Using the Computer as a Tool for Constructivist Teaching: A Case Study of Grade 7 Students Developing Representations and Interpretations of Mathematical Notation when Using the Software Grid Algebra

by

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DECLARATION OF AUTHORSHIP

I, the undersigned author of this work, hereby declare that this thesis is authentic and entirely my own effort. I also certify that this work is original and has not been previously submitted for any other award. This thesis was supervised by Dr Dave Hewitt and Dr Ian Jones, faculty members of the Mathematics Education Centre, School of Science, at Loughborough University.

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ABSTRACT

The aim of this research was to investigate how I engaged in constructivist teaching (CT) when helping a group of low-performing Grade 7 students to develop new meanings of notation as they started to learn formal algebra. Data was collected over a period of one scholastic year, in which I explored the teacher-student dynamics during my mathematics lessons, where students learnt new representations and interpretations of notation with the help of the computer software Grid Algebra. Analysing video recordings of my lessons, I observed myself continuously changing my teaching purpose as I negotiated between the mathematics I intended to teach and the mathematics being constructed by my students. These shifts of focus and purpose were used to develop a conceptual framework called Mathematics-Negotiation-Learner (M-N-L). Besides serving as a CT model, the M-N-L framework was found useful to determine the extent to which I managed to engage in CT during the lessons and also to identify moments where I lost my sensitivity to students' constructions of knowledge. The effectiveness of my CT was investigated by focusing on students' learning, for which reason I developed the analytical framework called CAPS (Concept-Action-Picture-Symbol). The CAPS framework helped me to analyse how students developed notions about properties of operational notation, the structure and order of operations in numerical and algebraic expressions, and the relational property of the equals sign. Grid Algebra was found to be a useful tool in helping students to enrich their repertoire of representations and to develop new interpretations of notation through what I defined as informal- and formal-algebraic activities. All students managed to transfer these representations and interpretations of notation to pen-and-paper problems, where they successfully worked out traditionally set substitution-and-evaluation tasks.

DEDICATION

To the loving memory of Maria Cassar.

We all wish you were here. Maybe, somehow, you are.

ACKNOWLEDGEMENTS

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LIST OF ABBREVIATIONS

APS	actions pictures, and symbols
CAPS	Concept Action Picture Symbol (framework)
CSAC	computer screen activity capture
CT	constructivist teaching
ES	the equals sign (or equality symbol)
HLT	Hypothetical Learning Trajectory
ICT	information and communications technology
IWB	interactive whiteboard
JNT	journal note-taking
L-N	Learner-to-Negotiation (shift)
LSA	learning support assistant
LVR	lesson video recording
MfS	(the) mathematics *for* students
M-N	Mathematics-to-Negotiation (shift)
M-N-L	Mathematics-Negotiation-Learner (framework)
MoS	(the) mathematics *of* students
N-L	Negotiation-to-Learner (shift)
RC	radical constructivism/constructivist
R_mC_n	Row *m*, Column *n* (e.g. R_3C_2 means Row 3 Column 2)
SWW	students' written work
VRI	video-recorded interviews
ZPD	zone of proximal development

Chapter 1

Introduction

1.0 Overview

In this chapter, I present the background which led to general research aims and briefly overview the content of the other chapters. Table 1.0.1 includes the section titles of this chapter.

Table 1.0.1   Chapter 1 section titles

	Section Title	Page
1.1	Background and Orientation	2
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1.1 Background and Orientation

In my first few days as a mathematics teacher in a Maltese secondary school, I became aware that students' minds operated in unique ways and their interpretations of whatever mathematics I presented in the classroom were as different as their facial features. Soon enough, I learnt that the only body of knowledge that made sense to the students was the kind that they could develop themselves, which fitted within the context of their individual experiences, and which was viable enough to explain their experiential realities. This was one of the reasons why I came to embrace radical constructivism (RC) as a general orientation towards reality, knowledge, and learning.

1.1.1 The Challenge of Constructivist Teaching

One of the lessons I learnt in twenty years of teaching was that bringing RC notions into school teaching practice was no mean feat. There were frequent moments when I found myself at odds with my RC beliefs when I realised I was paying closer attention to the mathematics that educational authorities handed me down as a body of *a priori* knowledge and expected me to teach, rather than the educational needs of my students and the knowledge they were constructing. My sensitivity to RC made me compensate for such moments by shifting my attention from my mathematics to the

students, by asking them questions about what they were thinking and how they were interpreting my mathematical representations.

As my teaching approach started to settle into a regular style, this toing and froing between my mathematics and the students became the norm. Being mindful of the mathematics I intended to teach whilst maintaining a sensitivity to students' conceptual constructions was my understanding of what some researchers (e.g. Steffe, 1991) referred to as constructivist teaching (CT). My interest to pursue research on CT, thus, originated from my own experience as a mathematics teacher and my beliefs in RC.

1.1.2 Computers as Tools that Assist Constructivist Teaching

This reflective, pedagogical journey happened during interesting times in Malta's education. Towards the end of the millennium, the Maltese Government took measures to promote the use of computers for teaching and learning. Secondary school mathematics curricula even required teachers to make use of computer software in their lessons (SEC Mathematics Syllabus, 2002). I welcomed the prospect of doing mathematics with computers and so did my students. Computer use encouraged students to participate actively in outcomes-based activities, something which is widely reported in the literature (e.g., Ramsay, 2001; Mathew, 2004; Fritz, 2005). Like teachers in Richardson's (1999) study, I frequently took on the role of a learner in the classroom because most students were much more computer-oriented than I was and were always willing to show me how to do things better and quicker. I learnt a great deal from my students, both inside and outside the classroom. For instance, I mastered some of the more advanced applications of MS Excel thanks to a Grade 10 student of mine who sacrificed a number of breaks for the sake of my learning.

I saw the computer as a promising tool to pass the teaching baton to my students. On a regular basis, I used to organise revision lessons conducted entirely by students with the help of computers. These student expositions, or presentations as we used to call them, had much in common with what Pask and Scott (1972) appropriately called the

teach-back method. Consisting basically of a group of students teaching me back what they had learnt about a topic, these student expositions had a number of benefits. They encouraged the students in charge of the presentation to consolidate their knowledge of that topic, served as a revision to the other students, and were an excellent source of feedback from which I could draw models of students' interpretations of my lessons.

Most mathematics teachers I knew, however, did not share my enthusiasm about the use of computers for teaching and learning and this was what motivated me to research teachers' attitudes towards computer applications (Borg, 2011). In the meantime, I continued to make regular use of computer software in my lessons because I saw it as indispensable tool for teachers in the 21st century. Consistent with research reports about teachers' use of computers for teaching (e.g., Dugdale, 2001; Kirshner & Wopereis, 2003; Hermans, Tondeur, van Braak, & Valcke, 2008), I saw the computer as a medium which enabled teacher-to-student power shifts, fostered students' active participation and knowledge construction, encouraged student-centred lessons, and, consequently, was in line with my RC philosophy and my understanding of CT. In particular, it facilitated my toing and froing between mathematics and learners mentioned earlier. It was therefore a natural choice for me to pursue the current research about the use of computer software to assist CT.

1.1.3   Developing a Broad Research Question

At the time when I was thinking about starting this research, I was teaching five groups of students (Grades 7-11, one class per grade) in a boys' secondary school. Two of these were low-performing groups at Grades 7 and 8. The school's policy was to have a very small student-teacher ratio in such classes (average of 8 students and a maximum of 10). I considered that research about the use of computer software to support CT would be advantageous if conducted with one of these classes because the small number of students would mean that I could pursue a qualitative research with the possibility of investigating the conceptual developments of each student. Furthermore, Grade 7 offered the opportunity to investigate how my CT facilitated the conceptual developments of students whom I had not taught before.

Algebra was the branch in the Grade 7 curriculum that interested me the most, not only because I had researched students' understandings of algebra before (Borg, 1997) but also because I considered algebra, in the broad sense of the word, to be key in students' awareness of mathematical structures (Gattegno, 1988). As I discuss further on, students' representations and interpretations of notation were crucial in their initial encounters with formal algebra. This was a determining factor when narrowing the focus on the teaching and learning topic. Looking into computer software with the potential to help students to learn about notation, it seemed that Grid Algebra[1] (GA) had many favourable characteristics. Consequently, I chose GA as the main medium of the lessons I intended to investigate.

Along with the experiences discussed earlier, these research interests merged into the following broad research question:

- How do I engage in CT by making use of GA to help Grade 7 students develop concepts about notation?

This question required a review of literature related to:

(i) constructivism and notions of CT, and

(ii) the teaching and learning of algebra.

I discuss this review in Chapters 2 and 3, where I present more specific research questions based on these two facets of the study. As I discuss in Chapter 4, I decided that the best way to address these questions was to pursue a case study of myself as a teacher attempting to engage in CT, and the students I helped to begin learning formal algebra by developing meanings of notation. Lessons about the methods employed in this research were mostly learnt during a pilot study which I carried out in the year prior to that in which data for the main study were gathered. This is discussed in Chapter 5. Data analysis corresponds to the two aspects of the study mentioned

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[1] Developed by Dave Hewitt and distributed by the Association of Teachers of Mathematics.

above, i.e. CT and the learning it facilitated. These were discussed respectively in Chapters 6 and 7. A more detailed overview of the other chapters of this thesis is included below.

1.2   Overview of the Thesis Chapters

In this section I give an overview of the other chapters of this thesis. The chapter titles are included in Table 1.2.1.

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6	Analysis and Discussion of Constructivist Teaching	181
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Chapter 2 includes a review of mostly conceptual literature related to CT. I start by discussing the contrasting characteristics of constructivism and objectivism. Then, I focus on RC from which I draw my standpoint about the nature of mathematics knowledge and the process of learning. Against this backdrop, I discuss mathematical representations and interpretations, emphasising particularly the works of Kaput (1991) and Bruner (1966). Then I focus on experiential learning, where I discuss mostly the theories of Dewey (1938) and Kolb (1984). Having discussed mathematics and learning, I discuss the teacher's role where I review Jaworski's (1994) teaching triad from which I elicit a key concept that permeates my research: sensitivity to students, particularly their constructions of knowledge. This creates a context for a discussion about CT through theories of Freire (2000), Dewey (1902), Glasersfeld (1991b), Steffe

(1991), Simon (1995), and others. This review leads to the first set of research questions, those regarding CT.

The second part of the literature review resumes in Chapter 3, where I turn my focus on algebra and its learning. I start by discussing the nature and definition of "algebra" where I bring in algebra theorists including Mason (1996), Kaput (2008), Radford (2014), Kieran (1996), and Gattegno (1988). This creates a context for my definition of algebra where I distinguish between what I call informal- and formal-algebraic activities. I follow this by a review of mostly research-based literature about the learning of algebra where I focus on students' difficulties. Bringing evidence from the literature, I show that notation is a make-or-break issue in students' success in learning algebra and thus emphasise the importance of students being helped to enrich and extend their representations and interpretations of notation. I present the use of ICT as one possible way forward, where I bring in literature reports about computer software that was found effective in helping students learn about notation and algebra. These include microworlds, Logo, spreadsheets, and grid-based environments including GA. This leads to the second set of research questions, those regarding students' representations and interpretations of notation with the help of GA.

After ending my literature review with the specific research questions, I set out, in Chapter 4, to discuss the research methodology and method I adopted to investigate those questions. I start by presenting a rationale for a qualitative methodology and follow this by a discussion of the factors which made me choose case study as an inquiry approach. Then I turn my focus on the student participants, where I describe the school and educational context and provide a brief profile of each student. This is followed by a detailed description of the GA software, where I give an overview of its key features, especially those relevant to the research data. I also discuss the GA lessons and other means of data generation. Then I describe my data-gathering methods and rationalise my choice of tools and method of analysis. I consider the issues about my dual role of a teacher researcher and also how I addressed concerns about reliability and validity. I end this chapter by discussing ethical considerations required for such a study conducted with young students.

Chapter 5 is a short chapter in which I briefly review the pilot study I undertook in the scholastic year preceding that of the main study. I discuss how this pilot was intended to refine the research questions and get information about technical matters and data-collecting methods. As I show, valuable lessons were also learnt about the participants and their learning and about myself as a teacher researcher. The pilot was also useful in helping me come to know what to expect of the main study and develop possible hypotheses.

In Chapter 6, I turn my attention back to the main study where I report and discuss data related to the first set of research questions, those about CT. I analyse the dynamics of CT in the GA lessons, paying attention to the way my focus seemed to "oscillate" between the students and the mathematical content I intended to teach. Identifying patterns in the way I changed my teaching purpose leads to the development of the Mathematics-Negotiation-Learner (M-N-L) framework which serves to define and characterise my CT. This framework is built around constructivist theories such as those of Dewey (1902) and Steffe (1991). I proceed to show that this framework could also be used as an analytical tool to investigate the extent to which I managed to engage in CT. Finally, I discuss how the M-N-L framework was instrumental in identifying instances during the lessons where I seemed to lose my sensitivity to students' constructions of knowledge.

Data related to the second set of research questions are analysed and discussed in Chapter 7. I start by drawing up a second analytical framework by amalgamating Bruner's (1966) theory of mathematical representations and the signifier-signified theory of Kaput (1991). I call this framework CAPS (Concept-Action-Picture-Symbol) and I use it to analyse how students developed concepts about notation through action, picture, and symbol representations. I proceed to use this framework to help me analyse how students enriched their representations and extended their interpretations of notation, namely, operational symbols, numerical and algebraic expressions, unknowns and variables, and the equals sign. I show how the CAPS framework was used to focus on an important aspect of CT – the communication of ideas from students

to teacher. I present evidence to show that students transferred concepts learnt within GA to pen-and-paper problems.

In the conclusion of Chapter 8, I recapitulate the aims and outcomes of this research. I revisit and answer each research question bringing evidence from data analysed and discussed in Chapters 6 and 7. After describing the limitations of the research, I set out to discuss the significance of this study for the mathematics education community. This is followed by recommendations for future research and actions.

In the following section, I include some notes about the writing style I adopted throughout this research report.

1.3 Writing Style

Being both the researcher and the teacher involved in the case study, I had to be careful which hat I was wearing when writing this thesis. In general, I took the stance of a researcher, but there was not a single moment in the duration of this research where I managed to take off the teacher hat completely. The first reason for this was that I carried out this research while working as a full-time teacher, so I tended to think of myself as a teacher throughout the study. This influenced the way I interpreted the literature and also my decisions regarding methodology. The second reason was that during the analysis, I was investigating data in which I was acting as a teacher and was reminded of what was going on in my mind during the data generation process, when my role was mainly that of a teacher. So, while I wrote this thesis as a researcher, I did not exclude the possibility that the teacher component may be felt when one reads my discussions.

Patton (2002, p. 65) argues that 'writing in the first person, active voice communicates the inquirer's self-aware role in the inquiry'. This is a prerequisite of researchers conducting qualitative research since they need to assume a subjectivist stance. Thus, I used the first person to refer to myself in this write-up. This contributes to my

acknowledgement of researcher bias and of my possible influence on the data, being also the teacher involved in the case study.

With regards to tense I followed this rule: For theoretical arguments and claims, I used the present tense to imply the assumption that the quoted authors would still make those statements today. For any results reported in the literature and in my research, I used the past tense.

In my writing, I strived to employ gender-fair language (NCTE, 2002) by using gender-neutral pronouns. The only times when this was not possible was when reporting data about my students (all boys) and in the rare occasions where an important direct quote included the male pronoun.

Since the participants were grouped in mathematics lessons according to their performance in the Grade 6 examinations, I occasionally used the term "low-performing" or its derivatives. I intentionally avoided using terms like "low ability", or even "low-attaining" because I believe that ability and attainment may not always be reflected in the way students perform in examinations and tests. However, I have used this term very sparingly because, as I report later on, the students taking part in this study have shown that they do not even deserve the "low-performing" adjective. Their performance throughout the year and in the examination was far more than expected in their curriculum and comparable to what was expected of high-performing students.

Chapter 2

Literature Review Part 1: Constructivism and Constructivist Teaching

2.0   Overview

As Gergen (1995) asserts, beliefs about knowledge inform, justify, and sustain our educational practices. This first part of the literature review includes discussions of theories which form a backdrop for my own epistemological views of teaching and learning and, consequently, my understanding of constructivist teaching (CT). Table 2.0.1 gives an overall view of the sections in this chapter.

Table 2.0.1   Chapter 2 section titles

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A literature map is presented in Figure 2.0.1 (overleaf). I start this review by contrasting objectivist and constructivist epistemologies. Favouring the latter, I discuss the radical constructivist (RC) perspective of knowing and learning from which I draw my views about the nature of mathematics. I discuss mathematical representations and interpretations as espoused by Kaput (1991) and Bruner (1966) and show how these complement each other in the context of RC. I follow this by focusing on experiential learning, eliciting lessons from the works of Dewey (1938) and Kolb (1984), both of whom emphasise learning through action and reflection. In the context Brousseau's (1986) didactic triangle, I bring in the teacher's role and review Jaworski's (1994) teaching triad with an emphasis on teachers' sensitivity to students, in particular to students' construction of knowledge. This sets the scene for a discussion of CT, with reference to key theories such as Dewey (1902), Steffe (1991), and Simon (1995). This part of the literature review leads to the first set of research questions, those related to CT.

Figure 2.0.1   Literature topic map – Part 1
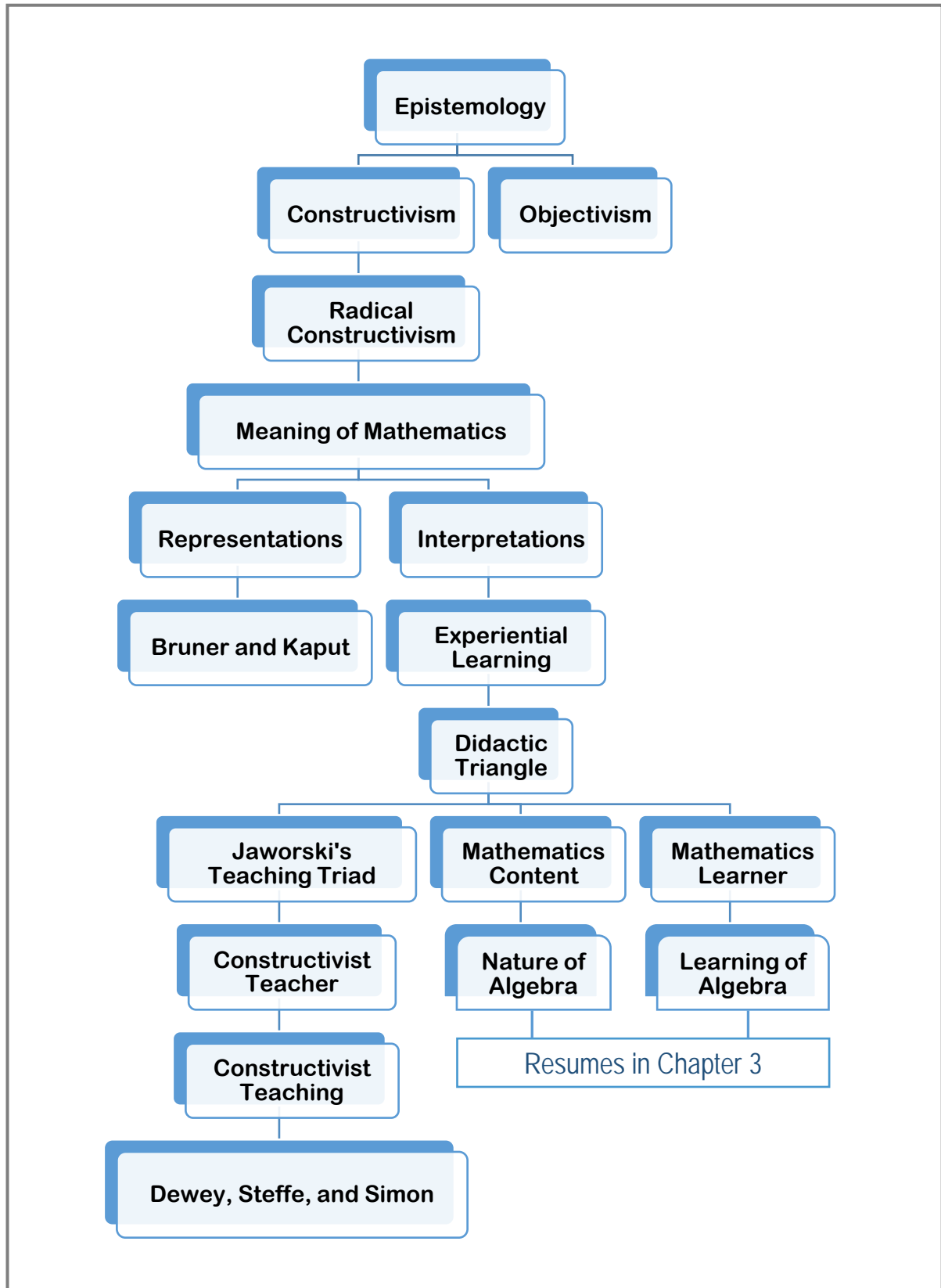
2.1 Radical Constructivism: A Paradigm for Knowing and Learning

In this section, I discuss RC as a paradigm of knowing and learning. I set the context by discussing constructivism in general, starting by contrasting objectivist and constructivist epistemologies.

2.1.1 Differences between Objectivist and Constructivist Epistemologies

Synthesising the works of Thorndike (1913) on objectivist learning psychology and Glasersfeld's (1989) studies of constructivism, Reeves (1997) captured five aspects of these epistemologies which set them apart. These contrasts are summarised in Table 2.1.1.

Table 2.1.1   Contrasts between the constructivist and objectivist epistemologies

	Objectivist Epistemology	Constructivist Epistemology
Nature of knowledge	Knowledge exists separate from and independently of knowing.	Knowledge does not exist outside the bodies and minds of human beings.
Existence of reality	Reality exists regardless of the existence of sentient beings.	Although reality exists independently, what we know of it is individually constructed.
Acquisition of knowledge	Humans acquire knowledge in an objective manner through the senses.	Humans construct knowledge subjectively based on prior experience and metacognitive processing or reflection.
Definition of learning	Learning consists of acquiring truth.	Learning consists of acquiring viable assertions or strategies that meet one's objectives.
Assessment of learning	Learning can be measured precisely with tests.	At best, learning can be estimated through observations and dialogue.

(Adapted from Reeves, 1997)

The table shows that while objectivism emphasises the existence of knowledge and reality in their own right, constructivism never accepts any notion of knowledge or reality without reference to the person who is coming to know or who is constructing a picture of reality.

For objectivists, both the world and its meaning exist objectively, independent of the human mind, and external to the knower (Jonassen, 1992; Lakoff, 1987). Jonassen (1991) affirms that constructivists do not refute the existence of an external reality. What they oppose is the objectivists' notion that people may have access to an external reality that exists independently of the senses. In fact, many constructivists, like Glasersfeld (1984) and Bruner (1986), have identified the roots of constructivism in the philosophy of Kant (1781) who postulates that an external, physical world (*noumena*) exists but we do not have a direct access to it. Rather, what we know and learn about it are the objects and occurrences we construe by way of our senses (*phenomena*). It follows that we can never really understand the world around us without referring to what our senses have processed in order to form our memories and experiences. This is a central standpoint of RC (Glasersfeld, 1984) as I discuss later.

Objectivists' and constructivists' perceptions of reality and knowledge entail major contrasts in their beliefs about learning. For objectivists, learners acquire knowledge about what already exists (Brown, Collins, & Duguid, 1989; Duffy, 1992) and learning may be gained objectively (Rand, 1966). In contrast, constructivists suggest that humans construct (rather than acquire) knowledge subjectively by reflecting on their perceptions and experiences. Glasersfeld (1984, p. 24) explains that coming to know is 'an ordering and organisation of a world constituted by our experience'. This emphasis on experience and reflection echoes the educational philosophy of Dewey, some of whose theories will be discussed in Section 2.3.

Furthermore, objectivists believe that the act of learning consists of acquiring truth, by grasping the meanings of words, i.e. the entities that words denote in an objective reality (Rand, 1966). On the other hand, constructivists hold that learning takes place when humans decide that an idea is *viable* and that their strategies serve them well in meeting their objectives. Glasersfeld (1995a, p.114) argues that learning 'comprises action schemes, concepts, and thoughts, and it distinguishes the ones that are considered advantageous from those that are not.'

With regards to assessment, objectivists believe that learning can be measured accurately and objectively, principally by means of tests (Reeves, 1997). Their view of assessment is goal-driven, where the evaluator specifies a set of clear objectives which indicate the expected observable behaviour of the learner (Vrasidas, 2000). On the other hand, constructivists believe that learning can only be *presumed* through observations and dialogue (Glasersfeld, 1989) in a context that is relevant to the learners (Brooks & Brooks, 1993). Constructivists are concerned with assessing the knowledge construction process (Vrasidas, 2000), using a range of methods and techniques (Burry-Stock, 1995; Zahorik, 1995) which include inquiry (Brooks & Brooks, 1993; Yager, 1991). They do not exclude the traditional test but this is not the only assessment measure (Cunningham, 1992).

Thus, my standpoint as a constructivist teacher has a bearing on the way I look at the nature knowledge and learning. Furthermore, the distinctive ways in which objectivists and constructivists look at assessing knowledge and learning entail parallel differences in the ways they look at gathering and analysing research data. These will be discussed in Chapter 4, where I give a rationale for my research methodology. I will now turn my attention to the constructivist paradigm with particular emphasis on RC.

2.1.2 Radical Constructivism

Altghough there are several strands of constructivism (Neimeyer & Raskin, 2001), Fox (2001) puts forward five claims on which constructivists seem to agree:

  (i)   Learning is an active process.
  (ii)   Knowledge is constructed, rather than innate or passively absorbed.
  (iii) Knowledge is invented not discovered.
  (iv)   Learning is essentially a process of making sense of the world.
  (v)   Effective learning requires meaningful, open-ended, challenging problems for the learner to solve.

Claim (iii) requires further clarification. Glasersfeld (1984, p. 25) asserts that learning is not the discovery of 'an independent, pre-existing world outside the mind of the knower'. However, this does not totally exclude the notion of "discovery". Glasersfeld's (1984) own references to the process of learning by phrases like 'get to know' and 'come to know' (pp. 28, 36 respectively) may be associated with Bruner's (1966, p. 90) 'internal discovery', by which he means learners' discovery of connections between concepts they had developed for themselves. Livio (2011) explains that the learning of mathematics involves the abstraction of concepts from the world around us and the discovery of connections among those concepts. Rather than the discovery of an objective reality, this is more akin to making sense of mentally constructed notions by linking them to others.

Although RC shares Fox's (2001) claims with other branches of constructivism, it distinguishes itself in two ways. The first distinction is that between trivial constructivism and RC. Glasersfeld (1991c, p. 16) says that 'those who merely speak of the construction of knowledge, but do not explicitly give up the notion that our conceptual constructions can or should in some way represent an independent, "objective" reality, are still caught up in the traditional theory of knowledge'. Riegler (2001) refers to this perspective as trivial constructivism and contrasts it with RC which holds that reality is subjective since the only reality we can gain access to is that which we see through our experiential worlds.

The second distinction is that between social constructivism and RC. While the former holds that knowledge is socially constructed, RC holds that all knowledge is personal and idiosyncratic. However, RC and social constructivism embrace similar notions when it comes to the construction of knowledge through peer collaboration. One of these is Vygotsky's (1978) *zone of proximal development* (ZPD) which he describes as 'the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers' (p. 86). ZPD infers that learners benefit from the guidance of a teacher when constructing knowledge.

From a RC standpoint, Vygotsky's (1978, p. 86) 'more capable peers' can be regarded as other "teachers" who may help a learner to develop concepts for her/himself. For example, students engaged in group work alternate in roles between being learners and being teachers, i.e. between listening to, reflecting upon, and making sense of what their group mates are saying (learners) and making their own contributions by guiding or demonstrating (teachers). Hence, there does not seem to be any contradiction in adopting the notion of ZPD within the context of RC, where social interaction assists the individual construction of knowledge. ZPD is in line with the RC contention that:

> [T]he "others" with whom social interaction takes place, are part of the environment, no more but also no less than any of the relatively "permanent" objects the child constructs within the range of its lived experience.
>
> (Glasersfeld, 1995a, p. 12)

RC is built on two principles about knowledge and cognition which Glasersfeld (1990a) claims to have surmised from Piaget, the first of which is shared by all branches of constructivism.

1a. Knowledge is not passively received either through the senses or by way of communication;

b. Knowledge is actively built up by the cognizing subject.

2a. The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability;

b. Cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality.

(Glasersfeld, 1990a, p. 22)

Glasersfeld's inclusion of the word "objective" when he mentions discovery seems to support my earlier argument that RC does not exclude the possibility of discovery as long as it is assumed to be a personal, invented conclusion – an *invented reality* (Glasersfeld, 1984). Another key point is Glasersfeld's likening of the development of thinking with biological evolution. Glasersfeld (1984) argues that just as the environment constrains living organisms and compels them to find ways in which they can find a viable existence, so does the experiential world serve as a testing ground for our concepts.

The relationship between biological and cognitive evolution (2a) is derived from the theories of Piaget who applied the biological concept of adaptation to epistemology (Glasersfeld, 1996). At the core of Piaget's teachings lies his theory of equilibration (Piaget, 1952, 1957, 1967, 1971, 1975, 1978, 1985; Inhelder & Piaget, 1958). Piaget proposed that knowledge in the mind is stored as groups of *schemas* – mental structures which represent some aspect of the experiential world. When children experience a new object (or situation or event) they try to deal with it by using an existing mental schema (*assimilation*). When the object does not fit in that schema a contradiction in children's thought occurs and this creates what Piaget calls *disequilibrium*, a kind of discomfort of thought which prompts them to transform or reconstruct their schema to be able to accommodate that object (*accommodation*). This enables assimilation and a stable equilibrium of the schema is regained. This is the process of equilibration, which, according to Piaget, is what drives intellectual growth.

Building on Piaget's idea of equilibration, Glasersfeld (1984) emphasises the viability aspect of mental schemas. He argues that only if our knowledge stands up to our experiences and proves itself reliable enough to help us predict, bring about, or avoid certain occurrences will it be considered as useful, relevant and viable. In this way, RC is a pragmatic paradigm to understand the creation, retention, modification, and disposal of knowledge. Our mental structures are constantly exposed to and tested within our personal experiential worlds from which we derive them and they either hold out or they do not.

In the context of evolutionary biology, Maturana and Varela (1980, 1992) invented the term *autopoiesis* – the notion that living organisms are, by definition, self-creating and self-sustaining systems (Maturana & Varela, 1980, 1992). Autopoiesis played an important part in the development of RC by giving an evolutionist slant to the notion of knowledge viability. Maturana (1970) claims that living systems tend to maintain that knowledge that has worked in the past can be expected to work again. Glasersfeld (1984) argues that the affinity between the RC epistemology and the theory of evolution is evident in the way humans respond to their environment. Just as living organisms mutate in such a way as to eliminate variants which are unfeasible within environmental constraints, humans continuously develop and modify ideas according to their experiences. In this way, our notions and theories about the world are proven or disproven on grounds of their viability and reliability.

Another similarity between biology and RC is the theory that living systems share common perceptions of the world they live in. Maturana and Varela (1980) and Glasersfeld (1991a) use the term *consensual domain* to describe areas of relative conformity in the way beings deal with their environment. Maturana and Varela (1980) describe the phenomenon where members of a species create shared ecological niches, a consensual domain of interaction and communication with their surroundings. Similarly, Glasersfeld (1991a) argues that although individual humans have different experiential worlds, they are able to agree and communicate through the possibility of building a consensus in certain areas of their subjective realities rather than through an observation of an objective reality. However, the development of a consensual domain does not mean that an absolute reality has been established. Glasersfeld (1991a) puts it plainly:

> If two people or even a whole society of people look through distorting lenses and agree on what they see, this does not make what they see any more *real*.
>
> (Glasersfeld, 1991a, p. xvi, original emphasis)

Hence, the consensual domain of humans is their accomplishment in reaching an area of relative agreement of understanding their *experiential* rather than absolute reality.

In Figure 2.1.2.1, I present a simplified illustration of how different persons may share a common consensual domain by which they seek to understand a viable explanation of particular aspects in their experiential world.

Figure 2.1.2.1   Consensual domains
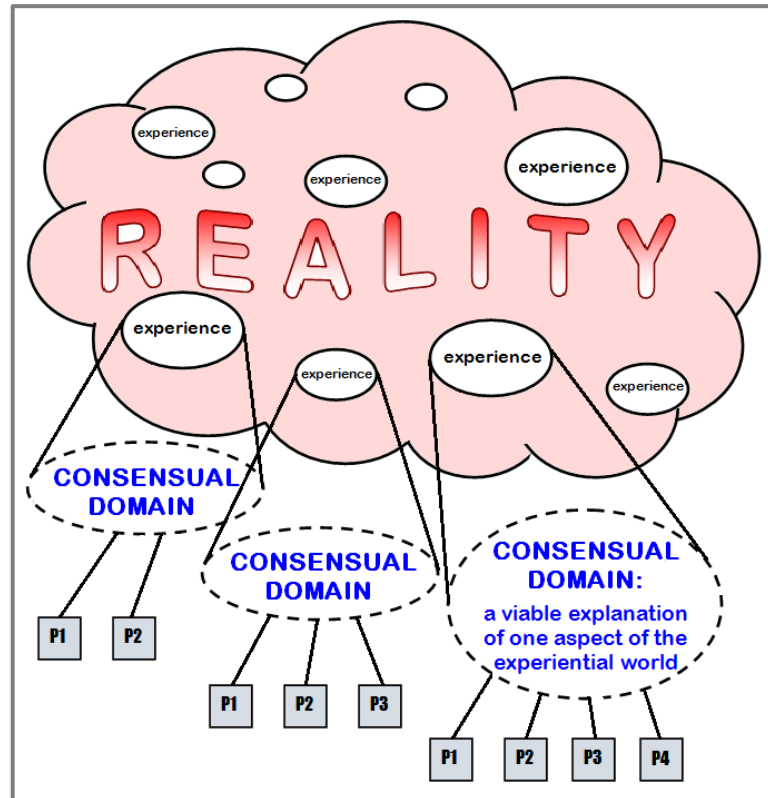
Although people (P1, P2, …) cannot have direct or objective access to an external reality, they can form common viable explanations for different aspects of it through their experiences, i.e. their experiential realities (Glasersfeld, 1991a). These frameworks of explanations are the consensual domains which are shared by two or more persons – a community. The diagram oversimplifies the complexity of such frameworks, however, because consensual domains may be subsets of or have common elements with other consensual domains.

In a classroom community, the teacher and the learners form a consensual domain about the topic of the lesson. For this to occur, the teacher needs to interpret and reflect about the learners' representations and the learners need to interpret and reflect about the teacher's and each others' representations. In Section 2.2, I review literature about

representations and interpretations of mathematics which, as Glasersfeld (1991a) says, is one of the oldest consensual domains – the domain of numbers.

2.2 Mathematical Representations and Interpretations

In this section, I discuss Kaput's (1991) conceptual framework of treating the interpretation and representation of mathematical notation as core elements of the mathematical consensual domain. I will then discuss Bruner's (1964, 1966) theory of mathematical representations and show its relevance for the teaching and learning of notation.

2.2.1 Kaput's Signifier-Signified Theory of Notation Usage

Kaput (1991) argues that while a person has no straightforward knowledge of reality, an observer of that person can still perceive and hypothesise on that person's interactions with experiential reality. He builds on his previous work (Kaput, 1985) and asserts that mathematical notation contributes to users' organisation of their thinking processes about an experiential, rather than external, world. This is in line with the RC claim that humans do not have access to the kind of reality which is detached from their experiences. Kaput (1991) distinguishes between:

(i) *mental structures*, which are the means by which an individual organises and manages the flow of experience, and

(ii) *notation systems*, which are the cultural and linguistic artefacts that are materially realised by a cultural or language community.

Kaput (1991) explains that individuals use notation systems to manage the creation and development of their own mental structures. Albeit limited in amount, notation systems are used throughout mathematics to express relationships whose variety is infinite and whose potential for generality is enormous. This activity is very similar to the way we use a finite number of words to create sentences and combination of sentences of infinite variety.

Material notation, things we interpret through reading and hearing and which we produce through writing and speaking, include alpha-numeric, pictorial, diagrammatic, and aural symbols. Such notation can either be consensual (such as language or conventional mathematical notation) or else idiosyncratic (personal symbols or marks). In both cases, a person interprets notation to create or elaborate mental structures and represents concepts by producing further notation. The person moves back and forth between the interpreted notation and the represented concept several times while reading and writing notation.

The first reference to the two-way link between external representations and internal interpretations was made by Saussure (1966)[2] who came up with a dichotomous model of the sign. He defined a sign as being composed of:

- the "signifier" - the external form which the sign takes, and

- the "signified" - the internal meaning or concept the sign represents.

Since he was working in the context of linguistics, Saussure's (1966) "sign" was the representation and interpretation of a verbal utterance, a sound-image, which was the signifier of the concept behind the word or phrase (Figure 2.2.1.1).

Figure 2.2.1.1   Saussure's signifier-signified model
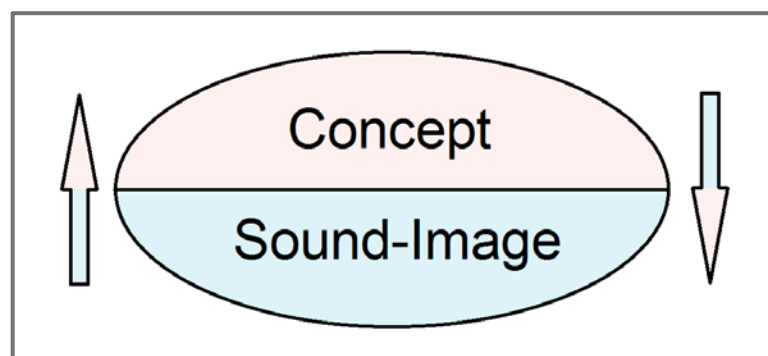
(Adapted from Saussure, 1966, p. 65)

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[2] Originally published in 1916.

Vergnaud (1987) takes up the signifier-signified notion and uses it for the first time in a mathematical context. He argues that a person observes and thinks about a referent in her/his experiential world, takes it to the signified (mental) level where 'invariants are recognised, inferences drawn, actions generated, and predictions made' (Vegnaud, 1987, p. 229), and externalises her/his thoughts with a signifier from a repertoire of symbolic syntax. Kaput (1991) builds on Vergnaud's (1987) use of "signifier" and "signified" to refer to mathematical notation and mental conception respectively. Figure 2.2.1.2 summarises the cyclical process, suggested by Kaput (1991), between physical operations observed through notation (signifier) and the mental operations associated with or evoked by those physical operations (signified).

Figure 2.2.1.2   Signifier physical operations and signified mental operations
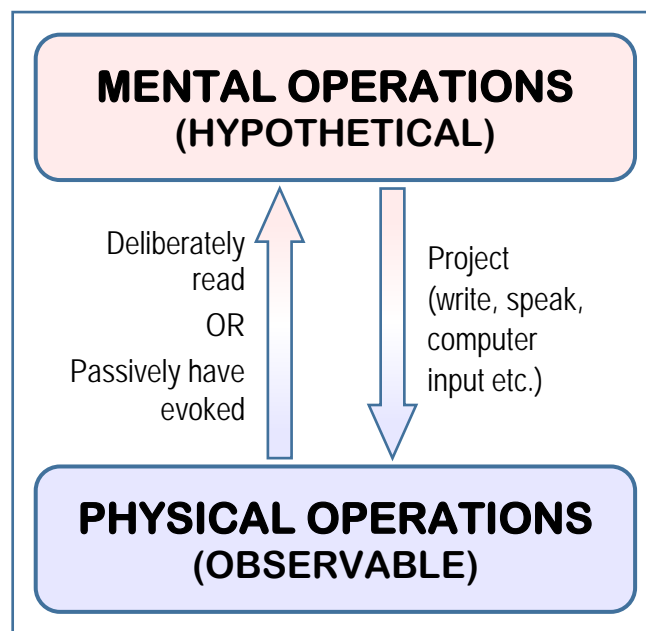
(Adapted from Kaput, 1991, p. 57)

It is through agreements about *observable* physical operations − the projected notation − that people can form a consensual domain, the repository of knowledge that a community, such as students and their teacher in a mathematics classroom, takes to be true on a particular occasion. Although physical operations of a member of that community are observable by other members in that community, the latter can only *hypothesise* about the mental operations of that member. Kaput (1991) argues that we constantly make inferences about other persons' thought processes by observing the

way they interact with external artefacts. He explains that we may observe the interactions of a person with notation and use those observations to hypothesise about that person's thinking (hypothesised mental states) and about how that person's thinking may be affecting and affected by that same notation (hypothesised interactions). This is demonstrated in Figure 2.2.1.3.

Figure 2.2.1.3   Our view of notation-users
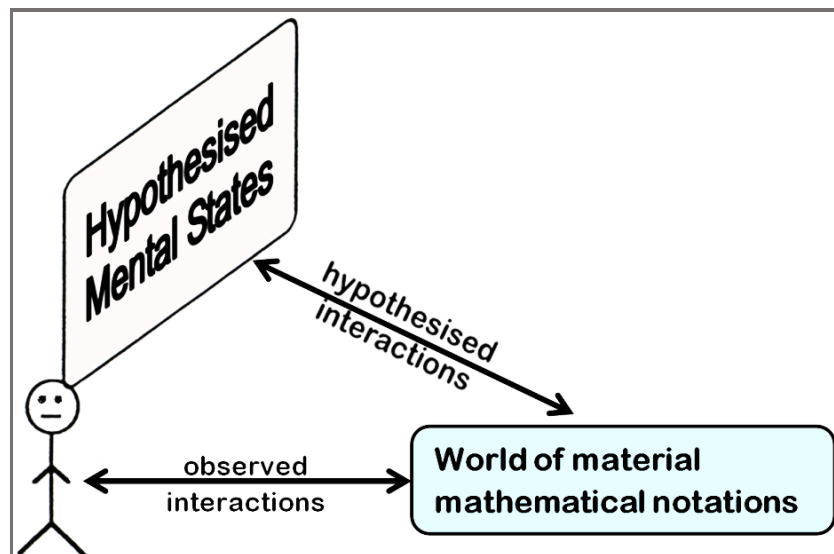
(Adapted from Kaput, 1991, p. 54)

In this way, Kaput (1991) explains:

(i)   the interplay between mathematical notation and the mathematical concepts it evokes or is derived from, and

(ii)   the possibility of other persons observing and making suppositions about the observable interactions between a person and mathematical notation.

Besides notation (i.e. written symbols), there are other mathematical representations which may be used to signify mental operations and which may thus be crucial in establishing a mathematical consensual domain in a teaching-and-learning setting. Such representations were studied by Bruner (1964, 1966), whose theory is discussed next.

2.2.2   Bruner's Theory of Knowledge Representation

At the core of Bruner's theory of instruction (Bruner, 1966) lies his theory of knowledge construction which he develops from his earlier work (Bruner, 1964). Bruner (1966) shows that knowledge can be represented in three ways which supplement spoken language:

(i)   **ENACTIVE REPRESENTATION**. A set of students' ***actions*** aimed at achieving a certain result (e.g. a child can learn the basic principles of a balance beam either by climbing on a see-saw or by experimenting with a balance and weights);

(ii)   **ICONIC REPRESENTATION**. A set of summary ***images*** or ***graphics*** which describe a concept without fully defining it (e.g. the balance beam could be illustrated on paper and its principles may be conveyed by studying the diagram and contemplating the concepts it evokes);

(iii) **SYMBOLIC REPRESENTATION**. A set of ***symbolic*** or ***logical propositions*** which are derived from a symbolic system governed by rules for the formation and transformation of such propositions (e.g. the balance concept may be used in physics to write down equations on moments or in mathematics to solve equations by using the inverse-and-balancing method).

Judge (1984) says that these modes of representation co-exist and that, for Bruner, the deepening of understanding comes by a spiral motion of transitions between one mode and the other rather than by a neat rectilinear fashion. In an interview with Shore (1997), Bruner rejects the idea that these representations are hierarchical stages and claims they can be incorporated in one another. He explains that humans' first interactions with ideas are intuitive and approximate. Then, when they find that their intuitions are incorrect, they feel the need to construct alternative ways of thinking about those ideas.

Thus, humans make transitions from one mode of representation to the other so that they can correct their understanding of that idea. This resonates with Piaget's (1975) equilibration theory, where learners constantly find they have to accommodate their

schemas when they find that these are incompatible with new experiences. Bruner states that when learners reflect about a concept these three modes are somehow 'all there and they gradually differentiate and get arrested' (Shore 1997, p. 11).

Bruner's (1966) Enactive-Iconic-Symbolic theory of mathematical representations complements Kaput's (1991) Signifier-Signified theory about the link between external representations (signifier) and conceptual interpretations (signified). While Kaput (1991) seems to focus entirely on symbolic representations (notation), Bruner's (1966) construct gives equal status to iconic and enactive representations each of which can serve as signifiers which evoke or are projected by mental operations. In Chapter 7, I show how I amalgamated these theories to form an analytical framework which was useful in investigating how students' mathematical representations helped me to develop models of their conceptual interpretations during the lessons.

Bruner's (1966) three modes of representation, especially the enactive, emphasise the importance of learning by doing, a standpoint which may be linked to a pragmatist philosophy. In fact, in his interview with Shore (1997), Bruner declares that he derived his representations model from Peirce, who defines 'what is tangible and conceivably practical as the root of every real distinction of thought, no matter how subtle it may be' (Peirce, 1878, p. 293). The importance of tangible, practical, and active experience in the process of learning was avidly promoted by Dewey (1938). This and the theory of experiential learning proposed by Kolb (1984) will be discussed in the next section.

2.3 Experiential Learning: Dewey's Theories and Kolb's Four-Stage Model

In this section, I discuss teaching and learning which centre on the learner's experience. I will consider two major models of experiential learning, those of Dewey (1938) and Kolb (1984), where the former is more theoretical and extensive while the latter takes on a more practical and specific form.

2.3.1 Dewey's Model of Experiential Learning

Core and influential constructivist theories of learning may be traced back to the teachings of Dewey who began his campaign for a more active and self-directed style of learning in schools over a century ago (Papert, 1993b). Elements of Dewey's philosophy of education may be found, among other works, in Piaget and Inhelder's (1962) theories of active learning, Bruner's (1966, 1967) theories of teaching and learning, Vygotsky's (1978)[3] definition of teaching as assisting the child's accomplishments, and Freire's (2000)[4] appeal not to treat the child as an empty depository of knowledge. In his extensive work about the potential of ICT to revolutionise teaching and learning, Papert espouses most, if not all, of these theories (e.g., Papert, 1993a, 1993b, 1996, 1998; Papert & Harel, 1991). He sums up Dewey's voluminous works by saying that it was 'Dewey's idea that children would learn better if learning were truly a part of living experience' (Papert, 1993b, p. 15).

For Dewey, learning is a consequence of experience, where learners experiment, invent, and test whether their actions are successful or not. In this context, Dewey sees mistakes as a natural component of action and experience. Dewey (1916) argues that teachers should not forbid learners to make mistakes. Rather, they should see mistakes as a requirement for learning the lessons of life:

> Opportunity for making mistakes is an incidental requirement. Not because mistakes are ever desirable, but because overzeal to select material and appliances which forbid a chance for mistakes to occur, restricts initiative, reduces judgement to a minimum, and compels the use of methods … remote from the complex situations of life.
>
> (Dewey, 1916, p.197)

Dewey's (1916) view of mistakes as a necessary incident in learning from experiences is today exhibited through a growing body of research which suggests that

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[3] Originally published in 1934.
[4] Originally published in 1968 and first published in English in 1970.

spontaneous classroom mistakes can be used by teachers to support learning (e.g. Lee, 2007; Radford, Blatchford, & Webster, 2011; Ingram, Pitt, & Baldry, 2015). Working in the field of mathematics education, Ingram et al. (2015) bring evidence to show that teachers may use mistakes as generators of classroom discussions and as tools in helping students develop and review mathematical concepts. This is one way in which teachers can convert classroom incidents (like spontaneous mistakes) into educative experiences.

However, Dewey (1938) warns that not all experiences are genuinely educative. He labels 'miseducative' (p. 25) those kinds of experiences which obstruct or discourage the growth of further experience. Consequently, Dewey acknowledges teachers' delicate task of providing learners with experiences which they find agreeable but which also stimulates their motivation to engage in more experiences.

Dewey (1938) defines experiential learning as *a formation of purpose* which involves:

(i)   Observation of the situation;

(ii) Knowledge of consequences of similar situations in the past which allow learners to attempt to foresee or anticipate the consequences of the current action to be taken;

(iii) Judgement, which synthesises observations and knowledge of past and of probable current consequences and what they both signify.

Step (iii) involves the decision about the success or failure of the current action. This entails the identification of and learning from mistakes which, according to Dewey (1916), are an experiential learning requirement. In any case, learning occurs if the knowledge reflected about in (ii) proves to be viable or unviable to explain the experience of (i) and which informs the decisions that follow from (iii). This tendency of humans to construct ideas according to their viability in explaining experiential phenomena (Kant, 1781) is in line with RC (e.g. Glasersfeld, 1984, 1990a). However, Dewey (1938) adds in a further element to his experiential learning theory. This is the role of impulse, the original desire to act upon observing a particular situation. Dewey

suggests that learners should channel the force created by these impulses to drive the process of experiential learning. He also suggests that teachers should guide learners to go through steps (i) – (iii) even though they may have a desire to take immediate action. In Figure 2.3.1.1, I illustrate Dewey's (1938) model of experiential learning. Step (i), observation, can be considered to be the input of a four-item process involved in (ii) and (iii):

- Reference to past experiences;
- Foresight / anticipation of consequences;
- Judgement;
- Action plan.

The output of this process is the final action to be taken where a new experience is gained. The energy of the original impulse and desire to bypass this process may be used to create a drive or momentum which maintains this process.

Figure 2.3.1.1   Dewey's model of experiential learning
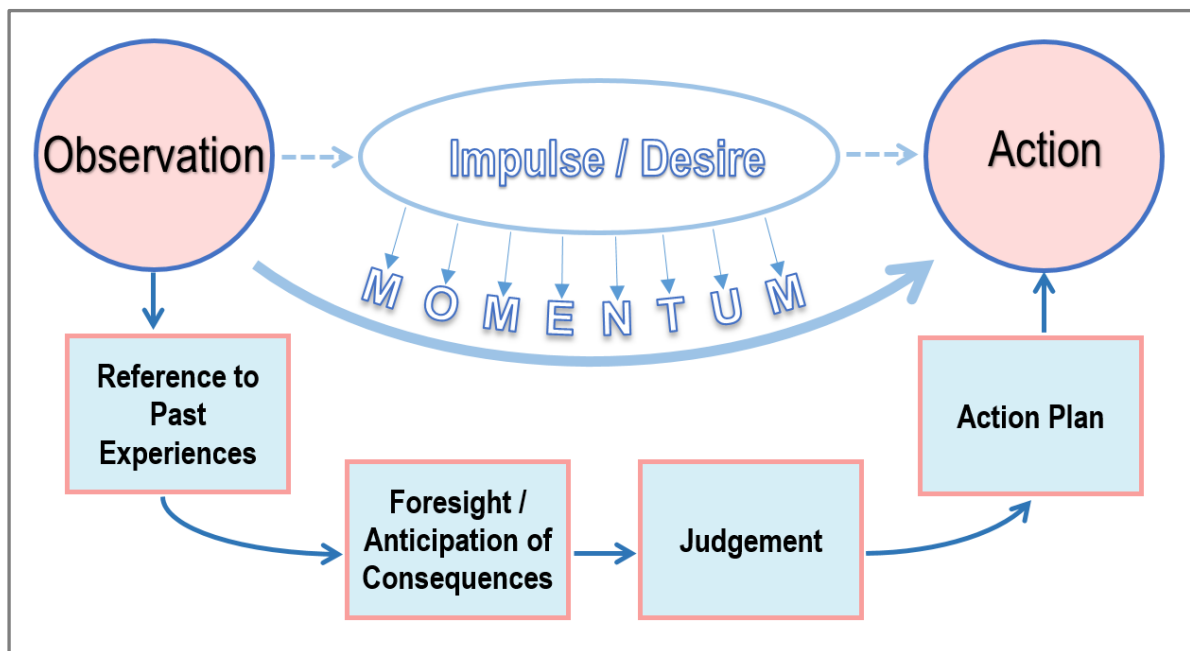
(Adapted from Dewey, 1938)

The process of experiential learning has one key element: reflective thinking. Dewey (1910) says that reflection involves consequential thinking, where thinking is not simply a sequence or trail of thoughts but a logical ordering of ideas in such a way that each

segment of thought 'determines the next as its proper outcome, while each in turn leans back on its predecessors' (p. 3). Reflection is one of the core features of Kolb's (1984) cycle of experiential learning, to which I will now turn.

2.3.2   Kolb's Model of Experiential Learning

In developing his model of experiential learning, Kolb (1984) acknowledges that he draws extensively from three theories:

(i)   Dewey's (1938) theory of learning through experience and reflection,

(ii)   Lewin's (1948) theory of group dynamics, and

(iii)   Piaget's (1970) theory of cognitive development.

However, Kolb (1984) presents his model in the context of a discussion about learning. He gives a short and powerful definition of experiential learning as follows:

> Learning is the process whereby knowledge is created through the transformation of experience. Knowledge results from the combination of grasping and transforming experience (Kolb, 1984, p. 41).

This definition is based a number of constructivist claims about the nature of knowledge and learning:

(i)   Learning is best conceived as a process, not in terms of outcomes;
(ii)   Learning is a continuous process based on experience;
(iii)   Learning is a process that requires the resolution of conflicts between dialectically opposed modes of adaptation to the world;
(iv)   Learning is a holistic process of adapting to the world;
(v)   Learning involves transactions between the learner and the environment;
(vi)   Learning is a process of creating knowledge.

Like Dewey (1938), Kolb (1984) attempts to describe how humans learn by thinking about and acting upon concrete experiences. He describes this learner-experience interaction in terms of a four-step cycle, a simplified version of which is shown in Figure 2.3.2.2.

Figure 2.3.2.2   Kolb's experiential learning cycle
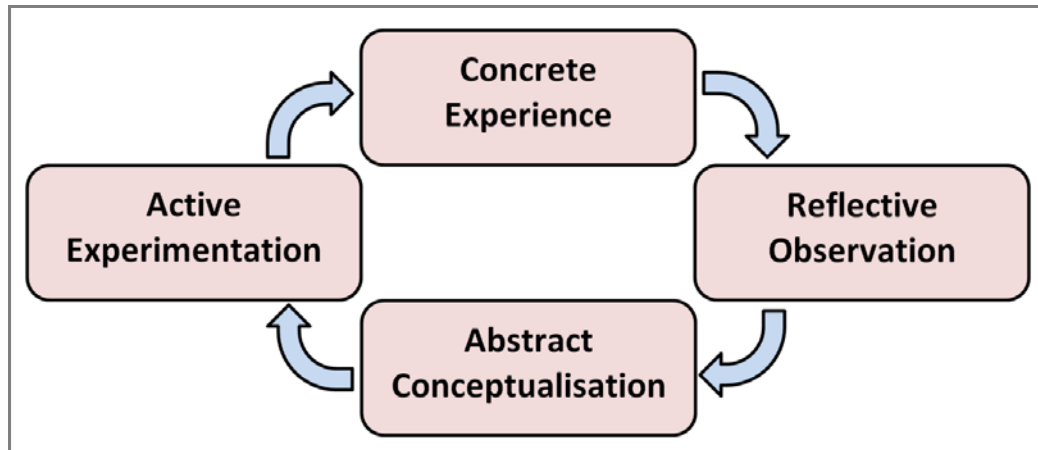
(Adapted from Kolb, 1984)

Stage 1. Kolb's (1984) cycle starts with a **concrete experience**. Learners are given the opportunity to be actively involved in an immediate experience. The use of "concrete" is there to distinguish between this type of experience, i.e. the first-hand, 'immediate, personal experience' (Kolb, 2015, p. 32) from the reflective/abstract experience which occurs when one thinks about first-hand experiences.

Stage 2. The second stage of the cycle is **reflective observation**. It means pausing to step back from the task at hand and reviewing *what* has been done and *what* has happened. Learners may need to exchange ideas with teacher and peers, but they can also opt to think and reflect individually.

Stage 3. Learners then make sense of the experience by finding relations between what has happened, what they reflected upon, and what they already know. They may draw upon past experiences, reflections, and understandings. It is the stage where new concepts are made or old concepts are modified. This is thus the stage of **abstract conceptualisation**. This involves generalisation of rules and/or formation of theories about the subject at hand.

Stage 4. The final stage of Kolb's learning cycle is **active experimentation** where learners consider how they can put what they have learnt into practice. If they are going to continue working on the same task, learners will refine or revise the way they will handle the task. In this way learning may be defined as the transformation of experience (Strauss, 2013).

The process between *observation* and *action* included in Dewey's (1938) theory of experiential learning may be associated with Kolb's (1984) Stages 3 to 4 where learners respectively ask the questions *What?, So what?,* and *Now what?* (Strauss, 2013). Perhaps more than Dewey (1938), Kolb (1984) emphasises the cyclical nature of experiential learning, where *active experimentation* leads to a new *concrete experience* and hence a new sequence of the four stages is generated.

Van Soest and Kruzich (1994), and Raschick, Maypole, and Day (1998) argue that some learners may have deductive orientations and may prefer to start the cycle from the *abstract conceptualisation* stage. This mode has similar properties to Kolb's (1984) "Stage 1". In fact, Kolb and Kolb (2005) assert that *concrete experience* and *abstract conceptualisation* are two dialectically related modes of grasping experience. On the other hand, they say that the two intermediary stages of *reflective observation* and *active experimentation* are two dialectically related modes of transforming experience. It seems, therefore, that the Kolb (1984) cycle may start from one of the two modes of *grasping* experience.

Similar to Dewey's (1938) theory, reflection plays a key role in the Kolb (1984) learning model. Whatever stage students choose to start their experiential learning, it is crucial that they reflect on that experience in a way that they can develop and transform their interpretation of current and related past experiences, where teachers take on the role of facilitators of this reflection. In this respect, Kolb (1984, p. 28) states that 'one's job as an educator is not only to implant new ideas but also to dispose of or modify old ones.'

How do educators go about facilitating the experiential learning process? In what situations, conditions, and context do teachers operate? What are the relationships between learners, knowledge, and teachers? The following section is dedicated to answering these questions in the context of mathematics education.

2.4 Learners, Knowledge, and Teachers: The Didactic Triangle

The didactic triangle (Figure 2.4.1.1), was used by Brousseau to study the dynamics between learners, teachers, and mathematical content (Brousseau, 1984, 1986, 1997; Brousseau & Otte, 1991). It has been used to structure and analyse research on teaching and learning inside the mathematical sphere (e.g., Steinbring, 1998, 2005; Hersant & Perrin-Glorian, 2005; Scherer & Steinbring, 2006; Pauli & Reusser, 2010) and also outside it (e.g., Tiberghien, Jossem, & Barojas,1998; Berglund & Lister, 2010).

Figure 2.4.1   The didactic triangle
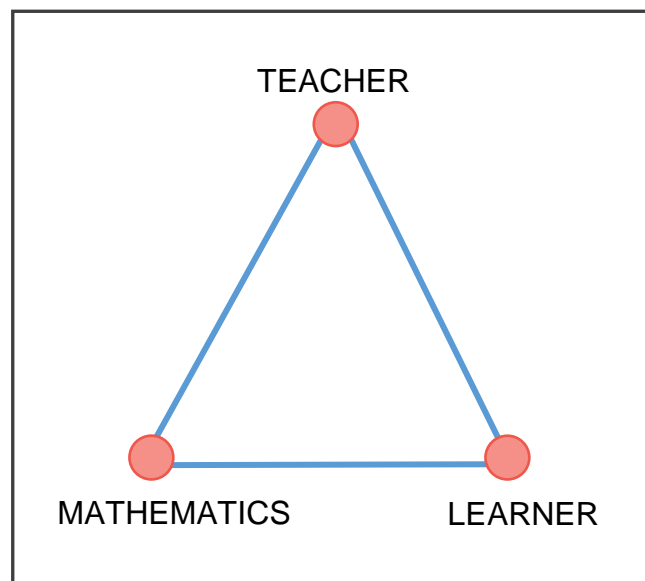
2.4.1 Didactical Situations

The central notion behind the didactic triangle is that of a *didactical situation* which Warfield (2006) defines as follows:

> A Situation describes the relevant conditions in which a student uses and learns a piece of mathematical knowledge. At the basic level, these conditions deal with three components: a topic to be taught, a problem in the classical sense and a variety of characteristics of the material and didactical environment of the action.

(Warfield, 2006, p. 105)

This seemingly mundane picture of what constitutes a mathematical teaching-and-learning situation is very appropriate for researchers in the field of mathematics education to ask questions about the three nodes of the didactic triangle such as, 'How does the teacher mediate between the learner and mathematics, shaping the learner's developing understanding of mathematics?' (Schoenfeld, 2012, p. 587). This question is very pertinent to the study of CT as I will show in section 2.5.

The didactic triangle makes it possible to isolate one of the nodes of the triangle and concentrate on it in order to elicit and expand its meaning and clarify its links with other nodes. Such is the work of Jaworski (1994, 2012) which I discuss below.

2.4.2   Focus on the Teacher Node of the Didactic Triangle: Jaworski's Teaching Triad

Jaworski (2012) focuses on the teacher node of the didactic triangle and identifies three interlinked activities that mathematics teachers carry out in their lessons. She calls these the *teaching triad*:

(i)   **Management of Learning.** This is the organisation of the classroom and the students, the set tasks, and the overall dynamics and interactions which teachers encourage in their lessons. It involves teachers' standpoints vis-à-vis curriculum, institutional standards, and assessment.

(ii)   **Sensitivity to Students.** This is inherent in teacher-student relationships and is the effort teachers make to become aware of learners' knowledge and thinking patterns and tendencies while striving to make their learners feel respected, included and cared for.

(iii)   **Mathematics Challenge.** This arises from teachers' epistemological standpoint and is the manner in which they present the mathematical problem to their learners in a way that interests them, motivates them to learn, and promotes participation and cognitive engagement.

This triad is almost totally dependent on the beliefs and the person of the teacher. What is special to me about the teaching triad is the way Jaworski (2012) portrays teachers not as clones of some ideological model derived from a set of philosophical beliefs but as individual, unique, human beings with their own personal characteristics and viewpoints (experiential realities), who are part of specific cultural settings, and who are subject to a range of influences of the communities in which they operate.

The teaching triad was developed from the realities of one teacher, Clare, who featured in an earlier study (Jaworski, 1994). The teaching triad was created as an analytical tool to characterise the teaching traits of Clare, where the three domains listed above were a synthesis of several other categories and which captured what Jaworski (1994) considered as important elements of Clare's teaching. Jaworski (1994) regards the triad as a strongly linked set of domains which are interdependent in such a way that some actions of a teacher may easily fall into the intersection two or three domains (Figure 2.4.2). She states that the three domains 'are closely interrelated, yet individual in identity, and have potential to describe the complex classroom environment'. (Jaworski, 1994, p. 108).

Figure 2.4.2.1   Jaworski's teaching triad
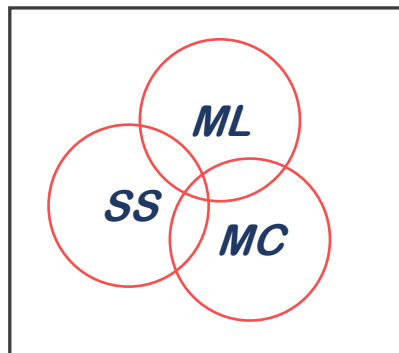
(Adapted from Jaworski, 1994, p. 107)

Jaworski (2012) says that one aspect of *management of learning* is teachers' interpretation of mathematical content. This suggests that mathematics is not the "out there" knowledge to be conveyed or received but an interpretation or construction of ideas. The mathematics that teachers present to their students is thus their own interpretation of concepts rather than an *a priori* body of knowledge.

Furthermore, as Chevallard (1988) argues, knowledge is inherently a tool to use rather than concepts to teach and learn. He claims that it is thus an artificial enterprise to teach a body of knowledge and that societies therefore delve into the arduous task of transforming knowledge from a tool to be put to use to something to be taught and learnt. He termed this as 'didactic transposition of knowledge' (p. 6). This makes teachers' presentation of the *mathematical challenge* (Jaworski, 1994) crucial if mathematics is to be seen as a useful subject, relevant to each student's experiential world. Kang and Kilpatrick (1992, p. 5) state that it is the teachers' duty 'to recontextualize and repersonalize the knowledge taught to fit the student's situation.'

Key in Jaworski's (1994) *sensitivity to students* are teachers' efforts to learn about the mathematics of their students. Only by sensitising themselves to students' exhibited representations and possible interpretations can constructivist teachers attempt to make mathematics relevant and meaningful to the students. According to Steffe (1991), learning about students' mathematics is one of the main tasks of RC teachers. This will be discussed in more detail in the following section, where I focus on this aspect of *sensitivity to students*: teachers' sensitivity to students' constructions of ideas.

2.5 Teaching with Constructivist Sensitivities

A principle which has significant bearing on my interpretation of CT is that without learning there is no teaching. I derive this from Freire (1998) who argues that if we agree that teaching is not simply the act of transferring knowledge but the creation of possibilities for the construction of knowledge, then we need to adhere to the philosophy that teaching and learning are so intertwined and interdependent that there is no teaching without learning. Freire puts it plainly:

> [T]here is no valid teaching from which there does not emerge something learnt and through which the learner does not become capable of recreating and remaking what has been taught. In essence, teaching that does not emerge from the experience of learning cannot be learnt by anyone.
>
> (Freire, 1998, p. 31)

The discussion that follows is presented in the context of this contention, that for teaching to exist it must bring about learning.

2.5.1 Constructivist Teachers and Their Teaching

In the context of mathematics education, Simon (1994) puts forward what he calls "myths" in constructivism:

Myth 1. *There is a specific kind of teaching called "constructivist teaching."* Simon (1994) argues that constructivism is a theory of learning and it does not stipulate any particular teaching style or method. Constructivists believe that any learning that results from teaching is a construction of the learner, regardless of the teaching style or method. Simon (1994) suggests that the question for the constructivist teacher is not, "Is my teaching constructivist?" but, "Is my teaching effective in bringing about learning?" He says that teachers who believe in constructivism may develop certain sensitivities about what their learners may be thinking or feeling that may make them more considerate of the learners' knowledge construction when they plan and carry out their lessons.

Myth 2. *Teachers with a constructivist perspective have no agenda for what their students will learn.* This myth may stem from models of learning which emphasise the activity of the learner while the role of the teacher is disregarded. Simon (1994) argues against this myth by saying that teachers with constructivist sensitivities usually spend much time planning how to create an environment that stimulates learning and that is 'designed to increase the probability that students will generate powerful ideas' (Simon, 1994, p. 74).

While finding no objection with Simon's (1994) second myth, I find some arguments he makes about the first almost contradictory. Even though he argues that CT is not a specific teaching approach, he still implies that there is much to be said about constructivist teachers' planning and implementing their lessons. Simon (1995, p. 117)

objects to the idea that CT translates into 'one set notion of how to teach'. Along similar lines, Engström (2014) objects to the use of the term CT on the grounds that constructivism is a theory of learning and not of teaching, and what is usually intended in the literature by CT is actually a progressive mode of teaching. I agree with both Simon (1995) and Engström (2014) that being a constructivist teacher does not translate into a set of particular stages that form a teaching *method* called CT, and that progressive methods have a tendency to be equated or at least associated with CT in the literature (e.g. Gash, 2014). I argue that even in what may seem to be a traditional classroom setting (a plenary approach), teachers may exhibit constructivist sensitivities when they stop to elaborate on a student's comment or question, ask students what they think about possible approaches in the solution of a problem, or encourage students to participate in classroom discourse and exhibit their interpretation of the topic. Hence, I agree that if CT were to be tied down to a particular teaching method I would rather, like Simon (1994), have it called a "myth". Ultimately, it would be a contradiction in terms if someone claims to promote RC philosophy and does not celebrate the uniqueness of teachers' mental constructions about what effective teaching is about and the diversity of their preferred teaching methods.

However, I argue that CT is plausible if it is attributed to teachers'

   (i)   adopting a constructivist stance on the nature of knowledge and knowing, and

   (ii)   being sensitive to students' active and subjective construction of knowledge, and taking actions because of that sensitivity.

These two characteristics permeate the literature about RC teachers' beliefs and classroom actions. This literature is discussed below.

2.5.2   Implications of Radical Constructivism for Teachers and Teaching

I support Freire's (1998) contention that teaching is dependent on learning to the extent that it only exists if it brings about learning. This means I cannot be both a constructivist and a teacher without allowing my constructivist beliefs and sensitivities about learning to have a bearing on my teaching. My experience as a teacher has taught me that my

perspective of the nature of knowledge, the process of learning, and learners' actions are omnipresent in my planning of the lessons, my interactions with the learners, and my own reflections during and after the lesson.

Glasersfeld (1991b, pp. 177-178) says that RC has many implications for teaching and he lists eight examples which I discuss below by referring to respective literature.

(i) *Constructivist teachers should not consider their learners as blank slates.* Glasersfeld's (1991b) warning is found in Freire's (2000) disapproval of the *banking concept of education* where he rebukes the kind of teaching which treats learners as '"containers," …"receptacles" to be "filled" by the teacher' (p. 72). If teachers are sensitive to constructivist notions, they cannot bear to act as if their job is to fill empty minds with knowledge. If they believe learners build up their own knowledge from personal experiences and reflections (Reeves, 1997), then it is a requirement for constructivist teachers to keep in mind that by the time they have come to their class, learners have already found many viable ways of dealing with their experiential worlds. Consequently, constructivist teachers should make it their business to get some idea of what concepts their learners have developed and how they relate to them.

(ii) *Constructivist teachers should refrain from telling learners that their response is wrong.* Glasersfeld (1991b) argues that whatever learners say in answer to a question or to a posed problem it is what makes sense to them at that moment. Constructivist teachers should take their responses seriously as such, regardless of how odd or "wrong" they might seem to them. Rather than discouraging their learners by saying that what they said is wrong, they should enquire what made the learners respond in that way. This goes beyond acknowledging that a response is mistaken and using the mistake to support teaching (e.g. Radford et al., 2011; Ingram et al., 2015). Glasersfeld (1991b) points out that given the way learners may have interpreted the question, their answer may turn out to be correct. Bruner (1986, p. 92) warns about this when he discusses the notion of psychological reality which is invented and modified

according to the 'psychological processes that people use in negotiating their transactions with the world.'

(iii) *Before modifying individual learners' concepts and conceptual structures, constructivist teachers should try to build up a model of those learners' reasoning.* Glasersfeld (1991b) says that models of how learners think can be generalised, but before assuming that a learner fits the general pattern, teachers should have some solid evidence that this is a viable assumption in that particular case. This seems to imply that although there exist various models of thinking styles (e.g., Briggs Myers, 1980; Kolb, 1984; Harrison & Bramson, 1984; Rancourt, 1988; Sternberg, 1988, 1997; Gardner, 1991) teachers should not simply pigeonhole learners because they fit in one particular thinking style in one particular occasion. It may be that learners may shift between preferred thinking styles according to the situation or simply change them over time (Dunn & Griggs, 1995).

(iv) *Constructivist teachers should seek to discover thinking patterns of their learners by asking them to explain how and why they got to their answer.* In this way, teachers may develop second-order experiential models (Steffe et al. 1983; Steffe & Ulrich, 2013) of learners' concepts which they may use to coordinate their interactions with them. These models are "second-order" because they are hypothetical models of what other people may be thinking and "experiential" because they are models of possible experiences of those people, in this case, the learners. Glasersfeld (1991b) suggests that when learners see that their teacher values their reasoning they will be more receptive to the idea that a particular answer or reasoning they have may not retain its viability in different situations.

(v) *Constructivist teachers should foster learners' motivation to ask and learn by creating problems which learners find pleasure in solving.* Questions that teachers pose in relation to the topic of the lesson may not be of any particular interest to the learners. Glasersfeld (1991b) warns that telling learners that they are answering the questions correctly does very little for learners' conceptual development if they were not interested in the questions in the first place. On

the other hand, if the questions arise naturally from a problem-solving situation which the learners find enjoyable, this is sure to stimulate interest to delve into further questions and further learning. For Dewey (1938) this is what constitutes an educative experience.

(vi) *For constructivist teachers, successful thinking is far more important than "correct" answers.* Teachers should praise learners' successful thinking even if they hold that it was based on unconventional premises. This results from constructivist teachers' investment in seeking to discover factors about the thinking patterns of their learners. This is consistent with Bruner's (1966) theory of instruction:

> We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself [sic.], to consider matter as an historian does, to take part in the process of knowledge-getting. Knowing is a process, not a product.
>
> (Bruner, 1966, p. 72)

(vii) *Constructivist teachers must have a very flexible mind in order to understand and appreciate learners' thinking.* Glasersfeld (1991b) advises teachers to be aware that learners may start from premises that seem inconceivable to them and so it is very easy for teachers to deduce that learners are wrong to think in that manner or conclude that statement. An infinitely flexible mind about the topic at hand is therefore needed for teachers to grasp the logical trail of ideas (Dewey, 1910) of individual learners.

(viii) *Constructivist teachers can never justify what they teach by claiming that it is "true".* Glasersfeld (1991b) says that mathematics teachers can only justify what they teach by taking it as a derivative of conventional definitions and operations. This concurs with the theory that mathematical truth is a convention (Quine, 1936; Quinton, 1963) which holds that mathematical statements are true by virtue of the meanings of the terms they contain (Ernest, 1991). This stance may help teachers to be more amenable to alternative reasoning of their learners and more willing to delve into discussions with learners who challenge the mathematical "truth" which they are bound to do if the meanings they construct do not agree with those posed by their teachers.

With the exception, perhaps, of characteristic (viii), all of the above seem to focus on constructivist teachers' duties vis-à-vis the learner. Teachers however, have other obligations to attend to. For instance, the integrity of teachers would be put in question if they disregard the subject content which they are entrusted to teach. Once again, I find Dewey's teachings to be very relevant: this time in thinking about constructivist teachers' obligations towards both the learner and the curriculum.

2.5.3 Constructivist Teachers' Obligations towards the Learner and the Curriculum

Being a pragmatist, Dewey insisted that education should be both practical and useful. In *The Child and the Curriculum,* Dewey (1902) neatly captured the two basic factors that necessitate education – the learner and the lessons to be learnt:

> [A]n immature, undeveloped being; and certain social aims, meanings, values incarnate in the mature experiences of the adult. The educative process is the due interaction of these forces.

> (Dewey, 1902, p. 2)

Dewey (1902) discusses two extreme ways of going about education. The first is a pedagogy centres on the curriculum. For Dewey, this system is unacceptable because the learner is inactive: 'the child is simply the immature being who is to be matured; he [sic.] is the superficial being who is to be deepened' (p. 13). Dewey advocates a pragmatic pedagogy with constructivist sensitivities which presents the curriculum in a way that students can see its relevance and usefulness by setting it against the backdrop of their own individual experiences. At the same time, however, Dewey argues that if teachers are too much focused on the learners, they may easily lose sight of what knowledge they have been entrusted to teach. As educators, we must simultaneously 'take our stand with the child and our departure from him [sic.]' (p. 13). Hence, Dewey argues that teachers must strike a balance between providing appropriate learning opportunities for the learners and being sensitive to learners' interests and experiences.

> The child and the curriculum are simply two limits which define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction.

> (Dewey, 1902, p. 16)

Figure 2.5.3.1   Dewey's construct about curriculum, learner, and teaching process

(Adapted from Dewey, 1902)

Figure 2.5.3.1 illustrates Dewey's construct about the function of teaching: to bring the curriculum to the learners and vice versa. Teaching must be defined by curriculum and learners just as roads are defined by possible journeys between two places. Driver (1995) argues that, from a constructivist standpoint, teachers must reconsider the traditional view of the curriculum as the body of knowledge which society deems important to pass on to its youngsters. In line with Dewey's (1902) teaching construct, Driver (1995) says that while acknowledging factors like social aims and values and subject content structure, the constructivist teacher needs to take into account the experiences, ideas, and purposes which learners bring to classroom while providing learners with new experiences. I associate this view with what I have called "toing and froing" between teacher and learners (Chapter 1), or rather between the teacher's knowledge and that of the learners.

Dewey's (1902) construct may be observed in most teaching models based on constructivist beliefs, including:

- Karplus's (1977) Learning Cycle,

- Driver and Oldham's (1986) Constructivist Instruction Model,

- Van Hiele's (1986) Phases of Instruction,

- Black and McClintock's (1995) Interpretation Construction Model, and

- Gagnon and Collay's (2006) Constructivist Learning Design.

In various shades and emphases, these teaching models have a number of core elements in common. Sunal (1995) maintains that such frameworks aim to help learners to:

(i)   become aware of their previous knowledge,

(ii)   experience a cooperative and safe learning environment,

(iii)   compare new alternatives and perspectives to their prior knowledge,

(iv)   connect the new perspectives to what they already know,

(v)   construct their own "new" knowledge, and

(vi)   apply their knowledge in diverse situations.

The most strikingly distinguishing feature of such teaching frameworks is, perhaps, an additional characteristic suggested by Driver (1995):

> [E]xperience by itself is not enough. It is the *sense* that students make of it that matters. If students' understandings are to be changed ... then intervention and negotiation with an authority, usually the teacher, is essential. From this perspective, teaching is also a learning process.

<div align="right">(Driver, 1995, p. 399, original emphasis)</div>

While supporting Dewey's (1938) experiential learning theory, Driver's (1995) statement echoes what has been stressed before (Steffe, 1991; Freire, 1998), that constructivist teachers need to consider themselves students of their students and negotiators between their knowledge and that of their students, thus drawing Dewey's (1902) connecting "line" between curriculum and learners. A teaching model based on such constructivist perspectives in mathematics education is that proposed by Simon (1995). This is discussed next.

2.5.4   Simon's Model of Teaching Mathematics with a Constructivist Perspective

There are extensive reports in the literature about how a constructivist mathematics learning environment can be fostered (e.g., Davis, 1984; Schoenfeld, 1985; Lampert, 1988; Steffe, Cobb, & Glasersfeld, 1988; Steffe, 1991; Wood, Cobb, & Yackel, 1991; Fennema, Franke, Carpenter, & Carey, 1993; Ball, 1993; Jaworski, 1994; Carpenter, Fennema, & Franke, 1996; Prawat & Jennings, 1997). These studies attest to several common features of the lessons such as:

- encouraging students to come to an answer in different ways,
- valuing students' interventions in the lesson and inviting them to articulate their interpretations of the mathematics at hand,
- allowing students to describe their methods and engaging them in debates which help them refine and adjust their strategies and understandings, and
- learning about students' mathematical meanings through reflection on classroom experiences.

One such study was presented by Simon (1995) who analysed his lectures with university students in a teacher training course. As a result of this analysis, he proposes the teaching model illustrated in Figure 2.5.4.1.

Figure 2.5.4.1   Simon's constructivist model of teaching mathematics
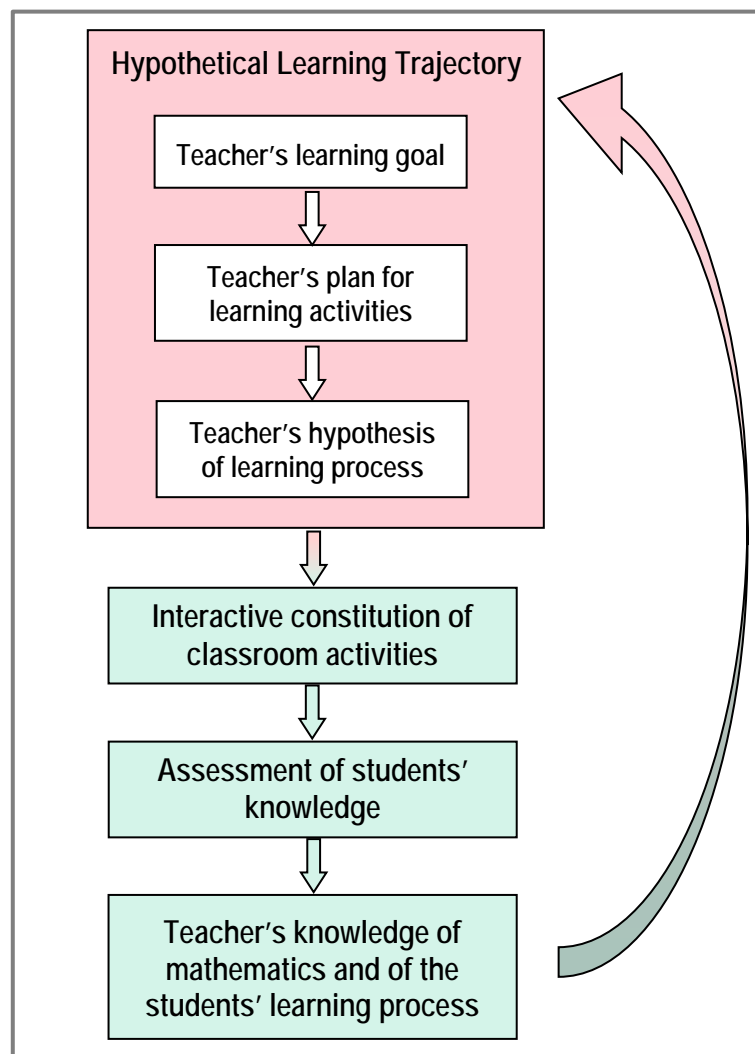
(Adapted from Simon, 1995, p. 136)

Constructivist teachers have what Simon (1995) calls a *hypothetical learning trajectory* (HLT). It is the way they predict the path through which learning might proceed. It is made up of the teacher's

(i)   learning goal which defines the direction of the lesson,

(ii)   plan of activities aimed to achieve the learning goal, and

(iii)   hypothesis of the learning process, i.e. the predictions of how students' thinking and understanding may evolve during the lesson.

Simon (1995) explains that HLT is hypothetical because the actual learning trajectory is not knowable in advance. I see the inclusion of the adjective "hypothetical" as an acknowledgement of the fact that any learning objectives teachers may have in mind before the lesson may be changed by what they learn from their students. This is perhaps what classifies this kind of model as constructivist. Teachers' actions from here onwards are geared towards learning about their students' knowledge and interests and about the subject itself as much as towards helping students understand more about the subject matter.

Simon (1995) classified the ensuing process into three steps:

Stage 1.   Teachers *interact* with students through classroom activities aimed at helping them gain more insight about the topic at hand. Teachers do not simply lecture but exchange ideas with the students (Driver & Oldham, 1986) and help students to generate new ideas or modify old ones. Steinbring (1998, p. 158) speaks of 'learning offers', i.e. teachers' classroom representations intended to bring about learning. One way how mathematics teachers can present learning offers is by taking non-contextualised mathematical ideas and embed them in a context conducive to learners' investigations (Brousseau, 1986). Besides helping students link what they already know with what the teacher intends them to learn (Gagnon & Collay, 2006) the teacher-student exchange of ideas helps teachers to evaluate their students' knowledge and ways of knowing.

Stage 2.   Simultaneous with this teacher-student interaction comes teachers' *assessment* of the way their students think and come to know. This brings about a modification in teachers' ideas and knowledge of what is happening in the classroom.

Stage 3.   Teachers use this assessment to *reconstruct* their knowledge of the students' learning process and of the subject content which will give rise to a revised hypothetical learning trajectory.

The model shows a two-way-traffic type of teaching, where both teacher and student are learners and both teacher and student are teachers (Freire, 1998). Although planning is ultimately done by the teacher, the teaching-learning process is student-centred: the assessment of students' knowledge may lead teachers to reject their premises and start anew. On their part, students construct their own knowledge by drawing upon their own cognition and interacting with each other and with the teacher.

It seems, therefore, that although Simon (1994) says that it is a myth to think of CT as a specific kind of teaching, Simon's (1995) own teaching model may well be identified with CT, as pointed out by Steffe and D'Ambrosio (1995). Furthermore, although Simon's (1995) model includes a teacher learning aspect, it seems to emphasise more the process where students are learning something from the teachers. In this model, teachers' *assessment of students' knowledge* seems to serve only to inform teachers about the appropriateness of the teaching approach. It does not really add anything to teachers' own mathematics. The possibility of teachers' content knowledge becoming enriched in the process of CT is an important aspect in Steffe's (1991) CT principles, which I discuss below.

2.5.5   Steffe's Constructivist Teaching Principles

There have been extensive contributions to the literature about RC and CT in mathematics education by Steffe (e.g. Steffe, Glasersfeld, Richards, & Cobb, 1983; Steffe & Blake, 1983; Steffe, Cobb, & Glasersfeld, 1988; Steffe & Wiegel, 1992; Steffe & D'Ambrosio, 1995; Steffe & Thompson, 2000a, 2000b, 2000c; Steffe & Ulrich, 2013;

Steffe, 1991, 2007, 2016). A common theme in Steffe's writings is that constructivist teachers must see themselves as learners when they engage in CT. His theories are in line with Freire (1998, p. 31) who states that 'whoever teaches learns in the act of teaching, and whoever learns teaches in the act of learning', but Steffe's notion of teachers' learning during CT goes beyond the construction of pedagogical knowledge. Steffe (1991) argues that constructivist teachers' *main goal* is to learn from interactions with students and with other colleagues and from observing students' mathematical interactions. In this way, the focus of mathematics education is shifted from the teacher, not to the students per se, but to the intellectual *interactions* between teachers and students, among students, and among teachers (von Foerster, 1984; Bauersfeld, 1988; Glasersfeld, 1990b). Steffe (1991) lists ten principal goals for CT. Teachers need to learn:

   (i)   how to communicate mathematically with students;

   (ii)   how to engage students in goal-directed mathematical activity;

   (iii)   the mathematics *of* the students they teach;

   (iv)   how to organise possible mathematical environments;

   (v)   the mathematical experience of students;

   (vi)   the mathematics *for* the students they teach;

   (vii)   how to foster reflection and abstraction in the context of goal-directed mathematical activity;

   (viii)   how to encourage students to communicate mathematically among themselves;

   (ix)   how to foster student motivation and sustain learning over a period of time;

   (x)   how to communicate pedagogically as well as mathematically with other mathematics educators.

<div align="right">(Adapted from Steffe, 1991, p. 191, original emphases)</div>

For Steffe (1991), constructivist mathematics teachers need to redirect their pedagogical goals towards communicating and reflecting on the nature of mathematical concepts and techniques with their students and colleagues. In classroom interactions, teachers may form second-order experiential models (Steffe et al., 1983; Steffe & Ulrich, 2013) of students' mathematics (goal iii). CT thus means that teachers invest themselves in developing hypothetical models of students' mental operations (Kaput's 1991). In a personal correspondence, Steffe asserts that constructivist teachers' own mathematical understandings can benefit from their search to understand students' mathematics:

> [T]he constructivist teacher sets out to learn students' mathematics and includes and synthesizes that mathematics with his or her own in ongoing teaching and learning. Does that mean that the constructivist teacher abandons his or her own mathematics? Not at all. In fact, a search for understanding students' mathematics can lead to conceptual analysis and enrichment of one's own mathematics.
>
> (Steffe, personal communication, October 7, 2015)

Like Simon (1995) and Steinbring (1998), Steffe (1991) holds that teachers' learning about students' mathematical understandings informs the mathematics they intend to teach but Steffe (1991, 2015) goes a step further: teachers' own mathematics can be enriched by their observations of and reflections on students' mathematical representations. This view may seem revolutionary for teachers operating in a culture upholding a didactical contract (Brousseau, 1984; Brousseau & Otte, 1991), i.e. the unwritten teacher-learner classroom pact that the teacher is there to teach and the learner is there to learn. In such classroom cultures, teachers seem to be the only ones with information and students seem to be the only ones who can learn (Kansanen & Meri, 1999). Like he does in most of his contributions on CT, Steffe (1991) compensates for this teacher-learner asymmetry and his CT principles are almost exclusively focused on teachers' *learning* from their observations and reflections of students' interactions.

I find Steffe's ideas about CT (1991) complementary to Simon's (1995) model of teaching from a constructivist perspective. While Simon emphasises teacher-to-student interactions and teachers' presentation of learning offers, Steffe (1991)

emphasises teachers' learning of students' mathematics and the synthesis of this knowledge with their (the teachers') mathematics, and with their expertise in teaching particular areas of mathematics, or what Shulman (1986, p. 9) refers to as 'pedagogical content knowledge'.

2.6   Research Questions about Constructivist Teaching

The combination of Steffe's (1991) and Simon's (1995) theories consolidates my notion of CT as a toing-and-froing exercise between the curriculum and the learners (Dewey, 1902). Such an exercise requires teachers to deal with what Windschitl (2002) refers to as the pedagogical dilemma. This dilemma stems from constructivist teachers' wanting to honour learners' attempts to think for themselves while needing to remain faithful to the accepted disciplinary notions of the subject. A satisfactory compromise between these two aspects is a considerable challenge for any teacher. Part of this challenge is due to the RC rejection of the notion of curriculum as a body of *a priori* knowledge. This view is espoused by Dewey (1902) as he urges teachers to

> abandon the notion of subject-matter as something fixed and ready-made in itself, outside the child's experience; cease thinking of the child's experience as also something hard and fast; see it as some-thing fluent, embryonic, vital.

(Dewey, 1902, p. 11)

This statement implies that RC teachers need to appreciate that

(i)   the curriculum consists of their own interpretations of subject matter,

(ii)   they need to relate this subject matter with learners' experiences, and

(iii)   these learners' experiences are ever-changing.

Thus, Dewey's (1902) curriculum-learner construct goes well with RC epistemology. It provides an overarching notion of practising forms of teaching compatible with constructivist beliefs (Cobb, Wood, & Yackel, 1990), like those proposed by Steffe (1991) and Simon (1995). It also reaffirms the possibility of defining CT as the endeavour of teachers to establish links between their interpretations of subject content (curriculum) and the learners.

As I mentioned in Chapter 1, one of the main objectives of this research was to investigate how I engage in CT and possibly develop a framework which helps me understand the dynamics of such a teaching approach. The literature reviewed in this chapter helped me to prepare for such an investigation, by which I seek to answer the first set of research questions:

(i)   How do I engage in CT and what are the distinguishing characteristics of such a teaching approach?

(ii)   What, if any, are the moments when I fail to engage in CT?

Since teaching is dependent on learning (Freire, 1998), and since I intended to investigate lessons in which I introduced 7th graders to formal algebra, a substantial part of my literature review consisted of studying the nature and learning of algebra. This review is presented in the chapter that follows.

Chapter 3

Literature Review Part 2: The Nature and Learning of Algebra

3.0   Overview

The second part of the literature review is subdivided into the sections shown in Table 3.0.1 as follows.

Table 3.0.1   Chapter 3 section titles

	Title	Page
3.1	The Nature of Algebraic Thinking and Activities	56
3.2	Algebra Difficulties	71
3.3	Notation as a Key Factor in Algebraic Activities	80
3.4	A Way Forward: Use of Computers for Algebraic Thinking	95
3.5	Research Questions about Students' Representations and Interpretations of Notation	109

The second part of the literature topic map, illustrated in Figure 3.0.1 (overleaf), resumes from the first part (Chapter 2, p. 13). The literature review starts by a discussion about the nature and definition of algebra, where I synthesise work of theorists including Mason (1996), Kaput (2008), Radford (2014), Kieran (1996), and Gattegno (1988). This helps me to establish a standpoint regarding the nature of algebraic thinking and differentiate between informal- and formal-algebraic activities.

This is followed by a review of research-based literature about the learning of algebra, where I focus on students' difficulties in algebra. Research evidence shows that such difficulties are usually caused by problems stemming from generality, arithmetic, and notation. Focusing on the latter as a make-or-break issue in algebraic activities, I discuss students' representations and interpretations of notation. I present the use of computers as a possible way forward for teachers to help students to enrich their meanings of notation. I discuss software which has been found effective in this respect including microworlds, Logo, spreadsheets, and other grid-based environments. One of the latter is Grid Algebra, with which I end my literature review. This second part of the literature review leads me to the second set of research questions, those regarding students' representations and interpretations of notation with the help of Grid Algebra.

Figure 3.0.1   Literature topic map – Part 2
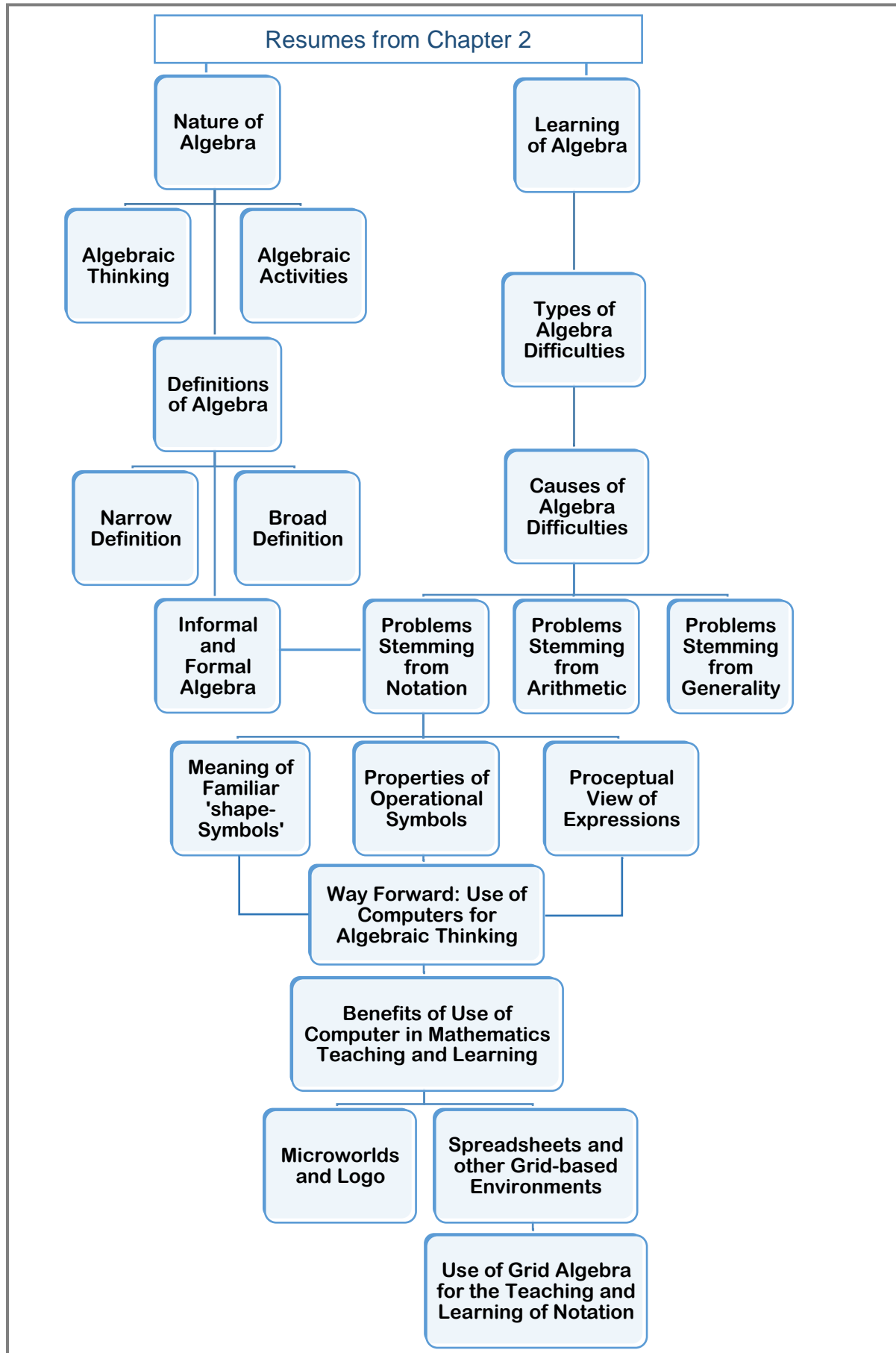
3.1 The Nature of Algebraic Thinking and Activities

Wheeler (1996, p. 319) says that algebra is difficult to define because 'it always seems to comprise rather more than any simple story suggests'. In this section, I discuss algebraic thinking and activities and establish my standpoint about each of these.

3.1.1 Algebraic Thinking

It seems that all theorists in the algebraic field agree that generality and the process of generalisation are key to what is usually attributed to algebraic thinking (e.g., Sfard, 1995; Usiskin, 1995; Mason, 1988, 1991,1996; Mason, Graham, Pimm, & Gowar, 1985). By generality, I share Usiskin's (1995) understanding that it is the search for general rules by observing and reflecting on differences, similarities, patterns, and classifications of numbers. Usiskin (1995, p. 31) says that students engage in algebraic thought when they formulate a rule such as the following: '*To multiply two fractions, multiply their numerators to get the numerator of the product, and then multiply their denominators to get the denominator of the product'*. Usiskin points out that this statement may be represented as:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

The above string of symbols is another way of writing the previous sentence using mathematical symbols and, as Skemp (1971, p. 95) says, 'basically these are a verbal shorthand'. Notation will be discussed further on, but for now it suffices to observe, even in this one example, how crucial notation can be in understanding and communicating algebraic concepts.

The ability to generalise makes students proficient in many areas. Bednarz, Kieran, and Lee (1996) identify four such areas. Each area is emphasised by other researchers when discussing the nature of algebra:

(i) Generalisation of numerical and geometric patterns and laws governing numerical relations (Arzarello, 1991a; Mason, Grahan, Pimm, & Gowan, 1985; Charbonneau, 1996);

(ii)   Solution of specific problems or classes of problems (Puig & Cerdán, 1990; Rubio, 1990, 1994);

(iii)   Focus on the concepts of variable and function (Confrey, 1992; Garaçon, Kieran, & Boileau, 1990; Heid & Zbiek, 1993; Kieran, 1994);

(iv)   Modelling of physical phenomena (Chevallard, 1989; Filloy & Rojano, 1989).

Some studies include all of these areas (e.g., Usiskin, 1988; Kaput 1995b, 1998, 1999, 2008; Blanton & Kaput, 2005; NCTM, 1998), but all of them depend on the processes of generalising and pattern-seeking, such as those identified by Mason (1996, p. 83) when he says: 'Detecting sameness and difference, making distinctions, repeating and ordering, classifying and labelling…are the basis of what I call algebraic thinking.' Mason (1996) goes on to say that algebra without generality is little more than a dead subject.

In his extensive studies on the teaching and learning of algebra, Kaput repeatedly stresses that algebraic reasoning is not confined to rule-based operations on symbolic syntax. For instance, Blanton & Kaput (2005) define algebraic reasoning as follows:

> a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways.
>
> (Blanton & Kaput, 2005, p. 413)

Along similar lines, Kaput (2008) proposes two core aspects of algebraic reasoning:

(i)   The generalisation and expression of generality in increasingly systematic and conventional symbol systems;

(ii)   The syntactically guided reasoning and actions on generalisations expressed in conventional symbol systems.

Kaput (2008) argues that the mathematical debate which divides mathematicians and mathematical educators over the definition of algebra is actually their emphasis on these core aspects. Before I delve into that debate, I will now turn to another aspect of the teaching and learning of algebra: the nature of algebraic activities.

3.1.2 Algebraic Activities

Kieran (1996) describes algebraic activities by subdividing them into three categories:

(i) **Generational activities.** These involve the *generation* of algebraic objects, i.e. expressions and equations. Kieran (1996) argues that much of the meaning-building for algebraic objects occurs within such algebraic activities.

(ii) **Transformational activities.** These are those rule-based activities concerned with *transforming* expressions such as substitution, simplification, expansion and factorisation.

(iii) **Global, meta-level, mathematical activities.** These are activities for which algebra is used as a *tool* but which are not exclusive to algebra. They include problem solving and modelling, analysing and generalising relationships, and proving mathematical statements. Kieran (1996) suggests that learners can engage in these activities without using formal algebraic notation but they can elaborate on them at any state by including conventional notation.

Sharing most of Kieran's (1996) views about the nature of algebraic activities, Bell (1996, p. 174) defines algebra as 'any manipulable language by which relations or compositions are handled in conceptual fields of space, number, or elsewhere in mathematics'. Bell points out three algebraic processes:

(i) Expressing mathematical relationships using algebraic notation;

(ii) Manipulating symbolic expressions into different forms to reveal new aspects of these relationships;

(iii) Applying the knowledge of using and manipulating algebraic expressions for specific activities such as forming and solving equations, working with formulae, etc.

Algebraic activities are characterised by specific conditions. Radford (2014) identifies three such conditions:

(i) **Indeterminacy**. This is the recognition of the use of indeterminate values (usually expressed as letters) in the form of variables, unknowns and parameters.

(ii) **Denotation**. This involves the symbolisation of the indeterminate values of the problem at hand, that can include the use of natural language, gestures, signs, and written symbols.

(iii) **Analyticity.** This is the skill of manipulating the indeterminate quantities as if they were known values.

Kieran (1996), Bell (1996), and Radford (2014) have captured the essence of what theorists usually regard as algebraic activities. Yet, some researchers seem to confuse algebraic *activities* with algebraic *thinking*, where they use interchangeably the ideas associated with these two terms. This is sometimes the cause of discord in researchers' definitions of algebra, as I will discuss next.

3.1.3   Two Definitions of Algebra

There seems to be two predominant schools of thought which differ in their definitions of algebra. I discuss these in the following two subsections.

A Narrow (Traditional) Definition of Algebra

On one side of the debate about what classifies as "algebra", there are those who make a clear distinction between algebra and arithmetic. Some of these do not even regard as algebraic some activities which are widely associated with algebra, such as the solution of equations. One of these is Saul (2001), who argues that the fact that students understand and apply the concepts of variable or function in solving a mathematical problem does not mean that they are engaged in algebra. He brought evidence from a case study of a student who knew that $2x + 1 = 7$ was true for $x = 3$ and false for any other value of x by substitution strategies but who could not engage in a transformational algebraic activity (Kieran, 1996) to bring x to be the subject of the equation. Saul (2001) argued that this student could operate arithmetically but not algebraically.

For others within this school of thought, algebra may not be present even in transformations of certain equations. For instance, Filloy and Rojano (1989) regard as "arithmetical" equations such as: $Ax \pm B = C$, $A(Bx \pm C) = D$, $\frac{x}{A} = B$, and $\frac{x}{A} = \frac{B}{C}$. The reason they give is that the unknown appears just once on one side, while the other side can be viewed as the result of the operations done on that unknown. Filloy and Rojano argue that in each of these equations, the value of x can be found by inverting or undoing the arithmetical operations one by one and hence only arithmetical processes are involved.

Filloy and Rojano (1989) contrasted these types of equations with others which they call "non-arithmetical" (p. 19), such as: $Ax \pm B = Cx$ and $Ax \pm B = Cx \pm D$. In such equations, the arithmetical processes of inverting successive operations are not sufficient. To solve these equations, it would be necessary to operate on the unknown. Drawing evidence from an earlier study (Filloy & Rojano, 1984) the authors affirm that between arithmetical activities (solving arithmetical equations) and algebraic activities (solving non-arithmetical equations) there is what they call a *didactic cut* which can only be addressed through instructional interventions.

On similar lines, Herscovics and Linchevski (1994) argue that between arithmetic and algebra there is a clear demarcation by what they call a *cognitive gap*. In their study with high-performing 7th graders, they presented students with equations such as: $63 - 5n = 28$, $11n + 14n = 175$, and $5n + 12 = 3n + 24$. Herscovics and Linchevski reported that the participants engaged in arithmetical activities to solve the equations. The students competently used number grouping techniques, systematic substitution, and inverse operations to solve the equations. Like Filloy and Rojano's (1984) participants, these students were unable to operate with or on the unknown to transform the equations and, thus, Herscovics and Linchevski (1994) declared that these students could not yet engage in algebraic tasks. Hence, they asserted the presence of a *cognitive gap* between arithmetic and algebra. They claimed that this gap could only be addressed through teaching and learning of algebraic concepts and techniques.

Other studies which support the idea of a demarcation between arithmetic and algebraic activities call for the need of a transition between arithmetic and algebra (e.g., Collis, 1975; Bednarz & Janvier,1996; Boulton-Lewis et al., 1998; Kilpartick, Swafford, & Findell, 2001; Kieran, 2004; Sadovsky & Sessa, 2005). These studies seem to make two interlinked assumptions:

(i)   Arithmetic and algebra are distinct disciplines. There is demarcation between the skills required to solve arithmetic problems and those required to solve algebraic problems and the transition from arithmetic to algebra can occur through instruction.

(ii) A number of years of instruction in arithmetic are required before starting an algebra course.

(iii) The generalisation and expression of generality, a core algebraic aspect according to Kaput (2008), may exist within arithmetic and without algebra.

The definition of algebra discussed in this section may be considered traditional, since this is what curriculum planners often refer to when they mention "algebra" in educational programmes (e.g. DfE, 2013; DLAP syllabus, 2014a, 2014b). Such a view of algebra is narrow when compared to that of the second school of thought which is discussed next.

A Broad Definition of Algebra

On the other side of the algebra definition debate there are theorists who claim that algebra exists even in problems that are normally associated with arithmetic or other mathematical domains. Foremost among these is Gattegno (1974, p. 82) who asserts that 'rather than teach mathematics we should strive to make people into mathematicians'. Gattegno (1988) shows that mathematics teachers and learners need to become aware of what they are assuming, doing, and achieving when they engage in mathematical processes. It is one thing to be aware that you are doing $6 + 9$ and achieving 15 as a result. This is just "awareness-in-action" (Mason, 1998, p. 255). It is quite another thing to be aware that $6 + 9$ and $9 + 6$ constitute the same quantity,

that $6 + 9$ may be expressed as $3 \times (2 + 3)$ and so 3×5, and that the result, 15, is an element in the same set (integers) as the numbers involved in performing those operations (closure property). Gattegno (1988) refers to this second type of awareness as "awareness of awareness" (p. 172), the awareness of an algebraic structure, 'a set together with an internal operation which associates to any pair of elements of the set another element of the same set' (Gattegno, 1987, p. 61).

Using Cuisenaire rods, Gattegno used what became known as Gattegno-Cuisenaire method (see Chambers, 1964) to instruct teachers how to teach number concepts to primary school children. Gattegno (1974) referred to this as "the algebra of arithmetic" (p. 61), where he demonstrated how Cuisenaire rods may be used to make children aware of properties of the addition operation like commutativity and associativity. Gattegno advocated that teachers should give their students the opportunity to encounter algebra *before* arithmetic by becoming aware that the "games" embody implicit mathematical structures and relations.

Gattegno's (1988) *awareness of awareness* identifies with algebra any attempt to regulate, systematise, and generalise numerical properties, operations, calculations, and relationships irrespective of whether a letter is used or manipulated on. Associating algebraic thinking with making sense of mathematical structures and relationships and the consequent pursuit of generality is widely reported in the literature (e.g., Usiskin, 1988; NCTM Algebra Task Force, 1993; NCTM Algebra Working Group, 1997; Kaput, 1995a, 1995b; Mason, 1996; Blanton & Kaput, 2005). This stance is central to discussions and research about the inclusion of algebra in the primary curriculum (e.g., Carraher, Schliemann, & Brizuela, 2000; Carraher, Schliemann, & Schwartz, 2008; Mason, 2008) because it promotes the explicit teaching of algebraic thinking from an early age, as soon as students start to wonder whether a pattern or rule might exist for particular calculations.

Rather than creating a stark demarcation between arithmetic and algebra, this definition of algebra tends to seek an area of overlap between the two, making them seem less and less as distinct disciplines. Algebra is thus seen to be *embedded* in the

very structure of arithmetic. Hewitt (1998) argues that algebraic thinking is actually necessary for arithmetic procedures to be carried out. Devoid of algebra, arithmetic would be little more than a recall of answers learnt by rote. 'Arithmetic is concerned with getting answers. Algebra shifts attention from answers to what is required to be done to get an answer' (Hewitt, 1998, p. 21). Algebraic notation is, then, the encapsulation (Dubinsky, Elterman, & Gong, 1988) of the structure behind arithmetical processes and products.

Using this line of thought, Carraher, Schliemann, Brizuela, and Earnest (2006, p. 89) argue that 'algebraic concepts and notation need to be regarded as integral to elementary mathematics'. They object to the idea of a developmental readiness which, they say, implies the basic assumption of studies that propose a gap between arithmetic and algebra (such as Filloy & Rojano, 1984, 1989; Herscovics & Linchevski, 1994). Carraher et al. (2006) back their claims by results of an earlier study (Carraher, Schliemann, & Brizuela, 2000) in which they reported that Grade 3 children were able to understand and use algebraic notation (e.g. $n \rightarrow n + 3$), and generalise how two series of numbers (e.g. n and $2n + 1$) were interrelated. Support for early algebra was again presented by Carraher, Schliemann, and Schwartz (2008) with evidence from a similar longitudinal study with students in Grades 2–4. The authors claimed that students engaged in algebraic activities by using algebraic representations and notation (e.g. $100 + W = 3W$) to make sense of practical situations such as the number of candies in a box and the amount of money in a wallet.

However, in their studies, Carraher et al. (2000, 2008) did not specify the number of students who managed to make use of such algebraic notation, and hence the extent of their claims is unclear. Furthermore, when introducing the use of letters as unknowns, they always used initials of the name of the unknown (e.g. n for number or W for wallet money) rather than making an arbitrary choice of letters. As I discuss in Section 3.2, this can be a crucial factor for determining the meaning children give to literal symbols. Nevertheless, whether Carraher et al.'s (2000, 2008) early graders could make sense of literal symbols is not a determining factor as to whether they could engage in algebraic activities. I will develop this argument in the next subsection.

The definition of algebra discussed in this section is much broader than the first because, while encompassing the activities deemed algebraic by the first school of thought, it extends beyond the application and manipulation of letters in expressions and equations. The proponents of such a broad definition hold that any attempt to generalise rules and techniques to find any answer (even those deemed arithmetical) classifies as algebraic thinking. The role that algebraic activities and algebraic thinking play in the discord between these two schools of thought is discussed next.

3.1.4   Reconciling the Two Schools of Thought

In this section, I argue that differences in the two schools of thought in their definition of algebra seem to stem from two factors:

- their emphasis on one of Kaput's (2008) core aspects of algebraic reasoning, and
- the confusion of algebraic thinking with algebraic activity.

I will also attempt to reconcile the two perspectives by addressing these two factors.

With regards to the first factor, it seems that those who give a narrow definition of algebra emphasise Kaput's (2008) core aspect (ii): the *syntactically* guided reasoning and actions on generalisations expressed in *conventional symbol systems*. Filloy and Rojano (1984, 1989), Herscovics and Linchevski (1994), and the others are seen to associate algebra with Kieran's (1996) *generational* and *transformational activities* each of which involve the use and manipulation of algebraic syntax.

While acknowledging that these activities form *part* of algebra, those who advocate a broad definition of algebra insist that algebra also exists where there is a search for patterns, rules, and generality which may eventually lead to the use of conventional notation. This is an emphasis on Kaput's (2008) core aspect (i): The generalisation and expression of *generality* in *increasingly systematic* and *conventional* symbol systems.

Gattegno (1987), Hewitt (1998), Kaput (2008) himself, and others adopting this perspective emphasise the applicability of algebra as a *tool* for what Kieran (1996) described as the *global, meta-level, mathematical activities*.

To address the second factor, the confusion of algebraic *thinking* and algebraic *activity,* I present Hewitt's (1998) problem of counting an array of matchstick-squares (Figure 3.1.4.1).

Figure 3.1.4.1   The matchstick-squares array
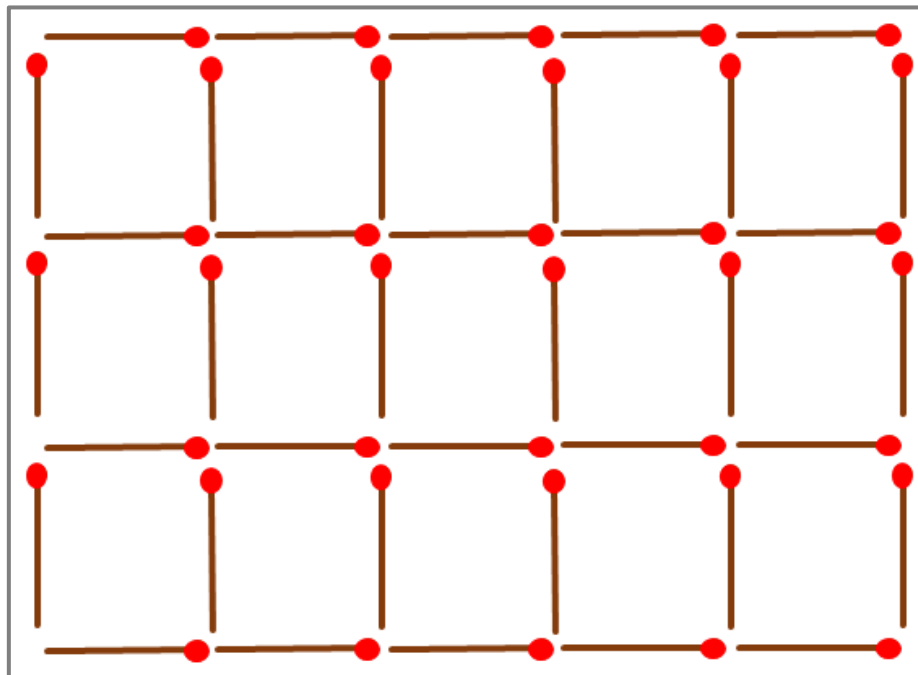
(Adapted from Hewitt, 1998, p. 19)

Hewitt (1998) argues that to count the matchsticks economically a student would need 'to work algebraically' (p. 20). This means engaging in a systematic, organised way of counting, which may lead to the determination of the number of matchsticks in hypothetical arrays (without even seeing them). I agree with Hewitt (1998) that such an activity requires algebraic thinking because it involves the generalisation of a rule that goes beyond the first few cases (Dienes, 1961).

Now, consider two students, S1 and S2, where S1 has not yet learnt about algebraic notation (expressions with literal symbols) and S2 has. S1 may reason algebraically that the rule to count the matches is equal to the addition of 5 times one more than the number of square rows (horizontal matches) and 6 times the number of square rows (vertical matches). S2 may make the same deduction but then moves on to write the formula $n = 5(r + 1) + 6r$, where n is the number of matchsticks in r square rows. S2 has an advantage over S1, not only due to the use of notation as verbal shorthand (Skemp, 1971) but also due to the possibility of transforming the formula into a more simplified form, such as $n = 11r + 5$.

Undoubtedly, both schools of thought about the definition of algebra would consider S2 as being engaged in an algebraic activity. However, albeit *thinking* algebraically, S1 may not be regarded by proponents of the narrow definition of algebra as being engaged in an algebraic *activity* because s/he is not engaged in any *syntactically guided reasoning* and her/his generalisations are not expressed in *conventional symbol systems* (Kaput's 2008, core aspect (i)). If S1 learns that one can arbitrarily represent (Hewitt, 1999) a variable by a letter and becomes comfortable with this use of letters, there would only be one thing left to learn in order to reach the level of S2: the convention of *transformational activities* (Kieran, 1996) which may be used to change the formula into a more convenient one.

Nevertheless, both students are engaged in *generational activities* (Kieran, 1996). To me this is a necessary and sufficient condition for algebra and, in this respect, I tend to side with those taking a broad definition of algebra. At the same time, I would agree with a proponent of a narrow definition who argues that S2's thinking and activities are not at the same level as those of S1. However, while differentiating between the two students' thinking and activities, I maintain that both are engaged in algebra. This is possible by introducing some "new" nomenclature. In Table 3.1.4.1, I propose such nomenclature which I define by using Hewitt's (1998) matchsticks array example. This nomenclature and its definition will attest to my own perspectives about algebraic thinking and activity, and will be used in this write-up.

Table 3.1.4.1   *Solely-arithmetic, informal-algebraic, and formal-algebraic thinking and activities*

	Task→	Count the number of matchsticks in a 3×5 squares array	Count the number of matchsticks in a 4×5 squares array	Count the number of matchsticks in a 100×5 squares array
Learner #1	Solely-arithmetic thinking→	"I will count the matchsticks one by one. So…"	"I will continue to count the matchsticks one by one after 38. So…"	"I will continue to count the matchsticks one by one after 49. So…"
	Solely-arithmetic activity→	$1, 2, 3, …, 38$ *Answer:* 38	$39, 40, 41, …, 49$ *Answer:* 49	$50, 51, 52, …, 1105$ *Answer:* 1105
Learner #2	Informal-algebraic thinking→	"I will use multiplication to count the horizontal and then the vertical matchsticks. So…"	"I will again use multiplication to do the same thing, just adding another line of horizontals and another row of verticals. So…"	"I notice that I am multiplying 5 by one more than the number of square rows, and 6 by the number of square rows, and then adding the results. So…"
	Informal-algebraic activity→	$5 \times 4 + 6 \times 3$ $20 + 18$ *Answer:* 38	$5 \times 5 + 6 \times 4$ $25 + 24$ *Answer:* 49	$5 \times 101 + 6 \times 100$ $505 + 600$ *Answer:* 1105
Learner #3	Formal-algebraic thinking→	"Let number of rows be r and number of matchsticks be n. Then n is 5 times $r + 1$ added to 6 times r. r is 3, so…"	"The number of columns is unchanged so I will use again the same formula. This time r is 4, so…"	"I will use again the same formula. However, I can simplify the formula and then substitute $r = 100$ afterwards, so…"
	Formal-algebraic activity→	$n = 5(r + 1) + 6r$ $n = 5(4) + 6(3)$ $n = 20 + 18$ *Answer:* $n = 38$	$n = 5(r + 1) + 6r$ $n = 5(5) + 6(4)$ $n = 25 + 24$ *Answer:* $n = 49$	$n = 5r + 5 + 6r$ $n = 11r + 5$ $n = 11(100) + 5$ *Answer:* $n = 1105$

Table 3.1.4.1 gives hypothetical thinking processes of three learners, #1, #2, and #3, and the corresponding activities involved in solving the problem of counting the number of matches in an $r \times 5$ matchstick-squares array (where r is the number of rows). The learners are required to find the number of matches in 3×5, 4×5, and 100×5 matchstick-squares arrays. The learners are assumed to think and act as follows:

#1. Solves the problem without any attempt to be economical, i.e. without trying to organise the counting in a way that can be feasible for large arrays. I am naming this kind of thinking and activity as *solely-arithmetic*;

#2. Solves the problem by attempting to organise the counting by the use of multiplication and addition operations. I am naming this kind of thinking and activity as *informal-algebraic*;

#3. Solves the problem by using standard algebraic syntax through the use of letters to stand for variables. This learner uses a formula to substitute respective values for the number of rows (r) and obtaining a value for the number of matchsticks (n), and also simplifies the formula by manipulating the terms in r. I call this kind of reasoning and working *formal-algebraic*.

One can find similar terms in the literature. Van Amerom (2003) uses the terms formal and informal strategies of students when solving equations. However, she reserves the term "informal" for strategies which she claims to be arithmetical. So, although she uses terms like 'formal algebraic approach' (p. 67) to denote the standard manipulations of equations, she never uses the term "informal algebraic". Linchevski (1995, p. 114) uses the term 'formal algebra' to denote what I mean by *formal-algebraic* thinking and activity, including Kieran's (1996) *generational and transformational* activities.

In the solely-arithmetic, informal-algebraic, and formal-algebraic thinking cells I am including hypothetical, typical thoughts about problem-solving strategies, including typical representations of those planned strategies. The reasoning and working of each case are hierarchical in nature, and I assume that students may progress from #1 stage to #3 stage. This assumption is backed by literature which attests to the possibility of such a successful transition (classically referred to as the arithmetic-to-algebra transition) if teachers address particular learner needs (e.g., Kieran, 2004).

There are a number of observations to be drawn from Table 3.1.4.1:

(i)   The use of "solely" in the term "solely-arithmetic" carries an implicit suggestion that in subsequent reasoning and activities arithmetic may still be present. Going by Hewitt's (1998) contention that arithmetic is about getting answers, I do not exclude that the utilitarian arithmetical aim of obtaining an answer at the end of a series of problem-solving steps will still prevail in most, if not all, mathematical thoughts and actions.

(ii)   In line with RC, I believe I have 'no direct access to the knowing or thinking of others' (Ulrich et al., 2014, p. 329) so the phrases within quotes shown in the "thinking" sections of the table are only second-order models (Steffe et al., 1983) of possible conceptual processes of each learner.

(iii)   Type #2 learners cannot simplify their numerical formula since they are not operating on the unknown and this makes their thinking and activity less economical than #3. This is one of the factors which places informal-algebraic at a lower hierarchical level than formal-algebraic.

(iv)   Mason (2008) defines an algebraic solution as the one which reveals similarities in structure. This is in line with my definition of informal- and formal-algebraic reasoning because both methods lend themselves well to revealing patterns in the data. Mason brought evidence to show that some learners are initially more inclined to apply arithmetical techniques to solve problems. The methods used by such learners are unstructured as shown in the solely-arithmetic thinking and activities of Table 3.1.4.1. These are bound to become more laborious as numbers get larger. Informal- and formal-algebraic thinking are more economical. Gattegno (1986, p. 43) says that, 'in algebra…one thought process is placed upon another precisely for the purpose of performing more for less'.

(v)   Schools of thought which demarcate arithmetic from algebra, usually promote the need for "pre-algebra" courses such as those given in the US in middle school (around Grade 7). According to Linchevski (1995, p. 119), pre-algebra is 'a stage of transition from the environment of arithmetic to that of formal algebra'. Linchevski and her team identified a number of areas to be addressed in pre-algebra courses. These fall under the category I call informal-algebraic activities but, as Linchevski suggested, they also include formal-algebraic aspects, such as the introduction of literal symbols (e.g x or $3x$) to stand for unknowns and variables in simple equations.

Without entering into the debate about the best time to introduce formal-algebraic activities in schools, I use this model to propose that these can only occur if learners have mastered thinking and working at the informal-algebraic level, and that the latter requires a basis of arithmetic. Specifically, for a successful transition through these levels learners need to be helped to develop concepts about:

(i)   ordinal and cardinal numbers,

(ii)   properties of addition and multiplication and their inverses,

(iii)   numerical expressions as processes of operations,

(iv)   numerical expressions as singular mathematical entities,

(v)   literal symbols to denote unknowns and variables, and

(vi)   substitution of letters for numbers and vice versa.

(i) is needed for solely-arithmetic, (i)–(iv) are needed for informal-algebraic, and (i)–(vi) for formal-algebraic. All concepts involve the interpretation and representation of notation. As I presently discuss with reference to the literature, notation is particularly crucial in determining the success or otherwise of learners' engaging in meaningful informal- and formal-algebraic reasoning and activity.

Like many other school curricula, the Maltese mathematics curriculum leaves formal-algebraic activities to be introduced at the start of the secondary school. One of the concerns I sought to address in my research was how children in their first year of secondary school (Grade 7), particularly those who did not perform well in past mathematical assessments, can be introduced to formal-algebraic reasoning and activities. Given my assumption that success at one level depends on success at the preceding level, I also needed to address these children's informal-algebraic reasoning. Research shows that this reasoning may be the origin of most difficulties that secondary students have in their learning of formal algebra, to which the next section is dedicated.

3.2   Algebra Difficulties

A review of the literature seems to point out five major areas of difficulties that children find in school algebra:

(i)   **Solving equations** (e.g., Filloy & Rojano, 1984, 1989; Herscovics & Linchevski, 1991, 1994; Kieran, 1988; Gallardo & Rojano, 1988; Wagner, 1977);

(ii)   **Manipulating algebraic expressions** (e.g., Booth, 1984; Sleeman, 1986; Borg 1997);

(iii) **Solving problems** (e.g., Bishop, Filloy, & Puig, 2008; Clement, 1980,1982; Clement, Lochhead, & Monk, 1981; Kieran, Booker, Filloy, Vergnaud, & Wheeler, 1990);

(iv)   **Conceptualising literal symbols** (e.g., Küchemann, 1981; Booth, 1984, 1988);

(v)   **Interpreting answers** (e.g., Collis, 1974; Davis, 1975; Sfard & Linchevski, 1994; Booth 1988).

I discuss each of these difficulties in the subsections that follow.

3.2.1   Difficulties in Solving Equations

The solution of equations has been one of the common focus areas where researchers investigated algebra difficulties. For instance, Linchevski and Herscovics (1996) found that Grade 7 students had difficulties in solving one variable equations with more than one term containing the unknown, such as $89 - 5n = 7n + 5$.   Such equations required students to operate on the unknown. The researchers reported that there were some recurring mistakes, even after a series of lessons. These were:

(i)   *detaching a term from the indicated operation*, such as simplifying $10n - 5n + 3n$ by working out $10n - (5n + 3n)$,

(ii)   *jumping off from a term with the posterior operation*, such as grouping the n terms in the equation $19n + 67 - 11n - 48 = 131$ as $30n$, and

(iii) *being unable to select the appropriate operation for the partial sum in an equation*, such as grouping the numerical terms in the equation $19n + 67 - 11n - 48 = 131$ correctly, i.e. 19, and writing -19 instead of $+19$.

Although Linchevski and Herscovics (1996) concluded that these mistakes were due to an insufficient preparation in arithmetic, it seems that their participants did not lack the skills required to add or subtract but they were misinterpreting the algebraic syntax and hence this was more a problem of notation interpretation.

Similar errors were found by Kieran (1988) when investigating Grade 8-11 students' solutions of linear equations. Students made computational errors due to a misuse of positive and negative numbers. Other errors were caused by students' reluctance to divide a number by a larger number. Kieran (1988) identified problems in arithmetic, particularly in students' understanding of integers and fractions. Similar arithmetical problems were found in 12- to 13-year-old students' solution of equations (Gallardo & Rojano, 1988) where students had trouble working with and interpreting negative numbers. Likewise, in Chaiklin and Lesgold's (1984) study, students were found to be unable to judge the equivalence of expressions like $685 - 492 + 947$ and $947 - 492 + 685$ without recourse to computation. Students' failure to see the possibility of swapping the first and last numbers indicates a limited interpretation of the numerical expression and hence, also a problem of notation.

In studies involving the use of letters, problems due to interpretation of notation were reported more explicitly. In a study with 12- to 17-year olds, Wagner (1977) found that some learners had not yet developed concepts about what the letter in an equation stood for. When she asked participants whether the equations $w + 22 = 109$ and $n + 22 = 109$ would yield different solutions, some of them said that the solution of the first one is greater than the second because w comes later than n in the alphabet. Others said that they could only know when they solved the equation. Similar findings were reported by Steinberg, Sleeman, and Ktorza (1991) who presented Grade 8-9 students with a list of pairs of equations such as:

- $x + 2 = 5$ and $x + 2 - 2 = 5 - 2$;

- $3x = 5 + 4$ and $3 + x = 5$;

Steinberg et al. (1991) found out that almost half of the Grade 8 students and even some of the Grade 9 students generally gave an incorrect reason to judge the equivalence of equations. The researchers claimed that these reasons show misunderstandings of basic concepts, like not distinguishing between $3x$ and $3 + x$. Such misconceptions may be attributed to students' misinterpretation of formal-algebraic notation rather than concepts of multiplication or addition. In spoken language "and" and "plus" usually represent the same meaning and hence learners may consider the expression ab (linking a *and* b) to mean the same as $a + b$ (Tall & Thomas, 1991; Stacey & MacGregor, 1994). Radford (2000) argues that studies about symbolic expressions should not only investigate learners' interpretations but also what notation *enables* learners to do. For some students in Steinberg et al.'s (1991) study, notation was a barrier, rather than an enabler, when these students worked with expressions or equations.

3.2.2 Difficulties in Manipulating Algebraic Expressions

Difficulties in algebra were also reported in learners' manipulations of algebraic expressions. Most errors seemed to stem from learners' overgeneralisations of certain rules, constructing what Sleeman (1986) called *mal-rules*, like applying the distributive property also in exponentiation over addition, e.g. $(x + y)^2 = x^2 + y^2$.

Similarly, Matz (1980, p. 95) found that students used 'extrapolation techniques…to bridge the gap between known rules and unfamiliar problems,' such as applying the cancellation property in $\frac{ax}{bx} = \frac{a}{b}$ to the expression $\frac{a+x}{b+x}$, simplifying the latter to $\frac{a}{b}$. The mal-rules (Sleeman, 1986) behind such misinterpretations were therefore overgeneralisations of conventional rules that students had previously learnt. Such errors usually occur due to failure of correctly constructing *structure sense* (Linchevski & Livneh, 1999, 2002), i.e. the conceptualisation of expression structure, rendering them errors of notation

interpretation. The rules behind the conceptualisations of such structures may be difficult for learners to grasp, especially when they are not *visually salient* (Kirshner & Awtry, 2004). For example, reading from left to right:

$$\frac{w}{x} \times \frac{y}{z} = \frac{wy}{xz} \quad \text{and} \quad \frac{w}{x} + \frac{y}{z} = \frac{wz+xy}{xz}$$

one may easily remember the first identity because all that needs to be done is to conjoin the numerators and the denominators, which makes the rule visually salient. The lack of visual salience in the second identity means that learners require more effort to conceptualise structure sense of the expression on the right and its equivalence to that on the left.

In a study with a group of 15-year-olds at Grade 11 (Borg, 1997), I found that students had similar difficulties when changing the subject of formulae. Students' mal-rules (Sleeman, 1986) were not just caused by mistaken extrapolation techniques (Matz 1980). My participants were observed to overgeneralise equation transformation techniques which revealed themselves to be little more than blind memorisation of rules without reason (Skemp, 1971). One such technique was the *change-side-change-sign* rule, a one-step shortcut of cancelling a term by applying the inverse operation and balancing out the equation. A typical mistake I identified was that done by one student (S16) who transformed $L + E = (VP)^2$ into $\frac{L+E}{\sqrt{P}} = V$. He reasoned that 'we skipped it [*meaning* P^2] over the equals so that V becomes by itself and from P-squared it became square root of P'. (Borg, 1997, p. 123). Like many others, even from another school, this student reasoned that just like a positive term becomes negative, and multiplied term becomes a divisor, then something which is squared becomes square-rooted on the other side of the equation. A similar mistake was transforming $b = \frac{1}{2}a + \frac{1}{2}c$ into $\frac{b}{-1/2} = a + \frac{1}{2}c$, a mistaken application of the *change-side-change-sign* rule.

Other misinterpretations in the Borg (1997) study were due to actions that students were accustomed to take when they encountered particular notation. For instance, it was evident that most students saw the brackets symbol as code for "expand". Most students expanded the brackets of the formula $n = (mc - r)^2$ as a first step towards

bringing r the subject of the formula, which made the formula unwieldy to handle when expanded correctly. These students were interpreting the brackets as a process to be performed and did not seem to view $(mc - r)$ as a unified object. Again, this is a notation interpretation problem, this time stemming from the *product-process dilemma* (Sfard & Linchevski, 1994) which will be discussed later.

3.2.3   Difficulties in Solving Problems

Apart from the difficulties mentioned above, word problems present learners with another hurdle: the translation from words to an algebraic statement. Bishop, Filloy, and Puig (2008) stated that this is the primary source of difficulty for students in solving algebraic word problems. It involves understanding the problem, examining the relationships between variables, assigning variables, and expressing the relationship in algebraic syntax which includes the variables, the constants, operational notation, and relational notation (usually the equals sign). Bishop et al. claimed that students' difficulties in translating from natural language to algebra and vice versa was one of the main difficulties that generally arose in such situations.

During a series of video-recorded interviews, Clement (1980) became aware that even college science students experienced this difficulty. In a large-scale study with freshmen engineering majors, Clement, Lochhead, and Monk (1981) found that approximately 40% of the students made mistakes when creating formulae from statements involving ratios between two variables. The most quoted example of such statements was: *There are six times as many students as professors.* Taking S and P to denote the number of students and professors respectively, many students converted this into $6S = P$. Clement et al. (1981) identified two sources for this error. The first was what Paige and Simon (1966) referred to as *syntactic translation,* a literal, direct mapping of the words from English language to algebraic syntax. The second was what Clement et al. (1981, p. 288) referred to as the 'the "static-comparison" method', where students would correctly interpret the statement and draw something like that shown in Figure 3.1.2.1. The mistake would occur when the space between the S-circles and the P-circle is translated into an equals sign.

Figure 3.1.2.1 Static-Comparison

Difficulties in transforming words to algebraic syntax (statements including letters) usually make students resort to an informal-algebraic approach. Consider the problem: *When 4 is added to 3 times a certain number, the sum is 40.* Kieran, Booker, Filloy, Vergnaud, and Wheeler (1990) reported that when students are left to their own devices, they would find it simpler to find the unknown number by subtracting 4 from 40 and dividing by 3. If they were asked to form an equation in x (the unknown), students would first need to represent the relationships in the statement, rather than directly perform operations to find the answer. Mason (2008) argues that a formal-algebraic approach may not seem viable for most students if all they require is to get a computational result for a one-off problem.

3.2.4   Difficulties in Conceptualising Literal Symbols

Kieran (2004) stated that a letter in an algebraic statement, or a literal symbol, may represent one of these concepts:

(i)   a variable, such as x in $ax^2 + bx + c$;

(ii)   an unknown, such as x in $ax^2 + bx + c = 0$,

(iii)   a parameter (or coefficient), such as $a, b,$ and c in the above examples.

Building on Usiskin (1988) and Küchemann (1978), Philipp (1999) added four more categories:

(iv)   a label, such as $3f = 1y$ to denote "3 feet make 1 yard",

(v)   a special constant, such as π and e,

(vi)   a generalised number, such as a and b in $a + b = b + a$, and

(vii) an abstract algebra element, such as e and x in $e * x = x$

Secondary students are expected to encounter and make use of (i) – (vi) and their interpretation of literal symbols may make or break their success in formal-algebraic activities. In a large-scale project with students in Grades 8 to 10, Küchemann (1981) identified six categories of students' conceptualisations of literal symbols which he presented in a hierarchical conceptual order.

Table 3.2.4.1   Küchemann's hierarchy of students' conceptualisations of letters

	Interpretation	Example
1	Letter Evaluated	*Solve the equation $a + 5 = 8$.* • a assigned random values – answer obtained by trial-and-error.
2	Letter Not Used	*If $a + b = 43$, $a + b + 2 = ?$* • $a + b$ ignored – answer obtained by matching.
3	Letter as Object	*Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If b is the number of blue pencils bought, and r is the number of red pencils bought, what can you write about b or r?* • b and r seen as labels – typical incorrect response: $b + r = 90$.
4	Letter as Specific Unknown	*What is the perimeter of a shape where there are n sides altogether of length 2?* • n is given a specific value – answer obtained by comparison.
5	Letter as Generalised Number	*What can you say about c if $c + d = 10$ and c is less than d?* Typical answer: "c can be 4, 3, 2, and 1."
6	Letter as Variable	*Which is greater $2n$ or $n + 2$?* • Typical answer: "Depends. If $n < 2$ then $2n < n + 2$. If $n > 2$ then $2n > n + 2$."

The results of the Küchemann's (1981) research led to a follow-up project by Booth (1984) who also investigated Grade 8-10 students' interpretations of formal-algebraic expressions. Misconceptions of literal symbols were common to those revealed by Küchemann (1981). For example, a common conceptual error was that students

interpreted letters as labels. A 15-year old student said that $3 + 5y$ could mean 'eight yachts' (Booth, 1984, p. 28). Besides the interpretation of y as an initial letter of a word, this statement shows problems in this student's interpretation of the expression structure. Booth (1984) also reported that some students thought that different letters had to have different values. She also found out that 32% of the students did not make the connection between the expression $m + m + m$ and $3m$ and 49% did not recognise that the expression $4m$ meant $4 \times m$. In addition, 31% said that since 2 lots of x is written as $2x$, then two lots of 7 may be written as 27. All errors identified by Booth (1984) could be linked to a notational misinterpretation.

Items from Küchemann's (1981) and Booth's (1984) projects were reused in many other research studies (e.g., Coady & Pegg, 1993; Trigueros & Ursini, 1999, 2003; MacGregor & Stacey, 1997; Fujii, 2003; Hodgen et al., 2008). I will highlight two main findings of these studies. The first is the disquieting conclusion by Trigueros and Ursini (1999) that students did not seem to achieve better or fuller interpretation of literal symbols as they progress through algebra courses. The researchers take this to imply that 'instead of promoting a deep understanding of variable and the development of intuitive algebraic ideas, current teaching practices seem to obstruct them' (p. 280). The second is that one particular phenomenon stands out in all these studies: the importance that context plays in determining the role of literal symbols (Wagner, 1981, 1983). This comes as no surprise, because context is crucial for the interpretation of experiences. Mercer (2000) says that words gain meaning from the 'company they keep' (p. 67). It seems that this statement can be extended to and must be stressed for the use of literal symbols in algebraic syntax.

Given the difficulties learners encounter when interpreting and representing literal symbols, it would seem almost inevitable that some difficulties may be found in interpreting answers consisting of formal-algebraic expressions. This is discussed next.

3.2.5 Difficulties in Interpreting Answers

Replicating Küchemann's (1981) study in a large-scale survey with students in Grades 7-9, Hodgen, Küchemann, Brown, and Coe (2008) asked the following question: *If $e + f = 8$, then $e + f + g =$*? A student who had answered $8g$, gave the following reasoning for his answer: '$8g$ just seems like an answer…but $8 + g$, you still think, "Oh, what will it equal?"' (Hodgen et al., 2008, p.39). This student's response seems to show difficulty of accepting *lack of closure* (Collis, 1974), i.e. accepting that formal-algebraic answers may contain an operational symbol. Collis (1974) found that beginning algebra students viewed expressions such as $8 + g$ or $x - y$ as incomplete due to their refusal to hold unevaluated operations in suspension. Collis (1974, 1975) argued that success in algebra requires the perception of such expressions as mathematical objects in their own right.

In their first encounters with formal-algebraic expressions, students often experience what Tall and Thomas (1991) call the *expected answer obstacle*. Kieran (1981a) argues that prior to their experience of literal symbols in algebra, learners become accustomed to obtaining a single numerical answer and this leads them to expect the same thing when working in formal-algebraic contexts. Booth (1984, p. 35) reported what a student, Wendy, told her interviewer when she discovered that $11 \times y$ was the expected response: 'I thought you wanted the answer.' Wendy experienced the *name-process dilemma* (Davis, 1975) or, as it is sometimes referred to, the *product-process dilemma* (Sfard & Linchevski, 1994). She interpreted the symbol $\times$ in the expression $11 \times y$ only in terms of a process to be performed, and by "answer" (or product) she probably intended a single term like 11.

Students usually settle the process-product dilemma by conjoining the two terms being separated by the operator such as $8 + g = 8g$ (Hodgen et al., 2008) or $2a + 5b = 7ab$. Pimm (1987) reported that some teachers attempt to show that this is a mistake by referring to $2a + 5b$ as 2 apples and 5 bananas. It turns out teachers themselves are are making two mistakes here. Firstly, they are encouraging the notion of letters as labels (Küchemann, 1981; Booth, 1984, 1988) which leads to structural

misconceptions (Booth,1988). Secondly, as Pimm (1987) argued, students may be actually encouraged to simplify $2a + 5b$ as $7ab$, by thinking of a fruit bowl having 2 $a$pples and 5 $b$ananas as a fruit bowl having 7 fruits: 7 apples-and-bananas. Moreover, conjoining letters to denote addition is conventional in non-mathematical contexts (e.g. in chemical equations). It is, thus, no exaggeration when Sfard and Linchevski (1994, p. 212) claim that 'the transition from purely operational to a dual process-object outlook is…likely to be a quantum leap'.

In the next section, I discuss the issue of notation, including the importance for students to develop process-object notions of expressions and to extend their meanings of familiar operational and relational symbols.

3.3 Notation as a Key Factor in Algebraic Activities

The literature seems to reveal three factors which are detrimental to formal-algebraic thought and activities:

(i) A weak basis of *arithmetic* (e.g., Gallardo & Rojano, 1987; Linchevski & Herscovics, 1996; Warren, 2003; Baroudi, 2006);

(ii) A reluctance or difficulty to express *generality* (e.g. Lee & Wheeler, 1987; Neria & Amit, 2004; Mason, 1996; Cooper & Warren, 2008);

(iii) A non-conventional interpretation and representation of *notation* (e.g. Booth, 1984; Kieran, 1981b; Kirshner, 1989; Borg, 1997; Van Amerom, 2003)

Students' manifestations of difficulties in algebra discussed in Sections 3.2.1–3.2.5 may be due to one of these issues. Furthermore, I argue that the interpretation and representation of notation is key in addressing these difficulties. What researchers report as problems in arithmetic or generality may be traced back to notation. For example, consider the arithmetic problem $3 + 7 = \boxed{} + 3$ and the generality $x + y = y + x$. To answer $3 + 7 = \boxed{} + 3$, students need to know that the addition notation implies

commutativity and that the equality notation signifies sameness. To develop such informal-algebraic statements into the formal-algebraic generality $x + y = y + x$, students need, then, to learn that generalised numbers (Philipp, 1999) may be represented by letters.

Adopting this argument, I observe problems of notation to permeate most of the research reporting arithmetical problems. For example, Falkner, Levi, and Carpenter (1999) presented 6th graders with this problem: $8 + 4 = \boxed{} + 5$. All students wrote 12 or 17 in the box. Falkner et al. (1999) concluded that the students had a restricted meaning of the equals sign, that of a unidirectional symbol indicating an operation to be performed on the left with an answer to appear on the right as in $4 + 3 = \boxed{}$. This problem is extensively documented in the literature (e.g., Behr, Erlwanger, & Nichols, 1976; Kieran, 1981; Herscoviscs & Linchevski, 1994; Linchevski, 1995; McNeil et al., 2006). This is actually a problem of notation since it stems from students' limited interpretations of a mathematical symbol, i.e. the equals sign. Another example of such a problem was reported by Warren (2003). In a large-scale study, Warren asked 7th and 8th graders which of the signs $+$, $-$, $\times$, and $\div$ could replace the symbol ♣ in statements such as $2♣3 = 3♣2$. She found that some students considered subtraction and division symbols to denote a commutative relationship.

Therefore, in order to progress from solely-arithmetic to informal-algebraic and formal-algebraic thinking (Table 3.1.4.1), students need to develop concepts about notation, namely:

(i) extend of the meaning of familiar "shape-symbols" (shapes, like ( ), which are not numbers, letters, or standalone operators like $+$ or $\times$);

(ii) understand the properties of operational symbols, and

(iii) learn that an expression may represent both a process and an object.

These are elaborated in the subsections that follow.

3.3.1 Extending the Meaning of Familiar "Shape-Symbols"

Serfati (2005) distinguished between three features of mathematical symbols: the materiality, the syntax, and the meaning. In this subsection, I discuss three symbols whose materiality is a shape (hence "shape-symbols") which need to take on a fuller, more extensive meaning for progress in informal- and formal-algebraic thinking:

- the meaning of equality symbol or equals sign (ES),

- the use of division line of a fraction, and

- the use of brackets.

Extended Meaning of the Equals Sign

Studies about students' notions of ES are well documented. In school mathematics, ES is used to denote eight types of relationships. These are shown in Table 3.3.1.1, where the first five were identified by Usiskin (1988) and the other three were added by Jones and Pratt (2012).

Table 3.3.1.1   The several uses of the equals sign

	Use of the Equals Sign	Example
1	Equation with one unknown	$2x + 5 = 3x - 6$
2	Relationship between two variables	$y = 2x + 3$
3	Identity	$2(x + 4) = 2x + 8$
4	Formula	$A = LW$
5	Property	$3 + 4 = 4 + 3$
6	Indicator of a computational result	$2 + 3 = 5$
7	Function	$f(x) = 3x - 2$
8	Substitution	$k = 2$

Each of these applications of ES entails a spectrum of difficulty levels. For example, use (8), substitution, may involve:

(i)   substituting a given value for a letter in an equation or formula,

(ii) choosing values and substituting them systematically in an equation to solve one equation (trial-and-error), or

(iii) using a trial-and-error method as in (ii) to solve a system of equations such as that reported by Filloy, Rojano and Solares (2003) when studying secondary students' solutions of simultaneous equations.

Furthermore, each of the uses in Table 3.3.1.1 can take one or both of two meanings of ES:

(i)   "results in" or "makes" (an operational view) and

(ii) "has the same value as" or "is the same as" (a relational view).

In order for students to progress through informal- and formal-algebraic reasoning they need to be able to extend the meaning of ES towards a relational view (ii) which is required for the formation of structure sense (Linchevski & Livneh, 2002) and for generational and transformational activities (Kieran, 1996) involving equality statements.

The research literature suggests that the most common conception that primary students have of ES is that of an indicator of a computational result. Behr, Erlwanger, and Nichols (1976) found that students in Grades 1-6 viewed ES as a 'do something signal' (p. 10). Students got confused with equalities such as $3 = 3$. When asked whether $3 = 3$ made sense, one student replied, 'Nope … you could fix that by going like this [*changes* $3 = 3$ *to* $0 + 3 = 3$] 0 plus 3 equals 3.' (Behr et al., 1976, p. 4). Like many others, to this student ES should have followed an operation and should have been followed by the result of that operation. When Falkner, Levi, and Carpenter (1999) asked a similar question about $8 = 8$, a 1st grader said, 'Well, yes, 8 equals 8, but you just shouldn't write it that way.' (p. 235). It seemed that while this student accept a *verbal representation* of the

relational meaning of ES, she rejected a *notational representation* of the same relational meaning.

Similar findings were reported by Behr et al. (1976). While students accepted verbal representations like *four plus five equals three plus six*, they refused the validity of the notational representation: $4 + 5 = 3 + 6$, arguing that in writing, the left of ES must contain a problem and the right of ES must contain the answer. One student argued that, 'if you went into writing you'd go like this [*writes* $4 + 5 = 9; 3 + 6 = 9$] (Behr et al., 1976, p. 9). Such students would view the $3 + 6$ on the right of ES as another problem: it presented a lack of closure (Collis, 1974) rather than the single number they were accustomed to write after writing ES.

Similarly, Ginsburg (1977) found that, in statements like $3 + 5 = 8$, primary students tended to interpret both $+$ and $=$ as operational symbols, the former as a signifier *to add* the two numbers 3 and 5 and the latter as a signifier of what this addition *makes*. This led students to finding it difficult to interpret equalities such as $3 = 3$. Furthermore, Ginsburg (1977) found that when he presented students with problems like: $\square = 3 + 4$, they said that it was written backwards. For students in Ginsburg's study, ES should:

- indicate that an answer is going to be written (operational view), and

- be read from left to right (unidirectional view).

Students' conception of ES as an indicator of an operation to be performed on its left and a single number to appear on its right has been widely reported (e.g., Kieran, 1979, 1981b; Herscovics & Kieran, 1980; Erlwanger & Berlanger, 1983; Herscoviscs & Linchevski, 1994; Linchevski, 1995; Anenz-Ludlow & Walgamuth, 1998; Pillay et al., 1998; McNeil et al., 2006). This should come to no surprise since the predominant use of ES in primary schools is exactly that of an indicator of a computational result.

Given this restricted interpretation of ES, one may understand why some students may write "false" equality statements (Kieran, 1981b) like $3 + 4 = 7 + 2 = 9$. Such students

see no errors in writing operations in the order in which they were being thought and in keeping a running-total, a common pattern observed in primary school children's mathematics (Kieran, 1979). This pattern of reasoning and working was probably what caused students in Falkner, Levi, and Carpenter's (1999) research to put a 12 in the box when presented with the equality statement: $8 + 4 = \boxed{} + 5$. Given that arithmetic is only concerned with getting a correct answer (Hewitt, 1998) one might almost understand why some teachers fail to regard such statements as incorrect. In this way, 'misconceptions about equality can become more firmly entrenched' (Falkner, Levi, & Carpenter, 1999, p. 233). Moreover, McNeil (2008) found that primary school teachers' use of ES only for typical arithmetical problems may limit students' meanings of ES to an operational view. Falkner et al. (1999) argue that given the limited use of ES in primary school, children are correct to think of the equals sign as a signal to compute. On the other hand, informal- and formal-algebraic activities are concerned with relationships (Scandura, 1971), and this points to the importance for students to develop relational views of ES.

Nevertheless, it is still the convention to indicate the answer of a computation with ES and therefore, unsurprisingly, students usually retain the operational meaning even when they develop relational meanings of ES (McNeil, 2008; Rittle-Johnson et al., 2011). Although late primary and early secondary students were found to exhibit operational views of ES which sometimes hinder their understanding of higher-order mathematics topics (Kieran, 1981b; Knuth et al., 2006; McNeil & Alibali, 2005b; McNeil et al., 2006), they were sometimes found to simultaneously exhibit relational ideas about equations in certain contexts (McNeil & Alibali, 2005a; McNeil et al., 2006) and perform well when solving problems involving equivalence (McNeil, 2007). Rittle-Johnson et al. 2011, p. 97) argue that, 'describing children as having an operational or relational view of equivalence is overly simplistic'. Rittle-Johnson et al. identified a continuum of knowledge progression from a rigid operational view of ES to a comparative relational view of ES. During this progression, students start developing relational meanings of ES while retaining an operational view, as reported by McNeil (2008).

Research has shown that teaching aimed at helping students to extend students' conceptions of ES had the desired impact for average-performing students (McNeil & Alibali, 2005b; Rittle-Johnson & Alibali, 1999) and also for low-performing students (Powell & Fuchs, 2010). These studies were not aimed at teaching students that an operational view of ES is incorrect. What these studies suggest is, rather, that teaching can and should help students to extend or elaborate their concept of ES (Herscovics & Linchevski, 1994; Kieran, 2004) so that it incorporates the relational aspect.

Extended Meaning of Brackets and Division Line of a Fraction

The two other shape-symbols whose meanings need to be elaborated by students as they progress through informal- and formal-algebraic activities are the division line of a fraction and the brackets. Rubenstein (2008) stated that one of the major challenges that mathematical symbols present is that the same mathematical concept may be represented by more than one symbol. She said that algebraic activities require students to start denoting divisions like $12 \div 3$ by $\frac{12}{3}$ and multiplications like 3×4 by $3(4)$. Rubenstein (2008) said that besides having to learn new notation for familiar operations (such as division and multiplication), students are faced with the challenge of giving different meanings to the same shape-symbols. For example, $\frac{12}{3}$ may denote 12 thirds and 12 divided by 3. Similarly, brackets may denote a multiplication, e.g. 3(4), and a means to specify the order of operations, e.g. $12 - (5 + 3)$. In addition, a notation like $(3, 4)$ may signify a point, an open interval between 3 and 4, and the vector $3\mathbf{i} + 4\mathbf{j}$.

Anghileri (1995) pointed out that most students are accustomed to seeing a computation such as $12 \div 3$ denoted by $3\overline{)12}$ which they sometimes read as 3 "*divided into*" 12. Unsurprisingly, as Hewitt (2009) pointed out, students are confused in their first encounters with the new notation of division. Students who were familiar with the notion of $\frac{12}{3}$ as being a mathematical *object*, a fraction, need to accommodate an extension of its meaning to include $12 \div 3$ or $3\overline{)12}$ which to them is a *process*. As I

presently discuss, it is crucial for students to learn that an expression like $\frac{12}{3}$ is indeed both a *process* and an *object* (e.g. Gray & Tall, 1994).

Similar confusions arise when students first encounter concatenations such as $5n$. Herscovics and Linchevski (1994) showed that when asked to use the substitution $n = 2$ in $5n$ students write 52 rather than $5(2)$ or 5×2. Moreover, due to their rejection of lack of closure (Collis, 1974) students may want to "simplify" something like $5 + n$ as $5n$ (Hewitt, 2012). In primary school, students learn that a mixed number like $5\frac{3}{4}$ means $5 + \frac{3}{4}$. It probably causes a disequilibrium (Piaget, 1975) in their notational schema when they learn that $5\left(\frac{3}{4}\right)$ means $5 \times \frac{3}{4}$ and not $5\frac{3}{4}$.

This extension of meaning for the brackets notation may be a further complication for some students who already have issues with the use of brackets. Kieran (1979) found that children typically do not use the brackets because they think that the written sequence of operations is what determines the order in which the computations should performed. This was corroborated by Booth (1984) who found that 88% of the students in her study failed to appreciate the need for brackets and so carried out the operations in the order they were written.

Primary school teachers usually address the issue of order of operations by teaching mnemonics like BIDMAS[5] (e.g., Headlam & Graham, 2009). Thus, before starting to learn that brackets may signify multiplication, students may have become accustomed to see the brackets as a *prompt* to work out what lies within them. This may be one of the causes of their difficulty in accepting the lack of closure (Collis, 1974) of expressions such as $6(n + 2)$. Further on, when solving equations like $5(x + 2) + 4(2 - x) = 7$ students are taught to expand the brackets before proceeding with the transposition of the equation. This technique may lead to complications in problems where expanding the brackets is counterproductive, such as bringing r the

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[5] Brackets first, then Index, Division and Multiplication, Addition and Subtraction

subject of the formula in $n = (mc - r)^2$ (Borg, 1997). While striving to help students to develop meaningful notions of notation, teachers need to be careful not to lead students to think that a technique that is viable in one context will necessarily work in another.

Extending concepts of familiar shape-symbols like ES, brackets, and division line is necessary but not sufficient for informal- and formal-algebraic activities. Students also need to develop concepts about the properties of operational symbols which are usually used in conjunction with such shape-symbols in expressions. This is discussed next.

3.3.2   Understanding the Properties of Operational Notation

Difficulties in informal- and formal-algebraic thinking and activities have been found to stem from limitations in students' knowledge about properties of operational notation. Some of these limitations manifest themselves in mistakes or inefficient solutions when solving problems.

Mistakes Involving Operational Notation Properties

I have already mentioned Warren's (2003) study which revealed that some Grade 7 and 8 students thought of subtraction and division as commutative. This finding is not uncommon especially for younger students (e.g. Carpenter and Levi, 2000). However, what may seem as students' overgeneralisation of the commutativity property may be caused by other factors. Booth (1988) reported that some students used $3 \div 12$ and $12 \div 3$ interchangeably. Although this may have been due to students' thinking that division is commutative, Booth pointed out that this mistake could have originated from students' experiences of division where the larger number was always divided by the smaller. Another reason could be that some students read divisions like $3\overline{)12}$ as "3 divided by 12" and give the (correct) answer of 4. Left untackled, such verbal representations may be translated into expressions like $3 \div 12$, which may, in turn, be interpreted as $12 \div 3$.

Other problems which have been associated with students' applying the commutative property to subtraction were identified by Chaiklin and Lesgold (1984) in their study with Grade 6 students. Students were asked to decide whether expressions with three numbers were equivalent and made mistakes such as thinking that $597 - 648 + 873 = 648 + 873 - 597$. Besides thinking that subtraction was commutative, these students may have been confused when transposing operators along with numbers. Similarly Herscovics and Linchevski (1994) found that half of the 7th graders in their study failed to solve the equation $4 + n - 2 + 5 = 11 + 3 - 5$, where one common mistake was to group the numbers on the left as $4 + n - 7$. Herscovics and Linchevski hypothesised that this may have been due to students' failure to use commutativity of $4 + n$ to obtain $(n + 4) - 2 + 5$ and then associativity to obtain $n + (4 - 2) + 5$. However, this may also have been caused by students' inexperience of negative numbers and the first operator in $-2 + 5$ did not make sense to them and so their unary parsing was compromised. Another possible cause could have been that these students had a tendency to group numbers in a calculation without resorting to brackets (as reported by Kieran, 1979; Booth, 1984) and thus interpreted $-2 + 5$ as if they were $-(2 + 5)$. Such mistakes in transformational activities may sometimes be caused by failing to apply inverse properties correctly. Gallardo and Rojano (1987) found that when solving equations like $x + 1568 = 392$ and $13x = 39$ secondary students subtracted 392 from 1568 and divided 13 by 39 respectively.

Limitations of Application of Operational Notation Properties

Students' inexperience in the use of operational notation properties does not always translate itself into mistakes. Sometimes it leads to limitations when engaging in informal- and formal-algebraic activities. Stacey and MacGregor (1997) found that while students were confident in using commutativity, associativity, and inverse properties in small whole numbers, they were unsure whether such properties applied to unfamiliar numbers. I argue that if students confirm that $3 \times 5 = 5 \times 3$ but are unsure about $3.7 \times 4.6 = 4.6 \times 3.7$, they may be engaged in solely-arithmetic thinking, i.e. recalling the answer of each side of the first equation to verify equivalence. In the

second equation, it would be difficult for them to compute each side, hence the uncertainty. Unless students moved beyond such thinking and progressed to informal-algebraic thinking, their concept of commutativity may not have been developed. Likewise, Chaiklin and Lesgold (1984) reported that some students could not decide about the validity of statements like $685 - 492 + 947 = 947 - 492 + 685$ without computing each side.

Similar limitations were found with regards to other properties of operational notation, such as associativity, distributivity, and inverse. In the Stacey and MacGregor (1997) study, one student stated that she knew that division can be "undone" by multiplication and knew that $18 \div 3 \times 3 = 18$, but she was unsure whether the same applied for 16. She argued that $16 \div 3 = 5r1$ and she did not think that $5r1 \times 3 = 16$. Again this seems to imply that this student was resorting to computing operations in order, rather than using the multiplicative inverse property.

Another type of limitation was that reported by Norton and Cooper (1999), where the vast majority of Grade 9-10 students could not work out expressions like $\frac{36+24}{6}$ and $5 \times (6 + 7)$ when they were not allowed to compute the addition first. The researchers concluded that students did not seem to be aware that multiplication and division were distributive over addition. However, it may have been that students felt that it was unacceptable not to follow the BIDMAS rule, especially in the second expression. Norton and Cooper's claim about students' limitations due to associativity is more convincing. Only a quarter of the students knew how to evaluate $\blacksquare + (\blacktriangle + 7)$ given that $\blacksquare + \blacktriangle = 11$.

The role that unfamiliar examples play in such investigations is crucial. In fact, what might be reported as a limitation may be more of a hesitation due to unusual questions or numbers. Carpenter and Levi (2000) found that while 1st graders were confident in switching numbers in an addition to start counting from the largest, they failed to do so when given very large numbers. Although this may have been due to a limited development of the commutativity concept, it may also have been a matter of students not being inclined to try out their technique with strange numbers.

Teaching Aimed at Tackling Operational Notation Issues

There are two conclusions I draw from studies about operational notation. The first is that teaching aimed at tackling operational notation issues such as the ones reviewed above are bound to make a difference in students' interpretations of notation. Pillay, Wilss, and Boulton-Lewis (1998) found that while Grade 7-8 students had limited understanding of commutativity and distributivity, Grade 9 students showed competence in these concepts when solving linear equations. Pillay et al. (1998) suggest that it may have been the very introduction to algebra (formal-algebraic activities) which helped students to develop these concepts. Such activities shift students' attention from products to processes (Hewitt, 1998), something Dreyfus (2002) insists is required for advanced mathematical thinking. After all, as Bruner (1966) argues, knowing is a process not a product.

The second conclusion is that teachers need to be careful observers of their students' representations if they want to form experiential models (Steffe et al. 1983; Glasersfeld, 1991b) of their students' conceptual constructions. They should ask students to read mathematical statements and listen attentively to their verbal representations. As Usiskin (1996, p. 236) says, 'if a student does not know how to read mathematics out loud, it is difficult to register the mathematics'. Teachers need to discuss with students why $3\overline{)12}$ should not be read as "3 divided by 12".

When reading students' work, teachers should not just be after "correct" answers but should strive to be the agents for the product-to-process shift (Hewitt, 1998). They should give weight to seemingly minor mistakes like:

- keeping a running total (e.g. Kilpatrick et al., 2001; Vergnaud et al., 1979; Kieran, 1979) like $7 + 3 = 10 + 2 = 12$ and
- interpreting verbal representations incorrectly (e.g. Subramaniam & Banjeree, 2004) like writing $7 - x$ for "seven less than x" .

As Glasersfeld (1991b, p. 178) maintains, 'for constructivist teachers, successful thinking is far more important than "correct" answers'.

So far, I have discussed two important ways in which students can develop higher-order concepts of notation:

  (i)   extending meanings of familiar "shape-symbols", and

  (ii)   understanding the properties of operational notation.

The third issue is the one which, in my experience as a teacher, is quite difficult to address. This is discussed below.

3.3.3   Proceptual View of Expressions

I have previously discussed how students' errors in their interpretation of algebraic answers (or expressions) may stem from their refusal to accept lack of closure (Collis, 1974). I have shown how students get confused when faced with the *name-process* (Davis, 1975) or *product-process* (Sfard & Linchevski, 1994) dilemma and they usually interpret expressions like $x + 2$ as a process (Kieran, 1979). Studies have repeatedly shown that in order to engage in algebraic activities students need to be able to interpret mathematical expressions both as a process and as an object.

The Process-Object Dilemma

Piaget (1975/1985, p. 49) says that one important aspect of knowledge construction occurs when 'actions or operations become thematized objects of thought or assimilation'. When explaining how mathematical schemas are formed, Piaget and García (1989, p. 105) explain that 'after a process…the particular notion used becomes an object of reflection, which then constitutes itself as a fundamental concept'. Building on Piaget's theories, Dienes (1971) uses grammatical terms to explain how mathematical processes become objects of other processes, saying that the object of a predicate becomes the subject of another predicate. Davis (1975) builds on Diene's (1971) work and identifies what he called the *name-process* dilemma faced by students when interpreting expressions. Davis (1984, p. 29) explains that 'the procedure itself becomes an entity - it becomes a *thing.* It, itself, is an input or object of scrutiny.'

The conceptual reconstruction of an expression resulting from a process into a mathematical entity is well documented in the literature, where different authors use different metaphors to describe the process-object unification. The *encapsulation* (Dubinsky, Elterman, & Gong, 1988; Ayers, Davis, Dubinsky, & Lewin, 1988; Dubinsky, 1991), *reification* (Sfard, 1989, 1991, 1992, 1995; Sfard & Linchevski, 1994), *integration operation* (Steffe & Cobb, 1988), or *entitication* (Harel & Kaput, 1991) of a process into a *conceptual entity* (Greeno, 1983) enables students to conceptualise a string of mathematical symbols as both a process and a mathematical concept, or what Gray and Tall (1991, 1994) called a *procept*.

Gray and Tall's "Procept"

Most relevant for my study, is the work of Gray and Tall (Gray & Tall, 1991, 1993, 1994, 2001; Gray, 1991; Gray, Pitta, & Tall, 2000; Tall, Thomas, et al. 2000; Tall, Gray, et al., 2001; Tall, 1991, 1994, 1995; Tall & Thomas, 1991) because they are the only ones who regard notation to be key in avoiding having to decide between process and object:

> It is through using the notation to represent either process or product, whichever is convenient at the time, that the mathematician manages to encompass both – neatly sidestepping the problem.
>
> (Gray & Tall, 1991, p. 73)

Gray and Tall (1991, p.73) give several examples where students can take a proceptual view of notation, i.e. interpreting an expression as a process and an object (or concept), such as:

- The process of *counting all* or *counting on* and the concept of *addition* ($5 + 4$ evokes both the counting on process and its sum, 9);
- The process of *division* of whole numbers and the concept of *fraction* (e.g. $3/4$);
- The process of adding 2 to $3x$ and the concept of the resulting sum evoked by the expression $3x + 2$.

Using data from Gray's (1991) study with students of aged 7 to 12, Gray and Tall (1994) showed that one characteristic which made some students able to progress through higher-order problems was their development of a proceptual view of expressions. This confirms the categories of thinking and activities I posited in Section 3.1.4, using the matchsticks-array problem (Hewitt, 1998). Solely-arithmetic thinkers use only counting to find the number of matchsticks and the task becomes more complicated as the array gets larger. Informal-algebraic thinkers manage to *encapsulate* (Ayers et al., 1988) repeated counting as multiplication. For them, 5×101 and 6×100 are respectively the *reification* (Sfard & Linchevski, 1994) of the process of adding 5 for 101 times and adding 6 for 100 times. However, informal-algebraic reasoning does not cater for the *entitication* (Harel & Kaput, 1991) of the sum of these two products a process into a single conceptual entity (Greeno, 1983). Formal-algebraic thinkers collapse the algorithm for finding the number of matchsticks into one entity, i.e. the expression $5(r + 1) + 6$ or $11r + 5$, an expression which they interpret proceptually:

(i)   the process of multiplying 11 by the number of rows and adding 5, and
(ii) the concept of the number of matchsticks in an array of r rows.

It seems that progress from solely-arithmetic to informal- and formal-algebraic thinking is possible by developing a proceptual view of notation and becoming flexible in proceptual reasoning. Gray and Tall (1994) argue that:

> The existence of flexible proceptual knowledge means not only that the number 5 can be seen as $3 + 2$ or $2 + 3$ but that if 3 and something makes 5, then the something must be 2. In proceptual thinking, addition and subtraction are so closely linked that subtraction is simply a flexible reorganization of addition facts.
>
> (Gray & Tall, 1994, p. 125)

This implies that a well-developed proceptual view may require students to acknowledge properties of operational notation, the absence of which may cause difficulties in algebraic thinking, such as commutativity of addition (e.g., Chaiklin & Lesgold, 1984; MacGregor, 1996; Warren, 2003) and additive inverse (e.g., Gallardo & Rojano, 1987; Herscovics & Linchevski, 1994). Furthermore, a proceptual interpretation of expressions is required for successful generational and transformational (Kieran, 1996) algebraic activities.

Extending the meaning of familiar shape-symbols, learning about the properties of operational notation, and obtaining a proceptual view of expressions are as challenging for students as they are crucial for the learning of algebra. Davis (1975, p. 29) says that 'many major cognitive adjustment are required…(to) start seeing the equal sign in new ways, and even seeing $\frac{3}{x}$ as an "answer" instead of a problem'. In the next section, I discuss a possible way forward in helping students to make such cognitive adjustments so that they can develop algebraic thinking skills and be successful in algebraic activities.

3.4 A Way Forward: Use of Computers for Algebraic Thinking

One of Kaput's long-term commitments to make mathematics accessible to all children was to promote the dissemination of algebra throughout the K-12 curriculum. In his last published work, Kaput (2008, p. 6) speaks about 'the highly dysfunctional result of the computational approach to school arithmetic and an accompanying isolated and superficial approach to algebra'. Several researchers support this contention, not least of which is Carraher who, together with his colleagues, argues for the inclusion of formal-algebraic activities in primary curricula (Carraher, Schliemann, & Brizuela, 2000, 2001, 2006; Carraher, Schliemann, Brizuela, & Earnest, 2006; Carraher, Brizuela, & Earnest, 2001; Carraher, Schliemann, & Schwartz, 2008). In Malta, like in many other curricula, formal-algebraic activities only start at Grade 7 (DLAP Syllabus, 2014a). Nevertheless, Maltese primary mathematics teachers can still pay heed to Kaput's (2008) criticism by engaging students in informal-algebraic activities where they emphasise processes rather than products (Dreyfus, 2002) This was suggested by National Council of Teachers of Mathematics (NCTM, 1998, 2000) and found to be both possible and effective (e.g. Falkner et al., 1999; Carpenter & Levi, 2000; Hunter, 2015). Grade 7 teachers need to ensure that their students have had the opportunity to engage in such activities before they introduce formal algebra. Further discussion of such informal-algebraic activities is included below.

3.4.1 Preparation for Formal-Algebraic Activities

Bell (1996) argues that a good preparation for formal-algebraic activities requires students to have been given the opportunity to become fluent in handling notation. Along the same lines, Boulton-Lewis et al. (1998) stress that students need to be given the opportunity to develop concepts about several aspects required for formal-algebraic activities including properties of operations and notion of the equals sign (ES).

Several researchers suggest that such preparation could be achieved through the kind of arithmetic which does not just focus on answers but on strategies and concepts involved in the process of getting those answers, in other words, informal-algebaic reasoning and activities. Livneh and Linchevski (2007) refer to 'arithmetic for algebraic purposes' (p. 217) and activities that are 'algebra compatible' (p. 219), such as those where students focus on the order of operations, use of brackets, and interpretation of ES. The authors reported that students were able to transfer informal-algebraic structural knowledge to formal-algebraic contexts.

Such informal-algebraic activities were suggested by Fujii and Stephens (2001) who devised a set of mathematical statements with numerical expressions which remain true whatever numbers were used. For instance, a series of equations like $78 - 49 + 49 = 78$ were meant to help students to develop concepts about additive inverse. Such informal-algebraic statements would later lead to a the formal-algebraic generality $a - b + b = a$. A study by Swafford and Langrall (2000) with Grade 6 students revealed the possibility of helping students to start making number generalisations by writing informal equations to represent problem situations. Swafford and Langrall demonstrated that students were able obtain a proceptual view of expressions (Gray & Tall, 1994). Similar recommendations were made by Pillay et al. (1998) who claimed that misconceptions they found in arithmetical knowledge (cited in Section 3.3.2) pointed to the need to help students to develop concepts about commutativity, distributivity, and the relational aspect of the equals sign.

Such concepts are bound to be limited if students are not taught how to appreciate processes and relationships rather than products and answers. My experience with beginning Grade 7 students taught me that most students come with preconceived notions that mathematics is all about finding answers, with little or no regard to the methods applied to obtain those answers. This comes as no surprise, given that the Maltese primary mathematics curriculum (DCM, 2014) assumes a computational slant. As Balacheff (1986) warned, a curriculum which emphasises computation rather than argumentation is bound to instil this kind of attitude in students because their main goal is to achieve an answer rather than to construct mathematical knowledge for themselves. Consequently, in mathematics curricula like ours, Grade 7 teachers need to make sure that students appreciate notions like the ones discussed earlier before introducing formal-algebraic tasks. In the following section, I will present information and communications technology (ICT) as one possible way forward to help students to appreciate mathematical relationships in order to make the journey to informal- and formal-algebraic reasoning.

3.4.2   Use of Computers in Mathematics Teaching and Learning

Educational organisations, policy makers, and curriculum developers have been reiterating the benefits of using ICT for mathematics teaching and learning (e.g. NCTM, 2000; DfES, 2004). In Malta, the integration of ICT in the teaching and learning of mathematics has been taken very seriously. In the late 1990s, specific use of ICT was put in requisition in the secondary level mathematics curriculum leading to the Secondary Education Certificate (SEC) public examinations. Mathematics teachers were required to use spreadsheets, dynamic geometry software, computer algebra systems, and Logo. These requirements, which are still valid today (SEC Mathematics Syllabus, 2017), were introduced almost concurrently with governmental policies aimed to disseminate the use of ICT in schools. Policies in favour of embedding ICT in teaching and learning were being adopted internationally towards the turn of the century (e.g., Kankaanranta & Kangassalo, 2003; Fung & Pun, 2001; Bucky, 2000; Sakamoto, 2003; Oldknow, 2006). In most European countries, core subject teachers were and still are continuously encouraged, through central level recommendations, to

apply a variety of ICT hardware and software in their lessons (EACEA/Eurydice, 2011). Such recommendations and suggestions were, at least in part, spurred by studies which suggested that ICT can bring about a positive change in the way school subjects are taught and learnt.

Benefits of ICT Applications in Mathematics Education

Benefits of computer software applications for mathematics teaching and learning are widely documented. One of the most reported claims is that ICT applications in mathematics lessons help students to understand and perform better. In particular, academic benefits were reported in all four types of computer software that have been included in the Maltese mathematics secondary education curriculum:

(i) ***Spreadsheets*** were found to help students to deeply explore mathematical concepts, construct multiple representations of a concept, and strive for generality (e.g., Healy & Sutherland, 1990; Rojano, 1996; Sutherland & Balacheff, 1999; Filloy, Rojano, & Rubio, 2000; Dugdale, 2001; Friedlander & Tabach, 2001; Hershkowitz, et al., 2002; Ainley, Bills, & Wilson, 2004).

(ii) ***Logo*** (discussed in Section 3.4.3) was found to help students to develop more analytic thinking skills, learn about geometric properties, generalise, learn about variables, predict and test mathematical theories (e.g., Hillel & Samurcay, 1985; Lehrer & Smith, 1986a, 1986b; Watson, 1993; Clements & Battista, 1997; Clements and Sarama, 1997; Vincent, 2001).

(iii) ***Dynamic geometry software*** was found to benefit students in learning geometric concepts, link dynamic visual representations of standard shapes, construct rigorous Euclidean proofs, and appreciate the dynamic nature of changing variables short of doing a field test (e.g., Vonder Embse & Yoder, 1999; Gerretson, 2004; Forsythe, 2007; Patsiomitou, 2008; Myers, 2009).

(iv) ***Computer algebra systems*** were found to give students the opportunity for systematic exploration, prompting rich algebraic discussions (e.g., Shoaf-Grubbs,1995; Penglase & Arnold, 1996; Drijvers, 2001, 2003; Pierce, Ball, & Stacey; 2008; Cedillo & Kieran, 2003; Meagher; 2012).

Apart from such academic gains, students have been reported to describe learning mathematics with ICT as an enjoyable experience (Judah, 1999; Ramsay, 2001; Scher, 2002; Lugalia, 2015). Mumtaz (2001) urged teachers to observe how children enjoy playing computer games at home and to find ways how to use computers to make learning resemble playing in a way as to 'enable children to work on activities *they* find valuable, motivational and worthwhile' (p. 347, my emphasis). Heath (2002) and Scher (2002) claimed that teaching mathematics with ICT has benefits for the learners which they cannot gain with traditional teaching approaches.

These educational benefits are neither automatic nor unconditional. Kaput (1992) elucidates four major principles that teachers and educational leaders need to follow if the use of ICT in mathematics education is to be beneficial. These are elaborated further by Hoyles and Noss (2007):

(i) *Attend to representational infrastructure*. Educators must seek to find ways how ICT can represent mathematical ideas and help students who do not seem to deal adequately with conventional representational systems.

(ii) *Work for infrastructural change*. The infrastructure of mathematical curricula needs to be changed, otherwise teachers would simply treat ICT as an ill-fitted add-on to the content they are required to teach (Borg, 2009). Computer software itself needs to be adapted to the needs and goals of mathematics educators and learners (Hoyles & Noss, 2003).

(iii) *Outsource processing to the computer but attend to the implications*. Mathematics educators should outsource processing to ICT but reveal *layers* (Hoyles & Noss, 2007) of calculation algorithms to help students to understand the mathematical theories behind those procedures.

(iv) *Exploit connectivity to encourage sharing and discussion*. ICT contributes to the emergence of a *cultural* infrastructure (Hoyles & Noss, 2007), one where students, teachers, and researchers share mathematical information and learn from each other.

Kaput's (1992) principles imply that ICT needs to be seen as a tool to bring *reform* and *innovation* in both teaching approaches and learning experiences (Kirschner & Wopereis, 2003), rather than to dress old teaching methods in new clothing. If ICT does not bring a positive change in the way students learn, it is bound to be regarded as expendable (Sutherland, 2005) and its use is likely to wane, especially when curricular and time constraints make it difficult for teachers to use ICT in their lessons (Borg, 2011). One of the main proponents of educational reform through ICT was Papert, whose ideas and work is discussed below.

3.4.3   Papert, Logo, and Other Microworlds for the Learning of Algebra

Together with colleagues and students at the MIT Artificial Intelligence Laboratory, Papert developed Logo in 1967. Logo is a programming language best known for its turtle graphics feature which was added by Papert in the later stages of its development. This allows users to create their own drawings on the computer screen by writing a series of commands. Papert (1993a)[6] explains how Logo can be used by students to learn mathematics, 'for example the mathematics of space and movement and repetitive patterns of action' (p. 54). He envisaged Logo as a model how children can learn through what he refers to as *constructionist* pedagogy. Papert and Harel (1991) explain how constructionism adds to constructivism the idea that students learn best when they create 'a public entity, whether it's a sand castle on the beach or a theory of the universe' (p.1). According to Ackermann (2001), while Piaget's developmental theory tends to overlook the role of context and individual learning needs, Papert's *constructionism* lays particular emphasis on learning conditions and circumstances. Papert's pedagogy also gives more importance to learning by doing. It is therefore 'both more *situated* and more *pragmatic* than Piaget's constructivism' (Ackermann, 2001, p. 89, original emphasis).

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[6] Originally published in 1980.

Almost idealistically, Papert (1993a, p. 182) depicts lessons with Logo as 'alternatives to traditional classrooms and traditional instruction'. For him, Logo is not a mere substitute for conventional teaching but a *model* of an alternative teaching style, one which allows students to learn by doing, by creating projects which make sense to them and which they enjoy. For Papert, rigid linear school curricula do not really reflect children's learning preferences and patterns. Rather, he argues, learning takes place when learners perform trials, errors, and improvements, a process he calls "debugging", which is the rule of the day when working with computer environments like Logo.

Like Dewey (1907), Papert (1993b) felt that the radical change he was expecting in education did not seem to be happening. Papert (1993b, p. 2) lamented: 'Why, through a period when so much human activity has been revolutionized, have we not seen comparable change in the way we help our children learn?' There is a striking affinity between this statement and Dewey's (1907, p. 22) disillusionment about the lack of a radical change in educational practices following the industrial revolution, when he exclaimed: 'That this revolution should not affect education in other than formal and superficial fashion is inconceivable.' The onset of Papert's (1993a) ICT revolution never occurred in schools. This was probably due to a stagnation in the educational infrastructure (Kaput, 1992; Hoyles & Noss, 2007), as Papert (1998) himself suggested. Nevertheless, there are research reports attesting to the educational benefits of Logo which started to regain popularity in the mid-nineties.

Teaching and Learning with Logo

Logo was first introduced in schools in the early eighties. Major projects like the Logo Maths Project in the UK (Sutherland, 1989) explored Logo's potential to help children in their constructions of mathematical concepts, namely the concepts of space and variable. The benefits of Logo for teaching and learning geometrical concepts are widely reported (e.g., Hoyles & Noss, 1988; Hoyles, Healy, & Sutherland, 1991; Battista & Clements, 1988, 1991; Yusuf, 1994; Lehrer, Randle, & Sancilio, 1989; Clements & Battista, 1989,1990, 1997; Lehrer & Smith, 1986a, 1986b; Watson, 1993).

Nevertheless, Logo was also found to be beneficial for the development of algebraic reasoning. Some studies attest to the power of Logo to enhance informal-algebraic thinking. For example, Hoyles and Noss (1989) reported that students developed additive strategies when using Logo commands. Logo was also found useful to introduce students to formal-algebraic activities. Clements and Sarama (1997) argued that the use of Logo immerses students in the use of two notions that according to Noss (1986) are crucial in the algebraic domain: variables and functions. Students write commands involving arbitrary (Hewitt, 1999) symbolic representations and watch the screen turtle move according to those commands.

Research studies such as those of Hillel and Samurcay (1985), and Sutherland (1989) showed that Logo tasks together with a series of lessons which emphasise links between Logo and algebra can lead students to develop, formalise, and generalise the concept of variable. In fact, Clements and Sarama (1997) suggested that Logo can be used to provide students with an "entry" to the use of the powerful tool of algebraic thinking. This contradicted findings of some studies (e.g., Johnson, 1986) claiming to have observed no significant differences between Logo and control groups.

Moreover, most of the students in Sutherland's (1989) study were also able to transfer knowledge about the concept of variable to pen-and-paper (algebraic) contexts. Again this contradicted findings by other researchers (e.g., Gurtner, 1992; Kurland, Pea, Clement, & Mawby, 1986), claiming that students did not transfer knowledge from a Logo environment to pen-and-paper problems. As Vincent (2001) argues, such failure may be due to a lack of teacher guidance for students to make explicit connections between Logo variables and those used in pen-and-paper algebra. Clements and Meredith (1993) stress that exposure to Logo without teacher guidance often yields little learning. This is partly due to the fact that the connections between Logo programming and traditional pen-and-paper problems are not as explicit as they are in other microworlds such as the ones discussed below.

Microworlds

"Microworlds" are those 'small, interactive, and programmable models of real world environments' (Ennis-Cole, 2004, p. 55). Besides the Logo turtle graphics, there are several microworlds available for classroom use and I am going to review some of the more significant research studies which attest to the application of microworlds for the teaching and learning of algebra.

One such microworld is *MathWorlds*, which offers students the possibility to work with multiple representations of moving objects. Doerr (2001) studied the way 15-17 year olds used *MathWorlds* to study functions. The students generated conjectures, provided explanatory arguments, and made symbolic representations of observable phenomena. Grade 7-9 students working with the Java version, *Simcalc MathWorlds* (Hegedus & Kaput 2003) were reported to have made statistically significant pre- and post-test gains on standard test items. Interestingly, the Grade 9 students who were low-performing, "at-risk" students made the highest gain with the help of the software. The researchers believed that *MathWorlds* had contributed to helping students better understand fundamental algebraic ideas by forming realistic identity-relationships with the mathematical objects they constructed in the microworld.

A significant amount of research evidence about the use of microworlds for the teaching and learning of algebra was provided by Noss and his colleagues. One such microworld was *Mathsticks*, a computer environment in which students assemble different objects made of matchstick patterns. The designers of the software, Noss, Healy, and Hoyles (1997), showed how *Mathsticks* helped secondary students engage in informal-algebraic activities where they described generalisations of how the number of matchsticks could be calculated. Hoyles, Noss, and Adamson (2002) similarly showed how *Mathsticks* could be used to help students to appreciate number patterns as functional relationships and formulate algebraic structures.

A longitudinal research programme in the UK, called the *MiGen* project, set out to design and evaluate a pedagogical and technical system to help students to develop a propensity to strive for algebraic generalisation. Mavrikis, Noss, Hoyles, and

Geraniou (2012) reported the results of the first three years of this project which consisted of several large-scale studies with 11-14 year old students and their teachers using the microworld *eXpresser*. With *eXpresser* students learnt how to create general rules to generate tile patterns. The number of tiles in patterns were expressed as a numerical expression in one window, and *eXpresser* matched this with an algebraic expression in another window, called the "general model". Mavrikis et al. (2012, p. 231) claim that *eXpresser* can be used by teachers to 'sow the seeds' in students to understand and appreciate the purpose of searching for algebraic generalisations. In a more recent analysis of part of the *MiGen* project, Geraniou and Mavrikis (2015, 2016) reported that students applied knowledge they constructed when working with *eXpresser* to pen-and-paper formal-algebraic problems. However, they argued that such a transfer was only possible through activities which were specifically designed to bridge microworld representations to those written on paper.

This finding challenges claims expressed by the EACEA Eurydice Report, (2011) that students rarely use ideas, concepts or strategies they develop through their interaction with such technologies in other contexts. The concern is that while students might know how to use ICT procedurally, they may fail to understand the mathematical concepts behind those procedures. Such concerns are less evident in the literature about students' use of grid-based environments like spreadsheets, maybe due to the closer resemblance of mathematical representations in such environments to conventional numerical and algebraic expressions.

3.4.4   Spreadsheets and Other Grid-based Environments

Several studies (e.g., Filloy, Rojano, & Rubio, 2000; Kieran, 1992; Rojano, 1996; Sutherland & Balacheff, 1999) showed that spreadsheets can help students to engage in algebraic thinking by generalising arithmetic. Research has shown that the use of spreadsheets can help students to develop powerful algebraic thinking skills such as generalisation, symbolisation and functional relationships (e.g., Lannin, 2003; Battista & Van Auken Borrow, 1999; Healy & Sutherland, 1990). The use of spreadsheets to generate recursive rules for number patterns was found to be one of the main benefits

of utilising spreadsheets for the learning of algebra (e.g., Ploger, Klingler, & Rooney, 1999; Friedlander, 1999). Another reported benefit was that with spreadsheets students engage in systematic approaches to problem-solving activities (e.g. Hersberger & Frederick, 1999; Filloy, Rojano, & Rubio, 2000). Such attempts to systematise calculations is one important attribute of algebraic thinking, as Gattegno (1988) explains when he discusses *awareness of awareness.*

Ainley, Bills, and Wilson (2004) showed how Grade 7 students made use of *carefully planned* activities on spreadsheets to construct meaning for the concept of a variable. One particular spreadsheet activity was called "The Hundred Square task", consisting of finding patterns in 3×3 cross-shapes from within a 100 square created on a spreadsheet (Figure 3.4.4.1).

Figure 3.4.4.1   The Hundred Square Task

Students were observed to discuss operations on the central number in the cross and they accepted the use of the cell as a placeholder for a general number (Ursini & Trigueros, 2001). Ainley et al. (2004) claimed that this was an indication of students'

constructions of meaning for the concept of variable as a placeholder for the middle number of the cross. They argued that such activities helped students appreciate the usefulness of algebra since

(i) it provided an *intermediate language,* a bridge between students' normal language and formal-algebraic notation, and

(ii) it gave immediate and meaningful feedback to the students in their activities.

The grid-based structure of a spreadsheet with cells containing numbers or expressions is a feature of at least two other grid-based environments designed for mathematics learning. One of these is *Structured Variation Grids* (*SVGrids*) which was developed by Mason[7] (see Mason, 2005a). *SVGrids* are two dimensional grids of cells based on the multiplication table. A typical interface is shown in Figure 3.4.4.2.

Figure 3.4.4.2   A typical SVGrids interface

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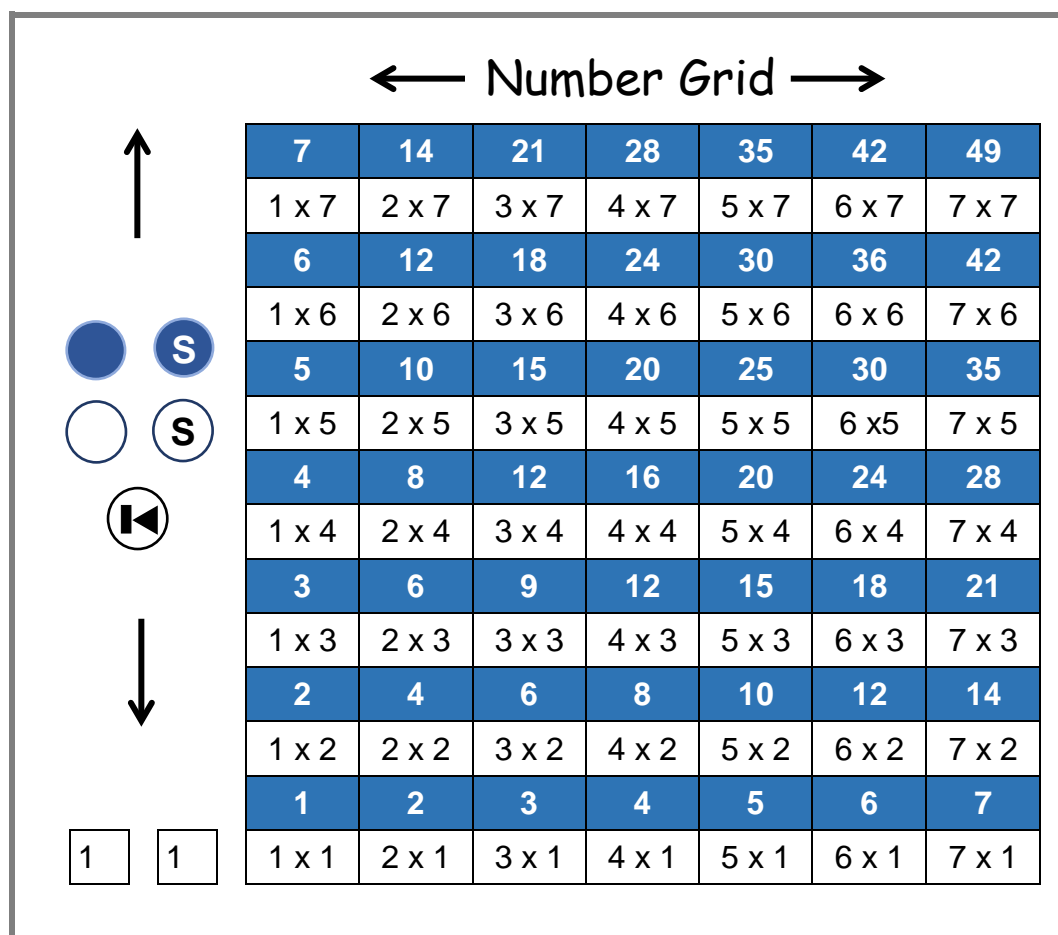
[7] In collaboration with Tom Button.

As shown Figure 3.4.4.2, *SVGrids* cells have two parts, one (light background) containing the expression or calculation and another (dark background) containing the result of that calculation. The former could take form of an expression in x, such as $(x + 1)(x + 3)$. Each cell is a particular case of a general rule for patterns of numerical or algebraic expressions. Mason (2005b) explained how *SVGrids* could be used to help students to use formal-algebraic expressions, detect and predict patterns, and discuss conjectures about the generalities of those patterns.

The other grid-based computer environment is *Grid Algebra*, for which I review the research literature below.

3.4.5   Grid Algebra[8]

Grid Algebra (GA) is a grid-based computer environment which is based on the multiplication grid like the ones mentioned above, but which allows movements between the cells. A typical GA interface is shown in Figure 3.4.5.1.

Figure 3.4.5.1   A typical GA interface

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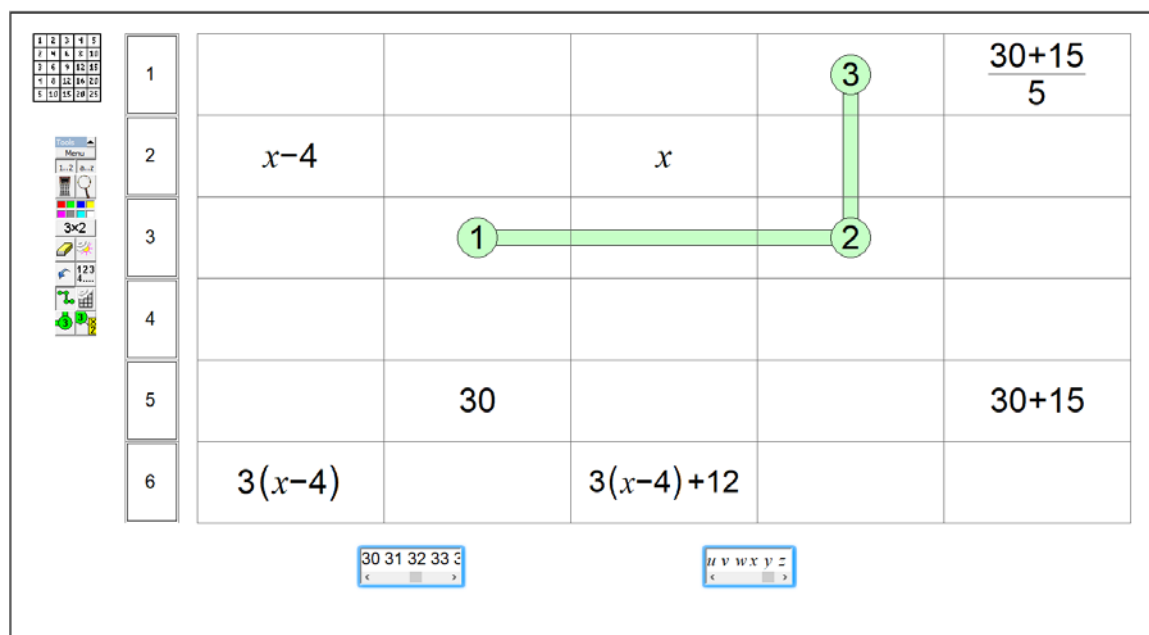
[8] Section 4.3 includes a fuller description of the Grid Algebra software. Here I will only discuss research literature in which Grid Algebra was used as a primary teaching medium.

GA enables users to create and build numerical and algebraic expressions either by moving a cell and its contents from one place to another or by typing it directly with respect to its place in the multiplication grid and in relation to other expressions existing in the grid. As shown in Figure 3.4.5.1, GA displays conventional notation which is useful to help students to become familiar with the way they are expected to represent mathematical expressions. Furthermore, it gives students the possibility to trace the movements of expressions around the grid, such as the 1-2-3 journey shown in Figure 3.4.5.1. Hewitt (2001), the designer of GA, maintains that GA encourages students to focus on the operations involved in algebraic expressions rather than the calculation required to evaluate that expression. Hewitt (2009) reported that Grade 5 students using GA were able to meaningfully learn formal-algebraic notation by referring to the journeys made by expressions. He showed how sometimes students focus on the expression as an *object* to be operated on and at other times they would pay more attention to the *process* by which a new expression is formed. This hints that one possible benefit of GA is that it offers students the opportunity of obtaining a proceptual view of expressions (Gray & Tall, 1994). Hewitt (2012) focused on another aspect of his earlier study (Hewitt, 2009). He showed how GA helped students learn the order of operations, not by memorising and following a mnemonic such as BIDMAS, but by observing how expressions were being built up one operation at a time with successive movements of the expressions in the cells around the grid.

GA offers three interlinked representations: the action of dragging the cells, the picture of journeys and cells, and the symbolic notation appearing in the cells. These are reminiscent of Bruner's (1966) enactive, iconic, and symbolic representation modes. Hewitt (2014) argues that students develop concepts by making connections between these representations. This echoes Dreyfus (2002) who asserts that multiple-linked representations give students the flexibility required for mathematical problem-solving.

Like other software, GA was found to boost students' motivation to learn mathematics. Lugalia (2015) investigated the learning attitudes and algebraic attainment of secondary students of varying ages in Kenya when taught with the help of GA. She reported that GA boosted 'students' engagement, enjoyment, new confidence, and

eagerness in mathematics' (p. 190). She also found that the majority of students improved their algebraic attainment and connectivity of ideas (Skemp, 1976; Dreyfus, 2002). The positive effect of GA on students' motivation and, subsequently, attainment was also reported in a UK secondary school (Lugalia, Johnston-Wilder, & Johnston-Wilder, 2011). Foy (2008) described similar experiences of using GA as a mathematics teacher of Grade 7 students. She said that GA provided 'an excellent kinaesthetic and visual approach to algebra, enriching students' overall understanding' (p. 41). However, she highlighted the importance for teachers to take the time to introduce the software well to their students.

Besides the reasons I gave in Chapter 1 for my choice of GA as the main teaching medium for my research, I believed there were aspects of GA which merited further investigation, namely:

- the use of GA as a tool which facilitates constructivist teaching, and

- students' use of GA to interpret and represent notation.

This leads me to the second set of research questions, the ones dealing with students' representations and interpretations of notation.

3.5 Research Questions about Students' Representations and Interpretations of Notation

In this second part of the literature review, I was concerned with the subject matter of my research lessons and how learners related to it. I explored the several views of "algebra" and used the terms solely-arithmetic, informal-algebraic and formal-algebraic to help me present my own viewpoint about algebraic thinking and activities. I discussed students' relationships with algebra and focused on the difficulties they encountered when engaging in algebraic activities. I identified the limited interpretation and representation of notation to be one major cause of such difficulties. Exploring ways in which this could be addressed, I presented the use of computer software, specifically GA, as a tool which could be used to help students to develop meanings about notation. Freire's (1998) contention that teaching is dependent on and linked to

learning meant that I could not investigate my constructivist teaching without reference to whether and how it facilitated learning.

Thus, I present my second set of research questions, the ones regarding students' representations and interpretations of notation. For the sake of completeness, I am also including the first set of research questions which were presented at the end of the previous chapter:

1. (i)   How do I engage in CT and what are the distinguishing characteristics of such a teaching approach?

   (ii)   What, if any, are the moments when I fail to engage in CT?

2. (i)   How do students represent and interpret mathematical notation as they start Grade 7?

   (ii)   How does GA help students to enrich their representations and extend their interpretations of mathematical notation?

   (iii) How do students transfer representations and interpretations of notation they develop when working with GA to pen-and-paper problems?

Questions 1(i)–(ii) are overarching, theory-seeking questions, where I seek to build a conceptual framework about CT in a mathematical context. Questions 2(i)–(iii) are more evaluative, where I investigate the effectiveness of my CT in terms of helping students to develop concepts about notation with the help of GA.

In the next chapter, I set out to discuss the methodology and method I employed to investigate the data collected for the purpose of addressing these questions.

Chapter 4

Methodology and Method

4.0   Overview

In this chapter, I describe and contextualise the research methodology and method I adopted to gather and analyse the research data. Table 4.0.1 gives an overview of the section titles.

Table 4.0.1   Chapter 4 section titles

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4.1   Research Design

In this section, I explain the rationale for my research methodology, which was a qualitative approach, and my research method, which was a case study. I will do this with reference to the two sets of research questions listed at the end of Chapter 3.

4.1.1 Rationale for the Research Methodology: A Qualitative Approach

The rationale for adopting a qualitative approach for my research was influenced by two factors, namely, the methodological stance and a number of pragmatic considerations. Regarding the former, Burrell and Morgan (1979) identify four assumptions which differentiate between subjectivist and objectivist perspectives. These are the ontological, the epistemological, the human nature, and the methodological assumptions. In the following subsections, I discuss each one of these assumptions as they stood in my research approach.

My Ontological Assumption

In line with Glasersfeld (1991a), my belief about reality is that it is the result of individual cognition, experiential rather than absolute. My view of reality is relativist rather than realist: I reject the possibility of making observations about the world which are unconnected to the observer's subjective interpretations of what his or her senses are receiving. I support Merriam's (2002) claim that

> [t]he world, or reality, is not the fixed, single, agreed upon, or measurable phenomenon that it is assumed to be in…quantitative research.

> (Merriam, 2002, p. 3)

My research results were thus presented as a subjective, interpretative description of my participants' *unique* experiences and mine rather than evidence of some universal truth or generality. This does not mean, however, that readers may not make their own inferences and/or apply lessons learnt from this study to their own situations. This issue will be discussed further on.

My Epistemological Assumption

In line with radical constructivism (RC), I assume that knowledge is not independent of the knower and, as such, it is internally created rather than passively received from external sources. The act of learning is an attempt of the learner to regain equilibrium

in the intellectual journey of understanding the world (Piaget, 1975/1985) and therefore knowledge makes sense to the knower only if it is viable (Glasersfeld, 1984) in comprehending experience. During the data collection process, my standpoint about my search for answers to the research questions, about the learning of my students and my endeavours to help them learn, I have regarded the nature of knowledge as such: a subjective construction of ideas which serve to better understand the experiential world. Once again, this does not mean that my students' experiences and mine are necessarily segregated from the experiences of readers: hopefully, some readers may share our consensual domain (Glasersfeld, 1991a) and find, within this study, viable explanations for their own unique experiences.

My Human Condition Assumption

My belief that learners construct their own knowledge stems from my assumption that humans take an active role to create and modify their environment. Human beings' actions are not simply a product of external circumstances but rather they are how they respond to their environment, in the sense of Maturana and Varela (1992): The motivation that drives humans to construct knowledge is the urge to make sense of their experiences by finding viable explanations of what they observe and sense. My view of the participants' actions and my own was, therefore, voluntaristic rather than deterministic. During the data-collection process, my students and I were using our environment to test whether our ideas were viable or not (Glasersfeld, 1984).

My Methodological Assumption

A methodology which stems from such beliefs must acknowledge the importance of details in the particular and individual in the quest for answers to the research questions. Such a methodology is ideographic (Cohen, Manion & Morrison, 2011), a process of investigating a small group of individuals in depth and detail to achieve a unique understanding of them. My choice for a qualitative study was therefore partially due to this set of assumptions about the nature of social reality.

Nevertheless, my choice for a qualitative paradigm was also due to pragmatic considerations of my research aims which I shall now discuss.

Pragmatic Considerations

As Muijs (2011) says, many researchers adopt a pragmatic approach on the basis of their research aims. Besides the small number of participants, which did not allow for non-parametric quantitative tests, I had to consider other, more fundamental, pragmatic factors which had to do with the nature of my research objectives.

In this research, I wanted to capture the subjective experiences of individual learners as I strived to engage in constructivist teaching (CT). Hence, I intended to analyse a small number of participant students in depth so that I could find explanations which were viable and made sense within the situatedness of the classroom, i.e. the interconnections between the social practices of the lessons and the teaching-and-learning process (Lave & Wenger, 1991). Context was thus very important for this study. As Sutton (1993, p. 413) claims, researchers need to be 'careful not to undermine the validity of observations by isolating them from the environment that gives them meaning'.

My research questions concerned my own students' learning through mathematical journeys and my own teaching when helping them make those journeys. When I set out to collect and then analyse the data I was fully aware of my bias in favour of the teaching method I was adopting and the teaching tools I was using. This acknowledgement of self-bias necessitated that any findings of this study needed to be sought and reported in an interpretive and pluralistic manner. I have therefore reported my findings in the spirit that:

(i) these were my own interpretations of my students' representations (verbal or otherwise) and may be different from what other people might have observed or concluded, and

(ii) there might be contrasting research findings of similar situations which are equally valid as mine.

The subjective nature of this approach led me to choose a qualitative methodology because I was investigating teaching and learning 'in their natural settings, attempting to make sense of or interpret phenomena in terms of the meanings people bring to them' (Denzin & Lincoln, 2011, p. 3) while assuming that meanings derived from the findings were plural and open (Bruner, 1993).

4.1.2   Rationale for the Research Method: A Case Study

After carefully considering the different types of qualitative research methods, I concluded that a case study was probably the best route for my research aims. Nevertheless, what I planned to do with the data contained elements from other qualitative disciplines as well. My active participation in the generation of the data may be attributed to an ethnography (LeCompte & Preissle, 1993; Hammersley & Atkinson, 2007) and the fact I intended to answer the research questions by investigating my students' lived experiences and mine may be associated with phenomenology (van Manen, 1990; Greene, 1997). Furthermore, the account I wanted to give of the participants' learning journey throughout a whole scholastic year, and of my efforts to accompany them in that journey, had the story-like characteristics of a narrative inquiry (Clandinin & Connelly, 2000). My intention to give meaning to these experiences and observations and to generate theoretical explanations within specifically designed conceptual frameworks may be regarded as an exercise in grounded theory (Corbin & Strauss, 2000, Creswell, 2013). However, none of these qualitative methods by themselves captured the entirety of my research aims as well as a case study, as I discuss next.

Appropriateness of a Case Study

MacDonald and Walker (1975) define a case study as 'the examination of an instance in action' (p. 2). The instance is that of a bounded system (Smith, 1974; Adelman, Kemmis, & Jenkins, 1980) such as a group of people surrounded by similar circumstances (Cohen, Manion, & Morrison, 2011). In such a group, the whole is more than a sum of parts (Nisbet & Watt, 1999) and such an inbuilt wholeness necessitates in-depth investigation (Sturman, 1999).

Hitchcock and Hughes (1995) mention a number hallmarks of a case study. In particular, a case study

(i)   blends a description of events with the analysis of them,

(ii)   focuses on individual actors and seeks to understand their perceptions of events,

(iii)   highlights specific events that are relevant to the case, and

(iv)   involves the researcher as an integral part of the case.

Furthermore, Hitchcock and Hughes (1995) suggest that a case study is set in temporal, geographical, organisational, institutional, and other contexts that enable boundaries to be drawn around the case and the participants' roles, functions, and characteristics within those contexts may form a reference point for the definition of the case. Moreover, a case study may be exploratory, descriptive, or explanatory (Yin, 2012) but always set in real life situations where researchers are able to provide a rich detail of those situations (Ary, Jacobs, Razavich, & Sorensen, 2006).

Bassey (1999) identifies three types of case studies in educational research:

(i)   Theory-seeking and theory-testing;

(ii)   Story-telling;

(iii)   Evaluative.

Bassey warns that such a categorisation 'is a dangerous game' (p. 64) because some case studies may have overlapping characteristics. These features may be derived from the weight that researchers give to the reasons why they take on the case study.

Cohen, Manion, and Morrison (2011, p. 129) give four reasons why researchers may choose to adopt a case study:

(i)   To portray, analyse and interpret the uniqueness of real individuals and situations through accessible accounts;

(ii)   To catch the complexity and situatedness of behaviour;

(iii)   To contribute towards action and intervention;

(iv)   To present and represent reality – to give a sense of "being there".

The appropriateness of a case study as my research method was revealed after considering how it fitted my research aims. The case I intended to study was that of a bounded system (Smith, 1974) of six learners and their teacher (myself) rendered closed and unique (Hitchcock & Hughes, 1995) by the learners' age, grade, course, achievement level, country and school context, their teacher, and their particular scholastic year. I also intended this to be a descriptive case study, which, according to Yin (2012), 'presents a complete description of a phenomenon within its context' (p. 5). I regarded my case study to be principally theory-seeking (Bassey, 1999) since I wanted to develop a theory that described CT in a mathematics classroom (research questions 1(i)–(ii)). However, this would not have been possible without telling the story of students' intellectual journeys and evaluating the extent to which they were able to develop mathematical concepts with my assistance and with the use of GA (research questions 2(i)–(iii)). Hence, my research also contained elements of "story-telling" and of evaluative case studies (Bassey, 1999).

I reported the research outcomes as unique but significant lessons in education through the experiences and interpretations (Cohen, Manion, & Morrison, 2011) of my students and mine while acknowledging my own methodological assumptions, philosophical bias, and a dual function of being both the researcher and the teacher in the case study. Being a longitudinal study, the research allowed for actions and interventions (Cohen et al., 2011) on my part as a teaching researcher while providing possibilities of catching the complex and situated nature (Cohen et al., 2011) of classroom dynamics.

It is worth noting here that my case study had some elements in common with action research, especially when I was collecting and analysing data to investigate my own teaching. McNiff and Whitehead (2005, p. 3) define action research as that which 'is done by people who are studying themselves and their work, and asking questions

about what they are doing, why they are doing it, and how they can improve it'. However, the main aim of this study was not to improve my teaching but to portray my teaching in the light of students' construction of ideas and to attempt to give theoretical explanations of this portrayal.

Concerns about a Case Study

Hall (2008) identifies four concerns which may render case studies inferior than other research methods:

(i) Concern over having lack of rigour;

(ii) Concern over not having clearly defined research questions and employing loosely defined concepts;

(iii) Concern over taking too long and generating too much unstructured data;

(iv) Concern over attempting to generalise from a single case.

To address these concerns Yin (2013) suggested that case study researchers should try to be as methodical as possible throughout the whole study, follow systematic procedures, and not to allow ambiguous data to affect the direction of findings and conclusions. Interestingly, Yin's (2013) suggestion emulates the natural science method where rigour is usually equated with internal validity, construct validity, external validity (or generalisability), and reliability (e.g. Gibbert, Ruigrok, & Wicki, 2008) which may seem at odds within a subjectivist and interpretivist paradigm.

I will, however, discuss how I viewed and tackled issues of reliability, validity, and generalisability in subsequent sections. These will be discussed in the light of a further concern, that of adopting the dual role of a teacher and a researcher. These discussions require a description of the research context and data-collecting methods, where I will start with matters regarding the students involved in this study.

4.2 Matters Regarding the Participants

In this section, I am concerned with matters regarding the student participants. I will introduce pseudonyms to refer to the school, the students, and other factors which may compromise the identity of the participants. This measure was one of the ethical considerations I discuss later on.

4.2.1 The School Context

In Malta there are three types of secondary schools. The vast majority are state schools which are run by the government, followed by Church schools which fall under the jurisdiction of the Diocesan Curia, and Independent schools which are the only profit-making schools in Malta. The school in which I was employed as a full-time teacher will be given the pseudonym "St. George's College". It is a Maltese Church secondary school for boys, with a yearly cohort of 52 students at Grade 7 level. At St. George's College, students are divided into three sets according to their performance levels in examinations and assignments. This is only done for three core subjects: Maltese, English, and Mathematics. I will call the sets Set A, Set B, and Set C, with the latter being the set for the lowest performing students.

The performance level of new students in St. George's College is determined from a national benchmark examination which almost all students in Malta sit for at the end of Grade 6. The number of students in Set C is usually low (6-10) to facilitate more individual attention. The performance level of a new student is decided according to the standard deviation of his Grade 6 benchmark examination grade in each of the three core subjects with reference to the cohort. As a rule, a Grade 7 student in Set C would have a score of less than 1 standard deviation below the mean of the scores of the new cohort of the school. In the 2014 mathematics examination, the mean was 79.6%, the median was 81%, and the standard deviation was 14.016. The participants of this research were the 2014-15 Grade 7 Set C (Mathematics) group ("Grade 7C Maths") with the highest benchmark score being 65% (standard deviation $= -1.04$) and the lowest score being 35% (standard deviation $= -3.18$).

4.2.2   The Participants

Five out of six of the student participants had specific learning needs. Between them they had two learning support assistants (LSAs) who used to sit with them during the lessons and help them to understand and to manage their classwork. Having observed that each of these five students managed quite well in mathematics lessons without the support of an LSA, I decided that the LSAs would not be present for the lessons involved in this study. I also gave the LSAs and students' guardians specific instructions not to help them with their homework. This decision was made to minimise compromising data with regards to my CT and the way it helped the students learn. Table 4.2.2.1 gives the characteristics of individual students as given to me at the beginning of the scholastic year. I am including personal comments, some of which I perceived after the first few weeks of lessons.

Table 4.2.2.1   Initial descriptors of the participants

	Age	Bench. Score	Standard Deviation	Special Ed. Need	Comments
Dan	12	35%	−3.2	ADHD; Very poor literacy skills	• A year older than his peers; • Difficult home situation' lives in a boys' institution by court order.
Dwayne	11	57%	−1.6	Coeliac	• Coeliac condition is severe; • Suffers from headaches; • Very difficult periods in his life, both physically and psychologically.
Jordan	11	53%	−1.9	Speech/ Language difficulty	• Takes a long time to express himself; • Difficulty to express thoughts in words.
Joseph	11	65%	−1.0	ADHD; Slight Dyslexia	• Outgoing personality but clashes with peers when things do not go his way.
Omar	11	49%	−2.2	Dyslexia; Poor literacy skills	• Reads and writes very slowly.
Tony	11	60%	−1.4	Nil	• Easily distracted; • Tends to give the impression that he knows more than he actually does.

These characteristics were important during the planning of the research lessons and my interactions with the learners. The other aspect of the lessons was the main tool which I planned to use, the GA software. Below is a brief discussion of its main features.

4.3　　The Grid Algebra Software

GA is a dynamic software based on the multiplication grid in which students engage in mathematical activities involving numerical and algebraic expressions.

4.3.1　　Value of Grid Algebra Cells

In the interactive (*Run*) mode (Figure 4.3.1.1), GA opens an interface showing a blank multiplication grid (Figure 4.3.1.1a). Numbers can be picked up from a number menu and dragged into any cell, as long as they are multiples of the first number that appears at the beginning of the row. For example, 15 can be inputted anywhere in the 3rd row because it is a multiple of 3 (Figure 4.3.1.1b). Once that is done, the number that goes in the cell on its right is 18 since it must be $15 + 3$ (the next number in the 3-times table), the number on its left is 12 since it must be $15 - 3$, and so on.

Figure 4.3.1.1　　GA cells
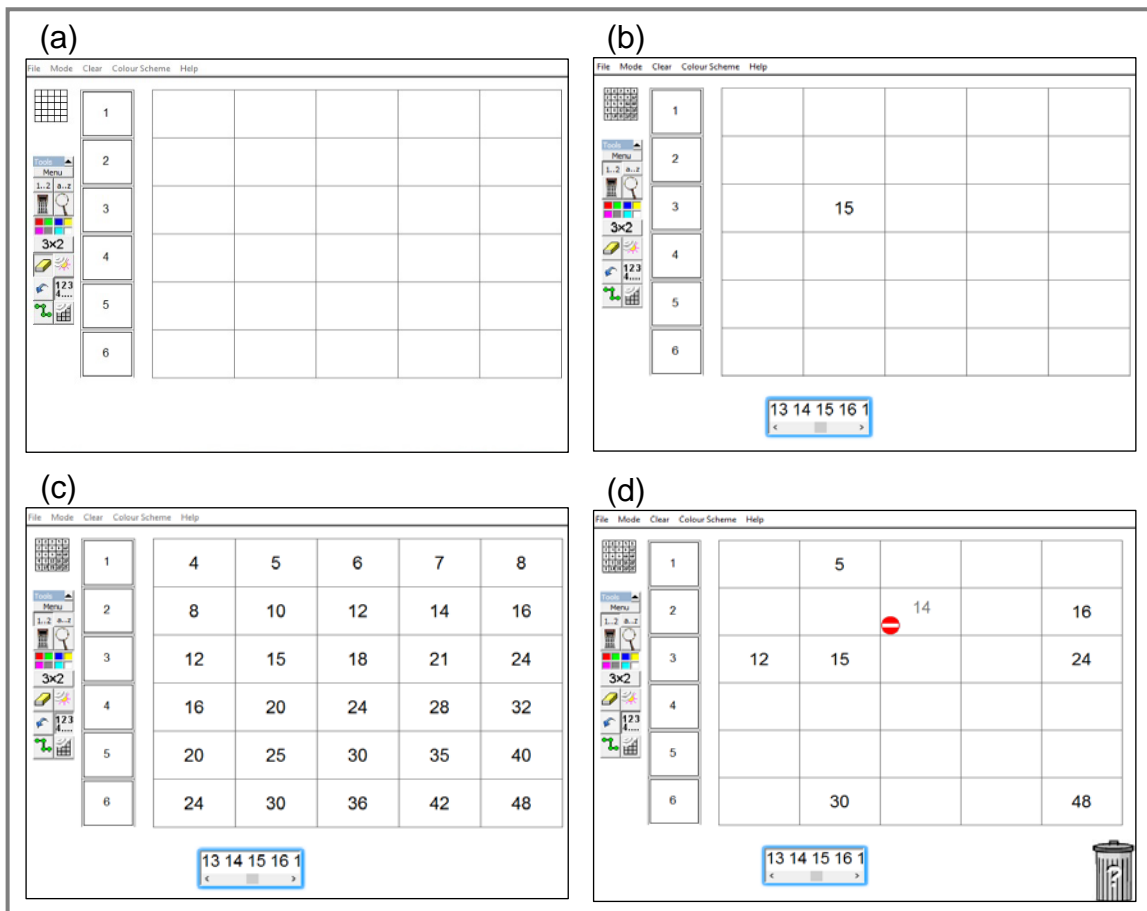
The numbers that can enter in the other rows are such that the grid is a snapshot taken from the multiplication table (Figure 4.3.1.1c). Numbers may be erased or entered into the grid as long as they correspond to the particular multiple of the row number. If students try to enter a number that does not have the value of that particular cell, GA would show a "no-entry" sign and a bin appears (Figure 4.3.1.1d). With a further click the inappropriate number is transported to the bin.

4.3.2 Movement of Grid Algebra Cells

In the *Run* mode, cells may be moved horizontally or vertically onto other cells. Horizontal movements represent addition or subtraction of multiples of a particular number and vertical movements represent multiplication and division. Figure 4.3.2.1 shows a series of screenshots as the cell representing 15 is moved three cells to the right in R_3 (row 3).

Figure 4.3.2.1   Movement of GA cells
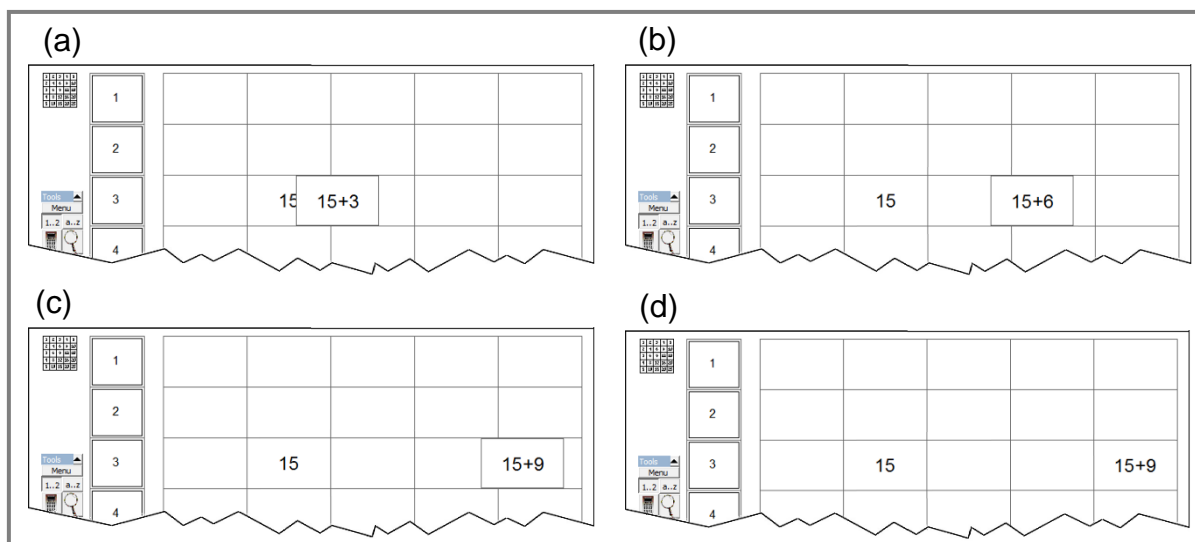
As the cell is dragged along R_3, GA changes the number into an expression containing the original number and the operation corresponding to that movement. As shown in Figure 4.3.2.1, the movement from R_3C_2 (cell in row 3 column 2) to R_3C_5 corresponds to +9. If the cell is let go on R_3C_5, GA does not show 24 but it shows the expression that corresponds to the whole action of picking 15 from R_3C_2 and moving it to R_3C_5,

i.e. it shows $15 + 9$. This is a key feature of GA since it focuses the attention of students on the operations involved and on the resulting expressions rather than computing the value of the cell (Hewitt, 2001).

4.3.3   Notation and the Meaning of Expressions

GA shows formal notation of addition, subtraction, multiplication, and division. Figure 4.3.3.1 shows how numbers and expressions are transformed when their cells are moved in the directions shown by the arrows[9].

Figure 4.3.3.1   Numerical expressions resulting from movements in GA
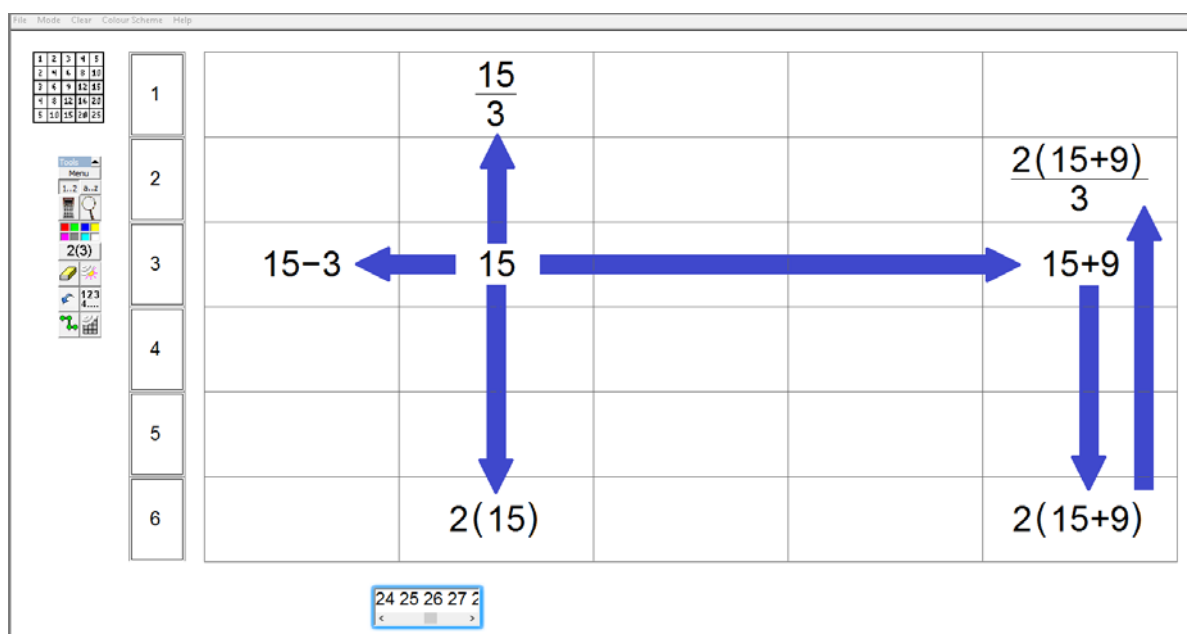
As shown in the diagram, when 15 in R_3C_2 is dragged onto R_6C_2, this movement corresponds to multiplication by 2 since the 6-times table is twice the 3-times table. Hence, the number that appears in R_6C_2 is $2(15)$ and if $15 + 9$ is dragged from R_3C_5 to R_6C_5 the number that appears in the latter is $2(15 + 9)$. If the latter is then dragged from R_6 to R_2, which corresponds to a division by 3, the expression that results is $\frac{2(15+9)}{3}$. In this way, GA introduces conventional notation which is challenging for many learners (Van Amerom, 2003). Furthermore, GA encourages the manipulation of non-

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[9] GA does not show arrows. These are added for the purpose of the description.

evaluated expressions. This may help students to become accustomed to the lack of closure (Collis, 1974) of such expressions, which will be crucial when dealing with algebraic expressions like $2x + 6$ which cannot be evaluated as a single number. Most importantly, this requirement helps students regard expressions both as processes and as manipulable objects, a proceptual interpretation (Gray and Tall, 1994) of notation.

As shown in Figure 4.3.3.2 (see R_1C_2), GA offers the feature that a letter may be placed in a cell where it represents the number which may reside in that cell. If no other numbers have been inputted beforehand, the letter represents a variable multiple of the row number. If at least one number has been inputted beforehand, the letter represents a constant, i.e. an unknown.

Figure 4.3.3.2   Successive cell movements producing more complex expressions in GA
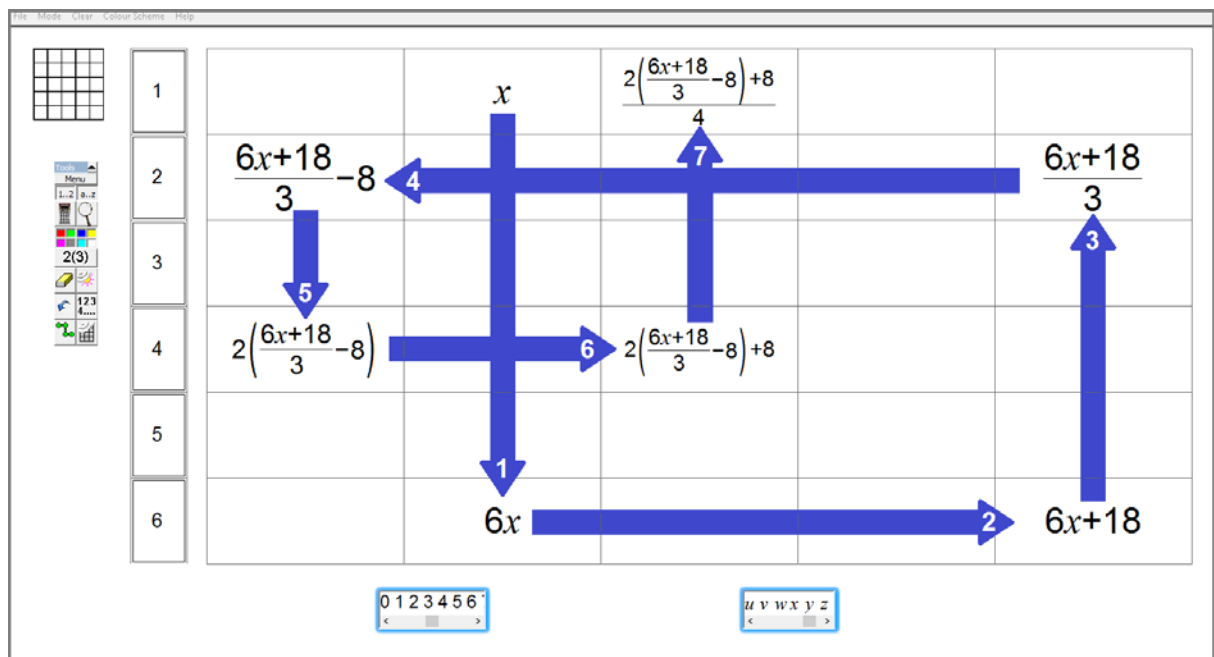
The transformations on numbers and numerical expressions resulting from cell movements may be done on letters of algebraic expressions, i.e. expressions containing a letter. Figure 4.3.3.2 shows how a letter, representing a variable multiple of 1 (since it is in R_1), undergoes transformations into increasingly complex algebraic expressions in a 7-movement journey which results in a 7-operation expression.

It is possible to track the journey made by the expression in a cell by choosing the journey button and clicking on successive cells as shown. Figure 4.3.3.3 shows a GA trail that is shown when this is done to the successive stages of the previous diagram.

Figure 4.3.3.3   Pictures of GA journeys
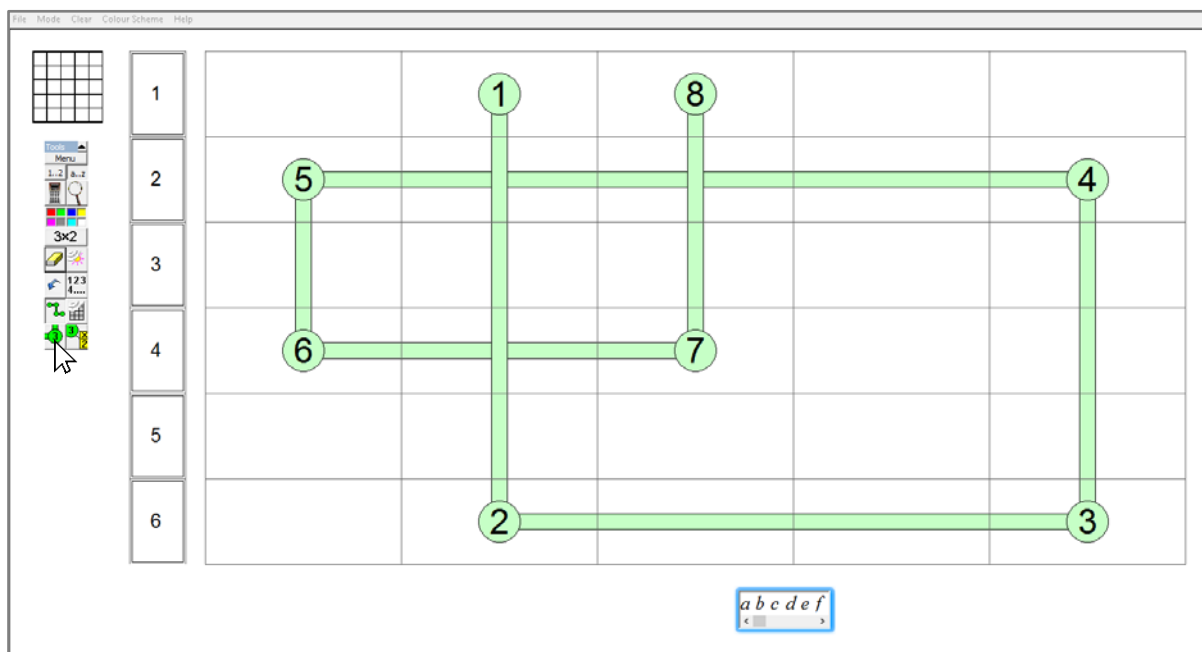
GA only forms such a journey picture if the cells involved in the stages of the journey are clicked in the right order, where it assigns ① to the 1st stage, ② to the 2nd stage (after the first step has been made), ③ to the 3rd stage, and so on. This feature gives students the opportunity to focus on the order of operations involved in an expression.

4.3.4   Equivalent Values in the Same Cell

A cell may contain two or more expressions as long as these are equivalent. When this occurs, a helpful tool is the "magnifier", an icon in the left menu which students can use to see what expressions are currently in that cell. If the magnifier icon is chosen and a cell is clicked on, a small window appears with an equation involving the expressions in that cell. Figure 4.3.4.1 displays three such equations.

Figure 4.3.4.1 GA magnifier
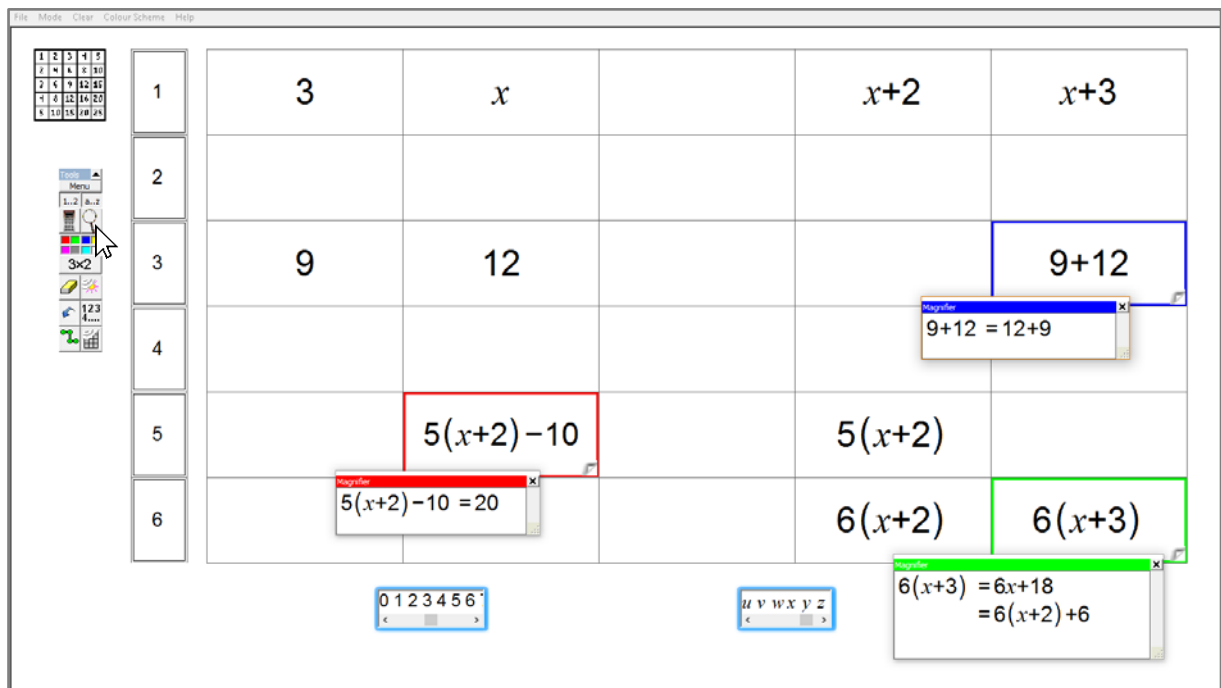
R_3C_5 contains the expressions $9 + 12$ and $12 + 9$ and choosing the magnifier and clicking on R_3C_5 displays a window with the equation $9 + 12 = 12 + 9$. A similar equation is formed in R_5C_2 which contains the expressions $5(x + 2) - 10$ and 20. The magnifier of R_6C_5 shows that there are three expressions in that cell. Clicking on the bottom right corner of the cell with multiple expressions alters the expression that is shown on top. This changes the order of the expressions shown in the equation of the magnifier. One possible use of the magnifier is to give students the opportunity to consider the sides of an equation separately and present the equals sign as a symbol that denotes a balance of quantities between the two sides of an equation (Kieran, 1981; Linchevski, 1995), i.e. a relational symbol, and not just an indicator that a computation has to be performed, i.e. an operational symbol.

There is another way of obtaining more than one expression in the same cell besides actually moving the cells. After entering a number, letter, or expression in a cell, students can imagine a journey of a number/letter which results in that cell and transform the movements of those journeys into operations. In Figure 4.3.4.2a, the number 28 is first entered in R_4C_2. An "expression calculator" icon is chosen from the

left menu and clicked on the cell. An expression resulting in 28 through movements in the grid may be entered in the calculator, for example $4(6 + 1)$. The latter is achieved by clicking on 6, $+$, 1, $\times$, 4. Every click of a number or operation adds to the expression being formed on the calculator, where clicking $\times$ shows the brackets enclosing $6 + 1$, and clicking 4 afterwards shows a 4 before the open bracket. Clicking "Enter" will place the expression in the cell if it is correct. Figure 4.3.4.2b shows that a magnifier chosen for that cell shows the equation $4(6 + 1) = 28$.

Figure 4.3.4.2   GA expression calculator
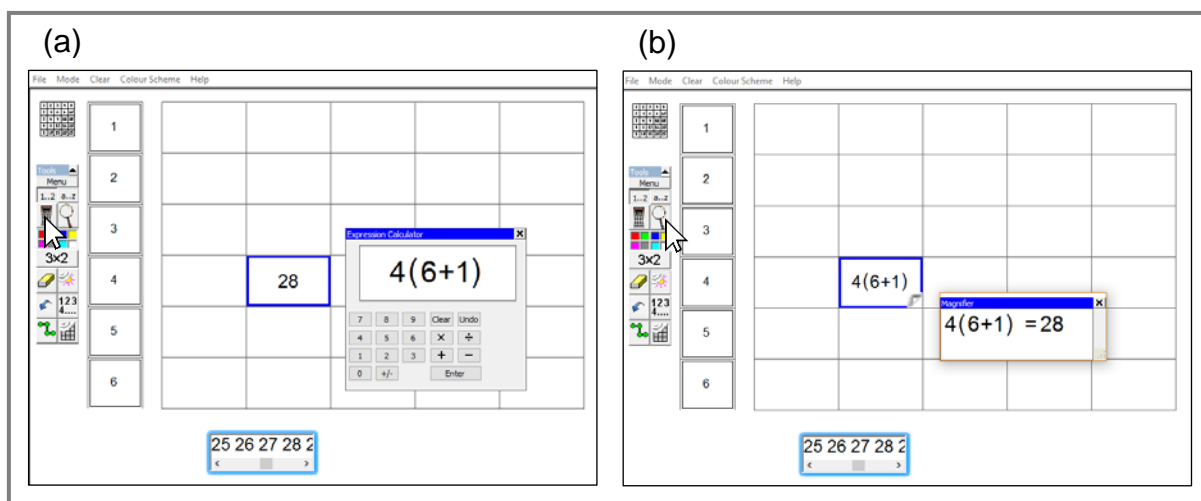
This is another feature which may be used to help students to get acquainted with new notation. It is also a useful in activities where the focus is the order of operations, where it encourages reading the expression in the order of operations and not just left-to-right. Like all the other features of the GA *Run* mode, the expression calculator is included in computer-generated tasks which are outlined below.

4.3.5   Computer-Generated Tasks

From the main menu screen, students can choose one of the 26 structured, interactive computer-generated tasks, including:

- placing numbers in designated cells,
- calculating the value of numerical expressions,

- finding the journey that a number/letter has to make to become a numerical/algebraic expression,

- constructing a numerical/algebraic expression by moving a number/letter around the grid,

- using the expression calculator to make equivalent expressions which can be in the same cell, and

- substituting numbers for letters in algebraic expressions in order to evaluate them.

All the tasks are presented in the context of the GA grid and a window containing instructions appears as the task is started. As an example, Figure 4.3.5.1 shows screenshots of Task 13, *Find the Journey (letters)*, in which students are required to trace the path that a letter has to make in the GA grid in order to be transformed into a given algebraic expression.

Figure 4.3.5.1   Task 13 – Find the Journey (letters)
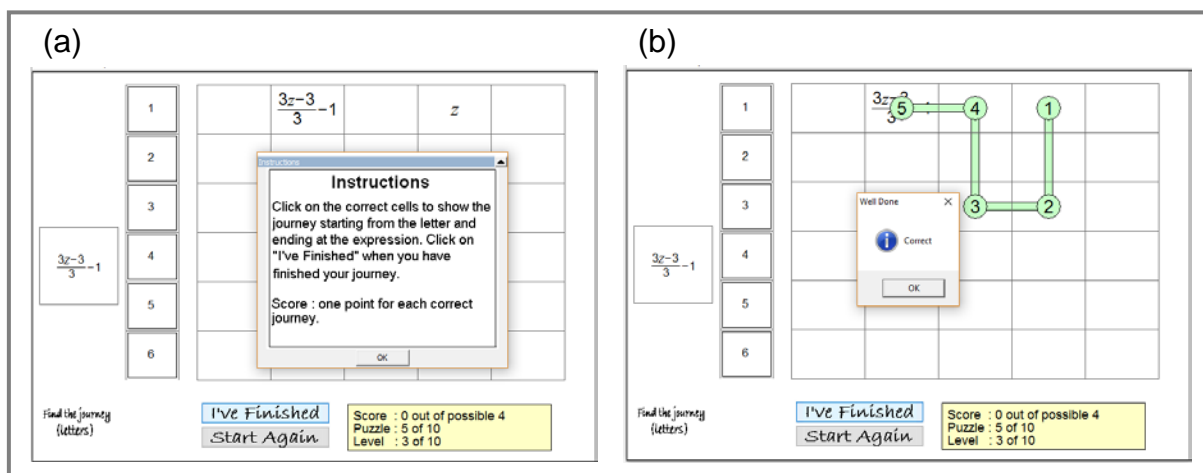
All GA tasks have many levels of difficulty and students may choose the level that challenges them while being attainable. This feature makes them low threshold, high ceiling tasks (McClure, 2011): they are quite easy to begin but allow considerable development and sophistication. An overview of the GA tasks used in this research is included in Appendix 1.

4.4 The Grid Algebra Lessons

The GA lessons took place over the scholastic year spanning from October 2014 until June 2015. There were twenty double lessons (80 min each) which typically consisted of three main parts:

  (i)   a plenary discussion (c. 40 min.) of GA demonstrations on the IWB;

  (ii)   students' pair work (c. 30 min.) on GA activities, and

  (iii) teacher's explanation of written homework (worksheets).

4.4.1 The Thinking behind Plenary Discussions

In the plenary discussions, I sought to help students to develop concepts through demonstrations of GA activities. NCTM (2000) underlines the importance of creating a classroom atmosphere conducive to communication about the mathematics that learners are studying and to facilitate discussions where students reflect and gain insights into their thinking. This was the first of two main purposes of the plenary discussions. The other main purpose was for me to learn about the mathematics of my students (Steffe, 1991) by seeking to develop experiential models of their conceptual processes (Steffe & Ulrich, 2013).

Figure 4.4.1.1   A typical plenary discussion of a GA activity
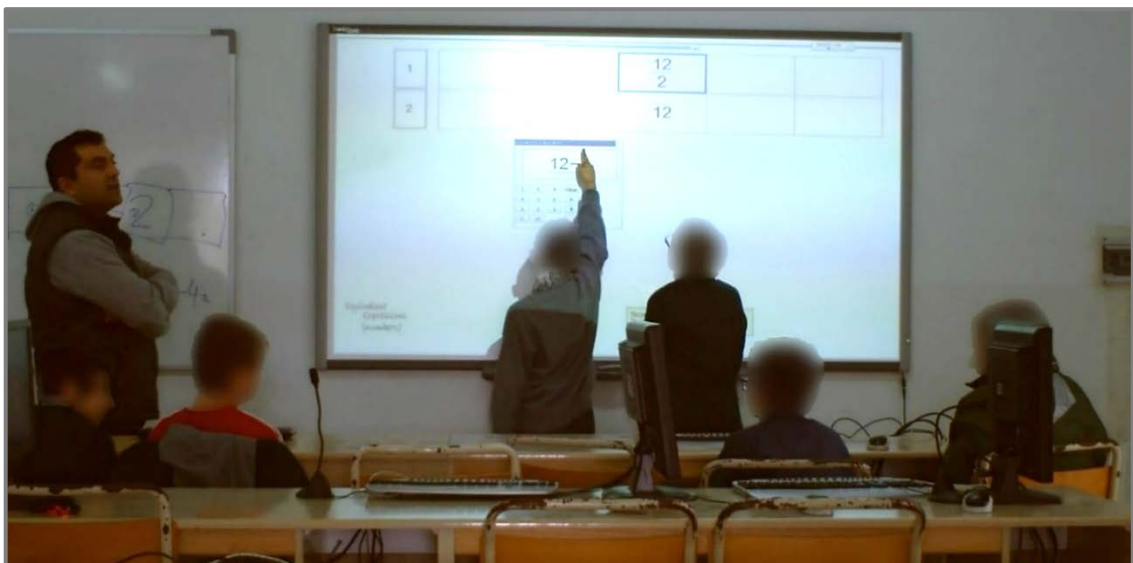
During these discussions, students played an active role and were encouraged to build and reflect on their own and each other's statements. Students were given the opportunity to work on GA activities on the IWB themselves, as shown in Figure 4.4.1.1. The small size of the class made it seem as though we were discussing in one sizeable group where all students had the chance to share their views. The discussions had features of investigative teaching which, according to Collins (1988, p. 1), encourages 'students to actively engage in articulating theories and principles that are critical to deep understanding of a domain'. Jaworski (1994, p. 207) identifies inquiry-based learning with 'students actively doing mathematics together, talking about mathematics, sharing mathematical ideas, and learning from each other'. These were all elements found in the plenary discussions.

However, it was I who asked most of the questions in class discussions. This was mainly due to my efforts to demonstrate GA features and prompt students to speak up. While teacher questioning may be regarded as the backbone of classroom communication (e.g., Dymoke & Harrison, 2008), mainly due to its potential to stimulate learners' thinking (Ellis, 1993; Wood & Anderson, 2001), I was aware that I also needed to stimulate students' questions. Incidentally this is Collins's (1986, p. 5) second aim of an inquiry-based approach: 'to teach students questioning skills so that they can learn new domains or solve novel problems on their own'. This awareness was part of the thinking behind students' pair-work which I will now discuss.

4.4.2 The Thinking behind Students' Pair-Work

Students' pair-work was principally aimed at achieving three benefits identified by Good, Reys, Grouws, and Mulryan (1990):

(i) Enhanced motivation and enthusiasm;
(ii) Positive peer interaction;
(iii) Advanced mathematical thinking.

With reference to benefit (ii) Good et al. (1990) presented evidence that when working in groups, students were more ready to exchange mathematical ideas. This was also true for my participants. When they worked in pairs they were compelled to communicate and to ask questions.

The pairs were divided according to compatibility of characters and working rate, as identified before the onset of the GA lessons. Omar and Jordan seemed to take a longer time than the others to complete classroom tasks so I decided to pair them up. Joseph and Tony had a similar working rate but they often quarrelled when I asked them to do something together. I decided that Dwayne was a much better partner for Tony. That left Dan as the remaining possible partner for Joseph. Although they were both quite loud and talkative, they had very similar and fast working rates and I found it was a good decision to pair them up. I asked each pair to come up with names of animals and these are the names they chose for themselves:

- "**Pandas**" – Tony and Dwayne;
- "**Chimps**" – Omar and Jordan;
- "**Sharks**" – Dan and Joseph.

During GA lessons, students used to pair up right after the class discussion where they started working on a particular GA activity or task. My role from then onwards was changed into that of monitoring the students' progress and being there for them should they ask me questions. Figure 4.4.2 shows a typical pair-work setting.

Figure 4.4.2   Pairs working out GA activities and tasks on the computer
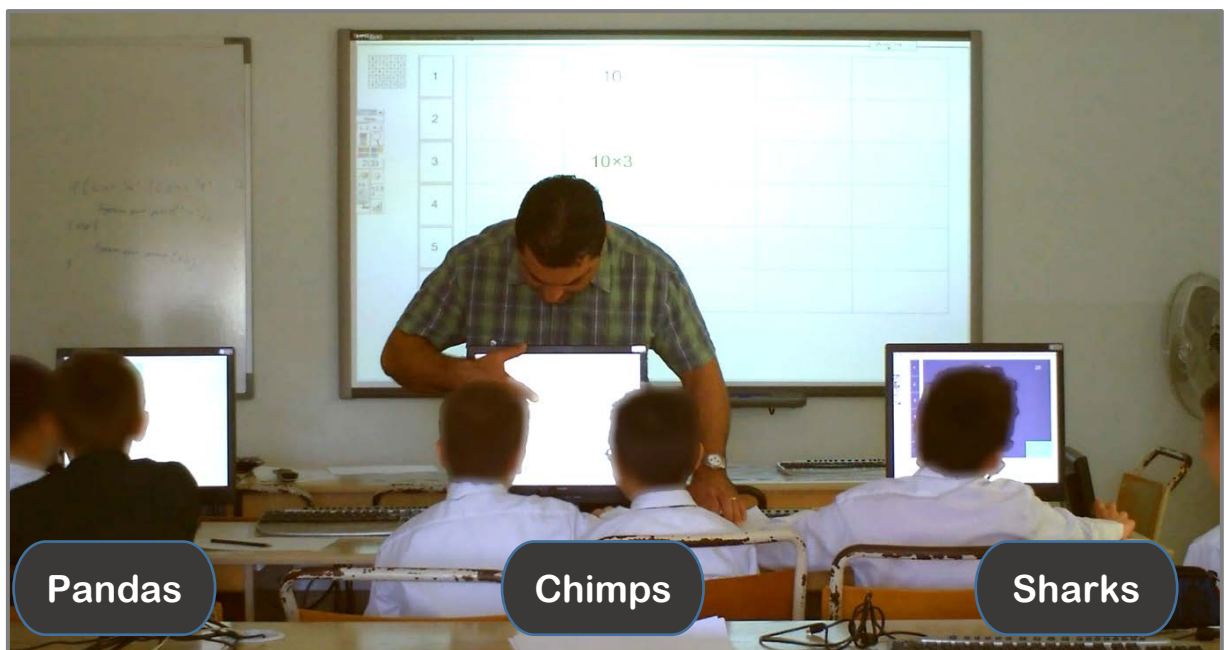
4.4.3   Rationale of the Series of Lessons

The series of GA lessons were planned and ordered according to five principal objectives:

(i) **Coming to know about the GA grid and how it works.** These lessons served as a platform for GA activities and also to revise previously learnt concepts.

(ii) **Coming to know about new notation and the order of operations.** These lessons were aimed to help students to learn new notation, such as $3(5)$ to mean 3×5 and also extend the meaning of familiar notation. Another aim was to help students to learn the order of operations in expressions.

(iii) **Coming to know about the use of a letter**. Students were given the opportunity to learn that a letter may stand for a generalised number (variable) or a particular number (unknown).

(iv) **Coming to know how letters may be used in expressions**. These lessons were aimed to help students to get a proceptual view (Gray & Tall, 1994) of algebraic expressions.

(v) **Coming to know about properties of the operational notation and how letters may be substituted with numbers**. This set of lessons were aimed to help students to reflect on the commutative and inverse properties of addition and multiplication and the non-commutativity of subtraction and division. This was useful for the second objective, that of learning how letters can be substituted with numbers, where one can evaluate, say, $2 + 3r$ by inverting the terms but this cannot be done with expressions like $2 - 3r$.

Table 4.4.3.1 shows how these aims were addressed through the use of GA applications and tasks. Appendix 2 includes an example of a typical GA lesson plan.

Table 4.4.3.1   GA applications and tasks used according to lesson aims

Aim	Lesson	GA Application or Task
1	1.1	RUN (2 rows, 3 rows, 4 rows, 6 rows). *Fill-in the cell*: R_1C_1 being 1
	1.2	RUN (2 rows, 3 rows, 4 rows, 6 rows). *Fill-in the cell*: R_1C_1 not 1
	1.3	RUN (2 rows, 3 rows, 4 rows, 6 rows). *Move the cell and reflect*
	1.4	RUN (6 rows). *Move the cell and reflect (Extension)*
	1.5	TASK 20: *Place the numbers* (two players)
2	2.1	TASK 16: *Make the expression* (Numbers – small grids)
	2.2	TASK 15: *Make the expression* (Numbers – large grids)
	2.3	TASK 10: *Find the journey* (Numbers – small grids)
	2.4	TASK 8: *Find the journey* (Numbers – large grids)
	2.5	TASK 5: *Equivalent expressions* (Numbers)
	2.6	RUN (6 rows). *Emphasis on the equals sign*
3	3.1	RUN (2 rows, 3 rows, 4 rows, 6 rows*). Letter as unknown*: What is the value of the letter?
	3.2	RUN (2 rows, 3 rows, 4 rows, 6 rows). *Letter as variable:* What may be the value of the letter?
4	4.1	TASK 14: *Make the expression* (Letters – small grids)
	4.2	TASK 13: *Make the expression* (Letters – large grids)
	4.3	TASK 9: *Find the journey* (Letters – small grids) TASK 7: *Find the journey* (Letters – large grids)
	4.4	TASK 4: *Equivalent expressions* (Letters)
5	5.1	RUN (use of the Magnifier function). *Properties of the basic operations*
	5.1	TASK 24: *Substitution* (small grids)
	5.2	TASK 22: *Substitution* (large grids)

As Farrugia (2006) asserts, in Maltese mathematics classrooms, English is the language of written texts, while for spoken language, technical words are usually retained in English. This is mainly due to limitations in the Maltese language for words like 'equals' or 'brackets'. However, the spoken language used in mathematics lessons varies according to teacher and learner preferences. In order to make sure that language was not a barrier for students' participation, I used mainly Maltese to communicate, but sometimes I code-switched to English to explain particular problems or instructions written in English.

GA lessons were a rich source of data generation but data was also collected outside the lesson context. In the following section, I describe and rationalise the methods I used to collect data.

4.5 Data-Gathering Methods

In this section, I describe the methods, tools, and frequency of data collection. I start by giving a brief rationale for the choice of data-gathering methods.

4.5.1 Rationale of the Choice of Data-Gathering Methods

The choice of data-gathering methods was determined by the qualitative nature of the research questions. In the following subsections, I describe and rationalise the methods with which I collected this type of data, namely:

(i)   video recording of the GA lessons,
(ii) screen and audio recording of students' computer activities,
(iii) video-recorded interviews,
(iv) students' written work, and
(v)   teacher journal[10].

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[10] Following the pilot study, this was discontinued for reasons discussed further on.

Video Recording of the Lessons

Cohen, Manion, and Morrison (2011) suggest that video recording might provide rich data and catch non-verbal communication. In fact, the use of video recording as a major means of classroom data collection is widely documented (e.g., Wood, Cobb, & Yackel, 1991; Pirie & Kieren, 1994). Since I was interested in capturing the complexity of interpersonal dynamics and communication gestures during the lessons, I decided that video recording was an appropriate means of collecting data in this respect. At the back of the classroom, I set up a digital video camera which captured the whole group of six students including myself during the lessons. The main aim of these videos was to record student-teacher discussions which provided data about my approach to CT and about students' meaning making. Key excerpts from the audio (transcripts) and from the video (screenshots) of the lessons were later used in conjunction for the analysis and discussion of CT.

Computer Screen Activity Capture

Imler and Eichelberger (2011) showed that video screen capture technology is a good way to track human-computer interaction. One of my aims was to gather data about students' working and reasoning as they worked on GA tasks and activities in pairs. In order to generate this kind of data, I used a computer screen activity capture (CSAC) software package called *BB Flashback Express* on each computer workstation. When the students went to their workstations to work out GA tasks, I activated this software which recorded what was happening on the screen (movements and clicks of the mouse pointer) and the audio of students' comments. Figure 4.5.1.1 shows a series of screenshots taken from the CSAC video of the Pandas' pair-work during one of the lessons. A dark circle around the mouse pointer appeared with every left click of the mouse.

Figure 4.5.1.1   Successive screenshots of Pandas' movements on GA activities

Video-Recorded Interviews

Patton (2002, p. 341) claims that 'the purpose of interviewing…is to allow us to enter into the other person's perspective'. Coming from an RC background, I do not believe that we can ever have access to another person's thoughts (Ulrich et al., 2014). At the same time, I believe interviews allow participants to offer verbal and other representations which can give the interviewer the opportunity to create models (Steffe et al., 1983) of their perspectives. With regards to interviews for educational research, Cohen, Manion, and Morrison (2011, p. 411) mention the need 'to evaluate or assess a person in some respect'. In my case, I needed to assess students' individual knowledge constructions about the topics discussed in Section 4.4.

Over the scholastic year, each student was interviewed five times, relating to the five objectives listed in Table 4.4.3.1. The interviews (see Appendices 3.1–3.5) were semi-structured to allow for individualised probing and elaboration. The interviews were about mathematical problems which the students were asked to work out and explain their reasoning. The questions I asked were:

  (i)   Can you tell me what you see?
  (ii)   Do you know how to find the answer?
  (iii)   (If answer to (ii) is yes) Work it out.
  (iv)   Can you explain how you did that?

Sometimes the question regarded a particular symbol or expression, where I asked:

  (i)   Have you ever seen that symbol?
  (ii)   What does it mean to you?

There were other questions which were given on the scripts themselves. I read these out and translated them to Maltese where necessary. In all the questions, I retained the flexibility to rephrase the questions or ask further questions in order to probe on an answer given by the interviewee. The interview problems were presented on the IWB on which the participants wrote and the filled electronic script was saved. Ample space was provided for any written work. At times, students were encouraged to write down examples to help expressing themselves. I opted for video-recorded interviews,

because, as Mishler (1986) claims, this is better than audio recording because the latter neglects the visual and non-verbal aspects of an interview. The video recording was later analysed together with the accompanying filled script which was saved as a PDF document from the IWB itself. Key excerpts from the audio (transcripts) and from the video (screenshots) of interviews with each student were later used in conjunction to discuss the outcomes of the interviews.

Figure 4.5.1.2   Interview 1 - Joseph
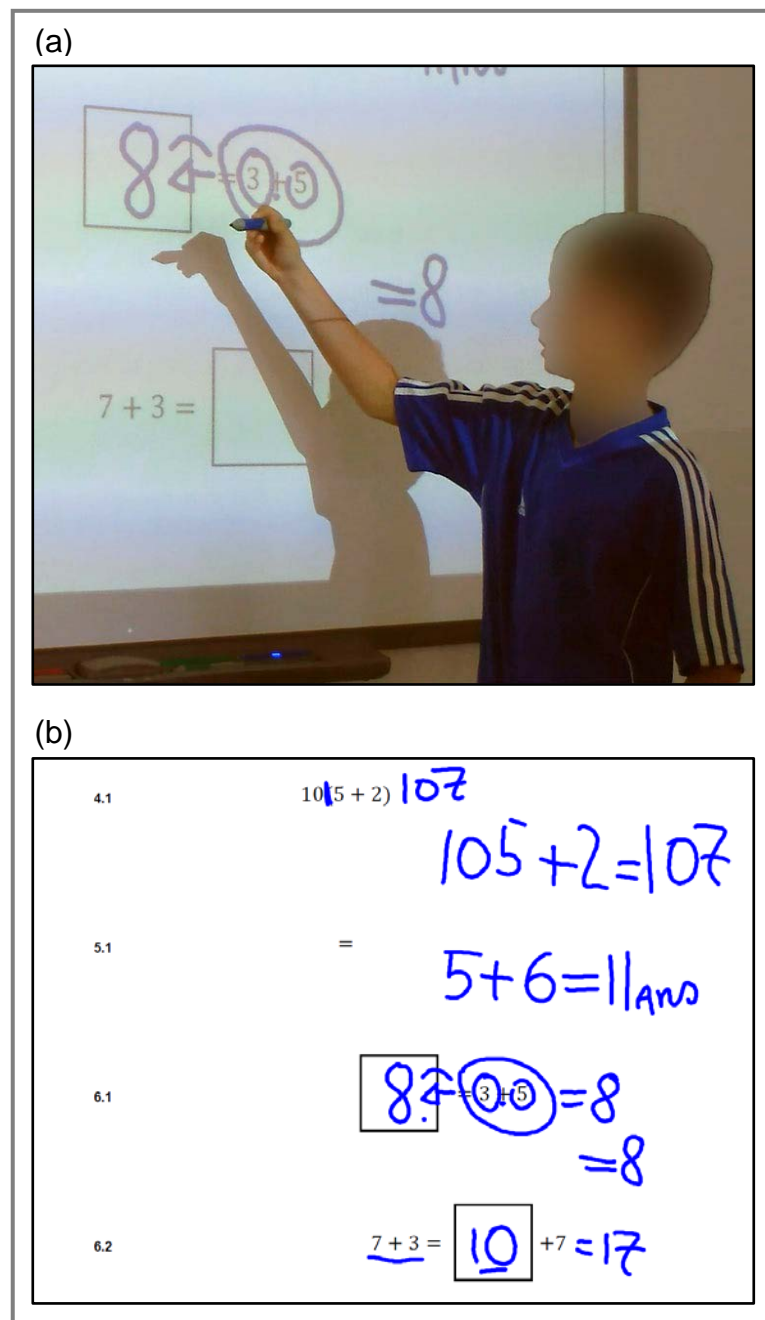
Figure 4.5.1.2a shows Joseph during Interview 1. The screenshot was taken at a crucial moment when he was showing me that he was reading the equality from right to left. Such a gesture would have been lost in an audio interview. Figure 4.5.1.2b shows a section of the accompanying script which provided a record of Joseph's written representations.

Table 4.5.1.1 shows the topics involved in each of the video-recorded interviews (VRI). The table also shows the positioning of the VRIs, according to lesson objectives which were discussed in Section 4.4.

Table 4.5.1.1   Topics of Video-Recorded Interviews

VRI	Positioning	Basic Arithmetic	Order of Operations	New Notation	Numerical Expressions	Algebraic Expressions	Substitution Tasks
1	**Before Objective 1**	●	◑	○			
2	**After Objective 1**		●	◑	○		
3	**After Objective 3**			●	◑	○	
4	**After Objective 4**				●	◑	○
5	**After Objective 5**					●	●

Codes: ○ Topics of questions asked not yet covered in class.   ◑ Topics of questions asked partially covered in class.   ● Topics of questions asked covered in class.

The development the five interviews was such that:
- the problems were progressively harder from one interview to the next, and
- there was a continuum along the interviews, with some overlapping questions (with altered numbers).

This development enabled the investigation of the learning journey of each participant and also revealed some emerging trends in the learning progress of the whole group.

Students' Written Work

D'Ambrosio (2013) argues that RC teacher-researchers need to use learners' writing as one of the sources which inform them about their learners' conceptual understandings. She says that 'the teacher-researcher reads hermeneutically the writing of students' (p. 249) in order to understand how students are constructing knowledge. Accordingly, I used students' written work to help me build models (Steffe et al., 1983) of students' intellectual journeys.

After every lesson, students were assigned written work on worksheets which was related to what they were doing in the lesson. Most worksheets mimicked the GA interface. Towards the end, worksheet questions were more of the traditional type. Samples of such worksheets can be found in Appendices 4.1–4.5. Every worksheet was collected and analysed before the next GA lesson. An important part of this analysis was devoted towards assessing whether concepts developed with GA on the computer were applied to pen-and-paper problems. Figure 4.5.1.3 shows the work of one student, Dwayne, on the written task following Lesson 20.

Figure 4.5.1.3   An excerpt from Dwayne's written work

Journal Note-Taking

As part of the data collection process in the pilot study (discussed in Chapter 5) I started keeping a journal of significant moments in the lessons which were relevant to the research questions. However, I soon found that the journal was not working well for me due to two main reasons:

(i)   The stops I was making throughout the lesson to jot down notes was interrupting my teacher frame of mind and influenced my decisions during the lesson. This could have compromised the data from the lessons.

(ii)   By the time I got the chance to sit down and develop further the points I wrote during the lesson (usually after a whole day's lessons), my thoughts were more of a reflection than a recollection of observations. I found that I could reflect better if I analysed the lesson video recordings.

In the end, I decided that it would be better for me not to use journal note-taking as one of the data-gathering methods for the main study.

4.5.2 Associating Data Collection Methods with Specific Research Questions

Table 4.5.2.1 summarises how and why lesson video recording (LVR), computer screen activity capture (CSAC), video-recorded interviews (VRI), and students' written work (SWW) were used to provide data associated with specific research questions.

Table 4.5.2.1 Associating Data Collection Methods and Research Questions

	Research Question	Data Collection Method
1(i)	How do I engage in CT and what are the distinguishing characteristics of such a teaching approach?	**LVR**: To investigate instances in GA lessons where I was teaching in a way that was sensitive to constructivist notions, particularly during the the toing and froing between my mathematics and that of my students. *Frequency: 20 (all lessons)*
1(ii)	What, if any, are the moments when I fail to engage in CT?	**LVR**: To investigate instances in GA lessons where I was teaching in a way that was not sensitive to constructivist notions. *Frequency: 20 (all lessons)*
2(i)	How do students represent and interpret mathematical notation as they start Grade 7?	**VRI**: To investigate individual students' representations and interpretations of notation involved in expressions they encountered before starting the GA lessons. *Frequency: 1 (Interview 1).*
2(ii)	How does GA help students **to** enrich their representations and extend their interpretations of mathematical notation?	**LVR**: To investigate the teaching and learning process while students are extending their meanings of notation with the help of GA. *Frequency: 20 (all lessons)* **VRI**: To investigate students' developments of new or extended representations and interpretations of notation with the help of GA. *Frequency: 4 times (VRI 2–5)* **CSAC**: To investigate students' interactions and progress through GA tasks as they work in pairs on their computers. *Frequency: 20 (all lessons)*
2(iii)	How do students transfer representations and interpretations of notation they develop when working with GA to pen-and-paper problems?	**SWW**: To investigate the way students transfer representations and interpretations of notation learnt during the GA lessons to pen-and-paper problems. *Frequency: 20 (after each lesson)*

4.6 Data Analysis

In this section, I describe how I set out to analyse the data by following Yin's (2013) advice that case study researchers need to be careful to engage in systematic procedures and not allow equivocal evidence to influence the direction of the findings and conclusions.

4.6.1 Discipline of Noticing

When revisiting classroom episodes, students' written work, and interview encounters I followed Mason's (2002) guidelines intended particularly for teachers who are researching their own practice. Mason (2002) advises us to engage in what he calls 'the discipline of noticing' (p. 61), a number of practices intended to help us not to miss opportunities to observe important elements of the data while avoiding to confuse speculations with observations:

(i) **Keeping accounts.** Mason (2002) distinguishes between giving account-*of* and accounting-*for*. The former means providing descriptions of what occurred. The latter is giving one's own interpretation and hypotheses of why certain things occurred. Mason stresses the importance that during noticing one should not 'dissipate energy in making judgements' (p. 14), and that interpretations of the collected data should occur only after a number of observations have been made and commonalities between the observed episodes could be detected. When reviewing the video recordings I kept an account *of* the events taking place during the lesson or interview. In subsequent reviews, I wrote notes and developed codes in order to keep an account *for* these events using techniques as shown in Section 4.6.2.

(ii) **Developing sensitivities**. Here Mason (2002) refers to sensitising oneself to observe factors which are pertinent to the research questions. I helped myself to do this by subdividing the research questions into specific subsidiary questions as shown in Table 4.5.2.1.

(iii) **Recognising choices**. Mason (2002) draws to our attention that teachers are all the time making choices according to situations occurring in the classroom. In my case, the discipline of noticing was particularly useful for the review of data in lesson video recordings, where I was on the alert for those significant moments in the lessons where, as a teacher, I had a choice of how to act or

proceed. Later I associated those choices with categories in the data which then developed into themes (see Section 4.6.2).

(iv) **Preparing and noticing**. Before even starting to review any data, Mason (2002) advises us to have a list of what to look for and to prepare ourselves to notice the items on that list at the instant they occur. This advice was very helpful during the review of the video clips because it focused my attention on capturing the relevant data from what would otherwise be perceived a rich but overwhelming collection of actions and representations.

(v) **Labelling**. Mason (2002) here is referring to the practice of data coding which I will describe in detail in Section 4.6.2.

(vi) **Validating with others**. Besides triangulating data by making use of diverse data-collection tools, I validated my observations through discussions with my supervisors and with the students themselves. In addition, I published preliminary findings in a peer-reviewed journal (Borg, Hewitt, & Jones, 2016a, 2016b) and also gave a presentation at Loughborough University[11] where I discussed the overarching conceptual framework (see Section 6.2.2) with faculty members and PhD students at the Mathematics Education Centre.

Engaging in these factors of disciplined noticing helped me to seek threads and similarities between the accounts (Mason, 2002) through a series of data analysis techniques which I discuss next.

4.6.2   Data Analysis Techniques

To analyse the data I followed a sequence of coding, categorising, classifying, and labelling patterns which is what Patton (2002) calls content analysis. According to Patton (p. 453), 'content analysis is used to refer to any qualitative data reduction and sense-making effort that takes a volume of qualitative material and attempts to identify core consistencies and meanings'. In my case, this consisted of systematically sifting through all the collected data so that I could choose the parts of it that best captured the experiential reality (Glasersfeld, 1991a) of the persons involved in the case and

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[11] This presentation was given at a meeting of the Curriculum, Pedagogy and Identity Group (CPIG) held by the Mathematics Education Centre.

which could answer the research questions in a detailed manner. Parlett and Hamilton (1976) use the term progressive focusing for such a process where the large amounts of raw data, which typically amasses from qualitative methods, is funnelled from the wide to the narrow.

This process was possible through a system of coding I developed and refined over the time I spent analysing the data. Saldana (2009, p. 3), defines a code as 'a word or short phrase that symbolically assigns a summative, salient, essence-capturing' descriptor for a portion of data. When coding the data, I created links between observable occurrences and the theories that gradually developed in relation to the literature and the research questions. The first stage of the coding process involved creating primary codes and classifying these into subcategories and then into categories. I did this by putting data 'into groups, subsets or categories on the basis of some clear criterion (e.g. acts, behaviour, meanings, nature of participation, relationships, settings, activities)' (Cohen et al., 2011, p. 558). This primary coding was followed by revisiting the literature and the primary data, and then by reviewing the codes and categories, allowing the latter to merge into theories that answered or were used to answer the research questions. This recoding process was what Miles and Huberman (1984) call s*econdary coding*. It was essentially a classificatory process (LeCompte & Preissle, 1993) that involved further reduction and selection of data that converged into a typological analysis.

A significant part of the work involved in answering the research question 1(i) was the creation of a conceptual framework which I will discuss in Chapter 6. For now, I will refer to this as the "CT framework". This framework was overarching, in that all research questions and their answers made sense within it. Research question 1(ii) could be answered by using the CT framework analytically. Research questions 2(i)–(iii), which were not directly related to my teaching approach, were asked to investigate what the students made of my CT, whether and how CT made them develop or extend mathematical concepts, and whether their learning was reflected in their written work. Hence, answering questions 2(i)–(iii) was a focus on one aspect of the CT theory.

When developing the CT framework, I made the journey from the concrete (data) to the abstract (theory), from the particular to the general. Figure 4.6.2.1 shows this journey as a streamlined codes-to-theory model derived from Saldana (2009).

The finalised codes regarding CT took the form of a three-point hierarchical acronym identifying theory, category, and subcategory, with the latter being part of the primary evidence I used to develop that theory. These codes, which are discussed further in Chapter 6, were logged into coding sheets which included detailed lesson event notes. An example of such a coding sheet is provided in Appendix 5.

Figure 4.6.2.1   Streamlined codes-to-theory model of analysing the data
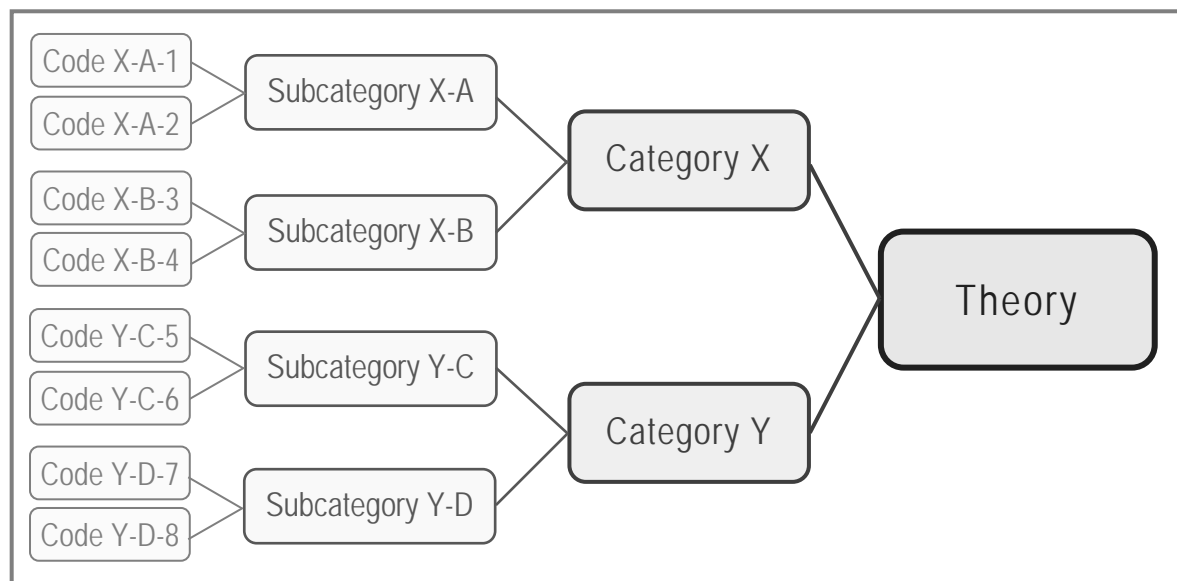
(Adapted from Saldana, 2009, p. 12)

Besides defining the dynamics in my teaching approach, the CT framework proved itself to be a viable analytical tool which helped me to investigate my teaching approach. Hence, answering questions 1(i)–(ii) was accomplished through the development and the analytical use of the CT framework. This analysis is discussed in Chapter 6.

When investigating data related to research questions 2(i)–(iii), I amalgamated two existing constructs to form a second framework (Section 7.1). This framework helped me to investigate students' construction of concepts about notation with the use of GA and facilitated by CT. This time, the codes I used to classify the data emerged from the framework itself, rather than formed it, as was the case of the CT framework. This coding enabled the data analysis which I develop in Chapter 7. This includes a discussion of how students' representations and interpretations fitted within the overarching CT framework.

In the following section, I discuss issues about being a teaching researcher which may have acted as enablers or barriers to the overall research design and development.

4.7 Issues about Being a Teaching Researcher

The term *practitioner research* is used in the literature to denote any 'research conducted by a practitioner/professional in any field ... into their own practice' (Wellington, 2000, p. 20). Stenhouse (1975) used the term *teacher as researcher* to specify practitioner research undertaken by teachers. Teacher researchers may embark on research projects

(i)   outside their school,
(ii)   in the same school but unrelated to their teaching, or
(iii)   in the same school and concerning their teaching.

To distinguish my situation (iii) from the others, I use the term "teach*ing* researcher" to convey the notion of data being collected *during and concerning* the teaching process. As I will argue later, this created issues which may not be present in situations (i) or (ii).

4.7.1 The Outsider and the Insider Doctrines

Mertin (1972) presents two extreme views with regard to research: The *Outsider* and the *Insider* doctrines. The Outsider doctrine holds that the only true research is that in which researchers are not in any way involved in the case being studied because living or working in a particular realm distorts researchers' perception of that realm. On the other hand, the Insider doctrine holds that only those who live or work in a particular field can truly understand what goes on inside that field. Being a teaching researcher myself, I cannot but side with this last standpoint. Nevertheless, I needed to be aware of certain pitfalls of insider research which may correspond to some epistemological claims of the Outsider doctrine (Mertin, 1972). Hence, I needed to acknowledge both advantages and problems of being a teaching researcher.

4.7.2   Advantages and Problems of a Teaching Researcher

Wellington (2000, p. 20) draws up a number of potential advantages and problems of practitioner research. These are presented in Table 4.7.2.1.

Table 4.7.2.1   Potential advantages and possible problems of practitioner research

	Potential advantages	Possible problems
(i)	Prior knowledge and experience of the setting/context (insider knowledge)	Preconceptions, prejudices
(ii)	Improved insight into the situation and people involved	Not as 'open-minded' as an 'outsider' researcher
(iii)	Easier access	Lack of time (if working inside the organisation) and distractions/constraints due to being 'known'
(iv)	Better personal relationships, e.g. with teachers, students	'Prophet in own country' difficulty when reporting or feeding back
(v)	Practitioner insight may help with the design, ethics and reporting of the research	Researcher's status in the organization, e.g. a school
(vi)	Familiarity	Familiarity

With the exception of problems (iv) and (v), I experienced all advantages and problems included in Wellington's (2000) table.

(i)   **Insider knowledge vs. Prejudice**. I cannot dispute Fraser's (1997) claim that teaching researchers may be biased towards establishing the effectiveness of a teaching programme for which they are responsible: as a teacher I strived to make my lessons effective. After the lessons, when I analysed the lesson videos as a researcher, I could observe moments of frustrations when the lesson was not going as I desired. However, those same frustrations became specific events that were relevant to the case (Hitchcock & Hughes, 1995) and what was frustrating for me as a teacher became interesting and pertinent data for me as a researcher. Nevertheless, I needed to acknowledge my bias in

favour of my choice of approach and tools (especially GA) as a teacher because these were approaches and tools I believed in. Nevertheless, as Chapter 6 will reveal, I did not hold back on self-criticism especially in those instances where I was found myself not implementing my pedagogical beliefs.

(ii) **Insight vs. Close-mindedness**. Elliott (1991) warns practitioner researchers about the problem of not having an outsider's vantage point into the research context and suggested 'dismantling the value structure of privacy, territory and hierarchy, and substituting the values of openness, shared critical responsibility and rational autonomy' (p. 67). I strived to be open with my student participants without influencing their behaviour and responses. However, I needed to be aware that school children are bound to regard their teacher as an authority figure and may not view themselves as research partners in school-based research.

(iii) **Access vs. Time-constraints**. Having easy access to the students and the classroom had many advantages. It enabled negotiating the time of the interviews, speaking individually to a participant, and preparing the room for the lessons with considerable ease to mention but a few. Being a teaching researcher brought about other than benefits, however. Sometimes, tensions were created due to time constraints, such as when I had to prepare the classroom for the data collection right after another lesson. Distractions from colleagues were minimal.

(iv) **Personal relationships vs. Teaching duties**. I found that establishing a formal but friendly rapport with the students helped them to be at ease during the lessons, and to behave in the way they usually do in my lessons. In fact, they completely ignored the video camera during the GA lessons. The same could be said for the interviews. However, I needed to keep in mind that besides a method of data generation, the GA lessons still formed a considerable portion of these students' yearly lessons and I could never dismiss the syllabus content I was entrusted to teach. While creating some constraints, this was an integral aspect of the CT theory I developed, as I discuss in Chapter 6.

(v) **Practitioner vs. Researcher status**. My status in the school was predominantly that of a teacher. At school, I only assumed a researcher status during the interviews. During the lessons, I strived not to assume a researcher status because I needed my lessons to be as "normal" as my other lessons. Only in this way I could ensure that my research analysis was not portraying a false picture of what would have happened if the lessons were not part of the data. Nevertheless, the analysis was necessarily an insider's report.

(vi) **Familiarity**. This was indeed a double-edged sword. On one side it helped students to be themselves in the lesson. It also helped to create a stress-free atmosphere during the interviews. However, although I was very satisfied with students' behaviour in the lessons, I suspect that at times it was this same familiarity which gave them an occasional misbehaving attitude.

4.7.3   Switching between the Researcher and Teacher Hats

Wellington's (2000) list excludes a dilemma I experienced. During the GA lessons and also during the interviews it was quite demanding for me to maintain a teacher-researcher simultaneity. In the lesson, the researcher in me was interested in developing a classroom discussion even if it was out of the lesson topic while the teacher in me was constantly aware that not much of the planned material was being covered for so much precious teaching time (GA lessons formed approximately 40% of the lesson time for a whole scholastic year). So sometimes I may have intervened when I felt that the lesson was taking on a different path than what was originally intended. Nolen and Putten (2007, p. 404) argue that 'the teacher cannot abandon the role of practitioner but must always exercise professional judgement and skill in the best interest of the student.'

My professional obligation as a teacher made me aware of the difficulties arising from what Nolen and Putten (2007, p. 404) call 'the practitioner-researcher duality,' even during the interviews. Sometimes, I felt awkward wearing my researcher hat during interviews when a student asked me to help him with a concept and I had to resist the teaching urge to assist him and keep on asking the questions as neutrally as possible.

On the other hand, during the lessons, I found it occasionally hard to concentrate on my teaching when I became aware that precious data collection time was being lost due to some students' misbehaviour even though, as a teacher, I usually expect such moments. Aware of Elliott's (1991, p. 66) warning that 'insider research tends to be viewed as a teaching versus research dilemma which gets resolved in favour of the former', I settled this role conflict issue by taking on a predominant teacher role during the lessons and a researcher role during the interviews, planning of the lessons, and analysis of data. As explained in Section 4.5.1, this was the main reason why I stopped writing a journal as a method of collecting data. Jotting down notes during the lesson itself was hampering my ability to focus on my teacher role.

In this section, I have given an account of the advantages and disadvantages of being a teaching researcher which necessarily entails elements of researcher bias which needed to be acknowledged for the sake of the robustness of the study. In the next section, I continue to discuss the issue of research rigour by showing how I strived for reliability and several types of validity (including generalisability).

4.8 Reliability, Validity, and Generalisability

Throughout the research design, the methodology, the timing, the choice of tools and methods, and the data collection and analysis I was careful to minimise compromising the reliability or validity of my research. In this section, I discuss the measures I took to reduce threats to reliability and validity. I also discuss the matter of generalisability as a means of external validity.

4.8.1 Measures to Maximise Reliability

There are several who contest the suitability of the term "reliability" for qualitative research (e.g. Lincoln & Guba, 1985; Winter, 2000; Stenbacka, 2001; Golafshani, 2003). In particular, Lincoln and Guba (1985) prefer terms such as "credibility", "consistency", and "trustworthiness" to replace "reliability" when it comes to qualitative research. One reason for this is that quantitative norms of reliability such as

consistency (stability), accuracy, predictability, equivalence of outcomes, replicability, and concurrence are absent when the interpretation of data is subjective, as it usually is in qualitative research.

While arguing that the basic tenets of reliability for quantitative research may be inapplicable for qualitative research, LeCompte and Preissle (1993) propose that qualitative research could still strive for replication through the generation, refinement, comparison, and validation of constructs by maintaining:

(i)   the status position of the researcher,
(ii)   the choice of participants,
(iii)   the social situation/condition,
(iv)   the analytic constructs used, and
(v)   the methods of data collection and analysis.

Throughout the whole scholastic year I spent generating and collecting data, I strived to maintain these five factors. My status of a mathematics teaching researcher with sympathies for RC epistemology and ontology were retained throughout the whole research process. The participants were always the same six students I started with. I strived to make the data collection process as stress-free as possible for the students to avoid any drop-outs, which I did not have.

The social situation and condition of the student participants and me was always that of a typical Grade 7C (Maths) class and their teacher at St George's College. As an extra measure of constancy I made sure to have all the GA lessons in the same classroom and retain the same pairs of students for computer work throughout the whole year. I was aware, however, of my limitations to maximise regularity of my students' mental conditions or even my own, especially when variables at home changed drastically. Just to mention two examples, there was a period when Dan was not himself due to serious trouble with his mother's health and there was a similar period for me when two very close family relatives passed away. So ensuring regularity in such a longitudinal research was virtually impossible since several background factors varied over time.

While the CT framework was developed as part of the analysis (Chapter 6), the analytical framework I used to help me investigate students' representations and interpretations of notation was always the same throughout the data analysis process (Chapter 7). Moreover, with the exception of journal note-taking which I abandoned after my pilot study, I kept the same data-gathering methods and tools and an organised coding system as can be seen in Appendix 5.

4.8.2   Measures to Maximise Validity

According to the Association for Qualitative Research (2015), validity is the capacity of research to actually measure what it sets out to, or to actually reflect the reality it claims to represent. Holding the RC ontological belief about the nature of reality that although reality may exist independently, what we know of it is individually constructed, I concur with Anderson and Jones's (2000, p. 44) claim that 'practitioners' accounts of their reality are themselves constructions of reality and not reality itself'. Nevertheless, I strived to maintain characteristics of a valid *qualitative* research such as those singled out by Cohen, Manion, and Morrison (2011): honesty, richness, authenticity, depth, subjectivity, catching uniqueness, and containing strength of feeling and idiographic statements. With the exception of the latter, all these characteristics depend on a choice of quality rather than method and I kept these in mind throughout the planning, implementation, and analysis stages of the research. Furthermore, the statements I will make in my analysis and interpretation will only be time-bound and context-bound claims. This does not exclude, however, that readers may not extrapolate lessons from this research to their own time- and context-defined realities. This will be discussed in the next section.

Like reliability, the concept of validity is drawn from the scientific tradition and needs specific interpretation in the context of qualitative research (AQR, 2015). I share Maxwell's (1992) notion that the term "understanding" would be more suited than "validity" for qualitative research. As a qualitative researcher I was aware that I was part of the same reality I was trying to understand and that my interpretations of the data were necessarily subjective. I was also aware that my participants' perspectives

of this reality were as valid as mine, and I needed to understand and report those perceptions, at least what I understood from their representations. Maxwell (1992) makes an argument for five factors of validating qualitative research through this notion of understanding. I am listing them here together with the measures I took to address them in my research:

(i) **Descriptive validity**. This is the factual accuracy of the account, which is not fabricated, selective, or distorted. Claiming subjectivity and refuting the *knowledge* (not the existence) of an external objective reality does not mean that RC researchers have a free pass to present speculation as factual occurrences or to present a fictitious narrative as a personal experience or observation. So even though I have approached the data analysis from a RC perspective, I still strived to give an accurate and complete account of my experiential reality (Glasersfeld, 1991a). One way of doing this was the employment of several methods of data generation and collection which served as a means of triangulation (see Table 4.5.2.1). The other way was to ensure that any relevant data which might enrich the answers to the research questions were not deliberately omitted during the analysis process. I also provided raw data for my supervisors to scrutinise and continuously discussed with them the issue of relevance of data.

(ii) **Interpretive validity**. This is the ability of research to capture meanings, interpretations, terms, and intentions that situations and events have for the participants themselves in *their* terms. In my research, particularly during the interviews, I made sure to give voice to the participants, allowing them all the time they required to express themselves and, where necessary, I probed to help them provide a more complete representation of what was going on inside their mind. This was one of the reasons I chose video over audio recordings, as discussed earlier. Here too, however, I need to claim subjectivity in my interpretations of what the students said and thought, even when I quote directly from transcripts. As Kvale (1996) suggests, the prefix "trans" indicates a change of state and thus transcription is, in itself, a selective transformation of data. It is therefore unrealistic to pretend that the data on transcripts are anything but *already interpreted* data (Cohen, Manion, & Morrison, 2011). This awareness was all the more relevant in my case because what I present here

as excerpts in English are my translations of Maltese exchanges. Once again, visual cues made possible through video recording helped me to make more accurate translations than if I was using only audio.

(iii) **Theoretical validity**. This is the extent to which theoretical constructions are able to explain phenomena. The theoretical framework I developed to describe and analyse my CT was inspired from seminal works discussed in Chapter 2 but was principally derived from my own work in the classroom. I can say that I found this framework viable in explaining the phenomena occurring in the classroom with regards to CT. Not only did this explanation make sense to me but I also seem to have persuaded others (my supervisors, my colleagues, and reviewers of the paper Borg et al., 2016a) that this framework is able to explain CT from a RC perspective.

(iv) **Evaluative validity**. This is the application of evaluative analysis, judgmental of that which is being researched, and not just a descriptive, explanatory or interpretive analysis of phenomena. As I show in Chapters 6 and 7, in my data analysis I assumed an evaluative and critical stance, especially in the extent to which I managed to engage in CT. My literature review, which in itself included elements of evaluation and argumentation, placed me in a position where I could analytically compare and contrast my research findings to those of other researchers in the field of mathematics education. This may be verified in my answers to the research questions (Chapter 8).

(v) **Generalisability.** This exists if the theory generated may be useful in understanding other similar situations. I discuss generalisability in more detail in the section that follows.

4.8.3   A Cautious Claim of Generalisability

One important criticism of case studies is that their focus on the singular renders them inappropriate to make claims for generalisability. Two long-standing critics of case study research, Atkinson and Delamont (1993, p. 38) base one of their criticisms on the fact that 'the proponents of case-study research often distinguish their enterprise from other research styles and approaches by stressing the unique, the particular,

(and) the "instance" ', a profoundly mistaken approach according to them. They argue that this might lead to an incapability of developing case studies into more general frameworks and reduce them into 'one-off affairs, with no sense of cumulative knowledge or developing theoretical insight' (p. 39).

I believe the most powerful answer to such a criticism was given almost two decades earlier:

> Case study is the examination of an instance in action. The choice of the word "instance" is significant in this definition, because it implies a goal of generalisation… Case study is the way of the artist, who achieves greatness when, through the portrayal of a single instance locked in time and circumstance, he [sic.] communicates enduring truths about the human condition.
>
> (MacDonald & Walker, 1975, pp. 2-3)

MacDonald and Walker claim that an in-depth study and portrayal of a singular case may yield insights of universal significance. As Hoepfl (1997) argues, this type of generalisability is not the causal determination, prediction, or generalisation of findings that are usually associated with quantitative research but rather the understanding, illumination, and extrapolation to similar circumstances.

There are two possible types of generalisation possible for case studies. I will describe them here with reference to how I sought to achieve them in my research.

(i)   The first type is what Yin (2013) terms *analytic generalisation*. This 'consists of a carefully posed theoretical statement, theory, or theoretical proposition. The generalisation can take the form of a lesson learnt, working hypothesis, or other principle that is believed to be applicable to other situations (not just other "like cases"). (p. 68). Yin (2013) argues that this type of generalisation is suitable for generating theory from a case study. As I discuss elsewhere (Borg, Hewitt, & Jones, 2016a), the theoretical framework regarding CT that I developed from the analysis of the research data may be found viable by other constructivist teachers and for other subjects. Thus, a claim for analytic generalisation (Yin, 2013) may be warranted.

(ii)   The second type is *naturalistic generalisation*, a term coined by Stake and Trumbull (1982) meaning the 'conclusions arrived at through personal engagement in life's affairs' (Stake, 1995, p. 86). It is the way humans create mental constructs from their own or other people's experiences and apply them to other situations as they expand their knowledge. Naturalistic generalisations in research can occur if readers identify with a vicarious experience (the research account), make it their own, and learn from it. When reporting the outcome of my case study, I have attempted to create such an account with which readers may empathise and create viable mental constructs to make sense of their own realities. My claim for naturalistic generalisability will be confirmed if readers of this work learn lessons which they can extrapolate to help them understand their own experiential worlds.

In the above sections I have discussed the way I collected, organised, and analysed data. Due to the nature of the research and the vulnerability of participants (age and power difference from researcher) I needed to take careful ethical considerations during the whole process of data handling. This is discussed next.

4.9   Ethical Considerations

This section includes ethical considerations I took in order to safeguard the welfare of the participants while making sure that the data collection proceeded as smoothly as possible. I was mainly guided by BERA's (2011) *Ethical Guidelines for Educational Research* and these considerations were vetted and approved by the Loughborough University Ethics Approvals (Human Participants) Sub-Committee.

4.9.1   Recruitment of Participants

The recruitment of participants was largely dictated by my situation as a full-time teacher and my interest in how CT may facilitate learning through the use of computer software. Given the amount of contact hours required for the collection of data for such a research project, the most viable option for me was to conduct the research in the

school in which I was teaching at the time of the research, i.e. St. George's College. In addition, I intended to examine the learning of informal- and formal-algebraic activities of Grade 7 students. Since the only Grade 7 I taught was composed of six prospective research participants, there was no question about the choice of participants. The small number of this group made it possible for me to include them all in the study of the case and hence any further discrimination in the choice of participants was eliminated.

The only question remaining with regards to recruitment was whether these prospective participants would accept my invitation to take part in this research and whether the persons responsible for their welfare would be willing to let them participate on the basis of voluntary informed consent.

4.9.2   Voluntary Informed Consent

According to BERA (2011, p. 5), voluntary informed consent is 'the condition in which participants understand and agree to their participation without any duress, prior to the research getting underway'. Before seeking such consent, I needed to keep in mind these factors:

(i)   The participants were minors and were my students.

(ii)   The participant-researcher relationship may have been in conflict with the student-teacher relationship.

(iii)   Most of the data was going to be collected during lessons in which I was the teacher (hence the authority figure).

(iv)   The whole research was conducted in a school setting in which the participants were new and in which I was not.

(v)   The overall wellbeing of all the students in the school was the responsibility of the head of school as well as that of their teachers.

Thus, I contacted three stakeholders from which I sought voluntary informed consent:

(i)   the head of school,

(ii)   the prospective participants, and

(iii)   the participants' guardians.

The first gatekeeper of these three parties was the head of school and I gave a consent letter to him first (Appendix 6.1). Once I obtained the head's consent, I gave a letter to each prospective research participant (Appendix 6.2), which I read out and explained to them face to face. The students were encouraged to ask questions to clear out any queries they might have had about the research. Thirdly, I sent a consent letter to the students' guardians (Appendix 6.3), including a form in which I required joint signatures from *both* guardians and students. All consent letters included:

  (i)   a short description of the aims of the research,

  (ii)   a description of the length of time when data will be collected,

  (iii)   the methods and tools of data collection,

  (iv)   the frequency of each type of data collection,

  (v)   possible benefits and risks,

  (vi)   a rationale for my choice of participants,

  (vii)   measures to ensure confidentiality and privacy,

  (viii)   assurance of voluntary participation throughout the whole research, and

  (ix)   the possibility of withdrawal without any consequence.

Following the letters, some guardians spoke to me on the telephone to inquire about some aspects of the research. Within a week, I had obtained voluntary informed consent from all parties involved.

4.9.3 Confidentiality of Data

Since the research was a case study where particular data needed to be associated with particular participants, anonymity was not applicable. However, I safeguarded the confidentiality of data by taking the following measures:

  (i)   Pseudonyms were used to refer to individual participants and the school;

  (ii)   When showing video screenshots, students' faces and the school uniform badge were blurred;

(iii)   Any information, such as the name of the school or its locality, which may reveal the identity of the participants was not included in any transcript.

In addition, throughout the whole research project, I kept all recorded data in a secure location and only the research supervisors at Loughborough University and I could have access to the data.

4.10   Time Frames

Table 4.10.1 shows the time frames for significant pieces of work in the research. Since I was a part-time researcher, this PhD project took 5 years to accomplish.

Table 4.10.1   Time frames for pieces of work in relation to the research

	JUN 2012	SEP 2012	JUN 2013	SEP 2013	JUN 2014	SEP 2014	JUN 2015	SEP 2015	JUN 2016	SEP 2016	JUN 2017	NOV 2017
Preparation of Proposal	■	■										
Refining Research Questions		■	■									
Journal of Pre-Literature		■	■									
Literature Review	■	■	■	■	■	■	■	■	■	■	■	■
Pre-pilot (GA trials)		■										
Ethical Review			■	■								
Pilot			■	■								
Main Study Data Collection						■	■					
Data Analysis						■	■	■	■	■		
Thesis write-up				■	■	■	■	■	■	■	■	
Submission of thesis												■

4.11   Summary

In this chapter, I gave a rationale for my research design and a description of the context and participants of the research. Included here is also an explanation of my use of the data-gathering tools, how the lessons were given, and the techniques I used for the data analysis. I included issues about being a teaching researcher and how I addressed them. I also discussed how I strived for reliability and validity in such a qualitative research context. Finally, I included the ethical considerations necessary for such a study that involved young students. In the next chapter, I briefly outline the pilot study, highlighting the lessons I learnt which helped me to make or change decisions in the data gathering and analysis of the main study.

Chapter 5

Pilot Study

5.0   Overview

In this chapter, I will give a brief review of the pilot study I undertook during the scholastic year 2013-2014, immediately preceding the scholastic year of the main study. The group of participants were the Grade 7C (lowest-performing set) mathematics students at St. George's College, whom I taught during that year. This was itself preceded by a pre-pilot study with my 2012-2013 Grade 7C students which served mainly to test how GA could be used to help students to engage in informal- and formal-algebraic activities.

The rationale of the pilot was to refine and, possibly, redefine the research questions and to get information about technical matters (Cohen et al., 2011). This chapter is mainly a discussion of the lessons I learnt from the pilot and how these affected the methodology and method of the main research study. Table 5.0.1 includes the section titles of this chapter.

Table 5.0.1   Chapter 5 section titles

	Title	Page
5.1	Description of the Pilot	164
5.2	Lessons about the Research Questions	167
5.3	Lessons about the Participants and their Learning	170
5.4	Lessons about Myself as a Teaching Researcher	176
5.5	Lessons about the Research Method and Tools	178
5.6	Possible Hypotheses	179

5.1   Description of the Pilot

The 2013-2014 Grade 7C (Maths) group consisted of 11 students all of whom turned 11 years of age by the end of 2013. The whole group took part in the pilot study but I only followed closely two of the students. I refer to these as the "case study students" with pseudonyms Alan and Manuel. Overall, the data was collected via:

(i)   lesson video recording (LVR),

(ii)   video-recorded interviews (VRI),

(iii)   computer screen activity capture (CSAC),

(iv)   students' written work (SWW), and

(v)   journal note-taking (JNT) during the GA lessons.

VRI and CSAC were only used with the two case study students. Table 5.1.1 gives an overview of the pilot study, showing when and how data was collected.

Table 5.1.1   Overview of the data collection process in the pilot

Week	Data Generation	Data Collection Methods				
		LVR	VRI	CSAC	SWW	JNT
1	Test 1: Arithmetic Skills (Pre-GA)	-	-	-	✓	-
2	Interview 1: Meanings for Notation (Pre-GA)	-	✓	-	-	-
3	GA Lesson 1: Getting used to the GA Grid	✓	-	-	-	-
4	GA Lesson 2: Moving the cells – Operations	✓	-	-	-	✓
5	Interview 2: GA Cell Representation – No Letters	-	✓	-	-	✓
6	GA Lesson 3: Numerical Expressions in GA	✓	-	-	-	✓
7	GA Lesson 4: Variable Expressions in GA	✓	-	-	-	✓
8	GA Lesson 5: More Practice on Expressions (1) Test 2: Arithmetic Skills (Post-GA)	✓	-	✓	✓	✓
9	GA Lesson 6: More Practice on Expressions (2) Interview 3: Meanings for Notation (Post-GA)	✓	✓	✓	-	✓
10	GA Lesson 7: Order of Operations on Letters (1)	✓	-	✓	-	✓
11	GA Lesson 8: Order of Operations on Letters (2) Interview 4: GA Cell Representation 2 – Letters	✓	-	✓	-	✓
12	GA Lesson 9: Constructing an Algebraic Expression	✓	-	✓	-	-
13	GA Lesson 10: GA to Paper – Substitution	✓	-	✓	-	-
14	GA Lesson 11: GA to Paper – Function (Processes) GA Lesson 12: GA to Paper – From Words to Symbols	✓	-	✓	-	-

The pilot was started after I got the ethical clearance in March 2014 and lasted until the end of that scholastic year. The wait for the ethical clearance gave me time to get to know the students and be in a better position to choose the two case study students. Alan and Manuel were strategically chosen to have different performance levels. Alan consistently showed a higher performance in his work than did Manuel and but both of them were quite able to express themselves verbally. It benefitted the pilot study that they agreed to take part in it because I got to know how students of different performance levels reacted to the GA software, the lessons, and the interviews. In addition, their verbal ability ensured that they could give me valuable feedback on their thought processes during the interviews.

All of the other students in the class agreed to act as "background participants". Their role was important because their participation provided me with information about the classroom dynamics during class discussions and also with insights about the way students represent and interpret mathematical expressions while working with GA. Their performance in written work made me aware of some common errors when dealing with expressions and their interpretations of the questions led me to think of better ways in which these could be worded.

The analysis of the pilot study enabled me to elicit lessons about

  (i)   the research questions,

  (ii)   the participants,

 (iii)   myself as a teaching researcher, and

 (iv)   the research method and tools.

In addition, I formed potential hypotheses which I set out to explore in the main study. In this chapter, I outline these lessons and hypotheses with reference to the research aims.

5.2 Lessons about the Research Questions

During the analysis of the lesson videos, I saw myself acting as a sort of catalyst to the students' thinking and reasoning as they interpreted and represented expressions generated by GA. During the analysis, I developed sensitivities (Mason, 2002) with regards to CT. I asked myself: *Is my teaching reflecting my RC beliefs? Am I being sensitive to students' constructions of knowledge?* During the analysis of the lesson videos I could identify instances when this seemed to be the case. As an example, I will present the three excerpts from Lesson 9 where learners were working on expressions that included a letter.

Excerpt 5.2.1   Lesson 9

    PB[12]:   What does this z mean?...
   Mario: Any number.
      PB: Any number. But is it any, *any* number? Can it be, say, one?
   Mario: No. In the 5-times table.
      PB: Any number in the 5-times table. Good.

This is an exercise in what Driver and Oldham (1986) call *elicitation*, an important step in their constructivist instruction model (CIM) where teachers help learners access and express their prior knowledge and current ideas on the topic. Here I was interested in learning the mathematics of my students, a crucial exercise in CT according to Steffe (1991). When I asked Mario to elaborate on his "any number" I was also aiming to restructure his and other students' ideas through clarification and exchange (Driver & Oldham, 1986).

While, in Excerpt 5.2.1, I seem to be interested in learning about a student's conceptual interpretation, in other moments it seemed I was trying to orient students' thinking (Glasersfeld, 1991b) towards some sort of conceptual agreement with my own notions of the subject matter. This was the case in the following episode.

---

[12] Philip Borg

Excerpt 5.2.2   Lesson 9

PB: [*Referring to the expression* $\frac{c-6}{6}$] …(W)hich is going to be performed first from it [*some students raise their hands*]…Let's see. There's someone else, Karl [*points towards Karl*].

Karl: c minus 6.

PB: Well done. Why are you realising that c minus 6 needs to be performed first [*Karl starts answering*] and not the division?

Karl: Because if you go up in another row you can't do minus 6 because there won't be enough boxes.

PB: All right. Because there aren't enough boxes to do minus six. But if you are just seeing this [*making a rotating motion on the expression*] and you're not seeing anything else, all right? Because now the grid will start to disappear, uh. We won't always have the grid…

Karl: Because you are in the 6-times table.

PB: …Why do I need to perform minus 6 before I do the division here?

Alan: [*Raises his hand*] Because there's that, that line separated for itself [*makes horizontal line gesture #1*].

PB: Good. Because this line [pointing to the division line] we are seeing that it is long [*makes horizontal line gesture #2*]…

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In the first part of this episode, I was interested to learn how students' were interpreting the expression $\frac{c-6}{6}$. When Karl answered correctly, I wanted to help all students to reflect on why it was -6 to be performed first. Karl's second reponse was not what I anticipated as a "correct" answer and I was aware that some "mistaken" answers are actually correct for particular interpretations of the question (Glasersfeld, 1991b). However, I wanted to orient students' reflections on the notation: "But if you are just seeing this…" When Alan gave the response I was after, I elaborated it by imitating and amplifying his horizontal line gesture.

Excerpt 5.2.2 shows a typical teaching approach during plenary discussions. Like other lessons, I observed myself here shifting my attention several times:

- interacting with the students,

- inquiring to learn about students' interpretations,

- going back to them with a new approach according to the feedback,

- acknowledging students' representations,

- reflecting on those students' representations, and

- reinventing my learning offer (Steinbring, 1998).

This toing and froing from students' mathematics (Steffe, 1991) to the mathematics I intended to teach and back to the students made me ask the question whether this was an indication of CT. These oscillations between learners and subject content reminded me of Dewey's (1902) metaphor that teaching is the line defined by two points: the child and the curriculum. These reflections helped me to refine my research question concerning CT (1(i)). If such acts of negotiation between learners and my mathematics were caused by my sensitivity to students (Jaworski, 1994), there might have been occasions where I lost this sensitivity. This led to research question 1(ii). As shown in the above excerpts, these classroom dynamics surrounded the subject matter (notation used in informal- and formal-algebraic activities) and facilitated by GA. This and my belief that my classroom actions could only be classified as teaching if they brought about learning (Freire, 1998) generated research questions 2(i)–(iii).

5.3 Lessons about the Participants and their Learning

As soon as I started the pilot study, I assigned a written classwork to test students' performance in simple arithmetic operations and evaluations of numerical expressions. While being able to perform simple arithmetical operations (e.g. 3×6 and $10 - 4$) through factual recall, most students evaluated numerical expressions by working out the operations in the order of appearance, a common mistake even among adults (e.g., Glidden, 2008).

An interview with Alan and Manuel before starting the GA lessons revealed that they had an operational conception of the equals sign (ES). Excerpts 5.3.1a and 5.3.1b provide a comparative view of how these students responded to some interview questions.

Excerpt 5.3.1a   Interview 1-Alan

(a) $3 + 4 =$ [7]

(b)   [3+2] $= 5$

(c) $7 =$ [3+4]

PB: [*Referring to empty box in (b)*] Can you put a single number and you do no plus or minus?...

Alan: [*Shakes his head (no)*]… Because I think that plus needs two numbers or more, so does minus, so does division and times, fraction I don't think so, percentage…I don't think so as well…

Excerpt 5.3.1a   Interview 1-Manuel

(a) $3 + 4 =$ [7]

(b)   [ ] $= 5$ $4 +$ [1] $= 5$

(c) $7 =$ [4] D

Manuel: [*Referring to (b)*] I get confused because it isn't like the other one (where) you have 3 plus something and then you are given the answer, or… for example 4 plus "hmm" [*empty box*] and then you are given the answer.

Note:
- M stands for "*Ma nafhiex*" (I don't know it);
- D stands for "*Dubjuż*" (I have doubts).

Both participants got (a) correct, but from their responses about (b) and their written answers to (c) it seems they were viewing ES as an operational symbol, which is widely reported in the literature (e.g. Rittle-Johnson et al., 2011). This may have been due to limited applications of ES in primary school (McNeil, 2008). It seemed, therefore, that Alan and Manuel had a limited interpretation of ES as they had for some other notation. For instance, Interview 1 revealed that they did not seem to be aware that $5(2 + 8)$ meant 5×10 or that $\frac{10}{2}$ could stand for $10 \div 2$.

Evidence from interviews excerpts after a series of lessons with GA suggests that these students seem to have extended their:

(i)   interpretations of notation,

(ii)   properties of operational symbols,

(iii)   meanings of ES (they obtained a relational view),

(iv)   interpretation and use of letters (variables and unknowns), and

(v)   knowledge about the order of operations.

Consistent with Hewitt (2012), by the end of the pilot study, GA had enabled Alan and Manuel to learn the order of operations in quite complex expressions, such as $\frac{6\left(\frac{10}{5}\right)+6}{2} + 3$, without having to resort to acronyms like BIDMAS.

Figure 5.3.1 shows screen shots from the work of Alan and Manuel who worked as a pair on GA Task 15. This task involved moving the original number, in this case 10, around the multiplication grid in the correct order of operations.

Figure 5.3.1   Alan and Manuel working as a pair on the order of operations with GA
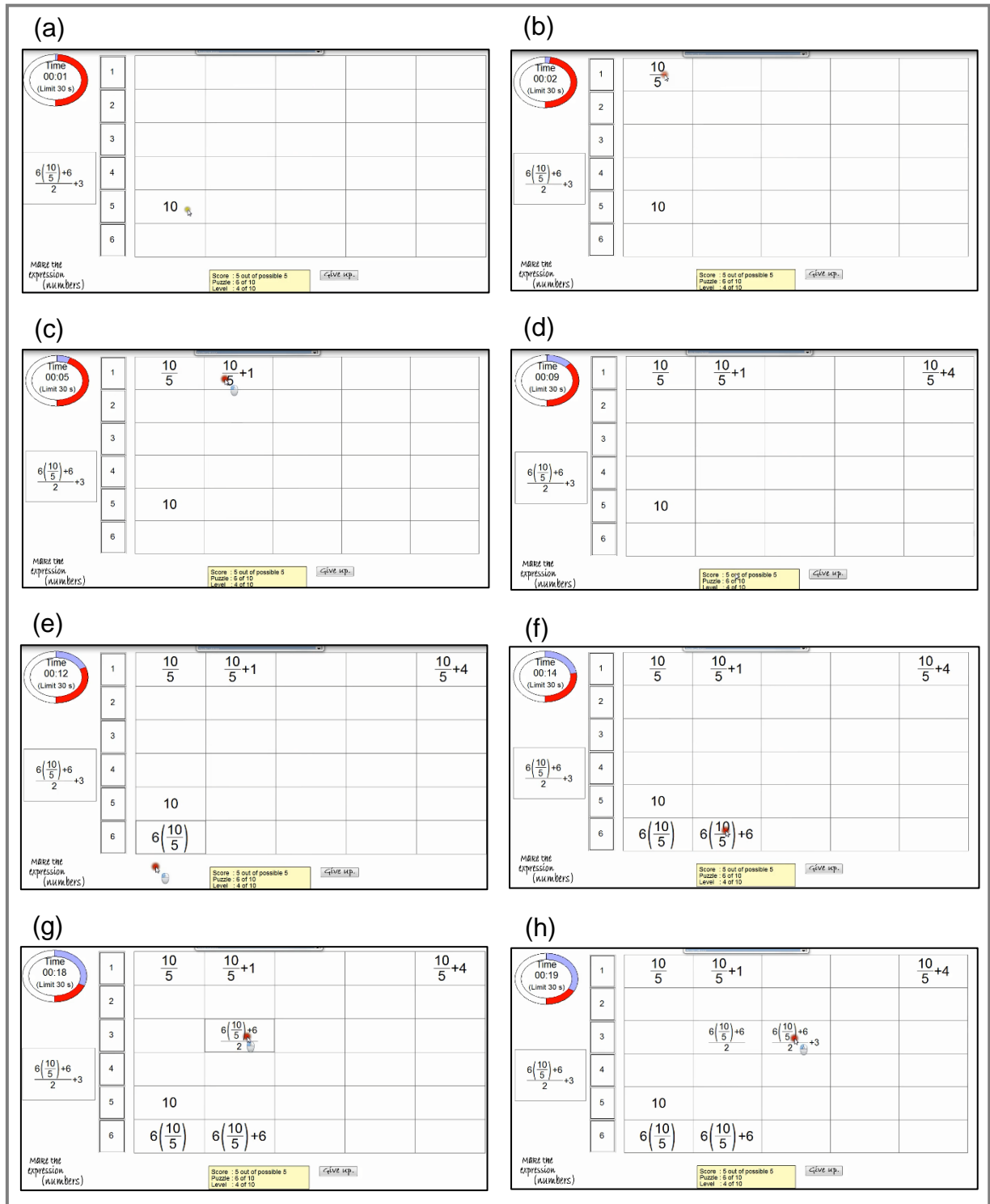
Screenshots (c) and (d) show that they were trying to perform $+6$ first, which is consistent with Glidden's (2008) finding that a common error is to perform operations from left to right first. By showing them the expressions that corresponded to these

actions, GA enabled Alan and Manuel to discover and correct their mistake. From (e) onwards, Alan and Manuel proceeded correctly. As shown on the timer, all this took place in just 19 seconds.

GA was found to provide all students with opportunities to enrich their representations of notation. Bruner's (1966) Enactive-Iconic-Symbolic construct to describe non-verbal mathematical representations was very relevant here. Students used GA to make representations in three forms:

(i)   **Actions.** Students represented operations by moving cells with expressions;

(ii)   **Pictures.** Students represented expressions by journey pictures provided by GA and also by particular cells (rectangles) of the grid;

(iii)   **Symbols.** Students represented expressions involving the four operations using standard conventional notation provided by GA.

My choice of "picture" over "icon" comes from my interpretation of the latter as being a picture that in itself has a characteristic linked to the notion it represents, e.g. $\overline{\Delta}$ may be an icon for "balance". In my case, students were representing a number with a rectangle (picture of a cell), not due to its shape but due to its position in the grid.

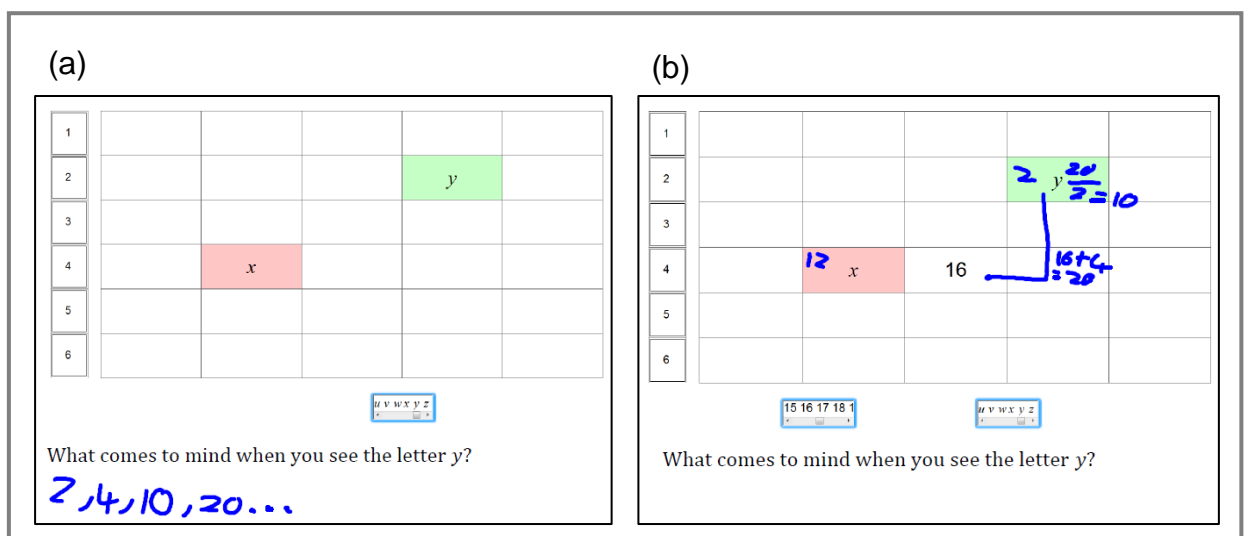*Figure 5.3.2   Manuel's interpretation of letters as variables and unknowns*

Figure 5.3.2 shows Manuel's written responses during Interview 2, where he used numbers and expressions to represent his conception of letters in particular cells in the GA grid. Figure 5.3.2a shows how the rectangular representation (cell) of the letter y in the grid helped Manuel to interpret it as a variable multiple of 2 (since it was in R$_2$). Figure 5.3.2b shows that the introduction of the number 16 in the grid rendered y a specific unknown, which, by checking its relation to 16, Manuel found to be 10. He also determined the value of the unknown x to be 12.

Such a repertoire of representations helped both case study students to construct new meanings of notation. They learnt about the use of brackets to denote multiplication and started to use conventional notation for division. They also learnt about the commutativity of addition and multiplication and came to view ES as a relational symbol. However, similar to what McNeil (2008) and Rittle-Johnson et al. (2011) reported, they still retained the operational view of ES. The case study students became competent in finding the order of operations in quite complex expressions. They also seemed to have started forming a proceptual view (Gray & Tall, 1994) of expressions. In addition, they constructed initial concepts of variables and unknowns. However, most of these conceptual developments were more exhibited by Alan than by Manuel.

In varying degrees, I could observe all students in the group making similar extensions in their interpretations and representations of notation, even those who had entered Grade 7 with a very low score in the Grade 6 benchmark exam. One such student was Noel (pseudonym) who had been diagnosed with ADHD and Dyslexia. Before starting the GA lessons he had difficulty in adding or multiplying small positive numbers. With the help of GA, he managed to make significant developments in his knowledge about new notation and the order of operations. Figure 5.3.2 shows Noel working on the expression $2(a + 6)$ on the IWB.

Figure 5.3.2   Noel learning about new notation and order of operations

The experience of Alan, Manuel, Noel and their peers taught me what I consider to be the most valuable lesson about these "low-performing" participants: Performance may be a function of the environment of the problem and the tools to solve it. With dedication, effective tools, and sufficient time, these students proved that they were not low-performing at all. The following were the other lessons I learnt about my participants and their learning:

(i)   Students started Grade 7 with a limited understanding of operational notation and ES. They were not familiar with the use of the brackets and the divisor line of a fraction to denote multiplication and division respectively.

(ii)   GA gave students the opportunity to enrich their representations of notation through actions, pictures and symbols.

(iii)   GA helped students to develop new and extended interpretations of notation. In particular, they learnt about

- the commutativity of addition and multiplication,
- the order of operations in complex expressions,
- the use of letters as variables and unknowns,
- the relational concept of ES, and
- the dual interpretations of expressions as procepts (Gray & Tall, 1994).

5.4   Lessons about Myself as a Teaching Researcher

In the following subsections, I include lessons about my dual role as a teaching researcher which I learnt by reflecting on the pilot study.

5.4.1   Lessons about Myself as a Teacher

Two Main Roles in the Grid Algebra Lessons

There were two principal roles I played as a teacher. The first was coordinating plenary discussions in the first part of the lessons and the second was monitoring students' work and intervening only when my help was required. It seemed to me that the first role was more appropriate to be analysed for CT, for two reasons:

(i)   During the plenary discussions, it was more challenging for me to maintain sensitivity to students' constructions of knowledge because I was concerned with presenting learning offers;

(ii) The students-teacher dynamics were more rich and continuous in the plenary discussions and were more suitable to generate trends and patterns in my approach, rather than the intermittent interventions I made whilst monitoring students working on their computers.

Questioning Technique

In my journal log of the week starting 5th May 2014, I pointed out three kinds of questioning I frequently used in plenary discussions:

(i)   **Open-ended question**. E.g., *"*What do you notice?", "Why do you think…?"

(ii) **Closed-ended question**. E.g., "What number can go here?", "How many…?"

(iii) **Unfinished closed-ended statement**. E.g., "The movement for an addition is…?"

Usually, when no one answered a question I would go for a more closed question. Interesting discussions were created in questions type (i) and type (ii), but all questions were intended to be productive (Eltgeest, 1985), in that they were aimed at stimulating students' conceptual processes.

During my questions, I found out that students were picking on visual or verbal cues to determine whether they were responding correctly or not. These included raising my eyebrows, looking away, or asking a question with a particular tonality when I thought an answer was incorrect and increasing my nodding or smiling when I thought an answer was correct. Such cues could have compromised the data, especially during interviews, and I was careful to avoid them in the main study.

5.4.2   Lessons about Myself as a Researcher

The conflicting functions of being a teacher and a researcher during the data collection process were discussed in Chapter 4 but I only learnt about the difficulties in maintaining a dual role of a teaching researcher during the pilot study. Here are three lessons I learnt about being a researcher from this preliminary study:

(i)   I was made more aware that I am not what can be called a multitasking person (Rosen, 2008). During the lessons, I focused only on teaching and if I had to tend to issues about data gathering this proved distracting. This was the main reason why I had to stop the journal note-taking during the lessons.

(ii)   Since technical assistance was hard to get, I had to rely on myself on issues that regarded computer hardware and software. One way of working around this was to prepare the room well before I started the first lesson of the day and check that everything was working fine and do the necessary replacements if not.

(iii)   I became aware of the massive amount of data that was being generated. This emphasised the need to organise the data according to time, type, and participants and to develop viable data analysis techniques as discussed in Chapter 4.

5.5   Lessons about the Research Method and Tools

There were quite a number of lessons I learnt about the research method, and about the data-gathering process and tools of the pilot study. The following were the more significant:

(i)   The case study proved to be an effective research method. It enabled an investigation of conceptual journeys made by the students where I could compare their conceptual developments. I felt that if I included more students in the case study it would be more beneficial to identify trends and similarities. This and the fact that the Grade 7C (Maths) group of the following year happened to be small were factors which encouraged me to include all six students in the main case study.

(ii)   The research design and my knowledge were simultaneously and continuously evolving during the pilot. Besides elaborating and refining the research questions, the pilot helped me to realise that unpredictable themes would emerge that necessitated further review of literature. In fact, literature reference and review occurred throughout the whole duration of the research.

(iii)   More GA lessons were needed to introduce the use of letters as variables and unknowns. The pilot helped me to realise that students benefit more if they have a good number of lessons using only numerical expressions (informal-algebraic activities) before introducing letters and algebraic expressions (formal-algebraic activities). This and the need to have more data from which I could elicit trends and similarities in students' developments made me opt for a longitudinal data collection process lasting a whole scholastic year with 20 double GA lessons in the main study.

(iv)   The pilot made me more aware that technological tools like the video camera and the computer screen activity capture (CSAC) software may malfunction. In the main study, I always carried an extra video camera which twice proved to be a lifesaver. I also made use of a more stable CSAC software which worked well for all the lessons.

(v)   The pilot eased one of my concerns about the data collection process: that students would act out during the lessons because they were on camera. On the contrary, students seemed to ignore the camera and forget that their actions were being recorded. The classroom situation was as "normal" as one would expect.

(vi)   The seating arrangement in the classroom during the pilot GA lessons was not found to be convenient. In order to be directed towards the IWB, the video camera only captured a fraction of the students and some of these were hidden behind others. In the main study, I gained access to another computer room with a more suitable seating arrangement, where the camera could capture the IWB completely, part of the whiteboard, and all students from behind.

(vii)   Analysing video recordings made me realise the importance of audio. Sometimes it was hard to decipher

- quiet communication during students' computer pair work, and

- simultaneous speech from two or more persons during the lessons.

In the main study, I installed better microphones with the computers and positioned the video camera close enough to pick class discussions clearly.

5.6 Possible Hypotheses

The pilot study helped me to make the following three hypothesis about CT and GA:

(i)   CT is an activity where the teacher's focus "oscillates" between: the subject matter (mathematics) she/he intends to teach and the learners' conceptual constructions, while the teacher maintains a sensitivity to constructivist notions of learning. CT is defined by the connection of these two factors in the same way that a line is defined by two points (Dewey, 1902).

(ii)   GA can help students to extend their conceptual interpretations of mathematical notation by providing them with rich and varied representations in the form of actions, pictures, and symbols.

(iii)   GA may be used as a tool for CT, where the teacher presents learning offers by creating mathematical representations on the GA grid and learns about students' mathematics by observing their representations on the same grid.

Together with a continuous literature search and review, these hypotheses helped me to develop sensitivities (Mason, 2002) about issues related to my research aims and questions.

I now return to the main research study, where I dedicate each of the following two chapters to an analysis and discussion of the data. In the first analysis chapter, I focus on my teaching, and in the second I focus on my students' learning. In Chapter 6, I am concerned with data regarding CT which is mostly related to research questions 1(i)–(ii).

Chapter 6

Analysis and Discussion of Constructivist Teaching

6.0   Overview

This chapter includes data analysis related to constructivist teaching (CT). This analysis investigates the dynamics of CT as observed in the Grid Algebra (GA) lessons and leads to the development of a conceptual framework which is later used analytically to detect and describe classroom situations in which I succeed or fail to engage in CT. This addresses the first set of research questions, those which concern CT. Table 6.0.1 includes the section titles of this chapter.

Table 6.0.1   Chapter 6 section titles

	Section Title	Page
6.1	Analysing the Lessons in Terms of Teaching Purpose	182
6.2	The Mathematics-Negotiation-Learner Framework	205
6.3	Overall Descriptive Statistics of the Shifts of Teaching purpose	210
6.4	Complete M-N-L Cycles in the Grid Algebra Lessons	214
6.5	Roadblocks between Mathematics and Learners	222
6.6	Summary and Conclusion	235

6.1   Analysing the Lessons in Terms of Teaching Purpose

In the GA lesson analysis, I focused mainly on the sections where I was involved in group discussions since my interactions with students when they were working on their own was very limited. When analysing my role in the various phases of class discussions as observed in the lesson videos, I investigated the nature of my teaching purposes against a backdrop of beliefs I adopted from constructivist literature, most prominently:

(i)   Dewey's (1902) claim that the content in the curriculum and the mental state of the learner should define the teaching and learning process;

(ii)   Steffe's (1991) claim that the distinguishing feature of CT is the way in which constructivist teachers strive to learn about the cognitive processes of their students;

(iii)   Simon's (1995) claim that constructivist teachers hypothesise about a possible learning trajectory and interact with students to facilitate that trajectory.

These claims provided a frame of reference to analyse my teaching against CT principles, but they were too broad to characterise the finer details of my purpose in the lessons. Thus, I set out to describe and categorise my purposeful actions and reactions during the GA lessons. A viable way for me to do this was to identify shifts in my focus which suggested a change of purpose. I observed eleven different shifts during the GA lessons:

(i)   From anticipating didactic processes to interacting with students;

(ii)   From associating students' mathematics with mine to interacting with students;

(iii)   From interacting with students to facilitating students' experiences;

(iv)   From interacting with students to facilitating students' reflections;

(v)   From facilitating reflection to creating a model of students' mathematics;

(vi)   From facilitating reflection to reviewing my learning offer;

(vii)   From creating a model of students' mathematics to reviewing my learning offer;

(viii) From reviewing my learning offer to associating my mathematics with students' representations;

(ix)   From reviewing my learning offer to adapting my mathematics to incorporate that of the students;

(x)   From creating a model of students' mathematics to associating my mathematics with students' representations;

(xi)   From creating a model of students' mathematics to adapting my mathematics to incorporate that of the students.

These shifts of teaching purpose were coded to facilitate analysis. The final version of the codes presented in this chapter resulted from a process of modification and

refinement of codes until I felt they captured the changes I was observing. The shifts were classified into four categories which are discussed in detail in Sections 6.1.1–4, where I provide lesson excerpts of typical examples of these changes of purpose.

During the analysis of these excerpts, I am not claiming that I am recalling the exact occurrences of what was going on in the lessons. I am interpreting video observations with the benefit of hindsight and of acting as a second-order observer: a researcher rather than a teacher. These interpretations are informed by my awareness of my teaching approach and the thoughts that usually precede or follow actions I do as a teacher. This does not exclude that the very analysis may be reminding me of actual cognitive processes I was experiencing during the lessons.

6.1.1   Teaching Purpose Shift 1: From Intention to Interaction

Whenever I had an intention to provide students a learning offer (Steinbring, 1998), I needed to anticipate the didactic processes facilitated by my interactions with the students. In excerpts such as the following, my focus changed from thinking about and anticipating possible didactic processes to actually engaging with the students in an interaction intended to facilitate those processes.

Excerpt 6.1.1.1   Lesson 2

PB: [*Looking at the IWB and setting up a new empty GA grid with four rows.*] I am going to create a new one [*grid*] for you. So… Now… In the last lesson, [*pointing to the cells of first column of the empty grid in order*] here there was 1, here there was 2, here there was 3, here there was 4. But now I'm going to insert another number – not 1, 2, 3, or 4. I'm going to invent…[*dragging the number 12 in R_3C_1[13] and not letting it in yet*]. Can I?

Joseph: You can.

PB: [*Letting the 12 stay in the cell and this was allowed by GA*] Why can I put a 12 there?

---

[13] Recall that GA allows only multiples of 3 in R_3, multiples of 4 in R_4 etc.

In this episode, I used the phrase "I am going to" three times, suggesting anticipation. I also used phrases like "create a new one" and "invent", indicating that I anticipated a new learning offer. This anticipation became interaction when I started to encourage students to participate in the setting up of the GA grid and asked them whether I could insert the number 12 in R_3C_1. I shifted my purpose through two simultaneous actions:

- *doing something* – dragging the number 12 from the number menu into the grid but not letting it go, and

- *saying something* – asking the students whether I could insert a 12 in that cell.

When Joseph confirmed that I could do it, I repeated the two actions (doing and saying) while my focus was on interacting with the students. I took off the cursor and asked the students another question: "Why can I put a 12 there?" This question was meant to kick-off a class discussion about multiples of 3 which took place after this episode. I decided to label such a change of teaching purpose as **Anticipate>Interact**, where my focus changed from (1) a planning and *anticipation* phase into (2) an *interaction* and discussion phase.

A similar but distinct change of focus occurred when I skipped the anticipation phase altogether and reacted with a direct interaction. This is what happened in episodes like the following, where I asked Dan to tell me what GA would show if I dragged the number 8 from R_1C_2 to R_2C_2. By now students knew that R_2 represented doubles of corresponding numbers in R_1. I was expecting the response "8×2" but this was not what Dan replied.

Excerpt 6.1.1.2   Lesson 4

> Dan: [*After PB asked him what R_2 will show if he dragged 8 into it.*] 8 plus 8.
>
> PB: All right, 8 plus 8, but it is going to show me… [*moving to the board and dragging the 8 in R_1C_2 to R_2C_2 and GA showing 8×2*]…
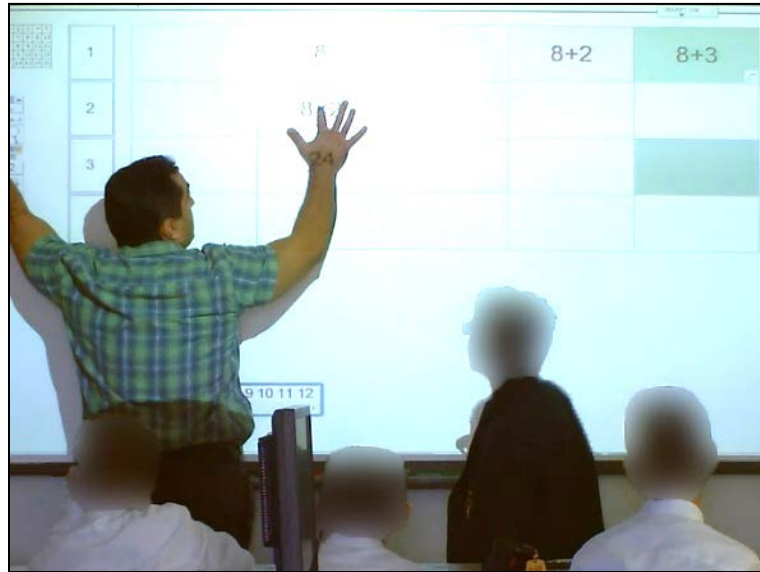
Dan: [*Talking at the same time as PB*] 8 times 2.

PB: …8 times 2, since 8 plus 8 is the same as 8 times 2.

Both $8 + 8$ and 8×2 mean "the double of 8" but I knew that GA would show the latter, so I associated Dan's $8 + 8$ with GA's 8×2 and immediately interacted with Dan by dragging the cell in a way that he could observe what GA would show. Once Dan confirmed that it was 8×2, I resumed my interaction to help him make the same association I had made earlier by telling Dan that GA had showed this because "8 plus 8 is the same as 8 times 2."

I coded such instances as **Assoc>Interact** where my purpose changed directly from (1) *associating* students' representations with the mathematics I intended to teach to (2) *interacting* with students in order to help them appreciate that association.

The similarity of the above changes of teaching purpose lay in the initial and final mental states of the teacher, the former being the teacher's intention (a section of mathematics intended to be taught) and the latter being the teacher's interaction with the students according to that intention.

6.1.2 Teaching Purpose Shift 2: From Interaction to Learner Experience and Reflection

The second type of shift of teaching purpose occurred when I was interacting with the students and my focus shifted to what students were experiencing. By "experiencing" I mean perceiving, interpreting, and possibly contributing mathematical representations. As shown in the following episode, sometimes I shared students' experience in order to encourage reflection.

Excerpt 6.1.2.1   Lesson 13

PB: [*GA on Run mode*] Come out Dwayne. [*Dwayne comes out near the IWB.*] Throw in another letter, not a c, anywhere you like. [*Dwayne inserts the letter e in R_6C_5.*] Now… that e…What is the number that that e is symbolising?

Dwayne: [*Working something out with his fingers*] Fifty-four.

PB: Come. Let me see. [*Nodding towards the board.*] Put it in and then tell us how you worked it out… [*Dwayne puts 54 from the number menu onto the cell containing e and GA accepted it.*] …Good boy! How did you realise it was a 54 over there? … So, first of all, get the magnifier. … [*Dwayne clicked on the magnifier icon and GA showed a window containing the equation $54 = e$ (see figure below).*]
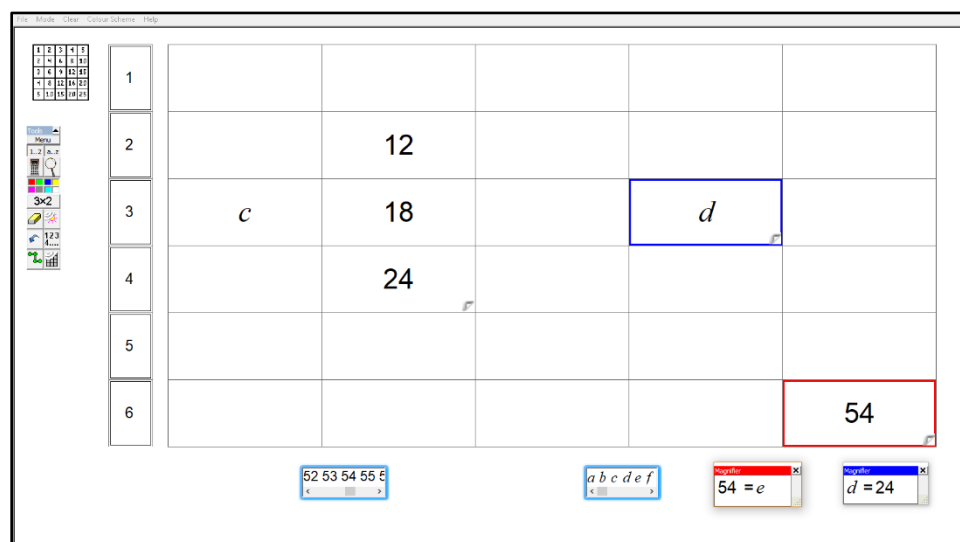
	… Let us see. What is it showing me? [*PB clicks on the cell repeatedly and the magnifier window alternated between* $54 = e$ *and* $e = 54$].
Dwayne:	54 equals e and e equals 4 [*probably meaning* 54].
PB:	Now explain to us how it came to be 54.
Dwayne:	Here you have… You get a 54 there [*pointing to the cell containing* 54] because here [*pointing to* C_2] you have the 6-times table, and here [*pointing to* R_6] you have the 6-times table as well.
PB:	OK.
Dwayne:	And then to make it shorter I did it from here [*points to* R_3C_2 *(*18*) and moving his hand to the right*].
PB:	All right. Come, tell us how.
Dwayne:	So then you have the 6-times table [*points to* R_1C_2] then here is the 7-times table [*pointing to* R_1C_3], then 8 [*points to* R_1C_4], and then 9 [*points to* R_1C_5].
PB:	All right
Dwayne:	And then I counted one, two, three, four, five [*pointing respectively to* R_1C_5 *to* R_5C_5], here [*points to* R_5C_5] it's 5 times 9 and then after it [*points to* R_6C_5] comes 54.

When I called out Dwayne, I was switching my focus from my previous interaction with the students to what this particular student might experience about a letter he introduced into the grid himself. It was as though I was playing the part of the curious classmate asking Dwayne to do something to see what will happen: "Come. Let's see… Get the magnifier… What is it showing…?" I immersed myself in Dwayne's experience and limited myself to suggesting things to do and asking questions:

- "What is the number that that e is symbolizing?";

- "How did you realise it was a 54 over there?";

- "…explain to us ...";

- "…tell us how".

These questions were meant to encourage Dwayne and others to reflect on mathematical representations. The whole episode can be regarded as an experience-reflection cycle. I labelled such changes in teacher focus as **Interact>Experience** where my attention shifted from (1) what I was saying, working, or demonstrating (*interaction*) to (2) what students were *experiencing* as a result of that interaction, which usually led to students' reflection, as shown in the next excerpt.

Sometimes, it seemed that my focus shifted immediately to students' reflection. In the following excerpt, Dan was solving a problem from GA Task 16 which presented the challenge of moving a cell with a number through a journey which would result in a target expression. Dan needed to take the cell containing 4 on a two-step journey resulting in the target expression $\frac{4+2}{2}$. Instead, his journey resulted in $\frac{4}{2}+2$. Dan himself was first to notice that his expression was incorrect because he was checking the syntax of his expression against that of the target expression.

Excerpt 6.1.2.2   Lesson 10

PB: Hang on, Dan… Let us concentrate because we learn more from the mistake because if he got it correct we applaud him and we do another one. Let me see who will tell me what he did incorrect. What can you see that makes it incorrect? [*pointing to the expression.*]
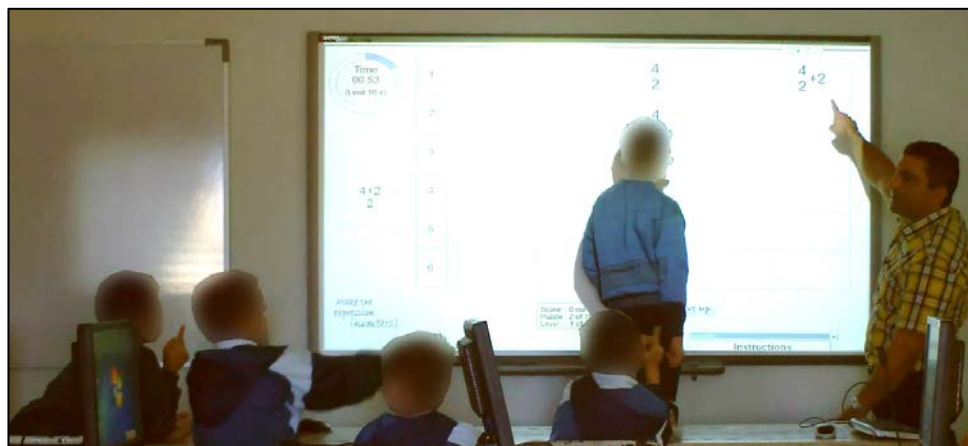
[*Points at Dwayne*]. Tell me.

Dwayne: Sir, it looks wrong because there [*pointing to Dan's expression*] where there is 4 all over 2, there should be $4 + 2$ all over 2.

Excerpt 6.1.2.2 shows how GA activities gave students the opportunity to make mistakes and learn from them (Dewey, 1916). Evidently, I saw Dan's making the mistake as a better learning opportunity than if he did not: "…we learn more from the mistake because if he got it correct we applaud him and we do another one." At the same time I avoided direct correction to help students to think independently (Radford, Blatchford, & Webster, 2011). Asking students to reflect about "what…makes it incorrect" was an attempt to use Dan's error as a starting point for reasoning and argumentation (Ingram, Pitt, & Baldry, 2015). I changed my purpose from interacting with Dan in his execution of the task to encouraging students' reflection that was triggered off by Dan's mistake. I intended not only to help students identify the mistake but also to appreciate why the action was considered as mistaken. It was desirable for Dan to identify the error himself, but it turned out to be Dwayne who did. Following this episode, Dwayne went on to show Dan what he should have done (correct order of operations) which led Dan to make subsequent correct moves in the GA grid.

The excerpt gives an example of the many instances where my interaction led straight to a reflection rather than to an action, that is, it was meant to encourage students to think back and reflect, rather than think forward and anticipate. Such instances were therefore coded **Interact>Reflect**, because my focus changed from (1) what I was saying or doing (*interacting* with students) to (2) what the students were *reflecting* and reasoning.

6.1.3 Teaching purpose Shift 3: From Learner Reflection to Evaluation

The third shift of teaching purpose I observed occurred when I diverted my attention from facilitating students' experience and reflections to forming a model of students' mathematical thinking processes. This shift occurred frequently in class discussions but was predominant when students were working on their own. The following excerpt is from the second part of GA Lesson 17, where students were working on Task 4 in pairs. This task required them to make three expressions corresponding to the journey that a letter in one cell of the grid needed to make to arrive in another highlighted cell. As I was moving among the students, I noticed that Omar was getting frustrated.

Excerpt 6.1.3.1   Lesson 17

PB: [*PB notices that Omar got frustrated.*] What's the matter there, mate? [...] [*PB comes to look at Omar's work on his computer (seen in the following computer-screen snapshot – numbers 1-3 are superimposed for reference purposes). Omar had already done one correct expression that could take the letter d from R_1C_4 to R_2C_5 – 2(d+1).*]
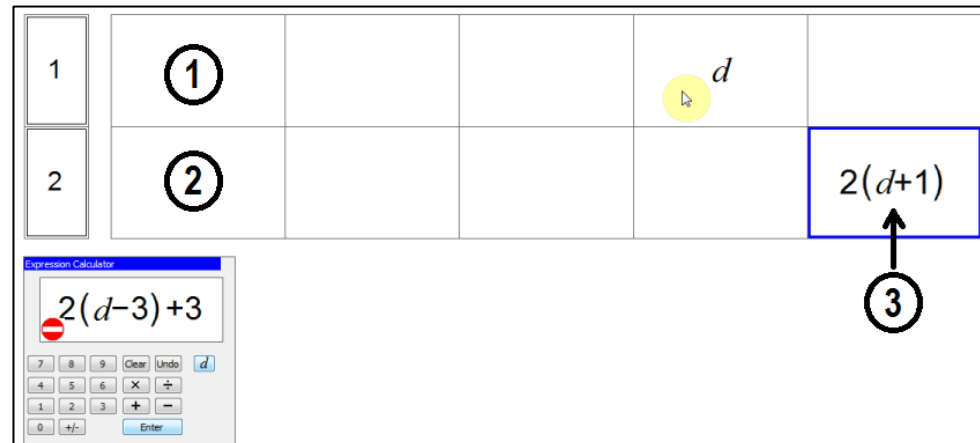
Omar: I, actually, I did the d, I will go here...

PB: Show me with the mouse [...] you said, d will go there [①], good.

Omar: I'll move over here [①], go down here [②],...

PB: Good.

Omar: And then I go over here [③]

PB: Good. But... tell me how much plus you do from here [②.]

It seemed that Omar was making two mistakes in moving from ② to ③: counting in 1's instead of 2's and not counting the last cell. Significant in the excerpt is the sequence of questions I asked Omar so that I could construct a model of his thinking processes:

(i) "What's the matter, mate?" – A general enquiry meant to kick-start communication where the word "mate" (in Maltese "*xbin*" - pronounced "*shbeen*") helped to establish a friendly, non-formal exchange.

(ii) "Show me…" – A specific enquiry meant to confirm the model I was building of Omar's thinking and hence the source of his mistake.

(iii) "But…tell me how much…" – A follow-up enquiry meant to confirm this model and also to start interacting with Omar to help him realise what he did wrong.

These enquiries were meant to help me generate possible interpretations (Glasersfeld, 1987) of Omar's conceptual patterns. I was trying to build a model of his reasoning that was both hypothetical and experiential (Steffe, Glasersfeld, Richards, & Cobb, 1983; Steffe & Ulrich, 2013). I could only *hypothesise* what Omar was thinking but observing the way he was *experiencing* the mathematical problem enabled me to make an educated guess of his thought processes. Further interaction between us led to Omar's detection of both of his mistakes and to a successful completion of the task. I coded such a shift as **Reflect>Model** – my purpose as a teacher changed from being (1) a facilitator of Omar's *reflection* on his actions to being (2) an evaluator of Omar's mathematics by building a *model* of his thinking patterns.

There were several instances where the model I formed about a *particular* student's thought processes was used to review my learning offer and decide whether it matched the *collective* thought processes of the whole group. The following episode is a typical example where I felt that such a match had been established. The conversation was about the expression $4 + 5 \times 2$.

Excerpt 6.1.3.2   Lesson 18

> PB: …So, to work this one out [*pointing to the expression*], to work this one out, what do you need to work out first?
>
> Dwayne: The 5 times 2.
>
> PB: *Bravi!* [*Maltese for "Well done to you (plural)!"*].

In this episode, I replied to Dwayne's response by exclaiming "*Bravi!*", (Maltese for "Well done to you (plural)!") rather than "*Bravu*", ("Well done to you (singular))". When teaching, I usually use the plural form when I am thinking of the class as a group. Congratulating the whole group rather than just Dwayne tells me that my model of Dwayne's thinking process at that time represented to me the *collective* thought

processes of the whole group, however diverse these might have been. This was not uncommon in the course of the lessons, and I became aware that I usually rested on one or two responses to suppose what a *representative* student in that group might be thinking. As I argued elsewhere (Borg, 2016, p. 108), teachers tend 'to think of those few students who speak up as a "sample" '. I am not stating here that this portrays a "true" picture of what goes on in the diversity of minds subjectively interpreting classroom representations, but teachers have to rest on whatever feedback individual students offer in order to form models of the thinking patterns of the whole group.

My exclamation of "*Bravi!*" seems to imply that Dwayne's response served me as an indication of what the *whole group* might have been making of my learning offer. Since Dwayne came up with the response I was after, I regarded my learning offer to be successful for him and possibly his classmates. Hence, the model I created of Dwayne's mathematics led to a positive review of the mathematics I was representing for students to interpret. Usually, creating a model of students' reasoning took more time than the episode shown in Excerpt 6.1.3.2, since I typically followed a student's statement by asking questions to the whole class to establish whether other students were reasoning along similar lines. This usually led to a review of how the learning offer was being interpreted by the whole group.

In any case, I labelled such shifts of focus as **Model>Review** because my purpose changed from (1) building a *model* of the mathematics that students were constructing to (2) *reviewing* the appropriateness of the current learning offer.

Purpose shifts leading to a review of the learning offer were more commonly observed when these resulted from my participation in students' reflection. The following excerpt shows the final moments of a discussion on GA Task 7, where students were given a letter in a cell and had to create a journey corresponding to the operations involved in a target expression. Joseph made the required journey and I asked him to reflect on its stages.

Excerpt 6.1.3.3   Lesson 16

PB: [*Pointing to stage ⑤ of Joseph's journey*].

What does this five mean, the five?

Joseph: The five? $6c$ plus 6 division by 2 plus 3.

PB: And this piece of journey over here? [*Pointing to the connector between ⑤ and ⑥ and looking at the other students.*] Come on. Pay attention.

Joseph: Division by 3.

PB: [*Pointing to stage ⑥*]

And the final result here, six ?

Joseph: $6c$ plus 6 division by 2 plus 3 division by 3.

PB: What is that called [*pointing to the target expression*]?

Joseph: Journey!

PB: No. Ok, the journey is all this that we did [*pointing to the journey*],...

Dwayne: Expression! Expression! [*The others clapped.*]

In this episode, I was asking questions to help Joseph and his classmates to reflect on the meaning behind the journey picture. For each question, Joseph came up with an anticipated response. During this time, I was continuously looking at his classmates for signs of confirmation, which I seem to have been getting. Dwayne's reply that the string of symbols was called an "expression" and the apparent approval of his peers (clapping), seemed to confirm to me that the other students had, like Joseph, made sense of my learning offer. In fact, right after this episode, I asked the students to tackle Task 7 on their own because I was confident they had developed the mathematics required for the activity they had been reflecting on.

I coded such a shift of teaching purpose as **Reflect>Review** because my focus changed from (1) *encouraging* and facilitating reflection about the mathematical problem at hand to (2) *reviewing* the appropriateness of the learning offer.

6.1.4 Teaching purpose Shift 4: From Evaluation to Intention

I detected an intermediary stage between (1) my purpose of building a model of students' mental processes and reviewing the learning offer and (2) anticipating and engaging in the didactic process. This intermediary stage was a quick but significant "check-in" to the mathematics I intended to help my students learn. This often involved matching students' representations with my own mathematics, an association which sometimes occurred after reviewing the learning offer.

The following excerpt shows what happened after Dwayne solved a problem in GA Task 8 (similar to Task 7 but with no letters involved), where R_3C_4 contained 12 and the target expression was $12 \times 2 - 6$.

Excerpt 6.1.4.1 Lesson 9

PB: [*Referring to Dwayne's GA journey.*]
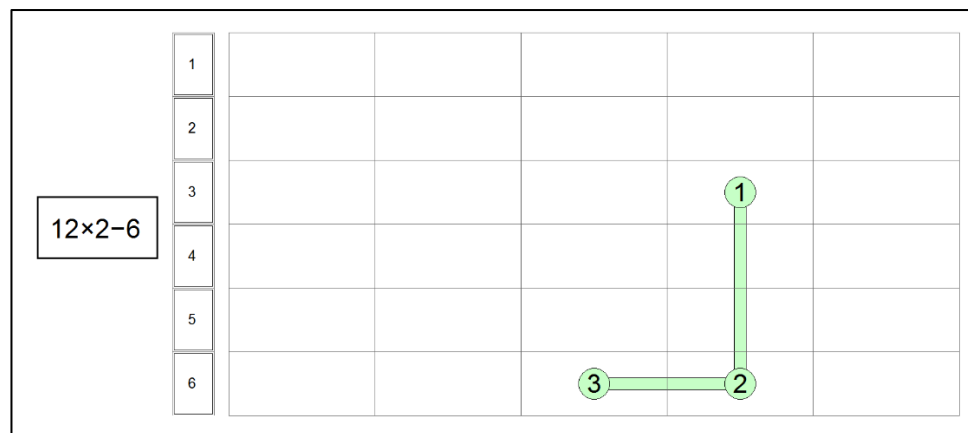
So, explain to us what you did, mate. Explain to us what you have and what you did.

Dwayne: Uh, I have the 12 [*pointing to the target expression*], I have the 12 times the 2 and then minus the 6. So, 12 was over here [①], the number...

PB: Yes.

Dwayne: And then, over here [②], the 12, it [*the computer*] told me to do it times 2, and the times is down here [②].

PB: OK. And how…?

Dwayne: [*Pointing to the row number of R_3.*] Since 3 times 2 becomes 6…

PB: Well done! These are very important things to remember. Here [*pointing to the row number of R_3*]. Have you heard what he said? It was a very important thing Dwayne said. He said [*pointing from ① to ②*] it becomes times 2 because here [*pointing to the row number of R_3*], the 3, the 3-times table, times 2 becomes 6 [*pointing to the row number of R_6*].

When I observed that Dwayne had completed the challenge, I formed a model of his conceptual process which matched what I was expecting students to learn. Dwayne linked the operations $\times 2$ and -6 to movements (and hence relationships) of numbers in the grid. However, I needed to review whether students were developing the mathematical concepts I was trying to teach. So, I asked Dwayne some probing questions to elaborate on his thoughts while working out the journey: "Explain to us what you did…"; "And how…?".

Dwayne's reasoning seemed to be "in harmony" with my mathematics and my immediate reaction was to exclaim, "Well done!", congratulating Dwayne for constructing mathematical concepts as I intended. Simultaneously, I was congratulating myself for a favourable self-review of my learning offer. I went on to repeat what he said to the others, "Have you heard what he said?…" I coded such instances as **Review>Assoc** because my focus changed from (1) *reviewing* the learning offer to (2) *associating* my mathematics with students' representations.

Often, however, such associations seemed to occur directly after building a model of students' conceptualisations. In the following representative episode of such a shift, I was helping students appreciate the inverse property of addition. Prior to this lesson, some students evaluated expressions like $497 + 2014 - 2014$ by computing addition and then subtraction, seemingly unaware of additive inverse. GA offers an interesting metaphor for such an inverse process. When the cell representing a number was added and subtracted by the same number, it underwent two inverse movements giving the impression of a number going somewhere and then coming back. In the following excerpt Omar had just completed a challenge of GA Task 10 on the IWB which involved the inverse operation $-1 + 1$.

Excerpt 6.1.4.2   Lesson 8

PB: [*Pointing to the last two stages of the journey, from R_1C_2 to R_1C_1 and then back to R_1C_2*]…Then we did minus 1 plus 1 and we came back to same place again. Why do you think we have come back to the same place again?
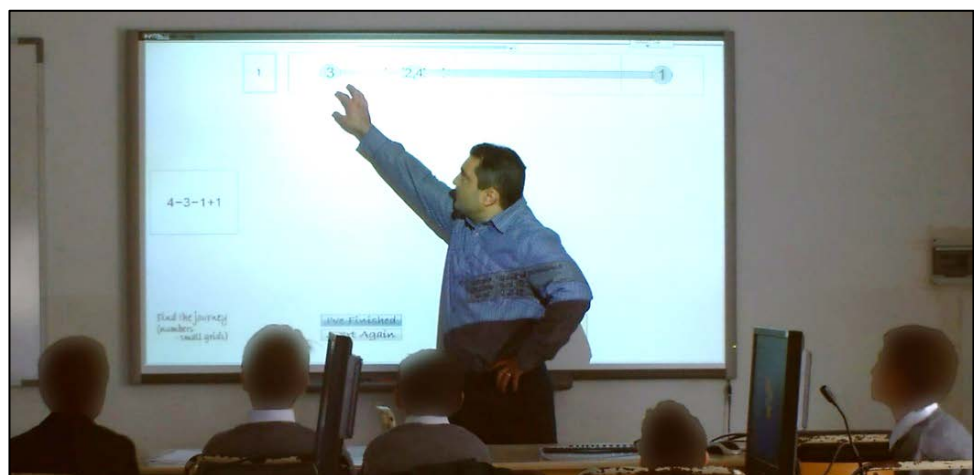
Omar:	[*Pointing to the target expression.*] Because you have it there written in the grid.
PB:	OK, because you have the grid but why did the grid force me to go back to the same place again? [*Dwayne wiggling his finger*] ...
Dwayne:	Sir, can I tell you?
PB:	Come on, tell me.
Dwayne:	Because the sum, it tells you that you do minus 1 and you put it there [*pointing to R_1C_2 and making a gesture of moving to the left*] and then again… the 1, you add it again [*making a rotating movement with his fingers*], it comes to the same place.
PB:	So if I have a number, say, 100, and I do minus 1 plus 1 what will it become?
Many students:	Hundred.

In contrast to Dwayne, Omar did not seem to have been focusing on what I intended students to see when I made those left and right movements. Dwayne made two gestures to explain the inverse operation:

- a right to left movement to show the actual movement of the cell, and

- a rotating movement with both of his index fingers to show a sort of cycle where something re-assumes its original position.

Here, I formed a model that his thinking was similar to mine but instead of praising him, I continued to expand the discussion. I coded such instances as **Model>Assoc** because my purpose changed from (1) forming a *model* of students' conceptual processes to (2) *associating* my mathematics with students' representations.

Following this, the discussion developed to include computations with awkward numbers which necessitated the use of additive inverse. Students worked out the value of expressions like $231789 - 9993 + 9993$, as shown in Figure 6.1.4.1 where Jordan was explaining his reasoning to me and to his classmates.

Figure 6.1.4.1   Jordan explaining the inverse property of addition
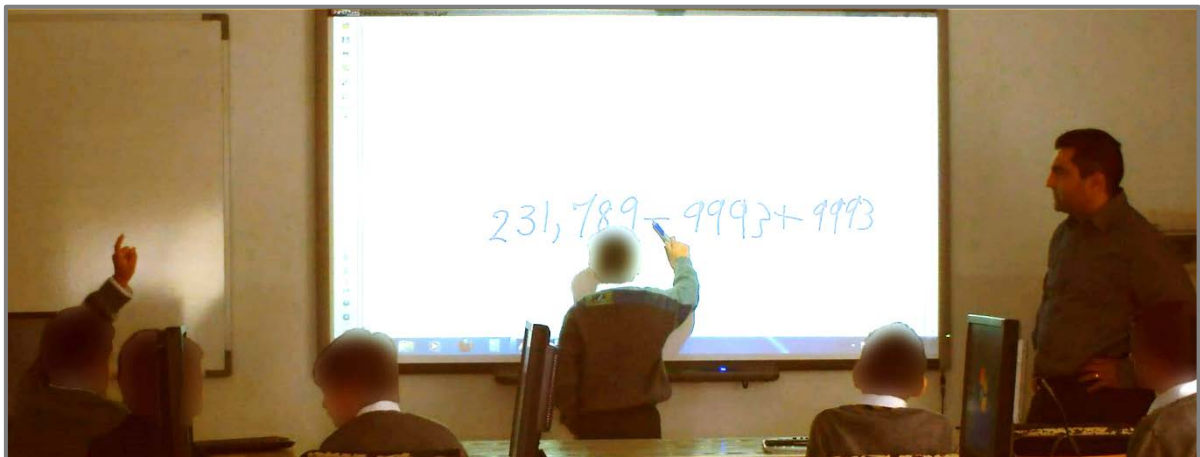
As I often do in my lessons, I used what Pask and Scott (1972) call *teach-back* where students are required to teach back what they learnt. In Figure 6.1.4.1, I was observing Jordan's teach-back in order to build a model of his mathematics and comparing it to my own. Sometimes, I needed to adapt my mathematics to fit in the students' representations. In the following episode, my adaptation occurred due to the latter. I was discussing the concepts of unknowns and variables represented by letters on the GA grid.

Excerpt 6.1.4.3   Lesson 14

      PB: [*Referring to the letters in the GA grid.*] That c: What can it be? [*Nods towards Jordan who had raised his hand.*] Tell me.
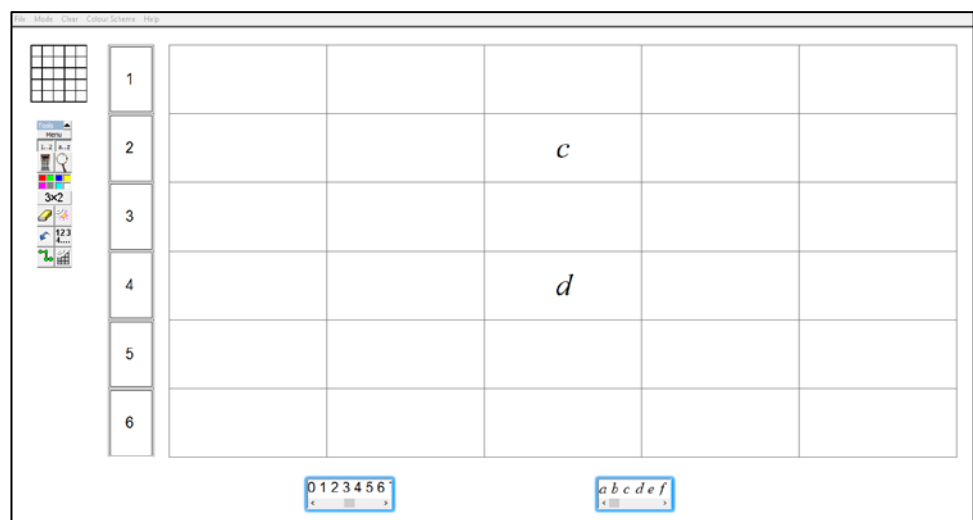
Jordan: It means, you can, um, you do top division by 2.

PB: I do top division by 2 where? Top [*points to R_1C_3*] division by 2 so that it comes here [*points to R_2C_3*]? Or...?

Jordan: Where there is... From d to c.

PB: [*Points from R_4C_3 to R_2C_3*]. From d to c you do division by 2.

Jordan: Uh-huh [*agrees*].

PB: … So what *(Jordan)* is telling me is that … if I make a 20 here [*inserts the number 20 from the number menu to R_4C_3 containing d then points to R_2C_3*]…

Tony: Ten.

PB: There [*pointing to R_2C_3*] should be a…?

Tony and
Jordan: Ten.

The response I anticipated for my initial question "That c: What can it be?" was "Any multiple of 2." Jordan's take on the question was not simply *what c* could be but *how* it was related to the other letter in the grid, d. When he said that division by 2 was happening "from d to c" two things happened:

(i) I reviewed the current learning offer (discussing possible values for letters on the grid) and saw that Jordan's conceptual processes were ahead of it (he seemed to be thinking about the *relationship* between those letters).

(ii) I adapted, or rather upgraded, my current mathematical thinking to include Jordan's more advanced thinking.

I coded such instances as **Review>Adapt** because my purpose shifted from (1) *reviewing* the learning offer to (2) *adapting* my mathematics to include students' conceptualisations. Subsequently, this adaptation enabled me to associate Jordan's mathematics with the mathematics I had intended to teach.

Similar to the Model>Assoc shift, forming a model of my students' mathematical reasoning was often followed by an adaptation of my own mathematics for it to incorporate students' representations. This required flexibility on my part because

usually I had to temporarily abandon my current train of thought. Such was the case I present in the following episode, where I had been discussing that subtracting a larger number from a smaller number resulted in the negative of the number that results when subtracting the smaller number from the larger number.

Excerpt 6.1.4.4   Lesson 19

> PB: … If I do 4 [*moving 4 to the left*] minus 2, minus 4, minus 6 [*stopping on R_2C_1*]…
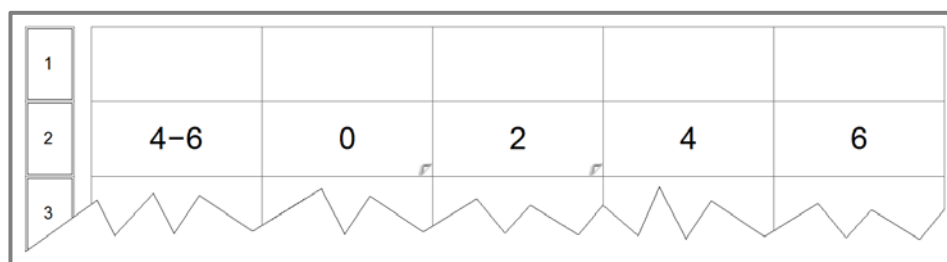
> Tony: [*Talking while PB is talking*] Negative 2.
>
> PB: If I do 4 minus 6...?
>
> Tony: Negative 2.
>
> PB: [*Pointing to Tony*] How did you work it out so that it resulted in negative 2? What did you do mentally?
>
> Tony: Minus 6.
>
> PB: How?
>
> Tony: I did 4 minus 6.
>
> PB: Yes, and how did you do it? What did you do mentally to get negative 2?
>
> Tony: Uh, because when I did the minus I did not stop at 0. I continued to walk backwards.
>
> PB: [*Looking at the others*] Good? So 4 minus 4 resulted in 0 and then you still got [*counting on his fingers*] minus 5, minus 6.
>
> Tony: Two.
>
> PB: So, when I did the minus, when I got to 0 I still had two left to reduce [*making a gesture of moving downwards in steps*] and I got to minus 1, minus 2.

The first half of this exchange shows that I considered my purpose to be developing a model of Tony's method. A representation he offered was "walking backwards", something which is enabled and encouraged in GA, i.e. representing mathematical operations with movement. In doing so, he provided me with an alternative method of subtracting a larger number: to count down to zero and count the remaining amount to the negative number. I modified my mathematics ($4 - 6 = -(6 - 4) = -2$) to make room for Tony's ($4 - 6 = 4 - 4 - 2 = -2$) and elaborated on it for the sake of Tony's classmates. I coded such a shift of teaching purpose as **Model>Adapt** because my purpose changed immediately from (1) forming a *model* of students' intellectual processes to (2) adapting my mathematics to incorporate students' mathematics.

6.1.5　Making Sense of Changes of Teacher Focus and Purpose

Changes in focus were not intermittent but continual, suggesting to-and-fro oscillations between forces attracting my attention during lessons. These changes reminded me of Dewey (1902) who regarded the educative process as the interaction between two forces: the needs of the child (learner) and the social aims of adults (mathematics) as discussed in Section 2.5.3. Consequently, I saw my teaching as being the negotiation between these two forces (Figure 6.1.5.1).

Figure 6.1.5.1　Teaching as negotiation between mathematics and learner
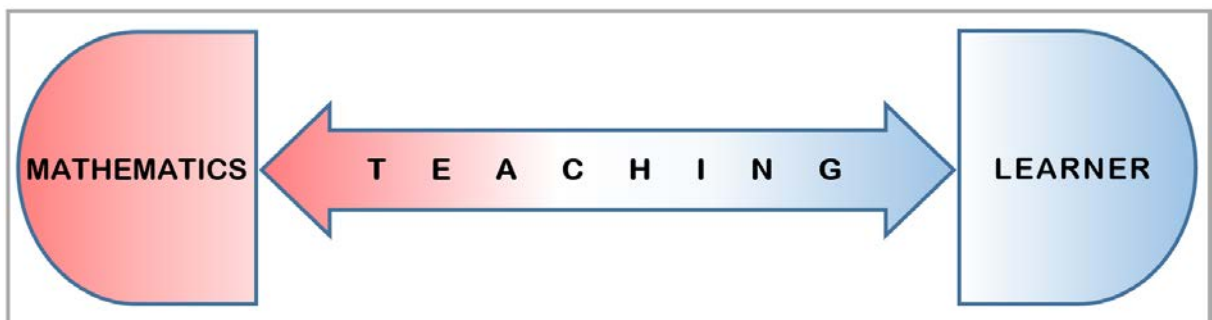
The changes of teaching purpose discussed in the above sections were my attempts to attend to the learners without abandoning my mathematics, and to attend to my mathematics without abandoning the learners. The left-right arrow indicates the dual

dimension of my teaching: rightwards showing where I was concerned with thinking about and creating a learning environment for the students and leftwards showing where I was concerned with creating models of students' mathematics and letting it inform my own mathematics, i.e. the mathematics I intended to teach. As I argued in Chapter 1, some CT frameworks emphasise the rightwards arrow, arguing that teachers are providers of learning offers (Steinbring, 1998) which students interpret subjectively through hypothetical learning trajectories (Simon, 1995). Other CT frameworks emphasise the leftwards arrow, arguing that main aim of constructivist teachers is to learn about the knowledge of their students and to attempt to combine it with their own (Steffe, 1991).

When analysing my teaching, I concluded that both directions of the negotiation arrow were crucial for CT:

(i)   If I were to teach mathematics without intending to learn about my students (rightwards arrow only), I would have been representing knowledge irrespective of what and whether learning was occurring. Such a teaching approach would have been deprived of any sensitivity to *constructivist* notions.

(ii)   If I were to learn about my students without intending to teach mathematics (leftwards arrow only), I would have been acting as a researcher not a teacher. Although learning might still have occurred, it would not have been facilitated by my actions and hence I would not have been *teaching*.

Both scenarios would have meant that CT did not occur. Thus, I avoided scenarios (i) and (ii) by continually changing my purpose in the lessons in order to negotiate pathways between my mathematics and my learners. I regarded these changes of purpose as the basic elements on which I could build a working model of CT.

Figure 6.1.5.2 shows how this CT framework was developed. Following Saldana's (2009) codes-to-theory method of analysing the data (Chapter 4), I started by coding the data and subdividing codes into Purpose Shifts 1-4 discussed above. I labelled these subcategories of shifts respectively as Mathematics-to-Negotiation (M-N),

Negotiation-to-Learner (N-L), Learner-to-Negotiation (L-N), and Negotiation-to-Mathematics (N-M) as shown in the diagram.

Figure 6.1.5.2   Development of the CT framework from codes to theory
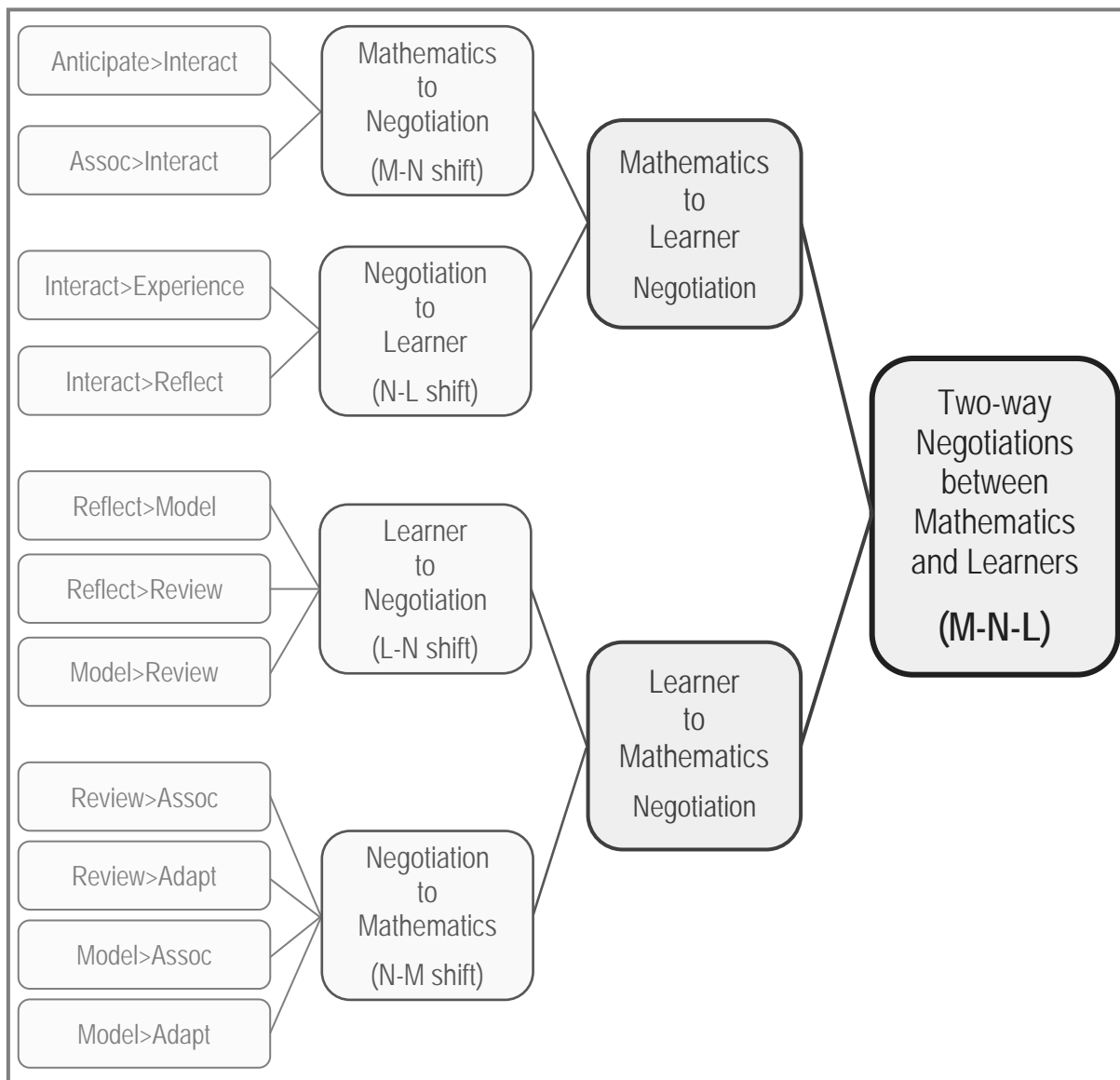
M-N and N-L shifts were further categorised as Mathematics-to-Learner Negotiation while L-N and N-M shifts were categorised as Learner-to-Mathematics Negotiation. These two categories of negotiation formed the core of the Mathematics-Negotiation-Learner conceptual framework which is discussed in the next section.

6.2 The Mathematics-Negotiation-Learner Framework

The Mathematics-Negotiation-Learner (M-N-L) framework was developed as a synthesis of the shifts of teaching purpose in my attempts to engage in CT. A detailed discussion of this framework may be found in Borg, Hewitt, and Jones (2016a, 2016b). The development of the M-N-L framework was an exercise in mapping these shifts in the broader Curriculum–Teaching–Learner construct envisaged by Dewey (1902). I changed "curriculum" to "mathematics" to focus on the subject matter relevant to my research, and changed "teaching" to "negotiation" to emphasise the dual role of teachers when taking into account both subject matter and learners' conceptualisations.

This mapping was inspired by constructivist literature which emphasised the negotiation of ideas directed towards the learner (e.g. Simon, 1995) and negotiation directed towards the teacher (e.g. Steffe, 1991). I refer to the former as forward-negotiation and to the latter as backward-negotiation. In this section, I describe the M-N-L framework, starting by defining some key terms.

6.2.1 Definitions of Key Terms used in the Negotiation Process

Against the backdrop of RC discussed in Chapter 2, I define four key terms I use to describe the negotiation process in the M-N-L framework:

(i) "**Anticipation**" is the act of expecting an outcome with reference to personal experiences. Teachers' anticipation of didactic processes in the classroom are built on their experiences of the students in that classroom and also of past students having similar characteristics to those of the current students.

(ii) "**Interaction**" is the act of representing internal concepts through various external expressions (such as utterances, actions, pictures, and symbols) which are meant to be interpreted by other persons with reference to their own

personal experiential realities. When interacting with students, teachers offer representations which students may relate to because they are linked to their own experiences. In the interaction process, teachers maintain an openness to students' own representations.

(iii) "**Mathematics *of* Students**" (MoS) is the mathematics inside students' minds which is only accessible to students themselves. This is a narrower definition than that given by Steffe (2016) in his commentary to Borg et al. (2016a), which encompasses any student construction 'that could be thought of as mathematical simply because they are human beings' (p. 77). As explained in Borg et al. (2016b), my definition of MoS encompasses the mathematical concepts which are subjectively constructed in students' minds but which are represented by and interpreted from conventional mathematical language and notation (e.g. "multiply three by two", 3×2). Teachers may form second-order experiential models (Steffe et al., 1983; Steffe & Ulrich, 2013) of MoS by observing students' representations and hypothesising about the mental processes they form during mathematical experiences. RC teachers hold that such experiences are unique and subjective to individual students, and so is MoS. However, teachers may hypothesise about possible thinking patterns of their students by making inferences from the models they create of individual students' mathematics.

(iv) "**Mathematics *for* Students**" (MfS) is the mathematics teachers intend to teach to a particular group of students. In the case of school teachers, as was the case in my research, MfS is likely to form part of the school's curriculum. RC teachers believe that no knowledge can exist outside the mind of the knower (Glasersfeld, 1984; Lerman, 1989) and hence they hold that the mathematical topics included in their syllabus is their own conceptualisations of those topics. Hence, MfS is a selection of this mathematics which teachers deem relevant to their students and which they seek to represent in order to create environments conducive to students' constructions of MoS. Teachers make decisions about the suitability of MfS by referring to models they build of MoS.

6.2.2 Description of the M-N-L framework

M-N-L portrays teachers as negotiators between their mathematics and their learners (hence the dashes in M-N-L). This negotiation involves creating:

(i) an environment conducive to students' constructions of mathematics, and

(ii) models of MoS with which they determine the suitability of MfS.

In M-N-L, teachers' reflections on MoS deepen and enrich their own mathematics, concurring with Freire (1998) that teachers are also learners and learners are also teachers. Figure 6.2.2.1 illustrates the dynamics of such a negotiation, where negotiations (i) and (ii) listed above, are respectively presented as a forward-negotiation road (rightwards arrows) and a backward-negotiation road (leftwards arrows) connecting mathematics and learner.

Figure 6.2.2.1   The Mathematics-Negotiation-Learner Framework
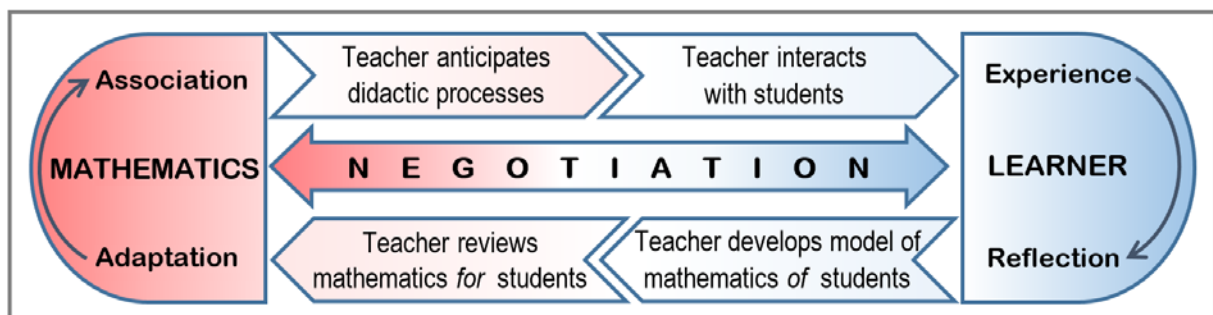
As discussed in Section 6.1.5, M-N-L builds on Dewey's (1902) Curriculum-Teaching-Learner construct by using the metaphor of a two-road link representing teachers' negotiations in the classroom. Each of these roads consist of two stages corresponding to shifts of teaching purpose. The following is a description of the stages starting from the upper left-hand arrow going from mathematics to learner:

1. Forward-Negotiation Road

The forward-negotiation road involves teachers' actions aimed at presenting a mathematical learning offer for the students:

(i) Starting with the left arrow, teachers build on models of MoS to *anticipate* possible didactic processes which may help the current students to partake in MfS, an element of teachers' mathematics. Simon (1995) calls this a hypothetical learning trajectory since the teacher has no means of knowing in advance the actual didactic processes that may occur.

(ii) Then, teachers *interact* with students by making representations of MfS intended for the creation of MoS. Teachers write, draw, and demonstrate mathematical representations (pictures and symbols), make utterances and gestures to express and represent thought processes, set up goal-oriented activities and discussions, ask questions to stimulate communication and reflection, answer students' questions, give feedback, and elaborate on students' actions and statements in order to help them think more deeply about their constructions of MoS. Thus, "interaction" includes teacher exposition and teacher-coordinated activities which, as argued in Chapter 1, are not necessarily teacher-centred or non-constructivist.

2. Learner

The "Learner" section of Figure 6.2.2.1 shows how this forward-negotiation road leads to students' *experience* of mathematical representations on which teachers encourage students to *reflect*. Students become learners by making abstract conceptualisations through an interplay of experience and reflection. This is reminiscent of Kolb's (1984) experiential learning construct but with an emphasis on teachers' actions after observing students' mathematical representations.

3. Backward-Negotiation Road

(i) The Learner-to-Mathematics arrow on the right shows that teachers build hypothetical, experiential *models* of MoS. Steffe emphasises that the

constructivist teacher must be a keen observer in order 'to construct the mathematical knowledge of his or her students'. (Steffe, personal communication, October 7, 2015). Since MoS is unique and subjective to individual students, teachers cannot assume a homogeneity of MoS among all the students in the class. However, models of individual MoS may serve teachers to make inferences about the possibility of similar MoS for the rest of the class. Teachers need to decide whether the model they create of individual MoS can serve as an indication of what a typical student in the class may be construing at a particular moment in the lesson.

(ii) The arrow that follows on the left shows that teachers use these models of MoS to *review* their intended MfS. This means that MoS serves as an assessment of whether the learning offer presented along the forward-negotiation road was appropriate for the students.

Each activity involved in the backward-negotiation road is a learning experience for teachers.

4. Mathematics

The mathematics end of the M-N-L diagram shows that teachers revisit their own mathematics, to decide whether MoS can be *associated* with it either directly or by going through some kind of *adaptation*. This enables teachers to go back to their students with a renewed MfS and a revised anticipation of the didactic processes with which they start constructing a new forward-negotiation road.

6.2.3 Application of M-N-L to Characterise and Analyse Constructivist Teaching

I consider teachers' deliberate shifts of purpose between the four elements described above to be an indication of CT. Although some exponents of CT (e.g., Steffe et al., 1983; Steffe, 1991) tend to focus almost exclusively on teachers' learning from their students (backward-negotiation road), I argue that teachers are duty-bound to teach and cannot study the learning of students without intervening to facilitate it. Conversely,

I argue that constructivist teachers cannot just present learning offers and, like Steinbring (1998), claim that mathematics teaching is an autonomous system. CT is dependent on students' feedback and on teachers' actions based on that feedback.

Teachers' ability to balance forward- and backward-negotiations is key to sustaining regular transitions from one stage to another of the M-N-L cycle, thus maintaining the two roads which bring together mathematics and learners. CT may be analysed by studying teachers' transitions between successive stages of the M-N-L cycle through changes of focus in their teaching. The extent to which teachers manage to complete M-N-L cycles may be an indication of their success to engage in CT. This means when teachers fail to complete M-N-L cycles it may indicate a failure to engage in CT. Constructivist teachers may momentarily create roadblocks in the negotiation process which hinders the changes of teaching purpose necessary to complete M-N-L cycles. In Section 6.5, using the M-N-L framework, I describe two such roadblocks. Being aware of these roadblocks may help teachers be more vigilant in striving to engage in CT.

In the following sections, I use the M-N-L framework to analyse my teaching. I start by presenting descriptive statistics of my teaching during the plenary discussions of the GA lessons with respect to the M-N-L cycle.

6.3 Overall Descriptive Statistics of the Shifts of Teaching Purpose

From 20 double lessons, each 80 minutes long, there was 745 minutes of lesson time devoted to plenary discussions. This was usually taken up by roughly the first half of each of the double lessons. In the second half of the double lessons students worked on GA tasks in pairs on their computers. During this period, I supervised students' work (saw that students were doing the tasks they were supposed to be doing) and only intervened when I saw that students needed my assistance.

Charting complete M-N-L cycles when students were working alone would have been impractical because:

- although I was silently forming models of students' conceptualisations, I found it impossible to record such instances while maintaining supervision,

- the learning offers were not being initiated by me (forward-negotiation) but by GA, and

- backward-negotiation possible during my interventions to help students was intermittent and rare since they usually relied on feedback from each other and from GA.

Hence, I decided to use the 745 minutes of plenary discussions with the students to map the diverse changes of teaching purpose as described in Section 6.1. Overall there were 1105 changes of purpose, each coded as described previously. Figure 6.3.1 shows the percentages of each of the four shifts occurring over the 20 lessons.

Figure 6.3.1   Overall percentages of the four shifts of teaching purpose

Negotiation->Mathematics
18.3%
(202 shifts)

Mathematics->Negotiation
23.7%
(262 shifts)

Learner->Negotiation
23.3%
(258 shifts)

Negotiation->Learner
34.7%
(383 shifts)
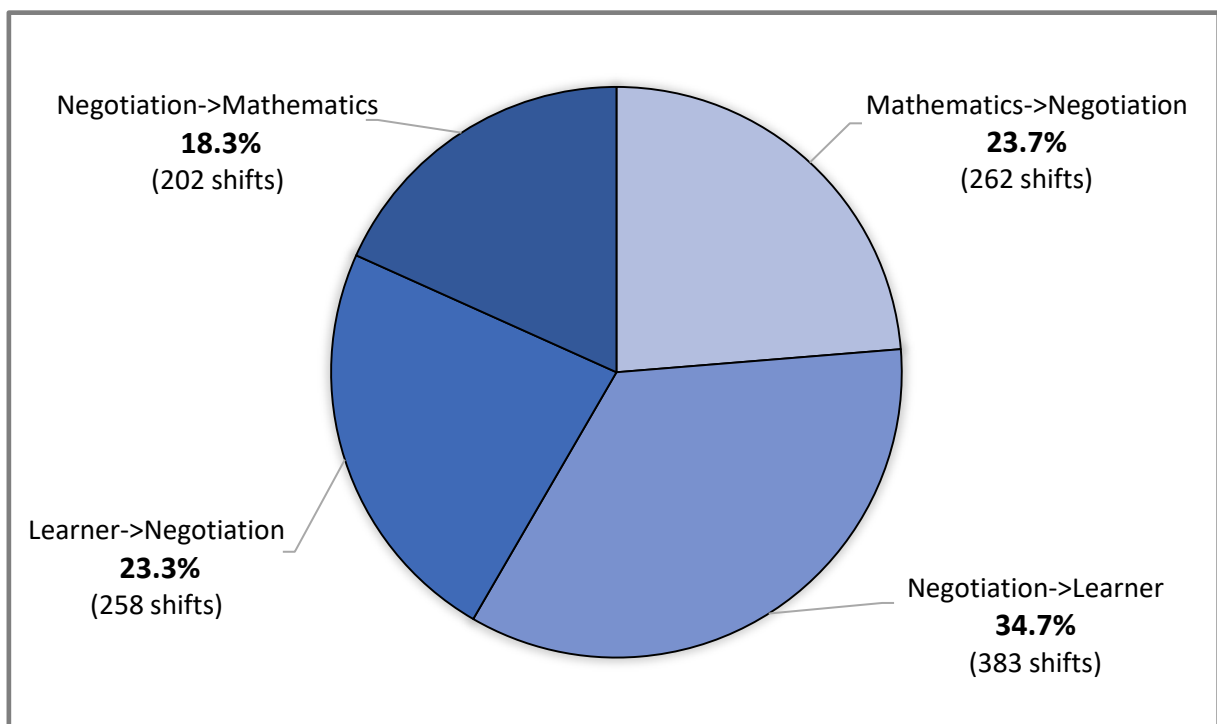
The following are my interpretations of this pie chart:

(i)   If each M-N-L cycle consisted only of four categories of shifts of purpose (one shift per category), and lessons could be divided into a whole number of M-N-L cycles, then one would expect this pie chart to be divided into 25% sectors. This was not the case as shown by the large amount of N-L shifts. A possible explanation could be that there were many instances where after an Interaction>Experience (N-L) shift, I re-interacted with students to help them reflect, entailing an Interaction>Reflection (N-L) shift. In such cases, there would be two N-L shifts for one M-N-L cycle. Although other shifts were occasionally observed to be repeated, repetitions were mostly pronounced in N-L shifts.

(ii)   There were 202 N-M shifts and data show that the number of complete M-N-L cycles was a bit less (c. 180), an average of one cycle per 4 minutes of plenary discussion. This adds to data patterns suggesting that:

- N-M shifts were only done after the other cycles were completed, and

- N-M shifts triggered a renewed M-N-L cycle (unless it was time to end the plenary discussion and students started working on their own).

It seems, therefore, that revisiting my mathematics with reference to models of students' conceptualisations (N-M shift) seems to have triggered and concluded M-N-L cycles. If the generation and completion of M-N-L cycles are taken to be indicative of CT, this would mean that N-M shifts are crucial for CT. This implies that CT hinges on the willingness and capacity of teachers to let students' representations inform and sometimes challenge their own mathematical knowledge and the way they envisage interacting with students to help them develop mathematical concepts.

(iii)   The percentages of M-N and L-N shifts were roughly equal, a pattern replicated in individual lessons. Thus, there seems to have been a good balance between instances where I initiated a negotiation from my mathematics and instances where I initiated a negotiation from the learners, the former to provide a learning offer, and the latter to learn from my observations of students' representations.

Figure 6.3.2 shows lesson-by-lesson descriptive statistics substantiating these interpretations. I am presenting here the percentages of each shift in each of the 20 double lessons.

Figure 6.3.2   Lesson-by-lesson comparison of percentages of shifts of teaching purpose
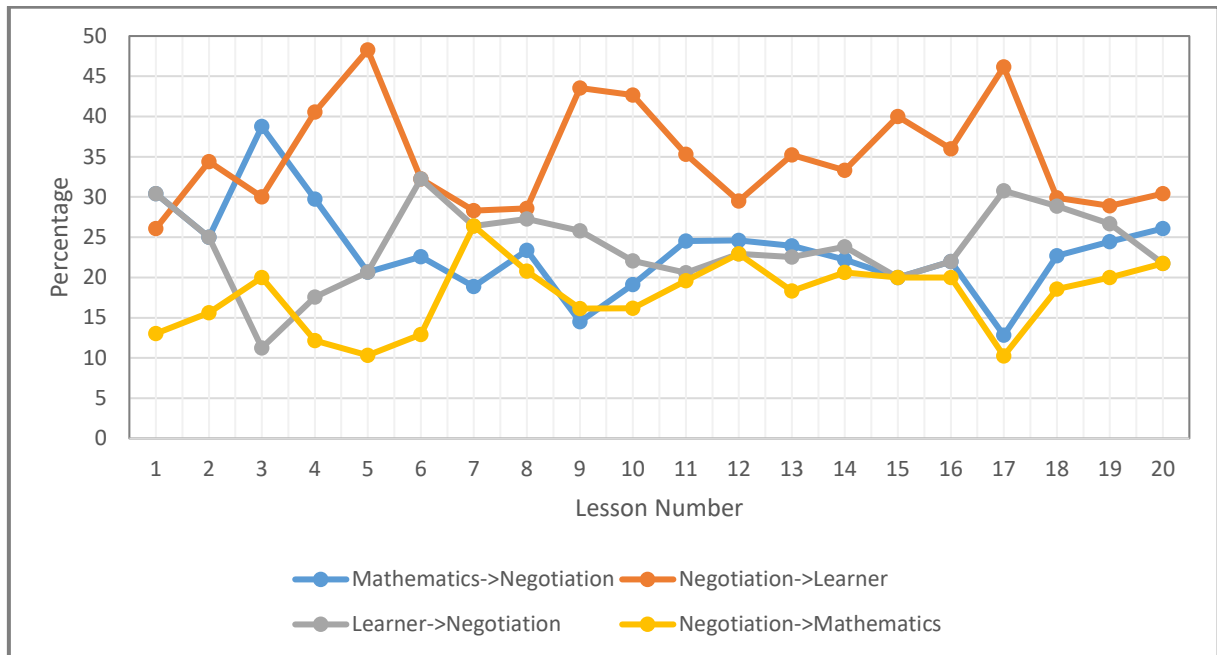
Figure 6.3.2 shows that the percentages presented in Figure 6.3.1 reflected the trend of the percentages of the shifts in individual lessons:

(i) Except for lessons 1, 3, and 6 the percentage of N-L shifts was the highest, due to repeated N-L shifts in M-N-L cycles;

(ii) M-N and L-N percentages varied only slightly in 8 lessons and were equal in the remainder, exhibiting a balance between starting a forward-negotiation process and starting a backward-negotiation process.

(iii) Increases and decreases in the percentages of N-L shifts were matched respectively by decreases and increases in percentages of the N-M shifts, due to the equality of percentages in the other two shifts as shown in (ii);

(iv) Except for 3 lessons, the number of N-M shifts was the least since these were very rarely repeated in single M-N-L cycles.

In the following section, I present a descriptive analysis of an episode where I was involved in two successive M-N-L cycles. This will contribute to my discussion of the M-N-L framework and how M-N-L cycles could be an indication of CT.

6.4   Complete M-N-L Cycles in the Grid Algebra Lessons

This section is devoted to giving a deeper look at an episode involving two successive cycles. The episode occurred two minutes into Lesson 13 in which students were introduced to the use of letters in the GA grid. I started the lesson by presenting students with the use of letters as unknowns and moved on to the use of letters as variables.

As often happened when introducing a new GA task, the plenary discussion consisted mainly of demonstrations of mathematical representations and students' reflections on those demonstrations. This meant students' experiences were limited to what GA was showing on the screen. As earlier excerpts show, this was not the case in all class discussions and students usually played a more active role. However, it is during exposition that maintaining sensitivity to constructivist notions is most challenging because teachers may be inclined to pay more attention to the subject matter than learners' conceptual constructions. I chose this episode precisely to show that teachers can strive for CT even during exposition.

In this episode, a number of mathematical concepts were discussed, namely:

- multiples of 3,

- letters standing for numbers and values of numerical expressions, and

- a substitutive meaning of the equals sign (e.g. $d = 24$).

For easier reference during the analysis, paragraph symbols are inserted in strategic places of the excerpt.

Excerpt 6.4.1   Lesson 13

(§1)

PB: …I am going to place the number 18 here. [*Drags* 18 *to R₃C₂ - #1.*]

… It will let me do it.

Joseph: Because it is in the 3-times table.

…

PB: Well done! Well done! Now, if I picked a letter at random from here [*picks the letter d and drags it to R₃C₄*] and I place it over here *[Joseph raises his hand]*, that d, first of all, what is it symbolising? [*Pointing at Joseph…*] Come, let's see.

Joseph: Uh, what it is, what the answer should be. Like if you do 18 plus 3 plus 3, that is plus 6, which becomes 24, it is d equals 24.

(§2)

PB: [*Nodding…*] All right, so what we're saying here is that d is, like, the answer of when [*points to respective cells*] 18 makes plus 3 plus 3. In fact, if you do like this [*drags the* 18 *to R₃C₃ to obtain* $18 + 3$] and like this [*moves* $18 + 3$ *to R₃C₄ obtaining* $18 + 3 + 3$ *on the same cell as d*] – all right? – we see [*clicking on the cell to show alternately the expressions* $18 + 3 + 3$ *and d*] d here and [*choosing the magnifier icon*] if we see … with the magnifier here, it is telling me exactly [*pointing to Joseph - #2*] like you told me that [*pointing to d*] d [*points to equals sign*] is [*points to respective numbers*] 18 plus 3 plus 3. [*Clicks on the cell to alter the expression from* $d = 18 + 3 + 3$ *to*

$18 + 3 + 3 + 3 = d$.] If I alter here it will tell me that [*points*] 18 plus 3 plus 3 equals d.

#2
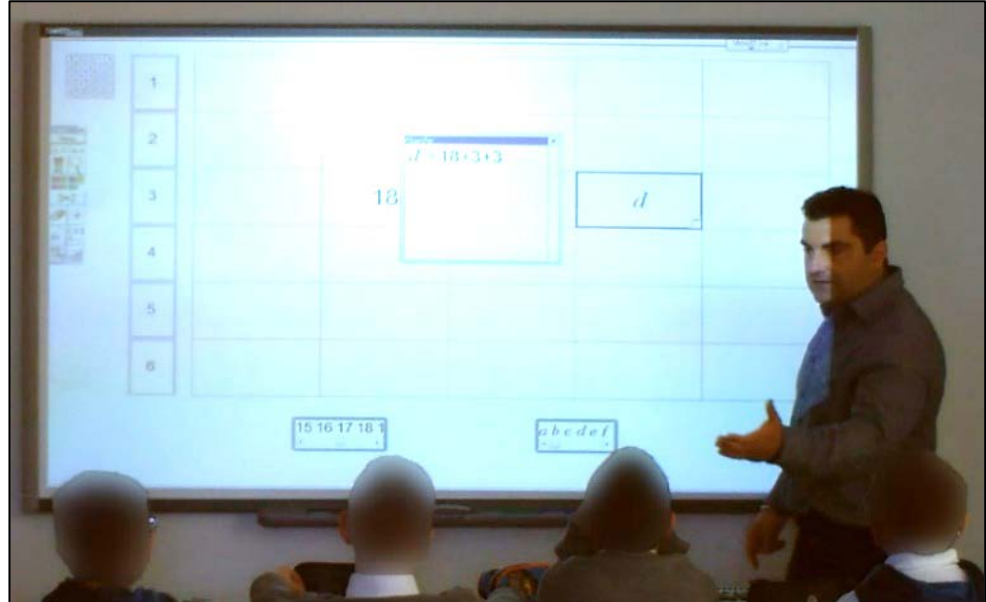
(§3)

PB: But if I want, instead of doing 18 plus 3 plus 3, I can, if I want to, erase here [*erases all expressions except* 18 *and* d] – OK? – I can just bring up [*pointing to the number menu*] that unique number that can be here [*the cell containing* d], a single number… What is the number?

Joseph: Twenty-four.

PB: Do we agree that it is 24?

Joseph: Yes [*the others nodding*].

(§4)

PB: Because we're in the 3-times table and we're doing plus 3 plus 3, all right? … I bring up the 24 … I'll pick the 24 from here [*drags* 24 *from the number menu to* R_3C_4 *containing* d] … And when I go with the magnifier there it is telling me d equals 24. … So, d equals 24 and [*clicks on the cell to alter the expression from* $d = 24$ *to* $24 = d$] 24 equals d…

Joseph: The same.

(§5)

> PB: … As such, we are not seeing an answer. When you say "answer" it's like you have done some calculation, some plus, minus…
>
> Joseph: 18 plus 3 plus 3.
>
> PB: We don't have any calculation, nothing, here. So now, I cannot quite say that "equals" is "answer". [*Jordan shaking his head.*] So what can I say that it means there [*pointing to $d = 24$ - #3*]?

#3
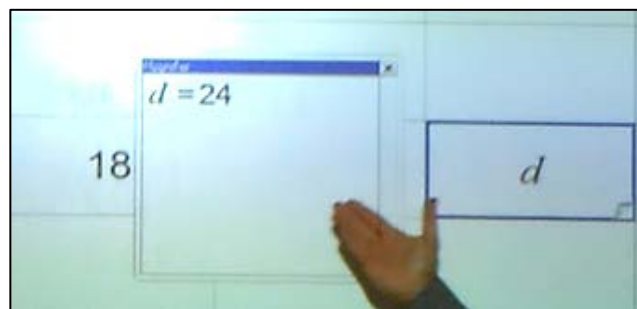
> The equals?
>
> Joseph: Equal to [*says it in English*].
>
> Dwayne: They are the same in size.

In this episode, the initial mathematics for students (MfS) was the appreciation of the difference between variables and unknowns. I anticipated that students were prepared to develop notions of letters as unknowns in the GA grid by referring to neighbouring cell values. This anticipation may be detected by phrases like "I am going to…", and "…it will let me" (**§1**).

With this anticipation in mind, I changed my focus (M-N shift) to start interacting with the students. This interaction started with 18 in R_3C_2, where I started asking students questions to help them reflect on that experience. Joseph immediately pointed out that this was accepted by GA because it was a multiple of 3. This was a cue for me that I could place a letter in the grid and I inserted d in a neighbouring cell (R_3C_4) and asked the students what that letter symbolised.

Here I shifted my focus to another purpose. From interacting with students to encouraging them to reflect on mathematical artefacts (N-L shift). This encouraged Joseph to suggest a meaning for d: "like if you do 18 plus 3 plus 3". Placing d in the neighbourhood of 18 (Figure 6.4.1) seemed to help Joseph interpret the symbol d, aided by the representation of its "container": the cell in the context of the grid. The interplay between conceptual interpretations and pictorial, symbolic, and kinaesthetic representations will be discussed in Chapter 7. It suffices for the moment to point out that Joseph's interpretation of the symbol d in association with the neighbouring cells and values is an example of Mercer's (2000) claim that symbols (like words) gain meaning from the company they keep.

Figure 6.4.1   Letter gaining meaning of from the company it keeps
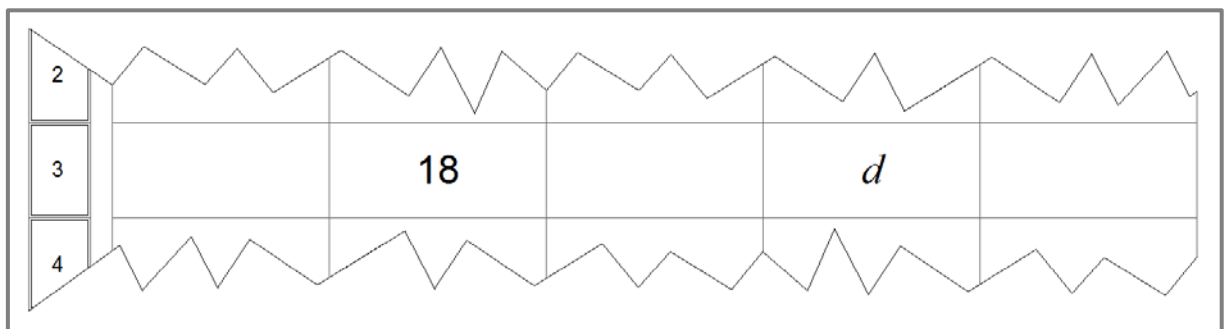
It was my turn to interpret Joseph's representations and I changed my focus from encouraging reflection to forming a model of his MoS (L-N shift). At first (**§2**), I confirmed aloud what Joseph seemed to be thinking: "…so what we're saying here is that…". I also made cell movements corresponding to Joseph's calculation of $18 + 3 + 3$ ending on the cell containing d, and used GA's magnifier to help the other students see that what Joseph seemed to be implying was that $d = 18 + 3 + 3$ or that $18 + 3 + 3 = d$.

This was followed by reviewing my MfS (**§3**), i.e. appreciating the circumstances making d an unknown. Joseph made valid mathematical statements but to him d signified the answer of a calculation, rather than a single fixed number. My focus changed again from reviewing the learning offer to associating MoS with my

mathematics (N-M shift). In order to do this, I had to adapt my idea of unknown as a fixed single number to accommodate Joseph's concept of unknown as "answer".

This shift prompted a new M-N-L cycle, with a renewed MfS: the appreciation of the connection between

- a letter as a single (unknown) number due to its being the value of an expression (Joseph's MoS) and

- a letter as a single fixed (unknown) number due to its neighbourhood in the GA grid (the original MfS).

I anticipated how students could make these connections as I started off the new M-N-L cycle.

I shifted my focus from anticipating these connections to interacting with students to help them develop mathematical appreciations of these connections (M-N shift). I erased all the expressions, except 18 and d (Figure 6.4.1) and hoped students would make the link between what was in the cell R_3C_4 a moment earlier ($18 + 3 + 3$) and the single number that GA could accept in the same cell. Students were competent in assigning single numbers in GA cells, so I figured the empty cell R_3C_4 could invoke the single number 24 in the minds of the students due to its position in relation to 18.

I asked students what was the "unique number that can be" in R_3C_4. Here my purpose had changed from interacting by erasing the expression $18 + 3 + 3$ to encouraging students to reflect on the single number which could be entered in that empty cell (N-L shift). Unsurprisingly, Joseph himself mentioned the number 24. He had already thought about it and even mentioned it earlier (end of **§1**) because he was thinking of it as the answer to $18 + 3 + 3$. Nevertheless, I wanted to orient students' thinking (Glasersfeld, 1991b) towards thinking of d as being 24 without the need to think of it as the answer to a calculation. So during the experience-reflection stage, I confirmed Joseph's statement by dragging 24 into the cell containing d (**§4**) and proceeded to help students to observe and consider the mathematical statement $d = 24$ which was enabled by the magnifier.

I knew that some students still found difficulty in conceptualising the equals sign (ES) unless it followed a computation or preceded an answer. So, during the reflection exercise, I focused on the meaning of ES in $d = 24$ (**§5**). When I asked what d "equals" 24 meant, Joseph explained himself by saying in English "equal *to*". His change from "equals" to the more exact "equal to" and his inclusion and emphasis of the preposition "to" gave ES a more a relational meaning. Dwayne immediately picked up on this and gave the anticipated response: "They are the same in size."

Dwayne and Joseph's feedback made me change my focus from helping students to reflect on the learning offer to forming a model of these students' MoS (L-N shift). I confirmed Dwayne's response, and repeated his statement in other words. I also said "Good", indicating a favourable review of Dwayne's statement and simultaneously of the effectiveness of my learning offer. In accepting that $d = 24$ meant d *has the same size of* 24, Dwayne and possibly Joseph, seemed to have constructed notions about a letter that could stand for a constant unknown irrespective of whether that constant was the answer of a calculation.

Once again, my purpose changed from reviewing the outcome of the learning offer to reflecting on my mathematics, i.e. my interpretation of $d = 24$ (N-M shift). I knew that the neighbouring 18 meant that d *could not be anything but* 24. This was a concept included in the original MfS, including the idea that, without any other numbers in the grid, d would be a *variable* multiple of 3 and hence the statement $d = 24$ would be viable if it were interpreted as one possible equality from various possibilities such as: $d = 21, d = 24, d = 27,$ etc. This prompted the onset a new M-N-L cycle in which I intended to help students to develop the notion of d as a variable.

The following table summarises how these two M-N-L cycles occurred by mapping each event to the respective teaching purpose.

Table 6.4.1   Teacher's forward- and backward-negotiation roads

Mathematics ⇐	Negotiation	⇒ Learner	
The notion of unknown represented by a letter in a GA environment.	Teacher anticipates students will appreciate the notion of unknown when this is contrasted with a variable.	Teacher interacts by placing 18 in R_3C_2 and d in R_3C_4. He asks students what this letter may be symbolising.	Joseph says that 18 is possible because d is a multiple of 3. Then he says that d is the answer of an operation done on 18.
The "answer" of a calculation may be thought of as an unknown. This holds also when the calculation is not meant to be expressed as a single number, e.g. $n = 3 + \sqrt{2}$	Teacher reviews the original MfS and seeks a way to incorporate the notion of an unknown with/within the notion of "answer".	Teacher builds an unexpected model of MoS concerning the letter d: the "answer" to a computation.	
	Teacher anticipates that students will link the notion of "answer" and unknown if they can see a particular example with the help of GA.	Teacher interacts by using Joseph's explanation and shows that d may be seen as the "answer" of $18 + 3 + 3$. He asks the students to tell him which number he could drag in that cell from the number menu.	Joseph says this number is 24 and the teacher dragged it in the cell in which it was accepted. Teacher helps students reflect on the statement $d = 24$. Joseph and Dwayne elaborate on the meaning of ES.
Teacher associates students' interpretations of d with his notion of d as variable.	Teacher reviews MfS. Dwayne and Joseph seem to interpret d as being *equal to* a constant.	Teacher builds a model of Joseph's and Dwayne's meaning for ES as "the same in size".	

Table 6.4.1 shows the continual and fast toing and froing between my mathematics and my students as I strived to engage in CT. Keeping in mind the MfS I intended to teach and the MoS of individual and groups of students was a challenging process fraught with the risk of failing to make valid learning offers or to attend to students' conceptual needs. In the following section, I discuss moments where I failed to maintain one of the negotiation roads between mathematics and learners.

6.5 Roadblocks between Mathematics and Learners

Analysing lesson videos against the M-N-L conceptual framework identified two ways in which a teacher could fail to engage in CT by creating a barrier in one of the two negotiation roads between the subject matter (mathematics) and the learners. Such a barrier could be created:

(i) on the learner side, blocking the backward-negotiation path, or

(ii) on the mathematics side, blocking the forward-negotiation path.

Figure 6.5.0   Barriers in the negotiation roads linking mathematics and learners
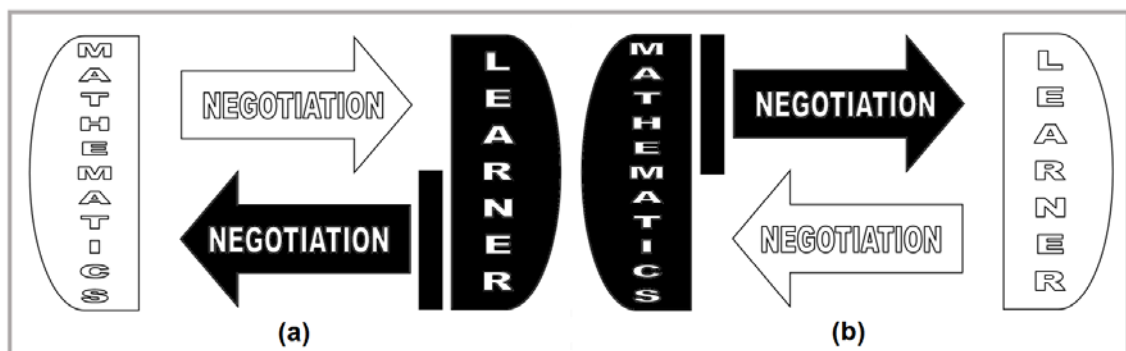
The first roadblock (Figure 6.5.0a) occurs when teachers fail to create models of the students' learning. This originates from teachers' exclusive focus on mathematics which alienates them from developing models of possible students' conceptualisations. I have discovered five types of such roadblocks in my lessons and I discuss these presently. The second roadblock (Figure 6.5.0b) occurs when teachers do not intend to anticipate how students might interpret learning offers. I did not identify any such barriers in the lessons but the M-N-L structure led me to hypothesise that such a barrier might exist.

In the following subsections, I present lesson excerpts to briefly discuss five types of backward-negotiation (Figure 6.5.0a) barriers which were identified during the analysis of the GA lessons. The first two barriers were quite common and, for these, I chose excerpts of what I believe to be typical situations where such roadblocks occurred. The other three barriers occurred only once and hence only one excerpt was available to illustrate and discuss them.

6.5.1 Backward-Negotiation Block Type 1: Failure to Elaborate on Students' Responses

There were instances where I observed myself failing to elaborate on a student's response, making it impossible to associate students' representations with my mathematics. I created one such barrier in Lesson 10 while demonstrating and discussing GA Task 5. A cell with a number was given and another cell was highlighted. Students had to enter three expressions equivalent to the journey that the cell with the given number had to make to end up on the highlighted cell. I encouraged students to attempt multi-step journeys rather than a single-step, one-operation journey.

Excerpt 6.5.1.1   Lesson 10

Omar: [*Pointing to* 5 *in #1…]* Five minus 1 [*moving his finger to the left*] plus three [*moving his finger to the right towards the highlighted cell*].

#1
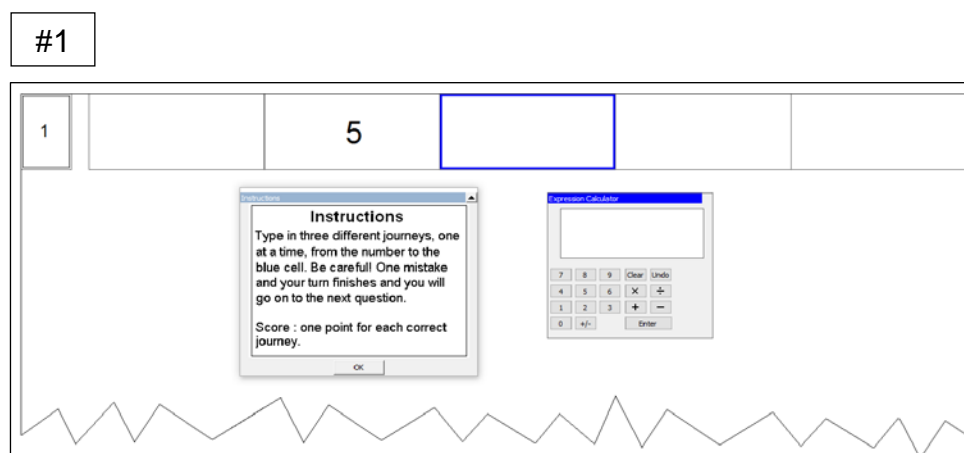
[*PB entered Omar's expression in the calculator.*]

Dwayne: Sir, you can't. [*At the same moment GA showed a no-entry symbol.*]

PB: 5 minus 1 plus 3, I don't know where you ended up. [*Dwayne said something in Omar's ear and made a symbol of* 2 *with his fingers*].

Dwayne: Sir…[*while PB was speaking*].

PB: So, [*presses button*] enter. Look what's going to happen to me. [*GA showed a dialogue box like that of #2*].

#2

> **Results**   ×
>
> This is an incorrect expression
>
> You get 0 out of a possible 3 points
>
> Try the next puzzle
>
> `Next Question`
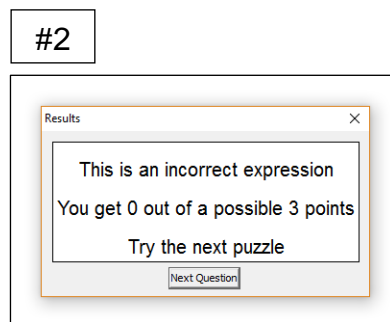
Incorrect. Zero. That's it.

For some reason, I was too impatient to stop and reflect on what seemed to me a mistake and hence failed to capitalise on Omar's apparent misconception to help him and others to make suitable interpretations. I forgot Dewey's (1916) claim that giving children the opportunity to learn from their mistakes is a requirement for active learning. I coded such barriers as **xL-xN_NoElab>NoAssoc** because it was a Learner-to-Negotiation barrier (xL-xN) where failure to elaborate a student's representations entailed failure to associate MoS with my mathematics.

6.5.2   Backward-Negotiation Block Type 2: Failure to Ask Questions

This backward-negotiation block originated from my failure to ask students what they thought about a situation. Once again, this entailed a failure to capitalise on students' mathematical representations. Unlike Type 1, this was not a failure to elaborate on a student's response, but a failure to *ask* questions.

A typical episode where this barrier occurred happened well into Lesson 11. A discussion about inverse operations was being facilitated by the GA *Run* mode where students moved numbers from one cell to another and back in order to obtain expressions with inverse operations. In this episode, Jordan was going to drag 4 from R_2C_2 to R_6C_2 and he was guessing the operation required for that movement.

Excerpt 6.5.2.1   Lesson 11

PB: [*Addressing Jordan*], what do you think will happen from 4 to 12?
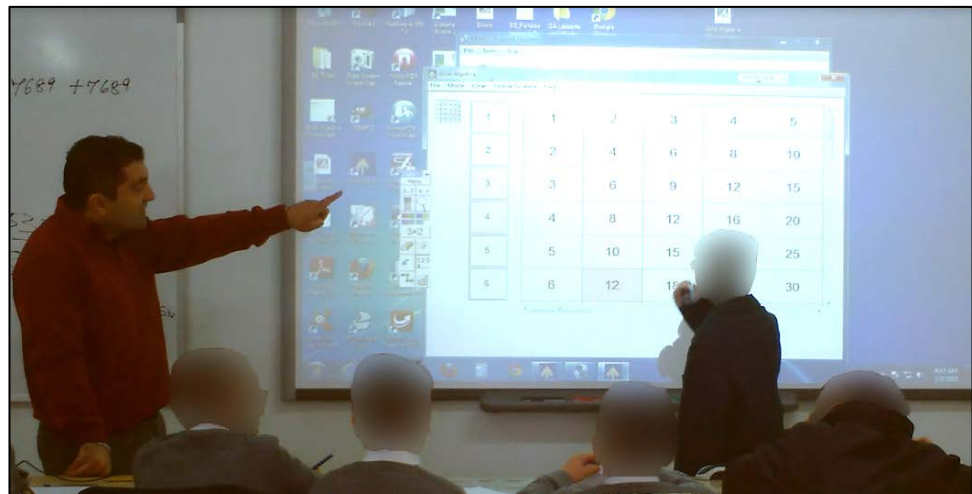
Jordan: Um, I think…

PB: If you don't know you can ask your friends to help you.

Jordan: …times 4.

PB: 4? Come on [*nodding towards the board*], try it out. Let's see. [*Jordan moves the cell and GA showed 4×3*]. Times 3, mate, to make it 12. Right? [*Jordan seemed focused on continuing the activity rather than seeing why 4×4 was wrong.*]

Jordan was not usually one to make such mistakes and it would have been beneficial for me to learn why he said 4×4 instead of 4×3. My failure to ask Jordan why he thought it was 4×4 led to a poor (hypothetical) model of his mathematical interpretation (MoS). The most significant failure here was that I just showed Jordan that his answer was unacceptable. Glasersfeld (1991b) warned constructivist teachers against telling students that their answer is wrong because it could actually be right if the problem is seen from their perspective. Telling Jordan to check whether the dragging of the cell resulted in his suggested expression and letting him realise that GA writes something else (which I knew it would) was almost identical to simply telling him that his expression was wrong without first trying to see how he interpreted my question. Such incidents were coded **xL-xN_NoAsk>PoorModel** where failure to ask students to explain their representations led to a poor model of MoS.

The following three barriers occurred only once in the course of the lessons but they are still significant for learning about situations where CT may be compromised.

6.5.3 Backward-Negotiation Block Type 3: Stopping a Student from Starting a Discussion

One of the three one-off backward-negotiation blocks occurred a few minutes into Lesson 1, when I was showing students what numbers could be allowed to exist in the first row of the GA grid by scrolling along the different numbers of the first row. I had forgot to set GA to exclude 0 and negative integers which were not part of the intended MfS.

Excerpt 6.5.3.1 Lesson 1

PB: …Over there we are going a bit further away as well. [*Scrolls a bit backwards and GA showed* 0 *and some negative numbers. Scrolls quickly back to* 1].

I am interested from 1.

Joseph: Or from minus 1.

PB: From 1. But I'm interested from 1. Do not take notice that we did, that there was 0 as well. When the time comes we will do that as well. [*Continues with what he was originally doing.*]

Glasersfeld (1991b) says that constructivist teachers should 'try and build up a model of the particular student's own thinking' (p. 178). In the above episode I failed to do exactly that. I missed the opportunity

(i)   to learn about Joseph's MoS, and possibly to review MfS,

(ii)   to extend or modify Joseph's conceptual structures, and

(iii)   to foster this student's motivation (Steffe 1991) to talk and possibly learn about 0 and negative numbers.

This momentary roadblock between learner and negotiation originated from the nature of the comment of the learner (Joseph) which, at the moment, I felt as threatening the anticipated didactic process. I coded this roadblock as **xL-xN_Stop>NoAdapt** since stopping a student from changing the intended MfS led to a failure to adapt my mathematics to incorporate MoS. My awareness of this failure led me to adopt a different attitude further on in the same lesson. When the issue of negative numbers came up again, I dedicated some time to discuss the need for negative numbers.

6.5.4   Backward-Negotiation Block Type 4: Failure to Let Go of the Software to Focus on Students' Statements

This roadblock and the next occurred in Lesson 4 and were both due to my failure to let go of my focus on what the GA software would show and concentrate more on students' mathematical statements, the validity of which I only appreciated during the lesson video analysis. The first roadblock occurred when I was explaining a game intended to help students to become familiar with GA representations of multiple operations by means of cell movements. On the IWB there was 18 in R_3C_2. This was moved to R_3C_3 (shaded yellow) to obtain $18 + 3$. I was asking Jordan to tell me what R_3C_4 (shaded blue) would show when the yellow cell with $18 + 3$ was dragged onto it.

Excerpt 6.5.4.1   Lesson 4

> PB: …You need to tell me also when the $18 + 3$ [*making a circle around the cell with $18 + 3$*], I take it there [*pointing to the empty coloured cell*].
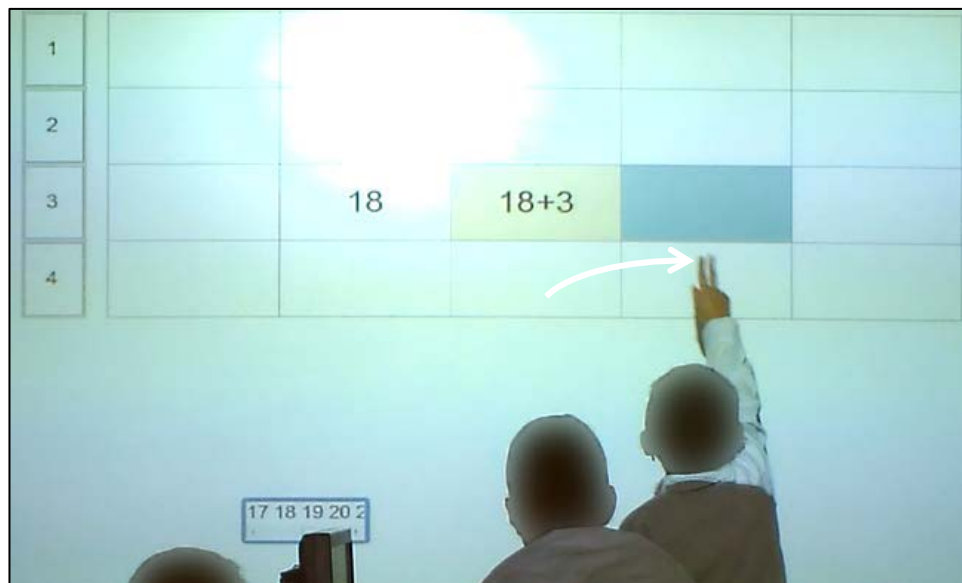
Jordan: So. Um, this *[pointing two fingers at R₃C₃ representing 18 + 3]*, if you put it there *[moving his hand to R₃C₄]*, a piece of it is going to be the same and then plus something…which I think it will be 6.

PB: So, I am going to move this *[making a circle and grabbing gesture on R₃C₃]*, all of this, I am going to move it here *[pointing to R₃C₄]*. Not the 18 *[pointing to 18]*, I am going to move this one *[pointing to 18 + 3]* here. So you said, it becomes…?

[Jordan seems to hesitate in answering my question. Joseph wanted to suggest the answer.]

PB: *[Nodding towards Joseph…]* Come on, what do you think.

Joseph: I think it is going to become 18 plus 3 plus 3.

Following this, I explained that the expression $18 + 3$ increased by 3 to become $18 + 3 + 3$. Jordan may not have learnt anything new from this because, as the excerpt shows, he had previously pointed two fingers to $18 + 3$ and moved his hand to the right, meaning that he was aware that it was $18 + 3$ that increased by 3. He thought that the final cell would show $18 + 6$. It is a particular feature of GA that if you stop dragging a cell, let go of the mouse, and then click and drag again to another cell, the latter would not show the same expression as if you clicked-and-dragged directly from the first cell. The students had to familiarise themselves with the fact that multiple movements of cells resulted in expressions with multiple operations. Here, I was too focused on what GA would show to appreciate that a similar software could have actually been programmed

to show $18 + 6$, a viable expression for the result of adding 3 twice to 18. In the analysis, I coded this roadblock as **xL-xN_Software>NoAssoc** because I let a feature of the software distract me from a possible association of MoS to my mathematics.

6.5.5   Backward-Negotiation Block Type 5: Failure to be Flexible by Focusing only on Software Representations

The other one-off L-N block occurred 14 minutes into Lesson 4, at the moment where students were guessing the correct numerical expression that resulted when a cell and its expression was moved horizontally or vertically. I was asking students what they would see if the cell with number 24 in R$_3$ (multiples of 3) was dragged up to R$_1$ (multiples of 1). The students had done similar work in Lesson 3, where we had an extensive discussion about a new notation for division, e.g. $\frac{24}{3}$ instead of $24 \div 3$. I am presenting an extended excerpt of the video surrounding the moment when this roadblock occurred to show how such a roadblock disturbs the negotiation process in the M-N-L cycle.

Excerpt 6.5.5.1   Lesson 4

§1

PB:　Tell me, Dwayne, how does 8 result over there [*pointing to the cell*].

Dwayne:　It results in 8, and you need to do [*PB moving his finger up and down the column*] the 24, you need to do it times,… wait…
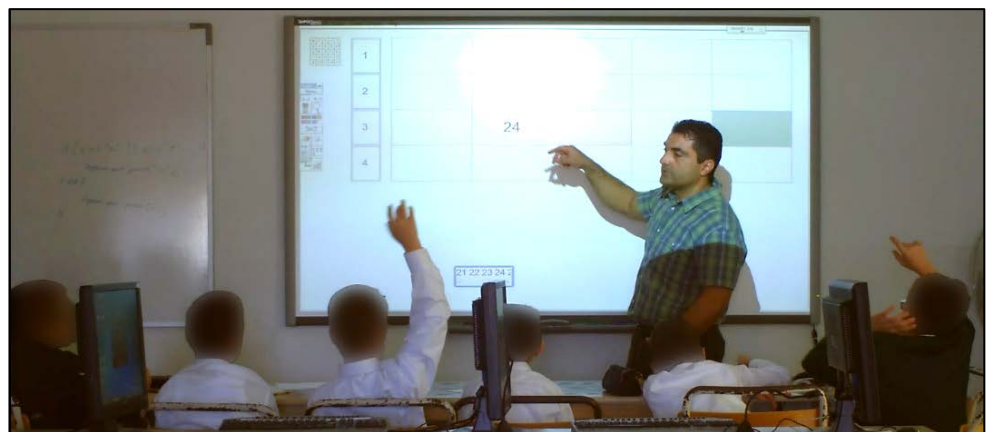
PB:　Upwards movement. You need to look over here [*pointing to row numbers on the left*]. What is happening to these?

Dwayne: 8. Division by 8.

PB: Division by 8, 24 division by 8 becomes 3. The 3 [*pointing to row number* 3 *and the rest of the row numbers*], what do you do to it so that it becomes 1? Division by…?

[*Dwayne doesn't answer and Tony raises his hand.*]

PB: [*Nodding towards Tony*]. Tell me. Let's see.

Tony: 8.

PB: The answer becomes 8 but I want to know what you do to 3 to get it 1.

Joseph: [*Speaking while PB is speaking, without raising his hands, and keeping his head rested on his hand…*] Division by 3.

§2

PB: [*Dan raises his hand. PB turns his attention to Dan and points at him.*] Dan.

Dan: 24 minus 16.

Dwayne: Division by 3.

PB: [*Keeping his attention on Dan*] OK, an upward or downward movement? …

Dan: I told you 24 minus 16…

PB: [*Sounding unconvinced…*] 24 minus 16. So I… Don't forget,…

Dan: It becomes 8.

PB: All right it becomes 8…

Dan: That's why.

§3

PB: But I asked you this question. Try to understand the question and answer the question. [*Moving his hand up and down C_2…*] The 24…

Dwayne: Now I know it [*raising his hand and wiggling his finger*].

PB: (The 24)…by how much does it need to be divided to get over there? [*Moving his hand up to R_1*].

Dwayne: Divided by 3.

PB: Division by 3. Because [*pointing to the row numbers on the left*] 3 division by 3 to become 1…

Dwayne: Uh-huh [*agrees*].

PB: …and so 24 division by 3 [*drags the cell*] to end up over there [*GA shows $\frac{24}{3}$*] and we see the fraction. We see that symbol that we talked about [*gesturing with his hands*] 24 over 3.

§1 shows my attempts to orient students' attention towards the *transformation* of 24 when its cell was moved from R_3 to R_1. It seemed they were not understanding because first Dwayne and then Tony were focusing on the value of the target cell in R_1, i.e. 8 rather than the operation of division by 3 required for 24 to become 8. Joseph, however, stated my anticipated response, "Division by 3," but his timing was not right. He spoke without permission and while I was speaking. Maybe for this reason I seem not to have acknowledged his response.

When Dan raised his hand to speak (**§2**) I gave him my full attention. He seemed to be answering the question, "How does 8 result over there?" (**§1** line 1), but he did not seem to me to be focusing on vertical cell movement. So when he said "24 minus 16" I immediately thought that he did not understand my question. Even when Dwayne said the anticipated response, "Division by 3," I kept my attention on Dan because I felt I needed to address Dan's apparent misinterpretation before endorsing Dwayne's answer. In earlier lessons, I had been discussing that subtraction and addition were horizontal movements on GA so I wanted Dan to realise his (apparent) mistake by asking him whether his "24 minus 16" was "an upward or downward movement" (**§2**).

Dan held fast to his original statement and argued that "24 minus 16 … becomes 8". I was so focused on the fact that GA showed a division symbol for an upward movement that I was not flexible enough to notice that Dan's statement could actually have been correct *even* if he was thinking of an upward movement. Had I not been focused on the parameters of the software, I could have connected Dan's $24 - 16$ with GA's $\frac{24}{3}$ by discussing the operation of subtracting $^2/_3$ of a quantity: $24 - \frac{2}{3}(24) = \frac{24}{3}$. I labelled this roadblock as **xL-xN_Software>NoFlex** because I let my focus on the software affect my flexibility to associate MoS with my original MfS.

Since the backward-negotiation road was blocked from the Learner side of M-N-L, I had to start another M-N-L cycle (**§3**) by anticipating students' interpretations of a reworded question about the movement and related operation of 24: "…by how much does it need to be divided…?" It seems that I was one step away from telling students the anticipated answer myself! This was the perfect timing for Dwayne to say again what he had stated when I was talking to Dan, "Divided by 3." I proceeded to endorse and accentuate his response by confirming it by moving the cell and finally obtaining the expected $\frac{24}{3}$.

The following table summarises how the M-N-L cycle got interrupted by this backward-negotiation block and how I was forced to start a new cycle.

Table 6.5.5.1   Teacher's failure to capitalize on students' responses
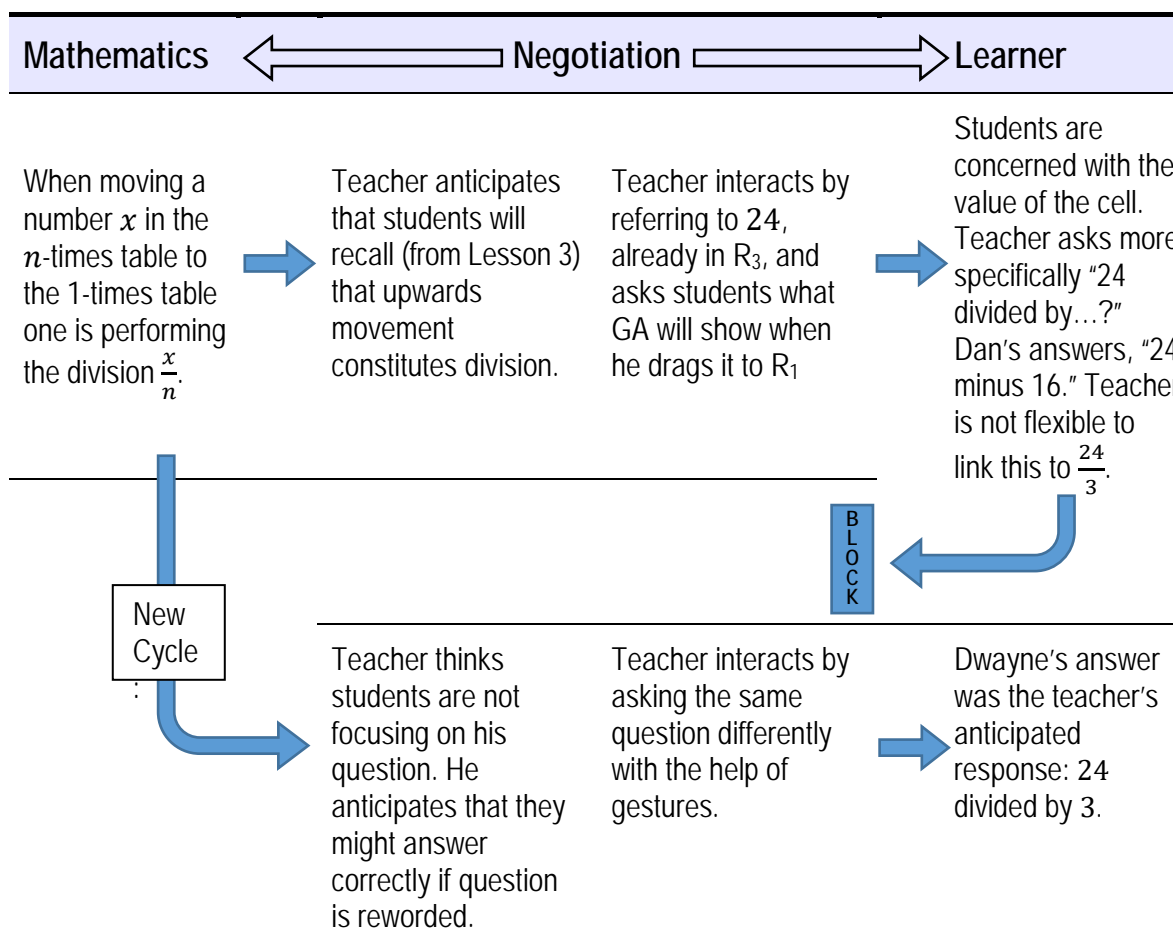
Table 6.5.5.1 shows how a backward-negotiation block usually contributed to an additional M-N and N-L shifts required to restart a new M-N-L cycle. These additional shifts contributed to the disproportionate percentages shown in the pie chart of Figure 6.3.1.

The next subsection gives an overview of the roadblocks discussed in Sections 6.5.1–6.5.4 and their frequency throughout the lessons.

6.5.6 An Overall Picture of the Backward-Negotiation Blocks

Overall, I identified a total of 23 backward-negotiation L-N blocks. Table 6.5.6.1 gives the frequencies of each type of block discussed above.

Table 6.5.6.1 Frequencies of backward-negotiation roadblocks divided by categories

Learner-Negotiation Roadblock	No Elab. > No Assoc.	No Ask > Poor Model	Stop > No Adapt.	Software > No Assoc.	Software > No Flex.
Frequency	7	13	1	1	1

The vast majority of these roadblocks (20/23) fell under the first two categories, i.e. NoElab>NoAssoc and NoAsk>PoorModel, meaning these were the areas where it was most challenging for me to sustain M-N-L cycles to engage in CT. These barriers occurred either because I was too hasty to take the time to elaborate on students' representations or because I did not assume a constructivist view of students' mistakes. In the case of the latter, there were two mental statements I could have made when I was getting responses other than those I was after:

(i) "They are not understanding me. I need to rephrase my question/explanation."

(ii) "Their mathematics may be correct. They just need to make the right connections."

I consider it a failure in my CT endeavours to have chosen (i) over (ii) in some of the roadblocks I mentioned. However, the most deliberate of roadblocks, Stop>NoAdapt, was a one-off and it was quickly rectified in the same lesson (negative numbers).

Although the numbers given above constitute a rough average of one roadblock per lesson, a lesson-by-lesson look at the frequencies of the roadblocks shows that there might have been an encouraging downwards trend (Figure 6.5.6.1).

Figure 6.5.6.1   Lesson-by-lesson frequencies of backward-negotiation roadblocks
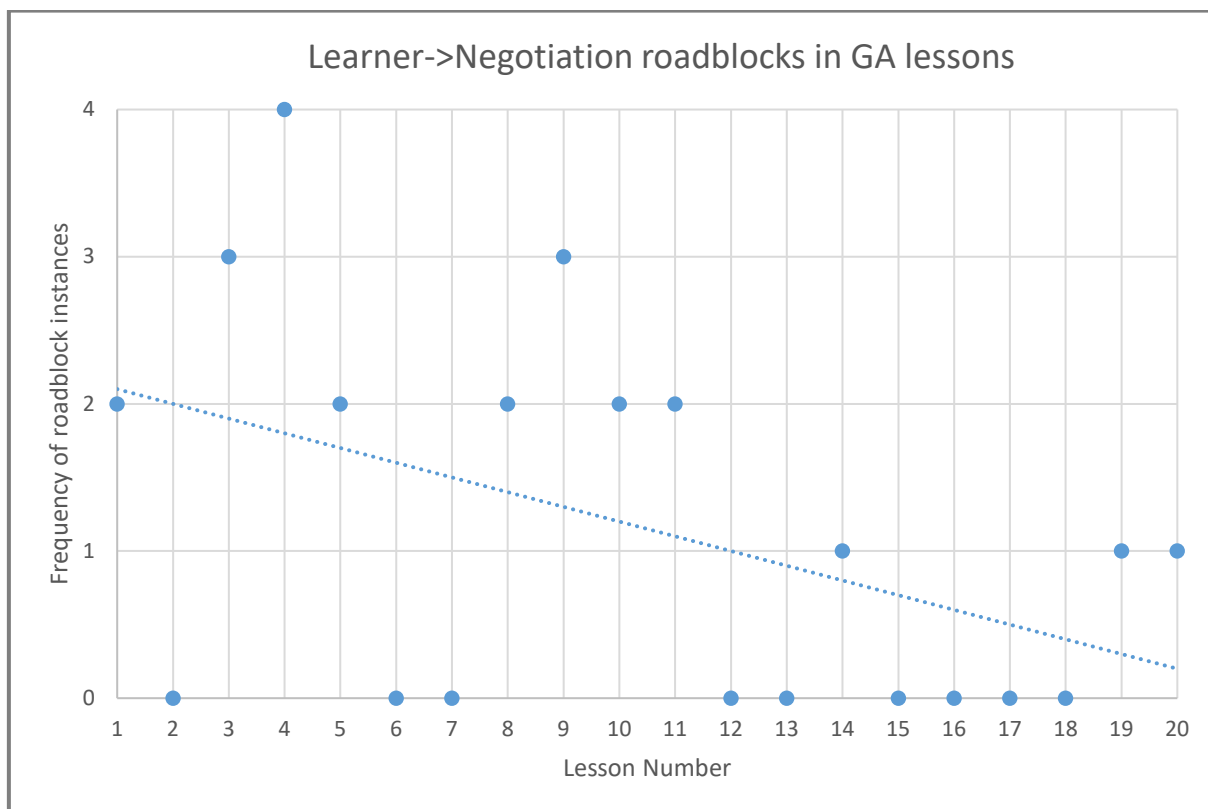
The downwards trend line[14] shows that the frequency of (backward-negotiation) roadblocks decreased along the scholastic year. In fact, their occurrence was more prominent in the first 10 lessons than in the second 10. Reasons for this trend may be several but I identified three:

(i)   I became more aware of my CT failures and conscious of how my constructivist beliefs should be reflected in my teaching;

(ii) In the first few lessons, I had to adopt mostly a teacher exposition approach to demonstrate GA's features, and hence I was more susceptible to focus exclusively on my mathematics at the expense of losing sight of the learners;

---

[14] Gradient and *y*-intercept of the line of best fit was calculated by the Least-Squares method.

(iii) Later plenary discussions were built around students' attempts of GA tasks and hence it was less difficult for me to maintain a focus on the learners.

6.6 Summary and Conclusion

I have discussed how changes of focus on my purpose during the lessons formed the basis for the development of the M-N-L framework. I have shown that this conceptual framework may serve to analyse how teachers may attempt to engage in CT. M-N-L may be found viable to analyse situations where teachers negotiate roads linking mathematics and learners, a process where teachers are also learners and learners are also teachers.

M-N-L helped me to identify possible roadblocks in the negotiation process that brings together learners and mathematics. Besides validating the usefulness of M-N-L to analyse CT, these roadblocks show that even a self-proclaimed constructivist teacher like myself is in danger of occasionally creating barriers which obstruct the subject-learner negotiation process. Such roadblocks originated from a strict focus on the subject matter at the expense of learners' needs. Being aware of these barriers makes us constructivist teachers more vigilant to avoid them as much as possible in our endeavours to engage in CT.

Furthermore, I do not believe M-N-L is exclusive to mathematics teaching and learning. Although the framework originated from a mathematics education research, it may be found viable to investigate CT of other curricular subjects. "Mathematics" in M-N-L may be changed into any taught subject.

One question remains: In what way did the students represent their mathematical interpretations to enable me to build experiential models of their conceptual structures as an integral part of the M-N-L cycles? This is a question which I attempt to address in Chapter 7, where I concentrate on whether and how CT helped the students to develop mathematical concepts with the help of GA.

Chapter 7

Analysis and Discussion of Students' Representations and Interpretations of Notation

7.0   Overview

In this chapter, I analyse data related to research questions 2(i)–(iii), which concern students' representations and interpretations of notation. Students' representations served me to develop models of their conceptual interpretations, i.e. MoS. This was a crucial stage in the backward-negotiation road of the M-N-L cycle discussed in Chapter 6. Table 7.0.1 shows the section titles of this chapter.

Table 7.0.1   Chapter 7 section titles

	Section Title	Page
7.1	Representations, Interpretations, and the CAPS Framework	238
7.2	Notation for Addition and Subtraction	241
7.3	Notation for Multiplication and Division	255
7.4	Notation for Unknowns and Variables	277
7.5	Notation for Equality	291
7.6	Students' CAPS Enabling M-N-L Cycles	308

In Section 7.1, I discuss the framework used for this analysis. In each of Sections 7.2–7.5, I analyse students' initial representations and interpretations of notation before the GA lessons, how GA helped students to enrich their representations of notation, and how this led to students' extending and developing conceptual interpretations of notation. I analyse data obtained from the interviews, computer screen activity capture of students' work, students' pen-and-paper work, and also lesson videos. My choice of data to be included as evidence of this analysis depends upon its being:

(i)   typical – representative of the student/s being discussed,

(ii)   convenient – clear and easy to follow by the reader, and

(iii)   comprehensive – some data from each student is included.

The following section includes a brief discussion of the analytic framework used.

7.1 Representations, Interpretations, and the CAPS Framework

In the literature, the words "representation" and "interpretation" take on many meanings and are sometimes interchanged. To avoid confusion, by "representation" I refer to any external manifestation of a mental schema, which, for convenience, will be referred to as "concept". "Conceptual interpretation", or "interpretation" is taken to mean the mental association of personal experiences with particular concepts. These experiences include perceptions of representations expressed by others and also by the self.

Two key theories discussed in Chapter 2 were:

(i) Kaput's (1991) Signifier/Signified theory about the connection of representations (signifiers) and interpretations (signified) and how these cause and are caused by each other;

(ii) Bruner's (1966) Enactive/Iconic/Symbolic theory about the three representation modes which supplement and sometimes replace verbal representation.

In Chapter 6, I analysed how I negotiated between mathematics and learners by making learning offers built upon models I formed of students' conceptual interpretations. In this chapter I focus on students' non-verbal representations, supplementing their speech, which I used to build those models. Hence, Bruner's (1966) theory was very significant. Since I was linking students' representations to their conceptual interpretations, Kaput's (1991) theory was also appropriate.

I found that the amalgamation of these constructs created something more than the sum of their parts. Signified concepts are interpretations of external representations manifested through signifier actions, pictures, and symbols. I combined these theories into a single framework I refer to as CAPS (Concept, Action, Picture, and Symbol), a detailed discussion of which is included in Borg and Hewitt (2015). Table 7.1.1 includes my definition of these four items as they are used in this framework, along with some examples of each item to illustrate these definitions.

Table 7.1.1   Definitions and examples of CAPS items
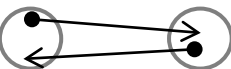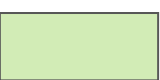
	Definition	Examples
Concept	Mental interpretation of experiences constituting a developing mental schema. Different people may establish a consensual domain about similar experiences where they agree about similarities in their personal mental schemata. Communication of a concept occurs through perception and manifestation of representations of this concept.	1. Balancing concept of equality. 2. Concept of additive inverse.
Action	Kinaesthetic representation of a concept including movements involved in activities, gestures, role-play, and virtual environments.	1. A gesture with both hands showing balance. 2. Step forward and backward showing inverse process.
Picture	Diagrammatic representation of a concept, drawn physically or virtually, including hand and computer-assisted shapes and figures. These pictures may be iconic such as examples 1 and 2 to represent concepts of balance and inverse repectively. They may also not be iconic, without any similar characteristics between the concept and the diagram, such as a rectangle that represents a number (3).	1. 2. 3.
Symbol	Formal (conventional) notational representation of a concept, including mathematical symbols and numbers and groups of these in expressions.	1. $4 = 4$ 2. $10 + 6 - 6$

Figure 7.1.1 shows how these four items are linked together. Action, picture, and symbol (APS) representations and conceptual interpretations originate from each other (continuous arrows in Figure 7.1.1). Simultaneous connections of multiple APS representations with conceptual interpretations form associations between representations themselves (broken arrows in Figure 7.1.1). In the context of constructivist teaching, teachers interpret students' APS representations to form experiential models of their conceptual interpretations. From a radical constructivist (RC) perspective, I consider any interpretation or representation to be subjective but individuals can agree on elements of these representations and interpretations to establish a consensual domain (Maturana & Varela, 1980; Glasersfeld, 1991b).

Figure 7.1.1   The CAPS framework
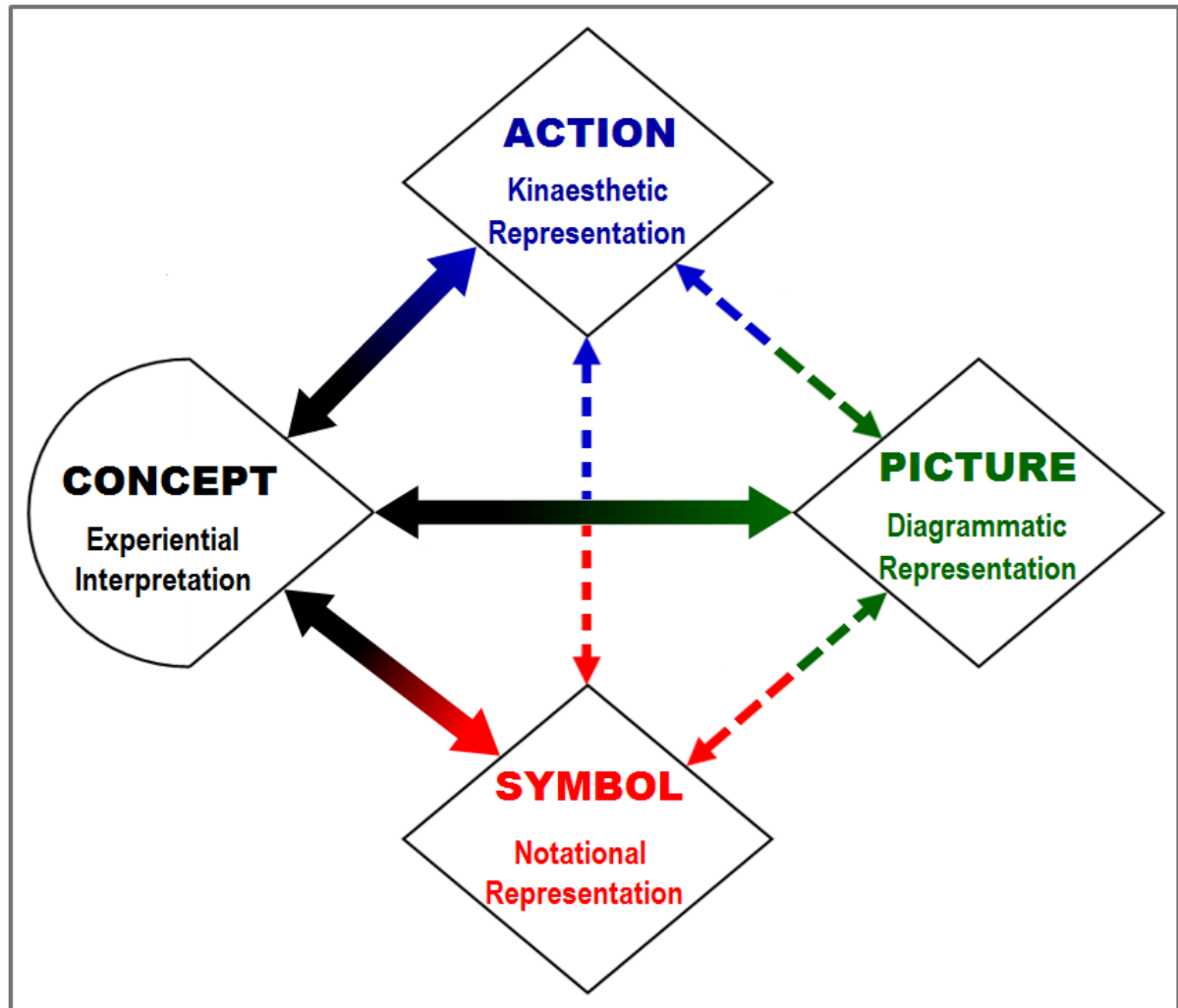
The investigation I discuss in this chapter focuses on students' conceptualisations of the notation of:

  (i)   addition and subtraction,
  (ii)   multiplication and division,
  (iii)   unknowns and variables, and
  (iv)   equality.

These are analysed respectively in Sections 7.2–5, where I discuss how students developed notions about notation with the help of GA. In these discussions, I will often make reference to APS representations by showing, in brackets, which of these three representations I was observing (e.g. finger counting (action), arrow (picture), written expression (symbol), etc.). I analyse how these representations helped me, as a

teaching researcher, to build models about students' interpretations of mathematical artefacts in order to form an assessment of students' conceptual development during the lessons and throughout the scholastic year. At the end of each section, using the CAPS framework, I give a summative analysis of students' representations and interpretations of notation as observed in the lessons and in the interviews.

7.2 Notation for Addition and Subtraction

In this section, I focus on students' representation and interpretation of the addition and subtraction notation. I discuss their understandings of the symbols $+$ and $-$ and how their concepts developed with the help of GA lessons.

7.2.1 Students' Initial Interpretations and Representations

Interview 1 revealed that all students demonstrated basic knowledge of how to add and subtract two numbers. They correctly worked out the sums and differences of two numbers less than 10. They also showed competence in carrying out vertical algorithms for adding or subtracting large numbers. This interview showed that for the symbols $+$ and $-$ students had developed concepts of a process.

Joseph, Dwayne, Tony, and Jordan worked out additions and subtractions like $4 + 3$ and $8 - 3$ mentally. Probing their interpretations of such expressions revealed that they gave answers through factual recall. For example, Dwayne said there was no particular process going on in his mind. Since these students used only verbal representations to express their understanding of addition and subtraction it was hard for me to hypothesise about their conceptual interpretations of such expressions besides factual recall.

It was a different matter for Omar and Dan, who made use of finger counting to work out additions and subtractions like $4 + 3$ and $8 - 3$. Omar's counting was explicit, extending successive fingers as he counted on or back (Figure 7.2.1.1a). Dan's use of his fingers was hardly noticeable because he counted by touching successive fingers to his chest (Figure 7.2.1.1b).

Figure 7.2.1.1 The action of finger counting
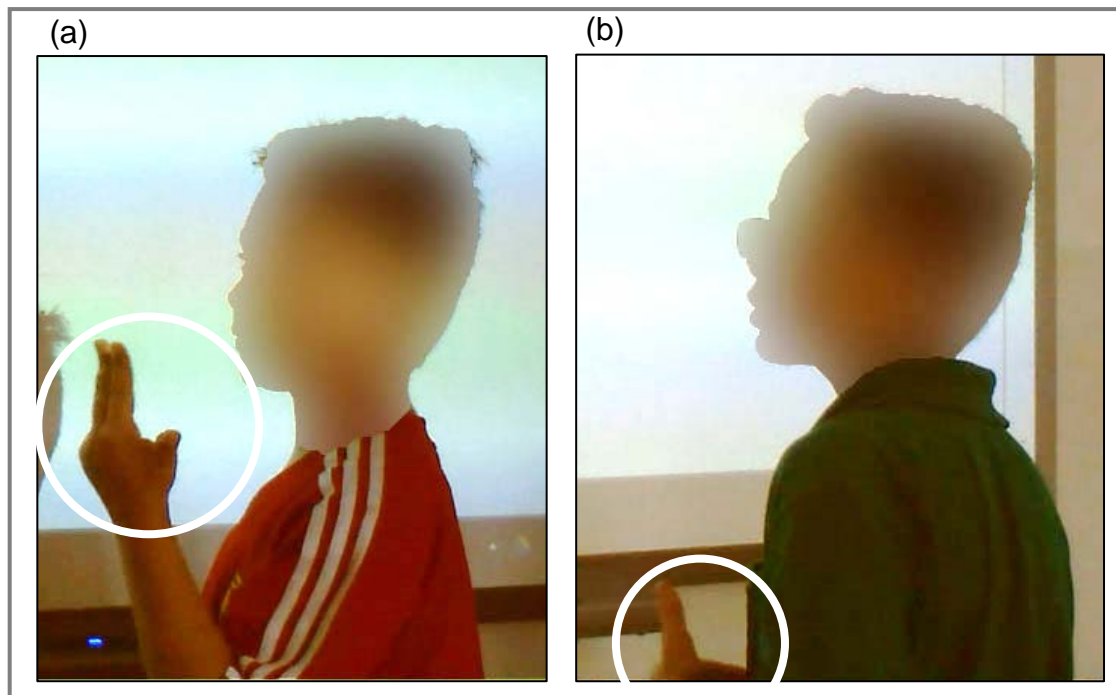
For these students, the addition $4 + 3$ signified adding 3 units to the previous number and they represented this concept by the action of counting on from 4 to 7 with their fingers to obtain the answer. Similarly, the subtraction $8 - 3$ signified subtracting 3 units from the previous number and they both represented this concept by the action of counting back from 8 to 5. It was an interplay of signifier-signified-signifier, from perceived symbol – to conceived concept – to represented action.

When asked what the processes addition and subtraction meant to him, Joseph showed me by counting on with his fingers but he said, "I just did it straightforwardly, mentally." It was evident he did not need to do finger counting any more because he already knew the answers through factual recall. It seems that the concrete action of counting backwards and forwards had become redundant to Joseph since he had already encountered and memorised the value of these expressions. However, to explain *to me* what he meant by addition, Joseph felt that he needed to represent his thoughts through finger counting. This is an example of how alternative representations, such as a kinaesthetic action, may become necessary to supplement verbal expression to explain one's thoughts during communication.

This interview also revealed that none of the students made use of the additive inverse property and they evaluated expressions like $5 + 4 - 4$ by adding up the first two numbers and then subtracting the third. This was consistent with findings of Galardo and Rojano (1987) and Herscovics and Linchevski (1994). Students' failure to make use of additive inverse meant that for expressions like $497 + 2014 - 2014$ they all engaged in long vertical algorithms which sometimes resulted in mistakes.

In the course of the lessons, I intended to help students to develop their understanding of inverse relationships, but first I needed to help them develop further their interpretations of the addition and subtraction notation by learning more ways of how to represent the concepts evoked by the symbols $+$ and $-$. As seen in the following subsection, GA proved to be a viable tool in helping me to achieve this goal.

7.2.2   Enriching Students' Interpretations and Representations

After some lessons of working with GA, students started getting used to the idea that when they dragged one cell, or rather its contents, to another place in the grid, GA would not simply change the number into another but it would show the expression with the necessary operation required for that mapping to occur. If a cell containing 15 in R$_3$C$_2$ was moved to R$_3$C$_4$, GA would show $15 + 6$. This helped the students to become accustomed to lack of closure (Collis, 1974) and to the idea that expressions like $15 + 6$ could be perceived as both as a process and a manipulable concept, or what Gray and Tall (1994) called a *procept*.

The first few lessons taught me that most students persisted in giving a single value for numerical expressions, even when I asked them specifically what GA would *show* if the value in a cell was transported to another cell. The following is an episode from Lesson 3, in which students had just seen what will happen to cells in GA when they are dragged.

Excerpt 7.2.2.1   Lesson 3

> PB: What shall I see if I grab this [*pointing to* 40 *in* R_1C_4] and I move it to here [*pointing to* R_1C_2].
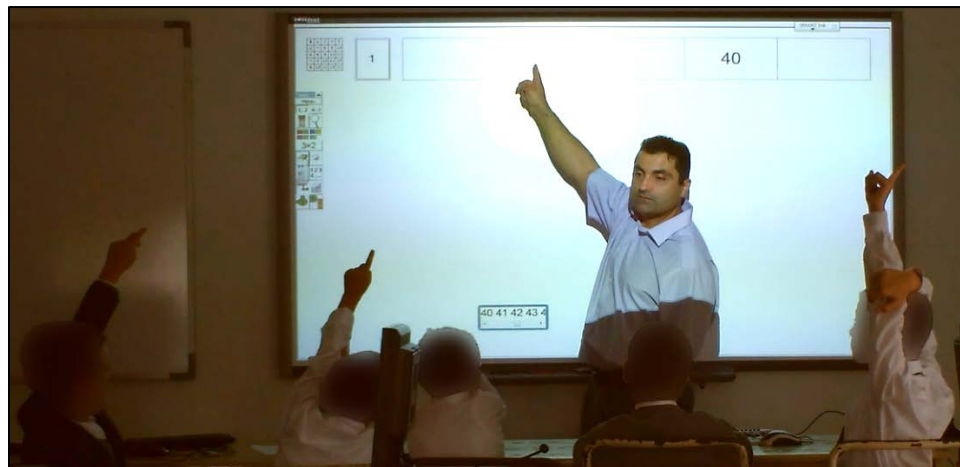
> Jordan: 38?
>
> PB: 38 would be its answer. But how did that answer come to be?
>
> Jordan: Uh. [*Jordan pauses, thinking. Other students raise their hands.*]
>
> PB: Give him a chance. Give him a chance…
>
> Jordan: You do minus 2.
>
> PB: So, what will I see completely [*pointing again to* R_1C_2]?
>
> [*Pause. Jordan thinking.*]
>
> PB: [*Nodding*] Good. But what will I see completely when the 40 I do it minus 2 [*gesturing as if grabbing the* 40 *and moving it to* R_1C_2]? I will see…?
>
> Jordan: I see… [*Long pause. Jordan thinking.*]
>
> PB: We are saying it. You have 40 [*pointing to it*], and we'll do minus 2 [*moving his hand to* R_1C_2]. So, you shall see…?
>
> Jordan: 38.

It seems that Jordan had been aware that leftwards movement in R_1 meant subtracting and he seemed also aware that the value of the destination cell would be 2 less than 40. However, he failed to say that the cell would show $40 - 2$. It seems that though Jordan was familiar with regarding this expression as a process he was not prepared to represent it as a product. This was quite common for the whole group in the first few GA lessons. It was interesting to analyse Jordan's progress from Lesson 3 to Lesson 6, where students worked on GA Task 16 – *Make the Expression (Numbers)*, where

they had to move a cell for a number of times until they obtained a given numerical expression. Figure 7.2.2.1 includes a series of screenshots showing Jordan's cell movements as he worked with Omar ("Chimps") to convert 8 into $8 + 1 - 2 - 1$.

Figure 7.2.2.1   Jordan's movements to represent an expression
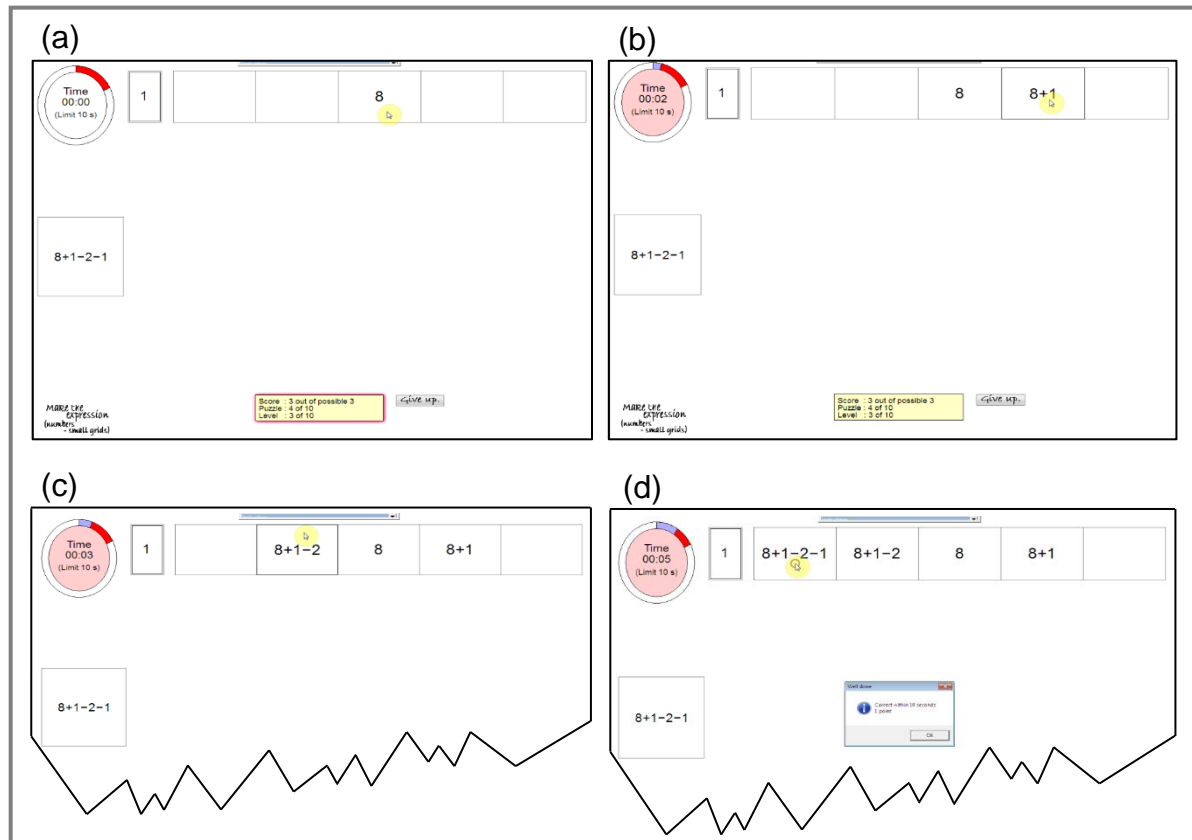
When Jordan made his first move (Figure 7.2.2.1a), and obtained $8 + 1$ (Figure 7.2.2.1b), GA required him to move the whole $8 + 1$ by two units to the left to obtain $8 + 1 - 2$ (Figure 7.2.2.1c). In this way, GA could have helped Jordan to interpret $8 + 1$ proceptually since:

(i) R_1C_4 did not show 9 but $8 + 1$, and

(ii) $8 + 1$ was treated as a single manipulable entity when it was moved leftwards.

After obtaining $8 + 1 - 2$, GA may have consolidated Jordan's proceptual interpretation of this expression when he had to move it yet again to obtain the target expression $8 + 1 - 2 - 1$ (d). This target expression was likely to be interpreted by Jordan as both a process and a product. The process was the set of movements (action) representing

the operations (concept) involved in the expression and the product was the target notation itself (symbols) represented consistently by a rectangle (picture) both as a given target and in the final destination cell R_1C_1. Similar challenges in Task 16 were successfully completed by all the other students.

GA offered more ways which may have helped students to develop a proceptual view of the addition and subtraction notation. In Lesson 8, we worked on GA Task 10 – *Find the Journey (Numbers)*. The task was similar to Task 16, where students clicked on successive cells to take a number in a given initial cell on a journey to obtain a given numerical expression in a destination cell. For each cell click, GA showed a circled number associated with a journey stage and it joined each number to the preceding one to give the impression of a route developing between the stages. The following excerpt and screenshots show Dwayne and Tony ("Pandas") collaborating on this task.

Excerpt 7.2.2.2   Lesson 8 – Pandas working on Task 10 (part 1)

Dwayne: Come on. [*Tony's turn.*]

Tony: So. Seven. One, two… [*Does the following three successive clicks very quickly.*]

#2
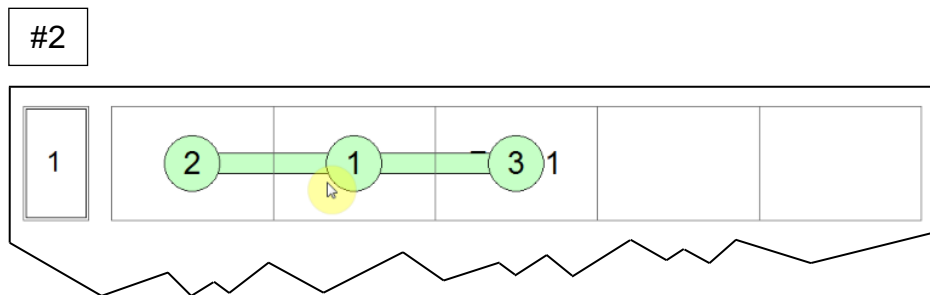
Dwayne: No, no, no! Plus! Try… Start again. [*Tony already clicking on the "Start Again" button. GA cleared the journey.*]

Tony: [*Immediately clicking on R_1C_2 containing 7. He stops his cursor on the expression and takes two seconds to consider it.*]

#3
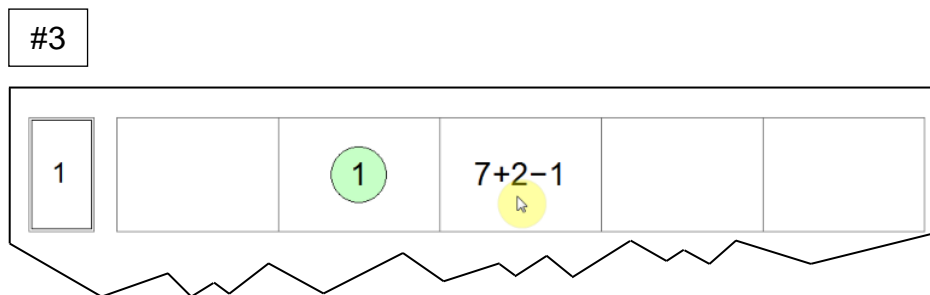
Ah! Plus 2 minus 1. [*Quickly does the second and third steps.*]

#4
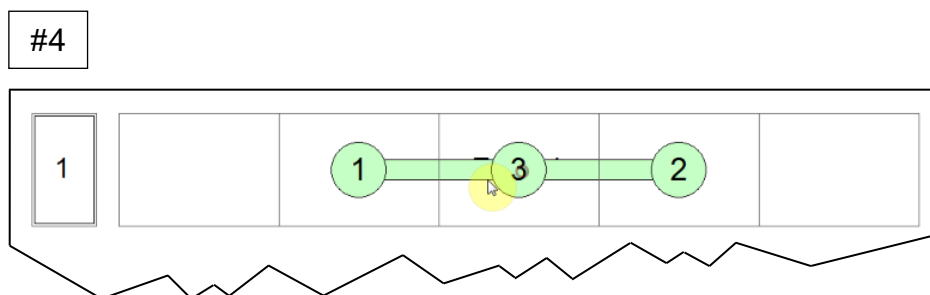
When detecting Tony's mistake, Dwayne seemed to be interpreting Tony's action and picture representations as addition and subtraction operations. When Tony moved leftwards from ① to ②, Dwayne corrected him by saying "Plus" rather than "Right". In his second attempt, Tony paused to consider the numerical expression and represented the plus and minus operations by rightwards and leftwards movements respectively. It seems that the cell representing $7 + 2$ in R_1C_4, marked by Tony as ② was interpreted proceptually by both Pandas since this was destined to be transported

to another cell (R$_1$C$_3$). These stills show two pictures representing mathematical concepts:

(i)   the rectangular cell representing a number or a procept, and

(ii)   the journey trail representing the successive operations.

Following Lessons 6 and 8, students were respectively assigned Worksheets 6 and 8, containing pen-and-paper exercises mimicking GA Tasks 16 and 10 with minor changes. Worksheet 8 contained more challenging tasks than did Worksheet 6. Figure 7.2.2.2 shows a problem in Worksheet 6 tackled by Jordan (a) and a problem in Worksheet 8 tackled by Dwayne (b).

Figure 7.2.2.2   *Pen-and-paper pictorial representations of addition and subtraction*
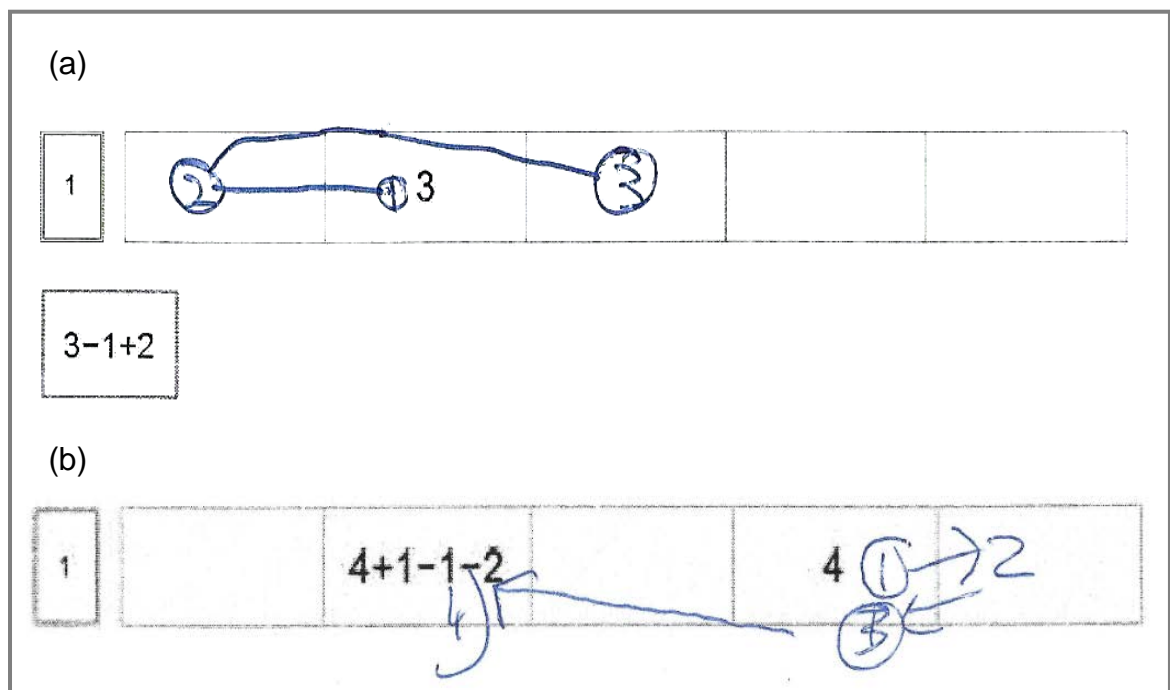
Evidently, students transferred their GA experience to pen-and-paper situations where the main difference was that they had to manually draw the journeys and the successive stops. The pictorial representations offered by GA for numerical expressions were maintained. Successive forward and backward movements, especially those resulting in landing on a previously used cell (as in steps ① to ③ of
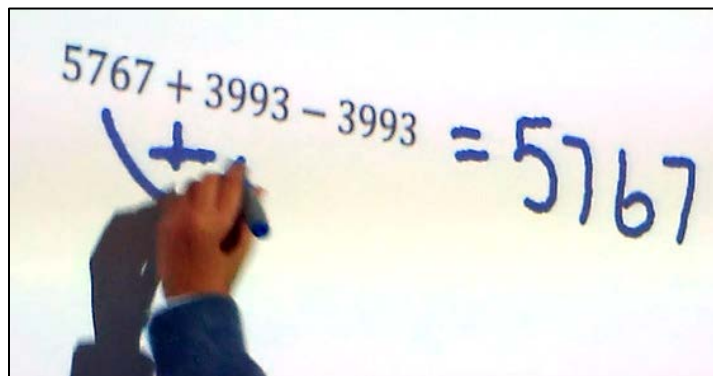
Figure 7.2.2.2b) played an important part in enriching students' interpretations of the inverse property of addition. This is discussed in the next subsection.

7.2.3 Extending Students' Concepts

As mentioned previously, all students started out Grade 7 without utilising additive and multiplicative inverse when evaluating numerical expressions. Each interview included difficult questions, such as $5767 + 3993 - 3993$ and $567 \times 123 \div 123$, which were intended to track students' developments in coming to know how to utilise additive and multiplicative inverse properties to evaluate such expressions. When observing how students worked out these problems, their APS representations helped me to create models of their conceptualisations, not only during lessons but also during interviews. In Interview 2, done before Lesson 8, Dan and Tony evaluated $5767 + 3993 - 3993$ as shown in the following excerpts.

Excerpt 7.2.3.1   Interview 2-Tony

> Tony: Because if you do these two plus [*making a U-shape underneath the first two numbers and writes + inside it*]…

> PB: OK.
>
> Tony: …uh, it becomes… [*started to work out the addition*].
>
> PB: You do not need to write it. Explain it to me without the working.
>
> Tony: [*Sighs.*] Because if you do these two plus [*pointing to the first two numbers*] you will get a number and if you subtract again this one [*pointing to the second* 3993], if you subtract again this one the same number will result, and so [*pointing to his answer* 5767].

Excerpt 7.2.3.2   Interview 2-Dan

> Dan: Because this [*underlined the first* 3993] I did it plus this [*underlined* 5767] and then this [*shaking his pen on the sum*] will become a number and then this I cancelled it [*pointing and cancelling the first* 3993]…
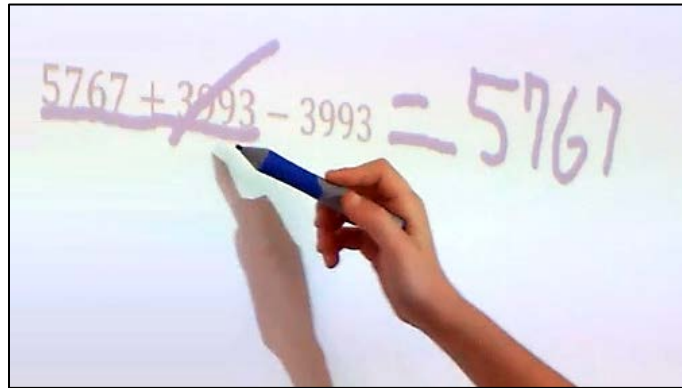
> …because we are doing minus that amount [*pointing to the second* 3993] and we were left with this one [*pointing to* 5767].

These students' symbolic/pictorial representations indicated that they had utilised concepts of additive inverse. Dan's mention of cancellation can be interpreted as his understanding that additive inverse was linked to additive identity. In fact, in Interview 4, he explicitly stated that such operations were equivalent to adding 0.

As a teacher, I felt I needed to work more with the rest of the students to make them aware of this important property of addition. In Lesson 8, I took the opportunity to discuss this property when Omar was working Task 10 on the IWB and got the following journey picture for the expression $4 - 3 - 1 + 1$.

Figure 7.2.3.1   Omar's picture containing an inverse journey
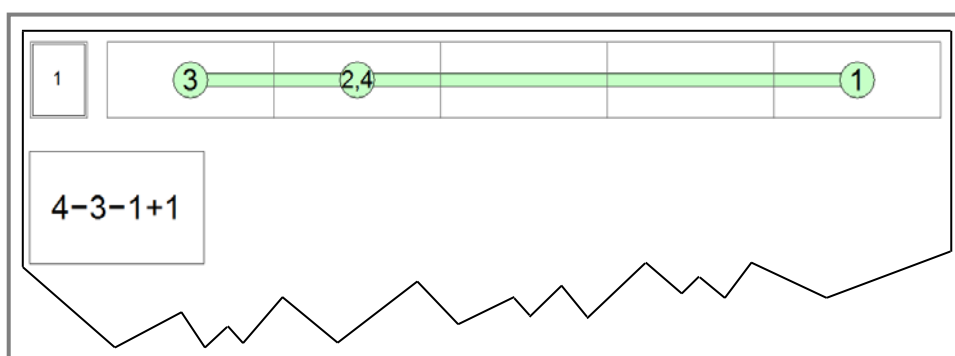
The first significant observation was the pictorial representation in GA resulting from an additive inverse. This picture resulted from a horizontal movement and the reverse movement with stage ④ of the journey overlapping stage ② on the same cell. Students may have interpreted this picture in two ways:

(i) an inverse operation follows the same path of the original operation but in reverse, or

(ii) the result of an operation and its inverse is the same as if no operation has been done.

The second observation is Dwayne's gesture (action) with which he described this inverse operation while saying "you add it again and it comes to the same place". He rotated the index fingers in a cyclical motion as seen in Figure 7.2.3.2.

Figure 7.2.3.2   Dwayne's gesture to represent inverse operation
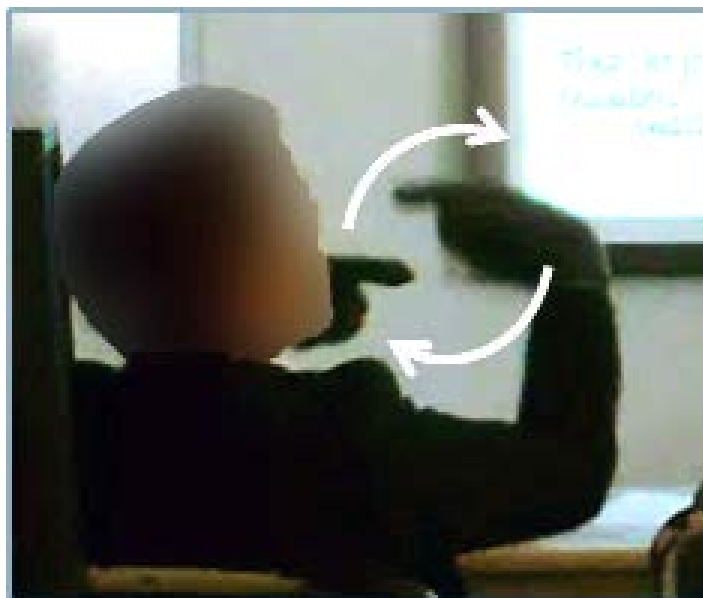
This action supplemented Dwayne's verbal representation of his interpretation of the additive inverse and, thus, helped me to form a better model of his conceptual interpretation of $-1 + 1$.

In Interviews 3 and 4, all students managed to answer the difficult additive inverse questions correctly. In the GA lessons before these interviews, students had been working on inverse movements with GA cells corresponding to inverse operations, such as that shown in Figure 7.2.3.1. GA seems to have helped the students develop concepts of inverse by giving them the opportunity to represent inverse processes by alternative action and picture representations. This supports the argument that diverse APS representations may enrich students' mathematical interpretations, in this case the concept of inverse evoked by the conventional notation (symbol) of adding and subtracting the same number. In addition, GA seems to have encouraged students to come up with their own APS representations and give them the possibility of using those representations with reference to their experiential worlds. In Interview 4, which was done after Lesson 18, when I asked Dwayne to explain how he got a straightforward value to the expression $5445 + 9997 - 9997$, he evoked the action of travelling between two places in Malta.

Excerpt 7.2.3.3   Interview 4-Dwayne

Dwayne: [*Starts drawing*]
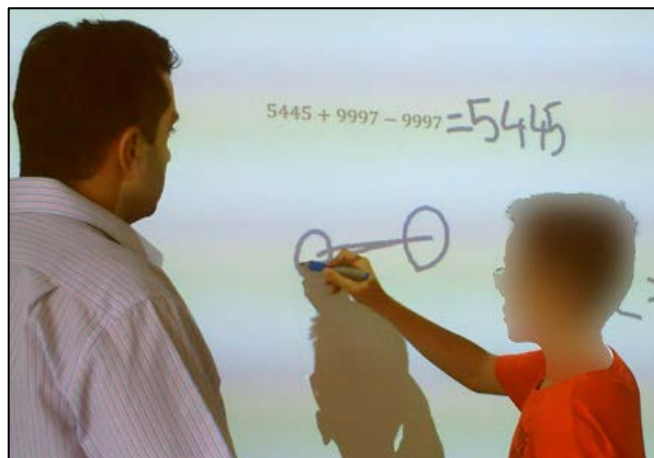
Here is, for example, Valletta [*draws a circle*] and here is Birzebbugia [*draws another circle*]. You go there [*draws a line from the first circle to the second*] and you come back again [*draws a line from the second circle to the first*].

After a number of lessons featuring journeys between cells in GA's grid, Dwayne may have associated these with actual journeys. He supplemented his verbal description by drawing a picture of two places and the return journey between them. His drawing

compared well to journeys between simulated GA grids he was required to draw as homework. Similar comparative representations were made by other students for the same question. Omar and Tony imagined the action of going somewhere and returning home and Joseph actually walked one pace forward and one pace backward. I argue that being flexible in switching between different representations for additive inverse helped these students to extend their meanings of it. Dreyfus (2002) maintains that rich mental schemata are formed of a multitude of representation systems and a flexibility to switch between them.

This does not imply, however, that students who externalised minimal representations had inferior interpretations. Dan and Jordan answered this question successfully using only symbolic representations. Both of them reasoned that since $9997 - 9997$ was equal to 0, then the expression was equivalent to $5445 + 0$. Relying only on symbolic representation, these students managed to express a conceptual interpretation which other students did not exhibit: a notion of additive identity.

Table 7.2.3.1 gives a summative analysis of students' representations and interpretations of the addition and subtraction notation with reference to the CAPS framework. This data was collected from the five interviews and from the twenty lessons throughout the year. The first column (**GA**) indicates when GA was used and this always happened during the lessons. The second column (**Symbolic**) gives typical expressions which students interpreted and represented. The third column (**Conceptual**) shows the concepts that students were signifying by means of their representations. These were the models of students' thinking processes that I (as a researcher) formed by observing students' APS representations and by listening to what they were saying to compliment those representations. As discussed in Chapter 6, I was forming similar models as a teacher during the lessons. The fourth column (**Active**) shows the kinaesthetic actions students were making as a means of representing their concepts. The fifth column (**Pictorial**) shows typical picture representations drawn by students manually or by using GA. Some of these were also generated by GA during the tasks. The sixth column (**Code**) shows the CAPS code I used to identify which type of representation was being expressed for each concept. Finally, the seventh column (**Students**), shows the students who were making those interpretations and representations.

Table 7.2.3.1   Students' CAPS for addition and subtraction

GA	Symbolic (notational) Representation	Conceptual (signified) Interpretation	Active (kinaesthetic) Representation	Pictorial (drawing/diagrammatic) Representation	Code	Students
	$4 + 3$ $8 - 3$	Process of counting on or back	Finger counting			▪ Dan ▪ Joseph ▪ Omar
	$4 + 3$ $8 - 3$	Process of adding or subtracting resulting in one value				▪ Dwayne ▪ Jordan ▪ Joseph ▪ Tony
✓	$6 + 2 - 1$	Proceptual view	Horizontal movement in GA grid			■ All
✓	$6 + 2 - 1$	Proceptual view	Horizontal movement in GA grid			■ All
	$3 - 1 + 3$	Proceptual view	Horizontal movement on paper grid			■ All
✓	$8 + 2 - 2$	Additive Inverse (process)	Horizontal inverse movement in GA grid			■ All
	$5445 + 9997$ -9997	Additive Inverse (process)	Return Journey			▪ Joseph ▪ Omar ▪ Tony
	$5445 + 9997$ -9997	Additive Inverse (process)	Return Journey			▪ Dwayne
	$5445 + 9997$ -9997	Addition of 0 (identity)				▪ Dan ▪ Jordan

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Table 7.2.3.1[15] shows that whenever GA was used, all students managed to use APS representations to express their conceptual interpretations of the addition and subtraction notation. Multiple representations were also possible in pen-and-paper work simulating a GA environment. With regards to preferred representations outside the context of a multiplication grid, I am aware that these were exhibited on particular moments. These were probably representations that seemed most viable to the students on those data gathering moments, and I do not exclude they might have been capable of exhibiting others.

The introduction of GA activities involving multiplication and division required the students to remember addition and subtraction representations analysed above. They needed to make sense of addition and subtraction notation in expressions containing also multiplication and division notation. The latter is discussed in the next section.

7.3   Notation for Multiplication and Division

For secondary students, the addition and subtraction notation remains the same as learnt in primary school. However, in secondary school, multiplication and division may be denoted in a new way. Secondary students need to learn that familiar notation encountered in primary school, namely the brackets and the divisor line of a fraction, may be used to denote multiplication and division respectively:

(i)   Absence of any sign before a bracketed expression like $2(3)$, or $2(3+6)$, would signify multiplication;

(ii)   Divisor lines of fractions like $\frac{4}{2}$ or $\frac{4+6}{2}$ would signify division.

These newly defined symbols presented the students with new challenges of using notation for expressions with a combination of multiplication, division, addition, and subtraction. This entailed the additional study of how students made sense of the order of operations when evaluating multi-operational expressions.

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[15] Similar summative analysis tables are presented at the end of Sections 7.2–5.

7.3.1 Students' Initial Interpretations and Representations

Interview 1 revealed that all students were competent in mentally evaluating simple expressions like 5×4 and $6 \div 2$. Other than Omar, students interpreted

- 5×4 as repeated addition of 5, and
- $6 \div 2$ as inverse of multiplication or halving 6.

Omar gave the answers through factual recall but he could not verbalise what he meant by multiplication or division, so I could not form a model of his interpretations of $\times$ and $\div$.

Dan, Jordan, Joseph, and Omar counted multiples on their fingers to represent multiplication and division. While Dan and Joseph used finger counting to supplement verbal expressions, for Omar and Jordan this seems to have been a necessary/helpful action to work out the answer. This does not mean an inferior level of thinking, however, as shown by the following excerpt taken from Jordan's interview.

Excerpt 7.3.1.1   Interview 1-Jordan

Jordan: 5 times 4. [*Looks down at his fingers.*] It results in [*writes* $= 20$] 20.

PB: OK. Tell me a bit how that 20 resulted.

Jordan: The 20 resulted, um, I did, I did the 5 times table and when I did 4 it resulted in 20.

PB: OK. So what does it mean when we say 5 times 4? Can you tell me "5 times 4" but you tell it to me in another way, the "5 times 4"?

Jordan: It is, it is like a type of plus but it moves quickly. Instantly.

This excerpt and what followed indicated that Jordan had a mental image of multiplication as skipping multiples rather than addition of units. In Jordan's words, 5×4 increases quickly: $5 + 5 + 5 + 5$, whereas $5 + 4$ increases at a slower rate: $5 + 1 + 1 + 1 + 1$. For me this was a simple but elegant way of comparing multiplication with addition.

Dwayne did not use finger counting during computation but to aid his verbal expressions of his interpretation of $\times$ he supplemented his verbal expressions with a hand action.

Excerpt 7.3.1.2   Interview 1-Dwayne

> Dwayne: We are, um, we're doing the 5… [*PB gestures for Dwayne to raise his voice.*] We take the 5 and we reuse 5 [*rotates his right hand in a clockwise sense and stops it under his left hand*]…
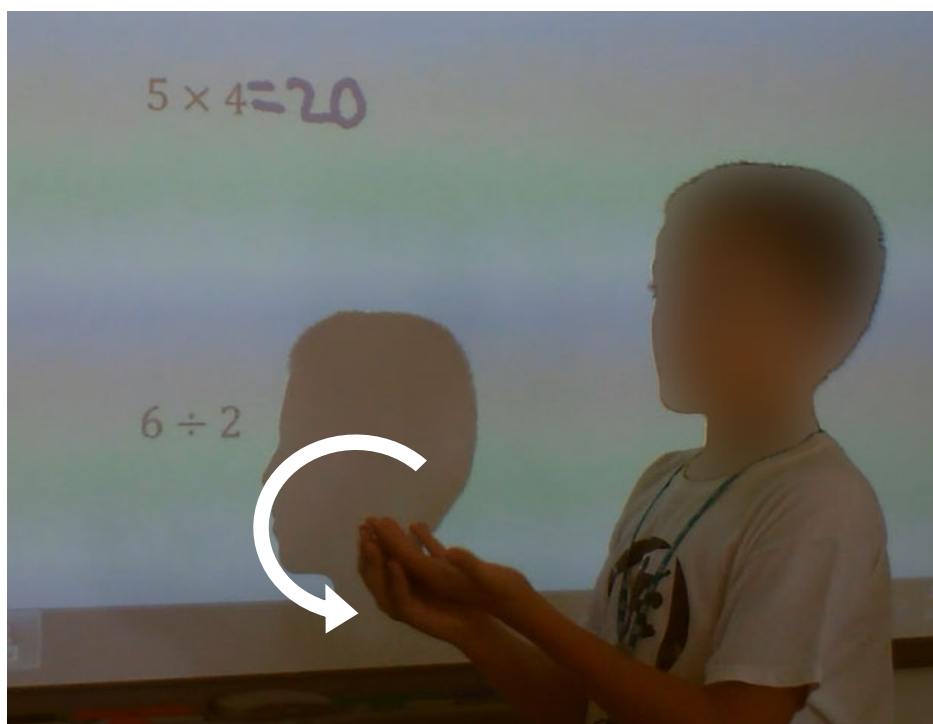
> …and we, sort of, do them plus.
>
> PB: Yes.
>
> Dwayne: And then we take 5 and with the answer we add another 5 [*repeats the previous gesture*]…
>
> PB: Another 5.
>
> Dwayne: And you keep on going like that.

As soon as Dwayne used the word "reuse" to refer to repeated addition, he felt he needed to reinforce his expression by showing me layer upon layer of groups of 5's represented by the fingers of his outstretched hand. By observing this action, I created a model of Dwayne's conceptual representation of multiplication by 5: the repeated addition of 5 distinct but grouped objects.

Similar to what I discovered about additive inverse, Interview 1 revealed that, in general, students did not make use of multiplicative inverse when evaluating expressions. For the expression $121 \times 350 \div 350$, only Dwayne knew how to give the answer with minimal calculation. He first divided 350 by 350, got 1, and multiplied 121 by 1. However, I expected students familiar with multiplicative inverse to give the answer 121 straightaway. This showed me that multiplicative inverse was something else I had to address as a teacher since, along with additive inverse, this would play an important part later on, when students would be required to solve or transform equations (Gallardo & Rojano, 1987; Herscovics & Linchevski, 1994).

Interview 1 also provided data about students' conceptualisation of multi-operational expressions. Only Dan, Dwayne, and Jordan evaluated $2 \times (3 + 1)$ as 8, all of them saying that the bracketed sum needed to be worked out first. Dwayne was the only one who mentioned "the BIDMAS rule", and said that it was due to this rule that brackets came first. However neither he nor any of the others evaluated $2 + 3 \times 10$ as 32. All students got 50, implying that they were working out operations in the order of appearance.

None of the students seem to have known that a number followed immediately by a bracketed expression meant multiplication. For the expression $10(5 + 2)$, Dan, Dwayne, and Jordan simply added 10 and 7, getting 17. Jordan said that "it came together with it", and made a gesture with both hands as if bringing two things together. Omar worked out the brackets, got 7 and stopped there, saying he did not know how to work it out. Joseph and Tony made drawings/markings to supplement their explanation and this served me well to form a model of what they were thinking.

Figure 7.3.1.1   Joseph's and Tony's representations of unfamiliar notation use
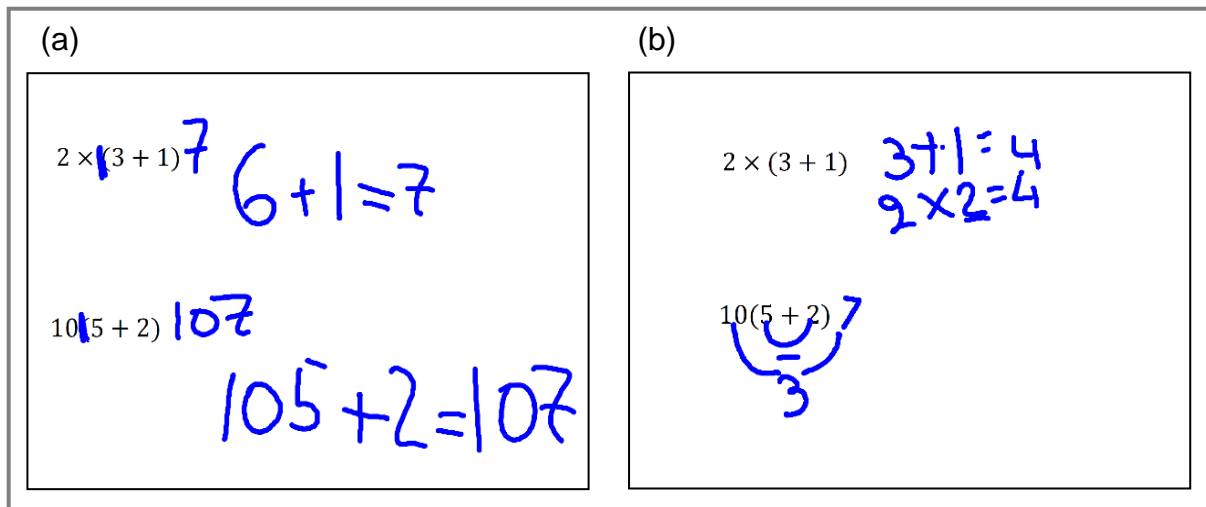
Joseph (Figure 7.3.1.1a) acted as if the first bracket did not exist and worked out $105 + 2$. This was consistent with his interpretation $2 \times (3 + 1)$ as $2 \times 3 + 1$. He supplemented his verbal explanation by crossing out the first bracket. Tony (Figure 7.3.1.1b) treated the brackets as if they were the place where the answer to a calculation was placed. He interpreted $10(5 + 2)$ as if it were $10 - ? = (5 + 2)$. He worked out the brackets first, got 7, and so his answer to this question was 3. This was consistent with his interpretation of $2 \times (3 + 1)$ as $2 \times ? = (3 + 1)$ and his answer was 2, emphasised by the underline. Without their drawings, I may not have formed an experiential model of what Joseph and Tony might have been interpreting when they perceived these expressions.

It seems none of the students was aware that a bracket following a number may represent multiplication, or that the divisor line of a fraction may be interpreted as division. Difficulties with algebra stemming from students' misrepresentations of notation are well documented (e.g. Booth, 1984; Kieran, 1981b; Kirshner, 1989; Borg, 1997; Van Amerom, 2003) and as a teacher I intended to help students to become acquainted with the new multiplication and division notation with the help of GA. This is discussed next.

7.3.2 Enriching Students' Interpretations and Representations

In Lessons 3 and 4, I introduced students to these two new symbols: a bracketed number signifying multiplication and a divisor line of a fraction signifying division. All students accepted these readily because that was what GA showed whenever they moved cells vertically. In Lesson 4, students' computer task consisted of entering a number in a cell and shading a cell in the same column or row for their teammate to guess what that number would be converted to if it was moved to the shaded cell. Figure 7.3.2.1 gives two screenshots showing how Joseph and Dan ("Sharks") represented multiplication through vertical movements in GA.

Figure 7.3.2.1 Sharks' GA representations of multiplication
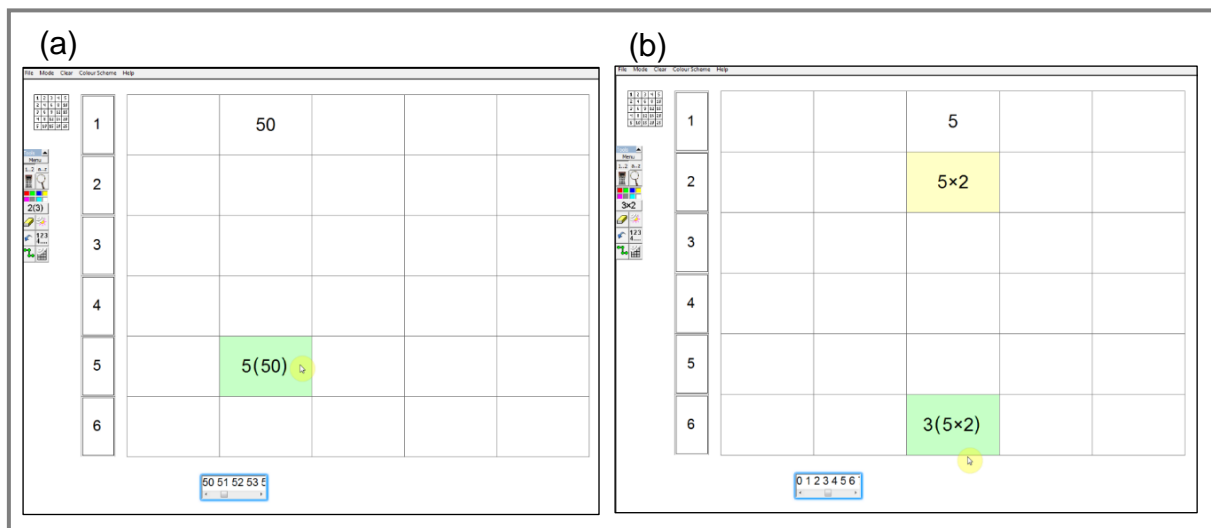
Figure 7.3.2.1a shows that when 50 was dragged from R_1C_2 to R_5C_2, GA showed $5(50)$. Besides being introduced to new notation, students seem to have found GA helpful in enriching interpretations of multiplication as:

(i)   the process involved in the shifting (action) of a number to a higher multiplication table, and

(ii)   the product represented by a shaded rectangle (picture) of R_5C_2.

Once again, this activity may have contributed to students' proceptual interpretations of expressions involving multiplication or division of two numbers. This interpretation was

probably consolidated when students were asked to shade yet another cell in the same column and repeat the process. In Figure 7.3.2.1b, Sharks chose the $\times$ notation option rather than the bracket notation option (as in (a)) from the menu on the left. In the first movement, 5 was converted to 5×2. The latter was moved again from R$_2$ to R$_6$, which meant multiplying by 3. This resulted in $3(5 \times 2)$ which may have helped students to:

(i)   interpret 5×2 proceptually, and

(ii) become flexible to switch between symbolic representations of multiplication.

Students gained similar experiences with the division notation. With the help of screenshots, Excerpt 7.3.2.1 shows Chimps' representations and interpretations of division during this activity.

Excerpt 7.3.2.1   Lesson 4 – Chimps' representations and interpretations of division

Omar: [*Addressing Jordan*] Come on.

Jordan: So. 8…, uh, all over 2. It becomes 8 all over 2.

Omar: Let me do it. [*Clicks on R_4C_2 (#2) drags the cell upwards to R_2C_2 (#3) and lets it go on R_2C_2 obtaining $\frac{8}{2}$ (#4).*]
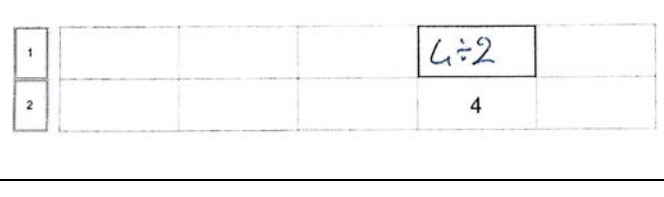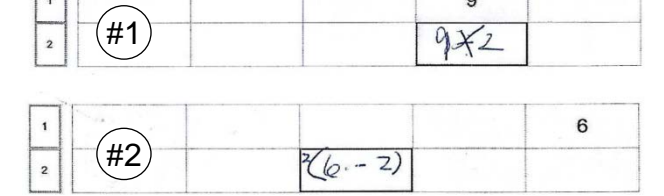
#2

#3

#4

Jordan: [*Triumphant*] Yes!

It is important to note how GA converted Omar's upward movement (action) into a symbolic representation. The content of the cell containing 8 was *being changed* along the movement (#3) and not simply shown on the destination cell (#4). Coupled with the fact that the rectangular cell (picture) representation of $\frac{8}{2}$ seems to have helped students to get a proceptual view of this expression, this feature has contributed to students collaboratively making connections between APS representations. Students took turns to make representations and interpretations of the division concept:

(i)   Omar shaded the cell, a pictorial representation, and asked Jordan to tell him what he interpreted by that cell;

(ii)   Jordan connected Omar's picture to the symbolic notation of "8 all over 2";

(iii) Omar moved the cell upwards, an action representation to check whether it matched Jordan's interpretation;

(iv)   GA converted Omar's action into a symbolic representation;

(v)   Jordan confirmed this was the symbolic representation he had predicted.

Students' interpretations of multiplication and division notation seem to have been enriched by such flexible switching between representations. Similar to addition and subtraction representations discussed earlier, students used GA Tasks 8 and 10 (see Appendix 1) which reinforced a proceptual view of expressions such as $5(50)$ and $\frac{8}{2}$. In Task 8, they also made use of GA journey trails to make pictorial representations of such expressions, just like they did with addition and subtraction, the only difference being that these involved also vertical journeys rather than just horizontal. As before, students reproduced such GA activities in pen-and-paper work. Table 7.3.2.1 shows typical students' written work taken from worksheets assigned at the end of GA lessons. The worksheet number is the same as the lesson number (e.g. Worksheet 3 was assigned at the end of Lesson 3, etc.). As shown by the CAPS codes, students represented multiplication and division symbols by interpreting and representing the corresponding pictures (e.g. journey picture to represent an operation, or cell to represent the resulting expression). The table includes some analytic comments showing the models of students' conceptualisations that I formed according to the representations I observed in their written expressions.

Table 7.3.2.1   Pen-and-paper multiplication/division interpretations and representations

Student & Code	Interpretation and Representation	Comments
Dwayne (w/sheet 3)		• Journey picture interpreted as division process; • Acceptance of "lack of closure" of expression in cell ②; • New division notation not used.
Jordan (w/sheet 4)		• Journey picture interpreted as a repeated division process; • New division notation used; • Cell ② interpreted proceptually as a manipulable concept; • Cell ③ interpreted as the result of a process performed on cell ②; • Mistaken division amount from ① to ② and from ② to ③.
Omar (w/sheet 5)		• Standard function machine diagram (non-GA); • Rectangle interpreted as proceptual output and input; • Expression 8x2 evaluated before second operation.
Joseph (w/sheet 6)		• $\frac{18}{2}$ interpreted as division; • Journey drawn to represent division process; • Cell ② possibly interpreted proceptually.
Tony (w/sheet 10)		• Acceptance of "lack of closure" in R_1C_4; • New division notation not used.
Dan (w/sheet 10)		• In (#1), new multiplication notation not used; • In (#2), subtraction preceding multiplication encouraged use of new multiplication notation; • In (#2), (6-2) of R_2C_5 interpreted proceptually.

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In each case, conceptual interpretation was related to a pictorial and a symbolic representation. Dwayne and Tony's retention of the old division notation implies that they:

- may have been more at ease when using old notation,
- still interpreted the picture of a cell proceptually, regardless of the notation they were using,
- showed flexibility in their interpretation of cells and representation of division, and
- were not relying on memorising what symbol GA would have shown.

Table 7.3.2.1 shows that although students' multiplication/division interpretations and representations had been enriched, they still needed more experience in:

- accepting expressions like 8×2 as a product which could be manipulated without being evaluated, and in
- using new notation.

Jordan's error in Worksheet 4, involving a mistaken division amount, was quite common among students in the first few lessons, where students were required to transform division/multiplication operations into vertical movements. The error consisted of dividing by the amount of rows counted from one cell to the next rather than referring to the row number. In the lessons that followed, I often engaged students in discussions to help them learn from such mistakes.

Dan's use of the old multiplication notation in 9×2 and his use of the new multiplication notation in $2(6 - 2)$ in the same worksheet hints that giving students the opportunity to combine multiple operations in single expressions may have encouraged them to use the new notation more frequently. This is discussed in the next section, where I analyse how students' concepts seem to have been extended with the help of GA.

7.3.3   Extending Students' Concepts

Table 7.3.3.1 gives an overview of interview data showing students' progress in evaluating multi-operational expressions. From Interview 3 onwards, such expressions started to appear only in the new multiplication and division notation.

Table 7.3.3.1   Students' learning progress in evaluating expressions

Interview	Expression	Correct	Comments about misinterpretations (if any)
1 (held before Lesson 1)	$2 + 3 \times 10$	☐ None	• All students worked operations in order of appearance.
	$2 \times (3 + 1)$	■ Dan ■ Dwayne ■ Jordan	• Dwayne mentioned "the BIDMAS rule". Except for "I", he knew what letters stood for. He did not use BIDMAS for $2 + 3 \times 10$.
	$10(5 + 2)$	☐ None	• Some students ignored brackets while others worked out $5 + 2$ first but did not multiply result by 10.
2 (held after Lesson 11)	$3 + 2 \times 5$	☐ None	• Students worked operations in order of appearance.
	$10 \times (2 + 6)$	■ Dwayne ■ Jordan ■ Joseph ■ Omar ■ Tony	• Dan worked operations in order of appearance.
	$2(3 + 7)$	■ Dan ■ Dwayne ■ Tony	• Similar mistakes as Interview 1.
	$\dfrac{4 + 6}{5}$	■ Dwayne ■ Jordan ■ Joseph ■ Tony	• Dan represented $\frac{10}{5}$ as an answer but did not interpret it as $10 \div 5$. • Omar worked out $4 + 6 = 10$ first but interpreted 5 as the answer to a division and he got confused.
3 (held after Lesson 13)	$4(5 + 1)$	■ All	
	$\dfrac{4 + 6}{5}$	■ All	
4 (held after Lesson 17)	$4(5) + 10$	■ All	
	$10 + 4(5)$	■ All	
	$\dfrac{12}{3} + 6$	■ All	
	$6 + \dfrac{12}{3}$	■ All	
	$2(3) - 5$	■ All	
	$5 - 2(3)$	☐ None	• All students interpreted this as $6 - 5$ rather than $5 - 6$.
	$\dfrac{10}{2} - 4$	■ All	•
	$4 - \dfrac{10}{2}$	■ Dan	• Other students interpreted this as $5 - 4$ rather than $4 - 5$. • At first Dan, made the same mistake, but when he considered what $4 - 5$ meant, he changed his answer.

Before starting the GA lessons, only Dwayne had mentioned the BIDMAS rule for the order of operations and as shown in Table 7.3.3.1 he failed to apply it in one of the two questions set in Interview 1. Consistent with Kieran (1979), when students failed to work out operations in the conventional order, they usually thought the operations followed the order in which operational symbols were written. In the lessons, I used GA to support students in learning about the conventional order of operations. I deliberately avoided using BIDMAS because I believed it could be misleading in expressions like $\frac{4+6}{5}$, where students following BIDMAS might divide by 5 before adding 6.

As shown in Table 7.3.3.1, students' interpretations of the order of operations and the new multiplication and division notation seem to have been developed concurrently. One possible hypothesis could be that new notation may have been more conducive to a proceptual view of certain expressions like $\frac{12}{3}$, making these appear more unified and, hence, more likely to be evaluated beforehand. Thus, it might have been more likely for students to interpret $6 + 4$ when they saw $6 + \frac{12}{3}$ than when they saw $6 + 12 \div 3$. Unfortunately, in pursuing my teaching aim to wean students off using old notation, once students started to get used to the new notation I stopped using the old notation even in the interviews. If the interviews retained some expressions with old notation, it would have been interesting to see whether the above hypothesis would have held.

Consistent with Hewitt (2012), GA was found to help students in evaluating multi-operational expressions in informal-algebraic activities without having to resort to mnemonics such as BIDMAS. An activity students enjoyed was GA Task 10, described earlier on, in which students transformed symbolic interpretations into action and picture representations. Excerpt 7.3.3.1 shows how Pandas did this with an expression involving the new division notation. It also shows how I occasionally intervened when I saw that students were finding difficulties.

Excerpt 7.3.3.1   Lesson 8 – Pandas' work on Task 10 (part 2)

Tony: Six, [*clicks on R_2C_4*]...

#1

			$\frac{6-4}{2}$			
1						
2				6		

$\frac{6-4}{2}$

...up here, [*clicks on R_1C_4*]

#2

			$\frac{6-4}{2}$			
1				2		
2				1		

$\frac{6-4}{2}$

[*Drags cursor to R_1C_2 and pauses a bit before clicking.*]

#3

PB: [*Disapproves*] Uuh! Uh-uh-uh-uh-uh-uh! How come you went up there? [*Tony presses the "Start Again" virtual button, GA removes his* ①-② *move, and he is going to start a new journey from* R_2C_4.]

Hang on. What would you do if you went up there?

Dwayne: There [*meaning if you go straight up*] it's division. There [*meaning the given expression*] it's telling you 6…

PB: [*Addressing Tony and speaking simultaneously with Dwayne.*] Isn't it division...?

Dwayne: …minus 4.

PB: Isn't it division by 2 there [*meaning if you go straight up*], my brother?

Tony: [*Starts again.*]

#4

[*Murmurs.*] …minus, uh… [*Drags cursor from* R_2C_4 *to* R_2C_2 *and goes back.*]

Dwayne: No. Here.

Tony: [*Drags cursor to R_2C_2*] Two [*clicks*]…

#5

…and then there. [*Drags cursor to R_1C_2 and clicks.*]

#6

[*Tony was going to press the "I've finished" button.*]

PB: So. Pay attention for a moment. Wait a bit. Let us reflect a bit on it… So. Six...

Dwayne: [*Interrupting*] Minus four…

PB: …What do you have first? Minus 4 and everything over 2, …?

Pandas: [*Interrupting*] Uh-huh [*agree*].

PB: …or, 6 over 2 by itself and minus 4. How is it?

Dwayne: No. No…

Pandas: 6 minus 4 all over 2.

This exchange shows how Tony engaged in the process described in Figure 7.3.3.1, showing continual toing and froing between APS representations and conceptual interpretations. It also shows how GA helped me to create a model of Tony's thinking.

Figure 7.3.3.1   The process of students' CAPS switching and associating with GA
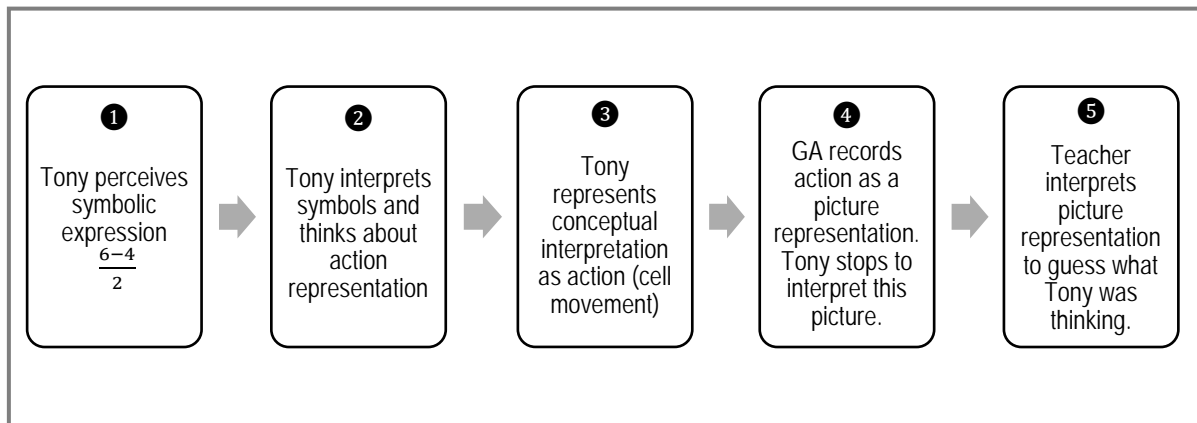
Had Tony not hesitated at the last moment, GA would have popped up an "Incorrect" message and presented a new challenge. Thus, I stopped him so that he could reflect on his actions, where Dwayne joined in to help him identify his mistake. At this moment it seems that Tony's APS representations triggered a Learner-to-Negotiation shift from my part: my teaching purpose changed from facilitating reflection to creating a model of Tony's interpretations. This was followed by a renewed interaction with Tony where I intended to help them learn from his mistakes. The interplay between symbolic interpretation and action representation resumed when Dwayne mentioned "6 minus 4" and followed this by showing Tony where to go: "No. Here." Tony went on to complete the task successfully and once again, I encouraged students to reflect on their experience.

Students made considerable progress in linking multi-operational expressions to movements of numbered cells in the multiplication grid. In Interview 3, when I asked them how they would move 45 in a mock GA grid to obtain $\frac{45-10}{5} - 1$, they all drew the journey they would have obtained in GA. Their drawings are included in Figure 7.3.3.2.

Figure 7.3.3.2   Students' pictorial representations of symbolic interpretation
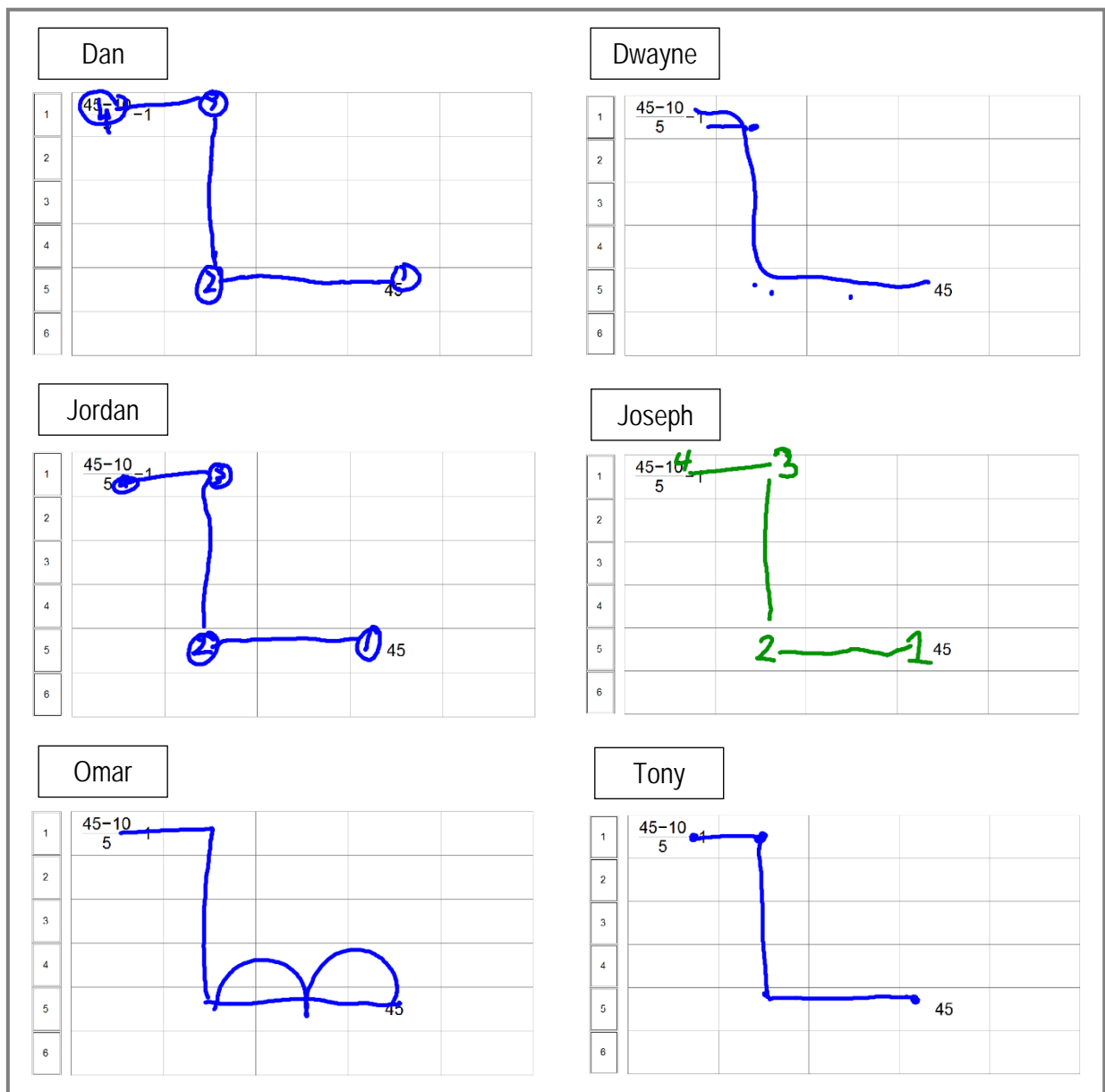
Dan, Jordan, and Joseph included the numbers for each of the journey stages. The others concentrated only on the journey. It seems that GA helped students to enrich their repertoire of representations and interpretations of numerical expressions like $\frac{45-10}{5} - 1$. This could have been key for learning about the conventional order of operations without resorting to BIDMAS, which may have been misleading here since $45 - 10$ was not enclosed in brackets. However, some students still felt uncomfortable to manipulate numerical expressions rather than single numbers. The next question in Interview 3 is shown in Figure 7.3.3.3.

Figure 7.3.3.3   Interview 3 – interpretation of a journey picture

This is a multiplication grid. The cell marked ① contains the number 5. What does the cell marked ④ contain after the number 5 goes through the journey ①→②→③→④ ?
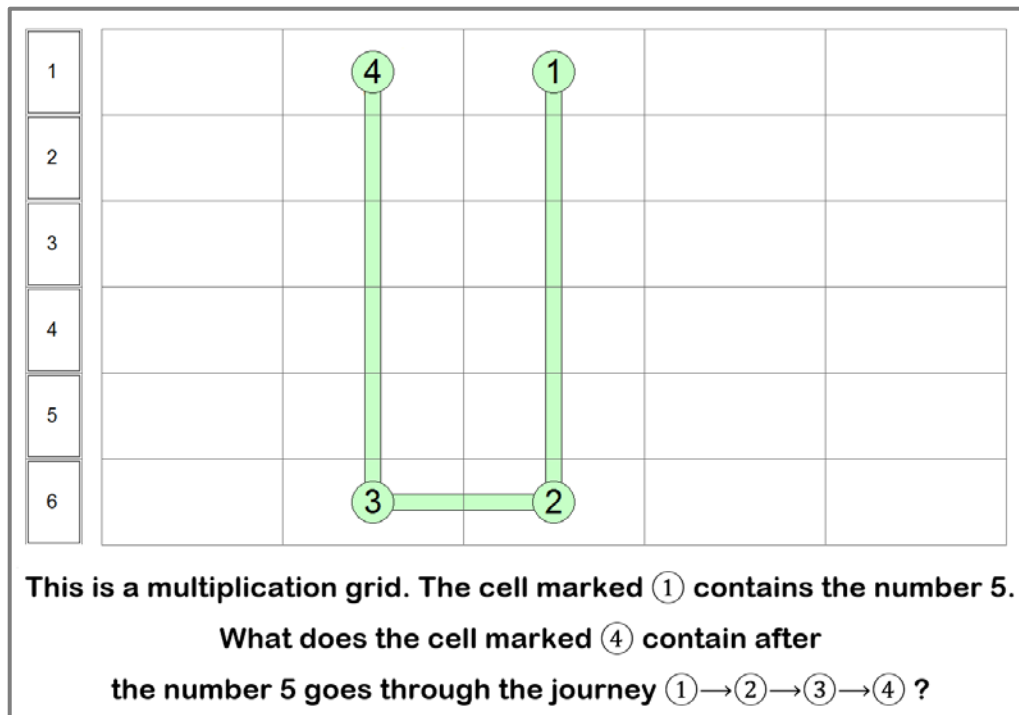
For stage ④, Dwayne's expression was $\frac{5 \times 6 - 6}{6}$ and Joseph's was $\frac{(5 \times 6) - 6}{6}$. Dan and Tony worked out according to the conventional order of operations but they both made a minor mistake when giving the final expression. Dan wrote the division line just beneath $5(6)$ and gave $\frac{5(6)}{6} - 6$ while Tony wrote $5 \times 6 - 6 \div 6$, ignoring the use of brackets altogether and confirming Kieran's (1979) finding that students sometimes think that brackets are redundant.

Jordan and Omar evaluated each of the four cells in this question, giving, say, 30 in stage ② instead of 5×6 or $5(6)$. Omar made a mistake from ② to ③ when he subtracted 1 instead of 6. All students interpreted movements between the cells as operations but only Dan, Joseph, and Tony gave numerical expressions rather than single numbers at each stage. Interestingly, Dwayne worked out the correct order of operations on the side (Figure 7.3.3.4) and although he told me that stage ④ would show $\frac{5 \times 6 - 6}{6}$, he still wrote the value of the cells at stages ② and ③, even when I asked him specifically to tell me what the cells would show when the previous number was *moved* to that cell.

Figure 7.3.3.4   Dwayne's interpretation of a journey picture
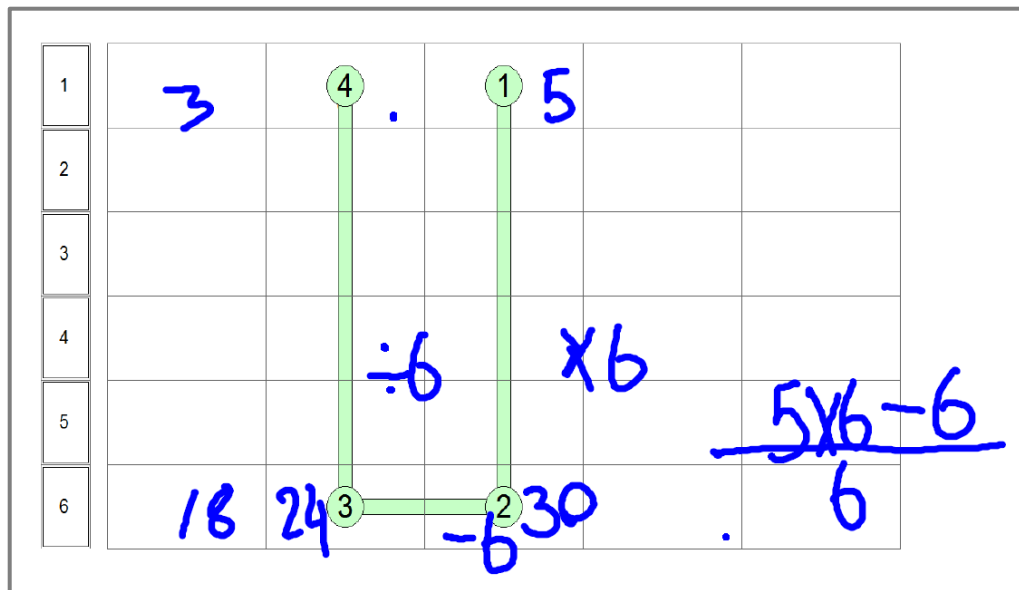
Jordan, Omar, and, to some extent, Dwayne seem to have refused to give unevaluated numerical expressions in stages ② to ④. This is consistent with Collis's (1974) finding that some students are bound to be uncomfortable with lack of closure and tend to give single values even when a numerical expression is expected. Nevertheless, this did not seem to hinder students' learning about the order of operations in multi-operation expressions.

Learning about the order of operations was the most, but not the only, significant conceptual development with regard to multiplication and division notation. Opposite vertical movements in the GA grid helped students realise that multiplication and division by the same amount retained the original value of an expression. The following points sum up the findings of Interviews 2-4 regarding multiplicative inverse:

- In Interview 2, only Dan knew how to evaluate $567 \times 123 \div 123$ straightaway. This time, Dwayne intended to work out long multiplication first.

- In Interview 3, all students but Dwayne knew how to give the value of $451 \times 999 \div 999$ without any calculation. Once again, Dwayne intended to evaluate the multiplication operation first.

- In Interview 4, students reaffirmed their knowledge of multiplicative inverse in their evaluation of $\frac{233 \times 676}{676}$ where Dwayne started doing the long multiplication 233×676 and when I asked him what he would do with the answer he said he was going to divide it by 676. If I presented him with $233 \left(\frac{676}{676}\right)$ he might have multiplied 233 by 1, as he did in Interview 1, but unfortunately I did not probe further.

Students using multiplicative inverse gave the following reasons for their working:

▶ Dan: "Because this [*making a circle around* $676/676$] stays as it was and so what remains is that [*makes a circle around* 233] answer";

▶ Jordan: "Times and plus are almost the same…"

▶ Joseph: "The same as that [*referring to* $5445 + 9997 - 9997$], if you move here [*takes a stride forward*] you do times or plus and if you do this [*takes a stride backward*] you do division or minus."

▶ Omar: "Uh, it's like…how shall I put it…it's like you're doing the same things. This is like, the times and division, like plus and minus [*points to* $5445 + 9997 - 9997$]."

▶ Tony: "Because he is telling you, give me something and I'll give it back to you. It's like when lend something. You give it back to him. It goes and comes back again."

Overall, GA seems to have helped students to extend their concepts of notation by enriching their APS representations of numerical expressions during informal-algebraic activities. Table 7.3.3.2 gives a summative analysis of students' representations and interpretations of the multiplication and division notation with reference to the CAPS framework.

Table 7.3.3.2   Students' CAPS for multiplication and division

GA	Symbolic (notational) Representation	Conceptual (signified) Interpretation	Active (kinaesthetic) Representation	Pictorial (drawing/diagrammatic) Representation	Code	Students
	5×4	Process of counting multiples	Finger counting or hand gesture			• Dan • Dwayne • Jordan • Joseph
	$6 \div 2$	Inverse of multiplying by 2 or halving				• Dan • Dwayne • Jordan • Joseph • Tony
	5×4 $6 \div 2$	(Representations not enough to suppose concept)	Finger counting			• Omar
✓	$\dfrac{12}{2} + 4$	Proceptual view; Order of operations	Vertical and horizontal movement in GA grid			■ All
✓	$\dfrac{12}{2} + 4$	Proceptual view; Order of operations	Vertical and horizontal movement in GA grid			■ All
	$\dfrac{2(1+3)+2}{2}$	Proceptual view; Order of operations	Vertical and horizontal movement on paper grid			■ All (varying degrees)
✓	$2\left(\dfrac{12}{2}\right)$	Multiplicative Inverse (process)	Vertical inverse movement in GA grid			■ All
	$\dfrac{233 \times 676}{676}$	Multiplicative Inverse (process)	Return journey			• Joseph • Omar
	$\dfrac{233 \times 676}{676}$	Multiplicative Inverse (process)	Borrowing and lending simile			• Tony
	$\dfrac{233 \times 676}{676}$	Multiplication by 1 (identity)				• Dan
	$\dfrac{233 \times 676}{676}$	Multiplicative Inverse like Additive Inverse				• Jordan • Joseph • Omar

Students' developments of their interpretation and representation of multi-operational expressions paved the way for the introduction of the letter notation and algebraic expressions. This is the focus of the next section.

7.4   Notation for Unknowns and Variables

In this section, I focus on students' coming to know about the use of letters in expressions. Research in the teaching and learning of school algebra has shown the importance of helping students to gain expression structure sense (Linchevski & Livneh, 1999, 2002) especially when rules behind such structures are not visually evident in algebraic expressions (Kirshner & Awtry, 2004). Students' failing to interpret algebraic expressions in the conventional manner includes misinterpretations of concatenations (Herscovics & Linchevski, 1994), improper simplification (Hewitt, 2012), misuse of brackets (Kieran, 1979; Booth, 1984), and misapplication of inverse operations (Borg, 1997).

I decided to introduce the letter on its own in a GA environment to help students to develop a sense of structure in algebraic expressions with single letters that could stand for unknowns or variables. My intention was to help students to enrich their representations and interpretations of the letter notation and manipulate algebraic expressions without having to deal with equations.

7.4.1   Students' Initial Interpretations and Representations

Before starting Grade 7, all students were familiar with the use of letters that stood for quantities. In particular, they used L and B to denote respectively the length and breadth of a rectangle which they used in formulas they learnt at Grades 5 and 6 for area and perimeter. In such circumstances, students had learnt to interpret L as "Length", i.e. the longer side of a rectangle, rather than the *value* of the length. This was evident during lessons where I taught the students about mensuration in the first months which were part of the Grade 7 syllabus but not part of the research data.

The first time I formally investigated students' representations of a letter in algebraic expressions was in Interview 3, where students had not yet encountered the use of letters in a GA setting. The question was an extension of the question shown in Figure 7.3.3.3, which required students to give the expression resulting when the number 5 made a 4-stage (3-step) journey in a simulated GA grid. As pointed out earlier, only Dan, Joseph, and Tony gave a numerical expression, rather than a single number, for every stage of the journey. The question that followed was the following:

Suppose instead of the number 5 you had "y".

What would the cell marked ④ contain after the journey ①→②→③→④ ?

Dan converted his previous $\frac{6(5)}{6} - 6$ (which was incorrect in that context) into $\frac{6(y)}{6} - 6$ and Tony converted his $6 \times 5 - 6 \div 6$ into $y \times 5 - 6 \div 6$. Joseph said that, since he did not know what number y was, he could not say what would appear. He seems to have been thinking that expressions could only contain numbers. None of the students who, in the previous question, evaluated the value of the cells at each stage of the journey came up with a possible answer to this question. One possible explanation for this could be that students who found it hard to accept expressions like $\frac{6(5)-6}{6}$ as an answer would similarly find difficulty in giving answers such as $\frac{6(y)-6}{6}$. It seemed, therefore, that students needed to continue learning to:

(i)   convert a series of operations on a number or letter into expressions with conventional notation, and

(ii)   become accustomed to giving answers which were not a single numerical value.

The more immediate teaching objective at this point was, however, to introduce the use of letters as unknowns/variables, which I intended to do with the help of GA. This is discussed in the next section.

7.4.2 Enriching Students' Interpretations and Representations

The first time I introduced the use of a letter in the GA environment was in Lesson 12, where letters were placed in a cell after a number was assigned to another cell, as shown in Figure 7.4.2.1. Since students were aware that GA cells represented numbers they readily accepted that since GA also allowed letters in its cells, then letters must stand for numbers. Since a number was previously entered in the grid, any letter entered in the grid constituted a "hidden" constant, an *unknown*.

Figure 7.4.2.1 Students entering numbers and letters in the same cell
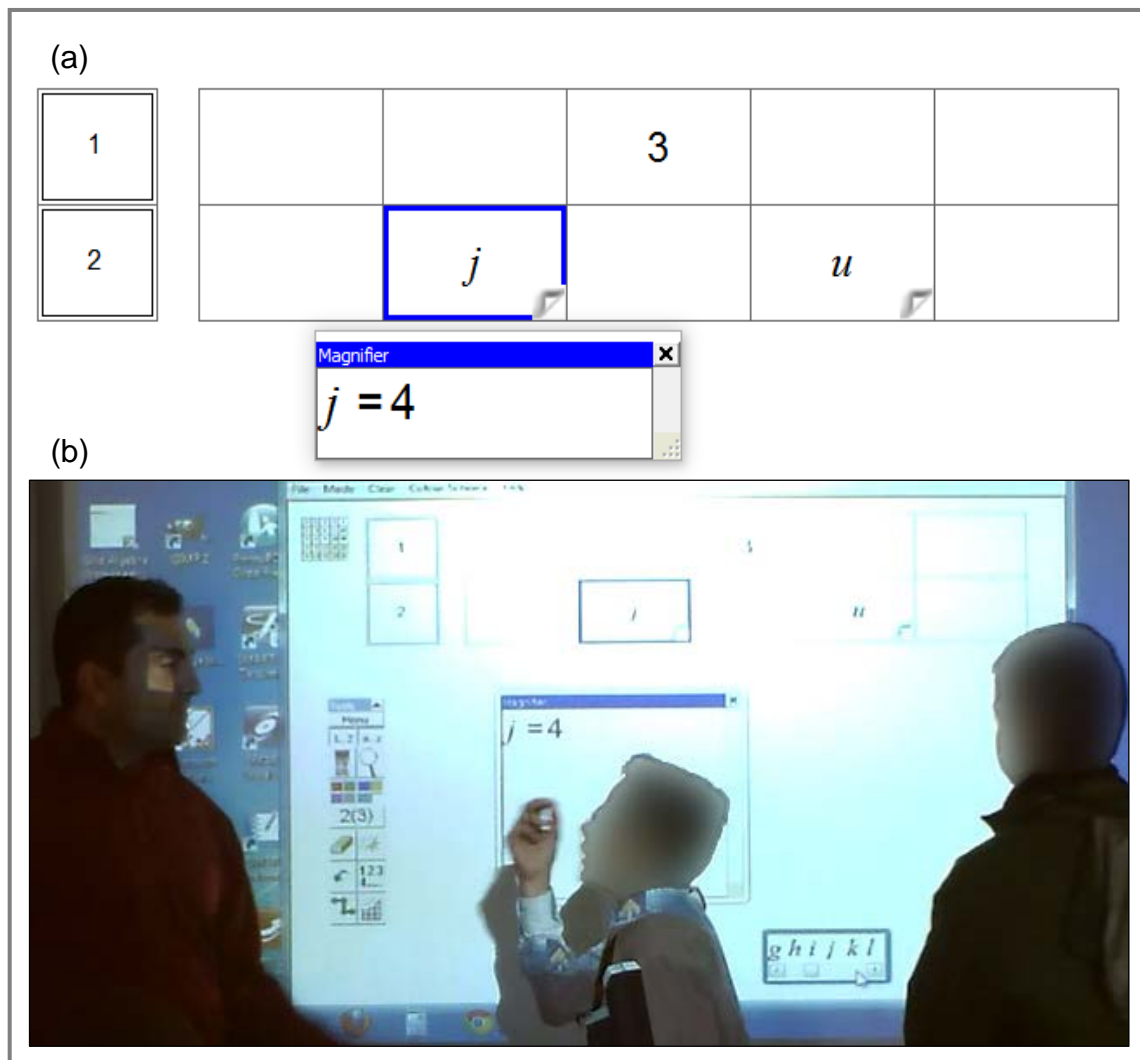
During the plenary session, students came out to work on the IWB and inserted possible numbers on the same cell as the ones having a letter. Figure 7.4.2.1a shows

one such context, where Sharks were collaborating in assigning numbers for the letters j and u. The magnifier on R$_2$C$_2$ showed $j = 4$. I asked the students why I could not place another j anywhere I wanted, say, in R$_1$C$_1$. Figure 7.4.2.1b shows Dan explaining to me why not: "Because j equals 4. Here [*pointing at R$_1$C$_1$*] equals 1."

In Lesson 13, I engaged students in similar discussions but without entering previous numbers in the grid. This made any letter entered in the grid a *variable* multiple of the row number. Excerpt 7.4.2.1 is taken from this discussion.

Excerpt 7.4.2.1   Lesson 13

PB: [*Dragged the letter k from the letter menu into an empty grid.*]
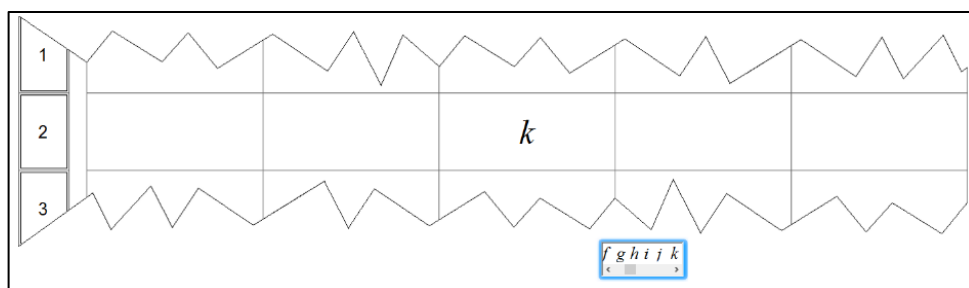
The k. What number is it symbolising? [*Dwayne raised his hand.*] Careful! Careful and reply…

Dwayne: [*Wiggling his finger frantically.*] Sir, I know. I know.

PB: …think a bit about it before replying. [*Joseph raised his hand.*]

Tony: [*Without asking for permission*] Any number [*opening his palms forward*].

Dwayne: [*Still waving frantically.*] Sir! Sir!

PB: [*Nodding towards Dwayne.*] Come. Let me see.

Dwayne: Every number there is… [*Joseph joins him.*]

Joseph and Dwayne: …in the 2-times table. [*Look at each other.*] Jinx!

Since we had already established that letters stood for numbers, I asked what *number* k symbolised. Tony said, "Any number," but Joseph and Dwayne elaborated that it had to be in the 2-times table. The symbol k gained meaning from its context (Mercer, 2000), where the absence of any number in the grid meant it could be any number, but since it was in R_2 it had to be a multiple of 2.

In GA, letters could be manipulated just like any other number. Once students got the idea that letters could either stand for unknowns or variables, they could proceed to transform these letters into algebraic expressions in the same way they had done with numerical expressions. In Lessons 15-17, all students successfully tackled formal-algebraic GA tasks which were similar to the ones discussed earlier but involving letters. Students had the opportunity to switch between APS representations of algebraic expressions. These activities are listed in Table 7.4.2.1 which includes screenshots of students' work.

Interestingly, students did not find it any harder to work with algebraic expressions than they did with numerical expressions. As Hewitt (2001) suggested, when students focus on the operations rather than on evaluations it makes only a very slight difference whether an expression originated from a number or from a letter. Rather than finding it more difficult to work with letters, students made fewer mistakes like those discussed in Section 7.3 since they had had more experience with GA. Table 7.4.2.1 shows examples of the rather complex expressions that students were working with.

Once more, GA was instrumental in helping students get accustomed to lack of closure (Collis, 1974) of algebraic expressions and to interpret these proceptually (Gray & Tall, 1994). This enabled them to manipulate complex expressions and to consolidate their knowledge about the conventional order of operations. Some activities, like GA Tasks 13 and 14 allowed students to adopt a trial-and-error approach which may have helped them to learn from and correct their own errors.

Table 7.4.2.1   Students' representations of the variable notation with GA

Tasks	Title	Screenshots
13 and 14	*Make the Expression – Letters* Screenshot from **Chimps** (Lesson 15)	
7 and 9	*Find the Journey – Letters* Screenshot from **Pandas** (Lesson 16)	
4	*Equivalent Expressions (Letters)* Screenshot from **Sharks** (Lesson 17)	

GA Task 4 – *Equivalent Expressions – Letters* required students to work on notation with minimal action and picture representations. Students were given a letter in a random cell and another empty cell was highlighted. They were also given an "expression calculator", a virtual calculator in which students could enter the necessary letters, numbers, and operators to form the expression that would occur in the highlighted cell if the given letter was transported to it. Students needed to enter three distinct but equivalent expressions, associated with different imagined journeys, for the highlighted cell.

The screenshot of GA Task 4, included in Table 7.4.2.1 above, shows the moment when Dan completed his expression. Figure 7.4.2.2 contains screenshots showing how he achieved it through successive clicks on the expression calculator. At each click, the calculator showed him the expression he was creating.

Figure 7.4.2.2   Sharks Creating Algebraic Expressions
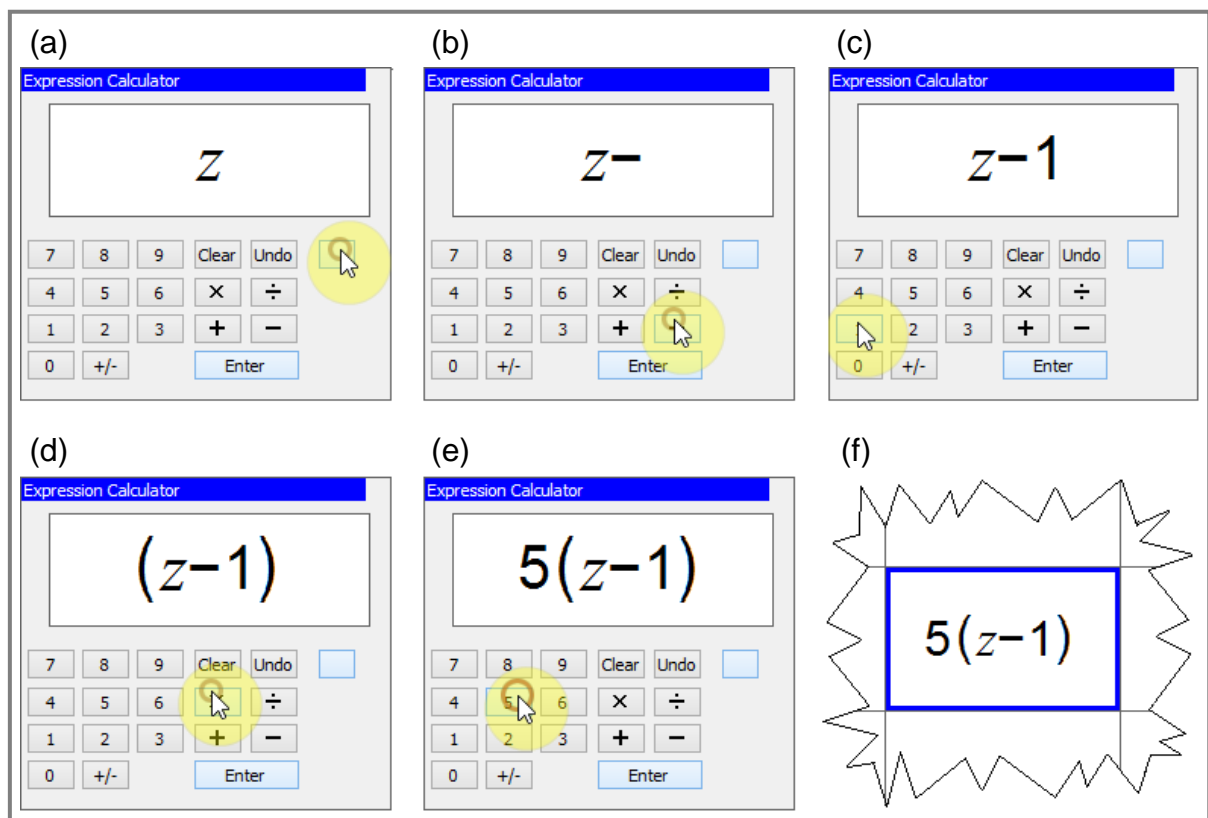
When Dan clicked on $\times$ to multiply the expression $z-1$, GA enclosed this expression in brackets (d), helping him to form a proceptual interpretation of $z-1$. Then he clicked on 5 and this appeared before the brackets (e), showing, therefore, a conventional notation of multiplication by 5. As soon as he pressed "Enter" the expression $5(z-1)$ disappeared from the calculator and was shown in the destination cell (f).

When Joseph took his turn to create the next equivalent expression he clicked: $z \times 9000 \div 9000 - 1 \times 5$ and the calculator showed him $5\left(\frac{9000z}{9000} - 1\right)$. In this way he was reusing Dan's expression by making inverse operations that kept the letter z in its original place. Then he attempted the third expression by using the same trick. He altered the imagined journey, got the required expression $5\left(\frac{9000z}{9000}\right) - 5$, and decided to elaborate it with repeated additions of $+5-5$. In this way, besides helping Sharks to enrich their representations and interpretations the letter notation, Task 4 seems to have helped them to consolidate concepts about the order of operations and about the additive and multiplicative inverse.

As before, students were assigned pen-and-paper work which mimicked a GA environment. This helped students to apply concepts they developed in computer environments to pen-and-paper problems. Table 7.4.2.2 gives a number of students' interpretations and representations of unknown/variable along with some analytic comments.

All students managed to complete these worksheets with varying degrees of performance. Jordan, Joseph, and Dwayne got all problems correct, Tony made a few mistakes, while Omar and Dan struggled with expressions involving three or more operations.

Table 7.4.2.2   Pen-and-paper interpretations and representations of unknowns/variables
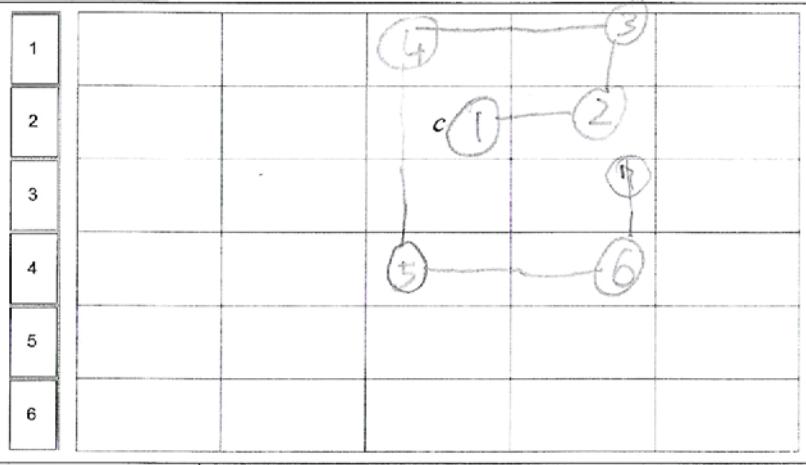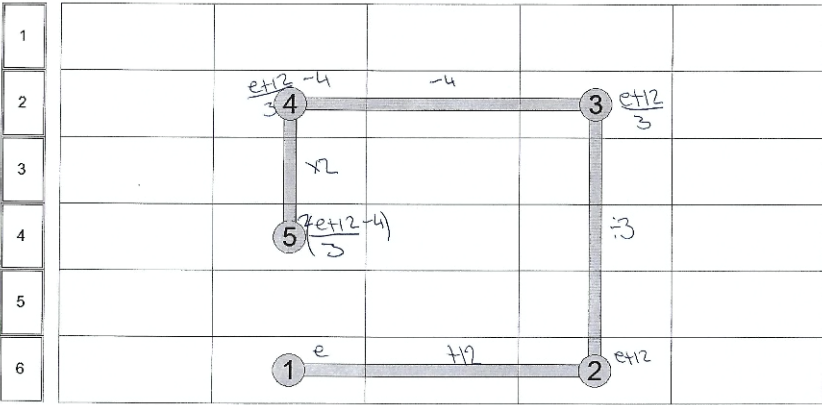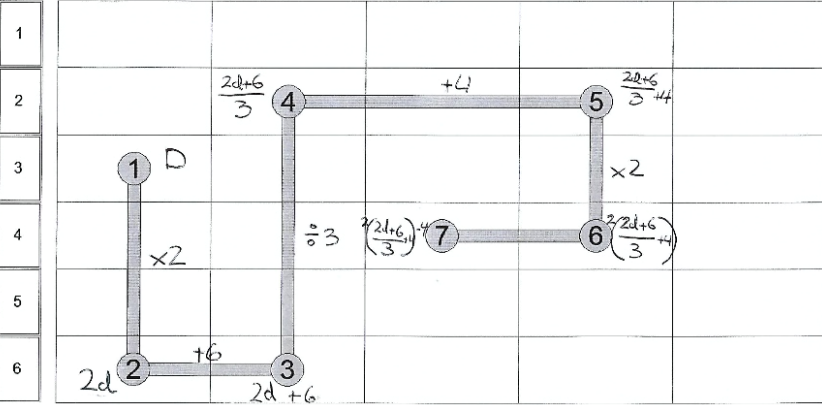
Student & Code	Interpretation and Representation	Comments
Jordan (w/sheet 13)		• Letter as unknown; • Cell interpreted as value of letter.
Omar (w/sheet 14)	$\dfrac{2(c+2)+4}{2}$	• Letter as variable; • Operations represented as journey picture • Algebraic expressions represented as journey stages • Omar makes mistake in last two operations
Tony (w/sheet 14)	$\dfrac{2\left(\dfrac{b}{3}+1\right)+2}{2}$	

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Table 7.4.2.2   (continued)

Student & Code	Interpretation and Representation	Comments
Dan (w/sheet 15)	$$\frac{4\left(\frac{c+2}{2}-1\right)+4}{2}$$	• Similar to Worksheet 15 but with 6-row grid
Dwayne (w/sheet 16)		• Journey picture interpreted as process; • Intermediary journeys interpreted as operations; • Each journey stage interpreted as an algebraic expression in conventional notation (procept); • No distinction yet between uppercase and lowercase letters
Joseph (w/sheet 16)		

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When making connections between grid journeys and algebraic expressions, students found it helpful to think of these as railway routes, where "train stations" ①, ②, ③ *etc.* represented manipulable algebraic expressions (proceptual view) and connections between stations represented operations performed on those expressions. The picture of the whole journey represented the series of operations (process) needed for the final station (product). Students were thus interpreting algebraic expressions as both processes and products. This was most evident in Worksheet 16 where, after seeing that students liked the railway route metaphor, I suggested they also write the intermediary operations to help them connect between one expression and the next, something which went beyond what GA would actually show. The railway route metaphor was only possible because each expression contained a single, unrepeated letter. It would have not been possible for an expression like, say, $2(d-3)+4d$ without changing this expression into $6d-6$ or $6(d-1)$ first, since the first "train station" was attributed to the value of the (single) letter.

I found the GA formal-algebraic activities and pen-and-paper tasks discussed above very helpful in broadening and developing students' repertoire of representations of the letter notation, where they became more conversant with conventional notation of algebraic expressions and flexible in switching between representations. This development occurred in parallel to a similar development of conceptual interpretations of the letter notation, which is discussed below.

7.4.3   Extending Students' Concepts

By the end of the GA lessons all students had become familiar with the use of letters in algebraic expressions. They distinguished between unknown and variables, even though they did not usually use this terminology. In Interview 4, all students interpreted $2a+6$ as an expression which yielded an answer. In a subsequent question, they also represented it as a journey which they drew on a mock GA grid. Except for Dan, all students drew the journey as it would have appeared in GA. Dan was a bit confused about the meaning of $2a+6$ because he evaluated it as $8a$. This was a one-off misinterpretation because he did very well in subsequent lessons and pen-and-paper assignments, as I will shortly demonstrate.

I got an interesting response when I asked Jordan what $2a + 6$ meant to him. Without any knowledge of the subsequent question, he drew a spontaneous picture of a multiplication grid (Figure 7.4.3.1) to represent to me what $2a + 6$ signified to him.

Figure 7.4.3.1   Jordan's representation of $2a + 6$
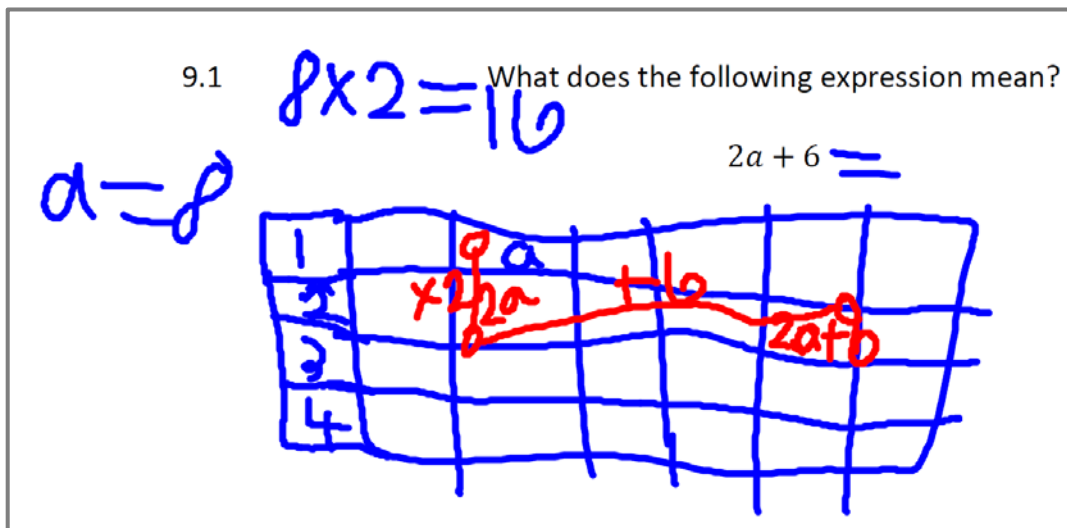
Jordan explained that a could be any number, say 8, and $2a + 6$ meant the result of its being multiplied by 2 and added to 6. The following excerpt shows his interpretation of the journey he drew.

Excerpt 7.4.3.1   Interview 4-Jordan

> PB: Now. Tell me a bit about this, uhm, [*pointing to the journey drawing*] the things in red that you made.
>
> Jordan: So. This is station "a" [*pointing to R_1C_2*] and then the road [*pointing to the route*] is called "times 2" and then station 2 [*pointing to R_2C_2*] is "station $2a$"…
>
> PB: Uh-huh.
>
> Jordan: …[*pointing to the route*] and then the road to station 3 is "plus 6"
>
> PB: Uh-huh.
>
> Jordan: …and then station 3 [*pointing to R_2C_5*] is called, um, "$2a$ plus 6"

The railway route metaphor helped Jordan to create a proceptual interpretation of $2a + 6$, without having to know the value of a and without having to evaluate $2a + 6$.
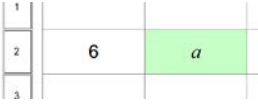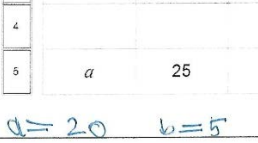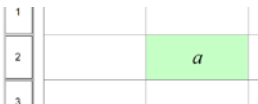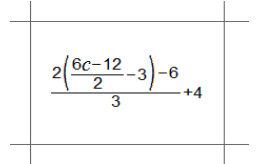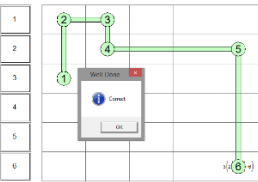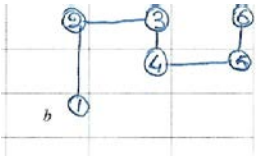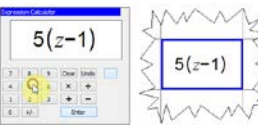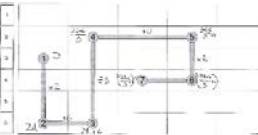
In Interview 5, all students except Dan interpreted $\frac{2(h+7)}{5}$ as h (any number) added by 7, then multiplied by 2, then divided by 5. Dan inverted the last two operations. Subsequent questions in Interview 5 were about evaluating algebraic expressions after substituting numbers for letters. These are shown in Table 7.4.3.1 which specifies whether such expressions were possible in GA and the students who evaluated them successfully. The table also includes comments about any remaining misconceptions.

Table 7.4.3.1　Students' substitution and evaluation of algebraic expressions

Expression	Possible in GA?	Students	Comments on Misconceptions (if any)
$\frac{2(h+7)}{5}$ $(h = 3)$	Yes (not with these numbers)	▪ Dwayne ▪ Jordan ▪ Joseph ▪ Omar ▪ Tony	• Dan interpreted this as $2\left(\frac{h+7}{5}\right)$.
$5 + 3e$ $(e = 4)$	No	◼ All	
$10 - (u + 1)$ $(u = 6)$	No	▪ Dan ▪ Dwayne ▪ Jordan ▪ Joseph ▪ Tony	• Omar worked out 10-7, obtained 3, but then got confused because he thought the bracket meant he had to multiply by something. Then he multiplied 3 by 7.
$\frac{5\left(\frac{y}{2}\right) + 5}{4}$ $(y = 6)$	Yes (not with these numbers)	▪ Dan ▪ Dwayne ▪ Jordan ▪ Joseph ▪ Tony	• Omar worked it out as if it were $\frac{5\left(\frac{y}{2}+5\right)}{4}$.
$\frac{3\left(10 - \frac{c}{3} + 2\right)}{6}$ $(c = 12)$	No	▪ Dan ▪ Jordan ▪ Joseph ▪ Omar ▪ Tony	• Dwayne used BIDMAS (as he did prior to the lessons) and worked it out as if it were $\frac{3\left(10-\left(\frac{c}{3}+2\right)\right)}{6}$, where he detached one of the terms $\left(\frac{c}{3}\right)$ from its sign. Such an error was identified by Linchevski and Herscovics (1996).

Albeit a few remaining misconceptions, it seems that GA formal-algebraic tasks helped students to extend their conceptual interpretations of the letter notation by enhancing their repertoire of APS representations of unknowns/variables in algebraic expressions. Table 7.4.3.2 gives a summative analysis of students' representations and interpretations of the letter and formal-algebraic expressions with reference to the CAPS framework.

Table 7.4.3.2   Students' CAPS for letters and algebraic expressions

GA	Symbolic (notational) Representation	Conceptual (signified) Interpretation	Active (kinaesthetic) Representation	Pictorial (drawing/diagrammatic) Representation	Code	Students
✓	a	Letter as unknown				■ All
	a	Letter as unknown				■ All (varying degrees)
✓	a	Letter as variable				■ All
✓	$\dfrac{2\left(\frac{6c-12}{2}-3\right)-6}{3}+4$	Variable; Proceptual View; Order of operations	Vertical and horizontal movements in grid			■ All
✓	$3\left(2\left(\dfrac{x}{3}+1\right)+6\right)$	Variable; Proceptual View; Order of operations	Vertical and horizontal movements in grid			■ All (varying degrees)
	$\dfrac{2\left(\frac{b}{3}+1\right)+2}{2}$	Variable; Proceptual View; Order of operations	Vertical and horizontal movements on paper grid			■ All (varying degrees
✓	$5(z-1)$	Variable; Proceptual View; Order of operations				■ All
	$2\left(\dfrac{2d+6}{3}+4\right)-4$	Variable; Proceptual View; Order of operations				■ All (varying degrees

Students' substitution and evaluation of algebraic expressions, such as those included in Table 7.4.3.1, were important representations and interpretations of letters as unknowns. However, these were purposefully not included in Table 7.4.3.2 because this work contributed substantially in helping students to extend their meaning of the equality notation. This is discussed in the following section.

7.5 Notation for Equality

The equals sign (ES) was an important notation in this study because I considered it to be a determining factor in students' success in formal-algebraic activities (McNeil et al., 2006). Lessons about ES occurred during GA activities and also in discussions that were outside the GA environment but still triggered by activities within GA.

7.5.1 Students' Initial Interpretations and Representations

Students' original conceptions of ES were mainly operational, where they interpreted equations like $4 + 1 = 5$ as "$4 + 1$ *makes* 5". This confirmed the results of forty years of research about ES (e.g., Behr, Erlwanger, & Nichols, 1976; Kieran, 1981b; Herscovics & Linchevski, 1994; McNeil, 2008), that primary school arithmetic inculcates perceptions of ES as an indicator of a computational result. Table 7.5.1.1 shows excerpts of students' responses in Interview 1 when presented with a standalone ES and asked what it meant to them.

Table 7.5.1.1   Interview 1 – initial interpretations of ES

Student	Meaning of ES	Script (if any)
Dan:	…To say what **the answer becomes**.	$=$ $4+1=5$
Dwayne:	To show you that that **becomes the answer**.	
Jordan:	… So that you are doing **the answer**, what it is **going to be**.	$=$ $3+4=7$
Joseph:	…So that the examiner would know that that is **the answer**.	$=$ $5+6=11_{ANS}$
Omar:	Equals means that **the answer is almost ready**.	
Tony:	The **last answer**.	

These responses, especially phrases in bold, suggest that students interpreted ES as an operational symbol. I do not regard this as a misinterpretation or misuse of ES, as reported by some researchers (e.g. Powell & Fuchs, 2010; Vincent, Bardini, Pierce, & Pearn, 2015). Jones and Pratt (2012) make us aware of the several uses of ES, one

of them being that of serving as an indicator of a computational result. In the absence of alternative conventional notation to symbolise the evaluation of an expression, it is reasonable to argue that these students' interpretation of ES was limited but not mistaken. All they needed was, therefore, to extend their current notion of ES.

Contrary to what some researchers reported (e.g. Falkner, Levi, & Carpenter,1999) for most of these students ES was not unidirectional. In a subsequent question in Interview 1 showing $\square = 3 + 5$, students were asked to fill in the box. Except Dan and Dwayne, all students wrote 8 in the box. Dwayne did not know what to do, but Dan immediately wrote 3. His explanation was that since there was nothing to show what he needed to do before ES, he wrote 3 since it would remain the same as the 3 on the right. This response seems to include elements of interpreting ES as a relational symbol but in Interview 2 Dan did not show a relational understanding of ES. For him, ES did not seem to be a symbol that signified a relation between the left and right hand sides of the equation. Rather, it signified that the expression on one side was required to be evaluated and the value was to be placed on the other side. This was an operational view of ES.

In Interview 1, only Omar and Tony wrote 3 in the box of: $7 + 3 = \square + 7$. Omar explained that since the numbers on the LHS made 10 he wanted those on the RHS to make 10 as well. Tony did not work out the sum on the LHS. His reasoning was that on the right it was written as in the left but in the opposite direction. These responses suggest that besides an operational view of ES, Omar and Tony held also a relational view. This implies that students may hold both an operational and a relational view of ES, which is consistent findings of McNeil (2008) and Rittle-Johnson et al. (2011). These studies show that the change from an operational understanding of ES to a relational one is not a switchover but a gradual transition through a continuum (Rittle-Johnson et al., 2011) in which students are usually reluctant to let go of the operational view completely (McNeil, 2008). In fact, Omar and Tony seemed to hold multiple views of ES that manifested themselves in different contexts.

The following subsection shows how GA served as a springboard for discussions to help students to enrich their representations and interpretations of ES. In the process, students learnt about the additive commutativity property which was important for them in making sense of some expressions which were not possible in GA.

7.5.2 Enriching Students' Interpretations and Representations

During a discussion on inverse operations in Lesson 11, I decided to focus on the equation $4 = 4 + 6 - 6$ shown by the GA magnifier. Jordan said that it seemed to be inverted and Joseph said it was like showing you where the answer 4 (on the LHS) came from. Both students seem to have interpreted ES as "makes" (operational) notation. I wrote this equation on the board and tried to see what students thought when I wrote the RHS as 4, transforming the equation to $4 = 4$.

Excerpt 7.5.2.1   Lesson 11

> PB: So, $4 = 4$ [*writes it and underlines it*], is it something valid?
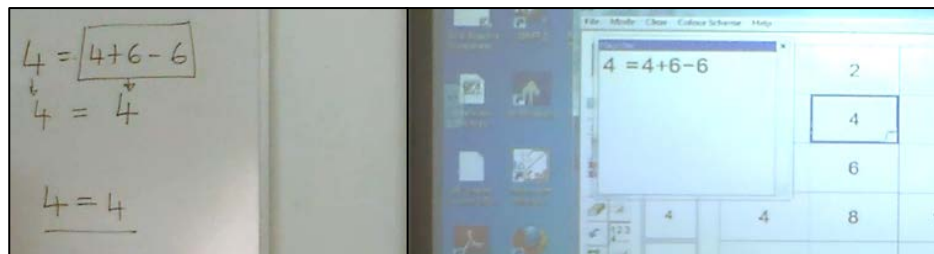
> Joseph: No.
>
> PB: Why not?
>
> Dwayne: Because there is no working.
>
> Joseph: Exactly.
>
> PB: But, [*pointing*] what does "equals" mean, then?
>
> Dwayne: The answer.
>
> Joseph: The answer.

On reviewing my learning offer, I decided that my attempt to get students to acknowledge the conventional validity of $4 = 4$ was unsuccessful. Thus, I modified my learning offer to include a balance representation (for ES) from students' experiences. I thought of mentioning the balance scale but suspected that some of them might never have seen one. The see-saw seemed to be a more promising metaphor. First, I focused on the word "equals" (which is what we use in Malta for ES). I emphasised that it should actually be read, "is equal to," and used the closest translation, "*huwa daqs,*" which literally translates to "is the same size as." Then, I suggested that the two

sides of ES were like a see-saw that had equal weights keeping it horizontal. I drew a balance beam on the board with 4 bags of flour (1 kg each) on each side keeping it in balance. I anticipated that students might appreciate the balance more if they were presented with a situation of imbalance, which would allow me to bring in $4 = 4 + 6 - 6$. The episode that follows was the discussion that ensued.

Excerpt 7.5.2.2   Lesson 11

PB: [*Referring to drawing #1 on the board.*].
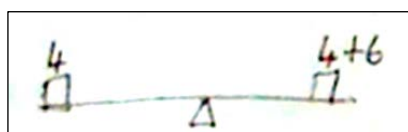
If I…added 6 kilos of flour, what happens to this side of the balance, uh, the see-saw?

Dwayne: It goes down, that of 4 plus 6 [*PB making an imbalance gesture*] and then the 4 goes up.

PB: Now, when I realise that it went down, if someone else comes and does minus 6 [*writes −6 next to the +6*]. Isn't minus 6 like decreasing…?

Dwayne: The same.

PB: ….decreasing those 6 bags that I added?

Joseph: Uh-huh. It becomes equal.

PB: What happens to it?

Joseph: It becomes equal.

PB: It becomes equal once again. It becomes balanced once again.

Dwayne: It would do that, Sir. It would be like that [*does gesture #2, then #3*].

Students seemed to be relating to the see-saw metaphor and this could have been helpful in their interpretation of ES as a balance between equal quantities. In the above episode, I emphasised the "balance" and "equal" similarity by swapping these words in the same sentence "It becomes equal/balanced once again." This may have prompted Dwayne to make a gesture of regaining balance. The discussion developed to include problems like $3 + 4 = \square + 3$ for which students found the see-saw metaphor helpful to guess what will happen if 7 was put in the box (a common mistake in Interview 1). They responded by making imbalance gestures (Figure 7.5.2.1).

Figure 7.5.2.1   Students' imbalance action representation

This action representation helped me to create a model of students' conceptualisation of ES: they seemed to appreciate the conditions in an equation for balance to be maintained. This led to a Learner-to-Negotiation shift, where I seemed to change my focus from coordinating discussion and reflection to hypothesising on learners' interpretations and reviewing the learning offer. This instigated further M-N-L cycles where further reflection on the balancing property of ES motivated Dan to come up with another ES metaphor.

Excerpt 7.5.2.3   Lesson 11

> Dan: Like those [*meaning weightlifters*], Sir, don't they lift that iron bar like that? [*Stands up and takes a weightlifter pose with his hands up.*]

PB: [*Nodding.*] The iron bar. Well done!

Dan: If on this side [*gestures towards his left hand*] you have much more, then this side [*left*] will topple [*makes a toppling movement*]...

...and he won't be able to keep the balance.

Dan's action representation helped me to form a model about his understanding of the notion of equality/inequality in terms of balance/imbalance. On reviewing the current learning offer (thus making a Learner-to-Negotiation shift), I felt that the concept of balance, which students seemed familiar with through their experiences of riding a see-saw or watching weightlifting, had been a viable interpretation for some students (at least Dwayne, Joseph, and Dan) to think of ES as having equal quantities on each side.

When students seemed to have developed balancing/sameness interpretations of ES, we returned to our original GA equation. This enabled me to discuss further examples of additive and multiplicative inverses, each time stressing the equality of the LHS and RHS of equations. The lesson seems to have made a complete cycle (Figure 7.5.2.2) where a discussion on ES was initiated by, taken out of, and taken back in a GA environment.

Figure 7.5.2.2   GA used as springboard for discussion
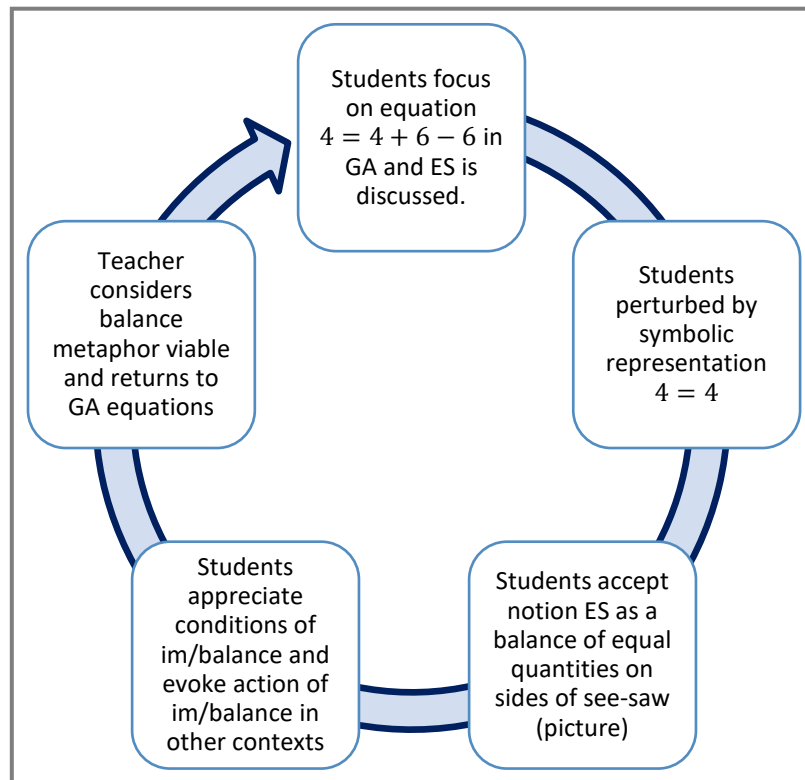
With reference to M-N-L shifts of focus, this cycle started when I changed my focus from interaction with the students to helping them reflect on the learning offer and ended when I shifted my focus from creating a model of MoS and to reviewing the learning offer. At the beginning, the only direct ES representations in GA were:

- the pictorial representation of the cell occupied by two expressions of equal quantity (4 and $4 + 6 - 6$) and

- the symbolic representation $4 = 4 + 6 - 6$ shown in the magnifier.

When GA was used as a springboard to expand the discussion, more representations of ES became possible. The drawing of the see-saw with flour bags, students' kinaesthetic simulations of an unbalanced see-saw, and Dan's weightlifting pose were all useful in students' enrichment of APS representations and conceptual interpretations of ES. On returning to the original GA equation, students used these representations to think about the notion of manipulating one side of ES to restore balance. Subtracting 6 was a corrective measure to restore the balance lost when adding 6. With this manipulation, the RHS of the equation became $4 + 6 - 6$ which

makes 4. This helped students construct a concept of *sameness* of quantities which was implied from the manipulation and computation of the RHS to have the same value as the LHS.

In Lesson 18 we discussed commutativity of addition. On the RUN function of GA, I moved the cell containing:

- 3 in R_1 by 2 units to the right achieving $3 + 2$, and

- 2 in R_1 by 3 units to the right achieving $2 + 3$.

Clicking on the cell containing $3 + 2$ and $2 + 3$ with the magnifier displayed the equation $3 + 2 = 2 + 3$. Excerpt 7.5.2.3 shows part of the class discussion that followed.

Excerpt 7.5.2.3   Lesson 18

PB: [*Pointing to each side of the equation.*]
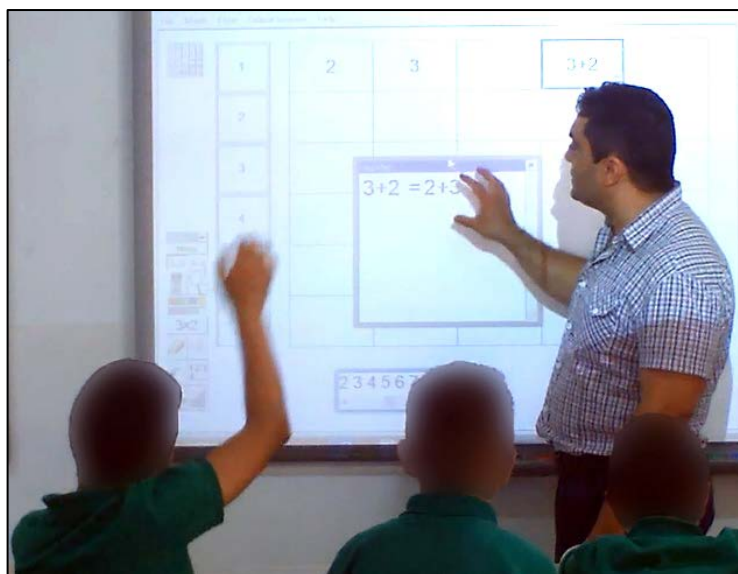
Now, tell me what you're noticing on both sides. What you're seeing. There's something special in both sides. [*Dwayne raises his hand; PB nods towards him.*] Tell me.

Dwayne: That their answer becomes the same…

PB: Their answer becomes the same. In what sense … In what sense is their answer the same?

Dwayne: Five. 3 plus 2, five and 2 plus 3, five.

It seems Dwayne still held a "makes" (operational) concept of ES. He did not just say each side "had the same value" (in Maltese "*indaqs*") but he was still thinking about "their answer". However, he seems to have extended his ES concept of "make" to include also situations where none of the two sides of the equation was a single number (as in $4 = 4 + 6 - 6$), but where each of the two sides of ES *made* the *same* value. Dwayne seemed to be using his "makes" (operational) interpretation of ES as a stepping stone to construct a same-value (relational) interpretation. The latter was what Rittle-Johnson et al. (2011) called *basic-relational* since Dwayne was still in a gradual transition from having an operational view of ES to having a relational view.

Lesson 18 resumed with a discussion of the additive commutativity property where I formed two further GA equations showing commutativity (Figure 7.5.2.3).

Figure 7.5.2.3   Equations showing additive commutativity property
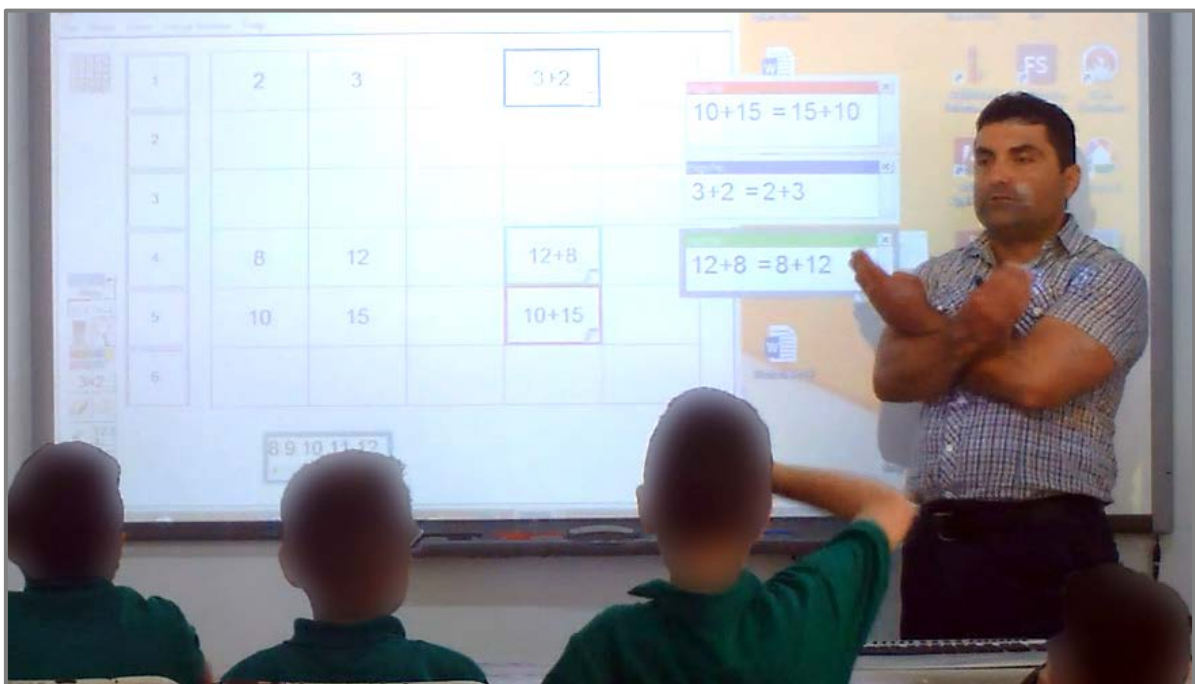
Students seemed to understand that an expression with two added numbers could be swapped (hence my gesture) and the new expression would be equal to the first. By then, students seemed to be accustomed to GA's lack of closure in cells and we started talking about equality of expressions without the need to evaluate them. While possibly helping students in taking proceptual views of expressions, each highlighted cell in

Figure 7.5.2.3 represented pictorially the quantity on each side of ES. This may have encouraged students to think about each side of ES individually and their comparison helped them to extend their ES interpretation from "makes" to the more comprehensive "has the same value as." Together with their knowledge about letters standing for generalised numbers, students made use of their modified interpretation of ES to make sense of equations like $a + b = b + a$. This generality helped them to fill in the blank of equations with awkward numbers like:

(i) $838{,}383{,}838 + 4{,}499{,}922 = \square + 838{,}383{,}838;$

(ii) $838{,}383{,}838 + 4{,}499{,}922 = \square + 4{,}499{,}922$

which they could not immediately solve by computing addition and subtraction. This interpretation of ES and an awareness of additive commutativity property were necessary in the second part of Lesson 18 where students learnt that expressions like $4 + 5 \times 2$ could be interpreted as $5 \times 2 + 4$.

In a *People Maths* activity (Bloomfield & Vertes, 2005), students acted out the role of numbers and operators as follows:

- Dwayne $\rightarrow$ 4,
- Tony $\rightarrow$ 2,
- Joseph $\rightarrow$ 5,
- Omar $\rightarrow$ Addition (hands outstretched like a + symbol), and
- Dan $\rightarrow$ Multiplication (hands on chest like a × symbol).

Standing in a straight line, students acted out the statement $5 \times 2 + 4$ (Figure 7.5.2.4a) where Dan (×) bonded Joseph (5) and Tony (2) together into a singular entity of value 10, where 5×2 could thus be interpreted proceptually. Jordan's role was to evaluate the expression embodied by his peers. He interpreted the first enactment as $10 + 4$. Figure 7.5.2.4b shows when students changed places to demonstrate $4 + 5 \times 2$. Jordan (the student sitting down facing the others) can be seen separating 5×2 from 4, enabling him to interpret this enactment as $4 + 10$.

Figure 7.5.2.4   *"People Maths" – Numerical expressions*

This action representation of equal values was later translated to symbolic form as:

$$5 \times 2 + 4 = 4 + 5 \times 2,$$

where each actor wrote his corresponding number/operator on the board. While helping students to interpret the conventional order of operations of expressions which were not possible on GA (such as the RHS), this exercise seems to have helped them to develop relational interpretations of ES by giving them opportunities to link action to symbolic representations of equivalent numerical expressions.

Work on representations and interpretations of the letter notation, algebraic expressions, and ES paved the way for *substitutive* interpretations of ES (Jones & Pratt, 2012). In Lessons 19 and 20, students worked on GA Tasks 22 and 24 – *Substitution,* where, given the value of a letter in a cell, they needed to enter the value of an algebraic expression with that letter in a destination cell. The following is an episode from Chimps' computer activity.

Excerpt 7.5.2.5   Lesson 19 - Chimps working on substitution

Omar: [*Jordan controlling the cursor (#1)*] So…9, and then minus 2: 8, 7.

#1

[*Jordan scrolls number menu.*] What do you think?

Jordan: [*Chooses* 7 *(#2).*] Good.

#2

[*Drags* 7 *onto* R_1C_3 *(#3)*].

#2

I did not know what Omar was doing when he said "So…9, and then minus 2: 8, 7." He may have been moving 18 on a grid journey or substituting 18 for c without referring to the grid at all. Thus, I checked the lesson video and observed that in the pause before he obtained 9, Omar was touching his forehead and closing his eyes. Thus, he seemed to have evaluated $\frac{18}{2}$ mentally. Then, after saying "minus 2", he clapped his hands twice while saying "8, 7" showing that he was counting backwards to do the subtraction. This was consistent with the observation I made earlier on with regards to Omar's apparent need to use his hands to count forwards or backwards when adding or subtracting. Nevertheless, it seemed that Chimps were not using the grid to make the substitution. In more difficult tasks, I observed Omar making use of a rough paper to work out the expressions, indicating that GA seems to have encouraged students to engage in pen-and-paper work. I found out that all the other students were working independently of the grid when they were evaluating algebraic expressions.

Students managed to solve substitution problems presented in Tasks 22/24 with varying performance levels. By the end of Lesson 20,

- Chimps progressed to algebraic expressions with 3 operations,

- Pandas managed expressions up to 5 operations, and

- Sharks evaluated expressions with 7 operations.

All students would have continued to progress through the levels if they were given more time. Since they were working independently of the grid, they all managed well when GA made the grid disappear and only the notation was left on the screen. This required students to substitute numbers for letters and evaluate algebraic expressions without referring to the grid structure. It seemed, therefore, that GA enabled students to wean themselves off pictorial representations (cells and journeys) of the grid and start making conceptual interpretations using only symbolic representations. With regard to ES, students seemed to have made a smooth transition from *sameness* to *substitutive* interpretations with minimal intervention from my part. These and other extensions of the concept of ES are discussed below.

7.5.3   Extending Students' Concepts

Diverse APS representations seem to have benefitted students in their conceptual interpretations of ES. Students' first significant development was when balancing representations of ES helped them to focus of each side of ES separately. Losing and restoring balance of an equation (adding and then subtracting 6 to the RHS of $4 = 4$) seemed to them that one side was being manipulated and computed to *make* the other side, i.e., to have the *same* value as the other side. It seems that to them:

$$
\begin{array}{ccc}
\text{Manipulating one side of ES} & & \text{Value of one side of ES} \\
\text{MAKES} & \Longrightarrow & \text{IS THE SAME AS} \\
\text{the other side} & & \text{the value of the other side}
\end{array}
$$

It seems that students assimilated the sameness component of ES (Jones, 2008) into their existing "makes" (operational) interpretation because *to them* the latter implied the former. This is consistent with my argument that an operational conception of ES is not a misconception but a limited conception. This argument may also explain why students do not seem to dismiss the operational conception just because they form a sameness-relational conception (Jones et al., 2013).

Jones (2008) distinguishes between the *sameness* component and *substitutive* component of equality. Collis (1975, p. 17) argues: 'If two expressions are equivalent then one may be used to replace the other at any time.' Based on my observations in this study, I find that the *sameness* and *substitutive* components seem to be necessary and sufficient conditions of equality, where Collis's argument makes sense the other way around too:

$$
\begin{array}{ccc}
\text{Value of one side of ES} & & \text{One side of ES} \\
\text{IS THE SAME AS} & \Longleftrightarrow & \text{MAY BE SUBSTITUTED FOR} \\
\text{the value of the other side} & & \text{the other side}
\end{array}
$$

Students in this study seem to have developed a sameness-relational interpretations of ES, together with interpretations of the letter notation, and algebraic expressions. This

helped them to incorporate a substituting meaning of ES (Jones & Pratt, 2012). They were capable of evaluating quite complex algebraic expressions, even in pen-and-paper work as shown in Table 7.5.3.1, which, except that of Worksheet 19 (Tony), were traditionally set substitution questions with expressions not possible in GA.

Table 7.5.3.1   Pen-and-paper interpretations and representations of ES
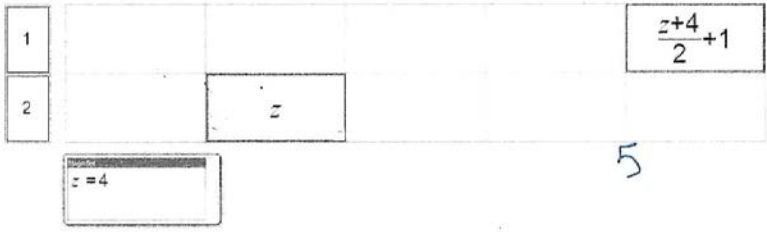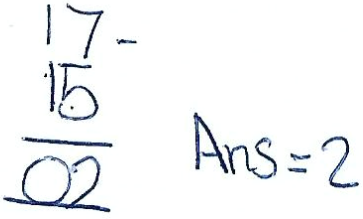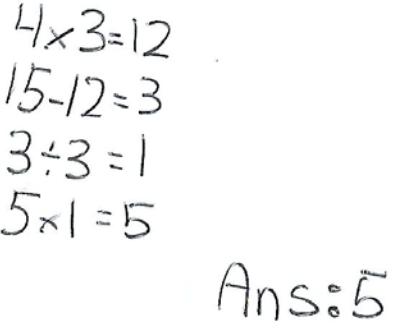
Student & Code	Interpretation and Representation	Comments
Tony (w/sheet 19)	$\frac{z+4}{2}+1$ z $z=4$ 5	• Tony worked the value menally. • May have been using the grid. • ES substitutive-relational.

Dwayne (w/sheet 20)	$17 - 3d$ $\quad\quad [d = 5]$
	
	$17 -$
	15
	$\overline{02}$ Ans = 2
		• ES substitutive-relational.
		• ES also "makes"-operational:
		  ➢ Dwayne: Ans=...;
		  ➢ Joseph: During steps.

Joseph (w/sheet 20)	$5\left(\frac{15-4n}{3}\right)$ $\quad\quad [n = 3]$ $4 \times 3 = 12$ $15 - 12 = 3$ $3 \div 3 = 1$ $5 \times 1 = 5$ Ans: 5	• Unlike Dwayne, Joseph careful not to write Ans=5 .

Table 7.5.3.1 (continued)

Student & Code	Interpretation and Representation	Comments
Dan (w/sheet 20)	$10 - \dfrac{1+\frac{12}{p}}{2}$ $[p = 4]$ $12 \div 4 = 3$ $3 + 1 = 4$ $4 \div 2 = 2$ $10 - 2 = 8$	• ES substitutive-relational. • ES also "makes"-operational during steps. • Dan used commutativity of addition to work out $1 + \frac{12}{4}$ but retained the given order in subtraction since this is not commutative.
Jordan (w/sheet 20)	$\dfrac{10-4(8-3r)}{2} = 1$ $[r = 2]$ $3 \times 2 = 6$ $8 - 6 = 2$ $4 \times 2 = 8$ $10 - 8 = 2 \div 2 = 1$	• Jordan keeps a running-total (Kieran, 1979) at the end.
Omar (exam)	$\dfrac{2(\overset{10}{\cancel{w}}-7)}{5}$ given that $w = 10$ $2 \times 3 = 6$ $6 \div 5 = 1$	• In pen-and-paper problems, Omar found it useful to cancel the letter to be substituted and place the number instead of it.

Students' performance in such pen-and-paper problems varied according to complexity. In Worksheet 20, Tony, Dwayne, and Joseph correctly evaluated expressions up to 4 operations, while Jordan and Dan managed up to 5 operations (Jordan's expression in Table 7.5.3.1). After assessing their work in Worksheet 20, I discovered that Omar and Dwayne were still sometimes confusing concatenated expressions like $2a$, where, if $a = 3$, they might interpret $2a$ as 23. Nevertheless, like the rest of the students in this study, Omar and Dwayne did very well in substitution questions with 2-3 operations set in the annual examination (e.g. Omar's expression in Table 7.5.3.1) where they both scored $9/10$ (mean $= 9.1/10$).

Students' developments of ES interpretations are included in Table 7.5.3.2 which shows students' responses when presented with a standalone ES and asked what it meant to them.

Table 7.5.3.2   Students' interpretations of standalone ES

	Interview 1	Interview 2	Interview 3	Interview 4
Dan	Makes	Makes	Makes Same	Same
Dwayne	Makes	Makes	Makes	Same
Jordan	Makes	Makes	Makes	Makes
Joseph	Makes	Makes	Makes Same Subs	Same
Omar	Makes	Makes	Makes	Same
Tony	Makes	Makes Same	Same	Makes. Same
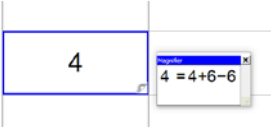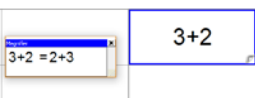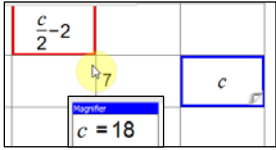
Key: *Makes→* ES indicates that a manipulation/computation "makes" an answer (operational)
Same→ ES indicates that the two sides are the same or have the same size (relational)
Subs→ ES indicates that one side may be substituted for the other (relational)

Consistent with the literature (e.g. Kieran, 1981; McNeil, 2008) students adopted a more relational understanding of ES as they started working on formal-algebraic tasks. However, some of them retained their previous operational interpretations even if they became competent in solving:

(i)   non-standard equations which were found to elicit a relational interpretation of ES (McNeil & Alibali, 2005), and

(ii)   substituting tasks which were found to help students to think about the substitutive and sameness components of ES (Jones et al., 2013).

With the exception of Jordan, who was very competent in these tasks, all students expressed a sameness interpretation of ES in Interview 4. Only Joseph mentioned substitution and when he did he also expressed sameness and operational interpretations. Students might have subsumed substitutive interpretations within their sameness interpretations. Nevertheless, students seem to have gained a flexibility in their interpretation and representation of ES which may have served them well in formal- and informal-algebraic tasks involved in this study and beyond. Table 7.5.3.3 gives a summative analysis of students' representations and interpretations of the equality notation with reference to the CAPS framework.

Table 7.5.3.3   Students' CAPS for ES

GA	Symbolic (notational) Representation	Conceptual (signified) Interpretation	Active (kinaesthetic) Representation	Pictorial (drawing/diagrammatic) Representation	Code	Students
	$=$	ES "makes"				■ All
✓	$4 = 4 + 6 - 6$	ES "makes"	Inverse movements on grid			■ All (class discussion)
	$4 = 4 + 6 - 6$	ES "makes" & balance, sameness	See-saw gesture			■ All (class discussion)
	$=$	ES balance	Weightlifter pose			▪ Dan
✓	$3 + 2 = 2 + 3$	ES "makes" & balance, sameness	Horizontal movements on grid			■ All
	$5 \times 2 + 4 = 4 + 5 \times 2$	ES "makes" & balance, sameness	"People Maths" role-play for each side of ES			■ All
✓	$\dfrac{c}{2} - 2,$ $c = 18$	ES sameness, substitution				■ All (varying degrees)
	$\dfrac{10 - 4(8 - 3r)}{2}$ $r = 2$	ES sameness, substitution				■ All (varying degrees)

Most pertinent to this study was the way APS representations helped me as a teacher to form better models of students' conceptual interpretations of ES, as they did in the other notation discussed before. For example, students making a see-saw gesture and Dan's weightlifter pose were actions that helped me to hypothesise that these students were linking ES with the concept of balance. Similarly, an action like dragging 18 to replace c in a GA cell as a response to the statement $c = 18$, helped me to hypothesise that students had developed a substitutive meaning of ES (Jones & Pratt, 2012), besides using the picture of a cell to interpret the letter c as a constant.

7.6   Students' CAPS Enabling M-N-L Cycles

In this section, I discuss how the CAPS framework was pertinent to my study of CT and how it fitted within the M-N-L framework. I start by summing up how CAPS was an analytical tool to help me investigate students' learning during the GA lessons.

7.6.1   Associations between Conceptual Interpretations and APS Representations

In Sections 7.2–7.5, I used the CAPS framework to analyse the way students developed interpretations and representations of notation by means of GA. Figure 7.6.1 summarises the associations that CAPS helped me to identify between students' representations and conceptual interpretations (continuous arrows) and among students' representations themselves (broken arrows).

Figure 7.6.1.1   CAPS associations of interpretations and representations with GA
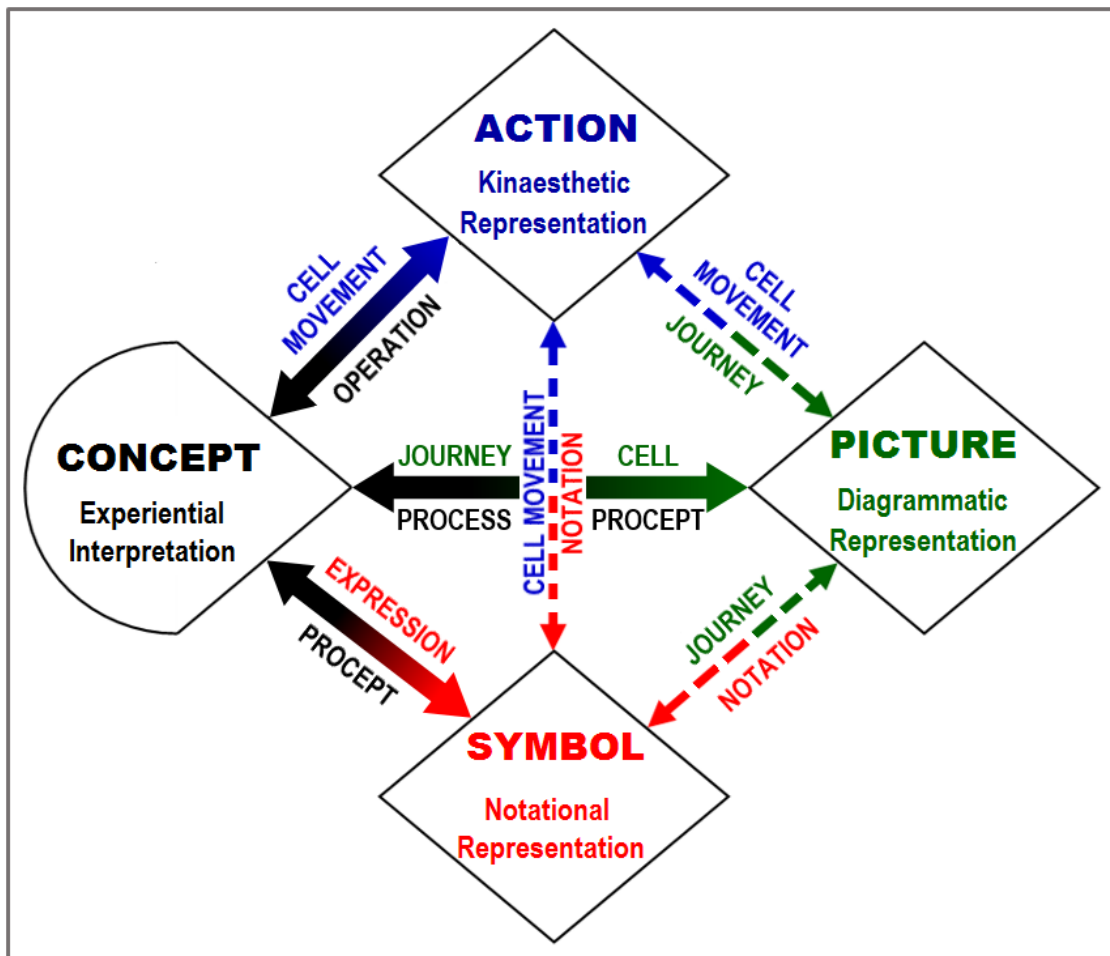
Students worked on informal- and formal- algebraic activities on GA and represented

   (i)   operations involved in expressions by the action of cell movements,

  (ii)   the process and object duality of expressions by journey and cell pictures, and

(iii)   a proceptual interpretation of expressions by conventional notation.

These associations are shown in Figure 7.6.1.1 by continuous arrows. Broken arrows show students' associations between symbols used in conventional notation, journey pictures, and cell movement (action) representations. In addition, CAPS helped me to identify similar connections between students' interpretations and representations of ES.

7.6.2   Students' Applying Concepts Learnt within GA in Pen-and-Paper Problems

By providing the opportunity to form multiple representations and switching between them, GA helped students to develop new conceptualisations of notation. Through informal- and formal-algebraic activities, all students seem to have extended their concepts of the structure of numerical and algebraic expressions, in particular by learning about new uses of notation and the conventional order of operations. GA encouraged students to develop proceptual interpretations (Gray & Tall, 1994) of expressions by requiring them to treat expressions as both processes and products. GA was also instrumental in serving as a springboard for discussions which helped students to modify their meanings of ES. Students seem to have started interpreting ES as a signifier of balance of same quantities and also of substitution, an idea which they applied to evaluate complex algebraic expressions ranging from 3 to 7 operations.

This attainment was well above curricular expectations of "low-performing" students at Grade 7: the national Core Curriculum Programme designed for such Grade 7 students (DLAP Syllabus, 2014b) does not even include algebraic expressions. The substitution-and-evaluation problems set in the annual examination, in which all students did very well (mean $= 9.1/10$), were comparable to those expected of *high-performing* students of their age group in the national Grade 7 syllabus. The only difference is that in the latter, students are expected to know how to 'substitute two positive inputs in simple expressions and evaluate' (DLAP Syllabus, 2014a, p. 5), rather than one input. Figure 7.6.2.1 shows a copy of the relevant page from the annual examination script of Tony, whose performance throughout the year tended towards the median of the group.

Figure 7.6.2.1 Tony's pen-and-paper work in the annual examination

As shown in Figure 7.6.2.1, Tony interpreted ES as a substituting symbol (relational meaning) when reading the values of the letters. However, in his working, he represented ES as a symbol that shows the result of a computation (operational meaning). He carefully worked out the operations in order, starting a new statement for each order to avoid keeping a running total. The latter is a common misuse of ES (e.g. Kilpatrick et al., 2001). While interpreting the new notation of multiplication and division correctly, he preferred to represent these operations with the old notation in his working.

With the exception of Jordan, who worked these examination questions correctly but kept a running total in the evaluation of successive expressions, the other students in Grade 7C exhibited very similar work to that of Tony. This came to no surprise, because, as I have shown in Sections 7.4 and 7.5 students were tackling multiple-operation expressions, both in GA and pen-and-paper environments, which were much more complicated than those I set in the examination.

In contrast to studies claiming that students rarely transfer mathematical ideas they develop within an ICT environment to paper-based questions (e.g. Gurtner, 1992; EACEA Eurydice Report, 2011), all students in this research have made a smooth transition from solving GA tasks to traditionally set pen-and-paper problems. One possible reason for this successful transition was that, in GA, students worked with formal, conventional notation and solved problems they later encountered in pen-and-paper problems. Another reason may have been that the GA worksheets were prepared in such a way that they started off simulating a GA grid but this simulation was gradually excluded as questions became more traditionally set. Such a bridging between ICT and pen-and-paper was found to be both necessary and effective in helping students to transfer mathematical concepts from a computer environment to paper-based questions (Geraniou & Mavrikis, 2015, 2016).

In traditionally set questions and especially in summative assessments, students rely heavily on symbolic representations to express their mathematical interpretations. The above analysis of Tony's examination work is a model I formed of his conceptualisations by observing the signifiers (Kaput, 1991) he chose to express through written symbolic representations. During the lessons, the students could express themselves better by complimenting symbolic expressions with actions and pictures which helped me to form more accurate models of their thinking processes. For example, in class, students expressed the order of operations they identified in algebraic expressions not only by writing successive computations (symbols), but also by drawing journeys (pictures) in the GA grid and by moving (action) cells with their expressions around the grid.

Nevertheless, whether expressed in a GA environment or written on paper, students' APS representations served me (as their teacher) to develop models of their conceptual interpretations during the lessons. This was a crucial aspect in the generation and continuation of M-N-L cycles, as I discuss next.

7.6.3   Zooming Out: Viewing CAPS as an Integral Part of M-N-L

The Mathematics-Negotiation-Learner (M-N-L) framework was developed on the idea that teachers negotiate between the mathematics for students (MfS), the mathematics intended to be taught, and the mathematics of students (MoS), the existing and developing mathematics inside students' minds. Teachers engage in Mathematics-to-Learner negotiations aiming to present students with learning offers (Steinbring, 1998) of MfS to help students to experience, reflect on, and conceptualise (Kolb, 1984) mathematics. Teachers then engage in Learner-to-Mathematics negotiations where they seek to learn about MoS (Steffe, 1991), by developing models of students' thinking processes and use them to review MfS to be able to make renewed Mathematics-to-Learner negotiations. I argued that these initiated and completed M-N-L cycles were indicative of my CT.

In this chapter, I used the CAPS framework to analyse students' conceptual interpretations of action, picture, and symbol (APS) representations. This was an investigation into students' learning without which there would not be any teaching (Freire, 1998). Thus, the CAPS framework makes sense and fits within the overarching M-N-L framework as shown in Figure 7.6.3.1.

Figure 7.6.3.1   Making sense of CAPS framework within M-N-L framework
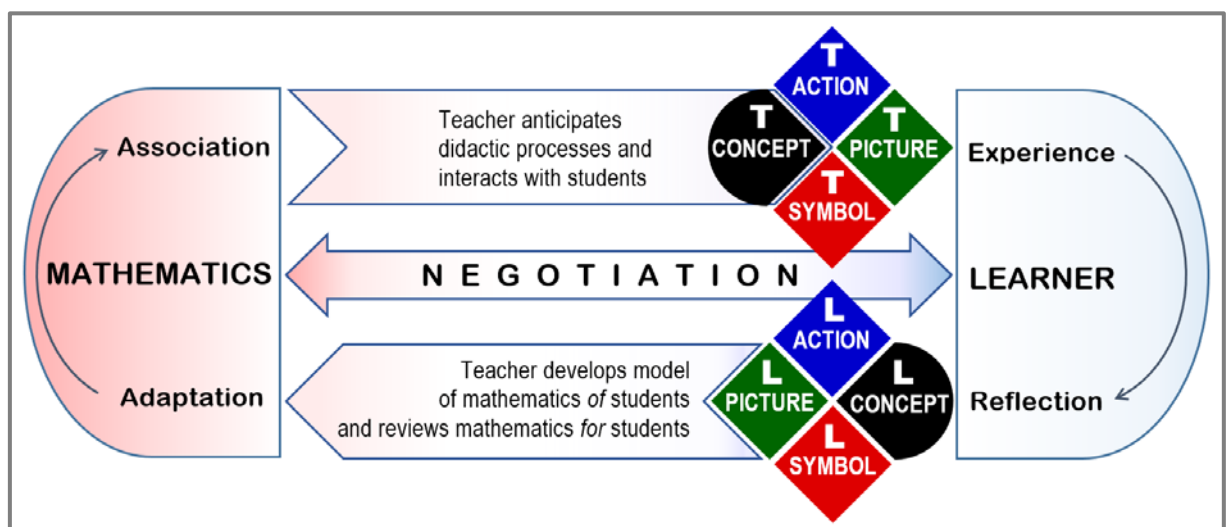
The relevance of the CAPS framework lies in the moment when teacher and learners exchange mathematical ideas to establish a mathematical consensual domain (Maturana & Varela, 1980; Glasersfeld, 1991b). In this consensual domain, teacher and students seek agreement about mathematical conventions (such as notation and structure of expressions).

The top rightwards arrow of Figure 7.6.3.1 shows how, as a teacher, I initiated M-N-L cycles by anticipating possible didactic processes and interacting with the students to communicate my mathematics (*T*-Concept), i.e. MfS, through my (*T*-) action, picture, or symbol representations. These *T*-representations were mostly facilitated by GA, where I adopted GA's inbuilt representations as my own learning offers to the students. GA also served as a springboard for discussions in which I came up with MfS representations outside the GA environment.

Learners experienced these representations, reflected on them, and formed personal conceptual interpretations (*L*-Concept) of the topic at hand. Learners then participated in classroom activities in which they created their own (*L*-) action, picture, or symbol representations, usually by adopting the representations offered in GA. This is the start of the bottom leftwards arrow of Figure 7.6.3.1, where, as a teacher, I used these *L*-representations to develop models of MoS (*L*-Concept) and used these models to review MfS. This led to an association of MoS with my own mathematics (sometimes requiring an adaptation) from which I then commenced another M-N-L cycle.

Throughout Chapter 7, I was mostly concerned with *L*- CAPS, i.e. students' conceptual interpretations and APS representations of notation as they engaged in informal- and formal algebraic activities. In Sections 7.2-7.5, I zoomed in on these to investigate how GA supported my CT in helping students to learn new representations of mathematical notation and how this enabled them to extend their conceptual interpretations. The latter was important in their learning about the order of operations, the meaning of symbols within the structure of numerical and algebraic expressions, and the multiple uses of ES. In this section, I zoomed out to demonstrate how the CAPS framework made sense within the M-N-L framework about CT, where students' (*L*-) action, picture,

and symbol representations enabled me, as a teacher, to form models of their conceptual constructions (*L*-concept). In Sections 6.1.3, 7.3.3, and 7.5.2 I have presented evidence that supports the idea that whenever students used APS representations to supplement their verbal expressions, I used those representations to make a Learner-to-Negotiation shift, where my focus of purpose changed from (1) encouraging students to reflect on the mathematics learning offer to (2) forming a model of their conceptualisations and reviewing the learning offer. Students' APS representations prompted me to shift my attention to start learning about students' mathematical knowledge (Steffe, 1991), not only to form models of the interpretations they seemed to be developing but also to enrich my own mathematics by adopting and synthesising students' mathematics with my own (Steffe, personal communication, October 7, 2015).

The development and application M-N-L and CAPS frameworks, and the analysis of how the latter fitted within and served the former, helped me to investigate and answer the research questions I aimed to address. These will be discussed in the concluding chapter that follows.

Chapter 8

Conclusion

8.0 Overview

In this chapter, I revisit the aims and outcomes of this research. Table 8.0.1 outlines the sections included in this chapter.

Table 8.0.1   Chapter 8 section titles

	Section Title	Page
8.1	Recapitulation of Aims and Outcomes	317
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8.4	Recommendations for Future Research and Actions	328
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8.1 Recapitulation of Aims and Outcomes

The aim of this research was to investigate the dynamics of constructivist teaching (CT) during regular school lessons in which I used Grid Algebra (GA) to help Grade 7 students in forming new representations and interpretations of notation through informal- and formal-algebraic activities. In these lessons, I aimed to coordinate and facilitate students' developments of concepts about:

- properties of operational notation,

- unknowns, and variables, and

- the equals sign (ES).

During these lessons, students learnt about new uses of operational notation and the order of operations in numerical and algebraic expressions.

In this section, I revisit the research questions which I answer by providing evidence presented and discussed in Chapters 6 and 7. I start with the first set of research questions, those regarding CT.

8.1.1　Answers to Research Questions Set 1

To answer the first set of questions, I made use of constructivist theories including Dewey (1902), Piaget (1975), Jaworski (1994), Simon (1995), Steinbring (1998), Kolb (1984), and Steffe (1991). My approach in addressing these questions was also influenced by my inclination towards radical constructivism (RC), as envisaged by Glasersfeld (1995a), from which I draw a subjectivist view of reality, knowledge, and learning.

Research Question 1(i)

How did I engage in CT and what were the distinguishing characteristics of such a teaching approach?

Addressing this question led to the design and development of the M-N-L framework (Borg et al., 2016a, 2016b). Inspired by theories mentioned above, particularly Dewey's (1902) curriculum-learner construct, I derived the M-N-L framework from patterns emerging from data collected during my GA lessons with Grade 7 students. In Section 6.1, I presented typical evidence of these patterns which were the continuous shifts of teaching purpose I made during the lessons to negotiate between my mathematics and learners' conceptual constructions. In Section 6.1.5, I discussed how primary data coded for such changes of teaching purpose were categorised into two types of negotiation: Mathematics-to-Learner and Learner-to-Mathematics (see Figure 6.1.5.2 for the codes-to-theory development). This dual negotiation was at the heart of the M-N-L framework which was discussed and developed in Section 6.2. I showed how M-N-L cycles were formed of these successive changes of teaching purpose during teachers' dual negotiations between mathematics and learners.

I thus presented M-N-L cycles as the distinguishing characteristic of CT. Consequently, I interpreted the extent to which I managed to generate and complete these M-N-L cycles to be indicative of my success to engage in CT. In Section 6.3, I showed that from 745 minutes devoted to plenary discussions in the lessons, I completed at least

180 M-N-L cycles, an approximate average of a cycle per 4 minutes of teaching (see Figures 6.3.1, 6.3.2). Given that every M-N-L cycle involves at least 4 shifts of teaching purpose, this rate confirms the rapid shifts of attention that teachers make during lessons attested by other researchers (e.g., Ingram, 2014). In Section 6.4, I brought evidence from the lessons to focus on how such M-N-L cycles were initiated and completed. I discussed each change of teaching purpose I made to create roads linking the mathematics *for* students (MfS) with the mathematics *of* students (MoS) (see summary in Table 6.4.1).

Research Question 1(ii)

What, if any, were the moments when I failed to engage in CT?

The M-N-L framework was effective in identifying moments where I failed to engage in CT by creating roadblocks in my negotiations between learners and mathematics. As shown in Section 6.5, this happened when I could have been using students' representations to form models of their interpretations in order to review the mathematics learning offer (Steinbring, 1998). Overall, I identified 23 such instances in which 20 happened in two ways. The first type (13/20) was when I failed to ask students to give their representational input and hence I missed the opportunity to develop models of their interpretations. The second type (7/20) was when I failed to elaborate on students' representational input and hence I failed to associate and synthesise their mathematics with my own.

All of the 23 instances of losing sensitivity to learners' conceptual constructions created a roadblock in the Learner-to-Mathematics negotiation. These roadblocks all originated from failing to let the students' mathematics affect my own. The structure of the M-N-L framework helped me to posit at least one other possible type of roadblock, one that affects the Mathematics-to-Learner negotiation. This might occur when a teacher has no intention of anticipating didactic processes or hypothesising about a learning trajectory. However, I did not identify any such barriers in my lessons.

8.1.2 Answers to Research Questions Set 2

Analysing data to answer the second set of research questions was guided by the amalgamation of Kaput's (1991) signifier-signified theory, and Bruner's (1966) enactive, iconic, and symbolic representations theory into the CAPS analytical framework discussed in Chapter 7.

Research Question 2(i)

How did students represent and interpret mathematical notation as they started Grade 7?

Interview 1 and the first lessons were crucial in answering this question, where all students were found to have limited interpretations of operational symbols and of ES. They all evaluated successfully small value additions, subtractions, multiplications and divisions. Consistent with the literature (e.g., Davis, 1975; Booth, 1984; Sfard & Linchevski, 1994), all students perceived numerical expressions as processes to be performed and did not view them as mathematical entities which could be manipulated without being evaluated beforehand. Moreover, none of them seemed to be aware that brackets could be used to denote multiplication or that a fraction may be regarded as a division. Their use of letters was limited to denoting mensuration quantities like the use of L for length of a rectangle, and they did not seem to be aware that letters could stand for generalised numbers.

None of the students seemed to have had a relational view of ES before the onset of the GA lessons. Their initial interpretation of ES was operational, a symbol that indicates that the numerical expression on its left makes the single number (answer) on its right. This was consistent with many research reports (e.g., Behr, Erlwanger, & Nichols, 1976; Kieran, 1981b; Herscovics & Linchevski, 1994; McNeil, 2008). However, unlike some research findings (e.g., Falkner, Levi, & Carpenter, 1999) Joseph, Jordan, Omar, and Tony could also read ES from right to left.

Research Question 2(ii)

How did GA help students to enrich their representations and extend their interpretations of mathematical notation?

GA was found to be very effective in helping students to extend their interpretations of notation by enriching their repertoire of representations. In Sections 7.2–4, I presented evidence from lesson videos and students' computer activities to show how GA provided all students with multiple-linked representations (Dreyfus, 2002) of notation, namely, actions, pictures, and symbols (APS). The picture of a cell within the GA grid represented a numerical or algebraic expression while the action of moving a cell onto another cell represented the operation on that numerical or algebraic expression. The series of operations on a number or a variable was represented by pictures of journeys around the GA grid, where intermediary "stations" represented mathematical objects which could be manipulated further. In this way, all students learnt new ways of representing notation involved in numerical and algebraic expressions (see, for example, Excerpts 7.3.2.1 and 7.3.3.1). In Section 7.5, I showed how GA activities helped students to represent ES as a balance between equal values represented by the same cell (picture). GA also triggered discussions which resulted in further representations of ES, where students acted out balance and imbalance to represent equality and inequality respectively (see Excerpts 7.5.2.2–3 for students' see-saw and weightlifting metaphors).

As a result, all students showed significant developments of their conceptual interpretations of notation with the help of GA: they learnt about the use of new notation (brackets to represent multiplication; fraction notation to represent division), the use of letters as unknowns or variables, the structure of numerical and algebraic expressions, and the order of operations in expressions. Tables 7.2.3.1, 7.3.3.2 and 7.4.3.2 summarise these extensions of students' interpretations of notation and the associated multi-linked representations described earlier. These tables also include ways in which students developed a proceptual view (Gray & Tall, 1994) of numerical and algebraic expressions.

Moreover, all students made considerable developments in their conceptualisations of ES. They learnt that besides indicating the answer of a calculation, ES could also be interpreted as a balance of same quantities and a symbol of substitution (Jones & Pratt, 2012). Excerpt 7.5.2.5 presents an example of the latter. Students' extensions of meanings of ES was in line with studies which reported that teaching aimed at helping students to extend their conceptions of ES had the desired impact (Rittle-Johnson & Alibali, 1999; McNeil & Alibali, 2005b; Powell & Fuchs, 2010). Table 7.5.3.3 summarises how APS representations provided or prompted by GA activities helped students to extend their interpretations of ES.

Research Question 2(iii)

How did students transfer representations and interpretations of notation they developed when working with GA to pen-and-paper problems?

All students were found to transfer representations and interpretations of notation from a GA environment to pen-and-paper problems. In Table 7.5.3.1, I provided samples of each students' work, showing that they managed to perform traditionally set substitution-and-evaluation tasks on paper. The range of performance varied between expressions with three operations (Omar) to five operations (Dan and Jordan). This contrasts with studies claiming that students do not usually transfer mathematical ideas they acquire within an ICT environment to paper-based questions (e.g. Gurtner, 1992; EACEA Eurydice Report, 2011).

This computer-to-paper transition was partly due to the nature of GA itself which encourages the use of formal notation and includes inbuilt questions that are typically present in paper-based problems. The transfer was also made possible through a series of worksheets with which I sought to bridge ICT-based to paper-based problems. Such bridging is necessary for successful transfer of mathematical applications from computer to pen-and-paper (Geraniou & Mavrikis, 2015). In these worksheets, I included questions that simulated a GA environment, both in the diagrams (context of a GA grid) and in the questions (similar to those of GA tasks). In such worksheets, all

students managed to reproduce in writing the actions, pictures, and symbol representations as shown in Table 7.4.2.2. Students' representations were based on their activities and discussions in the GA lessons. For example, Dwayne's and Joseph's representations in Worksheet 16, included in Table 7.4.2.2, were based on Lesson 16 where students compared GA-generated journey pictures to railway routes. In the lesson, they used the metaphor of train stations to think proceptually (Gray & Tall, 1994) about successive algebraic expressions being operated on and they compared the operations to the connections between stations. Such worksheets had the intermediary role of presenting students with a situation similar to the computer activity while familiarising the students with the writing of conventional notation when working on the order of operations. Gradually, these paper-based assignments contained more traditionally set questions, such as evaluating algebraic expressions by substituting values for letters (Appendices 4.1–4.5).

In addition, all students performed very well (mean = 9.1/10), in substitution-and-evaluation tasks I included in their annual examination (Figure 7.6.2.1). The difficulty level of these tasks was comparable to that expected of high-performing students at Grade 7 (DLAP Syllabus, 2014a). The national syllabus designed for students in the lowest performing set at Grade 7 (DLAP Syllabus, 2014b) did not even include algebraic expressions. These "low-performing" students have proved that given the right time, tools, and dedication they could meet the curricular expectations of high-performing students in paper-based problems. Although the issue of special educational needs did not feature in this research, this finding is all the more significant when one considers that, except for Tony, all the students had special educational needs (see Table 4.2.2.1) and their learning support assistants were never present in any GA lesson or assignment.

I took great care to address these research questions in an honest, meticulous, and trustworthy manner. However, this research is not without its limitations. These are outlined chronologically in the following section.

8.2 Limitations of the Research

(i) **Literature.** Although I did my best to make a thorough search of the most relevant literature, I am aware that the list of readings is not exhaustive. One of the reasons for this is that the literature I reviewed was limited to work which was published in English. However, besides literature originating from the UK, the US, Canada, and Australia, I also reviewed studies from other countries, especially from Europe and Asia.

(ii) **Stance.** In Section 4.7, I discussed the issues of being a teaching researcher including those of being biased towards establishing the effectiveness of my teaching, holding preconceptions about the participant students with reference to similar student groups I taught in the past, not being as "open-minded" as an "outsider" researcher, and time constraints and distractions due to gathering data while being on teaching duty. These limitations were due to a conflict of stance: I had to alternate between wearing the researcher hat and the teacher hat during the data collection stage which lasted a whole scholastic year. I settled this conflict by taking on a teaching role during the lessons and a researcher's role during the data analysis. However, I am aware that such a settlement is rather simplistic and I do not exclude that some of the issues listed above may have affected the research findings.

(iii) **Methodology.** From a case study of six students and their teacher, I could not expect to offer any kind of generalisation of findings which may have been possible in a large-scale study. I do hope, however, that I was careful enough when positing the M-N-L and CAPS frameworks as working hypotheses (Yin, 2013) to enable other teachers and researchers to adopt them as viable models for their own teaching or research. I also hope that the depth and richness of the data offers teachers and researchers the possibility of empathising with my narrative to create vicarious experiences (Stake, 1995) from which they can extrapolate lessons to their own situations.

(iv) **Data gathering.** The data gathering methods suffered a number of limitations.

- In the lesson video recording, some student gestures, facial expressions, and undertone utterances were lost due to students' facing away from the video recorder.

- In the interview video recording, sometimes the students went out of the field of vision of the video recorder while thinking about a problem or responding to a question. This may have caused some loss of data.

- During computer work, there were quite a number of long intervals where Pandas and Chimps worked on the tasks silently, taking turns to control the cursor and only muttering an occasional word or two. Although I could still observe what they were doing from the recording, lack of verbal communication was not helpful for me to infer which student was doing the screen movements and what kind of interaction was going on.

- Also during computer work, students (especially Sharks) sometimes spent time fooling around with the software and not working on the tasks I assigned them. While some of this activity (like moving the cells around the grid repeatedly) could have prepared them for the complexity of expressions, most of this activity (like colouring cells to make a chessboard pattern) was frivolous and wasted precious time. Since I was interacting with individual groups, I could not always detect such activities.

(v) **Data analysis.** In Chapter 6, I described how I made use of students' representations to hypothesise about their mathematical interpretations. Building a model of a person's thought processes from observing that person's representation was also relevant when I looked back at my own representations by studying lesson video recordings and built models about what I was probably thinking at particular moments during the lessons. In some cases, especially when an event had caused me substantial mental perturbation, I actually memorised what I had been thinking and feeling. In many other cases, I relied on my awareness of my teaching approach and on my teaching experience of what I usually do or say before or after thinking about something. However, I

do not exclude that I may have been inaccurate in inferring some shifts of focus and this may have affected the research findings to some extent.

8.3   Significance of the Research

This research contributes to the literature about CT, the introduction of algebra to students, and the use of ICT for the teaching and learning of mathematics.

The most significant contribution is, perhaps, the development of the M-N-L framework: a viable, working model resulting from my attempt to bring RC beliefs to daily teaching practices (Borg et al., 2016b). I am aware that the framework may seem simplistic when one considers the complexity of CT, especially when one is teaching from a RC stance, but I feel I have demonstrated how this framework can be used to analyse teaching against a constructivist backdrop. Its strength lies in emphasising a dual negotiation between subject matter and the learner. The fact that this framework was developed from data gathered from "normal" lessons set within a school's daily timetable adds to its credibility and viability.

Another important contribution of this study was the identification of CT barriers which hindered the negotiation process. Such roadblocks between mathematics and learners may occur from the learner side, when teachers impede the process of building models of students' conceptualisations in order to review their own mathematics and subsequent learning offers. They may also occur from the mathematics side when teachers do not seek to anticipate how students may interpret the learning offers. Each of these roadblocks are caused by excessive focus on the subject matter at the expense of learners' needs and interests. Complementing literature in which researchers (e.g. Glasersfeld, 1991b; Steffe, 1991) make recommendations about actions teachers *should do* in order to engage in CT, this research makes teachers aware of barriers they *should avoid* if they want to maintain CT in their lessons.

The research also shows how mathematics teachers use students' action, picture and symbol representations to build models of students' conceptualisations. In this study, the CAPS framework had a secondary, though crucial, role by serving as a focus within the M-N-L framework, located at the delicate stage of the M-N-L cycle where my students and I were exchanging ideas. However, I have shown how it can help teachers and researchers analyse data related to students' conceptual constructions as expressed through their representations. Thus, CAPS is a contribution by and of itself, which may be used as an analytical framework in studies that focus on the interplay between signifier representations and signified interpretations.

This research also contributes to the growing body of literature about helping low-performing students learn mathematics. The major research finding in this respect was that with the appropriate approach, resources, and time, "low-performing" students may not be not low-performing at all. Except for one, the participants of this study had special educational needs for which they were assigned full-time learning support assistants (LSAs) to help them during lessons and assignments. These LSAs were not present for any of the GA lessons and did not help the students with the written work that was part of the data. The students managed very well on their own, with the GA software being their only "LSA". They engaged in meaningful classroom discussions and in informal- and formal-algebraic activities on GA. By the end of the lessons, these students were evaluating complex algebraic expressions which were at or above the level recommended in the syllabus set for *high-performing* students (DLAP Syllabus, 2014a). Furthermore, this study corroborates research showing that teaching aimed at helping students to extend their conceptions of ES is effective for average-performing students (McNeil & Alibali, 2005a; Rittle-Johnson & Alibali, 1999) and low-performing students (Powell & Fuchs, 2010) alike.

Finally, this study continues to add to the body of research about the use of computer software in mathematics teaching and learning. GA has been found to be very practical in helping students increase their repertoire of notational representations and to extend their concepts signified by notation. Contrary to studies claiming that students seldom transfer concepts they develop within an ICT environment to pen-and-paper problems

(e.g. Gurtner, 1992; EACEA Eurydice Report, 2011), this research has shown that students can make a smooth transition from developing and applying concepts when solving computer-generated (GA) tasks to applying those concepts to solve typical pen-and-paper problems. Moreover, students' constant eagerness to attend what they used to call "algebra lessons", i.e. the GA lessons, confirms Lugalia's (2015) assertion that students' motivation, enjoyment, and engagement in mathematics learning were boosted by the use of GA in the lessons. Overall, GA has proven to be another success story of utilising computers in mathematics education and deserves further research, as I suggest in the section that follows.

8.4   Recommendations for Future Research and Actions

The work done in this study may be extended in further research in a number of ways.

(i) Educational researchers in the field of constructivism may use M-N-L to analyse CT at other levels and for other topics. Furthermore, even though the M-N-L framework has been devised from and for the analysis of mathematics lessons, it can be found viable for other school subject areas, where "M" in M-N-L may be replaced by the initial of any curricular subject.

(ii) The role of action, picture, and symbol representations in mathematics teaching and learning may be investigated in other topics such as geometry and data handling. Researchers may find the CAPS framework as a useful analytical tool to guide the analysis of data emerging from such studies.

(iii) The use of GA as a tool for mathematics teaching and learning merits further study. Besides its potential to assist the teaching and learning of informal algebra, researchers may explore other features of GA which did not form part of this study, such as the teaching of the distributive property of multiplication over addition by means of equivalent expressions and the use of inverse journeys to teach inverse functions and bringing a letter to be the subject of an equation.

In addition, this study calls for educational policies in favour of the dissemination of ICT applications in mathematics lessons. The success of GA in the lessons was partly due

to the design of the software and partly due to the time I devoted in familiarising myself and preparing classroom activities with it. While contributing to literature which supports policies in favour of ICT dissemination in schools and curricula, this study also calls for actions that promote teacher training in the use of specific computer software to assist the teaching and learning of mathematics.

8.5   Autobiographical Reflection

Undertaking a PhD research in which I investigated my own teaching and what my students made of my lessons has left its mark in my perspective as a teacher and a researcher.

As a teacher, I have become much more aware that if I have a classroom of twenty-five students, then there are probably twenty-five different shades of meanings being constructed for every learning offer I make. This has made me more cautious when making assumptions about students' knowledge constructions and while it has made teaching a more complex task than it seemed to me twenty years ago, I believe it has made me more sensitive to constructivist notions when interacting with my students.

As a researcher, this experience was a journey among contrasts: reading literature to develop knowledge and acquiring the humility of lacking it, handling large masses of data while valuing the smallest of details, delving into messy analyses and deriving neat themes and theories, being frustrated one day and excited the next, working with confidence while accepting critique, and striving for perfection while knowing that, like infinity, it is a status that can be approached but never reached. The self-discipline involved in navigating among these elements has given me invaluable lessons about what it takes to be a researcher.

Being first and foremost a teacher, I hope I will have the opportunity to share these lessons with others.

References

References

Ackermann, E. (2001). Constructivisme et Constructionisme: Quelle différence?. In *Constructivismes: Usages et perspectives en éducation.* (Volumes 1 et 2). Actes de Colloque. Geneva: Service de la recherche en éducation, Cahier 8, September 2001, pp. 85–94.

Adelman, C., Kemmis, S., & Jenkins, D. (1980). Rethinking case study: Notes from the Second Cambridge Conference. In H. Simons (Ed.) *Towards a Science of the Singular.* East Anglia: Centre for Applied Research in Education, University of East Anglia.

Ainley, J., Bills, L., & Wilson, K. (2004). Constructing meanings and utilities within algebraic tasks. In M.J. Høines and A.B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1-8). Bergen, Norway: Bergen University College.

Anderson, G. L., & Jones, F. (2000). Knowledge generation in educational administration from the inside out: The promise and perils of site-based, administrator research. *Educational Administration Quarterly, 36*(3), 428–464.

Anenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol*. Educational Studies in Mathematics, 35*, 153–187.

Anghileri, J. (1995). Language, arithmetic, and the negotiation of meaning. *For the Learning of Mathematics, 15*(3), 10–14.

Ary, D., Jacobs, L., Razavich, A., & Sorensen, C. (2006). *Introduction to research in education* (7th ed.). Belmont, CA: Thomas Wadsworth.

Arzarello, F. (1991a). Procedural and relational aspects of algebraic thinking. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference for the Psychology of Mathematics Education* (Vol. II, pp. 80-87). Assisi, Italy: PME Program Committee.

Association for Qualitative Research (2015). Glossary: Validity. Retrieved October 13, 2015 from http://www.aqr.org.uk/glossary/validity.

Atkinson, P., & Delamont, S. (1993). Bread and dreams or bread and circuses? A critique of case study research in evaluation. In M. Hammersley (Ed.) *Controversies in the Classroom*. Buckingham: Open University Press.

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P., (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education,19*(3), 243–259.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93,* 373–397.

References

Baroudi, Z. (2006). Easing students' transition to algebra. *Australian Mathematics Teacher, 62*(2), 28–33.

Bassey, M. (1999). *Case Study Research in Educational Settings.* Buckingham: Open University Press.

Battista, M. T., & Clements, D. H. (1988). A case for a Logo-based elementary school geometry curriculum. *Arithmetic Teacher*, 36, 11–17.

Battista, M. T., & Clements, D. H. (1991). *Logo geometry*. Morristown, NJ: Silver Burdett & Ginn.

Battista, M. T., & Van Auken Borrow, C. (1998). Using spreadsheets to promote algebraic thinking. *Teaching Children Mathematics,4,* 470–478.

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D. Grouws & T. Cooney (Eds.), *Perspectives on effective mathematics teaching* (pp. 27-46). Reston: National Council of Teachers of Mathematics.

Bednarz, N., & Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran, L. Lee (Eds.), *Approaches to Algebra: Perspectives for Teaching and Learning* (pp. 115-136). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Bednarz, N., Kieran, C., & Lee, L. (1996). Introduction: Approaches to algebra. In N. Bednarz, C. Kieran, L. Lee (Eds.), *Approaches to Algebra: Perspectives for Teaching and Learning* (pp. 3-12). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Behr, M., Erlwanger, S., & Nichols, E. (1976). How Children View Equality Sentences. *Project for the Mathematical Development of Children, Technical Report No. 3.* Florida State University, Tallahassee.

Bell, A. (1996). Problem-solving approaches to algebra: Two aspects. In N. Bednarz, C. Kieran, L. Lee (Eds.), *Approaches to Algebra: Perspectives for Teaching and Learning* (pp. 167-185). Dordrecht, The Netherlands: Kluwer Academic Publishers.

BERA (2011). Ethical Guidelines for Educational Research. Retrieved August 16, 2013 from https://www.**bera**.ac.uk/wp-content/.../**BERA**-**Ethical**-**Guidelines**-2011.pdf.

Berglund, A., & Lister, R. (2010). Introductory programming and the didactic triangle. *Proceedings of the 12th Australasian Conference on Computing Education* (ACE 2010, Vol. 103, pp. 35-44)*,* Brisbane, Australia. Darlinghurst, Australia: Computer Society Inc.

References

Bishop, A., Filloy, E., & Puig, L. (2008). *Educational Algebra: A Theoretical and Empirical Approach.* Boston, MA, USA: Springer.

Black, J. B., & McClintock, R. O. (1995). An interpretation construction approach to constructivist design. In Wilson, B. (Ed.). *Constructivist learning environments.* Englewood Cliffs, NJ: Educational Technology Publications.

Black, J. B., McClintock, R. O. (1995). An interpretation construction approach to constructivist design. In Wilson, B. (Ed.). *Constructivist learning environments.* Englewood Cliffs, NJ: Educational Technology Publications.

Blanton, M. L., & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education, 36*(5), 412−446.

Bloomfield, A., & Vertes, B (2005). *People Maths: Hidden Depths.* Association of Teachers of Mathematics.

Booth, L. R. (1982). Developing a teaching module in beginning algebra. *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education.* Antwerp.

Booth, L. R. (1984). *Algebra: Children's Strategies and Errors.* Windsor, UK: NFER-Nelson.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A.F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 20-32). Reston, Virginia: National Council of Teachers of Mathematics.

Borg, P. (1997). *Rote learning in Algebra: An investigation of students' reasoning in transforming equations at Form 5.* Unpublished B.Ed.(Hons.) dissertation, Univeristy of Malta.

Borg, P. (2009). Driving a car in a carriage road: Why don't Mathematics teachers use computers in their lessons? *Mathsline, 20,* 10–11.

Borg, P. (2011). The use of computers to help secondary level pupils learn mathematics: An investigation of teachers' attitudes before and after their first year of teaching. Unpublished M.Ed. dissertation, University of Malta.

Borg, P. (2016). The university lecture room and the school classroom: Does the stage affect the acting? *Constructivist Foundations, 12*(1), 107–108.

Borg, P., & Hewitt, D. (2015). Gaining meaning for expressions with Grid Algebra: Developing the CAPS framework. In: Adams, G. (Ed.) *Proceedings of the British Society for Research into Learning Mathematics, 35*(2),1–6.

Borg, P., Hewitt, D., & Jones, I. (2016a). Negotiating between learner and mathematics: A conceptual framework to analyze teacher sensitivity toward

constructivism in a mathematics classroom. *Constructivist Foundations, 12*(1), 59–69.

Borg, P., Hewitt, D., & Jones, I. (2016b). Authors' Response: The M-N-L framework: Bringing radical constructivist theories to daily teaching practices. *Constructivist Foundations, 12*(1), 83–90.

Bostock, L., Chandler, S., Shepherd, A., & Smith, E. (1991). *ST(P) Mathematics 1A.* Surrey: Stanley Thornes (Publishers) Ltd.

Boulton-Lewis, G.M., Cooper, T., Pillay, H., & Wilss, L. (1998). *Arithmetic, pre-algebra, and algebra: a model of transition.* Retrieved November, 2, 2012, from http://www.merga.net.au/documents/RP_BoultonLewis_Cooper_Pillay_Wilss_1998.pdf.

Briggs Myers, I. (1980). *Gifts differing: Understanding personality type.* Davies-Black Publishing

Brooks, J. G., & Brooks, M. G. (1993). *In search of understanding: The case for constructivist classrooms.* Alexandria, CA: Association for Supervision and Curriculum Development.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In Steiner, H. (Ed.), *Theory of Mathematics Education, Paper 54,* (pp. 110–119). Institut fur Didaktik der Mathematik der Universitat Bielefeld.

Brousseau, G. (1986). Fondemonts et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques, 7*(2), 33-115.

Brousseau, G. (1997). Theory of Didactical Situations in Mathematics: Didactiques des Mathématiques, 1970-1990. (Translated by N. Balacheff, M. Cooper, R. Sutherland, and V. M. Warfield). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Brousseau, G., & Otte, M. (1991). The fragility of knowledge. In A. Bishop, S. Olsen, & J. Van Dormolen (Eds.), *Mathematical Knowledge: Its Growth through Teaching* (pp. 13-36)*.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of Learning. *Educational Researcher, 18*(1), 32–42.

Bruner, E. M. (1993). Introduction: The ethnographic self and the personal self. In P. Benson (Ed.), *Anthropology and Literature* (pp. 1-26). Urbana: University of Illinois Press.

Bruner, J. S. (1964). The course of cognitive growth. *American Psychologist, 19,* 1−15.

References

Bruner, J. S. (1966). *Toward a theory of instruction.* London: Oxford University Press.

Bruner, J. S. (1967). *On knowing: Essays for the left hand.* Cambridge, Mass: Harvard University Press.

Bruner, J. S. (1986). *Actual minds, possible Worlds.* Cambridge, MA: Harvard University Press.

Bucky, M. (2000). *Learning Without Walls: A State of Learning.* Address by the Minister of Education and Children Services of South Australia in Fusion 2000 Global Learning Summit (Glasgow Scotland, September 29, 2000).

Burgess, R.G. (Ed.) (1989) *The Ethics of Educational Research.* Lewes: Falmer Press.

Burrell, G., & Morgan, G. (1979). *Sociological Paradigms and Organisational Analysis.* London: Heinemann Educational Books.

Burry-Stock, J. A. (1995). *Expert Science Teaching Evaluation Model (ESTEEM): Theory, Development, and Research.* (1st ed.). Kalamazoo, MI: Center for Research on Educational Accountability and Teacher Evaluation (CREATE), Western Michigan University.

Cajori, F. (1928). *A History of Mathematical Notations: Vol. I. Notations in elementary mathematics.* La Salle, IL: Open Court.

Carpenter, T. P., & Levi, L. (2000). Developing conceptions of algebraic reasoning in the primary grades. Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science. Retrieved, May 15, 2014, from *ncisla.wceruw.org/publications/reports/RR-002.PDF.*

Carraher, D. W., Brizuela, B. M., & Earnest, D. (2001). The reification of additive differences in early algebra. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 163-170). Melbourne, Australia: University of Melbourne Press.

Carraher, D. W., Schliemann, A. D., & Brizuela, B. M. (2001). Can students operate on unknowns? In *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 130-140). Utrecht, The Netherlands: Freudenthal Institute.

Carraher, D.W., Schliemann, A.D., & Schwartz, J.L. (2008). Early algebra is not the same as algebra early. In J.J. Kaput, D.W. Carraher, & M.L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 235-272). New York: Lawrence Erlbaum Associates

Carraher, D.W., Schliemann, A.D., & Brizuela, B.M. (2000). *Early Algebra, Early Arithmetic: Treating Operations as Functions.* Plenary address presented at the

References

22nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Tucson, AZ.

Carraher, D.W., Schliemann, A.D., Brizuela, B.M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*(2), 87–115.

Cedillo, T., & Kieran, C. (2003). Initiating students into algebra with symbol-manipulating calculators. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.) *Computer algebra systems in secondary school mathematics education*, (pp. 219-240). Reston, VA: NCTM.

Chaiklin, S., & Lesgold, S. (1984). *Prealgebra students' knowledge of algebraic tasks with arithmetic expressions*. Paper presented at the annual meeting of the American Research Association. Pittsburgh University, PA: Learning Research and Development Center.

Chaiklin, S., & Lesgold, S. (1984). *Prealgebra students' knowledge of algebraic tasks with arithmetic expressions.* Paper presented at the annual meeting of the American Research Association, April 1984, New Orleans, LA.

Chambers, C. E. (1964). The Cuisenaire-Gattegno Method of Teaching Mathematics: A Course for Teachers, Volume 1. Blackburn, Victoria: Cuisenaire Co. of Australia.

Charbonneau, L. (1996). From Euclid to Descartes: Algebra and its relation to geometry. In N. Bednarz, C. Kieran, L. Lee (Eds.), *Approaches to Algebra: Perspectives for Teaching and Learning* (pp. 3-12)*.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Chevallard, Y. (1988). On didactic transposition theory: Some introductory notes. Paper presented at the *International Symposium on Research and Development in Mathematics Education*, Bratislava, Czechoslavakia (August, 1988).

Chevallard, Y. (1989). Arithmétique, Algébre, Modélisation, Étapes d'une Recherché (Vol. 16). Aix-Marseille, France: IREM.

Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research.* San Francisco: Jossey-Bass.

Clement, J. (1980). Algebra word problem solutions: Thought processes underlying a common misconception. Paper presented at a meeting of the American Education Research Association, Boston, 1980.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception*. Journal for Research in Mathematics Education*, *14*, 16–30.

References

Clement, J., Lochhead, Monk, G. S. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly, 88*(4), 286–290.

Clements, D. H., & Battista, M. T. (1989). Learning of geometric concepts in a Logo environment. *Journal for Research in Mathematics Education,* 20, 450–467.

Clements, D. H., & Battista, M. T. (1990). The effects of Logo on children's conceptualization of angle and polygons. *Journal for Research in Mathematics Education,* 21, 356–371.

Clements, D. H., & Meredith, J. S. (1993). Research on Logo: Effects and efficacy. *Journal of Computing in Childhood Education, 4*, 263–290.

Clements, D. H., & Sarama, J. (1997). Research on Logo: A decade of progress. *Computers in Schools*, 14(1), 9–46.

Coady, C., & Pegg, J. (1993). An exploration of student responses to 'the more demanding Küchemann' test items. 16th Annual Conference of Mathematics Education Research Group of Australasia (pp.191-195), Australia: Brisbane.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14*(2), 83–94.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Monographs of the Journal for Research in Mathematics Education*, #4 (pp. 125-146). Reston, VA: National Council of Teachers of Mathematics.

Cochran-Smith, M., & Lytle, S. L. (1999). Relationships of knowledge and practice: Teacher learning in communities. In A. Iran-Nejad & C. D. Pearson (Eds.), *Review of Research in Education* (Vol. 24, pp. 249–306). Washington, DC: The American Educational Research Association.

Cohen, L., Manion, L., & Morrison, K. (2011). *Research Methods in Education* (7th Edition). Oxon: Routledge.

Collis, K. F. (1974). *Cognitive Development and Mathematics Learning.* Paper presented at the Psychology of Mathematics Workshop, Chelsea College, London.

Collis, K. F. (1975). *The Development of Formal Reasoning.* Report of a SSRC sponsored project carried out at the University of Nottingham School of Education during 1974. University of Newcastle: Newcastle, New South Wales.

Collins, A. (1986). *Different Goals of Inquiry Teaching: Technical Report.* Report N. 6458. Cambridge, MA: BBN Labs, Inc..

Confrey, J. (1992). Using computers to promote students' inventions of the function concept. In S. Malcolm, L. Roberts, & K. Sheingold (Eds.), *This Year in School*

Science, 1991 (pp. 131-161). Washington, DC: American Association for the Advancement of Science.

Cooper, T. J., & Warren, E. (2008). The effect of different representations on Years 3 to 5 students' ability to generalise. *ZDM The International Journal on Mathematics Education*, *40*(1), 23–37.

Corbin, J., & Strauss, A. (2007). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Creswell, J.W. (2013). Qualitative inquiry and research design: Choosing among five approaches. London: Sage Publications.

Cunningham, D. (1992). Assessing constructions and constructing assessments: A dialogue. In T. M. Duffy, & D. H. Jonassen (Eds.), *Constructivism and the Technology of Instruction* (pp. 35–44). Hillsdale, NJ: Lawrence Erlbaum Associates Inc.

D'Ambrosio, B. S. (2013). O professor-pesquisador diante da produção escrita de alunos. (Teacher-researchers' use of their students' written work). *Revista de Educação PUC-Campinas*,*18*, 249–258.

D'Ambrosio, B. S., & Steffe, L. P. (1994). O ensino construtivista. (Constructivist teaching. Em Aberto, 14(62), 23–32.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *Journal of Children's Mathematical Behavior, 1*(3), 7–35.

Davis, R. B. (1984). *Learning Mathematics: The Cognitive Approach to Mathematics Education.* Norwood, NJ: Ablex Publishing Company.

Denzin, N. K., & Lincoln, Y. S. (2011). Introduction: The discipline and practice of qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of Qualitative Research* (4th ed.)(pp. 1-19). Thousand Oaks, CA: Sage Publications.

Dettori, G., Garuti, R., & Lemut, E. (2000). From arithmetic to algebraic thinking by using a spreadsheet. In R. Sutherland, T. Rojano, A. Bell, & R. C. Lins (Eds.), *Perspectives on school algebra* (pp. 191-207). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Dewey, J. (1897). My pedagogic creed. *The School Journal*, LIV(3), 77-80.

Dewey, J. (1902). *The child and the curriculum.* Chicago: The University of Chicago Press.

Dewey, J. (1907). *The school and society*. Chicago: University of Chicago Press.

Dewey, J. (1910). *How we think*. Lexington, Mass: D.C. Heath.

Dewey, J. (1916). *Democracy and education.* New York: The Free Press.

References

Dewey, J. (1938). *Experience & Education*. New York: MacMillan.

DfE (2013). Mathematics Programmes of Study: Key Stage 3. Retrieved, June 4, 2014 from https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study.

DfES (2004). *ICT across the curriculum*. Retrieved, July 12, 2012, from https://www.education.gov.uk/publications/standard/publicationdetail/page1/DfES%200171%202004G.

Dienes, Z. P. (1961). On abstraction and generalization. *Harvard Educational Review, 31*, 281–301.

Dienes, Z. P. (1971). *Building Up Mathematics* (4th Edition). London: Hutchinson.

DLAP Syllabus (2014a). *Mathematics: Year 7 (Form 1) – Levels 7-8.* Directorate for Learning and Assessment Programmes, Ministry of Education and Employment, Malta. Retrieved September 12, 2015, from http://curriculum.gov.mt/en/Curriculum/Year-7/Documents/curric_f1/curric_f1_maths_guidelines_lev7to8.pdf.

DLAP Syllabus (2014b). *Mathematics: Year 7 (Form 1) – Core Curriculum Programme.* Directorate for Learning and Assessment Programmes, Ministry of Education and Employment, Malta. Retrieved September 12, 2015, from http://curriculum.gov.mt/en/Curriculum/Year-7/Documents/curric_f1/curric_f1_maths_ccp.pdf.

Doerr, H. M. (2001). Learning algebra with technology: The affordances and constraints of two environments. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.). *The future of the teaching and learning of algebra, The proceedings of the 12th ICMI Study Conference*. (pp. 199-205). Melbourne: Department of Science and Mathematics Education.

Dreyfus, T. (2002). Advanced mathematical thinking processes. In D. Tall (Ed.) Advanced Mathematical Thinking (pp. 25-41). London: Kluwer Academic Publishers.

Drijvers, P. (2001). The concept of parameter in a computer algebra environment. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.). *The future of the teaching and learning of algebra, The proceedings of the 12th ICMI Study Conference*. (pp. 221-227). Melbourne: Department of Science and Mathematics Education.

Drijvers, P. (2003). Algebra on screen, on paper, and in the mind. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.) *Computer algebra systems in secondary school mathematics education*, (pp. 241-268). Reston, VA: NCTM.

Driver, R. (1995). Constructivist approaches in science teaching. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 385-400). Hillsdale, NJ: Lawrence Erlbaum Associates.

References

Driver, R., & Oldham, V. (1986). A constructivist approach to curriculum development in science. *Studies in Science Education, 13*, 105–122.

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Dubinsky, E., Elterman, F., & Gong, C. (1988). The Student's Construction of Quantification. *For the Learning of Mathematics* 8, 2. 44–51.

Duffy, T. M. (1992). New implications for instructional technology. In T. M. Duffy & D. Jonassen (Eds.), *Constructivism and the Technology of Instruction: A Conversation*, Hillsdale, NJ: Erlbaum, pp. 1–16.

Dugdale, S. (2001). Order out of chaos: A Spreadsheet Excursion Into a Mathematical Frontier. *Journal of Computers in Mathematics and Science Teaching, 20*(3), 323-341.

Dunn, R., & Griggs, S. A. (1995). *Learning styles: Quiet revolution in American secondary schools.* Westport, CT: Praeger.

Dymoke, S., & Harrison, J. (2008). Reflective Teaching and Learning: A Guide to Professional Issues for Beginning Secondary Teachers. London: Sage Publications.

EACEA/Eurydice (2011). Key Data on Learning and Innovation Through the Use of ICT at School in Europe. Brussels: Eurydice. Retrieved, January 12, 2015, from https://webgate.ec.europa.eu/fpfis/mwikis/eurydice/index.php/Publications:Key_Data_on_Learning_and_Innovation_through_ICT_at_school_in_Europe.

Education, Audiovisual and Culture Executive Eurydice Report, (2011). Key data on learning and innovation through ICT at school in europe. Retrieved May 18, 2015, from http://eacea.ec.europa.eu/education/eurydice.

Efran, J. S., & Fauber, R. L. (1995). Radical constructivism: Questions and answers. In R. A. Neimeyer & M. J. Mahoney (Eds.), *Constructivism in Psychotherapy* (pp. 275-304). Washington, DC, US: American Psychological Association.

Elliott, J. (1991). *Action Research for Educational Change.* Buckingham: Open University Press.

Ellis, K. (1993). Teacher questioning behavior and student learning: What research says to teachers. Paper presented at the *Annual Meeting of the Western States Communication Association*, Albuquerque, NM.

Eltgeest, J. (1985). The right question at the right time. In Wynne Harlen (Ed.), *Primary Science: Taking the Plunge* (pp. 36-46). Portsmouth, NH: Heinemann.

Engström, A. (2014). RC is a theory of learning, not teaching. *Constructivist Foundations, 9*(3), 314–316

References

Ennis-Cole, D. (2004). Using microworlds in teaching and learning. *Proceedings of the AECT Professional Development Conference* (pp. 55-59). Denton: Texas. Retrieved, August 12, 2014 from http://www.aect.org/pdf/AECT_UNT_Proceedings_2004.pdf.

Erlwanger, S., & Berlanger, M. (1983). Interpretations of the equal sign among elementary school children. In *Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 250-259, Montreal.

Ernest, P. (1991). *The philosophy of mathematics education.* Bristol: The Falmer Press.

Falkner, K. P., Levi, L., & Carpenter, T. (1999). Building a foundation for learning algebra in the elementary grades. Retrieved May 12, 2016 from ncisla.wceruw.org/publications/briefs/fall2000.pdf.

Farrugia, M. T. (2006). *Medium and message: The use and development of an English mathematics register in two Maltese primary classrooms.* Unpublished Ph.D. Dissertation, University of Birmingham, UK.

Farrugia, M-T. (2006). Medium and Message: The Use and Development of an English Mathematics Register in Two Maltese Primary Classrooms. Unpublished Ph.D. Dissertation, University of Birmingham.

Filloy, E., & Rojano, T. (1984). From an arithmetical to an algebraic thought. In J.M. Moser (Ed.), *Proceedings of the 6th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 51–56). Madison: University of Wisconsin.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19−25.

Filloy, E., Rojano, T., & Rubio, G. (2000). Propositions concerning the resolution of arithmetical-algebra problems. In R. Sutherland, T. Rojano, A. Bell, & R. C. Lins (Eds.), Perspectives on School Algebra (pp. 155-176). Dordrech: Kluwer.

Filloy, E., Rojano, T., & Solares, A. (2003). Two meanings of the "equal" sign and senses of comparison and substitution methods. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th PME International Conference, 4,* 223−229.

Forsythe, S. (2007). Learning geometry through dynamic geometry software. *Mathematics Teaching Incorporating Micromath, 202*, 31–35.

Fox, R. (2001). Constructivism examined. *Oxford Review of Education 27*(1), 23–35.

Foy, M. (2008). A richer understanding of algebra. *Mathematics Teaching Incorporating Micromath*, *209*, 40–41.

References

Freire, P. (1998). *Pedagogy of Freedom: Ethics, Democracy, and Civic Courage.* Maryland: Rowman & Littlefield Publishers, Inc.

Freire, P. (2000). *Pedagogy of the oppressed.* Translated by Myra Bergman Ramos. New York: Continuum.

Friedlander, A. (1999). An EXCELlent bridge to algebra. *Mathematics teacher, 91*(5), 382-383.

Friedlander, A., & Tabach, M. (2001). Developing a curriculum of beginning algebra in a spreadsheet environment. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.). *The future of the teaching and learning of algebra, The proceedings of the 12th ICMI Study Conference.* (pp. 252-257). Melbourne Australia: Department of Science and Mathematics Education, University of Melbourne.

Fritz, M.L. (2005). *Students using handheld computers to learn collaboratively in a first grade classroom.* Unpublished Ph.D. thesis, Drexel University.

Fujii, T. (2003). Probing students' understanding of variables through cognitive conflict problems: Is the concept of a variable so difficult for students to understand? In N. Pateman, G. Dougherty, J. Zilliox (Eds.), *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education,* (Vol.1, pp. 49-65). Honolulu, Hawaii: PME Program Committee.

Fujii, T., & Stephens, M. (2001). Fostering understanding of algebraic generalisation through numerical expressions: The role of the quasi-variables. In H. Chick, K. Stacey, J.Vincent and J.Vincent (Eds.), *The future of the teaching and learning of algebra. Proceedings of the 12th ICMI study conference, Vol. 1*, (pp. 258-64). Melbourne: Australia.

Fung, A. C. W., & Pun, S. W. (2001). ICT in Hong Kong Education. *Journal of Southeast Asian Education, 2*(1), 165−180.

Gagnon, G. W., & Collay, M. (2006). Constructivist Learning Design: Key Questions for Teaching to Standards. London: Corwin Press.

Gallardo, A., & Rojano, T. (1987). Common difficulties in the learning of algebra among children displaying low and medium pre-algebraic proficiency levels. In J.C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education,* Montreal, Canada, 1, 301−307.

Gallardo, A., & Rojano, T. (1988). Difficulty areas in the acquisition of the arithmetic and algebraic language. *Recherches en Didactique des Mathematiques 9*(2) 155–188.

Garaçon, M., Kieran, C., & Boileau, A. (1990). Introducing algebra: A functional approach in a computer environment. In G. Booker, P. Cobb, & T. N. de

Mendicuti (Eds.), *Proceedings of the 14th International Conference for the Psychology of Mathematics Education* (Vol II, pp. 51-58). Oaxtepec, México: PME Program Committee.

Gardner, H. (1991) *The unschooled mind: How children think and how schools should teach.* New York: Basic Books Inc.

Gash, H. (2014). Constructing constructivism. *Constructivist Foundations, 9*(3): 302–310.

Gattegno, C. (1974). *The Common Sense of Teaching Mathematics.* New York: Educational Solutions.

Gattegno, C. (1986). *The Generation of Wealth.* New York: Educational Solutions Worldwide Inc.

Gattegno, C. (1987). *What We Owe children: The Subordination of Teaching to Learning.* NY: Educational Solutions Worldwide Inc. (Originally published in 1971 in London by Routledge and Kegan Paul).

Gattegno, C. (1988). *The Science of Education, Part 2B: The Awareness of Mathematization (2nd Edition).* New York: Educational Solutions Worldwide Inc.

Geraniou, E., & Mavrikis, M. (2016). Crossing the bridge: From a constructionist learning environment to formal algebra. In K. Krainer & N. da Vondrov, *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2494-2500). Prague, Czech Republic: CERME.

Geraniou, E., & Mavrikis, M. (2015). Building bridges to algebra through a constructionist learning environment. *Constructivist Foundations, 10*(3), 321–330.

Gergen, K. (1995). Social construction and the educational process. In L. Steffe & J. Gale (Eds.), *Constructivism in Education*, (pp.17-39). New Jersey: Lawrence Erlbaum Associates, Inc.

Gerretson, H. (2004). Pre-service elementary teachers' understanding of geometric similarity: The effect of dynamic geometry software. *Focus on Learning Problems in Mathematics, 26*(3), 12–23.

Gibbert, M., Ruigrok, W., & Wicki, B. (2008). What passes as a rigorous case study? *Strategic Management Journal (29)*13, 1465–1474.

Ginsburg, H. (1977). *Children's Arithmetic.* New York: Van Nostrand.

Glasersfeld, E. von (1996). Introduction: Aspects of constructivism. In C. T. Fosnot (Ed.), *Constructivism: Theory, perspectives, and practice* (pp. 3-7). New York, NY: Teachers College Press, Columbia University.

Glasersfeld, E. von (1984). An introduction to constructivism. In P. Watzlawick (Ed.) (1984),*The Invented Reality*. New York: Norton, pp. 17–40. English translation

of: Glasersfeld, E. von (1981) Einführung in den Radikalen Konstruktivismus. In P. Watzlawick (Ed.) *Die Erfundene Wirklichkeit*, Munich: Piper, pp. 16–38.

Glasersfeld, E. von (1989). Cognition, construction of knowledge, and teaching. *Syntheses*, 80(1), 121–140.

Glasersfeld, E. von (1990a). An exposition of constructivism: Why some like it radical. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Monographs of the Journal for Research in Mathematics Education*, #4 (pp. 19-29). Reston, VA: National Council of Teachers of Mathematics.

Glasersfeld, E. von (1990b). Environment and communication. In L.P. Steffe & T. Wood (Eds.), *Transforming early childhood mathematics education* (pp. 30-38). Hillsdale: Laurence Erlbaum.

Glasersfeld, E. von (1991a). Introduction. In E. Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education*, (pp. xiii-xix). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Glasersfeld, E. von (1991b). Questions and answers about radical constructivism. In M. K. Pearsall (Ed.), *Scope, sequence, and coordination of secondary school science, Vol. II: Relevant research,* (pp. 169–182). Washington, D.C.: The National Science Teachers Association.

Glasersfeld, E. von (1991c). Knowing without metaphysics: Aspects of the radical constructivist position. In F. Steier (Ed.), *Research and Reflexivity (Inquiries into Social Construction)* (pp. 12-29). London: Sage Publications.

Glasersfeld, E. von (1991d). Abstraction, re-presentation, and reflection. In L. P. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience* (pp. 45-67). New York: Springer.

Glasersfeld, E. von (1995a). *Radical constructivism: A way of knowing and learning*. London: The Falmer Press.

Glasersfeld, E. von (1995b). A constructivist approach to teaching. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 3-16). Hillsdale, NJ: Lawrence Erlbaum.

Glidden, P. L. (2008). Prospective elementary teachers' understanding of order of operations. *School Science and Mathematics, 108*(4), 130–136.

Golafshani, N. (2003). Understanding reliability and validity in qualitative research. *The Qualitative Report, 8*(4), 597–607. Retrieved October 12, 2015, from http://www.nova.edu/ssss/QR/QR8-4/golafshani.pdf.

Good, T. L., Reys, B. J., Grouws, D. A., & Mulryan, C. M. (1990). Using work-groups in mathematics instruction. *Educational Leadership, 47*(4), 56–62.

References

Gray, E. M. (1991). An Analysis of diverging approaches to simple arithmetic: Preference and its consequences, *Educational Studies in Mathematics*, *22*, 551–574.

Gray, E., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A "proceptual" view of simple arithmetic, *Journal for Research in Mathematics Education, 25*, 116–140.

Gray, E., & Tall, D. O. (1991). Duality, ambiguity and flexibility in successful mathematical thinking, *Proceedings of the 15th International Conference for the Psychology of Mathematics Education* (Vol. II, pp. 72-79). Assisi, Italy: PME Program Committee.

Gray, E., & Tall, D. O. (1993). Success and failure in mathematics: The flexible meaning of symbols as process and concept, *Mathematics Teaching*, *142,* 6–10.

Gray, E., & Tall, D. O. (2001). Relationships between embodied objects and symbolic procepts: an explanatory theory of success and failure in mathematics, *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol III, pp. 65-72). Utrecht, The Netherlands: PME Program Committee.

Gray, E., Pitta, D., & Tall, D. O. (2000). The nature of the object as an integral component of numerical processes**.** *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. I, pp. 115–130). Lanti, Finland: PME Program Committee.

Greene, M. (1997). The lived world, literature and education. In D. Vandenberg (Ed.), *Phenomenology & education discourse* (pp. 169-190). Johannesburg: Heinemann.

Greeno, G. J., (1983). Conceptual entities. In D. Genter & A. L. Stevens (Eds.), *Mental Models* (pp. 227-252). Hillsdale, NJ: Lawrence Erlbaum Associates.

Guba E. G., & Lincoln Y. S. (1989). *Fourth Generation Evaluation.* London: Sage Publications.

Gurtner, J.-L. (1992). Between Logo and mathematics: A road of tunnels and bridges. In C. Hoyles & R. Noss (Eds.), *Learning mathematics and Logo,* (pp.17-39). Cambridge MA: MIT Press.

Hall, R. (2008). Applied Social Research: Planning, Designing and Conducting Real-world Research. South Yarra, Victoria: Palgrave Macmillan.

Hammersley, M., & Atkinson, P. (2007). *Ethnography: Principles in practice* (3rd Edition). Taylor and Francis e-Library.

References

Harel, G., & Kaput, J. (1991). The influence of notation in the development of mathematical ideas. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 82-94). Dordrecht, The Netherlands: Kluwer.

Harrison, A. F., & Bramson, R. M. (1984). *The art of thinking*. New York: Berkeley Book.

Hart, K. (Ed.) (1981). *Children's understanding of mathematics;* 11-16. London: Murray.

Headlam, C., & Graham, T. (2009). Some initial findings from a study of children's understanding of the order of operations. In M. Joubert (Ed.) *Proceedings of the British Society for Research into Learning Mathematics, 29*(3), 37–41.

Healy, L., & Sutherland, R. (1990). *Exploring mathematics with spreadsheets.* Hemel Hempstead, UK: Simon & Schuster.

Healy, L., & Sutherland, R. (1991). *Exploring mathematics and spreadsheets.* London: Blackwell.

Heath, G.D. (2002). Using applets in teaching mathematics. *Mathematics and Computer education, 36*(1), 43–52.

Hegedus, S. J., & Kaput, J. J. (2003). The Effect of a SimCalc Connected Classroom on Students' Algebraic Thinking. Paper presented at the 27th International Group for the Psychology of Mathematics Education Conference, held jointly with the 25th PME-NA Conference (Vol. 3, pp. 47-54), Honolulu, HI: PME committee.

Heid, M.K., & Zbiek, R. M. (1993). Nature of understanding of mathematical modelling by beginning algebra students engaged in a technology-intensive conceptually based algebra course. In J. R. Becker & B. J. Pence (Eds.), *Proceedings of the 15th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 128-134). Pacific Grove, CA: PME-NA Program Committee.

Hembree, R. (1992). Experiments and relational studies in problem-solving: A meta-analysis. *Journal for Research in Mathematics Education*, *23*(3), 242-273.

Hermans, R., Tondeur, J., van Braak, J., & Valcke, M. (2008). The impact of primary school teachers' educational beliefs on the classroom use of computers. *Computers & Education, 51*(4), 1499–1509.

Hersant, M., & Perrin-Glorian, M.J. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactic situations. *Educational Studies in Mathematics*, *59*(1/3), 113–151.

Hersberger, J., & Frederick, B. (1999). Flower beds and landscape consultants: Making connections in middle school mathematics. In B. Moses (Ed.), *Algebraic thinking: Grades K-12* (pp. 247-251). Reston, VA: NCTM.

References

Herscovics, N., & Kieran, C. (1980). Constructing meaning for the concept of equation. *Mathematics Teacher, 73*, 572–580.

Herscovics, N., & Linchevski, L. (1991). Pre-algebraic thinking: Range of equations and informal solution processes used by seventh graders prior to any instruction. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference for the Psychology of Mathematics Education* (Vol. II, pp. 173-180). Assisi, Italy: PME Program Committee.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59−78.

Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., Tabach, M., & Schwarz, B. (2002). Mathematics curriculum development for computerized environments: A designer-researcher-teacher-learner activity. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 657-694). Mahwah, NJ: Lawrence Erlbaum.

Hewitt, D. (1994). *The Principle of Economy in the Learning and Teaching of Mathematics.* Unpublished PhD thesis. Milton Keynes: The Open University.

Hewitt, D. (1998). Approaching arithmetic algebraically. *Mathematics Teaching, 163*, 19−29.

Hewitt, D. (1999). Arbitrary and necessary part 1: A way of viewing the mathematics curriculum. *For the Learning of Mathematics, 19*(3), 2−9.

Hewitt, D. (2001). On learning to adopt formal algebraic notation. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.). *The future of the teaching and learning of algebra, The proceedings of the 12th ICMI Study Conference.* (pp. 305-312). Melbourne: Department of Science and Mathematics Education.

Hewitt, D. (2009), The role of attention in the learning of formal algebraic notation: The case of a mixed ability Year 5 using the software Grid Algebra. In *M.* Joubert (Ed.), *Proceedings of the British Society for Research into Learning Mathematics, 29(3),* Loughborough University, British Society for Research into Learning Mathematics, pp. 43–48.

Hewitt, D. (2012). Young students learning formal algebraic notation and solving linear equations: are commonly experienced difficulties avoidable? *Educational Studies in Mathematics, 81*, 139–159.

HEWITT, D. (2014). A symbolic dance: the interplay between movement, notation, and mathematics on a journey toward solving equations. *Mathematical Thinking and Learning, 16*(1), 1–31.

References

Hillel, J., & Samurcay, R. (1985). *Analysis of a Logo environment for learning the concept of procedures with variable.* Unpublished manuscript, Concordia University, Montreal.

Hodgen, J., Küchemann, D., Brown, M., & Coe, R. (2008). Children's understandings of algebra 30 years on. In: M. Joubert (Ed.) *Proceedings of the British Society for Research into Learning Mathematics, 28*(3), 36–41.

Hoepfl, M. C. (1997). Choosing qualitative research: A primer for technology education researchers. *Journal of Technology Education, 9*(1), 47–63.

Hoyles, C., & Noss, R. (1988). Formalising intuitive descriptions in a parallelogram Logo microworld. In A. Borbas (Ed.), *Proceedings of the 12th Annual Conference of the International Group for the Psychology of Mathematics Education* (pp. 417-424). Veszprem, Hungary: International Group for the Psychology of Mathematics Education.

Hoyles, C., & Noss, R. (1989). The computer as a catalyst in children's proportion strategies. *Journal of Mathematical Behavior, 8*, 53–75.

Hoyles, C., & Noss, R. (2003), What can digital technologies take from and bring to research in mathematics education? In A.J. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick, and F.K.S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 323-349). Dordrecht: Kluwer Academic Publishers.

Hoyles, C., & Noss, R. (2007). Next steps in implementing Kaput's research programme. *Educational Studies in Mathematics, 68*(2), 85-94.

Hoyles, C., Healy, L., & Sutherland, R. (1991). Patterns of discussion between pupil pairs in computer and non-computer environments. *Journal of Computer Assisted Learning, 7,* 210–228.

Hoyles, C., Noss, R., & Adamson, R. (2002). Rethinking the microworld idea. *Journal of Educational Computing Research*, 27(1&2), 29–53.

Hunter, J. (2015). Teacher actions to facilitate early algebraic reasoning. In M. Marshman, V. Geiger, & A. Bennison (Eds.), Mathematics education in the margins, *Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 58–67). Sunshine Coast: MERGA.

Imler, B., & Eichelberger, M. (2011). Using screen capture to study user research behaviour. *Library Hi Tech, 29*(3), 446–454.

Ingram, J. (2014). Shifting attention. *For the Learning of Mathematics, 34*(3), 19–24.

Ingram, J., Pitt, A., & Baldry, F. (2015). Handling errors as they arise in whole-class interactions. *Research in Mathematics Education, 17*(3), 183–197.

References

Inhelder, B., & Piaget, J. (1958). *The Growth of Logical Thinking.* Henley: Routledge & Kegan Paul.

Jaworski, B. (1994). Investigating Mathematics Teaching: A Constructivist Enquiry. London: The Falmer Press.

Jaworski, B. (2012). Mathematics teaching development as a human practice: Identifying and drawing the threads. *ZDM Mathematics Education, 44*(5), 613–626.

Johnson, P. A. (1986). *Effects of computer-assisted instruction compared to teacher-directed instruction on comprehension of abstract concepts by the deaf.* Unpublished doctoral dissertation, Northern Illinois University.

Jonassen, D. H. (1992). Objectivism versus constructivism: Do we need a new philosophical paradigm? *Educational Technology Research and Development, 39*(3), 5–14.

Jones, I. (2008). A diagrammatic view of the equals sign: Arithmetical equivalence as a means, not an end. *Research in Mathematics Education 10*(2),151–165.

Jones, I., & Pratt, D. (2012). A substituting meaning for the equals sign in arithmetic notating tasks. *Journal for Research in Mathematics Education, 43*(1), 2–33.

Jones, I., Inglis, M., Gilmore, C., & Evans, R. (2013). Teaching the substitutive conception of the equals sign. *Research in Mathematics Education, 15*(1), 34–49.

Judah, M-L. (1999). *'Occasioning selves' in the computer assisted mathematics classroom.* Unpublished M. Ed. Dissertation, University of Alberta.

Judge, H. (1984). 'In Search of Mind' by Jerome Bruner (Book review). *Oxford Review of Education, 10*(3), 305–307.

Kang, W., & Kilpatrick, J. (1992). *Didactic transposition in Mathematics textbooks. For the Learning of Mathematics, 12*(1), 2–7.

Kankaanranta, M., & Kangassalo, M. (2003). Information and Communication Technologies in Finnish Early Childhood Environments. *Childhood Education, 79*(5), 287−292.

Kansanen, P., & Meri, M. (1999). The didactic relation in the teaching-studying-learning process. In *TNTEE Publications* (Vol. 2, pp. 107 - 116).

Kant, I. (1781). *Kritik der reinen Vernunft. (Critique of Pure Reason).* Translated and edited by P. Guyer & A.W. Wood (1998). Cambridge, UK: Cambridge University Press.

Kaput, J. J. (1985). Representation and problem solving: Methodological issues related to modelling. In E. Silver (Ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Hillsdale, NJ: Erlbaum.

Kaput, J. J. (1991). Notations and interpretations as mediators of constructive processes. In E. Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education*, (pp. 53-74). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Kaput, J. J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *Handbook on research in mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Kaput, J. J. (1995b). A research base supporting long term algebra reform? In D.T. Owens, M.K. Reed, & G.M. Millsaps (Eds.), *Proceedings of the 17th Annual Meeting of PME-NA* (Vol. 1, pp. 71-94). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Erlbaum.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J.J. Kaput, D.W. Carraher, & M.L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 5-18). New York: Lawrence Erlbaum Associates.

Kaput, J. J. (1995a). Transforming Algebra from an Engine of Inequity to an Engine of Mathematical Power By "Algebrafying" the K-12 Curriculum. Paper presented at the 1995 National Council of Teachers of Mathematics.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19-55). New York: Lawrence Erlbaum Associates,

Kaput, J.J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In S. Fennell (Ed.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium* (pp. 25-26). Washington, DC: National Research Council, National Academy Press.

Karplus, R. (1977). Science Teaching and the Development of Reasoning. *Journal of Research in Science Teaching, 14,* 169–175.

Kieran, C. (1979). Children's operational thinking within the context of bracketing and the order of operations. In D. Tall (Ed.), *Proceedings of the Third International Conference for the Psychology of Mathematics Education, 1,* 128–133.

Kieran, C. (1981a). Pre-algebraic notions among 12- and 13-year-olds. In Equipe de Recherche Pedagogique Laboratoire (Eds.), *Proceedings of the 5th annual*

conference of the International Group for the Psychology of Mathematics Education (pp. 158-164). Grenoble, France: PME Program Committee.

Kieran, C. (1981b). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12,* 317–326.

Kieran, C. (1988). The equation-solving errors of novice and intermediate algebra students. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education* (pp. 141-146). Noordwijkerhout, The Netherlands: State University of Utrecht, Research Group on Mathematics Education and Educational Computer Centre.

Kieran, C. (1992). The learning and teaching of school algebra. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 390-419). New York: Macmillan.

Kieran, C. (1994). A functional approach to the introduction of algebra: Some pros and cons. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Plenary address, Vol. 1, pp. 157-175). Lisbon, Portugal: PME Program Committee.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *8th International Congress on Mathematical Education: Selected lectures* (pp. 271-290). Seville, Spain: S.A.E.M. Thales.

Kieran, C., Booker, G., Filloy, E., Vergnaud, G., & Wheeler, D. (1990). Cognitive processes involved in learning school algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 96-112). Cambridge UK: Cambridge University Press.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education, 35*(4), 224–257.

Kirshner, P., & Wopereis, I.G.J.H. (2003). Mindtools for teacher communities: A European perspective. *Technology, Pedagogy and Education, 12*(1)*,* 19–26.

Knuth, E. J., Stephens, A. C, McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37,* 297–312.

Kolb, A. Y., & Kolb, D. A. (2005). Learning styles and learning spaces: Enhancing experiential learning in higher education. Academy of Management Learning & Education, 4(2), 193–212.

Kolb, D. A. (1984). *Experiential Learning: Experience as the Source of Learning and Development.* New Jersey: Prentice-Hall.

Kolb, D. A. (2015). Experiential Learning: Experience as the Source of Learning and Development (2nd edition). New Jersey: Pearson Education.

Kosslyn, S. M., & Rabin, C. S. (1999). Imagery. In R. A. Wilson & F. C. Keil (Eds.), *The MIT Encyclopaedia of the Cognitive Sciences* (pp. 387–389). Cambridge, MA: MIT Press.

Kosslyn, S. M., Thompson, W. L., & Ganis, G. (2006). *The case for mental imagery.* New York: Oxford University Press.

Küchemann, D. E. (1978). Children's Understanding of Numerical Variables. *Mathematics in School, 7*, 23–26.

Küchemann, D. E. (1981). Algebra. In K. M. Hart (Ed.), *Children's Understanding of Mathematics, 11-16* (pp. 102-119). London: John Murray.

Kurland, D. M., Pea, R., Clement, C., & Mawby, R. (1986). A study of the development of programming ability and thinking skills in high school students. *Journal of Educational Computing Research, 2*, 429–458.

Kurz, T. L., Batarelo, I., & Middleton, J. A. (2009). Examining elementary preservice teachers' perspectives concerning curriculum themes for video case integration. *Educational Technology Research and Development, 57*(4), 461–485.

Kvale, S. (1996). *Interviews.* London: Sage Publications.

Kwan, K. P., & Jones, J. (1999). What students expect of their teachers. In J. Jones & K. P. Kwan (Eds.), *Evaluation of the Student Experience Project: Vol. 3. The Video Interview Project: Listening to Our Students Talk* (pp. 21-32). City University of Hong Kong, Centre for the Enhancement of Learning and Teaching.

Lakoff, G. (1987). *Women, fire, and dangerous things.* Chicago: University of Chicago Press.

Lampert, M. (1988). The teacher's role in reinventing the meaning of mathematical knowing in the classroom. *Proceedings of the North America Chapter of the Psychology of Mathematics Education Group* (pp. 433-480). Northern Illinois University.

Lannin, J. K. (2003). Developing algebraic reasoning through generalization. *Mathematics Teaching in the Middle School, 8*(7), 342–348.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge, UK: Cambridge University Press.

LeCompte, M., & Preissle, J. (1993). *Ethnography and Qualitative Design in Educational Research* (2nd Edition). London: Academic Press.

References

Lee, L., & Wheeler, D. (1987). *Algebraic Thinking in High School Students: Their Conceptions of Generalisation and Justification* (Research report). Montréal, Canada: Concordia University, Mathematics Department.

Lee, Y-A. (2007). Third turn position in teacher talk: Contingency and the work of teaching. *Journal of Pragmatics, 39*(6), 1204–1230.

Lehrer, R., & Smith, P. (1986a). *Logo learning: Is more better?* Paper presented at the meeting of the American Educational Research Association, San Francisco.

Lehrer, R., & Smith, P. C. (1986b). *Logo learning: Are two heads better than one?* Paper presented at the meeting of the American Educational Research Association, San Francisco.

Lehrer, R., Randle, L., & Sancilio, L. (1989). Learning pre-proof geometry with Logo. *Cognition and Instruction, 6*, 159–184.

Lerman, S. (1989). Constructivism, mathematics and mathematics education. *Educational Studies in Mathematics 20*(2), 211–223

Lewin, K. (1948*). Resolving social conflicts: Selected papers on group dynamics.* New York: Harper & Bros.

Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behaviour,* 14, 113−120.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics, 30*(1), 39−65.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts, *Educational Studies in Mathematics, 40*(2) 173–196.

Linchevski, L., & Livneh, D. (2002). The competition between numbers and structure: Why expressions with identical algebraic structures trigger different interpretations. *Focus on learning problems in mathematics, 24*(2), 20–35.

Lincoln, Y. S., & Guba, E. (1985). *Naturalistic Inquiry*. Beverly Hills, CA: Sage Publications.

Livio, M. (2011). Why math works. *Scientific American*, 81–83.

Livneh, D., & Linchevski, L., (2007). Algebrification of arithmetic: Developing algebraic structure sense in the context of arithmetic. *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Vol. III,* (pp. 217-224). Seoul: PME Program Committee.

References

Lugalia, A., Johnston-Wilder, S., & Johnston-Wilder, P. (2011). Mathematics and ICT: Using Grid Algebra, *INTED2011 Proceedings* (pp. 436-443). Valencia Spain: IATED.

Lugalia, M. (2015). *Pupils learning algebra with ICT in Key Stage 3 mathematics classrooms.* PhD thesis, University of Warwick. Retrieved March 13, 2016, from http://go.warwick.ac.uk/wrap/72804.

Macdonald, B. & Walker, R. (1975). Case study and the social philosophy of educational research. *Cambridge Journal of Education, 5*(1), 2–12.

MacGregor, M. (1996). Curricular aspects of arithmetic and algebra. In J. Gimenez, R. C. Lins, and B. Gomez (Eds.), *Arithmetics and algebra education: Searching for the future* (pp. 50-54). Tarragona, Spain: Universitat Rovira i Virgili.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in mathematics, 33,* 1−19.

Margolis, J., & Fisher, A. (2002). Unlocking the clubhouse: Women in computing. Cambridge, MA: The MIT Press.

Markauskaite, L. (2006). Gender issues in preservice teachers' training: ICT literacy and online learning. *Australasian Journal of Educational Technology*, *22*(1), 1–20.

Mason, J. (1988). *Expressing Generality.* Milton Keynes, UK: Open University Press.

Mason, J. (1991). *Supporting Primary Mathematics: Algebra.* Milton Keynes, UK: Open University Press.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, L. Lee (Eds.), *Approaches to Algebra: Perspectives for Teaching and Learning* (pp. 65-86). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1,* 243–267.

Mason, J. (2002). Researching your own practice: The discipline of noticing. London: RoutledgeFalmer.

Mason, J. (2005a). *Structured Variation Grids.* Retrieved January 11, 2015, from http://mcs.open.ac.uk/jhm3/SVGrids/SVGridsMainPage.html.

Mason, J. (2005b). Frameworks for learning, teaching and research: Theory and practice, Plenary presentation, in S. Wilson & ? (Eds.) *Frameworks That Support*

Research & Learning: Proceedings of PME-NA.

References

Mason, J. (2008). Making use of children's powers. In J.J. Kaput, D.W. Carraher, & M.L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 57-94). New York: Lawrence Erlbaum Associates.

Mason, J., & De Geest, E. (2010). How an experientially presented labelled-framework approach to professional development at a distance can influence ieachers' practice. In M. Pinto & T. Kawasaki (Eds.) *Proceedings of the 34th International Conference on the Psychology of Mathematics Education* (Vol.3 pp. 273-280), Belo Horizonte: PME Program Committee.

Mason, J., Graham, A., Pimm, D., & Gowar, N. (1985). *Routes to/Roots of Algebra.* Milton Keynes, UK: Open University Press.

Mathew, S.K. (2004). *An investigation into the implementation of computer-assisted education in outcomes-based education: A case study at Sterling Primary School in East London.* Unpublished M. Ed. mini-thesis, University of Pretoria.

Matthews, M. R. (2002). Constructivism and science education: A further appraisal. *Journal of Science Education and Technology 11*(2), 121–134.

Maturana, H. R. (1970). *Biology of Cognition* (Report #9.0). Urbana, Illinois: BCL, University of Illinois.

Maturana, H. R., & Varela, F. J. (1980). Autopoiesis and cognition: The realization of the living. In R. S. Cohen & M. W. Wartofsky (Eds.), *Boston Studies in the Philosophy of Science, (Vol. 42).* Dordrecht, The Netherlands: D. Reidel Publishing Company.

Maturana, H. R., & Varela, F. J. (1992). *The Tree of Knowledge: The Biological Roots of Human Understanding* (revised edition: R. Paolucci, Trans.). Boston: Shambhala.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior, 3,* 93–166.

Mavrikis, M., Noss, R., Hoyles, C & Geraniou, E. (2012). Sowing the seeds of algebraic generalization: Designing epistemic affordances for an intelligent microworld. *Journal of Computer Assisted Learning, 29*(1), 68–85.

Maxwell, J. A. (1992). Understanding and validity in qualitative research. *Harvard Educational Review*, 62(3), 279–300.

McClure, L. (2011). Using low threshold high ceiling tasks in ordinary classrooms. Retrieved July 12, 2015 from: nrich.maths.org/7701

McNeil, N. M. (2007). U-shaped development in math: Seven year olds outperform nine year olds on equivalence problems. *Developmental Psychology*, *43*, 687–695.

References

McNeil, N. M. (2008). Limitations to teaching children 2 + 2 = 4: Typical arithmetic problems can hinder learning of mathematical equivalence. *Child Development*, *79*, 1524–1537.

McNeil, N. M., & Alibali, M. W. (2005a). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6,* 285–306.

McNeil, N. M., & Alibali, M. W. (2005b). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development, 76,* 1–17.

McNeil, N., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E., (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction, 24*(3), 367–385.

McNiff, J., & Whitehead, J. (2005). *Action Research for Teachers: A Practical Guide.* Oxon: David Fulton Publishers.

Meagher, M. (2012). Students' relationship to technology and conceptions of mathematics while learning in a computer algebra system environment. *International Journal for Technology in Mathematics Education, 19(*1), 3–16.

Mercer, N. (2000). *Words and minds.* London: Routledge.

Merriam, S. B. (2002). *Qualitative Research in Practice: Examples for Discussion and Analysis.* San Francisco, CA: Jossey-Bass.

Merriam, S. B. (1988). *Case Study Research in Education: a Qualitative Approach.* San Francisco, CA: Jossey-Bass.

Miles, M. and Huberman, M. A. (1984) *Qualitative Data Analysis.* Beverly Hills: Sage Publications.

Muijs, D. (2011). *Doing quantitative research in education with SPSS* (2nd ed.). London: Sage Publications.

Mumtaz, S. (2001). Children's enjoyment and perception of computer use in the home and the school. *Computers & Education, 36*(4), 347–362.

Myers, R. Y. (2009). The effects of the use of technology in mathematics instruction on student achievement (Doctoral Dissertation). *FIU Electronic Theses and Dissertations,* Paper 136. Retrieved, June 8th, 2015, from http://digitalcommons.fiu.edu/etd/136.

National Council of Teachers of Mathematics (1998). Principles and Standards for School Mathematics: Discussion Draft. Reston, VA: National Council of Teachers of Mathematics.

References

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics Algebra Task Force (1993). *Report from the Algebra Task Force.* Reston, VA: National Council of Teachers of Mathematics

National Council of Teachers of Mathematics Algebra Working Group (1997). *A Framework for Constructing a Vision of Algebra: A Discussion Document.* Final report to the Board of Directors. East Lansing, MI: Michigan State University.

NCTE (2002). *Guidelines for Gender-Fair Use of Language.* Revised by Nancy Prosenjak, et al. National Council of Teachers of English. Retrieved August 19, 2017 from http://www.ncte.org/positions/statements/genderfairuseoflang.

Neimeyer, R. A., & Raskin, J. D. (2001). Varieties of constructivism in psychotherapy. In K. S. Dobson (Ed.), *Handbook of Cognitive-Behavioral Therapies* (2nd ed., pp. 393-430). New York: Guilford.

Neria, D., & Amit, M. (2004). Students' preference of non-algebraic representations in mathematical communication. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, Bergen, Norway, Bergen University College, pp. 409-416.

Nolen, A. L., & Putten, J. V. (2007). Action research in education: Addressing gaps in ethical principles and practices. *Educational Researcher, 36*(7), 401–407.

Norton, S. J., & Cooper, T. J. (1999). *Do our really have arithmetic knowledge to start algebra?: Analyzing misconceptions.* Published on the World Wide Web by Queensland Association of Mathematics Teachers (QAMT Inc.). Retrieved October 2, 2015 from https://www.researchgate.net/publication/2598719

Noss, R. (1986). Constructing a conceptual framework for elementary algebra through logo programming. *Educational Studies in Mathematics, 17*(4), 335–357.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics, 33,* 203–233.

Nyikos, M., & Hashimoto, R. (1997). Constructivist theory applied to collaborative learning in teacher education: In search of ZPD. *The Modern Language Journal*, *81*(4), 506–517.

Oldknow, A. (2006). Let's get the show on the road!*Mathematics Teaching Incorporating Micromath*, *196*, 16–21.

References

Paige, J. M., & Simon, H. A. (1966). Cognitive processes *in* solving algebra word problems (pp. 51–118). In B. Kleinmuntz (Ed.), *Problem Solving: Research, Method, and Theory*. New York: Wiley.

Papert, S. (1993a). *Mindstorms: Children, computers, and powerful ideas (2nd Edition).* New York: Basic Books.

Papert, S. (1993b). *The children's machine: Rethinking school in the age of the computer.* New York: Basic Books.

Papert, S. (1996). *The connected family: Bridging the digital generation gap.* Georgia: Lonstreet Press.

Papert, S. (1998, May 21). Whose fingers on the button? *The Independent* (London).

Papert, S. and Harel, I. (Eds) (1991). *Constructionism.* Norwood, NJ, Ablex Publishing Corporation.

Parlett, M., & Hamilton, D. (1976) Evaluation as illumination. In D. Tawney (ed.) *Curriculum Evaluation Today: Trends and Implications.* London: Macmillan.

Pask, G., & Scott, B. C. E. (1972). *Learning strategies and individual competence. International Journal of Man-Machine Studies, 4,* 217–253.

Patsiomitou, S. (2008). The development of students geometrical thinking through transformational processes and interaction techniques in a dynamic geometry environment. *Issues in Informing Science and Information Technology (5),* 353–393.

Patton, M.Q. (2002). *Qualitative research and evaluation methods (3rd Edition).* London: Sage Publications.

Pauli, C., & Reusser, K. (2010). Expertise in Swiss mathematics instruction. In Y. Li & G. Kaiser (Eds.), *Expertise in Mathematics Instruction: An International Perspective* (pp. 85–107). New York: Springer.

Peirce, C. S. (1878). How to make our ideas clear. *Popular Science Monthly, 7*, 286–302.

Penglase, M., & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. *Mathematics Education Research Journal, 8*(1), 58–90.

Philipp, R. A. (1999). The many use of algebraic variables. In B. Moses (Ed.), *Algebraic thinking, grades: 9-12* (pp.150- 156). Readings from NCTM's School Based Journals and Other Publications, Reston, VA: National Council of Teachers of Mathematics.

Piaget, J. (1952). *The Child's Conception of Number.* Henley: Routledge & Kegan Paul.

References

Piaget, J. (1957). *Construction of Reality in the Child.* London: Routledge & Kegan Paul.

Piaget, J. (1967). *Six Psychological Studies.* New York: Random.

Piaget, J. (1970). Piaget's theory. In P.H. Mussen (Ed.) *Carmichael's Manual of Child Psychology, 1* (pp. 703-732). New York: Wiley.

Piaget, J. (1971). *Biology and Knowledge: An Essay on the Relations between Organic Regulations and Cognitive processes.* Chicago: University of Chicago Press.

Piaget, J. (1975). *L' Equilibration des Structures Cognitives.* Paris: Presses Universitaires de France.

Piaget, J. (1978). *The Development of Thought.* Oxford: Blackwell.

Piaget, J., & García, R. (1989). *Psychogenesis and the History of Science* (Translated by H. Feider). New York: Columbia University Press. (Original work published in 1983).

Piaget, J., & Inhelder, B. (1962). *The Psychology of the Child.* New York: Basic Books.

Piaget, J., (1985), *The Equilibration of Cognitive Structures* (translation by T. Brown & K. J. Thampy), Harvard University Press, Cambridge MA (originally published in 1975).

Pierce, R. Ball, L., & Stacey, K. (2008). Is it worth using CAS for symbolic algebra manipulaion in the middle secondary years? Some teachers' views. *International Journal of Science and Mathematics Education, 7*(6), 1149–1172.

Pillay, H., Wilss, L., & Boulton-Lewis, G. (1998). Sequential development of algebra knowledge: A cognitive analysis. *Mathematics Education Research Journal, 10*(2), 87–102.

Pimm, D. (1987). *Speaking Mathematically: Communication in Mathematics Classrooms,* New York: Routledge and Kegan Paul.

Pirie, S. E. B & Kieren, T. E. (1994). Beyond metaphor: Formalising in mathematical understanding within constructivist environments. *For the Learning of Mathematics, 14*(1), 39–43.

Ploger, D., Klingler, L., & Rooney, M. (1999). Spreadsheets, patterns, and algebraic thinking. In B. Moses (Ed.), *Algebraic thinking: Grades K-12* (pp. 232-237). Reston, VA: NCTM.

Powell, S. R., & Fuchs, L. S. (2010). Contribution of equal-sign instruction beyond word-problem tutoring for third-grade students with mathematics Difficulty. *Journal of Educational Psychology, 102*(2), 381–394.

References

Prince, M. J., & Felder, R. M. (2006). Inductive teaching and learning methods: Definitions, comparisons, and research bases. *Journal of Engineering Education 95*, 123–138.

Pring, R. (1984). Confidentiality and the right to know. In C. Adelman (Ed.) *The Politics and Ethics of Education* (pp. 8-18). London: Croom Helm.

Project Tomorrow (2008). *21st Century Learners Deserve a 21st Century Education: Selected National Findings of the Speak Up 2007 Survey.* Retrieved May 20, 2010, from http://www.tomorrow.org/speakup/speakup_congress_2007.html.

Puig, L., & Cerdán, F. (1990). Acerca del charácter aritmetico o algebraico de los problemas verbales. In E. Filloy & T. Rojano (Eds.), *Memorias del Segundo Simposio Internacional sobre Investigación en Educación Matemática* (pp. 35-48). Cuernavaca, México: PNFAPM.

Quine, W. V. O. (1936). Truth by Convention. In O.H. Lee (Ed.), *Essays for A.N. Whitehead,* New York: Longmans; reprinted in Quine (1966).

Quinton, A. (1963). The a priori and the analytic. *Proceedings of the Aristotelian society, 64*, 31–54.

Radford, J. Blatchford, P., & Webster, R. (2011). Opening up and closing down: How teachers and TAs manage turn-taking, topic and repair in mathematics lessons. *Learning and Instruction, 21*(5), 625–635.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics, 42*(3), 237–268.

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal, 26,* 257–277.

Ramsay, G. (2001). Teaching and Learning with Information and Communication Technology: Success through a Whole School Approach. In NECC (2001) *Building the future*, National Educational Computing Proceedings (21st, Chicago, IL, June 25-27, 2002). Retrieved July 18, 2012, from http://www.eric.ed.gov/ERICWebPortal/recordDetail?accno=ED46294.

Rancourt R. (1988). *K.A.M.I.: Consultant's manual to style intepretation.* Ontario: Impact Inc.

Rand, A. (1966). *Introduction to Objectivist Epistemology.* New York: New American Library.

Raschick, M, Maypole, D. E., & Day, P. A. (1998). Improving field education through Kolb learning theory. *Journal of Social Work Education, 34*(1), 31–42.

Reeves, T. C. (1997). Evaluating what really matters in Computer-Based Education. Retrieved June, 8, 2013, from http://bengal.missouri.edu/~sfc42/reeves.html.

References

Richardson, C.R. (1999). *Exploring the changes in teaching strategies enabled by internet and communications technology.* Unpublished Ph.D. thesis, University of Toronto.

Riegler, A. (2001). Towards a Radical Constructivist Understanding of Science. *Foundations of Science, 6*(1–3), 1–30.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91,* 175–189.

Rittle-Johnson, B., Matthews, P. G., Taylor, R. S., & McEldoon, K. L. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach. *Journal of Educational Psychology, 103*(1), 85–104.

Rojano, T. (1996). Developing algebraic aspects of problem solving within a spreadsheet environment. In N. Bednarz, C. Kieran, L. Lee (Eds.), *Approaches to Algebra: Perspectives for Teaching and Learning* (pp. 137-146). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Rosen, C. (2008). The Myth of multitasking. *The New Atlantis, 20,* 105–110. Retrieved January 9, 2016 from http://www.thenewatlantis.com/docLib/20080605_TNA20Rosen.pdf.

Rubenstein, R. N. (2008). Mathematical symbolization: Challenges across levels. Paper presented at ICME 11, *TSG 31: Language and communication in mathematics education*. Retrieved November 15, 2014 from http://www.tsg.icme11.org/document/get/853.

Rubio, G. (1990). Algebra word problems: A numerical approach for its resolution (a teaching experiment in the classroom). In G. Booker, P. Cobb, & T. N. de Mendicuti (Eds.), *Proceedings of the 14th International Conference for the Psychology of Mathematics Education* (Vol II, pp. 125-132). Oaxtepec, México: PME Program Committee.

Rubio, G. (1994). Modelos didacticos para resolver problemas verbales aritmetico-algebraicos: Tesis teoricas y observacion empirica. Unpublished doctoral dissertation, CINVESTAV del Institutuo Politécnico Nacional, México.

Sadovsky, P., & Sessa, C. (2005). The adidactic interaction with the procedures of peers in the transition from arithmetic to algebra: A "milieu" for the emergence of new questions. *Educational Studies in Mathematics, 59*(1), 85–112.

Sakamoto, T. (2003). E-learning in Japanese Universities. *Industry and Higher Education, 17(6),* 397–404.

Saldana, (2009). *The coding manual for qualitative researchers.* London, Sage Publications Ltd.

References

Saul, M. (2001). Algebra: What are we teaching? In A. A. Cuoco & F. R. Curcio (Eds.), The roles of representation in school mathematics: 2001 yearbook (pp. 35-43). Reston, VA: National Council of Teachers of Mathematics.

Saussure, F. de (1916). *Cours de linguistique générale.* 3rd Edition (1966) Edited by C. Bally & A. Sechehaye in collaboration with A. Riedlinger, translated by W. Baskin. London: McGraw Hill.

Scandura, J. M. (1971). Mathematics: Concrete Behavioural Foundations. New York: Harper.

Scher, D.P. (2002). Geometry software, dynamic. *Mathematics.* Retrieved, February, 28, 2009, from http://www.encyclopedia.com/doc/1G2-3407500132.html.

Scherer, P., & Steinbring, H. (2006). Noticing children's learning processes— teachers jointly reflect on their own classroom interaction for improving mathematics teaching. *Journal of Mathematics Teacher Education, 9,* 157–185.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic Press.

Schoenfeld, A. H. (2012). Problematizing the didactic triangle. *ZDM–The International Journal of Mathematics Education, 44,* 587–599.

Schwartzman, S. (1977). Helping students understand the distributive property. *The Mathematics Teacher*, Oct. 1977, 594.

SEC Mathematics Syllabus (2017). Malta: MATSEC Department, University of Malta. Retrieved March, 2016, from https://www.um.edu.mt/matsec/syllabus/sec_level_2017.

Serfati, M. (2005). La révolution symbolique. La constitution de l'écriture symbolique mathématique. Paris: Pétra.

Sfard, A. (1989). Transition from Operational to Structural Conception: The notion of function revisited, *Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education* (pp. 151-158), Paris: PME Program Committee.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1–36.

Sfard, A. (1992). Operational origins of mathematical notions and the quandary of reification: The case of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Funciton: Aspects of Epistemology and Pedagogy* (MAA Notes, Vol. 25, pp. 59-84). Washington, DC: Mathematical Association of America.

References

Sfard, A. (1995). The development of algebra−Confronting historical and psychological perspectives. In C. Kieran (Ed.), New perspectives on school algebra: Papers and discussions of the ICME-7 algebra working group (special issue). *Journal of Mathematical Behavior, 14*, 15−39.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: the case of algebra. *Educational Studies in Mathematics, 26*, 191−228.

Shephard, R. N., & Metzler, J. (1971). Mental rotation of three-dimensional objects. Science, 1971(3972), 701–703.

Shoaf-Grubbs, M. (1995). Research results on the effect of the graphic calculator on female students' cognitive levels and visual thinking. In L. Burton & B. Jaworski. (Eds.), *Technology in mathematics teaching,* (pp. 213-227). Kent, UK: Chartwell-Bratt.

Shore, B. (1997). Keeping the Conversation Going: An Interview with Jerome Bruner. *Ethos, 25*(1), 7–62.

Shulman, L.S. (1986). Paradigms and research programs in the study of teaching. In M.C. Wittrock (Ed.), *Handbook of research on teaching.* New York: MacMillan.

Silverman, D. (2000). Doing Qualitative Research: A Practical Handbook. London: Sage Publications.

Simon, M. A. (1994). Learning mathematics and learning to teach: Learning cycles in mathematics teacher education. *Educational Studies in Mathematics, 26*(1), 71–94.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114−145.

Skemp, R.R. (1971). *The Psychology of Learning Mathematics.* Middlesex, UK: Penguin.

Sleeman, D. H. (1982). Assessing aspects of competence in algebra. In D. Sleeman and J. S. Brown (Eds.), *Intelligent tutoring systems.* London: Academic Press.

Sleeman, D. H. (1984). An attempt to understand students' understanding of basic algebra. *Cognitive Science, 8,* 387–412.

Sleeman, D. H. (1986). Introductory algebra: A case study of students' misconceptions. *The Journal of Mathematical Behavior, 5,* 25–52.

Smith, L. (1974). *An Aesthetic Education Workshop for Administrators: Some Implications for a Theory of Case Studies.* Paper presented at the American Educational Research Association (AERA) meeting, Chicago, Ill., 1974.

References

Smock, C. D., & Glasersfeld, E. von (1974). *Epistemology and Education: The Implications of Radical Constructivism for Knowledge Acquisition* (Report #14). Athens, Giorgia: Follow Through Publications.

Sooknanan, P., Melkote, S.R., & Skinner, E.C. (2002). Diffusion of an educational innovation in Trinidad and Tobago: The role of teacher attitudes and perceptions toward computers in the classroom. *International Communication Gazette64*(6), 557–571.

Stacey, K., & MacGregor, M. (1994). Algebraic sums and products: Students' concepts and symbolism. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Vol. 43, pp. 289-296), Lisbon, Portugal: PME Program Committee.

Stacey, K., & MacGregor, M. (1997). Building foundations for algebra. *Mathematics Teaching in the Middle School, 2*(4), 252–260.

Stake, R. E., & Trumbull, D. (1982). Naturalistic generalizations. *Review Journal of Philosophy and Social Science, 7*(1), 1–12.

Steedman, P.H. (1991). There is no more safety in numbers: A new conception of mathematics teaching. In E. Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education*, (pp. 1-13). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Steffe, L. P. (1991). The constructive teaching experiment: Illustrations and implications. In: E. Glasersfeld (Ed.) *Radical constructivism in mathematics education* (pp. 177-194). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Steffe, L. P. (2007). Radical constructivism and school mathematics. In M. Larochelle (Ed.), *Key Works in Radical Constructivism* (pp.279-290). Rotterdam: Sense Publishers.

Steffe, L. P. (2015). Personal Communication, October 7, 2015.

Steffe, L. P. (2016). Toward a model of constructivist mathematics teaching. *Constructivist Foundations, 12*(1), 75–77.

Steffe, L. P., & Blake, R. N. (1983). Seeking meaning in mathematics instruction: A response to Gagné. *Journal for Research in Mathematics Education, 14*, 210–213.

Steffe, L. P., & Cobb, P. (1988). *Construction of Arithmetical Meanings and Strategies.* New York: Springer.

Steffe, L. P., & D' Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education, 26*(2), 146−159.

References

Steffe, L. P., & Thompson, P. W. (2000a). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*, 191–209.

Steffe, L. P., & Thompson, P. W. (2000b). Teaching experiment methodology: underlying principles and essential elements (pp. 267−307). In: R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*. Mahwah, NJ: Erlbaum,

Steffe, L. P., & Thompson, P. W. (Eds.), (2000c). *Radical constructivism in action: Building on the pioneering work of Ernst Glasersfeld*. Falmer Press, London.

Steffe, L. P., & Ulrich, C. (2013). Constructivist teaching experiment. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education,* (pp. 102-109). Berlin: Springer.

Steffe, L. P., Cobb, P., & Glasersfeld, E. von (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Steffe, L. P., Glasersfeld, E. von, Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Prager.

Steinberg, R. M., Sleeman, D. H., & Ktorza, D. (1991). Algebra students' knowledge of equivalence of equations. *Journal for Research in Mathematics Education, 22*(2), 12–121.

Steinbring, H. (2000). Interaction analysis of mathematical communication in primary teaching: The epistemological perspective. *Zentralblatt für Didaktik der Mathematik, 5,* 138–148.

Steinbring, H. (2005). Analyzing mathematical teaching-learning situations: The interplay of communicational and epistemological constraints. *Educational Studies in Mathematics, 59*(1/3), 313–324.

Stenbacka, C. (2001). Qualitative research requires quality concepts of its own. *Management Decision, 39*(7), 551–555.

Stenhouse, L. (1975). An Introduction to Curriculum Research and Development. Oxford: Heinemann.

Sternberg, R. J. (1988). Mental self-government: A theory of intellectual styles and their development. *Human Development, 31,* 197–224.

Sternberg, R. J. (1997). *Thinking styles.* New York: Cambridge University Press

Strauss, R. (2013). Circle of Experiential Learning. Retrieved, December, 20, 2013 from http://www.gpccolorado.com/circle-experiential-learning/.

Subramaniam, K. and Banerjee, R. (2004). Teaching arithmetic and algebraic expressions. In M. J. Hoines and A. B. Fuglestad (Eds.), *Proceedings of the 28th International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 121-128). Bergen, Norway: PME Program Committee.

References

Sutherland, R. (1989). Providing a computer-based framework for algebraic thinking. *Educational Studies in Mathematics, 20*, 317−344.

Sutherland, R. (2005). ICT and learning mathematics: Developing a community of practice. *Mathematics in School, (34)*4, 4–5.

Sutherland, R., & Balacheff, N. (1999). Didactical complexity of computational environments for learning of mathematics. *International Journal of Computers for Mathematical Learning, 4*, 1–26.

Sutherland, R., & Balacheff, N. (1999). Didactical complexity of computational environments for the learning of mathematics. *The International Journal of Computers for Mathematical Learning, 4*, 1–26.

Sutton, B. (1993). The rationale for qualitative Research: A review of principles and theoretical foundations. *The Library Quarterly: Information, Community, Policy, 63*(4), 411–430.

Swafford, J. O., & Langrall, C. W. (2000*).* Grade 6 Students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education, 31*(1), 89–112.

Tall, D. (2001). Reflections on early algebra. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 1 pp. 149-152.) Freudenthal Institute, The Netherlands: PME Program Committee.

Tall, D. O. (1991). Reflections. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 251-260). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Tall, D. O. (1995). The Psychology of Symbols and Symbol Manipulators, Proceedings of the 7th Annual International Conference on Technology in College Mathematics Teaching (pp. 453-457), Addison-Wesley.

Tall, D. O., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M. O. J., Yusof, Y. (2001). Symbols and the Bifurcation between Procedural and conceptual thinking, *Canadian Journal of Science, Mathematics and Technology Education, 1*, 80–104.

Tall, D. O., Thomas, M. O. J., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of the encapsulation of a process?, *Journal of Mathematical Behavior, 18* (2), 1–19.

Thorndike, E. L. (1913). *Educational psychology: The psychology of learning* (Vol. 2). New York: Teachers College Press.

References

Tiberghien, A., Jossem, E. L., & Barojas, J. (1998). Making the results of research in Physics education available to teacher educators. In A. Tiberghien, E. L. Jossem, & J. Barojas (Eds.), *Connecting Research in Physics Education with Teacher Education.* International Commission on Physics Education. Retrieved October 29, 2015, from http://pluslucis.univie.ac.at/Archiv/ICPE/A1.html.

Trigueros, M., & Ursini, S. (1999). Does the understanding of variable evolve through schooling? *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 273-280), Haifa, Israel: PME Program Committee.

Trigueros, M., & Ursini, S. (2003). First-year undergraduates' difficulties in working with different uses of variable. In A. Selden, E. Dubinsky, G. Harel, & F. Hitt (Eds.), *Research in Collegiate Mathematics Education V* (Vol. 12, pp. 1-29). Providence, RI: American Mathematical Society.

Ulrich, C., Tillema, E. S., Hackenberg, A. J., & Norton, A. (2014). Constructivist model building: Empirical examples from mathematics education. Constructivist Foundations, 9(3), 314–316.

Ursini, S., & Trigueros, M. (2001). A model for the uses of variable in elementary algebra. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 4, pp. 335-342). Utrecht, The Netherlands: PME committee.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variable. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (1988 Yearbook of the National Council of Teachers of Mathematics, pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (1995). Why is algebra important to learn? *American Educator, 19,* 30–37.

Usiskin, Z. (1996). "Mathematics as a Language." In P. C. Elliott & M. J. Kenney (Eds.), *Communication in Mathematics, K-12 and Beyond* (pp. 231-243). Reston, VA: National Council of Teachers of Mathematics.

Van Amerom, B.A. (2003). Focusing on informal strategies when linking arithmetic to early algebra. *Educational Studies in Mathematics, 54*(1), 63−75.

Van der Veer, G. C. (1990). *Human-Computer Interaction: Learning, Individual Differences, and Design Recommendations.* PhD Thesis. Amsterdam, The Netherlands: Vrije Universiteit.

van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education.* Orlando, FL: Academic Press.

van Manen, M. (1990). *Researching Lived Experience: Human Science for an Action Sensitive Pedagogy*. Albany: State University of New York Press.

Van Soest, D., & Kruzich, J. (1994). The influence of learning styles on student and field instructor perceptions of field placement success. *Journal of Teaching in Social Work*, *9*, 49–69.

Vergnaud, G. (1987). Conclusion. In C. Janvier (Ed.), *Problems of Representation in Mathematics Learning and Problem Solving,* Hillsdale, NJ: Erlbaum.

Vergnaud, G., Benhadj, J., & Dussouet, A. (1979). *La coordination de l'enseignement des mathematiques entre le cours moyen 2e annee et la classe de sixieme.* France: Institut National de Recherche Pedagogique.

Vincent, J. (2001). MicroWorlds^TM and early algebra. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.). *The future of the teaching and learning of algebra, The proceedings of the 12th ICMI Study Conference.* (pp. 621-627). Melbourne: Department of Science and Mathematics Education.

Vincent, J., Bardini, C., Pierce, R., & Pearn, C. (2015). Misuse of the equals sign: An entrenched practice from early primary years to tertiary mathematics. *Australian Senior Mathematics Journal, 29*(2), 31–39.

von Foerster, H. (1984). On constructing a reality. In P. Watzlawick (Ed.), *The invented reality: How do we know what we believe we know?* (pp. 41-61). London: W.W. Norton & Company.

Vonder Embse, C., & Yoder, V. W. (1999). Multiple representations and connections using technology. In B. Moses (Ed.), *Algebraic thinking: Grades K-12* (pp. 225-231). Reston, VA: NCTM.

Vrasidas, C. (2000). Constructivism versus objectivism: Implications for interaction, course design, and evaluation in distance education. *International Journal of Educational Telecommunications, 6*(4), 339–362.

Vygotsky, L. S. (1978). *Mind and Society: The Development of Higher Psychological Processes.* Cambridge, MA: Harvard University Press.

Wagner, S. (1977). *Conservation of equation and function and its relationship to formal operational thought.* Paper presented at the annual meeting of the American Educational Research Association, New York, 1977.

Wagner, S. (1981). An analytical framework for mathematical variables. In C. Corniti & G. Vergnaud, *Proceedings of the 5th Conference of the Psychology of Mathematics Education* (pp. 165-170). Grenoble, France: Laboratoire I.M.A.G.

References

Wagner, S. (1983). What are these things called variables?, *Mathematics Teacher 76*, 474–479.

Wagner, S., Rachlin, S. L., & Iemsen, R. J. (1984). *Algebra learning project, final report.* Department of Mathematics Education, University of Georgia.

Warfield, V. M. (2006). *Invitation to didactique.* Seattle, Washington: University of Washington. Retrieved October 29, 2015 from https://www.math.washington.edu/~warfield/Inv%20to%20Did66%207-22-06.pdf.

Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal, 15*(2)*,* 122–137.

Watson, D. (1993). ImpacT: An Evaluation of the Impact of the Information Technology on Children's Achievements in Primary and Secondary School. London: King's College London.

Wellington, J. (2000). Educational Research: Contemporary Issues and Practical Approaches. London: Continuum.

Wheeler, D. (1996). Backwards and forwards: Reflections on different approaches to algebra. In N. Bednarz, C. Kieran, L. Lee (Eds.), *Approaches to Algebra: Perspectives for Teaching and Learning* (pp. 317-326)*.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Wilson, K., Ainley, J., & Bills, L. (2003). Comparing competence in transformational and generational algebraic activities. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA* (Vol. 4, pp. 427-434). Honolulu: Center for Research and Development Group, University of Hawaii.

Windschitl, M. (2002). Framing constructivism in practice as the negotiation of dilemmas: An analysis of the conceptual, pedagogical, cultural, and political challenges facing teachers. *Review of Educational Research, 72*(2), 131–175.

Winter, G. (2000). A comparative discussion of the notion of 'validity' in qualitative and quantitative research. *The Qualitative Report,4*(3/4). Retrieved October 12, 2015, from www.nova.edu/ssss/QR/QR4-3/winter.html.

Wood, A. T., & Anderson, C. H. (2001). The case study method: Critical thinking enhanced by effective teacher questioning skills. Paper presented at the *Annual International Conference of the World Association for Case Method Research & Application,* Lund, Sweden.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal, 28*, 587–616.
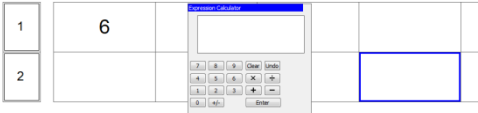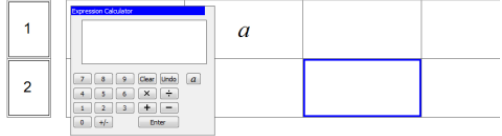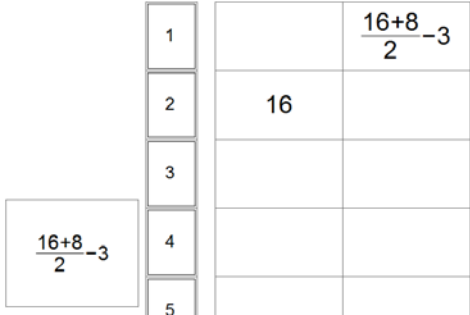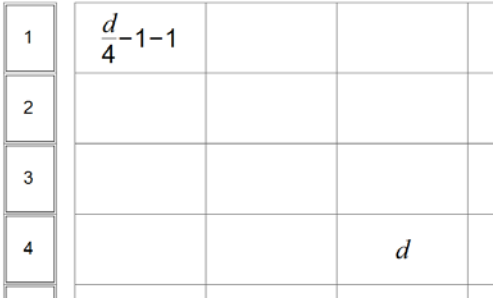
References

Yager, R. E. (1991). The Constructivist Learning Model: Toward Real Reform in Science Education. *The Science Teacher, 58*(6), 52–57.

Yin, R. K. (2012). *Applications of Case Study Research* (3rd ed.)*.* London: Sage Publications.

Yin, R. K. (2013). *Case Study Research: Design and Methods* (5th ed.). London: Sage Publications.

Yusuf, M. M. (1994). *Mathematics instruction with Logo tutorials and activities.* Paper presented at the meeting of the American Educational Research Association, New Orleans, LA.

Zahorik, J. A. (1995). *Constructivist Teaching (Vol. 390).* Bloomington: Phi Delta Kappa Educational Foundations.

Appendices

Appendix 1   Description of GA Activities and Tasks Used

GA Computer-Generated Tasks	
Tasks involving numbers only	*Tasks involving letters*
5 *Equivalent expressions (Numbers)* Given a selected cell and a starting number. User needs to make the expression which would be inside the cell by adding operations to the starting number by means of an expression calculator.	4 *Equivalent expressions (Letters)* Given a selected cell and a starting letter. User needs to make the expression which would be inside the cell by adding operations to the starting letter by means of an expression calculator.
8 *Find the journey (Numbers)* Given a numerical expression inside a cell and a starting number. User needs to click on the steps of the journey from the starting number to the given numerical expression by doing the correct operations in the correct order. A journey will form.	7 *Find the journey (Letter)* Given a formal algebraic expression inside a cell and a starting letter. User needs to click on the steps of the journey from the starting letter to the given expression by doing the correct operations in the correct order. A journey will form.
10 Same as Task 8 but only in small grids.	9 Same as Task 7 but only in small grids.

Tasks involving numbers only	Tasks involving letters
15 *Make the expression (Numbers)* Given a number inside the grid and a numerical expression outside. Users must drag the number to the right cells in the right order to obtain the expression. The activity is timed.	**13** *Make the expression (Letters)* Given a letter inside the grid and a formal algebraic expression outside. Users must drag the letter to the right cells in the right order to obtain the expression. The activity is timed.
16 Same as Task 15 but only in small grids.	**14** Same as Task 13 but only in small grids.
20 *Place the numbers (two players)* A number is shown on the grid. Two players take turns to choose a number from the number box and drag it to the correct cell. Each player has 5 turns. If a player chooses a number that appears twice that player gets double points, trice means triple points etc.	**22** *Substitution* Given a cell with an algebraic expression and another cell with the letter involved in the expression. Given also the value of that letter. Users must work out the value of the cell with the expression by substituting the value for the letter (cont./…)

	Tasks involving letters
22	(…/cont.) Users can flip the cell with the letter to see the number (value) and drag it mentally to the final cell in order to obtain the correct value at the end.
	Later in the task the grid disappears and users need to do the substitution and evaluation without the help of the grid.
24	Same as Task 22 but only in small grids.
25	What is the expression?
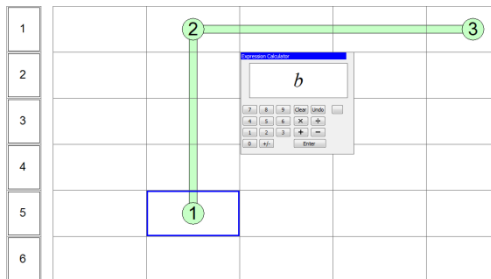
Users are given a journey and an expression calculator with a letter to stand in position 1. With the expression calculator users are expected to work out the expression in the final destination of the journey by making the correct operations in the correct order.

GA Activities with the *RUN* function

Activities involving numbers only	*Activities involving numbers and letters*

RUN: *Guess the number in the shaded cell. Number in Row 1, Column 1 is 1.*

RUN: *Letter as unknown: What is the value of the letter?*

A number from the number menu is picked up and inserted in a cell. If it is the correct number, it will stay there. If it is not, a bin will appear and a further click will dispose of the number and the user may try again. Students take turns shading a cell of their choice and their partner selects a number for it. Then they swap roles (each student shades with his own particular colour).

A number from the number menu is picked up and inserted in a cell. A student, chooses a letter from the letter menu and places it in a cell. Then he asks his partner to guess what number it represents (it is an unknown). After guessing the number, the other students drags that number from the number menu and checks if it stays there. If a bin appears he tries again until he gets it right. Then the roles are swapped and the last student places another letter in the same grid and so on.

RUN: *Guess the number in the shaded cell. Number in R_1C_1 is **not** 1.*

Same as above but the number in the top left corner is larger than 1.

Activities involving numbers only	Activities involving numbers and letters

RUN: *Move the cell and reflect.*

RUN: *Letter as variable: What may be the value of the letter?*

A picks a number from the number menu and inserts it in a cell. Then he asks his partner what would happen if that cell (i.e. that number) was mapped to another cell of his choice. After the other student guesses, the first one clicks and drags that cell to the designated cell and the other student verifies or negates his answer. Then the roles are swapped.
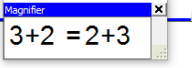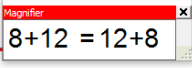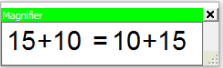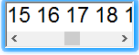
A student, chooses a letter from the letter menu and places it in a cell. Then he asks his partner to guess what number it *may* represent (it is a variable since the grid has no numbers yet). After guessing the number, the other student drags that number from the number menu and checks if it stays there. If a bin appears he tries again until he gets it right. The cell automatically takes the shape of a page with a corner rolled upwards, inviting the student to swap between cell values: e.g. x and 25. Then, a new grid is created and the roles are swapped.

Activity involving numerical expressions

RUN: *Use of Magnifier to show associativity of addition and multiplication*

1	2	3		3+2
				Magnifier ☒
2				3+2 = 2+3
3				
				Magnifier ☒
				8+12 = 12+8
4	8	12		8+12
5	10	15		15+10
				Magnifier ☒
6				15+10 = 10+15

15 16 17 18 1

< ▪ >

Numbers are placed in strategic places as shown in the above grid. Each of these numbers is moved to a cell on the right as shown. The cell containing the addition holds two equivalent numerical expressions. These are viewed by the magnifier. students appreciate that a numerical expression showing the addition of two numbers can have the numbers swapped and still retains its value, and hence the commutativity of addition. Moreover, the equals sign is seen here to represent an identity or equality rather than the precursor of the answer of a process.

This activity is repeated for other operations.

Activity involving algebraic expressions

RUN: *Order of operations: Process starting with a letter*

1	$\frac{m}{4}-2$		$\frac{m}{4}$		
2					
3		k			
4			m		
5					
6		$2k$		$2k+12$	

$$m \ n \ o \ p \ q \ r$$
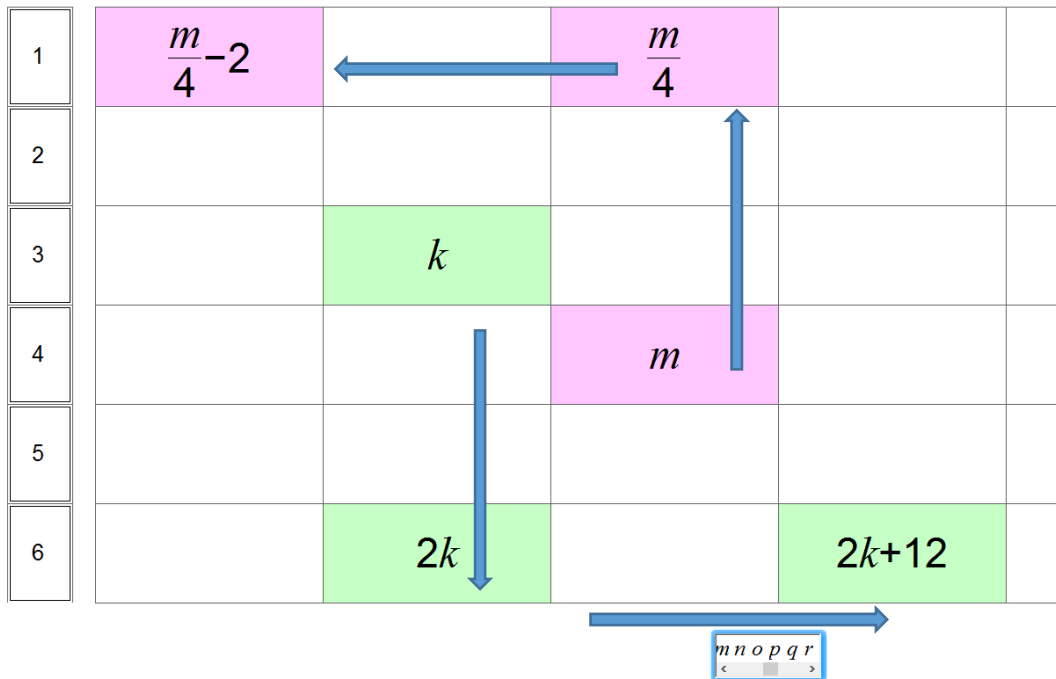
A letter is placed in a cell. It is moved vertically and then horizontally so that students appreciate that multiplication or division are performed before addition and subtraction in expressions such as $2k+12$ and $\frac{m}{4}-2$. Then together with the previous activity showing commutativity of addition students are led to appreciate that an expression such as $12+2k$ is still the same as $2k+12$ and hence, in $12+2k$ it is still the multiplication that is performed before even though it is written after addition.

Appendix 2   Example of a GA Lesson Plan

3 Form 1.1

RUN (2 - 6 rows). *Guess the expression: Numbers*

2 75 min

Activities on the Grid

Grid Algebra RUN 2X5, 3X5, 4X5, 5X5, and 6X5 NUMBERS only.
- Guess a correct number in a cell and guess the expression in the resulting cell when it is moved to a specified location (one move – same row or col.)

Learning Opportunities

1. Continue familiarisation with the GA grid.
2. Learn to associate movement of the GA cells with operations.
3. Regard expressions such as 3+6 as a mathematical object in their own right rather than just a process to be worked out.

Presumed Knowledge

1. Concept of addition, subtraction, multiplication and division.
2. Know how to add, subtract, multiply and divide simple numbers without remainder e.g. 56÷7, 16+20, 35-15 (pen and paper may be used when required).

Homework

- Worksheet 3: Fill in the expression.

Note: Pupils are given the chance to ask about any queries and, possibly, start off the exercise in class.

PUPIL

Teacher's Intervention

- Demonstrate a RUN with GA numbers and discuss movements of cells (one move only).
- Engage pupils in discussions during C and G activities.
- Assist and guide G activities.

Activity

1. After showing how GA makes movement expressions ask pupils to guess the expression in a predetermined cell (C) (35 min).
2. Pupils repeat the activity in pairs (G) (35 min).
3. The homework is explained (5 min).

Possible Difficulties

1. There might be difficulties when numbers move to another cell (x or ÷)
2. If the first number that is guessed is large, subsequent numbers may be larger and use of a calculator may be allowed.

Follow-up

- Task 20: Place the number: 2 players
- Finding the expression of a two-direction journey.

Notes:
- C≡ Whole Class Activity; G≡ Group Activity (pair)

- Self-critique:_____

Appendix 3.1   Interview 1 Problems (as presented to the students)

Name:_____

1.1 $4 + 3$

1.2 $8 - 3$

1.3 5×4

1.4 $6 \div 2$

2.1 $5 + 4 - 4$

3.1 $\qquad$ $2 + 3 \times 10$

3.2 $\qquad$ $2 \times (3 + 1)$

4.1 $\qquad$ $10(5 + 2)$

5.1 $\qquad$ $=$

6.1 $\qquad$ $\boxed{} = 3 + 5$

6.2 $\qquad$ $7 + 3 = \boxed{} + 7$

7.1 $497 + 2014 - 2014$

7.2 $121 \times 350 \div 350$

************************ **T H A N K   Y O U** ************************

Appendix 3.2   Interview 2 Problems (as presented to the students)

Name:_____

3.1 $\qquad$ $3 + 2 \times 5$

3.2 $\qquad$ $10 \times (2 + 6)$

4.1 $\qquad$ $2(3 + 7)$

4.2 $\qquad$ $\dfrac{8}{2}$

4.3 $\dfrac{4+6}{5}$

5.1 $=$

6.1 $\boxed{} = 5 + 4$

6.2 $10 + 7 = \boxed{} + 10$

6.3 $998877 + 1234 = \boxed{} + 998877$

6.4 $15 = \boxed{}$

7.1 $\qquad 5767 + 3993 - 3993$

7.2 $\qquad 567 \times 123 \div 123$

The expression

$4 + 8$

is multiplied by two.

8.1 What is the new expression?

Appendix 3.3   Interview 3 Problems (as presented to the students)

Name:_____

4.1 $4(5 + 1)$

4.2 $\dfrac{15}{3}$

4.3 $\dfrac{3 + 6}{3}$

5.1 $=$

6.1 $\boxed{} = 3 + 7$

6.2 $8 + 2 = \boxed{} + 8$

6.3 $229977 + 4321 = \boxed{} + 229977$

6.4 $26 = \boxed{}$

7.1 $6162 + 4994 - 4994$

7.2 $451 \times 999 \div 999$

8.2

1	$\dfrac{45-10}{5}-1$				
2					
3					
4					
5				45	
6					

8.3

1			④	①	
2					
3					
4					
5					
6			③	②	
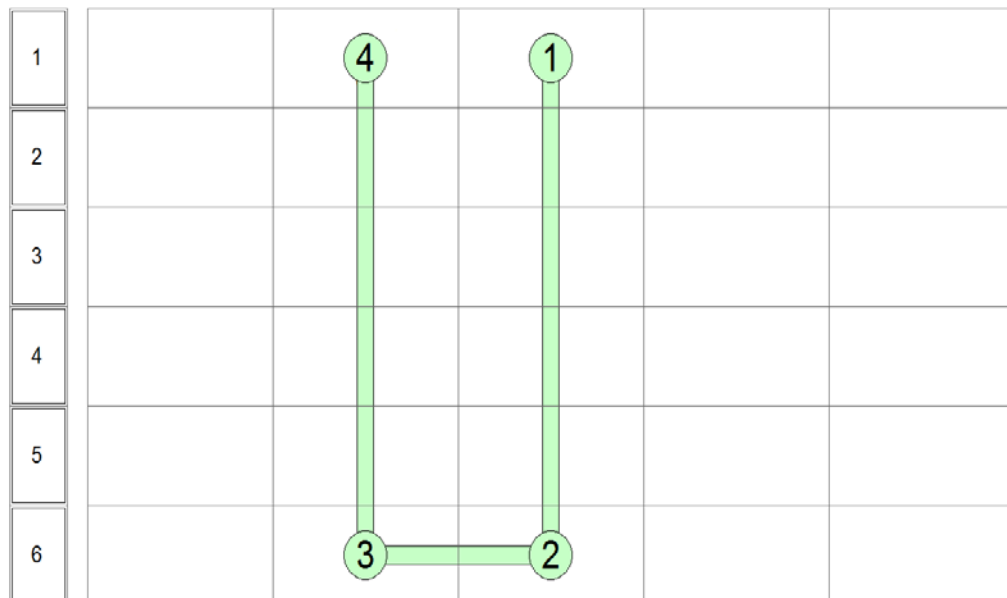
Suppose instead of the number 5 you had "y".

9.1   What would the cell marked (4) contain after the journey

$(1) \longrightarrow (2) \longrightarrow (3) \longrightarrow (4)$?

*********************** **T H A N K   Y O U** ***********************

Appendix 3.4   Interview 4 Problems (as presented to the students)

Name:_____

4.1 $4(5) + 10$

4.2 $10 + 4(5)$

4.3 $\dfrac{12}{3} + 6$

4.4 $6 + \dfrac{12}{3}$

4.5 $2(3) - 5$

4.6 $5 - 2(3)$

4.7 $\dfrac{10}{2} - 4$

4.8 $4 - \dfrac{10}{2}$

6.1 $6 + 3 = \boxed{} + 6$

6.2 $7 + 4 = \boxed{} + 4$

6.3 $399993 + 8228 = \boxed{} + 399993$

6.4

$$54 = \boxed{}$$

7.1

$$5445 + 9997 - 9997$$

8.1

$$\frac{233 \times 676}{676}$$

9.1   What does the following expression mean?

$$2a + 6$$

9.2   How would you help me understand the meaning of the above expression by using the GA grid below?

1					
2					
3		a			
4					
5					
6					

9.3   What does the following expression mean?

$$\frac{3x - 6}{2}$$

9.2   How would you help me understand the meaning of the above expression by using the GA grid below?

1				
2		x		
3				
4				
5				
6				

************************ **T H A N K   Y O U** ************************

Appendix 3.5   Interview 5 Problems (as presented to the students)

Name:_____

9.1   What does the following expression mean?

$$\frac{2(h + 7)}{5}$$

10.1   If $h = 3$, give a single value for

$$\frac{2(h + 7)}{5}$$

10.2   If $e = 4$, give a single value for

$$5 + 3e$$

10.3   If $u = 6$, give a single value for

$$10 - (u + 1)$$

10.4   If $y = 6$, give a single value for

$$\frac{5\left(\frac{y}{2}\right) + 5}{4}$$

10.5   If $c = 12$, give a single value for

$$\frac{3\left(10 - \frac{c}{3} + 2\right)}{6}$$

*********************** **THANK YOU** ***********************

Appendix 4.1 Worksheet 1 (given after GA lesson 1)[16]

Fill in the shaded cells with the correct number.

1	1				

1	1				
2					

1					
2	2				
3					

1					
2					
3					
4		8			

1					
2					
3			12		
4					
5					

1					
2					
3					
4					
5					
6					30

---

[16] In Appendices 4.1-4.5 the sizes of the diagrams and spaces in the tables were reduced from the original handouts to fit within this page border.

Appendix 4.2 Worksheet 5 (given after GA lesson 5)

You will be given a number in a box. This number is transformed by the operation shown on the arrow. The resulting expression is shown in the final box. Attention: ALL ARROWS POINT TO THE RIGHT even the ones that do not show +. Sometimes the expression is again transformed by another arrow. The first two problems are worked out for you.

$3 \xrightarrow{+2} 3+2$

$6 \xrightarrow{-4} 6-4 \xrightarrow{\times 2} 2(6-4)$

$8 \xrightarrow{\times 2} \boxed{}$

$4 \xrightarrow{\div 2} \boxed{}$

$6 \xrightarrow{+3} 6+3 \xrightarrow{+9} \boxed{}$

$7 \xrightarrow{+1} \boxed{} \xrightarrow{+2} \boxed{}$

$6 \xrightarrow{-4} \boxed{} \xrightarrow{-2} \boxed{}$

$9 \xrightarrow{+3} \boxed{} \xrightarrow{\times 2} \boxed{}$

$8 \xrightarrow{\times 2} \boxed{} \xrightarrow{+4} \boxed{}$

$9 \xrightarrow{-3} \boxed{} \xrightarrow{\div 2} \boxed{}$

Appendix 4.3   Worksheet 8 (given after GA lesson 8)

You will be given a number in a cell. In another cell you have a numerical expression formed by operations on that number. Your task is to show how the number in the cell is moved so that this expression is formed. Write ① on the first number, ② on the second position, ③ on the third position and so on. The first one is worked out for you.

First example (worked out):
Row 1: 2 ① ... $2+4-1$ ③ — ②

Second set, row 1: $4+1-1-2$ ... 4

Third example, row 1: $\dfrac{12}{2}+3$; row 2: 12

Fourth example, row 1: $\dfrac{16-4}{2}-1$; row 2: 16

Fifth example, row 2: $2\left(\dfrac{15}{3}+1\right)$; row 3: 15

Sixth example, row 3: 27 ; row 4: $4\left(\dfrac{27-3}{3}+1\right)$

Seventh example, row 3: 18 , $3\left(\dfrac{18-6+6}{3}\right)+3$

Eighth example, row 1: $\dfrac{3\left(\dfrac{27}{3}-1\right)-3}{3}-1$; row 3: 27
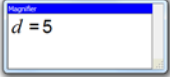
Appendix 4.4   Worksheet 19 (part) (given after GA lesson 19)

For each of the following grids make the substitution for the letter and evaluate the expression in the other cell.

1.

| 1 | | | $d-2-1+2$ | d | |

Magnifier
$d = 5$

2.

| 1 | | | x | |
| 2 | | | $2x+2$ | |

Magnifier
$x = 3$

3.

| 1 | | | | $\dfrac{z+4}{2}+1$ |
| 2 | | z | | |

Magnifier
$z = 4$

4

$3(y+2)+3$

Magnifier
$y = 3$

5.

$\dfrac{2a+4}{2}$

Magnifier
$a = 2$

6.
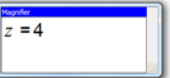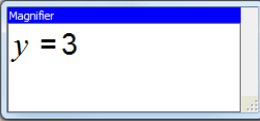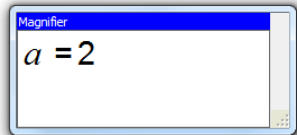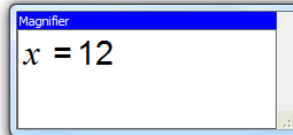
$\dfrac{x-2+4}{2}$

Magnifier
$x = 12$

Appendix 4.5   Worksheet 20 (given after GA lesson 20)

For each of the following expressions make the substitution shown in the brackets and evaluate the expression.

$5(a - 2)$	$[a = 3]$	$3(5 + b)$	$[b = 2]$	$4 + \dfrac{c}{3}$	$[c = 6]$
$17 - 3d$	$[d = 5]$	$4 + \dfrac{15}{e}$	$[e = 3]$	$2(1 + 5f)$	$[f = 2]$
$\dfrac{12 - 2g}{2}$	$[g = 5]$	$2\left(10 + \dfrac{h}{5}\right)$	$[h = 15]$	$\dfrac{2(2i - 6)}{4}$	$[i = 7]$
$3 + \dfrac{5(j-2)}{4}$	$[j = 10]$	$\dfrac{\frac{4+k}{2} + 6}{3}$	$[k = 2]$	$\dfrac{10\left(3 + \frac{m}{3}\right)}{5}$	$[m = 6]$
$5\left(\dfrac{15 - 4n}{3}\right)$	$[n = 3]$	$10 - \dfrac{1 + \frac{12}{p}}{2}$	$[p = 4]$	$\dfrac{10 - 4(8 - 3r)}{2}$	$[r = 2]$

Appendix 5 Example of a Coding Sheet Including Lesson Event Notes: GA Lesson 9

Codes and Frequencies

Maths-> Negotiation		Negotiation-> Learner		Learner-> Negotiation		Negotiation-> Mathematics		Forward Negotiation Block		Backward Negotiation Block	
M-N_Anticipate>Interact	3	N-L_Interact>Experience	10	L-N_Reflect>Model	12	N-M_Review>Assoc	4	xM-xL_FixedObject>Ignore	0	xL-xM_NoElab>NoAssoc	1
M-N_Assoc>Interact	6	N-L_Interact>Reflect	17	L-N_Reflect>Review	4	N-M_Review>Adapt	2			xL-xM_Stop>NoAdapt	0
				L-N_Model>Review	0	N-M_Model>Assoc	3			xL-xM_NoAsk>PoorModel	2
						N-M_Model>Adapt	1			xL-xM_Software>NoFlex	0
										xL-xM_Software>NoAssoc	0

GA Lesson 9

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
1(i)	M-N_Anticipate>Interact	04:57				I introduced GA Task 8: Find the Journey (Numbers). I started out with a journey with two operations. I reminded students what to do. I started out with the first expression. I asked the students to reflect on the order of operations as they see it in the expression before attempting the journey. I asked the students what they are going to do to the first expression: $\frac{32-4}{2}$.
1(i)	N-L_Interact>Experience	05:40				Dwayne said he'll do minus four, division and then times. I said division by two. Then asked where is the times? Dwayne said he made a mistake. So we continued. Joseph came out and started out the journey. He got it correct.
1(i)	L-N_Reflect>Review	06:38	Y			I was satisfied that Joseph remembered the task well, so I asked out Dan. In the meantime I remembered that someone made a comment when Joseph was

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[17] Key: **RQ**: Research Question; **Exr**, **Pct**, **Scr**: Transcript exerpt, Video still frame picture, Computer screenshot (respectively) could be included in write-up.

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
						working[18]. It turned out to be Dwayne. He asked whether division was an upward movement or a downward movement.
1(i)	N-M_Review>Adapt N-L_Interact>Reflect	06:44	Y			I adapted my present aim (to help the children go through the proper order of operations) to help Dwayne remember something that we had been doing for some time now. I asked the others. Joseph said, upwards[19]. I suggested to Dwayne to look at the row numbers (factors) what they do, the others do. I discussed this a bit with Dwayne who seemed a bit confused. I asked the others again, what 3 does to become 6. Some students said x 2. I confirmed and amplified.
1(i)	N-L_Interact>Experience	07:16				Dan started out his challenge. $\frac{42}{6} + 2$. He got it correct. I said well done.
1(i)	N-L_Interact>Reflect	07:30				I asked what does Dan's upward movement mean. Joseph said division by six. I said well done (but wait for permission to speak first).
1(i)	L-N_Reflect>Review	07:34				Satisfied with both Dan's performance and Joseph's statement I applauded Dan and asked another student out.
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	08:02				It was Jordan's turn. Jordan spent some seconds looking at the IWB. Omar told him to start with 48. I asked Jordan to tell me where he was going to start from and directed his attention to the target expression[20]. He said from 2. I asked, from 2?! Then I guessed he was saying from division by two and I said so (remember Jordan finds it difficult to speak out). I showed him that the first number to start from was 48 and he had to press that cell. I had to explain this further to him. Joseph said 6 division by 2 (he was right because 48 was in R6C2 and what 6 does it does – so Jordan had to press on R3C2 since 6/3=2). Jordan said 3. I reminded him of the "mother-and-children" metaphor. Jordan got the journey correct. Interestingly, in the last step, he did not go immediately to the designated cell but he counted in 3's until he got there.

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[18] I'm trying to keep up with individual needs of students. My review does not always rest on what I perceive about one student. If I feel that I can get a more holistic picture of the class I act on it. In this case I ask about an individual comment of another student.

[19] What is the difference **to Dwayne** between Joseph saying "upwards" and I confirm, and me simply saying "upwards".

[20] Orientation of students' thinking processes (von Glasersfeld, 1995)

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
1(i)	L-N_Reflect>Model N-M_Model>Assoc M-N_Assoc>Interact	09:48				I nodded and seemed to build a model of Jordan's reasoning that he was counting in 3's to get +9. I associated this with the relationship between rightwards movements of journeys and the addition operation. I considered this worth discussing.
1(i)	N-L_Interact>Reflect	09:50				I asked the other students what does it mean that Jordan did +3, +6, +9? Omar said that the +9 was divided into threes. Joseph added that it was because we were in R3, and that if we were in R4 we would do plus 4 or minus 4.
1(i)	L-N_Reflect>Model N-M_Model>Assoc M-N_Assoc>Interact	10:35				I got the idea that the students were thinking along my lines. I associated Joseph's comment with the diverse movements one has to do when doing plus and minus and then times and division. So I decided to remind the students of this.
1(i)	N-L_Interact>Reflect	10:37	Y			I asked the students to give me the different operations for different operations. I made the gestures for them and they told me the correct operations.
1(i)	L-N_Reflect>Review	11:07				The students' correct responses told me that we were on the same lines. I called out Omar.
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	11:33	Y			Omar's expression was $\frac{36+6}{3}$. He started off by clicking (1) on 36. I asked him why. He said it was because the first number was 36 and he wanted to convert it to the target expression. He said, 36 plus 6 division 3. I said, you said it correctly but why isn't it 36 division by 3 and then plus 6? He said because you have plus six here (points) and the division is underneath them. I took the discussion to the whiteboard. I asked him to write down how it would appear if it were 36 division by 3 and then plus 6. He wrote $\frac{36/3}{+6}$. Joseph said it's wrong. I asked tell us what's wrong about it. In the meantime Omar said I did an extra line and he rubbed off the division line. But the +6 remained underneath the 36/3. So I asked him whether if it was 36 divided by 3 first and then +6 the +6 needed to be underneath. I asked him why didn't he put it on the right. He said because I said so. I repeated what I said. He

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
						said, I think it's wrong.[21] Then I said it was right but usually GA writes +6 on the right. I said there's something else that's a bit so and so. Omar pointed at the slash of 36/3. Then with some guidance he arranged it to $\frac{36}{3} + 6$. I continued to ask, what was it then that made you notice that it was 36+6 and then divided by 3. He answered satisfactorily but he did not explain himself well. Joseph wanted to explain it and I let him. He said that the large line is like you have a brackets. I wrote this on the board. Then I said that it one could find old textbooks writing it like this: (36+6)/3. Omar suggested, 3(36+6). I said what would that mean. Joseph and Jordan said times. (Omar may have been thinking about the division as it is written in the primary but I did not elaborate).
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	17:18				I asked Omar to continue with the challenge. He went from R6 to R2 (division by 3). Joseph said it is wrong. I said give him a chance. I asked him what he did. He said times. I asked what does an upward movement mean. Someone said division. I asked them not to say the answer. I continued to discuss this with Omar. There was an issue about what is done from R6 to R2. He said division by 4. He ended up doing it correctly but he needed some guidance. I discussed each step with him.
1(i)	L-N_Reflect>Model	21:46				Omar seemed a bit shaky on the issue of what was to happen when we divide six by three. At this stage, where students were taking turns on the board I may have thought that he would gain further experience by watching others and then watching his own work on the computer.
1(i)	L-N_Reflect>Review N-M_Review>Adapt	21:48				Tony was next. Before he started he asked whether it was true that computers use slash and not the division symbol. I said only in some software because GA didn't. Dan mentioned that sometimes computers use a star. I said that only in some

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[21] Students, sometimes equate probing questions to them getting wrong answers: If the teacher is asking too many questions about my answer then something must be wrong with it. Actually Omar's notation was both correct and standard. But since GA writes it horizontally I felt I had to point this out. Once again I find myself having to conform with what I knew the software would show. But why? Because students might get confused? Because I might have to engage in such discussions which, sort of, disrupt the "smooth flow" of the lesson.

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
						software. Unfortunately I had to reprimand some students (Omar and Joseph) for misbehaving.
1(i)	N-L_Interact>Reflect	22:34				Tony did his challenge correctly. I asked him about what he did and why. He answered satisfactorily. He even referred to the row number (factor) to show me that they guided him as to what to do.
1(i)	L-N_Reflect>Model N-M_Review>Assoc	23:55				I was satisfied with Tony's responses because they matched my own way of thinking. I asked the last student to come out, Dwayne. He got it correct quickly.
1(i)	L-N_Reflect>Model N-M_Review>Assoc	24:05	Y			I asked Dwayne to explain his moves. He made an important statement about why he got to R6 from R3. I decided to amplify.
1(i)	M-N_Assoc>Interact	24:31	Y			I restated what Dwayne said with more emphasis.
1(i)	N-L_Interact>Reflect	24:57	Y			I asked him how he got his last step. His reply was again very satisfactory.
1(i)	L-N_Reflect>Model N-M_Review>Assoc M-N_Assoc>Interact	25:21				I was so satisfied with Dwayne's responses, which again matched the way I would explain, that I decided to raise the bar of the task. I went to level 5 and warned them that it would be difficult. I said I don't want the answer but the reasons why you would do certain moves.
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	25:57	Y			Joseph wanted to attempt the challenge. It was $2\left(2\left(\frac{4}{4}+3\right)-4\right)+12$. I gave him permission to come out and do it. He clicked (1) on the first number, 4. Omar exclaimed, "Mamma Mia" – "an Italian version of Oh My". I was asking Joseph what was next. He said division by 4. I emphasized that we should take it bit by bit. I asked Omar why he said it. He said it was because there was a lot to do. I said if you were to eat a plate of spaghetti you would do eat it bit by bit. So that's what you should do here. I asked Joseph to continue. Omar wanted to know what the double 2's were doing there. I said you have to wait. He said you skipped them. I said I'm still here and pointed at the $\frac{4}{4}$. I helped Joseph to continue by telling him what we have done and asking him what to do next. Some students were saying division when the operation was times. Evidently some students had forgot that 2(...) meant

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
						(...) times 2. I said there is no other sign for division. Joseph said it correctly however – times two. He completed the task correctly and I praised him.
1(ii)	xL-xM_NoAsk>PoorModel	27:16				I did not ask Omar why he was saying that I skipped the first double 2s. If I did I may have got a better model of the way he was thinking. The previous tasks were sequential – operations in the order of appearance. This wasn't. It had to start from the core of the expression. I may have addressed Omar's, and possibly someone else's problem, better.
1(i)	L-N_Reflect>Model	28:58				Although with some difficulty, I was confident that the students could attempt such difficult questions. So when some other students told me to try one I said you will all try one and asked Dan to come out next (I spent some time doing his tie ☺ because he would have continued to play with it).
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	29:50				Dan's target task was $\frac{6\left(\frac{25-5}{5}+2\right)-6}{3}$. As before, GA gave him the place of the first number, 25 on which he clicked (1). I asked him which was first. He said division. He was referring to the large division which was actually the last operation to be done.[22] I asked the others. What do you think, division first. Dwayne said no it's times. Dan pointed to the fact that all the numerator needed to be divided. I said first you need to work that numerator and then go about making that division. I emphasized that we need to work from the inside. I was making circling motions around the first part of the expression that needed to be worked out (CAPS). I said you need to start with 25 and move out of it. Dan suggested division by 5. I asked who thought differently. Dwayne said it should be minus 5. I asked why. He said because we were in the 5-times table. He said because the division has a long line. Dan started the journey by moving to the left (-5). Dan continued but each time I had to show him what we did in the expression and ask him what was next (times 6 then minus 6 then divided by 3)… He managed to get it correct.

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[22] Interestingly, such an expression would be entered as Dan was saying if one were to compute it by a modern calculator, or write it in a Word document. Could calculators actually be a source of confusion?

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
1(i)	L-N_Reflect>Model	32:04				I said we are making progress. It seemed I was satisfied that the students were getting there.
1(i)	L-N_Reflect>Model	32:15				Tony asked why we started the previous question in a cell which wasn't in the last column. He had to be reminded that the last the columns do not always start with the one-times table.
1(i)	N-M_Model>Adapt M-N_Assoc>Interact	32:47	Y			Jordan was next. His expression was $\frac{2\left(\frac{30-6}{3}-2\right)+8}{2}$. He was going to start off immediately to move the starting cell to the first part of the journey (from R6C3 to R6C2 meaning -6). I asked the students, If R6C3 was 30 then what were R6C2 and R6C1? The students agreed these were 24 and 18. I said Tony's question was why isn't R6C1 one times 6 i.e. 6. He asked the same thing. I said this is like a snapshot of the tables but the picture was taken in the middle of the tables. I wrote the table on the board and told them that the grid is like a snapshot from the column that was the 3-times table.
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	34:27				I asked Jordan to carry on. He started explaining his moves – first minus six (he made the move) then all over 3. I asked what does that "all over" mean. He said division. I said where. I asked him to look at the row number. I asked where would division by 3 end up? He said 2. So he moved it to the second row. He proceeded with the next three operations correctly. Dwayne said that there was a mistake. But when I asked him he realized that it was he (Dwayne) who was wrong and he said he thought the division was before the +8. Omar asked why Jordan went to the second row when he did division by 3. I asked him what would 6 result if it were divided by 3? He said 2. I said that's why. I asked Jordan to carry on and he concluded it correctly.
1(i)	L-N_Reflect>Model	36:43				It seems I was satisfied that the students were gaining confidence in this task. I asked Omar to come out and tackle one himself.
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	36:45				Omar's expression was $5\left(\frac{\frac{16}{2}+8}{2}-1+1\right)$. He started out correctly. He moved from R4 to R2 (division by 2) then counted the cells to do +8. He moved three cells

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
						not four (he was counting the initial cell too). I thought he was moving correctly . Joseph said there was a mistake. I said no there isn't. But I asked him what mistake it was. He pointed out that he was supposed to do +8 not +6. I counted the cells and told him he was right! I restarted the challenge for Omar and moved to the moment he had to do +8. I counted the cells for him +2+2+2+2. He made the correct move. I said well done to Joseph. Omar went to the target expression and made a circle with his hand on the expression he did. I asked what is next. He said, minus 1? I said no. I asked the others to help him. I said look, we did these (I circled the $\frac{16}{2} + 8$ with my hand). I asked what is next, minus 1? He said no division by 2. I agreed and he went to do it. He proceeded correctly. I emphasised a bit the back-and-forth movement of -1 followed by +1. He made the last move correctly. I asked about the last operation. They agreed that it was times five. I was going to say something about the expression but Omar had already cleared the board so I did not say what I was going to say.
1(i)	L-N_Reflect>Model	36:43				Once again it seemed to me that the students were getting more competent with this task so I asked out Tony to come out and do his challenge: $2\left(\frac{2(18-3)+12}{2}\right) - 6$
1(i)	N-L_Interact>Reflect	39:50				Dwayne said he had a difficulty knowing by how many cells to move down when he had a number like 18 in the middle of the grid like R3. I said this was a common difficulty. I asked the others to help him. I asked if a number is in R2 and we need to do it x2 where does it go. Joseph said it went to R4. I asked why. He said because 2x2=4. I agreed and reminded Dwayne and the others that we can be guided by the row number – the mother-and-children metaphor.
1(i)	N-L_Interact>Experience N-L_Interact>Reflect	41:28				Tony proceeded with the task. From the second step onwards, I asked the others to help him and join in the discussion. It was interesting that the students offered explanations as to why they were doing that operation. In the second to last operation, Dwayne said times 2. I said correct. He said, it's because there's the brackets. Tony was going to do the multiplication. He concluded the challenge successfully/

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
1(i)	N-M_Review>Assoc	43:13				I reviewed the situation and saw that as regards the GA task itself the students were doing well but I also saw that in the last two operations of Tony's challenge we were doing an inverse operation (times 2, division by 2). I decided to take the opportunity to discuss, once again inverse operations and how these were translated to reverse movements on GA (CAPS). I associated this with what I was doing in the previous lesson (see Lesson 8, 26:02 -> 231786 + 9993 -9993)[23].
1(i)	M-N_Anticipate>Interact	43:21	Y	Y		I drew a grid and inserted a considerably complicated/strange number 1272 in what I told students to be R6. I said I was going to move it to R3 and then I am going to move it again to R6. What is the answer going to be? I rephrased, asking what I was doing. Joseph said division by two. I said, if I were to multiply it again by two what would the answer be. Joseph and others (Dwayne and Tony) said 1272. I said, can someone tell me how would you realise that the answer is 1272 without first having to multiply by 2 and then divide by 2? Tony said that when you divide by 2 it's like you are halving it and then you are sort of doubling it. I asked whether there was another explanation. Omar said you will notice that you're going to have the same number because you moved up and then you moved down again (CAPS)
1(i)	L-N_Reflect>Model N-M_Model>Assoc M-N_Assoc>Interact	46:08	Y			Omar's answer was exactly what I intended the students to realise. I emphasised and elaborated on this answer. I said that's the easiest explanation. If you go to a place and then went back you will get the same answer (CAPS). I asked, so what do we learn about division by 2 and times 2? Omar said they are the same. I corrected him – you stay where you were. I decide to associate vertical reverse movement to horizontal reverse movement (same row – plus and minus). I started out to interact with the children by drawing a separate row.
1(ii)	xL-xM_NoElab>NoAssoc	46:34	Y			Dwayne, commenting on what we were saying two seconds earlier, said – it's like the airport, you come and go. In retrospect it was an excellent metaphor but for some reason I did not elaborate on his response so I could not associate what he was saying with what I was proposing. The reason for not taking notice or

[23] My association of classroom occurrences with my mathematics occurs also with reference to things I had in mind in previous lessons.

RQ[17]	Code	Time	Exr	Pct	Scr	Event Notes
						elaborating on Dwayne's response may have been because he talked without asking permission.
1(i)	M-N_Anticipate>Interact	46:33				I continued drawing R″ and placed a large number inside - 6329. I drew two arrows moving right by three cells and then left again.
1(i)	N-L_Interact>Reflect	47:03				I asked the students, what happened? I said this is row 1. Jordan said first you did plus 3 and then you did minus 3. I asked, what is the answer. Jordan said 6329. I said, do we have to work out the addition first, see the answer, then we do the subtraction. Or can we just write 6329 straightaway? Jordan said we can write 6329 straightaway. I said, why? He said, it's easy because the plus and the minus are the same. Most probably he meant that we were adding and subtracting the same amount but I interpreted this incorrectly....
1(ii)	xL-xM_NoAsk>PoorModel	48:38	Y			I did not ask Jordan to explain what he meant and I formed a poor model of what he was thinking.
1(i)	N-L_Interact>Reflect	48:38	Y	Y		I thought he was referring to plus and minus being the same, so I said, a better word for that would be… and I was moving my hands to-and-fro, gesturing inverse process (CAPS). Omar said they were the same like division and times. I said they are the opposite of each other. Omar repeated what I said as a sign and seemed to agree (48:43).

From then onwards the students started working on GA Task 8 on their own as pairs on their computers.

Appendix 6.1   Consent Letter to the Head of School

Head Teacher's Consent to Participate in a Research Project by Philip Borg

Dear [Name of Head of School],

As you know, I am currently reading for a PhD degree with Loughborough University. In my research, I will be conducting a case study of a group of students in which I will seek to find ways how computer software may help beginning secondary school students to bridge arithmetic with algebra.

I am therefore asking your permission to allow the following Form 1 [set C] Mathematics students of the scholastic year 2014-15 to participate in my research:
1. Name of prospective participant 1
2. Name of prospective participant 2
3. Name of prospective participant 3
4. Name of prospective participant 4
5. Name of prospective participant 5
6. Name of prospective participant 6

I will closely follow the six students listed above, where I will be monitoring their progress in making the necessary connections and transitions between arithmetic and algebra. In order to do this I will gather data by the methods described below.

Methods and tools of gathering information
I will gather data about the mathematical progress of the six case study students by using the following methods and tools. The following will be performed throughout Form 1 (Year 7):

a) **Video Recording of the Research Lessons** (20 double lessons). There will be one video camera in the classroom to capture what is going on the interactive whiteboard and the activities of all the students and their interactions between themselves and with the teacher (myself);

b) **Computer Screen Recording Software** (the second half of the same double lessons when video recording is used). There will be software installed on the computer of the case study students which records what they are doing on the computer and also what they are saying.

c) **Written Work** (handouts to fill in). This will be work assigned at the end of each lesson and collected in the lesson that follows.

d) **Interviews** (Five, approx. 30 – 45 min. each). These interviews will be conducted with the case study students during break time. I will ask the students questions about arithmetic and algebra. In these interviews I seek to gather data about the students' knowledge and understanding which may not be possible to detect in their written work. <u>Children's performance in these interviews will not affect their assessment scores in any way.</u>

All of the above will be spread over a period of twenty weeks (which is approximately one scholastic year excluding holidays and school activities).

Benefits and Risks

If children participate in this project they will have no direct benefits but they will help me understand their mathematical progress in a better and deeper way than the normal lesson situation. In addition they will be helping me as their mathematics teacher, my mathematics teacher colleagues, and also other researchers to understand how we can help students to learn mathematics through the use of computer software (apart from traditional methods).

I believe there is very little risk to the children in participating in this project but I will be taking measures to prevent possible harm to them, no matter how small or insignificant, namely:

- If a child becomes uncomfortable or stressed by answering any of the test questions, he can skip the question, take a break, stop the test, or withdraw from the project altogether.
- I will make sure that the data gathering process will not impede the learning of any child.
- I will take all the necessary precautions to prevent any loss of data which may compromise the confidentiality of the data.

Confidentiality and Privacy

Any written work from this research will never use the real name of a student, nor the name of the school. If any still images from video are used in published written work from this research, the faces of the students will be obscured. It is possible that video or audio clips may be used as part of a presentation at a professional conference or a teacher training session but no video or audio clips will be put up on a webpage or on any social media.

During this research project, I will keep all recorded data in a secure location. Only my supervisors at Loughborough University and I will have access to the data. Upon completion of the research, data will be preserved and accessible for a period of ten years.

Voluntary Participation

Participation in this research project is voluntary and subject to the main guardians' consent and the willingness of the child to participate. I am presenting you with a copy of the consent letter and form I will be giving to the main guardians, subject to your own consent. Children and their guardians can choose freely to participate or not to participate.

In addition, guardians can withdraw their permission and/or the pupil can stop participating without any consequence the child's education. The deadline for guardian's permission withdrawal or for children to stop participating is two months. If a pupil stops participating I will only use the data gathered until that time and further

data about him will not be gathered. Furthermore, he will be seated in a way so as not to be included in the camera angle during classroom video recordings.

I recognise that I am the researcher in this project and, at the same time, these children's teacher. Thus, I will ensure that children's participation or non-participation in my research project does not impact their learning, their grades, or our pupil-teacher relationship. I will also ensure that your approval or refusal to allow me to conduct this research in our school will not affect our collegiality.

Questions

If you have any questions about this research project, please contact me, Philip Borg, personally, via phone: […] or email: […]. You can also contact my supervisors at Loughborough University:

- Dr Dave Hewitt, via email […] or
- Dr Ian Jones, via email [...].

I thank you heartily in advance for your cooperation.

Please keep the above portion of this consent form for your records.
If you consent these children to participate in this project and to allow this project to take place in our school, please sign the following signature portion of this consent form.

✂ -------------------------------- cut here ---------------------------- ✂

Head of School's Signature for Consent

I give permission for all Form 1 [Set C] Mathematics students of the scholastic year 2014-15 to participate in the research project as detailed above. I am aware that this project will take place in our school and that data will be gathered from mathematics lessons and from interviews in break time as explained in the above letter. I understand that for a child to participate in this project both he and his main guardians must agree that he participates. I also understand that students, guardians, or I can change our minds about the participation of a pupil at any time, by notifying the researcher of our decision to end participation of that pupil in this project.

Name and Surname of Head of School:

_____

Signature of Head of School:

_____

Date:

_____

Appendix 6.2 Consent Letter to the Prospective Participants

Letter for Students' Permission to Participate in a Research Project

Dear students,

At present I am studying for a PhD in Education with Loughborough University. Throughout this year I will be gathering data for a research I am doing on the use of computer software for the learning of algebra. I am asking your permission to gather this data during some of our lessons. This will involve a video of twenty double lessons, recording of the computer activities, some homework given at the end of every double lesson, and a five interviews which we be spread over this scholastic year and which we will do during break time.

I want to include all of you in my research but you do not have to agree to take part in the research. If you don't agree I will make sure you don't appear in the videos and that the results of the tests will not be used for my research. Rest assured that I will not be insulted or hold it against you if you do not wish to take part. Our relationship as a teacher and pupil will remain intact. Furthermore, your education would not be affected in any way whether you agree to participate or not.

If you agree to take part in the research you will have no direct benefit or credit but you will help me to study new ways how mathematics can be taught. If you participate in the study no one except I and my research supervisors (not the Head or Assistant Head) will have access to the data you provide. In my research I will not write down your name or the school but I will use a fake name to identify your data. Also, during the first two months you can choose to stop participating in the research.

I will be sending a more detailed letter to your guardians. I ask you to please discuss this with your guardians and then, if you agree to join in, please sign the letter I am sending to your guardians.

I remain,

Yours sincerely,

Philip Borg
(Mathematics Teacher)

Appendix 6.3   Consent Letter to the Participants' Guardians

Main Guardian's Consent for Child to Participate in a Research Project

Dear [Name of Guardian],

I am Philip Borg, your child's mathematics teacher at [Name of School]. I am currently reading for a PhD degree with Loughborough University. In my research, I will be conducting a case study in which I will be looking into ways how computer software may help beginning secondary school students to bridge arithmetic with algebra.

I am asking your permission for your child to participate as one of the students in the case study. I will also ask your child if he agrees to participate in this project.

Together with the other students in the case study, I will closely follow your child throughout Form 1 where I will be monitoring his progress in making the necessary connections and transitions between arithmetic and algebra. In order to do this I will gather data by the methods described below.

Methods and tools of gathering information

I will gather data about the mathematical progress of the case study students by using the following methods and tools. The following will be performed throughout Form 1 (Year 7):

a) **Video Recording of the Research Lessons** (20 double lessons). There will be one video camera in the classroom to capture what is going on the interactive whiteboard and the activities of all the students and their interactions between themselves and with the teacher (myself);

b) **Computer Screen Recording Software** (the second half of the same double lessons when video recording is used). There will be software installed on the computer of the case study students which records what they are doing on the computer and also what they are saying.

c) **Written Work** (handouts to fill in). This will be work assigned at the end of each lesson and collected in the lesson that follows.

d) **Interviews** (Five, approx. 30 – 45 min. each). These interviews will be conducted with the case study students during break time. I will ask the students questions about arithmetic and algebra. In these interviews I seek to gather data about the students' knowledge and understanding which may not be possible to detect in their written work. Children's performance in these interviews will not affect their assessment scores in any way.

All of the above will be spread over a period of twenty weeks (which is approximately one scholastic year excluding holidays and school activities).

Benefits and Risks

If children participate in this project they will have no direct benefits but they will help me understand their mathematical progress in a better and deeper way than the normal lesson situation. In addition they will be helping me as their mathematics teacher, my mathematics teacher colleagues, and also other researchers to understand how we can help students to learn mathematics through the use of computer software (apart from traditional methods).

I believe there is very little risk to the children in participating in this project but I will be taking measures to prevent possible harm to them, no matter how small or insignificant, namely:

- If a child becomes uncomfortable or stressed by answering any of the test questions, he can skip the question, take a break, stop the test, or withdraw from the project altogether.
- I will make sure that the data gathering process will not impede the learning of any child.
- I will take all the necessary precautions to prevent any loss of data which may compromise the confidentiality of the data.

Confidentiality and Privacy

Any written work from this research will never use the real name of a student, nor the name of the school. If any still images from video are used in published written work from this research, the faces of the students will be obscured. It is possible that video or audio clips may be used as part of a presentation at a professional conference or a teacher training session but no video or audio clips will be put up on a webpage or on any social media.

During this research project, I will keep all recorded data in a secure location. Only my supervisors at Loughborough University and I will have access to the data. Upon completion of the research, data will be preserved and accessible for a period of ten years.

Voluntary Participation

Participation in this research project is voluntary. Your child (and you) can choose freely to participate or not to participate. In addition, during the first two months of the research you can withdraw your permission and/or your child can stop participating without any consequence to his education. If your child stops participating I will only use the data gathered until that time and further data about him will not be gathered. Furthermore, he will be seated in a way so as not to be included in the camera angle during classroom video recordings.

I recognise that I am the researcher in this project and, at the same time, your child's teacher. Thus, I will ensure that your child's participation or non-participation in my research project does not impact his learning, his grades, or our pupil-teacher relationship.

Questions

If you have any questions about this research project, please contact me, Philip Borg, personally, via phone: […] or email: […]. You can also contact my supervisors at Loughborough University:
- Dr Dave Hewitt, via email […] or
- Dr Ian Jones, via email [...].

I thank you heartily in advance for your cooperation.

Please keep the above portion of this consent form for your records.
If you consent for your child to participate in this project, please sign the following signature portion of this consent form and send it to me with your child.

✂ ------------------------------ cut here ------------------------------ ✂

Signature for Consent

I give permission for my child to participate in the research project as detailed above. I understand that, in order to participate in this project, my child must also agree to participate. I understand that my child or I can change our minds about participation, at any time, by notifying the researcher of our decision to end participation in this project.

Name and Surname of Child: _____

Signature of Child: _____

Name and Surname of Main Guardian: _____

Main Guardian's Signature: _____

Date: _____