real roots of the dispersion relation

$$4\omega^2 = g\kappa_0\tanh\kappa_0 h, \ 4\omega^2 = -g\bar{\kappa}_l\tan\bar{\kappa}_n h, \quad \kappa_l = i\bar{\kappa}_l, \quad l = 1,\ldots,\infty,$$
(20)

while the Γ-terms have expressions

$$\Gamma_1 = \left(\frac{Ag}{\omega}\right)^2\left(i\omega k_0 - \frac{3i\omega^5}{g^2}\right), \ \Gamma_2 = \omega^2 X_{11}^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{D_n D_m\cosh k_n h\cosh k_m h}{C_n C_m k_n k_m}\left(\frac{3i\omega^5}{g^2} - 2i\omega k_n k_m - i\omega k_n^2\right),$$

$$\Gamma_3 = -AgX_{11}\omega\sum_{n=0}^{\infty}\frac{D_n\cosh k_n h}{C_n k_n}\left(\frac{6\omega^4}{g^2} - k_0^2 - k_n^2\right), \Gamma_4 = -\frac{2iA^2 g^2 k_0^2}{\omega}, \Gamma_5 = 4iAgX_{11}\omega k_0\sum_{n=0}^{\infty}\frac{D_n\cosh k_n h}{C_n}.$$
(21)

The velocity potential due to forcing on the gate surface reads

$$\phi_{22}^F = -\sum_{l=0}^{\infty}\left(\frac{2\omega X_{22}D_l}{\kappa_l C_l} + \frac{\Delta_l}{\epsilon}\right)e^{i\kappa_l x}\cosh\kappa_l(h+z), \ \ \Delta_n = -\frac{i}{\kappa_l C_l}\int_{-h}^{0}dz\cosh\kappa_l(z+h)\phi_{11_{xx}}X_1,$$
(22)

in which C_l and D_l have the form (13) with wavenumber κ_l. Finally the gate response X_{22} is

$$X_{22} = \frac{\rho w\left\{-2i\omega\left[\int_{-h}^{0}dz\,\phi_{22}^S\big|_{x=0} - \sum_{l=0}^{\infty}\Delta_l D_l\right] + \int_{-h}^{0}dz\left[-i\omega\phi_{11_x}X_{11} + \frac{\nabla\phi_{11}\cdot\nabla\phi_{11}^*}{2}\right] - \frac{g\eta_{11}^2}{2}\right\}}{\epsilon\left(-4M\omega^2 + C - 2i\omega\nu_{pto} - 4i\omega^2\rho w\sum_{l=0}^{\infty}\frac{D_l^2}{C_l\kappa_l}\right)}.$$
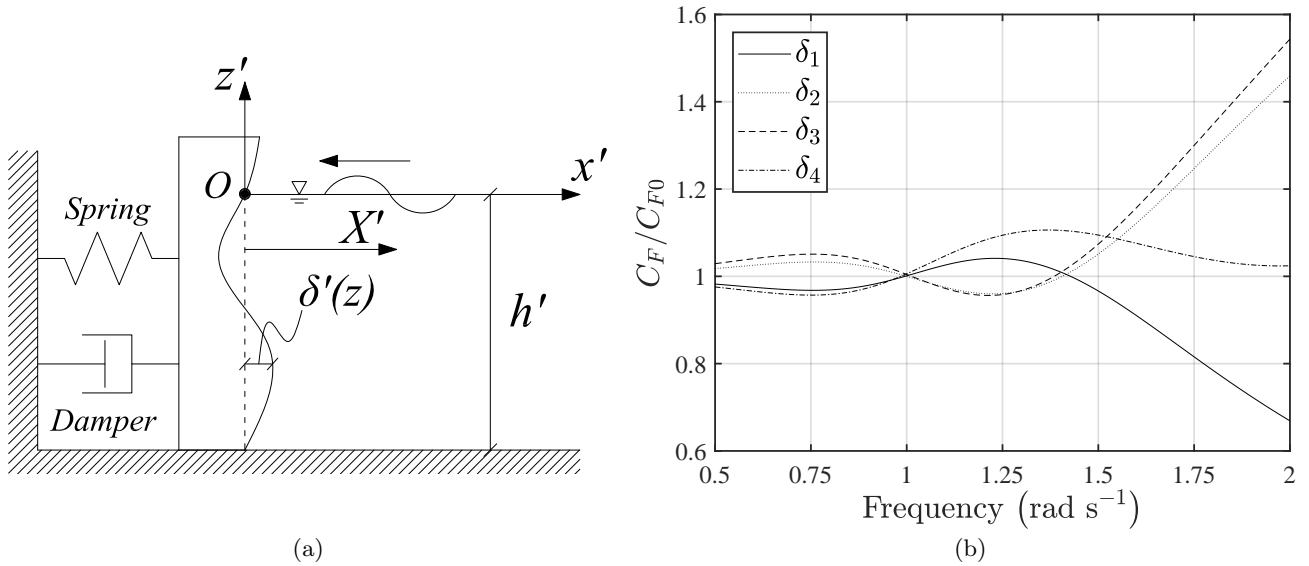(23)

Figure 1: (a) Geometry of the system in physical variables; (b) Ratio between the capture factor C_F of each configuration and the capture factor of a vertical flat-gate C_{F0}

Results and discussions

Since the gate response is $X = (X_{11} + \epsilon X_{21}) e^{-\mathrm{i}\omega t} + \epsilon X_{22} e^{-2\mathrm{i}\omega t} + *$, the average extracted power becomes

$$P = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathrm{d}t \left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2 \nu_{pto} = 2\omega^2 \nu_{pto} \left(|X_{11} + \epsilon X_{21}|^2 + 4 |\epsilon X_{22}|^2\right). \tag{24}$$

The efficiency of the system is finally assessed by considering the capture width ratio C_F defined as the ratio between the extracted power P and the incident wave energy flux per channel width w [1]. Now, let us compare the flat-gate ($\delta = 0$) with four different gate configurations, respectively

$$\delta_1 = A_g \sin \frac{\pi (z + h)}{h}, \ \delta_2 = -A_g \sin \frac{\pi (z + h)}{h}, \ \delta_3 = A_g \frac{2z + h}{h}, \ \delta_4 = -A_g \frac{2z + h}{h}, \tag{25}$$

where the water depth is $h = 10$ m and the maximum gate displacement is $A_g = h/10$. Let us assume the eigenfrequency of the system equal to $\omega = 1$ rad s^{-1} and the amplitude of the incident waves $A = h/10$. The power generated at $O(1)$ is maximized by assuming the PTO coefficient equal to the radiation damping in resonance conditions, i.e $\nu_{pto} = 7.9 \cdot 10^4$ kg m^2 s^{-1}. Figure 1(b) shows the ratio between C_F of each gate configuration and the capture factor of the flat-gate C_{F0}. Note that the curves cross each other in correspondence of the resonance frequency $\omega = 1$ rad s^{-1}. At low frequencies the effect of a vertical displacement is small. Constructive or destructive effects due to the vertical displacement δ arise mainly at large frequencies. Configurations 2 and 3 are more efficient than the flat gate, especially at large frequencies $\omega > 1.5$ rad s^{-1}. The least beneficial effects occur in Configuration 1. It should be noted that the optimization process depends on a large number of parameters even though the analytical model is 2-D. Work is currently in progress to use genetic algorithms able to search a family of vertical shapes which maximizes power output. Finally, we remark that this analytical model constitutes the basis for investigating nonlinear resonance phenomena occurring in more complex systems ([2, 3, 4]).

References

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