POLYNOMIAL MATRIX REDUCTION

TO

LINEARISED FORM

by

Y.L.Li

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Supervisor: Dr. A.C. Pugh,

Department of Mathematical Sciences

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Abstract:

In many aspects of system analysis it is required to consider a set of equations in order to infer the behaviour or, more simply, properties of the system. In many cases these equations will be complex and consequently difficult to analyse. It would be useful therefore from the analysis point of view if a similar but equivalent set of equations describing the system's behaviour could be found.

In the case of linear systems these equations describing the system may be represented by a general polynomial system matrix as has been proposed by Rosenbrock (1970). By reducing this system matrix to linear polynomial form the system can be more easily examined. In the conventional study of linear systems this linear polynomial form is taken to be the usual state space form of the system matrix, but if a generalised study is to be undertaken then the linear form will be the generalised state space form of the system matrix. There are therefore various ways of performing these reductions, all of which preserve particular properties of the original system. The dissertation addresses these issues.

The main contributions of the thesis are contained in chapters 6 and 7. Described in chapter 6 are three ways of system matrix reduction to linear polynomial form. Hayton et al. (1989) have formed matrix pencil equivalents from a general polynomial matrix, preserving the finite and infinite zero structure. This is based on the system matrix idea by Bosgra and Van der Weiden (1981). A further method discussed is the reduction of a polynomial matrix of a linear multivariable system to generalised state space form proposed by Vardulakis (1991). A final reduction is the linearisation described by Zhang (1989) which produces a strongly irreducible realisation for singular systems. Some comparisons of these methods are made.

In chapter 7 the Hayton et al. algorithm which permits the reduction of a general polynomial matrix to a similarly equivalent matrix pencil form is computerised. The key to the reduction is an efficient method of selecting linearly independent rows and columns from a block Toeplitz matrix. By using the program by Demianczuk (1985) which computes the infinite frequency structure of a given rational matrix from its Laurent expansion, the equivalent infinite zero property of the matrix pencil and the polynomial matrix can be verified directly.

CONTENTS

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Addendum

In chapters II, IV and VII it is mentioned that the Smith form and Smith McMillan form of certain matrices will be produced. Note that in fact these forms have not actually been found. This is because it can be seen from the working given what the finite and infinite system poles and zeros are. The Smith form and Smith McMillan form of the appropriate matrices will now be stated.

In examples 1 and 2 section II.4 (p.16, p.17), example 1 section IV.2 (p.43, p.45, p.48), example 1 section IV.3 (p.76) and in chapter VII (p.142), the Smith forms and Smith McMillan forms can be obtained from the forms given by elementary row and column interchanges.

Now consider example 1 section IV.2. The Smith form of the matrix $[T(s)\ U(s)]$ (p.42) is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & s(s+2) & 0 \end{bmatrix}$$

The Smith form of the matrix $[\mathfrak{T}(s)\ \mathfrak{U}]$ (p.44) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s(s+2) & 0 \end{bmatrix}$$

The Smith form of the numerator of $\left[\mathfrak{T}\left(\frac{1}{w}\right) \mathfrak{U}\right]$ (p.47) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1+2w & 0 \end{bmatrix}$$

Now consider example 3 section IV.2. The Smith form of $T(s)$ (p.53) is

$$\begin{bmatrix} 1 & 0 \\ 0 & s^2(s+1)(s+2) \end{bmatrix}$$

The Smith form of $P(s)$ (p.54) is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^2(s+1)(s+2)^2 \end{bmatrix}$$

The Smith form of the numerator of $\mathfrak{T}\left(\frac{1}{w}\right)$ (p.56) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & w(1+w)(1+2w) \end{bmatrix}$$

The Smith form of the numerator of $\mathfrak{P}\left(\frac{1}{w}\right)$ (p.58) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & (1+w)(1+2w)^2 \end{bmatrix}$$

Now consider example 1 section IV.3. The Smith form of the numerator of $\left[\mathfrak{T}\left(\frac{1}{w}\right)\ \mathfrak{U}\right]$ (p.71) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \end{bmatrix}$$

The Smith form of the numerator of $\begin{bmatrix} \mathfrak{T}\left(\frac{1}{w}\right) \\ -\mathfrak{V} \end{bmatrix}$ (p.73) is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence it can be seen that the results determined from above are the same as those given in the text.

CHAPTER I

INTRODUCTION

I
INTRODUCTION

Bosgra and Van der Weiden (1981) have given a procedure whereby a general polynomial system matrix may be reduced to an equivalent generalised state space form. The sense in which this is equivalent to the original system matrix is that the reduced system exhibits identical system properties, both at finite and infinite frequencies.

In this thesis, a computerised version of this algorithm is provided, and it will be seen that this permits the reduction of a general polynomial matrix to a similarly equivalent matrix pencil form (i.e. one which exhibits identical finite and infinite zero structure). The key to this reduction is an efficient method of selecting a set of linearly independent rows and columns from a block Toeplitz matrix. It will also be seen that this reduction algorithm is a full system equivalence transformation and a characterisation of this equivalence in a matrix transformational sense is provided. The computational procedures for the determination of the infinite frequency structure of rational matrices by Demianczuk (1985) are used, and this illustrates the identical infinite frequency property of the general polynomial matrix and its reduced form.

Chapters II-V contain relevant results from multivariable systems theory necessary for this thesis. In chapter II various results concerning rational matrices are discussed. These include definitions of finite and infinite poles and zeros of a rational matrix via the Smith McMillan form and via the Laurent expansion. Also, the McMillan degree of a rational matrix is defined, which plays an important role in the equivalence relation between a general polynomial matrix and its associated matrix pencil form.

Chapter III discusses the various system representations including the state space description. the transfer function matrix description, the Rosenbrock's system matrix and the matrix fraction description.

The system structure is discussed in chapter IV. System poles and zeros at both finite and infinite frequencies, and the matrix pencil are defined.

Finally chapter V describes the various equivalence relations between systems. In the conventional study of linear systems, the transformations are system similarity for state space models and extended strict system equivalence (e.s.s.e.) for polynomial models. In the generalised theory, the appropriate system transformations are complete system equivalence (c.s.e.) for generalised state space models and full system equivalence (f.s.e.) for general polynomial models.

Also discussed in this thesis in chapter VI are three methods of system matrix reduction to linear polynomial form. Hayton et al. (1989) have formed matrix pencil equivalents from a general polynomial matrix, preserving the finite and infinite zero structure. It can be seen how this reduction is based on the system matrix idea by Bosgra and Van der Weiden (1981). Another method discussed is the reduction of a polynomial matrix of a linear multivariable system to generalised state space form proposed by Vardulakis (1991). The final reduction discussed is the linearisation described by Zhang (1989) which produces a strongly irreducible realisation for singular systems. Finally some comparisons of these three methods of linearisation are made.

In the computer program the language used is Fortran 77 and the computer used is the Macintosh, Computer Centre of Loughborough University of Technology.

Note

The worked examples are intended to show the consistency of the theorems and definitions but, of course, do not prove or establish them.

CHAPTER II

RATIONAL MATRICES

II

RATIONAL MATRICES

II.1 Introduction

This chapter contains relevant results concerning rational matrices necessary for this thesis. These include definitions of poles and zeros at finite and infinite frequencies, which rely heavily on the Smith McMillan form (Rosenbrock 1970) of the system transfer function matrix. The idea of relative primeness of polynomial matrices is also introduced. Also discussed in this chapter is the Toeplitz matrix, and hence how to obtain the infinite poles and zeros of a rational matrix from its Laurent expansion at infinity. Finally, the McMillan degree is defined and ~~it will be seen in later chapters~~ the importance of the McMillan degree in the equivalence relation between a general polynomial matrix and its associated matrix pencil form will be seen in later chapters.

II.2 Polynomial matrices

A POLYNOMIAL FORM is an expression of the form

$$p(s) = a_0 + a_1 s + \ldots + a_n s^n \tag{2.1}$$

in which $a_0, a_1, \ldots, a_n$ belong to a field F (more generally, a ring R) and s is an INDETERMINATE such that s^r and as are defined whenever $a \in F$ and $as = sa$.

If $a_n \neq 0$, the number n is the DEGREE of $p(s)$. If $a_n = 1$, the polynomial is said to be MONIC. The values of s for which $p(s)$ takes the value $0 \in F$ are called the ROOTS or ZEROS of $p(s)$.

A polynomial $g(s)$ DIVIDES a polynomial $p(s)$ if there exists a polynomial $p_1(s)$ such that

$$p(s) = g(s)p_1(s) \tag{2.2}$$

If $g(s)$ divides $p(s)$ and $q(s)$, but no polynomial of higher degree than $g(s)$ divides $p(s)$ and $q(s)$, then $g(s)$ is a GREATEST COMMON DIVISOR of $p(s)$ and $q(s)$. If a greatest common divisor of $p(s)$ and $q(s)$ has degree zero, then $p(s)$ and $q(s)$ are RELATIVELY PRIME.

Let $\mathbb{R}[s]$ denote the ring of polynomials in the indeterminate s with coefficients in $\mathbb{R}$. Now define the following:

Definition 1

Let $M(s) \in \mathbb{R}[s]^{m\times m}$. Then $M(s)$ is said to be UNIMODULAR if $M^{-1}(s)$ exists and $M^{-1}(s) \in \mathbb{R}[s]^{m\times m}$.

A POLYNOMIAL MATRIX is a matrix whose elements are polynomials. Suppose that the matrix $[T \quad U]$ is reduced to

$$[T_2 \quad U_1] = Q(s)\,[T \quad U] \tag{2.3}$$

Then $Q^{-1}(s)$ is a polynomial matrix, and

$$\begin{aligned} T &= Q^{-1}T_2 \\ U &= Q^{-1}U_1 \end{aligned} \tag{2.4}$$

The polynomial matrix $Q^{-1}(s)$ has provided a left factorisation of T and U. $Q^{-1}(s)$ is called a LEFT DIVISOR of T and U.

Suppose that $\begin{bmatrix} T_2 \\ -V \end{bmatrix}$ is ~~further~~ reduced to

$$\begin{bmatrix} T_1 \\ -V_1 \end{bmatrix} = \begin{bmatrix} T_2 \\ -V \end{bmatrix} R(s) \tag{2.5}$$

where again $R^{-1}(s)$ is a polynomial matrix. This gives

$$\begin{aligned} T &= \left(Q^{-1}T_1\right)R^{-1} \\ V &= V_1 R^{-1} \end{aligned} \tag{2.6}$$

T, V are said to have a COMMON (RIGHT) DIVISOR R^{-1}.

If two matrices T, U have only unimodular common divisors on the left, they are called RELATIVELY (LEFT) PRIME, or (LEFT) COPRIME. Similarly, if T, V have only unimodular common divisors on the right, they are called RELATIVELY (RIGHT) PRIME, or (RIGHT) COPRIME.

II.3 Finite poles and zeros

Elementary row and column operations on any rational matrix $A(s) \in \mathbb{R}(s)^{m\mathrm{x}n}$, where $\mathbb{R}(s)$ denotes the field of rational functions, are defined as follows:

(a) interchange any two rows or columns of $A(s)$,

(b) multiply row or column i of $A(s)$ by a non-zero constant in $\mathbb{R}$,

(c) add to row or column i of $A(s)$ a multiple by any non-zero element $t(s) \in \mathbb{R}[s]$ of row or column j.

Let $P(m, l)$ denote the class of $(r+m)\mathrm{x}(r+l)$ polynomial matrices, where m, l are fixed positive integers but r is variable and ranges over all integers greater than max $(-m, -l)$. A relation between polynomial matrices is now stated as follows:

Definition 1

(a) Two $m\mathrm{x}l$ polynomial matrices $P_1(s)$, $P_2(s)$ are said to be UNIMODULAR EQUIVALENT (u.e.) if there exist unimodular matrices $M(s)$, $N(s)$ such that

$$P_1(s) = M(s)P_2(s)N(s) \tag{3.1}$$

(b) Let $P_1(s)$, $P_2(s) \in P(m, l)$. Then $P_1(s)$, $P_2(s)$ are said to be EXTENDED UNIMODULAR EQUIVALENT (e.u.e.) if there exist polynomial matrices $M(s)$, $N(s)$ of appropriate dimensions such that

$$M(s)P_1(s) = P_2(s)N(s) \tag{3.2}$$

or

$$\begin{bmatrix} M(s) & P_2(s) \end{bmatrix} \begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{3.3}$$

where $P_2(s), M(s)$ are relatively left-prime

$N(s), P_1(s)$ are relatively right-prime

Note that the extra generality of extended unimodular equivalence is achieved by its facility for allowing matrices of different dimensions to be related.

In chapter V, it will be seen that the above definition is useful in establishing a notion of equivalence between polynomial matrices.

The elementary row and column operations on any rational matrix $A(s)$ can be accomplished by multiplying the given $A(s)$ on the left (right) by elementary unimodular matrices which are obtained by performing the above elementary operations on the identity matrix $I_{m(n)}$.

By a combination of these elementary operations, an mxn rational matrix $A(s)$ can be reduced to its Smith McMillan form defined as follows:

Smith McMillan form (Rosenbrock 1970)

Let $A(s) \in \mathbb{R}(s)^{m\mathrm{x}n}$ with rank $A(s) = r$. By elementary row and column operations, $A(s)$ can be reduced to Smith McMillan form

$$S(s) = M(s)A(s)N(s) \tag{3.4}$$

where M, N are unimodular, and where

$$S(s) = \begin{cases} [Q(s) \quad 0_{m,n-m}] & n > m \\ Q(s) & n = m \\ \begin{bmatrix} Q(s) \\ 0_{m-n,n} \end{bmatrix} & n < m \end{cases} \tag{3.5}$$

and

$$Q(s) = \operatorname{diag}\left[\frac{e_1(s)}{f_1(s)}, \frac{e_2(s)}{f_2(s)}, \ldots, \frac{e_r(s)}{f_r(s)}, 0, \ldots, 0\right] \tag{3.6}$$

where e_i and f_j are relatively prime, monic polynomials where e_i divides e_{i+1} for $i = 1, \ldots, r-1$ and f_j divides f_{j-1} for $j = 2. \ldots, r$.

Equation (3.4) defines an equivalence relation on $\bar{I\!R}(s)^{m\times n}$ which is denoted $E^{I\!R}$. The Smith McMillan form $S(s) \in I\!R(s)^{m\times n}$ of a rational matrix $A(s)$ is a canonical form for $E^{I\!R}$ on $I\!R(s)^{m\times n}$.

If $f_i(s) = 1$, $i = 1, \ldots, r$, in (3.6) above, that is, if $S(s)$ is a polynomial matrix, then it is called the SMITH FORM of $A(s)$. Otherwise, if $A(s)$ is non-polynomial, then $f_i(s)$, $i = 1, \ldots, r$, ~~is~~ are non-constant, that is, $S(s)$ is also non-polynomial and is called the SMITH MACMILLAN FORM of $A(s)$.

The poles and zeros of a scalar transfer function are fundamental to the behaviour of the corresponding system since the poles typify the free response of the system while the zeros have implication for the forced response. In a multivariable system the concepts are of no less significance. The various definitions of multivariable poles and zeros at finite frequencies rely heavily on the Smith McMillan form (Rosenbrock 1970) of the system transfer function matrix and from this their complete effect on the transmission properties of the system may be ascertained quite readily.

Desoer and Schulman (1974) deduced that $p \in \mathbb{C}$ is a finite pole of a rational transfer function matrix if and only if some input creates a zero-state response of the form re^{pt} for $t > 0$ (where r is a constant) i.e. the presence of a pole in a given transfer function matrix changes a system from its zero state at $t = 0^-$ to a state at $t = 0^+$ which results in a purely exponential output for all $t > 0$. Also in the same paper it is deduced that $z \in \mathbb{C}$ is a finite zero of a rational transfer function matrix if it blocks the transmission of signals proportional to e^{zt} in that the corresponding output $y(t) \equiv 0$ for all $t > 0$.

Multivariable transfer functions can have poles and zeros at the same location. To reflect this fact, it is useful to rewrite the Smith McMillan form as

$$\operatorname{diag}\left[\frac{e_i(s)}{f_i(s)}\right] = \prod_{\alpha} M_\alpha(s) \tag{3.7}$$

where α ranges over the set of poles and zeros of $A(s)$, and each $M_\alpha(s)$ has the form

$$M_\alpha(s) = \text{diag}\left\{(s-\alpha)^{\sigma_1}, \ldots, (s-\alpha)^{\sigma_r}\right\} \tag{3.8}$$

The Smith McMillan form of a rational matrix $A(s)$ is now used to define the finite poles and zeros of $A(s)$ as follows:

Definition 2

$A(s)$ has a FINITE POLE at α of degree $-\sigma_i$ if σ_i is negative. The pole at α has multiplicity equal to the number of $(s-\alpha)^{\sigma_i}$, with $\sigma_i < 0$, present in the Smith McMillan form of $A(s)$.

Definition 3

$A(s)$ has a FINITE ZERO at α of degree σ_i if σ_i is positive. The zero at α has multiplicity equal to the number of $(s-\alpha)^{\sigma_i}$, with $\sigma_i > 0$, present in the Smith McMillan form of $A(s)$.

Example 1

Consider the rational matrix

$$A(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{1}{s+2} & \dfrac{1}{s+3} \end{bmatrix}$$

To produce the finite poles and zeros of $A(s)$, the Smith McMillan form of $A(s)$ is firstly found as follows:

$$\begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{1}{s+2} & \dfrac{1}{s+3} \end{bmatrix}$$

$$\xrightarrow[(row\ 2)\,-\,(row\ 1)]{new\ row\ 2\ =} \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{-1}{(s+1)(s+2)} & \dfrac{1}{s+3} \end{bmatrix}$$

$$\xrightarrow[\textit{row } 1 \text{ and } \textit{row } 2]{\textit{interchange}} \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & \dfrac{-1}{(s+3)} \\[2ex] \dfrac{1}{s+1} & 0 \end{bmatrix}$$

new row 1 = − row 1

$$\xrightarrow[(\textit{row } 2) - ((s+2) \text{ x } \textit{row } 1)]{\textit{new row } 2 =} \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & \dfrac{-1}{(s+3)} \\[2ex] 0 & \dfrac{s+2}{s+3} \end{bmatrix}$$

$$\xrightarrow[(\textit{col. } 2) + (s \text{ x } \textit{col. } 1)]{\textit{new col. } 2 =} \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & \dfrac{-2}{(s+1)(s+2)(s+3)} \\[2ex] 0 & \dfrac{s+2}{s+3} \end{bmatrix}$$

$$\xrightarrow[\substack{\textit{multiply new col. } 1 \\ \textit{by } -1/2}]{\substack{\textit{interchange} \\ \textit{col. } 1 \text{ and } \textit{col. } 2}} \begin{bmatrix} \dfrac{1}{(s+1)(s+2)(s+3)} & \dfrac{1}{(s+1)(s+2)} \\[2ex] \dfrac{-(s+2)}{2(s+3)} & 0 \end{bmatrix}$$

$$\xrightarrow[(\textit{col. } 2) - ((s+3) \text{ x } \textit{col. } 1)]{\textit{new col. } 2 =} \begin{bmatrix} \dfrac{1}{(s+1)(s+2)(s+3)} & 0 \\[2ex] \dfrac{-(s+2)}{2(s+3)} & \dfrac{(s+2)}{2} \end{bmatrix}$$

$$\xrightarrow[\substack{\textit{new row } 2 = \\ (\textit{row } 2) + ((s+1)(s+2)^2 \text{ x } \textit{row } 1)}]{\substack{\textit{multiply row } 2 \\ \textit{by } 2}} \begin{bmatrix} \dfrac{1}{(s+1)(s+2)(s+3)} & 0 \\[2ex] 0 & (s+2) \end{bmatrix}$$

Hence, the Smith McMillan form of $A(s)$ is

$$S(A) = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)(s+3)} & 0 \\[2ex] 0 & (s+2) \end{bmatrix}$$

Therefore, $A(s)$ has one finite zero at $s = -2$ of degree 1 and multiplicity 1, a pole at $s = -1$ of degree 1 and multiplicity 1, a pole at $s = -2$ of degree 1 and multiplicity 1 and a pole at $s = -3$ of degree 1 and multiplicity 1 .

II.4 Infinite poles and zeros

In multivariable systems theory there are several problems in which it is important to keep track of the behaviour at infinity. Poles at $s = \infty$ correspond to non-proper systems (or systems with differentiators), as may arise in constructing inverse systems. The zeros at infinity are important, for example, in studying the asymptotic behaviour of multivariable root loci. For scalar systems with a numerator of degree m, $m < n$, m of the closed-loop poles will converge towards the m finite zeros as the feedback gain goes to infinity. The remaining $n - m$ poles will converge to the $n - m$ zeros at infinity. It is clear that the concept of poles and zeros at infinity are important in the study of the properties of multivariable root loci.

The basis for the results derived in this section is the Smith McMillan form at infinity of a rational matrix, as developed by Vardulakis et al. (1982). In the following $\mathbb{R}[s]$ denotes the ring of polynomials in the indeterminate s with coefficients in $\mathbb{R}$, while $\mathbb{R}(s)$ denotes the associated field of rational functions and $\mathbb{R}_{pr}(s)$ denotes the ring of proper rational functions. Note that $\mathbb{R}_{pr}(s)$ is essentially the same as the polynomial ring $\mathbb{R}[s]$ in that a degree may be associated with proper rational functions. Now define the following:

Definition 1

Let $G(s) \in \mathbb{R}(s)^{m\mathrm{x}n}$ be an $m\mathrm{x}n$ rational matrix. $G(s)$ is said to be PROPER if

$$\lim_{s \to \infty} G(s)$$

exists. If this limit is zero, $G(s)$ is said to be STRICTLY PROPER, while if this limit is non-zero, $G(s)$ will be called EXACTLY PROPER.

Definition 2

The $m\mathrm{x}m$ rational matrix $W(s) \in \mathbb{R}_{pr}^{m\mathrm{x}m}(s)$ is said to be BIPROPER if and only if

(a) $\lim_{s \to \infty} W(s) = W_\infty \in \mathbb{R}^{m\mathrm{x}m}$

(b) det $W_\infty \neq 0$ **(4.1)**

Thus, $W(s)$ is biproper if and only if it is exactly proper (by (a)) and has an exactly proper inverse (by (b)).

Definition 3

The mxn rational matrices $G_1(s)$ and $G_2(s)$ are said to be EQUIVALENT AT INFINITY if there exist biproper rational matrices $W(s) \in \mathbb{R}_{pr}^{m\mathrm{x}m}(s)$, $V(s) \in \mathbb{R}_{pr}^{n\mathrm{x}n}(s)$ such that

$$W(s)G_1(s)V(s) = G_2(s) \quad \textbf{(4.2)}$$

In the previous section, finite poles and zeros of a rational matrix $A(s) \in \mathbb{R}(s)^{m\mathrm{x}n}$ were defined via its Smith McMillan form. It was seen in order that the finite pole-zero structure of $A(s)$ be preserved during the reduction process to the Smith McMillan form, row and column operations represented by polynomial matrices with no finite poles nor zeros (i.e. $\mathbb{R}[s]$-unimodular matrices) were used. These ($\mathbb{R}[s]$-unimodular) elementary operations will in general destroy the pole-zero structure of $A(s)$ at infinity since their polynomial matrix representation may have poles and zeros there. Thus, the reduction procedure to a diagonal matrix, which gives in a simple form the pole-zero structure at $s = \infty$ of a given rational matrix will be obtained by carrying out elementary row and column operations whose matrix representations have no poles nor zeros at $s = \infty$. These elementary operations are represented by square rational matrices whose elements are proper rational functions (and thus have no poles at $s = \infty$), are non-singular and have also no zeros at $s = \infty$.

These elementary row and column operations on any rational matrix $G(s) \in \mathbb{R}(s)^{m\mathrm{x}n}$ are defined as follows:

(a) interchange any two rows or columns of $G(s)$,

(b) multiply row or column i of $G(s)$ by a unit $u(s) \in \mathbb{R}_{pr}(s)$,

(c) add to row or column i a multiple by a $t(s) \in \mathbb{R}_{pr}(s)$ of row or column j of $G(s)$.

These elementary operations on a rational matrix $G(s)$ can be accomplished by multiplying $G(s)$ on the left (right) by elementary biproper matrices which are obtained by performing the above elementary operations on the identity matrix $I_{m(n)}$.

By a combination of these elementary row and column operations, an mxn rational matrix $G(s)$ can be reduced to its Smith McMillan form at infinity. A canonical form for a rational matrix under the equivalence relation of Definition 3 is its Smith McMillan form at infinity, $S^{\infty}(G)$:

Smith McMillan form at infinity (Vardulakis et al., 1982)

Let $G(s) \in \mathbb{R}(s)^{m\mathrm{x}n}$ with rank $G(s) = r$. Then there exist biproper rational matrices $W(s)$ and $V(s)$ such that

$$W(s)G(s)V(s) = S^{\infty}(G) \tag{4.3}$$

where

$$S^{\infty}(G) = \begin{cases} \begin{bmatrix} Q(s) & 0_{m,n-m} \end{bmatrix} & n > m \\ Q(s) & n = m \\ \begin{bmatrix} Q(s) \\ 0_{m-n,m} \end{bmatrix} & n < m \end{cases} \tag{4.4}$$

and

$$Q(s) = \mathrm{diag}\,[s^{q_1}, s^{q_2}, \ldots, s^{q_r}.0, \ldots, 0] \tag{4.5}$$

with
$$q_1 \geq q_2 \geq \ldots \geq q_k \geq 0 \geq q_{k+1} \geq \ldots \geq q_r \tag{4.6}$$

$S^{\infty}(G)$ is called the SMITH MACMILLAN FORM AT INFINITY of $G(s)$.

The Smith McMillan form at infinity of a rational matrix $G(s)$ is now used to define the infinite poles and zeros of $G(s)$, as follows:

Definition 4

If p_∞ is the number of q_i s in (4.5) with $q_i > 0$, then $G(s)$ has p_∞ POLES AT INFINITY, each having degree q_i, and where p_∞ is the multiplicity.

Definition 5

If z_∞ is the number of q_i s in (4.5) with $q_i < 0$, then $G(s)$ has z_∞ ZEROS AT INFINITY, each having degree $|q_i|$, and where z_∞ is the multiplicity.

The effect of poles and zeros at infinity on the transmission properties of a multivariable system may be ascertained from the Smith McMillan form at infinity. Verghese (1978) showed that the presence of an infinite pole in a given transfer function matrix causes the transmittance to contain a linear combination of impulse functions which are not present in the input. Thus, impulses are generated by a pole at infinity. Pugh and Krishnaswamy (1985) showed that the presence of an infinite zero in a given transfer function matrix causes the non-transmittance of the impulsive part of certain inputs. Thus, zeros at infinity enable a system to absorb certain impulses.

Proper rational functions have no infinite poles, biproper rational functions have no infinite poles nor zeros, and polynomial matrices have only poles at infinity, as it will now be seen.

Every $t(s) \in \mathbb{R}(s)$ can be written as

$$t(s) = s^{q_i} \frac{n_1(s)}{d_1(s)} \tag{4.7}$$

where $n_1(s), d_1(s) \in \mathbb{R}[s]$ with deg $n_1(s) =$ deg $d_1(s)$. This is because $t(s)$ can be written as

$$t(s) = \frac{n(s)}{d(s)} s^{-q_i} s^{q_i} = u(s) s^{q_i} \tag{4.8}$$

where

$$u(s) = \frac{n_1(s)}{d_1(s)} = \frac{n(s)}{d(s) s^{q_i}}$$

and

$$q_i = \deg n(s) - \deg d(s) \tag{4.9}$$

If $q_i \leq 0$, then $t(s)$ is called a proper rational function and if the inequality is strict, then $t(s)$ is called a strictly proper rational function. Thus, proper rational functions have no poles at $s = \infty$.

The set of proper rational functions $\mathbb{R}_{pr}(s)$ is a commutative ring with unity element (the real number 1) and no zero divisors. $\mathbb{R}_{pr}(s)$ is therefore an integral domain. Biproper rational functions $u(s) \in \mathbb{R}_{pr}(s)$ are those proper rational functions for which there exist a $u'(s) \in \mathbb{R}_{pr}(s)$ such that $u(s)u'(s) = 1$. This implies that $u(s) \in n(s)/d(s) \in \mathbb{R}_{pr}(s)$ is a unit if and only if $\deg n(s) = \deg d(s)$ i.e. if and only if $q_i = 0$. Thus, biproper rational functions have no poles nor zeros at $s = \infty$.

The rational matrix $t(s) \in \mathbb{R}(s)^{mxl}$ is polynomial if and only if it has no finite poles i.e. a polynomial matrix has all its poles at infinity. If $t(s)$ is polynomial, then it has a coprime factorisation

$$t(s) = I_m^{-1} t(s) \tag{4.10}$$

Hence, I_m is a denominator of $t(s)$, and clearly $t(s)$ has no finite poles. Conversely, suppose that

$$t(s) = D^{-1}(s)N(s) \tag{4.11}$$

is a coprime factorisation of $t(s)$. If $t(s)$ has no finite poles, then all denominators of $t(s)$ have no finite zeros. Thus the Smith form of $D(s)$ is I_m, and so $D(s)$ is unimodular. Consequently, $D^{-1}(s)$ is a polynomial matrix and so is $t(s)$.

Example 1

Consider the rational matrix

$$G(s) = \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix}$$

In order to find the infinite poles and zeros of $G(s)$, it is necessary to produce the Smith McMillan form at infinity of $G(s)$, as follows:

$$\begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix} \xrightarrow[\textit{row } 1 \text{ and } \textit{row } 2]{\textit{interchange}} \begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\textit{col.}\ 1 \text{ x } s/(s+1)]{\textit{new col.}\ 1 =} \begin{bmatrix} s & 1 \\ s/(s+1) & 0 \end{bmatrix}$$

$$\xrightarrow[(\textit{col.}\ 2) - (\textit{col.}\ 1 \text{ x } 1/s)]{\text{new } \textit{col.}\ 2 =} \begin{bmatrix} s & 0 \\ s/(s+1) & -1/(s+1) \end{bmatrix}$$

$$\xrightarrow[(\textit{row}\ 2) - (\textit{row}\ 1 \text{ x } 1/(s+1))]{\text{new } \textit{row}\ 2 =} \begin{bmatrix} s & 0 \\ 0 & -1/(s+1) \end{bmatrix}$$

$$\xrightarrow[-\textit{col.}\ 2 \text{ x } (s+1)/s]{\text{new } \textit{col.}\ 2 =} \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}$$

Hence, $G(s)$ has Smith McMillan form at infinity

$$S^{\infty}(G) = \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}$$

i.e. $G(s)$ has a pole at infinity of degree 1 and a zero at infinity of degree 1.

Example 2

Consider now the rational matrix

$$T(s) = \begin{bmatrix} s^3 & s^2 & 1 \\ -1 & 0 & 0 \\ -s & -1 & 0 \end{bmatrix}$$

Firstly produce the Smith McMillan form at infinity of $T(s)$, as follows:

$$\begin{bmatrix} s^3 & s^2 & 1 \\ -1 & 0 & 0 \\ -s & -1 & 0 \end{bmatrix} \xrightarrow[\substack{\textit{new col.}\ 3 = \\ (\textit{col.}\ 3) - (\textit{col.}\ 1 \text{ x } 1/s^3)}]{\substack{\text{new } \textit{col.}\ 2 = \\ (\textit{col.}\ 2) - (\textit{col.}\ 1 \text{ x } 1/s)}} \begin{bmatrix} s^3 & 0 & 0 \\ -1 & 1/s & 1/s^3 \\ -s & 0 & 1/s^2 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new } row\ 3 = \\ (row\ 3) + (row\ 1 \text{ x } 1/s^2)}]{\substack{\text{new } row\ 2 = \\ (row\ 2) + (row\ 1 \text{ x } 1/s^3)}} \begin{bmatrix} s^3 & 0 & 0 \\ 0 & 1/s & 1/s^3 \\ 0 & 0 & 1/s^2 \end{bmatrix}$$

$$\xrightarrow[(col.\ 3) - (col.\ 2 \text{ x } 1/s^2)]{\text{new } col.\ 3 =} \begin{bmatrix} s^3 & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/s^2 \end{bmatrix}$$

Hence, $T(s)$ has Smith McMillan form at infinity

$$S^{\infty}(T) = \begin{bmatrix} s^3 & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/s^2 \end{bmatrix}$$

i.e. $T(s)$ has a pole at infinity of degree 3 and a zero at infinity with multiplicity 2 and degrees $1, 2$.

II.5 The Laurent expansion at infinity

Van Dooren et al. (1979) used the Laurent expansion of a rational matrix $G(s)$ about a finite point and the corresponding Toeplitz matrix coefficients to determine the Smith McMillan form at s_0 of $G(s)$. In an analogous way the Smith McMillan form at infinity of $G(s)$, and hence the infinite pole and zero structure of $G(s)$, can be determined by considering the Laurent expansion at infinity of $G(s)$ and the corresponding Toeplitz matrices.

Suppose the rational matrix $G(s)$ has a Laurent expansion at infinity of the form:

$$G(s) = \sum_{i=-\infty}^{l} G_i s^i$$

$$= G_l s^l + G_{l-1} s^{l-1} + \ldots + G_0 + G_{-1} s^{-1} + \ldots \qquad \textbf{(5.1)}$$

Now define the Toeplitz matrices as follows:

Definition 1

The TOEPLITZ MATRICES AT INFINITY, $T_i^{\infty}(G)$, associated with $G(s)$ are defined as follows:

$$T_i^{\infty}(G) = \begin{bmatrix} G_l & G_{l-1} & G_{l-2} & \cdots\cdots & G_{-i} \\ & G_l & & & \vdots \\ & & G_l & & \vdots \\ & 0 & & \ddots & \\ & & & & G_l \end{bmatrix}, \quad i \geq -l \qquad \mathbf{(5.2)}$$

The information concerning the rank of the $T_i^{\infty}(G)$ will determine the rank indices at infinity of $G(s)$, which are defined as follows:

Definition 2

The RANK INDICES AT INFINITY of $G(s)$ are defined as

$$\rho_i^{\infty}(G) = \text{rank}\,[T_i^{\infty}(G)] - \text{rank}\,[T_{i-1}^{\infty}(G)], \quad i = -l, -l+1, \ldots, \qquad \mathbf{(5.3)}$$

where it is assumed that the non-existing T_i^{∞}, $i > l$, have rank zero.

These rank indices at infinity are invariant under the transformation of equivalence at infinity given by Definition 3 section II.4.

Theorem 1

Let $G(s)$ and $H(s)$ be two mxn rational matrices. If $G(s)$ ~~and~~ $H(s)$ are equivalent at infinity then they have the same rank indices at infinity.

Proof

See Pugh et al. (1989).

As a consequence of the above theorem, it follows that a rational matrix $G(s)$ has the same rank indices at infinity as its Smith McMillan form at infinity, $S_{-}^{\infty}(G)$. Thus, a direct relationship between the rank indices of $G(s)$ and its Smith McMillan form at infinity has been established, which makes it possible to deduce the Smith McMillan form at infinity of $G(s)$ from the rank differences of its Toeplitz matrices at infinity. To derive this relationship requires the following theorem:

Theorem 2 (Pugh et al., 1989)

Let $S^{\infty}(G)$ denote the Smith McMillan form of the rational matrix $G(s)$, and ρ_i^{∞} denote the rank indices of $G(s)$ constructed on the basis of its Laurent expansion about the point at infinity. Then

$$S^{\infty}(G) \triangleq \text{block diag}\{Q_i(s)\} \tag{5.4}$$

where $Q_i(s)$ is the $(\rho_i^{\infty} - \rho_{i-1}^{\infty})\text{x}(\rho_i^{\infty} - \rho_{i-1}^{\infty})$ matrix given by

$$Q_i(s) \triangleq \begin{bmatrix} s^{-i} & 0 & \dots & 0 \\ 0 & s^{-i} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{-i} \end{bmatrix} \tag{5.5}$$

for $i = -l, -l+1, \ldots$, and if $\rho_i^{\infty} - \rho_{i-1}^{\infty} = 0$ then the corresponding matrix $Q_i(s)$ is not present in (5.4).

The pole-zero structure at infinity may now be deduced as follows:

Corollary 1

If, in theorem 2, $\rho_i^{\infty} - \rho_{i-1}^{\infty} \neq 0$, then

(a) $G(s)$ will have $\rho_i^{\infty} - \rho_{i-1}^{\infty}$ POLES AT INFINITY of degree $|i|$ if $i < 0$, with multiplicity $(\rho_{-1}^{\infty} - \rho_{-l-1}^{\infty})$.

(b) $G(s)$ will have $\rho_i^{\infty} - \rho_{i-1}^{\infty}$ ZEROS AT INFINITY of degree i if $i > 0$, with multiplicity $(\rho_l^{\infty} - \rho_0^{\infty})$.

At $i = 0$, $G(s)$ will have no poles nor zeros at infinity.

Example 1

Consider example 2 section II.4. It was seen that the rational matrix

$$T(s) = \begin{bmatrix} s^3 & s^2 & 1 \\ -1 & 0 & 0 \\ -s & -1 & 0 \end{bmatrix}$$

had one pole at infinity of degree 3, one zero at infinity of degree 1 and one zero at infinity of degree 2. The infinite pole and zeros of $T(s)$ will now be found via its Laurent expansion at infinity.

$$T(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^3 + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= G_3 s^3 + G_2 s^2 + G_1 s + G_0$$

Now construct the Toeplitz matrices as follows:

$$T_{-3}^{\infty} = G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } T_{-3}^{\infty} = 1$$

$$\begin{aligned} \rho_{-3}^{\infty} &= \text{rank } T_{-3}^{\infty} - \text{rank } T_{-4}^{\infty} \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Therefore, there is one infinite pole of degree 3 since $\rho_{-3}^{\infty} - \rho_{-4}^{\infty} = 1 - 0 = 1$.

$$T_{-2}^{\infty} = \begin{bmatrix} G_3 & G_2 \\ 0 & G_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } T_{-2}^{\infty} = 2$$

$$\begin{aligned} \rho_{-2}^{\infty} &= \text{rank } T_{-2}^{\infty} - \text{rank } T_{-3}^{\infty} \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

There is no other pole since $\rho_{-2}^{\infty} - \rho_{-3}^{\infty} = 1 - 1 = 0$.

$$T_{-1}^{\infty} = \begin{bmatrix} G_3 & G_2 & G_1 \\ 0 & G_3 & G_2 \\ 0 & 0 & G_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } T_{-1}^{\infty} = 3$$

$$\begin{aligned} \rho_{-1}^{\infty} &= \text{rank } T_{-1}^{\infty} - \text{rank } T_{-2}^{\infty} \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

There is no other pole since $\rho_{-1}^{\infty} - \rho_{-2}^{\infty} = 1 - 1 = 0$.

$$T_0^{\infty} = \begin{bmatrix} G_3 & G_2 & G_1 & G_0 \\ 0 & G_3 & G_2 & G_1 \\ 0 & 0 & G_3 & G_2 \\ 0 & 0 & 0 & G_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } T_0^{\infty} = 4$$

$$\begin{aligned} \rho_0^{\infty} &= \text{rank } T_0^{\infty} - \text{rank } T_{-1}^{\infty} \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

The rational matrix $T(s)$ has no infinite poles nor zeros at $i = 0$.

$$T_1^{\infty} = \begin{bmatrix} G_3 & G_2 & G_1 & G_0 & G_{-1} \\ 0 & G_3 & G_2 & G_1 & G_0 \\ 0 & 0 & G_3 & G_2 & G_1 \\ 0 & 0 & 0 & G_3 & G_2 \\ 0 & 0 & 0 & 0 & G_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } T_1^{\infty} = 6$$

$$\begin{aligned} \rho_1^{\infty} &= \text{rank } T_1^{\infty} - \text{rank } T_0^{\infty} \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

Therefore, there is one infinite zero of degree 1 since $\rho_1^{\infty} - \rho_0^{\infty} = 2 - 1 = 1$.

$$T_2^{\infty} = \begin{bmatrix} G_3 & G_2 & G_1 & G_0 & G_{-1} & G_{-2} \\ 0 & G_3 & G_2 & G_1 & G_0 & G_{-1} \\ 0 & 0 & G_3 & G_2 & G_1 & G_0 \\ 0 & 0 & 0 & G_3 & G_2 & G_1 \\ 0 & 0 & 0 & 0 & G_3 & G_2 \\ 0 & 0 & 0 & 0 & 0 & G_3 \end{bmatrix}$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

rank $T_2^\infty = 9$

$$
\begin{aligned}
\rho_2^\infty &= \text{rank } T_2^\infty - \text{rank } T_1^\infty \\
&= 9 - 6 \\
&= 3
\end{aligned}
$$

Therefore, there is one infinite zero of degree 2 since $\rho_2^\infty - \rho_1^\infty = 3 - 2 = 1$.

Hence, $T(s)$ has one infinite pole of degree 3, one infinite zero of degree 1 and one infinite zero of degree 2 i.e. the poles and zeros at infinity of a rational matrix can be obtained via its Laurent expansion at infinity or via the Smith McMillan form at infinity.

All the infinite poles and zeros will have been found when

$$\rho_k^\infty = r = \text{rank } [G(s)] \tag{5.6}$$

for some k. This is because the rank difference of two successive Toeplitz matrices cannot exceed r, which means that if (5.6) holds then

$$\rho_{k+i}^\infty = r, \qquad i = 1, 2, \ldots \tag{5.7}$$

Thus in this case,

$$\rho_{k+i}^{\infty} - \rho_{k+i-1}^{\infty} = 0, \qquad i = 1, 2, \ldots \tag{5.8}$$

indicating that the search is complete.

This observation leads to the following test for the absence of infinite zeros in a rational matrix:

Result 1

The mxl rational matrix $G(s)$ of normal rank r will possess no infinite zeros if and only if

$$\text{rank}[T_0^{\infty}(G)] = \text{rank}[T_{-1}^{\infty}(G)] + r \tag{5.9}$$

II.6 The McMillan degree

In this section, the McMillan degree of both a polynomial matrix and a rational matrix will be defined. Firstly, the least order of a rational matrix will be discussed.

Let $T(s) \in \mathbb{R}(s)^{m\text{x}l}$ and let

$$T(s) = A_1^{-1}(s)B_1(s) = B_2(s)A_2^{-1}(s) \tag{6.1}$$

be respectively left and right coprime polynomial matrix fraction descriptions (MFD) of $T(s)$, where $A_1(s) \in \mathbb{R}[s]^{m\text{x}m}$, $B_1(s) \in \mathbb{R}[s]^{m\text{x}l}$, $B_2(s) \in \mathbb{R}[s]^{m\text{x}l}$, $A_2(s) \in \mathbb{R}[s]^{l\text{x}l}$. The least order of a rational matrix is defined as follows:

Definition 1

The LEAST ORDER of a rational matrix $T(s)$, denoted by $\nu(T)$, is defined as

$$\nu(T) = \deg|A_1(s)| = \deg|A_2(s)| \tag{6.2}$$

Equivalently, the least order is given by the following:

Definition 2

The LEAST ORDER $\nu(T)$ of $T(s) \in \mathbb{R}(s)^{m\mathrm{x}l}$ is the degree of the product of the denominator polynomials $f_i(s)$, $i = 1, \ldots, r$, appearing in the Smith McMillan form of $T(s)$ i.e.

$$\nu(T) = \deg\left[\prod_{i=1}^{r} f_i(s)\right] \tag{6.3}$$

Clearly, $\nu(T)$ is the number of finite poles of $T(s)$, multiplicities and degrees accounted for.

The least order of a rational matrix plays an important part in the definition of the McMillan degree of a rational matrix.

Let $T(s) \in \mathbb{R}(s)^{m\mathrm{x}l}$ and write it as

$$T(s) = T_{sp}(s) + T_{pol}(s) \tag{6.4}$$

where $T_{sp}(s) \in \mathbb{R}_{pr}^{m\mathrm{x}l}(s)$ is strictly proper (i.e. $\lim_{s\to\infty} T_{sp}(s) = 0$) and $T_{pol}(s) \in \mathbb{R}[s]^{m\mathrm{x}l}$.

Definition 3

The MACMILLAN DEGREE $\delta_M\Big(T(s)\Big)$ of $T(s)$ is defined by

$$\delta_M\Big(T(s)\Big) = \nu\Big(T_{sp}(s)\Big) + \nu\Big(T_{pol}(\frac{1}{w})\Big) \tag{6.5}$$

If $T(s)$ is entirely polynomial i.e. if in (6.4), $T_{sp}(s) \equiv 0$, then the following holds:

$$\begin{aligned}\delta_M\Big(T(s)\Big) &\equiv \delta_M\Big(T_{pol}(s)\Big)\\ &\equiv \nu\Big(T_{pol}(\frac{1}{w})\Big)\\ &\equiv \delta_M\Big(T_{pol}(\frac{1}{w})\Big)\end{aligned} \tag{6.6}$$

The following result of the McMillan degree of a polynomial matrix $P(s)$ is proposed:

Result 1

For an mxl polynomial matrix $P(s)$, the MACMILLAN DEGREE of $P(s)$, denoted $\delta_M\Big(P(s)\Big)$, is the total number of infinite poles of $P(s)$ (McMillan 1952), i.e.

$$\delta_M\Big(P(s)\Big) = \sum_{i=1}^{k} q_i \tag{6.7}$$

where $q_i,\ i = 1, \ldots, k$, are the orders of the poles at $s = \infty$ of $P(s)$ in the Smith McMillan form at $s = \infty$ of $P(s)$, or the highest degree of minors of all orders of $P(s)$ (Rosenbrock 1970).

Another characterisation has been noted by Barnett (1971) and is the following. *Let the mxl polynomial matrix $P(s)$ correspond to the matrix* polynomial defined by

$$P(s) \equiv P_0 + P_1 s + P_2 s^2 + \ldots + P_q s^q \tag{6.8}$$

where $P_i,\ i = 1, 2, \ldots, q$, are mxl constant matrices with

$$P_q \neq 0 \tag{6.9}$$

The McMillan degree of $P(s)$ is defined in the following lemma:

Lemma 1 (Pugh 1976)

$$\delta_M\Big(P(s)\Big) = \operatorname{rank}\begin{bmatrix} P_1 & P_2 & \ldots & P_{q-1} & P_q \\ P_2 & P_3 & \ldots & P_q & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ P_{q-1} & P_q & \ldots & 0 & 0 \\ P_q & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{6.10}$$

Consider now a proper rational matrix $T(s)$ and decompose it as in (6.4)

$$T(s) = T_{sp}(s) + E \tag{6.11}$$

where $E \in I\!R^{mxl}$. A constant matrix E has no finite nor infinite poles so its McMillan degree is zero. Thus,

$$\begin{aligned}\delta_M\Big(T(s)\Big) &= \nu\Big(T_{sp}(s)\Big) + \nu(E) \\ &= \nu\Big(T_{sp}(s)\Big) \\ &= \nu\Big(T(s)\Big)\end{aligned} \tag{6.12}$$

Also, for any rational matrix $T(s)$, (see Definition 3),

$$\begin{aligned}\delta_M\Big(T(s)\Big) &= \nu\Big(T_{sp}(s)\Big) + \nu\Big(T_{pol}(\frac{1}{w})\Big) \\ &= \nu\Big(T(s)\Big) + \delta_M\Big(T_{pol}(s)\Big) \quad \text{using (6.12) and (6.6)} \\ &= \deg\Bigg[\prod_{i=1}^{r} f_i(s)\Bigg] + \sum_{i=1}^{k} q_i \quad \text{using (6.3) and (6.7)}\end{aligned}$$

Now $\deg\Big[\prod_{i=1}^{r} f_i(s)\Big]$ gives the number of all finite poles of $T(s)$. Hence, the following results:

Result 2

The MACMILLAN DEGREE $\delta_M\Big(T(s)\Big)$ of a rational matrix $T(s)$ is equal to the total number of poles of $T(s)$ (finite ones and those at $s = \infty$, and multiplicities and degrees accounted for).

A technical result involving the McMillan degree of a polynomial matrix will now be stated. In chapter VI, it will be seen that the theorem is useful in establishing a notion of equivalence between a general polynomial matrix and its associated matrix pencil form.

Theorem 1

Let $P(s)$ be an mxl polynomial matrix.

(a) If $P_i(s)$ are mxl_i polynomial matrices $(i = 1, 2)$ such that

$$\delta_M\Big(\big[\,P(s) \quad P_1(s)\,\big]\Big) = \delta_M\Big(P(s)\Big) \tag{6.13}$$

then

$$\delta_M\Big(\,[P(s) \quad P_1(s) \quad P_2(s)]\,\Big) = \delta_M\Big(\,[P(s) \quad P_2(s)]\,\Big) \tag{6.14}$$

(b) If B is an lxn constant matrix such that

$$\delta_M\Big(P(s)B\Big) = \delta_M\Big(P(s)\Big) \tag{6.15}$$

then for any mxp polynomial matrix $C(s)$,

$$\delta_M\Big(\,[P(s)B \quad C(s)]\,\Big) = \delta_M\Big(\,[P(s) \quad C(s)]\,\Big) \tag{6.16}$$

Proof:

See the work by Hayton et al. (1989).

CHAPTER III

SYSTEM REPRESENTATIONS

III

SYSTEM REPRESENTATIONS

III.1 Introduction

In the study of control systems it is often sufficient to examine a linear model of the system in order to infer its behaviour. There are four principal types of linear model, or system description, that are used in multivariable control system studies. These are known as the state space description, the transfer function matrix description, the system matrix (the Rosenbrock system matrix and the generalised state space system matrix), and the matrix fraction description. Each of these will be considered in this chapter.

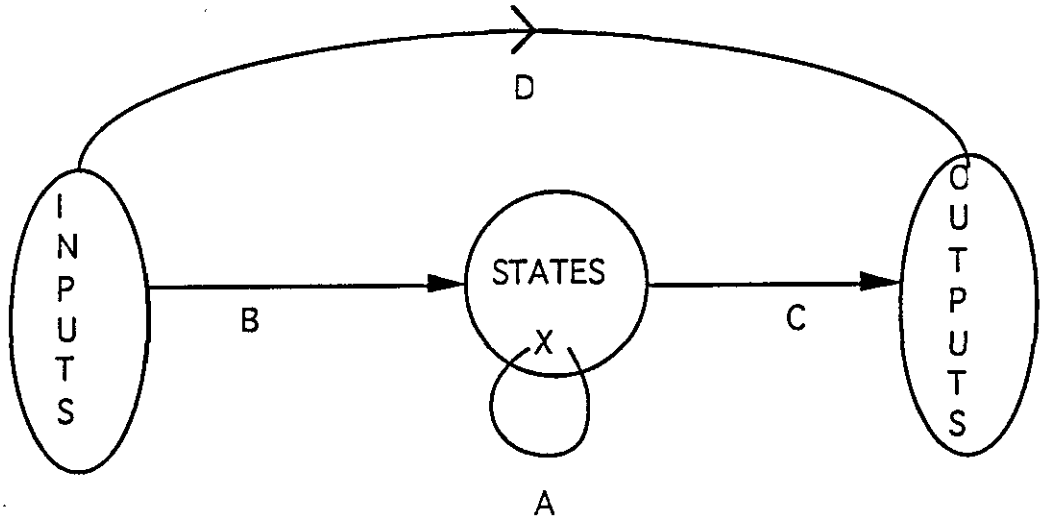
III.2 The state space description

Consider a control system whose defining equations are of the form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + D(s)u(t) \qquad \textbf{(2.1)}$$

where $x(t)$ is an nx1 vector of state variables, $u(t)$ is an lx1 vector of input functions and $y(t)$ is an mx1 vector of outputs. The matrices A, B, C are constant and are of dimension nxn , nxl , mxn respectively. They are called respectively the plant matrix, the input matrix and the output matrix. $D(s)$ is a polynomial matrix of dimension mxl. This system is said to be in STATE SPACE FORM.

This state space description of the system provides a picture of the system structure as shown below:



The internal variables $x_i(t)$ $(i = 1, n)$ interact with one another, the inputs $u_k(t)$ $(k = 1, l)$ affect the system states $x_i(t)$, and the outputs $y_j(t)$ $(j = 1, m)$ are obtained from various combinations of the state variables $x_i(t)$ and the inputs $u_k(t)$. This form of description is called an INTERNAL DESCRIPTION.

Note that the eigenvalues (or poles) of the system are given by the roots of $|sI - A| = 0$.

III.3 The transfer function matrix description

Consider again the system in state space form (see equations (2.1) section III.2). Assume that the initial conditions are zero i.e. $x(0) = 0$. Taking Laplace transforms gives

$$\begin{aligned} s\bar{x}(s) &= A\bar{x}(s) + B\bar{u}(s) \\ \bar{y}(s) &= C\bar{x}(s) + D(s)\bar{u}(s) \end{aligned} \tag{3.1}$$

Substituting the first equation into the second, eliminating $\bar{x}(s)$, gives

$$\bar{y}(s) = \left[C(sI - A)^{-1}B + D(s)\right]\bar{u}(s) \tag{3.2}$$

The mxl rational matrix

$$G(s) = C(sI - A)^{-1}B + D(s) \tag{3.3}$$

summarises the response of the system so far as its external behaviour is concerned. This matrix is called the TRANSFER FUNCTION MATRIX. $G(s)$ is an input-output map relating the Laplace transform of the vector of outputs $y(s)$ to the Laplace transform of the vector of inputs $u(s)$, with zero initial conditions, by the relationship

$$\bar{y}(s) = G(s)\bar{u}(s) \tag{3.4}$$

For an arbitrary ordered linear model, the elements $g_{ij}(s)$ of the matrix $G(s)$ are ratios of polynomials in s representing the transfer function seen between output y_i and input u_j. This form of description provides little real information about the internal structure of the system, and is known as an EXTERNAL DESCRIPTION. Only the transfer function matrix description is truly external. All other descriptions are realisations of $G(s)$ and so are internal to varying degrees of detail. The matrix fraction description (see section III.5) is the least detailed internal description and the generalised state space system matrix description (see section III.4) is the most detailed.

If it is assumed that no cancellations have occurred in the matrix product $C(sI - A)^{-1}B$ in (3.3), then the poles of $G(s)$ are the roots of $|sI - A| = 0$.

Also note that the matrix $G(s)$ is said to be STRICTLY PROPER if $G(s) \to 0$ as $s \to \infty$, and is said to be PROPER if $G(s) \to$ constant matrix as $s \to \infty$.

III.4 Rosenbrock's system matrix

The equations of a physical system may not always initially be in state space form. There are always procedures for converting a linear model to state space form. Often, however, the physical system is non-linear, and linear equations are obtained by considering small perturbations from a steady state.

In such a case the resulting equations will not usually be in state space form. For example, the equations which result from linearisation may be mixed algebraic and differential equations. Alternatively they may be in the form

$$Q\dot{x} = Ax + Bu \tag{4.1}$$

with Q singular. Therefore, it is desirable to have a more general way of describing a system.

Assume that the system satisfies linear algebraic and differential equations with constant coefficients. Taking Laplace transforms with zero initial conditions gives

$$\begin{aligned} T(s)\bar{z} &= U(s)\bar{u} \\ \bar{y} &= V(s)\bar{z} + W(s)\bar{u} \end{aligned} \tag{4.2}$$

where $z \in \mathbb{R}^r$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^l$. T, U, V, W are polynomial matrices of dimension rxr, rxl, mxr, mxl respectively. $|T(s)| \neq 0$ since otherwise the first of equations (4.2) would be indeterminate.

The ORDER, n, of the system (4.2) is the degree of the determinant of $T(s)$.

Now ~~ensure~~ assume that $r \geq n$. If this is not true, the polynomial matrices T, U, V, W are trivially expanded as follows:

$$\begin{aligned} T_1 &= \begin{bmatrix} I_{n-r} & 0 \\ 0 & T \end{bmatrix} & U_1 &= \begin{bmatrix} O_{n-r,l} \\ U \end{bmatrix} \\ V_1 &= [\, O_{m,n-r} \quad V \,] & W_1 &= W \end{aligned} \tag{4.3}$$

Therefore, assuming that $r \geq n$, the set of equations (4.2) can be written as

$$\begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} \begin{bmatrix} \bar{z} \\ -\bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{y} \end{bmatrix} \tag{4.4}$$

The $(r+m)$x$(r+l)$ polynomial matrix

$$P(s) = \left[\begin{array}{c|c} T_1(s) & U_1(s) \\ \hline -V_1(s) & W_1(s) \end{array} \right] \tag{4.5}$$

contains all the mathematical information about the system which is needed to discuss its behaviour, and is called a (ROSENBROCK) SYSTEM MATRIX in polynomial form.

Substituting the first equation in (4.2) into the second gives

$$\bar{y} = [\,V(s)T^{-1}(s)\bar{U}(s) \quad + \quad W(s)\,]\,\bar{u} \tag{4.6}$$

from which the transfer function matrix can be obtained as

$$G(s) = V(s)T^{-1}(s)U(s) + W(s) \tag{4.7}$$

Example 1

The system matrix

$$P(s) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & (s+1)^2 & s^3 \\ \hline 0 & -1 & 2-s \end{array}\right]$$

gives

$$G(s) = \frac{s^3}{(s+1)^2} + (2-s)$$

$$= \frac{3s+2}{(s+1)^2}$$

Notice that the non-vanishing of $W(s)$ does not prevent $G(s)$ from being proper. Indeed, there is no easy way to tell, from a general polynomial system matrix, whether $G(s)$ is proper or not.

Consider now the system in state space form (see (3.1) section III.3). Equations (3.1) may be written as

$$\begin{bmatrix} sI_n - A & B \\ -C & D(s) \end{bmatrix} \begin{bmatrix} \bar{x} \\ -\bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{y} \end{bmatrix} \tag{4.8}$$

The $(n+m)$x$(n+l)$ polynomial matrix

$$P(s) = \left[\begin{array}{c|c} sI_n - A & B \\ \hline -C & D(s) \end{array}\right] \tag{4.9}$$

contains all the mathematical information about the system which is needed to discuss its behaviour, and is called a (ROSENBROCK) SYSTEM MATRIX in state space form.

Consider again equation (4.7). For the special case of $P(s)$ in state space form, the transfer function matrix $G(s)$ becomes the relationship given previously in (3.3).

Conventional state space theory deals essentially with the strictly proper part of the transfer function matrix

$$\begin{aligned} G(s) &= G_{spr} + G_{pol} \\ &= C(sI - A)^{-1}B + D(s) \end{aligned} \tag{4.10}$$

Generalised state space theory considers the polynomial part of $G(s)$ on the same basis as the strictly proper part i.e.

$$G(s) = C(sE - A)^{-1}B + D \tag{4.11}$$

In linear systems theory, standard matrix theory may be considered deficient. This is the failure to consider the point at infinity on an equal basis with the rest of the points in the frequency domain. The generalised theory of linear systems arises from an attempt to consider the point at infinity on the same basis as the finite points of the frequency domain. This arose from the recognition that linear systems may exhibit significant infinite frequency behaviour of an impulsive nature.

The simplest systems exhibiting significant infinite frequency dynamical behaviour are the GENERALISED STATE SPACE SYSTEMS (gss systems). Such systems give rise to a system matrix of the form (Rosenbrock 1974, Verghese 1978)

$$P(s) = \left[\begin{array}{c|c} sE - A & B \\ \hline -C & D \end{array}\right] \tag{4.12}$$

where E, A, B, C are constant matrices, (E may or may not be singular), and $|sE - A| \neq 0$.

The generalised state space form (4.12) corresponds to the pair of equations

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{4.13}$$

The pair of equations (4.13) is also known as the descriptor form.

III.5 The matrix fraction description

One further form of multivariable system description, known as the matrix fraction description (or MFD) can now be introduced. This is an interesting form of system model, which has led to the development of improved design algorithms. It is a natural extension of the classical single-input single-output system transfer function model.

Assume that $G(s)$ is a strictly proper lxm matrix. Let $G(s)$ be expressed as

$$G(s) = \frac{N(s)}{d(s)} \tag{5.1}$$

where $N(s)$ is an lxm polynomial matrix, and $d(s)$ is the monic least common denominator of all elements of $G(s)$. $G(s)$ can be written as the MATRIX FRACTIONS

$$\begin{aligned} G(s) &= N(s)\,[d(s)I_m]^{-1} && \sim \text{right fraction} \\ &= [d(s)I_l]^{-1}\,N(s) && \sim \text{left fraction} \end{aligned} \tag{5.2}$$

Note that equations (5.2) are analogous to the form given by equation (4.7) with $U(s) = I_m$ and $V(s) = I_l$. Also note that it is always possible to write $G(s)$ in the form of (5.2) i.e. matrix fraction descriptions always exist.

Example 1

This example illustrates the construction of an MFD. Consider the strictly proper transfer function matrix

$$G(s) = \begin{bmatrix} \frac{1}{(s-1)^2} & \frac{1}{(s-1)(s+3)} \\ \frac{-6}{(s-1)(s+3)^2} & \frac{s-2}{(s+3)^2} \end{bmatrix}$$

$G(s)$ will now be expressed as

$$G(s) = \frac{N(s)}{d(s)}$$

Now,

$$\begin{aligned} d(s) &= (s-1)^2(s+3)^2 \\ &= s^4 + 4s^3 - 2s^2 - 12s + 9 \end{aligned}$$

Hence,

$$\begin{aligned} G(s) &= \frac{1}{d(s)} \begin{bmatrix} (s+3)^2 & (s-1)(s+3) \\ -6(s-1) & (s-2)(s-1)^2 \end{bmatrix} \\ &= \frac{1}{d(s)} \begin{bmatrix} s^2+6s+9 & s^2+2s-3 \\ -6s+6 & s^3-4s^2+5s-2 \end{bmatrix} \\ &= \frac{N(s)}{d(s)} \end{aligned}$$

Matrix fraction descriptions (MFD s) are not, in general, unique. There exist many left and right MFD s of a given $G(s)$. Consider the following:

$$G(s) = D_L^{-1}(s)N_L(s) \tag{5.3}$$

where

$$\begin{aligned} D_L(s) &= d(s)I_l \\ N_L(s) &= N(s) \end{aligned} \tag{5.4}$$

The degree δ of the denominator matrix $D_L(s)$ is

$$\delta\left\{|D_L(s)|\right\} = lr \tag{5.5}$$

where r is the degree of $d(s)$, and $D_L(s)$ has dimension lxl. Note also that the degree of an MFD is

$$\text{deg of an MFD} = \text{deg det } D_L(s) \quad (\text{or } \delta\left\{|D_L(s)|\right\}) \tag{5.6}$$

Now multiply D_L and N_L on the left by any non-singular polynomial matrix $W^{-1}(s)$ such that

$$\begin{aligned} \bar{D}_L(s) &= W^{-1}(s)D_L(s) \\ \bar{N}_L(s) &= W^{-1}(s)N_L(s) \end{aligned} \tag{5.7}$$

Then

$$\begin{aligned} G(s) &= D_L^{-1}(s)N_L(s) \\ &= \left(W(s)\bar{D}_L(s)\right)^{-1} W(s)\bar{N}_L(s) \\ &= \bar{D}_L^{-1}(s)\bar{N}_L(s) \end{aligned} \tag{5.8}$$

which is also an MFD of $G(s)$.

Now rewrite equations (5.7) as

$$\begin{aligned} N_L(s) &= W(s)\bar{N}_L(s) \\ D_L(s) &= W(s)\bar{D}_L(s) \end{aligned} \tag{5.9}$$

(Note here that $W(s)$ is a left divisor of $N_L(s)$ and $D_L(s)$.) Then,

$$\begin{aligned} \delta\left\{|D_L(s)|\right\} &= \delta\left\{|\bar{D}_L(s)|\right\} + \delta\left\{|W(s)|\right\} \\ \delta\left\{|D_L(s)|\right\} &\geq \delta\left\{|\bar{D}_L(s)|\right\} \end{aligned} \tag{5.10}$$

In other words, the degree of the MFD (i.e. the degree of the determinant of the denominator matrix) can be reduced by removing left divisors of the numerator and denominator matrices. Therefore, a minimum degree MFD can be obtained by removing the greatest common left divisor of $N_L(s)$ and $D_L(s)$.

An MFD $G(s) = D^{-1}(s)N(s)$ is said to be IRREDUCIBLE if $N(s)$ and $D(s)$ are left coprime. Irreducible MFD s are not unique, because if $D^{-1}(s)N(s)$ is irreducible, so is $D^{-1}(s)W(s)\left[N^{-1}(s)W(s)\right]^{-1}$ for any unimodular $W(s)$.

Similar results can be stated with respect to right MFD s.

CHAPTER IV

SYSTEM STRUCTURE

CHAPTER IV

SYSTEM STRUCTURE

IV.1 Introduction

This chapter discusses various results concerning the structure of a system. Firstly the idea of decoupling zeros is introduced, and the definition of system poles and zeros is given. It is noted that the Rosenbrock (1973) definition of the zeros of a system is incorrect, and the correct definition of the zeros of a system as given by Rosenbrock (1974) is stated. Also discussed is the infinite system zero, which is seen to be a natural extension of the finite system zero defined by Rosenbrock (1974). Finally, in this chapter the matrix pencil is defined.

IV.2 System poles and zeros

Consider a system which satisfies linear algebraic and differential equations with constant coefficients. Taking Laplace transforms with zero initial conditions gives

$$\begin{aligned} T(s)\bar{z} &= U(s)\bar{u} \\ \bar{y} &= V(s)\bar{z} + W(s)\bar{u} \end{aligned} \tag{2.1}$$

where $z \in \mathbb{R}^r$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^l$. T, U, V, W are polynomial matrices of dimension rxr, rxl, mxr, mxl respectively, and $|T(s)| \neq 0$. Let

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \tag{2.2}$$

be the Rosenbrock system matrix. In order to obtain definitions of poles and zeros at infinity consistent with the definitions of their finite counterparts, it is usual (Verghese 1978) to work with an alternative form of (2.2). This is called the NORMALISED FORM of the system matrix $P(s)$ which is denoted $\mathfrak{P}(s)$ and defined as follows. Equations (2.1) may be written in the following form:

$$\begin{bmatrix} T(s) & U(s) & O_{rm} \\ -V(s) & W(s) & I_m \\ O_{lr} & -I_l & O_{lm} \end{bmatrix} \begin{bmatrix} \bar{z}(s) \\ -\bar{u}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} 0_{rl} \\ 0_{ml} \\ I_l \end{bmatrix} \bar{u}(s)$$

$$\bar{y}(s) = [\,O_{mr} \quad 0_{ml} \quad I_m\,] \begin{bmatrix} \bar{z}(s) \\ -\bar{u}(s) \\ \bar{y}(s) \end{bmatrix} \tag{2.3}$$

The NORMALISED FORM $\mathfrak{P}(s)$ of the system matrix $P(s)$ is defined by

$$\mathfrak{P}(s) = \left[\begin{array}{c|c} \mathfrak{T}(s) & \mathfrak{U} \\ \hline -\mathfrak{V} & O \end{array}\right] = \left[\begin{array}{ccc|c} T(s) & U(s) & 0 & 0 \\ -V(s) & W(s) & I_m & 0 \\ 0 & -I_l & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right] \tag{2.4}$$

The following definitions may now be stated:

Definition 1 (Rosenbrock 1970, 1974 b, Verghese 1978)

The INPUT DECOUPLING (i.d.) ZEROS of a linear multivariable system Σ at $s_o = \infty$ (at $s_0 \in \mathbb{C}$) are the zeros at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) of the polynomial matrix

$$[\,\mathfrak{T}(s) \quad \mathfrak{U}\,]$$

Definition 2 (Rosenbrock 1970, 1974 b, Verghese 1978)

The OUTPUT DECOUPLING (o.d.) ZEROS of a linear multivariable system Σ at $s_o = \infty$ (at $s_0 \in \mathbb{C}$) are the zeros at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) of the polynomial matrix

$$\begin{bmatrix} \mathfrak{T}(s) \\ -\mathfrak{V} \end{bmatrix}$$

Definition 3 (Rosenbrock 1970, 1974 b, Verghese 1979)

The INPUT-OUTPUT DECOUPLING (i.o.d.) ZEROS of a linear multivariable system Σ at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) are those output decoupling zeros of Σ which disappear when the input decoupling zeros are eliminated.

Result 1

The finite i.d. and o.d. zeros of a normalised system are those of the original system, namely, the zeros of

$$[T(s) \quad U(s)] \text{ and } \begin{bmatrix} T(s) \\ -V(s) \end{bmatrix} \text{ respectively.}$$

Proof

From definitions 1 and 2, the finite i.d. and o.d. zeros of a normalised system are those of the polynomial matrices

$$[\mathfrak{T}(s) \quad \mathfrak{U}] \text{ and } \begin{bmatrix} \mathfrak{T}(s) \\ -\mathfrak{V} \end{bmatrix} \text{ respectively.}$$

From (2.4), it can be seen that the finite i.d. and o.d. zeros of a normalised system are those of the original system i.e. the zeros of

$$[T(s) \quad U(s)] \text{ and } \begin{bmatrix} T(s) \\ -V(s) \end{bmatrix} \text{ respectively.}$$

Theorem 1

For a linear multivariable system Σ,

$$\begin{aligned} \{\text{decoupling zeros at } s_0\} = &\{\text{input decoupling zeros at } s_0\} + \\ &\{\text{output decoupling zeros at } s_0\} - \\ &\{\text{input} - \text{output decoupling zeros at } s_0\} \quad \textbf{(2.5)} \end{aligned}$$

whether $s_0 = \infty$ or $s_0 \in \mathbb{C}$.

Example 1

Consider the system matrix

$$P(s) = \left[\begin{array}{cc|c} s^2(s+1) & s(s+2) & 0 \\ 0 & s+2 & (s+2)^2 \\ \hline 0 & 1 & 0 \end{array}\right]$$

In this example, the finite and infinite decoupling zeros of $P(s)$ will be found, and result 1 will be verified.

Firstly consider the finite decoupling zeros of $P(s)$. From definitions 1, 2 and 3, the finite decoupling zeros are found using the normalised form of the system matrix. However, it will now be seen that for the finite case, the decoupling zeros may be found using the original system matrix i.e. the following verifies result 1.

Firstly produce the finite decoupling zeros of $P(s)$ using the original system matrix. Produce the Smith forms of the matrices $[T(s) \quad U(s)]$ and $\begin{bmatrix} T(s) \\ -V(s) \end{bmatrix}$, as follows:

$$[T(s) \quad U(s)] = \begin{bmatrix} s^2(s+1) & s(s+2) & 0 \\ 0 & s+2 & (s+2)^2 \end{bmatrix}$$

$$\xrightarrow[(row\ 1) - (s \times row\ 2)]{new\ row\ 1 =} \begin{bmatrix} s^2(s+1) & 0 & -s(s+2)^2 \\ 0 & s+2 & (s+2)^2 \end{bmatrix}$$

$$\xrightarrow[[(s+2) \times col.\ 2] - (col.\ 3)]{new\ col.\ 3 =} \begin{bmatrix} s^2(s+1) & 0 & s(s+2)^2 \\ 0 & s+2 & 0 \end{bmatrix}$$

Hence, using the original system matrix, $P(s)$ has finite i.d. zeros, $\beta = 0, -2$. For $s = 0, -2$, the rank of $[T(s) \quad U(s)]$ is reduced.

$$\begin{bmatrix} T(s) \\ -V(s) \end{bmatrix} = \begin{bmatrix} s^2(s+1) & s(s+2) \\ 0 & s+2 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow[(row\ 1) - (s \times row\ 2)]{new\ row\ 1 =} \begin{bmatrix} s^2(s+1) & 0 \\ 0 & s+2 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{interchange \\ row\ 2\ and\ row\ 3}]{\substack{new\ row\ 2 = \\ (row\ 2) - [(s+2) \times row\ 3]}} \begin{bmatrix} s^2(s+1) & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, using the original system matrix, $P(s)$ has finite o.d. zeros, $\gamma = 0, 0, -1$. Putting $s = 0, 0, -1$ reduces the rank of $\begin{bmatrix} T(s) \\ -V(s) \end{bmatrix}$.

Now produce the finite decoupling zeros using the normalised form of the system matrix $P(s)$:

$$\mathfrak{P}(s) = \begin{bmatrix} \mathfrak{T}(s) & | & \mathfrak{U} \\ \text{—} & \text{—} & \text{—} \\ -\mathfrak{V} & | & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s^2(s+1) & s(s+2) & 0 & 0 & | & 0 \\ 0 & s+2 & (s+2)^2 & 0 & | & 0 \\ 0 & 1 & 0 & 1 & | & 0 \\ 0 & 0 & -1 & 0 & | & 1 \\ \text{—} & \text{—} & \text{—} & \text{—} & & \text{—} \\ 0 & 0 & 0 & -1 & | & 0 \end{bmatrix}$$

Produce the Smith forms of the matrices $[\mathfrak{T}(s) \quad \mathfrak{U}]$ and $\begin{bmatrix} \mathfrak{T}(s) \\ -\mathfrak{V} \end{bmatrix}$, as follows:

$$[\mathfrak{T}(s) \quad \mathfrak{U}] = \begin{bmatrix} s^2(s+1) & s(s+2) & 0 & 0 & 0 \\ 0 & s+2 & (s+2)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{new\ col.\ 3 = \\ (col.\ 3) +\ (col.\ 5)}]{\substack{new\ col.\ 2 = \\ (col.\ 2) - (col.\ 4)}} \begin{bmatrix} s^2(s+1) & s(s+2) & 0 & 0 & 0 \\ 0 & s+2 & (s+2)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[(row\ 1) -\ (s \text{ x } row\ 2)]{new\ row\ 1 =} \begin{bmatrix} s^2(s+1) & 0 & -s(s+2)^2 & 0 & 0 \\ 0 & s+2 & (s+2)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[[(s+2) \text{ x } col.\ 2] -\ (col.\ 3)]{new\ col.\ 3 =} \begin{bmatrix} s^2(s+1) & 0 & s(s+2)^2 & 0 & 0 \\ 0 & s+2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, using the normalised system matrix, $P(s)$ has finite i.d. zeros, $\beta' = 0, -2$.

$$\begin{bmatrix} \mathfrak{T}(s) \\ -\mathfrak{V} \end{bmatrix} = \begin{bmatrix} s^2(s+1) & s(s+2) & 0 & 0 \\ 0 & s+2 & (s+2)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{new\ row\ 3 = \\ (row\ 3) +\ (row\ 5)}]{\substack{new\ row\ 2 = \\ (row\ 2)\ +\ [(s+2)^2 \text{ x } row\ 4]}} \begin{bmatrix} s^2(s+1) & s(s+2) & 0 & 0 \\ 0 & s+2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{new\ row\ 2 = \\ (row\ 2) - [(s+2) \times row\ 3]}]{\substack{new\ row\ 1 = \\ (row\ 1) - (s \times row\ 2)}} \begin{bmatrix} s^2(s+1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Hence, using the normalised system matrix, $P(s)$ has finite o.d. zeros, $\gamma' = 0, 0, -1$.

Hence, it has been seen that the finite i.d. and o.d. zeros of a normalised system are those of the original system i.e. result 1 has been verified.

Now remove the i.d. zeros, $\beta = 0, -2$ from $P(s)$. At the same time this removes one of the o.d. zeros, $\gamma = 0$. Hence, $P(s)$ has one i.o.d. zero, $\delta = 0$. Therefore,

$$\begin{aligned} \{\text{finite decoupling zeros}\} &= \{0, -2\} + \{0, 0, -1\} - \{0\} \\ &= \{0, 0, -1, -2\} \end{aligned}$$

The infinite i.d. and o.d. zeros of the original system (2.1) are defined as those of the normalised system (2.3).

$$\mathfrak{P}(s) = \left[\begin{array}{c|c} \mathfrak{T}(s) & \mathfrak{U} \\ \hline -\mathfrak{V} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} s^2(s+1) & s(s+2) & 0 & 0 & 0 \\ 0 & s+2 & (s+2)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

The infinite i.d. zeros of $P(s)$ are the zeros at $w = 0$ of

$$[\mathfrak{T}(\tfrac{1}{w}) \quad \mathfrak{U}] = \begin{bmatrix} \frac{1}{w^2}(\frac{1}{w}+1) & \frac{1}{w}(\frac{1}{w}+2) & 0 & 0 & 0 \\ 0 & \frac{1}{w}+2 & (\frac{1}{w}+2)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{w^3}(1+w) & \frac{1}{w^2}(1+2w) & 0 & 0 & 0 \\ 0 & \frac{1}{w}(1+2w) & \frac{1}{w^2}(1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Now express the above matrix in relatively prime form as follows:

$$[\mathfrak{T}(\tfrac{1}{w}) \quad \mathfrak{U}] = \begin{bmatrix} \frac{1}{w^3} & 0 & 0 & 0 \\ 0 & \frac{1}{w^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w^3 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

The numerator is used to determine whether $P(s)$ has any infinite i.d. zeros. Perform elementary row and column operations to produce the Smith form of the numerator:

$$\begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new col. } 3 = \\ (\text{col. } 3) \;+\; (\text{col. } 5)}]{\substack{\text{new col. } 2 = \\ (\text{col. } 2) \;-\; (\text{col. } 4)}} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[(\cancel{\text{col.}}\,\text{row } 1) \;-\; (\cancel{\text{col.}}\,\text{row } 2)]{\text{new } \cancel{\text{col.}}\,\text{row } 1 =} \begin{bmatrix} 1+w & 0 & -(1+2w)^2 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

At $w = 0$, the rank of the above matrix is not reduced. Therefore, there are no zeros at $w = 0$ of $[\mathfrak{T}(\frac{1}{w}) \quad \mathfrak{U}]$. Hence, the system matrix $P(s)$ has no infinite i.d. zeros.

Similarly, the infinite o.d. zeros of $P(s)$ are the zeros at $w = 0$ of

$$\begin{bmatrix} \mathfrak{T}(\frac{1}{w}) \\ -\mathfrak{V} \end{bmatrix} = \begin{bmatrix} \frac{1}{w^3}(1+w) & \frac{1}{w^2}(1+2w) & 0 & 0 \\ 0 & \frac{1}{w}(1+2w) & \frac{1}{w^2}(1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Expressing the above matrix in relatively prime form gives

$$\begin{bmatrix} \mathfrak{T}(\frac{1}{w}) \\ -\mathfrak{V} \end{bmatrix} \doteq \begin{bmatrix} \frac{1}{w^3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{w^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} w^3 & 0 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The numerator is used to determine whether $P(s)$ has any infinite o.d. zeros. Perform elementary row and column operations on the numerator to give its Smith form:

$$\begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new row } 3 = \\ (\text{row } 3) \;-\; (\text{row } 5)}]{\substack{\text{new row } 2 = \\ (\text{row } 2) \;+\; [(1+2w)^2 \text{ x row } 4]}} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new row } 2 = \\ (\text{row } 2) \;-\; [w(1+2w) \text{ x row } 3]}]{\substack{\text{new row } 1 = \\ (\text{row } 1) \;-\; (\text{row } 2)}} \begin{bmatrix} 1+w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

At $w = 0$, the rank of the above matrix is not reduced. Therefore, there are no zeros at $w = 0$ of $\begin{bmatrix} \mathfrak{T}(s) \\ -\mathfrak{V} \end{bmatrix}$. Hence, the system matrix $P(s)$ has no infinite o.d.

zeros. Also, $P(s)$ has no infinite i.o.d. zeros. Therefore,

$$\{\text{infinite decoupling zeros}\} = \{\phi\}$$

The following definitions and results of system poles and zeros may now be given:

Definition 4 (Rosenbrock 1973)

The SYSTEM POLES of Σ in $\mathbb{C}$ are defined as the zeros of $T(s)$ in $\mathbb{C}$ and the SYSTEM POLES of Σ at $s = \infty$ are the zeros of $\mathfrak{T}(s)$ at $s = \infty$.

Result 2 (Rosenbrock 1973, 1974)

The set of SYSTEM POLES of Σ at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) is given as the union of the set of poles at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) of $G(s)$ and the set of decoupling zeros of Σ at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$), i.e.

$$\begin{aligned}\{\text{System poles at } s_0\} &= \{\text{Transfer function poles at } s_0\} + \{\text{i.d. zeros at } s_0\} \\ &\quad + \{\text{o.d. zeros at } s_0\} - \{\text{i.o.d. zeros at } s_0\} \qquad \textbf{(2.6)}\end{aligned}$$

whether $s_0 = \infty$ or $s_0 \in \mathbb{C}$.

Result 3 (Rosenbrock 1973, 1974, Ferreira P.M.G., 1980)

The set of SYSTEM ZEROS of Σ at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) is given as the union of the set of zeros at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) of $G(s)$ and the set of decoupling zeros of Σ at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$), i.e.

$$\begin{aligned}\{\text{System zeros at } s_0\} &= \{\text{Transfer function zeros at } s_0\} + \{\text{i.d. zeros at } s_0\} \\ &\quad + \{\text{o.d. zeros at } s_0\} - \{\text{i.o.d. zeros at } s_0\} \qquad \textbf{(2.7)}\end{aligned}$$

whether $s_0 = \infty$ or $s_0 \in \mathbb{C}$.

The zeros of a system ~~is~~ are now defined. Firstly consider Rosenbrock's (1973) definition as follows:

Definition 5 (Rosenbrock 1973)

(a) Let the $(r+m)$x$(r+l)$ system matrix

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \tag{2.8}$$

have Smith form

$$S(s) = \begin{bmatrix} \text{diag } [\varepsilon_i(s)] & 0 \\ 0 & 0 \end{bmatrix} \tag{2.9}$$

where some of the zero blocks may be absent. Then the SYSTEM ZEROS of Σ are the zeros of the polynomials $\varepsilon_i(s)$, taken all together.

The above definition of the zeros of a system is faulty. In particular, result 3 is incorrect with this definition. Rosenbrock (1974) proposed the following definition to replace definition 5:

Definition 6 (Rosenbrock 1974)

(a$'$) Consider the $(r+k)$x$(r+k)$ minors of the system matrix (2.8) which have the form

$$P^{1,2,\ldots,r,r+i_1,\ldots,r+i_k}_{1,2,\ldots,r,r+j_1,\ldots,r+j_k} \tag{2.10}$$

That is to say, the minors are formed from rows $1,2,\ldots,r,r+i_1,\ldots,r+i_k$ and columns $1,2,\ldots,r,r+j_1,\ldots,r+j_k$. Let p satisfying $0 \le p \le \min\,(l,m)$ be the largest value of k for which there is a minor of this form not identically zero. Let $\phi(s)$ be the monic greatest common divisor of all those minors (2.10) having $k=p$ which are not identically zero. Then the zeros of ϕ are the ZEROS OF THE SYSTEM.

This new set $\{a'\}$ includes the set $\{a\}$ as defined before, but is in general larger. If P is square and $|P(s)| \not\equiv 0$, the two sets coincide, which was the source of the error. It will now be seen in the following example that definition 5 is inconsistent with result 3.

Example 2

Consider the system matrix

$$P(s) = \left[\begin{array}{c|cc} (s-1)(s-2) & 1 & 1 \\ \hline s-1 & 1 & 0 \end{array}\right]$$

$P(s)$ has no i.d. zeros and one o.d. zero at $s = 1$. Also, $P(s)$ has no i.o.d. zeros. Now produce the transfer function zeros as follows:

$$\begin{aligned} G(s) &= VT^{-1}U + W \\ &= -\frac{(s-1)}{(s-1)(s-2)}[1 \quad 1] + [1 \quad 0] \\ &= \left[\tfrac{(s-3)}{(s-2)} \quad -\tfrac{1}{(s-2)} \right] \end{aligned}$$

Therefore, $P(s)$ has no transfer function zeros. Hence, result 3 gives

$$\{\text{Finite system zeros}\} = \{1\}$$

Now, from definition 5, the system zeros are the zeros of the polynomials $\varepsilon_i(s)$ in the Smith form of $P(s)$. $P(s)$ has Smith form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

which gives no system zeros. Therefore, definition 5 is inconsistent with result 3.

However, definition 6 gives the zeros of the system as the zeros of

$$\left[\begin{array}{c|c} (s-1)(s-2) & 1 \\ \hline s-1 & 0 \end{array}\right]$$

which has Smith form

$$\begin{bmatrix} 1 & 0 \\ 0 & (s-1) \end{bmatrix}$$

Hence, it can be seen that definition 6 gives a zero at $s = 1$ which is consistent with result 3. Therefore, the example shows that the correct definition of the zeros of a system is given by definition 6.

Now define the following:

Definition 7 (Rosenbrock 1970, Pugh and Ratcliffe 1979)

The TRANSMISSION POLES AND ZEROS of Σ at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) are the poles and zeros at $s_0 = \infty$ (at $s_0 \in \mathbb{C}$) of the transfer function $G(s)$.

Infinite system zeros will now be defined and a result is stated which is an extension for the zeros at infinity of Rosenbrock's (1974) definition of system zeros. Consider the normalised system matrix $\mathfrak{P}(s)$ defined as in (2.4):

$$\mathfrak{P}(s) = \begin{bmatrix} \mathfrak{T}(s) & \mathfrak{U} \\ -\mathfrak{V} & 0 \end{bmatrix} \tag{2.11}$$

Definition 8 (Ferreira 1980)

The INFINITE ZEROS of the original system are defined as the infinite system zeros of the normalised system matrix (2.11). Note that the system zeros are defined using the corrected definition of the zeros of a system as given in Rosenbrock's paper (1974) (see definition 6).

Result 4 (Ferreira 1980)

Consider $\mathfrak{P}(\frac{1}{w})$ with $\mathfrak{P}(s)$ defined as in (2.11). Let $r + l + m + q$, where $0 \leq q \leq \min\,(l, m)$, be the normal rank of $\mathfrak{P}(.)$. Let $\bar{N}(w)\bar{D}^{-1}(w)$ be a right coprime factorisation of $\mathfrak{P}(\frac{1}{w})$. Consider all the $(r + l + m + q)$-order minors of $\bar{N}(w)$ which contain the first $(r + l + m)$ rows and columns . Let $d(w)$ be a greatest common divisor of all these minors. Then

$$\left\{\begin{matrix}\text{Infinite system zeros of the original} \\ \text{(and of the normalised) system}\end{matrix}\right\} = \left\{\text{Zeros at the origin of } d(w)\right\} \tag{2.12}$$

Example 3

The finite and infinite system poles and zeros will now be constructed according to definitions 4, 6 and 8, and used to verify results 2 and 3.

Consider again example 1 with the system matrix

$$P(s) = \left[\begin{array}{cc|c} s^2(s+1) & s(s+2) & 0 \\ 0 & s+2 & (s+2)^2 \\ \hline 0 & 1 & 0 \end{array}\right]$$

As has been seen in example 1, $P(s)$ has the following decoupling zeros:

$$\{\text{finite decoupling zeros}\} = \{0, 0, -1, -2\}$$

$$\{\text{infinite decoupling zeros}\} = \{\phi\}$$

Firstly consider the finite system poles and zeros. From definition 4, the finite system poles are the zeros of

$$T(s) = \begin{bmatrix} s^2(s+1) & s(s+2) \\ 0 & s+2 \end{bmatrix}$$

which has Smith form

$$\begin{bmatrix} s^2(s+1) & 0 \\ 0 & s+2 \end{bmatrix}$$

Hence, $P(s)$ has finite system poles at $s = 0, 0, -1, -2$. From definition 6, the finite system zeros are the zeros of

$$\left[\begin{array}{cc|c} s^2(s+1) & s(s+2) & 0 \\ 0 & s+2 & (s+2)^2 \\ \hline 0 & 1 & 0 \end{array}\right]$$

Perform elementary row and column operations to give the Smith form of the above matrix:

$$\begin{bmatrix} s^2(s+1) & s(s+2) & 0 \\ 0 & s+2 & (s+2)^2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{(row 1)} \ - [s(s+2) \text{ x row 3}]]{\text{new row 1} =} \begin{bmatrix} s^2(s+1) & 0 & 0 \\ 0 & s+2 & (s+2)^2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{col. 2 and col. 3}]{\text{interchange}} \begin{bmatrix} s^2(s+1) & 0 & 0 \\ 0 & (s+2)^2 & s+2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{(row 2) − [(s + 2) x row 3]}]{\text{new row 2 =}} \begin{bmatrix} s^2(s+1) & 0 & 0 \\ 0 & (s+2)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $P(s)$ has finite system zeros at $s = 0, 0, -1, -2, -2$.

Now form the finite system poles and zeros via the transfer function poles and zeros, and decoupling zeros. The transfer function matrix $G(s)$ is formed as follows:

$$\begin{aligned} G(s) &= V(s)T^{-1}(s)U(s) + W(s) \\ &= [0 \quad 1] \begin{bmatrix} s^2(s+1) & s(s+2) \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ (s+2)^2 \end{bmatrix} \\ &= \frac{1}{s^2(s+1)(s+2)} [0 \quad 1] \begin{bmatrix} s+2 & -s(s+2) \\ 0 & s^2(s+1) \end{bmatrix} \begin{bmatrix} 0 \\ (s+2)^2 \end{bmatrix} \\ &= s+2 \end{aligned}$$

Therefore, $P(s)$ has a finite transmission zero at $s = -2$ and no finite transmission poles. Hence,

$$\begin{aligned} \{\text{finite system zeros}\} &= \{-2\} + \{0, 0, -1, -2\} \\ &= \{0, 0, -1, -2, -2\} \\ \{\text{finite system poles}\} &= \{\phi\} + \{0. 0, -1, -2\} \\ &= \{0, 0, -1, -2\} \end{aligned}$$

It has been seen that the finite system poles and zeros formed from their definitions are the same as those formed via the finite transmission poles and zeros, and finite decoupling zeros i.e. results 2 and 3 have been verified for the finite case.

Now consider the infinite system poles and zeros. Consider the normalised system matrix

$$\mathfrak{P}(s) = \begin{bmatrix} \mathfrak{T}(s) & \mathfrak{U} \\ -\mathfrak{V} & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} s^2(s+1) & s(s+2) & 0 & 0 & 0 \\ 0 & s+2 & (s+2)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

From definition 4, the infinite system poles are the zeros at $w = 0$ of

$$\mathfrak{T}(\frac{1}{w}) = \begin{bmatrix} \frac{1}{w^2}(\frac{1}{w}+1) & \frac{1}{w}(\frac{1}{w}+2) & 0 & 0 \\ 0 & \frac{1}{w}+2 & (\frac{1}{w}+2)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{w^3}(1+w) & \frac{1}{w^2}(1+2w) & 0 & 0 \\ 0 & \frac{1}{w}(1+2w) & \frac{1}{w^2}(1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Now express $\mathfrak{T}(\frac{1}{w})$ in relatively prime form as follows:

$$\mathfrak{T}(\frac{1}{w}) = \begin{bmatrix} \frac{1}{w^3} & 0 & 0 & 0 \\ 0 & \frac{1}{w^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} w^3 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The numerator is used to determine whether $P(s)$ has any infinite system poles. Perform elementary row and column operations on the numerator to give its Smith form:

$$\begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new row 2 } = \\ (\text{row 2}) + [(1+2w)^2 \text{ x row 4}]}]{\substack{\text{new col. 2 } = \\ (\text{col. 2}) - (\text{col. 4})}} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 \\ 0 & w(1+2w) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{interchange} \\ \text{col. 3 and col. 4}}]{\substack{\text{new row 1 } = \\ (\text{row 1}) - (\text{row 2})}} \begin{bmatrix} 1+w & 0 & 0 & 0 \\ 0 & w(1+2w) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

At $w = 0$, the rank of the above matrix is reduced. Therefore, there is a zero at $w = 0$ of $\mathfrak{T}(\frac{1}{w})$. Hence, $P(s)$ has an infinite system pole.

From definition 8, the infinite system zeros are the zeros at $w = 0$ of

$$\begin{bmatrix} \mathfrak{T}(\frac{1}{w}) & \mathfrak{U} \\ -\mathfrak{V} & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} \frac{1}{w^2}(\frac{1}{w}+1) & \frac{1}{w}(\frac{1}{w}+2) & 0 & 0 & 0 \\ 0 & \frac{1}{w}+2 & (\frac{1}{w}+2)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} \frac{1}{w^3}(1+w) & \frac{1}{w^2}(1+2w) & 0 & 0 & 0 \\ 0 & \frac{1}{w}(1+2w) & \frac{1}{w^2}(1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Now express the above matrix in relatively prime form as follows:

$$\begin{bmatrix} \mathfrak{T}(\frac{1}{w}) & \mathfrak{U} \\ -\mathfrak{V} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{w^3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{w^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} w^3 & 0 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The numerator is used to determine whether $P(s)$ has any infinite system zeros. Perform elementary row and column operations on the numerator to give its

Smith form:

$$\begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new col. 3 } = \\ (\text{col. 3}) + (\text{col. 5})}]{\substack{\text{new row 3 } = \\ (\text{row 3}) + (\text{row 5})}} \begin{bmatrix} 1+w & w(1+2w) & 0 & 0 & 0 \\ 0 & w(1+2w) & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new row 2 } = \\ (\text{row 2}) - [w(1+2w) \text{ x row 3}]}]{\substack{\text{new row 1 } = \\ (\text{row 1}) - [w(1+2w) \text{ x row 3}]}} \begin{bmatrix} 1+w & 0 & 0 & 0 & 0 \\ 0 & 0 & (1+2w)^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{interchange} \\ \text{col. 4 and col. 5}}]{\substack{\text{interchange} \\ \text{col. 2 and col. 3}}} \begin{bmatrix} 1+w & 0 & 0 & 0 & 0 \\ 0 & (1+2w)^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

At $w = 0$, the rank of the above matrix is not reduced. Therefore, there are no zeros at $w = 0$ of $\mathfrak{P}(\frac{1}{w})$. Hence, $P(s)$ has no infinite system zeros.

Now form the infinite system poles and zeros via the transfer function poles and zeros, and decoupling zeros. To find the infinite transmission poles and zeros, put $s = \frac{1}{w}$ in $G(s)$:

$$G(s) = s + 2$$
$$G(\frac{1}{w}) = \frac{1}{w} + 2$$
$$= \frac{1+2w}{w}$$

Therefore, $P(s)$ has no infinite transmission zeros but has an infinite transmission pole. Hence, it has been seen that the infinite system poles and zeros formed from their definitions are the same as those formed via the infinite transmission poles and zeros, and infinite decoupling zeros i.e. results 2 and 3 have been verified for the infinite case.

The following two terms may now be defined:

Definition 9

The GENERALISED ORDER f of Σ is defined as the total number of system poles of Σ in $\mathbb{C} \cup \{\infty\}$ or equivalently, the total number of zeros of $\mathfrak{T}(s)$ in $\mathbb{C} \cup \{\infty\}$.

Definition 10 (Verghese 1978)

A system as in (2.1) is called STRONGLY IRREDUCIBLE if and only if the compound polynomial matrices

$$[\mathfrak{T}(s) \quad \mathfrak{U}] \text{ and } \begin{bmatrix} \mathfrak{T}(s) \\ -\mathfrak{V} \end{bmatrix}$$

have no finite nor infinite zeros.

IV.3 A result for infinite system zeros

In the previous section, the relationship between infinite system zeros, infinite transfer function zeros and infinite decoupling zeros was given (see result 3 section IV.2). In this section, this result will be established.

Let $r+q$, where $0 \leq q \leq \min\,(l.m)$, be the normal rank of $P(s)$ given as in (2.2) :

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \qquad \textbf{(3.1)}$$

Also, let $r+l+m+q$ be the normal rank of $\mathfrak{P}(.)$ with the normalised system matrix $\mathfrak{P}(s)$ defined as in (2.4):

$$\mathfrak{P}(s) = \begin{bmatrix} \mathfrak{T}(s) & \mathfrak{U} \\ -\mathfrak{V} & 0 \end{bmatrix} \qquad \textbf{(3.2)}$$

In Rosenbrock's paper (1974) it is seen that the finite system zeros of the original system are the zeros of the greatest common divisor of all $(r+q)$-order minors of $P(s)$ which have the first r rows and columns of $P(s)$.

In Ferreira's paper (1980) it is seen that the infinite system zeros of the original system are the zeros at the origin of the greatest common divisor of all $(r+l+m+q)$-order minors of $\bar{N}(w)$ which contain the first $(r+l+m)$ rows and columns of $\bar{N}(w)$, where $\bar{N}(w)\bar{D}^{-1}(w)$ is a right coprime factorisation of $\mathfrak{P}(\frac{1}{w})$.

In the same Rosenbrock (1974) paper, he proved that for finite frequencies

$$\begin{aligned} \{\text{System zeros}\} &= \{\text{Transfer function zeros}\} + \{\text{i.d. zeros}\} \\ &\quad + \{\text{o.d. zeros}\} - \{\text{i.o.d. zeros}\} \end{aligned} \qquad \textbf{(3.3)}$$

In this section, the 'dual' result concerning infinite system zeros and infinite transfer function zeros is established, using Ferreira's paper (1980). This 'dual' result relies on the definition of infinite system zeros. The key for the adequate definition of infinite system zeros, as well as infinite system poles and decoupling zeros, is the notion of the normalised system, introduced by Verghese (1978). In Ferreira's paper (1980) this definition is not immediately clear and will be stated here as follows:

Definition 1 (Ferreira 1980)

The INFINITE ZEROS of the original system are defined as the infinite system zeros of the normalised system matrix (3.2), where the term 'system zeros' is defined using the corrected definition of the zeros of a system as given in Rosenbrock's paper (1974).

Using this definition and the result given above concerning infinite system zeros, which is a natural extension of the finite system zeros defined by Rosenbrock (1974), the relationship between infinite system zeros, infinite transfer function zeros and infinite decoupling zeros is now established.

Firstly, the infinite i.d. and o.d. zeros of the original system matrix $P(s)$ are given as follows. Let $N(w)D^{-1}(w)$ be a right coprime factorisation of $\mathfrak{T}(\frac{1}{w})$. The infinite i.d. zeros of the normalised system matrix (3.2) and hence the original system matrix (3.1) are equal in number to the zeros at the origin of

$$[\,N(w)D^{-1}(w) \quad \mathfrak{U}\,] = [\,N(w) \quad \mathfrak{U}\,]\begin{bmatrix} D(w) & 0 \\ 0 & I \end{bmatrix}^{-1} \tag{3.4}$$

Now, since

$$[\,N(w) \quad \mathfrak{U}\,] \text{ and } \begin{bmatrix} D(w) & 0 \\ 0 & I \end{bmatrix}$$

are right coprime, the zeros at the origin of $[\,N(w)D^{-1}(w) \quad \mathfrak{U}\,]$ are those of $[\,N(w) \quad \mathfrak{U}\,]$ (see Wolovich 1974). Now let $X(w)$ be a greatest common left divisor of $N(w)$ and $\mathfrak{U}$ and define $N_1(w)$ and $\mathfrak{U}_0(w)$ such that

$$[\,N(w) \quad \mathfrak{U}\,] = X(w)\,[\,N_1(w) \quad \mathfrak{U}_0(w)\,] \tag{3.5}$$

It is now clear that the number of zeros at the origin of $|X(w)|$ is equal to the number of infinite i.d. zeros of the original system.

Now consider the infinite o.d. zeros of the original system. Similarly, the infinite o.d. zeros of the normalised system matrix (3.2) and hence the original system matrix (3.1) are equal in number to the zeros at the origin of

$$\begin{bmatrix} N_1(w)D^{-1}(w) \\ -\mathfrak{V} \end{bmatrix} = \begin{bmatrix} N_1(w) \\ -\mathfrak{V}D(w) \end{bmatrix} D^{-1}(w) \tag{3.6}$$

with $\begin{bmatrix} N_1(w) \\ -\mathfrak{V}D(w) \end{bmatrix}$ and $D(w)$ right coprime. Now let $Y(w)$ be a greatest common right divisor of $N_1(w)$ and $\mathfrak{V}D(w)$ and define $N_0(w)$ and $\mathfrak{V}_0(w)$ such that

$$\begin{bmatrix} N_1(w) \\ \\ -\mathfrak{V}D(w) \end{bmatrix} = \begin{bmatrix} N_0(w) \\ \\ -\mathfrak{V}_0(w) \end{bmatrix} Y(w) \tag{3.7}$$

It is now seen that the number of infinite o.d. zeros (which are not simultaneously i.d. zeros) is given by the number of zeros at the origin of $|Y(w)|$.

The infinite transfer function zeros are now given. Consider the transfer function of the original system. It is the same as that of the normalised system.

$$\begin{aligned} G(s) &= V(s)T^{-1}(s)U(s) + W(s) \\ &= \mathfrak{V}\mathfrak{T}^{-1}(s)\mathfrak{U} \end{aligned} \tag{3.8}$$

Hence, the infinite zeros of the transfer function are determined by the zeros at the origin of $\mathfrak{V}\mathfrak{T}^{-1}(\frac{1}{w})\mathfrak{U}$. Now

$$\begin{aligned} G(\frac{1}{w}) &= \mathfrak{V}\mathfrak{T}^{-1}(\frac{1}{w})\mathfrak{U} \\ &= \mathfrak{V}D(w)N^{-1}(w)\mathfrak{U} \\ &= \mathfrak{V}_0(w)Y(w)Y^{-1}(w)N_0^{-1}(w)X^{-1}(w)X(w)\mathfrak{U}_0(w) \\ &= \mathfrak{V}_0(w)N_0^{-1}(w)\mathfrak{U}_0(w) \end{aligned} \tag{3.9}$$

Since $N_0(w)$ and $\mathfrak{U}_o(w)$ (resp. $\mathfrak{V}_0(w)$) are left (resp. right) coprime, the zeros of $G(\frac{1}{w})$ i.e. the infinite transfer function zeros are those of

$$\begin{bmatrix} N_0(w) & \mathfrak{U}_0(w) \\ \\ -\mathfrak{V}_0(w) & 0 \end{bmatrix}$$

i.e. the zeros of the greatest common divisor of all its $(r + l + m + q)$- order minors.

A matrix is now formed which is appropriate for the calculation of the infinite system zeros. From definition 1, the infinite system zeros are the zeros at $w = 0$ of

$$\mathfrak{P}(\frac{1}{w}) = \begin{bmatrix} N(w)D^{-1}(w) & \mathfrak{U} \\ -\mathfrak{V} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \end{bmatrix} \begin{bmatrix} D(w) & 0 \\ 0 & I \end{bmatrix}^{-1} \tag{3.10}$$

It will be shown here that $\begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \end{bmatrix}$ and $\begin{bmatrix} D(w) & 0 \\ 0 & I \end{bmatrix}$ are right coprime, although this is not shown in Ferreira's paper. Write the two matrices as follows and perform elementary row operations:

$$\begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \\ D(w) & 0 \\ 0 & I \end{bmatrix} \xrightarrow[\text{(row 1)} - (\mathfrak{U} \text{ x row 4})]{\text{new row 1} =} \begin{bmatrix} N(w) & 0 \\ -\mathfrak{V}D(w) & 0 \\ D(w) & 0 \\ 0 & I \end{bmatrix}$$

$$\xrightarrow[\text{(row 2)} + (\mathfrak{V} \text{ x row 3})]{\text{new row 2} =} \begin{bmatrix} N(w) & 0 \\ 0 & 0 \\ D(w) & 0 \\ 0 & I \end{bmatrix}$$

Now $N(w)D^{-1}(w)$ is right coprime. Therefore

$$\begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \end{bmatrix} \text{ and } \begin{bmatrix} D(w) & 0 \\ 0 & I \end{bmatrix}$$

are right coprime. Hence the matrix

$$\begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \end{bmatrix}$$

is appropriate for the calculation of the infinite system zeros.

The relationship between infinite system zeros, infinite transfer function zeros and infinite decoupling zeros can now be given . Firstly state the following lemma:

Lemma 1

Let A, B be matrices over a ring R, respectively lxn, nxm. Then the rxr minors of the product matrix $C = AB$ are

$$C^{i_1,i_2,\ldots,i_r}_{j_1,j_2,\ldots,j_r} = \sum\nolimits_k A^{i_1,i_2,\ldots,i_r}_{k_1,k_2,\ldots,k_r} B^{k_1,k_2,\ldots,k_r}_{j_1,j_2,\ldots,j_r} \tag{3.11}$$

where

$$1 \leq i_1 < i_2 < \ldots < i_r \leq l$$

$$1 \leq j_1 < j_2 < \ldots < j_r \leq m$$

and the sum is taken over all indices k_q satisfying

$$1 \leq k_1 < k_2 < \ldots < k_r \leq n$$

Proof

See Rosenbrock (1970, p. 5).

The above lemma will be used in the proof of the following theorem:

Theorem 1

The relationship between infinite system zeros, infinite transfer function zeros and infinite decoupling zeros is given by the following :

$$\begin{aligned} \text{Number of infinite system zeros} = &\big(\text{no. of infinite transfer function zeros}\big) \\ &+ \big(\text{no. of infinite i.d. zeros}\big) \\ &+ \big(\text{no. of infinite o.d. zeros}\big) \\ &- \big(\text{no. of infinite i.o.d. zeros}\big) \end{aligned} \tag{3.12}$$

Proof

The matrix

$$\begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \end{bmatrix}$$

is appropriate for the calculation of the infinite system zeros (see earlier). Now, the above matrix can be written as

$$\begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \end{bmatrix} = \begin{bmatrix} X(w) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} N_0(w) & \mathfrak{U}_0(w) \\ -\mathfrak{V}_0(w) & 0 \end{bmatrix} \begin{bmatrix} Y(w) & 0 \\ 0 & I \end{bmatrix} \tag{3.13}$$

Define $n = r + l + m$. Consider the minor of

$$\begin{bmatrix} N(w) & \mathfrak{U} \\ -\mathfrak{V}D(w) & 0 \end{bmatrix}$$

formed with the first n rows and columns plus the rows of order $n + i_1, n + i_2, \ldots, n + i_q$ (with $i_1 < i_2 < \ldots < i_q$) and the columns of order $n + j_1, n + j_2, \ldots, n + j_q$ (with $j_1 < j_2 < \ldots < j_q$). Denote this minor by

$$\bar{N}(w)^{1,2,\ldots,n,n+i_1,\ldots,n+i_q}_{1,2,\ldots,n,n+j_1,\ldots,n+j_q}$$

Denote the three matrices of the right -hand side of (3.13) by $X_0(w)$, $\bar{N}_0(w)$ and $Y_0(w)$ respectively. Using lemma 1,

$$\bar{N}(w)^{1,\ldots,n,n+i_1,\ldots,n+i_q}_{1,\ldots,n,n+j_1,\ldots,n+j_q} = \sum_{k=1}^{n+q} X_0(w)^{1,\ldots,n,n+i_1,\ldots,n+i_q}_{k_1,k_2,\ldots,k_{n+q}} \quad \mathrm{X}$$
$$\sum_{m=1}^{n+q} \bar{N}_0(w)^{k_1,k_2,\ldots,k_{n+q}}_{m_1,m_2,\ldots,m_{n+q}} Y_0(w)^{m_1,m_2,\ldots,m_{n+q}}_{1,\ldots,n,n+j_1,\ldots,n+j_q} \tag{3.14}$$

All the minors

$$X_0(w)^{1,\ldots,n,n+i_1,\ldots,n+i_q}_{k_1,k_2,\ldots,k_{n+q}}$$

are either zero or equal to $|X(w)|$. Similarly, all the minors of $Y_0(w)$ are either zeros or equal to $|Y(w)|$. Also, the zeros of $G(\frac{1}{w})$ are those of

$$\begin{bmatrix} N_0(w) & \mathfrak{U}_0(w) \\ -\mathfrak{V}_0(w) & 0 \end{bmatrix}$$

(see earlier). Hence, it is concluded that

$$\begin{aligned} \text{Number of infinite system zeros} = & \big(\text{no. of infinite transfer function zeros}\big) \\ & + \big(\text{no. of infinite i.d. zeros}\big) \\ & + \big(\text{no. of infinite o.d. zeros}\big) \\ & - \big(\text{no. of infinite i.o.d. zeros}\big) \qquad \textbf{(3.15)} \end{aligned}$$

Example 1

It will now be seen that the above result (3.15) holds true for a normalised system but not when applied to the original system matrix. Consider the system matrix

$$P(s) = \begin{bmatrix} s & 1 \\ -(s^2+1) & -s \end{bmatrix}$$

Firstly consider the original system matrix $P(s)$. To determine whether $P(s)$ has any infinite decoupling zeros, put $s = \frac{1}{w}$ in $P(s)$:

$$P(\frac{1}{w}) = \begin{bmatrix} \frac{1}{w} & 1 \\ -(\frac{1}{w^2}+1) & -\frac{1}{w} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{w} & 1 \\ -\frac{1}{w^2}(1+w^2) & -\frac{1}{w} \end{bmatrix}$$

The infinite i.d. zeros of $P(s)$ are the zeros at $w = 0$ of

$$[\,T(s) \quad U(s)\,] = [\,\tfrac{1}{w} \quad 1\,]$$

$$= \frac{1}{w}[\,1 \quad w\,]$$

Therefore, $P(s)$ has no infinite i.d. zeros. The infinite o.d. zeros of $P(s)$ are the zeros at $w = 0$ of

$$\begin{bmatrix} T(s) \\ -V(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{w} \\ -\frac{1}{w^2}(1+w^2) \end{bmatrix}$$
$$= \frac{1}{w^2} \begin{bmatrix} w \\ -(1+w^2) \end{bmatrix}$$

Therefore, $P(s)$ has no infinite o.d. zeros. Hence, using the original system matrix, $P(s)$ has no infinite decoupling zeros.

Now form the transfer function matrix

$$G(s) = V(s)T^{-1}(s)U(s) + W(s)$$
$$= \frac{(s^2+1)}{s} - s$$
$$= \frac{1}{s}$$

The infinite transfer function zeros are the zeros at $w = 0$ of $G(\frac{1}{w})$. Now

$$G(\frac{1}{w}) = w$$

Therefore, $P(s)$ has an infinite transfer function zero.

Now determine whether $P(s)$ has any infinite system zeros directly from the original system matrix. The infinite system zeros of $P(s)$ are the zeros at $w = 0$ of $P(\frac{1}{w})$. (Remember that when finding system zeros, the Rosenbrock (1974) definition of the zeros of a system is used.) $P(\frac{1}{w})$ needs to be expressed in the form $D^{-1}N$ where matrices D and N are relatively prime:

$$P(\frac{1}{w}) = \begin{bmatrix} \frac{1}{w} & 1 \\ -\frac{1}{w^2}(1+w^2) & -\frac{1}{w} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{w} & 0 \\ 0 & \frac{1}{w^2} \end{bmatrix} \begin{bmatrix} 1 & w \\ -(1+w^2) & -w \end{bmatrix}$$
$$= \begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w \\ -(1+w^2) & -w \end{bmatrix}$$

The relatively prime form of the two above matrices is now found. Express the two matrices as follows and perform elementary row operations:

$$\left[\begin{array}{cc|cc} w & 0 & 1 & w \\ 0 & w^2 & -(1+w^2) & -w \end{array}\right] \xrightarrow[\text{(row 1) + (row 2)}]{\text{new row 2 =}} \left[\begin{array}{cc|cc} w & 0 & 1 & w \\ w & w^2 & -w^2 & 0 \end{array}\right]$$

$$\xrightarrow[\text{(row 2)}/w]{\text{new row 2 =}} \left[\begin{array}{cc|cc} w & 0 & 1 & w \\ 1 & w & -w & 0 \end{array}\right]$$

Therefore,

$$P(\frac{1}{w}) = \begin{bmatrix} w & 0 \\ 1 & w \end{bmatrix}^{-1} \begin{bmatrix} 1 & w \\ -w & 0 \end{bmatrix}$$
$$= D^{-1}N$$

where D and N are relatively prime. The matrix

$$N = \begin{bmatrix} 1 & w \\ -w & 0 \end{bmatrix}$$

is used to determine whether $P(s)$ has any infinite system zeros. Perform elementary row and column operations to produce the Smith form of N:

$$\begin{bmatrix} 1 & w \\ -w & 0 \end{bmatrix} \xrightarrow[\text{(row 2) + }(w \text{ x row 1})]{\text{new row 2 =}} \begin{bmatrix} 1 & w \\ 0 & w^2 \end{bmatrix}$$

$$\xrightarrow[\text{(col. 2) - }(w \text{ x col. 1})]{\text{new col. 2 =}} \begin{bmatrix} 1 & 0 \\ 0 & w^2 \end{bmatrix}$$

From this, it can be seen that $P(s)$ has one infinite system zero of degree 2.

Hence, $P(s)$ has no infinite decoupling zeros, an infinite transfer function zero and an infinite system zero of degree 2 i.e. for the original system matrix,

$$\text{Number of infinite system zeros} \neq (\text{no. of infinite transfer function zeros}) + (\text{no. of infinite decoupling zeros})$$

Therefore, result (3.15) does not hold true when applied to the original system matrix. It will now be seen that result (3.15) does hold true for a normalised system.

Consider the normalised system matrix

$$\mathfrak{P}(s) = \left[\begin{array}{ccc|c} s & 1 & 0 & 0 \\ -(s^2+1) & -s & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

To determine whether $\mathfrak{P}(s)$ has any infinite decoupling zeros, put $s = \frac{1}{w}$ in $\mathfrak{P}(s)$:

$$\mathfrak{P}(\frac{1}{w}) = \left[\begin{array}{ccc|c} \frac{1}{w} & 1 & 0 & 0 \\ -(\frac{1}{w^2}+1) & -\frac{1}{w} & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} \frac{1}{w} & 1 & 0 & 0 \\ -\frac{1}{w^2}(1+w^2) & -\frac{1}{w} & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

The infinite i.d. zeros of $\mathfrak{P}(s)$ are the zeros at $w = 0$ of

$$[\mathfrak{T}(\tfrac{1}{w}) \quad \mathfrak{U}] = \begin{bmatrix} \frac{1}{w} & 1 & 0 & 0 \\ -\frac{1}{w^2}(1+w^2) & -\frac{1}{w} & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{w} & 0 & 0 \\ 0 & \frac{1}{w^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & w & 0 & 0 \\ -(1+w^2) & -w & w^2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w & 0 & 0 \\ -(1+w^2) & -w & w^2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Now express the two above matrices in relatively prime form as follows:

$$\left[\begin{array}{ccc|cccc} w & 0 & 0 & 1 & w & 0 & 0 \\ 0 & w^2 & 0 & -(1+w^2) & -w & w^2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{(row 1) + (row 2)}]{\text{new row 2 =}} \left[\begin{array}{ccc|cccc} w & 0 & 0 & 1 & w & 0 & 0 \\ w & w^2 & 0 & -w^2 & 0 & w^2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{(row 2)}/w]{\text{new row 2 =}} \left[\begin{array}{ccc|cccc} w & 0 & 0 & 1 & w & 0 & 0 \\ 1 & w & 0 & -w & 0 & w & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Therefore,

$$\begin{aligned}[\mathfrak{T}(\tfrac{1}{w}) \quad \mathfrak{U}] &= \begin{bmatrix} w & 0 & 0 \\ 1 & w & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w & 0 & 0 \\ -w & 0 & w & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\ &= D^{-1}N \end{aligned}$$

where D and N are relatively prime. The matrix

$$N = \begin{bmatrix} 1 & w & 0 & 0 \\ -w & 0 & w & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

is used to determine whether $\mathfrak{P}(s)$ has any infinite i.d. zeros. Perform elementary row and column operations on N to give its Smith form:

$$\begin{bmatrix} 1 & w & 0 & 0 \\ -w & 0 & w & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{(row 2) } + (w \text{ x row 1})]{\text{new row 2} =} \begin{bmatrix} 1 & w & 0 & 0 \\ 0 & w^2 & w & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{(col. 2)} + \text{(col. 4)}]{\text{new col. 2} =} \begin{bmatrix} 1 & w & 0 & 0 \\ 0 & w^2 & w & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{(col. 2) } - (w \text{ x col. 1})]{\text{new col. 2} =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & w^2 & w & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

At $w = 0$, the rank of the above matrix is reduced. Therefore, $\mathfrak{P}(s)$ has an infinite i.d. zero.

Similarly, the infinite o.d. zeros of $\mathfrak{P}(s)$ are the zeros at $w = 0$ of

$$\begin{bmatrix} \mathfrak{T}(\frac{1}{w}) \\ -\mathfrak{V} \end{bmatrix} = \begin{bmatrix} \frac{1}{w} & 1 & 0 \\ -\frac{1}{w^2}(1+w^2) & -\frac{1}{w} & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{w} & 0 & 0 & 0 \\ 0 & \frac{1}{w^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & w & 0 \\ -(1+w^2) & -w & w^2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w & 0 \\ -(1+w^2) & -w & w^2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Express the two above matrices in relatively prime form as follows:

$$\left[\begin{array}{cccc|ccc} w & 0 & 0 & 0 & 1 & w & 0 \\ 0 & w^2 & 0 & 0 & -(1+w^2) & -w & w^2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[\text{(row 1) + (row 2)}]{\text{new row 2 =}} \left[\begin{array}{cccc|ccc} w & 0 & 0 & 0 & 1 & w & 0 \\ w & w^2 & 0 & 0 & -w^2 & 0 & w^2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[\text{(row 2)}/w]{\text{new row 2 =}} \left[\begin{array}{cccc|ccc} w & 0 & 0 & 0 & 1 & w & 0 \\ 1 & w & 0 & 0 & -w & 0 & w \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

Therefore,

$$\begin{bmatrix} \mathfrak{T}(\frac{1}{w}) \\ -\mathfrak{V} \end{bmatrix} = \begin{bmatrix} w & 0 & 0 & 0 \\ 1 & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w & 0 \\ -w & 0 & w \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= D^{-1}N$$

where D and N are relatively prime. The matrix

$$N = \begin{bmatrix} 1 & w & 0 \\ -w & 0 & w \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is used to determine whether $\mathfrak{P}(s)$ has any infinite o.d. zeros. Perform elementary row and column operations to produce the Smith form of N:

$$\begin{bmatrix} 1 & w & 0 \\ -w & 0 & w \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[\text{(row 2) + (w x row 1)}]{\text{new row 2 =}} \begin{bmatrix} 1 & w & 0 \\ 0 & w^2 & w \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{new row 2 =} \\ \text{(row 2) + (w x row 4)}}]{\substack{\text{new row 1 =} \\ \text{(row 1) + (w x row 3)}}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

At $w = 0$, the rank of the above matrix is not reduced. Therefore, $\mathfrak{P}(s)$ has no infinite o.d. zeros. Also, $\mathfrak{P}(s)$ has no infinite i.o.d. zeros. Hence, $\underline{\mathfrak{P}}(s)$ has one infinite decoupling zero.

Now form the transfer function matrix

$$G(s) = V(s)T^{-1}(s)U(s) + W(s)$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 1 & 0 \\ -(s^2+1) & -s & 1 \\ 0 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \frac{1}{s} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -s \\ (s^2+1) & s & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s}$$

The infinite transfer function zeros are the zeros at $w = 0$ of $G(\frac{1}{w})$. Now

$$G(\frac{1}{w}) = w$$

Therefore, $\mathfrak{P}(s)$ has an infinite transfer function zero.

Now determine whether $\mathfrak{P}(s)$ has any infinite system zeros directly from the normalised system matrix. The infinite system zeros of $\mathfrak{P}(s)$ are the zeros at $w = 0$ of $\mathfrak{P}(\frac{1}{w})$. Again. remember that the Rosenbrock (1974) definition of the zeros of a system is used to find the system zeros. Express $\mathfrak{P}(\frac{1}{w})$ in the form $D^{-1}N$ where matrices D and N are relatively prime:

$$\mathfrak{P}(\frac{1}{w}) = \left[\begin{array}{ccc|c} \frac{1}{w} & 1 & 0 & 0 \\ -\frac{1}{w^2}(1+w^2) & -\frac{1}{w} & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

$$= \begin{bmatrix} \frac{1}{w} & 0 & 0 & 0 \\ 0 & \frac{1}{w^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & w & 0 & 0 \\ -(1+w^2) & -w & w^2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w & 0 & 0 \\ -(1+w^2) & -w & w^2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Now express the two above matrices in relatively prime form as follows:

$$\left[\begin{array}{cccc|cccc} w & 0 & 0 & 0 & 1 & w & 0 & 0 \\ 0 & w^2 & 0 & 0 & -(1+w^2) & -w & w^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{(row 1)} + \text{(row 2)}]{\text{new row 2} =} \left[\begin{array}{cccc|cccc} w & 0 & 0 & 0 & 1 & w & 0 & 0 \\ w & w^2 & 0 & 0 & -w^2 & 0 & w^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{(row 2)}/w]{\text{new row 2} =} \left[\begin{array}{cccc|cccc} w & 0 & 0 & 0 & 1 & w & 0 & 0 \\ 1 & w & 0 & 0 & -w & 0 & w & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right]$$

Therefore,

$$\begin{aligned} \mathfrak{P}(\frac{1}{w}) &= \begin{bmatrix} w & 0 & 0 & 0 \\ 1 & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & w & 0 & 0 \\ -w & 0 & w & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ &= D^{-1}N \end{aligned}$$

where D and N are relatively prime. The matrix

$$N = \begin{bmatrix} 1 & w & 0 & 0 \\ -w & 0 & w & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

is used to determine whether $\mathfrak{P}(s)$ has any infinite system zeros. Perform elementary row and column operations on N to give its Smith form:

$$\begin{bmatrix} 1 & w & 0 & 0 \\ -w & 0 & w & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow[\text{(row 2) } + (w \text{ x row 1})]{\text{new row 2} =} \begin{bmatrix} 1 & w & 0 & 0 \\ 0 & w^2 & w & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{(col. 2) } - (w \text{ x col. 1})]{\text{new col. 2} =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & w^2 & w & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{interchange} \\ \text{col. 3 and col. 4}}]{\substack{\text{new col. 2} = \\ \text{(col. 2)} + \text{(col. 4)}}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & w^2 & 0 & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[\text{(row 2) } + (w \text{ x row 4})]{\text{new row 2} =} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Therefore, the normalised system matrix $\mathfrak{P}(s)$ has a zero at infinity of degree 2.

Hence, it has now be seen that $\mathfrak{P}(s)$ has an infinite decoupling zero, an infinite transfer function zero and an infinite system zero of degree 2 i.e. for the normalised system matrix,

Number of infinite system zeros = (no. of infinite transfer function zeros)
+ (no. of infinite decoupling zeros)

Therefore, it has been seen in this example that result (3.15) holds true for a normalised system but not when applied to the original system matrix.

In this section, the relationship between infinite system zeros, infinite transfer function zeros and infinite decoupling zeros has been established using a result concerning infinite system zeros, which is a natural extension of the finite system zeros defined by Rosenbrock (1974), and the definition of infinite system zeros.

The key for the adequate definition of infinite system zeros is the notion of the normalised system, introduced by Verghese (1978).

IV.4 Matrix pencils

Recall that a generalised state space system is one whose defining equations give rise to a system matrix of the form (Rosenbrock 1974, Verghese 1978)

$$P(s) = \left[\begin{array}{c|c} sE - A & B \\ \hline -C & D \end{array}\right] \tag{4.1}$$

where E, A, B, C are constant matrices and $|sE - A| \neq 0$.

Definition 1

A MATRIX PENCIL is any mxl matrix of the form

$$T(s) = sE - A \tag{4.2}$$

where E, A are constant matrices. The pencil (4.2) is said to be REGULAR if $m = l$ and

$$|sE - A| \neq 0 \tag{4.3}$$

Otherwise the pencil is called SINGULAR.

The generalised state space system matrix (4.1) contains a number of interesting pencils:

(a) the POLE PENCIL $sE - A$, which determines the modes (finite and infinite) of the system,

(b) the INPUT PENCIL $[sE - A \quad B]$, which determines the controllability properties of the system,

(c) the OUTPUT PENCIL $\left[(sE - A)^T \; -C^T\right]^T$, which determines the observability properties of the system,

(d) the SYSTEM PENCIL

$$\begin{bmatrix} sE - A & B \\ -C & D \end{bmatrix} \tag{4.4}$$

which determines the transmission-blocking properties of the system.

It should be noted that a fine distinction is made between the system matrix (4.1), where full account is taken of the block structure, and the system pencil (4.4), where the block structure has no role. Also note that the pole pencil is always regular, while both the input and output pencils are not square and consequently are singular. Typically, the system pencil is singular but it could be regular, in which case the system transfer function matrix would be invertible.

A relation between matrix pencils is now stated as follows:

Definition 2

Two mxl matrix pencils $sE_1 - A_1$, $sE_2 - A_2$ are said to be STRICTLY EQUIVALENT (s.e.) if there exist constant non-singular matrices M, N such that

$$sE_1 - A_1 = M(sE_2 - A_2)N \tag{4.5}$$

CHAPTER V

SYSTEM EQUIVALENCE

CHAPTER V

SYSTEM EQUIVALENCE

V.1 Introduction

It is often advantageous for analysis and design purposes to consider an alternative but equivalent representation of a system model. That two system representations give rise to the same transfer function matrix is perhaps the most basic notion that the two representations be equivalent.

In this chapter it will be seen that in the conventional theory of linear systems the equivalence transformation for state space models is system similarity, and for general polynomial models it is extended strict system equivalence (e.s.s.e.). In the generalised theory the appropriate transformations are complete system equivalence (c.s.e.) for generalised state space systems and full system equivalence (f.s.e.) for general polynomial models.

It is ~~observed those~~ noted which system properties ~~which~~ are preserved under an equivalence transformation. In the conventional theory of linear systems all the essential finite zero and pole structures of a polynomial system matrix are invariant under (e.s.s.e.). In the generalised theory of linear systems (c.s.e.) preserves the finite and infinite zero / pole structures of a generalised state space system matrix while under (f.s.e.) all the essential finite and infinite zero / pole structures of a general polynomial system matrix are invariant.

The relation of full equivalence also has the useful property of permitting the given polynomial matrix to be reduced to an equivalent matrix pencil form. This will be seen in the next chapter.

V.2 The general form of system transformations

Let

$$P_i(s) \triangleq \begin{bmatrix} T_i(s) & U_i(s) \\ -V_i(s) & W_i(s) \end{bmatrix}, \qquad i = 1, 2 \tag{2.1}$$

where it is assumed that

$$\det T_i(s) \neq 0 \tag{2.2}$$

be two $(r_i+m)\mathrm{x}(r_i+l)$ Rosenbrock system matrices. These system matrices are called POLYNOMIAL or RATIONAL depending on whether the component matrices $T_i(s)$, $U_i(s)$, $V_i(s)$, $W_i(s)$ are polynomial or rational (Rosenbrock 1970). Let

$$G_i(s) = V_i(s)T_i^{-1}U_i(s) + W_i(s), \qquad i = 1, 2 \tag{2.3}$$

denote the associated transfer function matrices.

The most basic form of equivalence of system matrices is then

Definition 1

$P_1(s), P_2(s)$ of (2.1) are said to be INPUT- OUTPUT EQUIVALENT (i/o equivalent) if, and only if, they give rise to the same transfer function matrix i.e. if, and only if,

$$G_1(s) = G_2(s) \tag{2.4}$$

A question of fundamental importance in linear systems theory concerns the nature of the relationship between two system matrices which are input-output equivalent. Rosenbrock proposed an exact characterisation of this equivalence in system matrix terms.

Definition 2

If $P_1(s), P_2(s)$ of (2.1) are rational system matrices, one of which, $P_1(s)$ say, can be obtained from the other, $P_2(s)$, by a finite sequence of the following elementary operations:

(a) multiply any one of the first r_2 rows (respectively, columns) by a non-zero rational function,

(b) add a multiple, by a rational function, of any one of the first r_2 rows (respectively, columns) to any other row (respectively, column),

(c) interchange any two among the first r_2 rows (respectively, columns),

(d) add a row and column to $P_2(s)$ to form

$$\begin{bmatrix} 1 & 0 \\ 0 & P_2(s) \end{bmatrix},$$

then $P_1(s)$ and $P_2(s)$ are said to be SYSTEM EQUIVALENT.

It can be shown that the transfer function matrix is invariant under system equivalence.

Theorem 1

The transfer function matrix $G(s)$ is a standard form for system matrices under system equivalence.

Proof

Operate on P as follows:

$$\begin{aligned}
\begin{bmatrix} T & U \\ -V & W \end{bmatrix} &\longrightarrow \begin{bmatrix} I & T^{-1}U \\ -V & W \end{bmatrix} \\
&\longrightarrow \begin{bmatrix} I & T^{-1}U \\ 0 & VT^{-1}U + W \end{bmatrix} \\
&= \begin{bmatrix} I & T^{-1}U \\ 0 & G \end{bmatrix} \\
&\longrightarrow \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} \\
&\longrightarrow G
\end{aligned}$$

Note that the above operations can be reversed i.e.

$$G \longrightarrow \ldots\ldots \longrightarrow \begin{bmatrix} T & U \\ -V & W \end{bmatrix}$$

Theorem 2

Any two system matrices are system equivalent if, and only if, they give rise to the same transfer function matrix i.e. if, and only if, they are input-output equivalent.

Proof

If two matrices are system equivalent, they give rise to the same $G(s)$, as was noted above. If they give rise to the same $G(s)$, each can be reduced to $G(s)$ by system equivalence i.e.

$$\begin{bmatrix} T_1 & U_1 \\ -V_1 & W_1 \end{bmatrix} \longrightarrow \ldots\ldots \longrightarrow G_1 = G_2 \longrightarrow \ldots\ldots \begin{bmatrix} T_2 & U_2 \\ -V_2 & W_2 \end{bmatrix}$$

Hence, the desired result follows.

It is seen from the above result that system equivalence is an exact characterisation of input-output equivalence in system matrix terms in the sense that it provides a catalogue of permitted elementary operations that can be applied to the system matrix without affecting the associated transfer function matrix.

A relation between two system matrices which are input-output equivalent is now stated in the following theorem:

Theorem 3

Two system matrices of the form (2.1) are input-output equivalent if, and only if, there exist system matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \tag{2.5}$$

Relation (2.5) and system equivalence are both exact characterisations of input-output equivalence. It follows that the elementary operations defining (2.5) are precisely those of system equivalence. The relation (2.5) is thus the

GENERAL MATRIX FORM FOR ANY SYSTEM MATRIX TRANSFORMATION. It is the zero and unit blocks in the transforming matrices which guarantee the invariance of $G(s)$ irrespective of $M(s)$, $N(s)$, $X(s)$, $Y(s)$ being polynomial, rational, singular or non-singular (Pugh et al., 1989). Also note that writing the transforming matrices on both sides of the expression (2.5) permits system matrices of different dimensions to be related. Thus, the equivalence classes set up by such transformations are suitably enlarged and complete. The further imposition of the conditions of the various matrix transformations onto the general form (2.5) will generate the relevant system matrix transformations, as will be seen later.

V.3 Strict system equivalence

Let $P(m,l)$ denote the class of $(r+m)\mathrm{x}(r+l)$ polynomial matrices, where m,l are fixed positive integers but r is variable and ranges over all integers greater than max $(-m,-l)$.

Definition 1

Let $P_i(s) \in P(m,l)$, $i=1,2$. Let $M(s), N(s)$ be $r\mathrm{x}r$ unimodular polynomial matrices. Also let $X(s), Y(s)$ be polynomial matrices, respectively $m\mathrm{x}r$ and $r\mathrm{x}l$. If $P_1(s)$ and $P_2(s)$ are related by the transformation

$$\begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} = \begin{bmatrix} M(s) & 0 \\ X(s) & I_m \end{bmatrix} \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I_l \end{bmatrix} \quad \mathbf{(3.1)}$$

then $P_1(s)$ and $P_2(s)$ are said to be STRICTLY SYSTEM EQUIVALENT (s.s.e.).

Note that relation (3.1) is a special case of theorem 3, section V.2. Strict system equivalence is restrictive in that it only allows system matrices of the same dimension to be related. Also note that (s.s.e.) is the system version of unimodular equivalence (see definition 1(a), section II.3).

Under this type of transformation it is important to note those system properties which are preserved. It can be shown that under (s.s.e.) the dimension r_i of the matrices T, U, V, the order n of the system, and the corresponding transfer function matrix G are all invariant.

Theorem 1

Two system matrices which are strictly system equivalent have the same order and give rise to the same transfer function matrix.

Proof

Equation (3.1) gives

$$T_1(s) = M(s)T_2(s)N(s) \tag{3.2}$$

On taking determinants and noticing that M and N are unimodular it follows that $|T_1(s)|$ and $|T_2(s)|$ have the same degree. Therefore P_1 and P_2 have the same order.

On multiplying out, equation (3.1) gives

$$\begin{bmatrix} T_1 & U_1 \\ -V_1 & W_1 \end{bmatrix} = \begin{bmatrix} MT_2N & M(T_2Y + U_2) \\ -(V_2 - XT_2)N & XT_2Y - V_2Y + XU_2 + W_2 \end{bmatrix} \tag{3.3}$$

Therefore,

$$\begin{aligned} G_1 &= V_1T_1^{-1}U_1 + W_1 \\ &= (V_2 - XT_2)NN^{-1}T_2^{-1}M^{-1}M(T_2Y + U_2) \\ &\qquad + XT_2Y - V_2Y + XU_2 + W_2 \\ &= V_2T_2^{-1}U_2 + W_2 \\ &= G_2 \end{aligned}$$

Therefore, P_1 and P_2 give rise to the same transfer function matrix.

Any relation of strict system equivalence can be generated by elementary operations of the following types:

(a) multiply any one of the first r rows (respectively, columns) by a non-zero constant,

(b) add a multiple, by a polynomial, of any one of the first r rows (respectively, columns) to any other row (respectively, column),

(c) interchange any two among the first r rows (respectively, columns).

Comparison with definition 2 section V.2 shows that system equivalence includes strict system equivalence as a special case.

Theorem 2

Any polynomial system matrix $P_1(s)$ can be brought by strict system equivalence to the form

$$\begin{bmatrix} I_{r-n} & 0 \\ 0 & P_2(s) \end{bmatrix}$$

where $P_2(s)$ is a system matrix in state space form.

Proof

For the proof see Rosenbrock (1970, p. 53).

V.4 System similarity

Consider a system in state space form (see (2.1) section III.2):

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_1u_1 \\ y_1 &= C_1x_1 + D_1u_1 \end{aligned} \tag{4.1}$$

If a system matrix $P_1(s)$ in state space form is transformed by strict system equivalence, the result will not generally be in state space form. Given a state space description of a system, a new model may be obtained which evidently preserves the state space form. This may be achieved by a change of basis in the state space of the form

$$x_1 = Hx_2 \tag{4.2}$$

where H is a non-singular constant matrix. Then equations (4.1) become

$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y_1 = C_2 x_2 + D_2 u_2 \tag{4.3}$$

where

$$A_2 = H^{-1} A_1 H$$
$$B_2 = H^{-1} B_1$$
$$C_2 = C_1 H \tag{4.4}$$
$$D_2 = D_1$$

This transformation is called SYSTEM SIMILARITY. It can equally be defined in terms of Rosenbrock's system matrix in state space form by

$$\begin{bmatrix} sI - A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix} = \begin{bmatrix} H^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} sI - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I_l \end{bmatrix} \tag{4.5}$$

Strict system equivalence clearly includes system similarity as a special case. It follows that the system order and the transfer function matrix are both invariant under system similarity.

Theorem 1

Two system matrices $P_1(s)$ and $P_2(s)$ in state space form are system similar if, and only if, they are strictly system equivalent.

Proof

For the proof see Rosenbrock (1970, p. 56).

V.5 Extended strict system equivalence

The conventional theory of linear systems is built on the standard matrix theory of equivalence by unimodular matrices (u.e.). Let $P(m, l)$ denote the class of $(r+m)\mathrm{x}(r+l)$ polynomial matrices, where m, l are fixed positive integers but r is variable and ranges over all integers greater than max $(-m, -l)$.

In section II.3 unimodular equivalence (u.e.) and extended unimodular equivalence (e.u.e.) were defined. The essential properties of the transformation of (e.u.e.) can be summarised as follows:

Lemma 1

(a) Extended unimodular equivalence is an equivalence relation on $P(m,l)$.

(b) $P_1(s), P_2(s) \in P(m,l)$ are (e.u.e.) if, and only if, their Smith forms are related by a trivial expansion.

Proof

For the proof see Pugh and Shelton (1978).

It is now apparent from lemma 1(b) that (e.u.e.) is simply (u.e.) together with the addition operation of trivial expansion / deflation. It is also seen from the above lemma that the essential invariants under (e.u.e.) are the non-unit invariant polynomials or, equivalently, the finite elementary divisor or zero structure of the polynomial matrix under consideration. In fact, (e.u.e.) represents a complete description of the relationship which holds between any two polynomial matrices from $P(m,l)$ whose finite zero structures are identical. However, the infinite zero structure is not invariant under (e.u.e.).

Imposing the conditions of (e.u.e.) onto the general form for any system matrix transformation (see (2.5) section V.2) i.e.

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \tag{5.1}$$

(where $M(s), N(s), X(s), Y(s)$ are polynomial matrices) gives rise to the following system transformation:

Definition 1

Let $P_i(s) \in P(m,l)$, $i = 1,2$. $P_1(s), P_2(s)$ are said to be EXTENDED STRICT SYSTEM EQUIVALENT (e.s.s.e.) if there exists a transformation of the form (5.1) such that

$$M(s)T_1(s) = T_2(s)N(s) \tag{5.2}$$

is an (e.u.e.) transformation.

The following result shows that (e.s.s.e.) inherits the basic features of (e.u.e.):

Theorem 1

Under extended strict system equivalence (e.s.s.e.) all the essential finite zero and pole structures of a polynomial system matrix are invariant.

Note here the mechanism by which the results of theorem 1 above are achieved. (E.s.s.e.) induces transformations of (e.u.e.) directly on those sub-matrices of the system matrix used to define the various finite pole and zero structures.

V.6 Complete system equivalence

The conventional systems theory transformation of (e.s.s.e.) does not preserve the infinite frequency structure. A transformation is now introduced which leaves invariant all finite and infinite frequency properties of a generalised state space system.

Recall that a generalised state space system is one whose defining equations give rise to a system matrix of the form

$$P(s) = \left[\begin{array}{c|c} sE - A & B \\ \hline -C & D \end{array}\right] \qquad \textbf{(6.1)}$$

where E, A, B, C are constant matrices and $|sE - A| \neq 0$ (see (4.12) section III.4).

The underlying matrix transformation for such systems is one which relates matrix pencils. Recall that a matrix pencil is any mxl matrix of the form

$$T(s) = sE - A \qquad \textbf{(6.2)}$$

where E, A are constant matrices. Recall also that the pencil (6.2) is said to be

REGULAR if $m = l$ and

$$|sE - A| \neq 0 \tag{6.3}$$

Otherwise it is SINGULAR.

Let $P'(m,l)$ denote the class of $(r+m)\mathrm{x}(r+l)$ matrix pencils, where m,l are fixed positive integers but r is variable and ranges over all integers greater than max $(-m,-l)$. Also let $P'(0)$ denote the set of $r\mathrm{x}r$ regular matrix pencils, where again the integer r is variable but $r > 0$. Now define the following:

Definition 1

Two pencils $T_1(s) = sE_1 - A_1$, $T_2(s) = sE_2 - A_2$ in $P'(m,l)$ are said to be COMPLETELY EQUIVALENT (c.e.) if there exist constant matrices M, N such that

$$MT_1(s) = T_2(s)N \tag{6.4}$$

or

$$[M \quad T_2(s)] \begin{bmatrix} T_1(s) \\ -N \end{bmatrix} = 0 \tag{6.5}$$

where

$$\left.\begin{array}{ll} T_2(s),\ M & \text{are relatively left} - \text{ prime} \\ N,\ T_1(s) & \text{are relatively right} - \text{ prime} \end{array}\right. \tag{6.6}$$

and

$$\left.\begin{array}{c} [T_2(s) \quad M] \\ \begin{bmatrix} T_1(s) \\ -N \end{bmatrix} \end{array}\right\} \text{have no infinite zeros} \tag{6.7}$$

In the case of regular pencils i.e. $T_1(s), T_2(s) \in P'(0)$, the condition (6.6) may be replaced by the equivalent condition that the matrices occurring in (6.7) have no finite zeros and full normal rank.

It is noted that (c.e.) is a restriction of (e.u.e.) in that its transforming matrices are constant and its action with respect to the point at infinity is also constrained. The main properties of (c.e.) are then

Theorem 1

(a) Under (c.e.) the finite and infinite zero structures of a matrix pencil are invariant.

(b) In the case of regular pencils the finite and infinite zero structures form a complete set of independent invariants under (c.e.).

(c) (C.e.) is an equivalence relation on the set of regular pencils.

Rather in the way that unimodular equivalence of polynomial matrices induces an equivalence relation on the set of polynomial system matrices, called strict system equivalence (Rosenbrock 1970), so the equivalence relation (c.e.) of regular matrix pencils may be taken as a basis for an equivalence transformation of generalised state space system matrices of the form (6.1). This may be defined as follows:

Let $P_0'(m, l)$ denote the class of $(r+m)$x$(r+l)$ generalised state space system matrices of the form (6.1), where the integer $r > 0$ is variable and where $|sE - A| \neq 0$.

Definition 2

$P_1(s),\ P_2(s) \in P_0'(m, l)$ are said to be COMPLETELY SYSTEM EQUIVALENT (c.s.e.) if there exist constant matrices M, N, X, Y such that

$$\left[\begin{array}{c|c} M & 0 \\ \hline X & I \end{array}\right]\left[\begin{array}{c|c} sE_1 - A_1 & B_1 \\ \hline -C_1 & D_1 \end{array}\right] = \left[\begin{array}{c|c} sE_2 - A_2 & B_2 \\ \hline -C_2 & D_2 \end{array}\right]\left[\begin{array}{c|c} N & Y \\ \hline 0 & I \end{array}\right]. \tag{6.8}$$

where

$$M(sE_1 - A_1) = (sE_2 - A_2)N \tag{6.9}$$

is a statement of (c.e.).

Note that in the above definition, since the pencils $sE_1 - A_1$, $sE_2 - A_2$ are regular, the requirement that (6.9) be a statement of (c.e.) may be replaced by conditions analogous to (6.6) and (6.7) of the form that the matrices

$$[\, sE_2 - A_2 \quad M \,], \qquad \begin{bmatrix} -N \\ sE_1 - A_1 \end{bmatrix} \tag{6.10}$$

have neither finite nor infinite zeros.

The importance of (c.s.e.) is indicated by the following:

Theorem 2

Under complete system equivalence (c.s.e.) all the essential finite and infinite zero / pole structures of a generalised state space system matrix are invariant.

V.7 Full system equivalence

A transformation of system matrices is now introduced which plays the same role in the generalised theory of linear systems as (e.s.s.e.) does in the conventional theory. Let $P(m, l)$ denote the class of $(r+m)\mathrm{x}(r+l)$ polynomial matrices, where m, l are fixed positive integers but r is variable and ranges over all integers greater than max $(-m, -l)$.

Definition 1

$P_1(s),\ P_2(s) \in P(m, l)$ are said to be FULLY EQUIVALENT (f.e.) if there exist polynomial matrices $M(s), N(s)$ of the appropriate dimensions such that

$$[\, M(s) \quad P_2(s) \,] \begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{7.1}$$

where the compound matrices

$$[\, M(s) \quad P_2(s) \,], \qquad \begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} \tag{7.2}$$

satisfy the following:

(a) they have full normal rank,

(b) they have no finite nor infinite zeros,

(c) the following McMillan degree conditions hold:

$$\delta_M\Big(\begin{bmatrix} M(s) & P_2(s) \end{bmatrix}\Big) = \delta_M\Big(P_2(s)\Big) \tag{7.3}$$

$$\delta_M\left(\begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix}\right) = \delta_M\Big(P_1(s)\Big) \tag{7.4}$$

Note that (a) together with the condition that the matrices (7.2) have no finite zeros is equivalent to the relative primeness requirements of (e.u.e.). Hence, (f.e.) is a special case of (e.u.e.). A most important feature of this transformation is the following:

Theorem 1

If $P_1(s),\ P_2(s) \in P(m,l)$ are related by full equivalence then they possess identical finite and infinite zero structures.

Proof

For the proof see Hayton et al. (1988).

Consider again the general form for any system matrix transformation (see (2.5) section V.2):

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \tag{7.5}$$

where $M(s), N(s), X(s), Y(s)$ are polynomial matrices.

In the case of (e.s.s.e.) it is immaterial whether the conditions of (e.u.e.) are imposed on the whole of the transformation (7.5) or simply on the transformation that (7.5) induces on the associated $T(s)$ matrices i.e. the transformation

$$M(s)T_1(s) = T_2(s)N(s) \tag{7.6}$$

However, in obtaining a systems theory version of (f.e.), the manner in which the conditions of (f.e.) are imposed on (7.5) turns out to be crucial. Consider the following example:

Example 1

Consider

$$P_1(s) = \left[\begin{array}{c|c} s & 1 \\ \hline -(s^2+1) & -s \end{array}\right], \qquad P_2(s) = \left[\begin{array}{c|c} s & 1 \\ \hline -1 & 0 \end{array}\right] \tag{7.7}$$

These matrices can be related by

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline s & 1 \end{array}\right] \left[\begin{array}{c|c} s & 1 \\ \hline -(s^2+1) & -s \end{array}\right] = \left[\begin{array}{c|c} s & 1 \\ \hline -1 & 0 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{7.8}$$

which is a transformation of the form (7.5). The transformation induced on the $T(s)$- blocks of $P_1(s)$ and $P_2(s)$ is

$$1.s = s.1$$

or

$$\left[\begin{array}{c|c} s & 1 \end{array}\right] \left[\begin{array}{c} s \\ \hline -1 \end{array}\right] = 0 \tag{7.9}$$

(7.9) is an (f.e.) transformation between the respective $T(s)$-blocks. If the overall transformation (7.8) is considered, then it can be seen that this is not an (f.e.) transformation. The compound matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & s & 1 \\ s & 1 & -1 & 0 \end{array}\right]$$

does not satisfy the required McMillan degree condition (7.3). Further $P_1(s)$, $P_2(s)$ as matrices in their own right cannot be related by any (f.e.) transformation since they actually possess different infinite zero structures (and hence would violate theorem 1). Thus, it is seen that imposing the conditions of (f.e.) on the transformation of the associated $T(s)$-blocks of the system matrices will not necessarily imply that the overall transformation (7.5) is (f.e.). It will be impossible to guarantee the invariance of the infinite zero structure of the system matrix under such a transformation.

Consider now imposing the conditions of (f.e.) on the overall transformation (7.5). There will be two transformations arising from (7.5) depending upon whether the system matrix or its normalised form is chosen.

Definition 2

$P_1(s),\ P_2(s) \in P(m,l)$ are said to be NORMAL FULL SYSTEM EQUIVALENT (n.f.s.e.) if there exist polynomial matrices $\mathfrak{M}(s)$, $\mathfrak{N}(s)$, $\mathfrak{X}(s)$, $\mathfrak{Y}(s)$ such that the corresponding normalised forms $\mathfrak{P}_1(s), \mathfrak{P}_2(s)$ are related by

$$\begin{bmatrix} \mathfrak{M}(s) & 0 \\ \mathfrak{X}(s) & I \end{bmatrix} \begin{bmatrix} \mathfrak{T}_1 & \mathfrak{U}_1 \\ -\mathfrak{V}_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{T}_2 & \mathfrak{U}_2 \\ -\mathfrak{V}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{N}(s) & \mathfrak{Y}(s) \\ 0 & I \end{bmatrix} \tag{7.10}$$

where (7.10) is an (f.e.) transformation.

Definition 3

$P_1(s), P_2(s) \in P(m,l)$ are said to be FULL SYSTEM EQUIVALENT (f.s.e.) if there exist polynomial matrices $M(s), N(s), X(s), Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \tag{7.11}$$

where (7.11) is an (f.e.) transformation.

Although (f.s.e.) relates to a system matrix, it induces transformations of (f.e.) on the various submatrices of the normalised form rather than those of the system matrix itself. It is the submatrices of the normalised form which are used to define the finite and infinite zero / pole structures of the system. It thus follows that

Theorem 2

Under full system equivalence (f.s.e.) all the essential finite and infinite zero / pole structures of a general polynomial system matrix are invariant.

V.8 Conclusions

In this chapter a number of matrix transformations together with their systems theory analogs have been given. In the conventional analysis of linear systems the system transformations are system similarity for state space models and (e.s.s.e.) for general polynomial models. In the generalised theory the appropriate system transformations are (c.s.e.) for generalised state space models and (f.s.e.) for general polynomial models.

The system transformations arise in an identical way from the underlying matrix transformation. The conventional way of achieving this (i.e. placing restrictions on the transformation induced on the $T(s)$-blocks of the system matrix) does not carry through to the general case of the transformation (f.s.e.). The correct method of generating a system transformation is to apply the restrictions of the underlying matrix transformation to the basic structural form of system transformations.

Also in this chapter it is noted which system properties are left invariant under an equivalence transformation. In the conventional theory of linear systems, (e.s.s.e.) leaves invariant the finite zero and pole structures of a polynomial system matrix. In the generalised theory of linear systems, (c.s.e.) preserves the finite and infinite zero / pole structures of a generalised state space system matrix. Finally, under (f.s.e.) all the essential finite and infinite zero / pole structures of a general polynomial system matrix are invariant.

The relation of full equivalence also has the useful property of permitting the given polynomial matrix to be reduced to an equivalent matrix pencil form. This will be seen in the next chapter.

CHAPTER VI

SYSTEM MATRIX REDUCTION

TO

SINGULAR SYSTEM FORM

VI

SYSTEM MATRIX REDUCTION TO SINGULAR SYSTEM FORM

VI.1 Introduction

This chapter describes three methods of system matrix reduction to singular system form. Hayton et al. (1989) have formed matrix pencil equivalents from a general polynomial matrix, and it can be seen how this reduction is based on the system matrix idea by Bosgra and Van der Weiden (1981). In chapter VII, the algorithm which permits the polynomial matrix reduction to pencil form is computerised and discussed further. Also discussed in this chapter is the system matrix reduction by Vardulakis (1991) which transforms a polynomial matrix model of a linear multivariable system to generalised state space form. Finally, the system matrix linearisation by Zhang (1989) is discussed and all three methods of linearisation are compared via an example. The extent to which the resulting singular system is equivalent to the original is also discussed.

VI.2 Polynomial matrix reduction to pencil form

Bosgra and Van der Weiden (1981) have outlined a procedure whereby a generalised state space system may be obtained from a general polynomial system, preserving fundamental system properties at finite and infinite frequency. This algorithm may also be used to reduce any general polynomial matrix to an equivalent matrix pencil (in the sense of having identical finite and infinite zero structure). This will be seen later to be a special case of full equivalence (Hayton et al., 1988). The algorithm is described as follows:

Let the mxl polynomial matrix $P(s)$ correspond to the matrix polynomial defined by

$$P(s) = P_0 + P_1 s + P_2 s^2 + \ldots + P_q s^q \qquad \textbf{(2.1)}$$

where P_i, $i = 1, 2, \ldots, q$, are $m\mathrm{x}l$ constant matrices with

$$P_q \neq 0 \tag{2.2}$$

Define the following matrices:

$$\Pi(E) \triangleq \begin{bmatrix} P_2 & P_3 & \ldots & P_q \\ P_3 & P_4 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ P_q & 0 & \ldots & 0 \end{bmatrix}$$

$$\Pi(A) \triangleq \begin{bmatrix} P_3 & P_4 & \ldots & P_q & 0 \\ P_4 & P_5 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ P_q & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

$$\Pi(B) \triangleq \begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_q \end{bmatrix}$$

$$\Pi(C) \triangleq [P_2 \quad P_3 \quad \ldots \quad P_q] \tag{2.3}$$

Let $\rho(E) \triangleq \text{rank } (\Pi(E))$. A method is needed to determine $\rho(E)$ linearly independent rows (resp. columns) from $\Pi(E)$, although as will become clear the precise choice of rows (resp. columns) is largely irrelevant. Let $I \triangleq \{i_1, i_2, \ldots, i_{\rho(E)}\}$ (resp. $J \triangleq \{j_1, j_2, \ldots, j_{\rho(E)}\}$) be the positive sets of integers which define such a row (resp. column) selection, denoted I (resp. J).

Let P_E (resp. P_A) be that submatrix of $\Pi(E)$ (resp. $\Pi(A)$) formed from rows of the selection I and columns of the selection J. Let P_B be the submatrix of $\Pi(B)$ formed from the rows of the selection I, and P_C be the submatrix of $\Pi(C)$ formed from the columns of the selection J.

The matrix pencil $P_F(s)$ is formed as follows:

$$P_F(s) = \begin{bmatrix} P_E - sP_A & P_B s \\ -P_C s & P_1 s + P_0 \end{bmatrix} \tag{2.4}$$

One particular method for determining a row and column selection is described as follows:

Beginning with the last row and working upwards in the matrix $\Pi(E)$, select any row which is linearly independent of all other previously selected rows. Clearly if rows $h_1, \ldots, h_k$ are selected from the one block row of $\Pi(E)$, then these rows will always be selected when considering the next block row of $\Pi(E)$, plus any additional linearly independent rows from the correct block row. This particular row selection will be called the "natural row selection", and similarly the "natural column selection" for the columns can be defined by working from the last column to the first. This is discussed in more detail in the next chapter where some of the numerical advantages provided by this selection are also described.

Two examples illustrating the formation of an equivalent matrix pencil are as follows:

Example 1

Let $P(s)$ correspond to the matrix polynomial

$$P(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2$$

Now form the following matrices:

$$\Pi(E) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Pi(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Pi(B) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Pi(C) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The row selection $I = \{1\}$ corresponds to the linearly independent row from $\Pi(E)$. The column selection J is $J = \{1\}$. Therefore

$$P_E = 1$$

$$P_A = 0$$

$$P_B = [1 \quad 0]$$

$$P_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, the matrix pencil $P_F(s)$ is

$$P_F(s) = \begin{bmatrix} 1 & s & 0 \\ -s & s & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Example 2

Let $P(s)$ correspond to the matrix polynomial

$$P(s) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^3$$

Hence,

$$\Pi(E) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi(B) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Pi(C) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The row selection $I = \{1, 3\}$ corresponds to the set of linearly independent rows from $\Pi(E)$. The column selection J is $J = \{1, 3\}$. Therefore

$$P_E = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$P_C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, the matrix pencil $P_F(s)$ is

$$P_F(s) = \begin{bmatrix} 1-s & 1 & s & 0 \\ 1 & 0 & s & 0 \\ -s & -s & 0 & s \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The following results arise directly from the construction of P_E, P_A, P_B, P_C:

Lemma 1

Let P_E, P_A, P_B, P_C be constructed from the row selection I and column selection J. The following hold:

(a) P_E is non-singular

(b) $P_E - sP_A$ is unimodular

(c) $P_C(P_E - sP_A)^{-1}P_B = P_2 + P_3 s + \ldots + P_q s^{q-2}$ **(2.5)**

(d) If P'_E, P'_A, P'_B, P'_C is a construction corresponding to any other row selection I' and column selection J', then there exist constant non-singular matrices T_1 and T_2 such that

$$P_E - sP_A = T_1(P'_E - sP'_A)T_2, \quad P_B = T_1 P'_B, \quad P_C = P'_C T_2 \tag{2.6}$$

i.e two matrix pencils, each formed from a different row and column selection but from the same polynomial matrix. are related by strict equivalence (see definition 2 section IV.4):

$$\begin{bmatrix} P_E - sP_A & P_B s \\ -P_C s & P_1 s + P_0 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P'_E - sP'_A & P'_B s \\ -P'_C s & P_1 s + P_0 \end{bmatrix} \begin{bmatrix} T_2 & 0 \\ 0 & I \end{bmatrix} \tag{2.7}$$

Proof:

See Bosgra and Van der Weiden (1981).

An example illustrating the constructions described in the above lemma is as follows:

Example 3

Let $P(s)$ correspond to the matrix-polynomial

$$P(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} s + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} s^3$$

Hence,

$$\Pi(E) = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\Pi(A) = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Pi(B) = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\Pi(C) = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Firstly, let the row selection I be $I = \{2, 3, 5, 6\}$ which corresponds to a set of linearly independent rows from $\Pi(E)$. Also, let the column selection J be $J = \{1, 3, 4, 6\}$. Then

$$P_E = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and

$$P_E - sP_A = \begin{bmatrix} 1-s & 0 & 1 & 0 \\ 0 & 1-s & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$(P_E - sP_A)^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -(1-s) & 0 \\ 0 & 1 & 0 & -(1-s) \end{bmatrix}$$

It can be seen from this that lemma 1(a) and 1(b) hold.

Now,

$$P_C(P_E - sP_A)^{-1}P_B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -(1-s) & 0 \\ 0 & 1 & 0 & -(1-s) \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1+s \\ 1+s & 1 & 0 \\ 0 & 0 & 1+s \end{bmatrix}$$

$$= P_2 + P_3 s$$

i.e. lemma 1(c) holds.

Now consider another row selection $I = \{1, 2, 4, 5\}$ and column selection $J = \{1, 3, 4, 6\}$. Then

$$P_E' = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P_A' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_B' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_C' = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and

$$P_E' - sP_A' = \begin{bmatrix} 0 & 1-s & 0 & 1 \\ 1-s & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The following relation holds between $P_E - sP_A$ and $P_E' - sP_A'$:

$$\begin{bmatrix} 1-s & 0 & 1 & 0 \\ 0 & 1-s & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1-s & 0 & 1 \\ 1-s & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e.

$$P_E - sP_A = T_1(P_E' - sP_A')T_2$$

where T_1 and T_2 are the constant non-singular matrices

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, it can be seen that the following relations hold:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

i.e.

$$P_B = T_1 P_B'$$

and

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e.

$$P_C = P_C' T_2$$

i.e. lemma 1(d) holds.

Hence, it has been seen that two matrix pencils each formed from the same matrix polynomial but from different row and column selections are related by strict equivalence, and that the precise choice of linearly independent rows (resp. columns) from the matrix $\Pi(E)$ defined earlier is largely irrelevant.

The following theorem states additional properties of any construction P_E, P_A, P_B, P_C and it can be seen from the proof that the natural row and column selections are useful in establishing the results.

Theorem 1 (Hayton et al., 1989)

(a) The matrices $[P_B \quad P_A]$, $\begin{bmatrix} P_C \\ P_A \end{bmatrix}$ have full rank.

(b) $\delta_M\Big(P_C s(P_E - sP_A)^{-1} P_B\Big) = \delta_M\Big(P_C s(P_E - sP_A)^{-1}\Big) \qquad (= \rho(E))$ **(2.8)**

Proof:

(a) Consider the matrix

$$\begin{bmatrix} P_{C_N} \\ P_{A_N} \end{bmatrix} \tag{1}$$

which has dimension $(m+\rho(E))\text{x}\rho(E)$ and where P_{C_N}, P_{A_N} are matrices formed from $\Pi(C)$, $\Pi(A)$ respectively on the basis of the natural row and column selections. It is necessary to demonstrate the existence of $\rho(E)$ linearly independent rows in the matrix (1). Select from matrix (1) linearly independent rows in the manner of the natural row selection. This is equivalent to performing the natural row and column selection on the matrix

$$\begin{bmatrix} \Pi(E) \\ 0_{m,(q-1)l} \end{bmatrix}$$

since the columns of matrix (1) have already been selected in this way. This will result in the formation of the $\rho(E)\text{x}\rho(E)$ matrix P_{E_N} which is non-singular (lemma 1(a)). Thus matrix (1) contains P_{E_N} as a submatrix and so has $\rho(E)$ linearly independent rows.

Now suppose that

$$\begin{bmatrix} P_C \\ P_A \end{bmatrix} \tag{2}$$

is formed on the basis of an arbitrary row and column selection. By lemma 1(d), there exist non-singular constant matrices T_1, T_2 such that

$$\begin{bmatrix} P_C \\ P_A \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} P_{C_N} \\ P_{A_N} \end{bmatrix} T_2 \tag{2.9}$$

Hence, matrices (1) and (2) have identical rank and so matrix (2) has full rank, as required. Similarly, $[\,P_B \quad P_A\,]$ has full rank.

(b) From lemma 1(c), and lemma 1 section II.6,

$$\begin{aligned} \delta_M\left(P_C s(P_E - sP_A)^{-1}P_B\right) &= \delta_M\left(P_2 s + P_3 s^2 + \ldots + P_q s^{q-1}\right) \\ &= \text{rank } \Pi(E) \\ &= \rho(E) \end{aligned} \tag{2.10}$$

Consider now the matrix $P_C s(P_E - sP_A)^{-1}$. $P_E - sP_A$ is unimodular, so it belongs to the ring of polynomial matrices and so does its inverse. Hence $P_C s(P_E - sP_A)^{-1}$ is a polynomial matrix which means that it has only poles at infinity. Therefore its McMillan degree is simply the total number of such poles. To determine this, put $s = \frac{1}{w}$ in $P_C s(P_E - sP_A)^{-1}$. The resulting rational matrix is given by

$$P_C(wP_E - P_A)^{-1} \tag{3}$$

The only poles that $P_C s(P_E - sP_A)^{-1}$ possesses are at infinity. Thus the only poles that $(wP_E - P_A)^{-1}$ possesses are at $w = 0$, or alternatively the only zeros that $wP_E - P_A$ possesses are at $w = 0$. Hence $wP_E - P_A$ has full rank for all finite $w \neq 0$ and so

$$\begin{bmatrix} P_C \\ wP_E - P_A \end{bmatrix} \tag{4}$$

has full rank at $w \neq 0$. Also matrix (4) has full rank at $w = 0$ (theorem 1(a)). Hence (4) has full rank for all finite w and hence (3) is a relatively right prime factorisation of the rational matrix it represents. Therefore the zero structure of $wP_E - P_A$ reflects the pole structure of this rational matrix. Since P_E is non-singular, the total number of finite zeros of $wP_E - P_A$ (they are all at $w = 0$) is rank $P_E = \rho(E)$. Therefore

$$\delta_M\left(P_C s(P_E - sP_A)^{-1}\right) = \rho(E), \tag{2.11}$$

i.e.

$$\delta_M\left(P_C s(P_E - sP_A)^{-1} P_B\right) = \delta_M\left(P_C s(P_E - sP_A)^{-1}\right),$$

as required.

Example 4

To illustrate the properties in theorem 1 above of any construction P_E, P_A, P_B, P_C, consider example 1 again.

The matrices

$$[P_B \quad P_A] = [1 \quad 0]$$

$$\begin{bmatrix} P_C \\ P_A \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

obviously have full rank.

Now

$$P_C s(P_E - sP_A)^{-1} P_B = \begin{bmatrix} s \\ 0 \end{bmatrix} [1 \quad 0]$$

$$= \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$P_C s(P_E - sP_A)^{-1} = \begin{bmatrix} s \\ 0 \end{bmatrix}$$

The McMillan degree (see result 1 section II.6) of both forms is equal to 1. Hence theorem 1 holds true.

It will now be seen from the following theorem that the polynomial matrix reduction to pencil form is a full equivalence transformation. The properties in theorem 1 above are useful in establishing the proof of the following:

Theorem 2 (Hayton et al., 1989)

If $P(s)$ is an arbitrary mxl polynomial matrix with a corresponding matrix polynomial

$$P(s) = P_0 + P_1 s + P_2 s^2 + \ldots + P_q s^q, \tag{2.12}$$

then $P(s)$ is related to the matrix pencil $P_F(s)$ by full equivalence where

$$P_F(s) = \begin{bmatrix} P_E - sP_A & P_B s \\ -P_C s & P_1 s + P_0 \end{bmatrix} \tag{2.13}$$

where P_E, P_A, P_B, P_C are as previously defined.

Proof:

Matrix transformations relating $P(s)$ and $P_F(s)$ are seen to be

$$[P_C s(P_E - sP_A)^{-1} \quad I] P_F(s) = P(s) [0 \quad I] \tag{2.14}$$

or alternatively

$$\begin{bmatrix} 0 \\ I \end{bmatrix} P(s) = P_F(s) \begin{bmatrix} -(P_E - sP_A)^{-1} P_B s \\ I \end{bmatrix} \tag{2.15}$$

It remains to show that these are statements of full equivalence.

Equation (2.14) may be written as

$$[\,P_C s(P_E - sP_A)^{-1} \quad I \quad -P(s)\,]\begin{bmatrix} P_F(s) \\ [0 \quad I\,] \end{bmatrix} = 0 \tag{2.16}$$

The matrix

$$[\,P_C s(P_E - sP_A)^{-1} \quad I \quad -P(s)\,] \tag{5}$$

has no finite nor infinite zeros, by virtue of the identity block. It must now be verified that the McMillan degree of (5) is $\delta_M\,(P(s))$. Any constant columns will not contribute to this McMillan degree, so

$$\delta_M\Big(\,[\,P_C s(P_E - sP_A)^{-1} \quad I \quad -P(s)\,]\Big) = \delta_M\Big(\,[\,P_C s(P_E - sP_A)^{-1} \quad -P(s)\,]\Big) \tag{2.17}$$

From theorem 1(b),

$$\delta_M\Big(P_C s(P_E - sP_A)^{-1}\Big) = \delta_M\Big(P_C s(P_E - sP_A)^{-1} P_B\Big) \tag{2.18}$$

and so, using theorem 1(b) section II.6,

$$\delta_M\Big(\,[\,P_C s(P_E - sP_A)^{-1} \quad -P(s)\,]\Big)$$

$$= \delta_M\Big(\,[\,P_C s(P_E - sP_A)^{-1} P_B \quad -P(s)\,]\Big)$$

$$= \delta_M\Big(\,[\,P_2 s + P_3 s^2 + \ldots + P_q s^{q-1} \quad -(P_0 + P_1 s + \ldots + P_q s^q)\,]\Big)$$

(using lemma 1(c))

$$= \operatorname{rank}\begin{bmatrix} P_2 & -P_1 & P_3 & -P_2 & \ldots & 0 & -P_q \\ P_3 & -P_2 & P_4 & -P_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ P_q & -P_{q-1} & 0 & -P_q & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

(using lemma 1 section II.6)

$$= \operatorname{rank}\begin{bmatrix} P_1 & P_2 & \ldots & P_q \\ P_2 & P_3 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ P_{q-1} & P_q & \ldots & 0 \\ P_q & 0 & \ldots & 0 \end{bmatrix}$$

$$= \delta_M\Big(P(s)\Big) \tag{2.19}$$

Thus, matrix (5) satisfies the requirements of full equivalence. Now consider the matrix

$$\begin{bmatrix} P_F(s) \\ [0 \quad I] \end{bmatrix} = \begin{bmatrix} P_E - sP_A & P_B s \\ -P_C s & P_1 s + P_0 \\ 0 & I \end{bmatrix} \tag{6}$$

The McMillan degree condition of full equivalence is clearly satisfied ((6) has McMillan degree $\delta_M\ (P_F(s))$). Also matrix (6) has no finite zeros since $P_E - sP_A$ is unimodular. It remains to show that (6) has no infinite zeros. Since (6) is a matrix pencil, its McMillan degree $\delta_M\ (P_F(s))$ is

$$\operatorname{rank} \begin{bmatrix} P_A & P_B \\ -P_C & P_1 \end{bmatrix}$$

From theorem 1(a),

$$\begin{bmatrix} P_A \\ -P_C \end{bmatrix}$$

has full column rank. Therefore, there is some minor of (6) of degree $\delta_M\ (P_F(s))$ which incorporates all the columns in the first block column of (6). The unit matrix in the second block column then ensures that this minor can be extended to give a maximum size minor of (6) with degree equal to the McMillan degree of (6). This is the condition for no infinite zeros (Hayton et al., 1988). Hence, (2.14) is a relation of full equivalence between $P(s)$ and the matrix pencil $P_F(s)$. Similarly, the same applies to (2.15).

Hence, the result demonstrates that under fundamental equivalence it is possible to reduce any polynomial matrix to matrix pencil form.

Example 5

Consider example 1. It will now be shown that the polynomial matrix with corresponding matrix polynomial

$$P(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2$$

is related to the matrix pencil

$$P_F(s) = \begin{bmatrix} 1 & s & 0 \\ -s & s & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

by full equivalence.

$P(s)$ and $P_F(s)$ are related by

$$[\,M(s) \quad P_F(s)\,]\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix} = 0$$

or

$$M(s)P(s) = P_F(s)N(s)$$

where $M(s)$, $N(s)$ are polynomial matrices of the appropriate dimension i.e.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s+s^2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & s & 0 \\ -s & s & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} -s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It now needs to be shown that

$$[\,M(s) \quad P_F(s)\,] = \begin{bmatrix} 0 & 0 & 1 & s & 0 \\ 1 & 0 & -s & s & 1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix} = \begin{bmatrix} s+s^2 & 1 \\ -1 & 0 \\ s & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfy

(a) they have full normal rank,

(b) they have neither finite nor infinite zeros,

(c) the following McMillan degree conditions hold:

$$\delta_M\Big(\,[\,M(s) \quad P_F(s)\,]\,\Big) = \delta_M\Big(P_F(s)\Big)$$

$$\delta_M\left(\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}\right) = \delta_M\Big(P(s)\Big)$$

(a) It can be seen that

$$[\,M(s) \quad P_F(s)\,] \text{ and } \begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}$$

clearly have full rank.

(b) Perform elementary column operations on the matrix

$$[\,M(s) \quad P_F(s)\,] = \begin{bmatrix} 0 & 0 & 1 & s & 0 \\ 1 & 0 & -s & s & 1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[(col.\ 3)\ +\ (s \times col.\ 1)]{new\ col.\ 3 =} \begin{bmatrix} 0 & 0 & 1 & s & 0 \\ 1 & 0 & 0 & s & 1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{interchange \\ col.\ 1\ and\ col.\ 3}]{\substack{interchange \\ col.\ 1\ and\ col.\ 2}} \begin{bmatrix} 1 & 0 & 0 & s & 0 \\ 0 & 1 & 0 & s & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

It can now be seen that the matrices $[\,M(s) \quad P_F(s)\,]$ and $\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}$ have neither finite nor infinite zeros, by virtue of the identity block.

(c) It can be seen that the following McMillan degree condition holds, since constant columns do not contribute to the McMillan degree:

$$\delta_M\left(\begin{bmatrix} 0 & 0 & 1 & s & 0 \\ 1 & 0 & -s & s & 1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}\right) = \delta_M\left(\begin{bmatrix} 1 & s & 0 \\ -s & s & 1 \\ 0 & -1 & 0 \end{bmatrix}\right).$$

i.e.

$$\delta_M\Big(\,[\,M(s) \quad P_F(s)\,]\,\Big) = \delta_M\Big(P_F(s)\Big)$$

Also, the highest degree of minors of all orders of $P(s)$ is equal to the highest degree of minors of all orders of $\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}$ i.e. the following holds:

$$\delta_M\left(\begin{bmatrix} s+s^2 & 1 \\ -1 & 0 \\ s & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \delta_M\left(\begin{bmatrix} s+s^2 & 1 \\ -1 & 0 \end{bmatrix}\right).$$

i.e

$$\delta_M\left(\begin{bmatrix} P(s) \\ -N(s) \end{bmatrix}\right) = \delta_M\Big(P(s)\Big)$$

Hence, the polynomial matrix $P(s)$ and its associated matrix pencil form $P_F(s)$ are related by full equivalence.

An immediate consequence of this theorem in view of theorem 1 section V.7 is the following:

Corollary 1

If $P(s)$ is an arbitrary mxl polynomial matrix, then $P(s)$ and any matrix pencil $P_F(s)$ constructed as in (2.13) have an identical finite and infinite zero structure.

In this section, it has been seen how a general polynomial matrix may be reduced to its associated matrix pencil form based on an algorithm suggested by Bosgra and Van der Weiden (1981). The properties of any construction P_E, P_A, P_B, P_C from the row selection I and column selection J have been discussed, and it has been seen how the natural row and column selections have been useful in establishing the results. Finally, it has been seen that a relationship exists (Hayton et al., 1988), from the point of view of the matrix transformation, between two matrix polynomials which have identical finite and infinite zero structure. This is the relationship of full equivalence.

VI.3 Extension to system matrices

The polynomial matrix reduction to its associated matrix pencil form can be extended to system matrices. Consider a linear multivariable system represented by an $(r+m)$x$(r+l)$ polynomial system matrix (Rosenbrock 1970)

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \qquad \textbf{(3.1)}$$

where det $T(s) \neq 0$. $P(s)$ can be written as

$$P(s) = P_0 + P_1 s + P_2 s^2 + \ldots + P_q s^q \qquad \textbf{(3.2)}$$

where $P_i,\ i = 1, 2, \ldots, q$, are $(r+m)$x$(r+l)$ system matrices.

Bosgra and Van der Weiden (1981) have outlined a procedure whereby a generalised state space system matrix may be constructed from a general polynomial system matrix whilst preserving fundamental system properties at both finite and infinite frequencies. This algorithm is the same as that used to reduce any general polynomial matrix to a matrix pencil having the same finite and infinite zeros. Based on this algorithm, the matrix pencil can be slightly modified to give the generalised state space form:

$$P_R(s) = \left[\begin{array}{ccc|c} P_1 s + P_0 & P_C s & \begin{bmatrix} 0 \\ I_m \end{bmatrix} & 0 \\ P_B s & P_A s - P_E & 0 & 0 \\ [0 \ \ -I_l] & 0 & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right] \tag{3.3}$$

Example 1

Consider example 1 section VI.2 with $P(s)$ now as the system matrix

$$P(s) = \left[\begin{array}{c|c} s + s^2 & 1 \\ \hline -1 & 0 \end{array}\right]$$

The generalised state space form is

$$P_R(s) = \left[\begin{array}{cccc|c} s & 1 & s & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ s & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

(Here $m = l = 1$.)

Example 2

Consider example 2 section VI.2 with $P(s)$ now as the system matrix

$$P(s) = \left[\begin{array}{c|c} s^2(s+1) & s \\ \hline -1 & 0 \end{array}\right]$$

The generalised state space form is

$$P_R(s) = \left[\begin{array}{ccccc|c} 0 & s & s & s & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ s & 0 & s-1 & -1 & 0 & 0 \\ s & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

(Here $m = l = 1$.)

As before, it is possible to form another construction P'_E, P'_A, P'_B, P'_C corresponding to another row selection I' and column selection J'.

Two generalised state space forms, each formed from a different row and column selection but from the same system matrix, are related by strict system equivalence (see definition 1 section V.3) i.e. the following relation holds:

$$\left[\begin{array}{ccc|c} P_1 s + P_0 & P_C s & \begin{bmatrix} 0 \\ I_m \end{bmatrix} & 0 \\ P_B s & P_A s - P_E & 0 & 0 \\ [0 \quad -I_l] & 0 & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right] \tag{3.4}$$

$$= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & T_1 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \left[\begin{array}{ccc|c} P_1 s + P_0 & P'_C s & \begin{bmatrix} 0 \\ I_m \end{bmatrix} & 0 \\ P'_B s & P'_A s - P'_E & 0 & 0 \\ [0 \quad -I_l] & 0 & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right] \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

where T_1 and T_2 are constant non-singular matrices.

VI.4 System matrix reduction to generalised state space form by Vardulakis's method

Vardulakis has suggested a method of transforming a general polynomial matrix

$$\left[\begin{array}{c|c} T(s) & U(s) \\ \hline -V(s) & W(s) \end{array}\right]$$

of a linear multivariable system into the generalised state space form

$$\left[\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right]$$

at the same time preserving the transfer function matrix, system poles in $\mathbb{C} \cup \{\infty\}$, decoupling zeros in $\mathbb{C} \cup \{\infty\}$ and the generalised order.

Consider a system which satisfies linear algebraic and differential equations with constant coefficients. Taking Laplace transforms with zero initial conditions gives

$$
\begin{aligned}
T(s)\bar{z} &= U(s)\bar{u} \\
\bar{y} &= V(s)\bar{z} + W(s)\bar{u}
\end{aligned}
\tag{4.1}
$$

where $z \in \mathbb{R}^r$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^l$. T, U, V, W, are polynomial matrices of dimension $r\text{x}r$, $r\text{x}l$, $m\text{x}r$, $m\text{x}l$ respectively, and $|T(s)| \neq 0$.

This can be written in normalised form (Verghese et al., 1981):

$$
\begin{bmatrix} T(s) & U(s) & O_{rm} \\ -V(s) & W(s) & I_m \\ O_{lr} & -I_l & O_{lm} \end{bmatrix} \begin{bmatrix} \bar{z}(s) \\ -\bar{u}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} O_{rl} \\ O_{ml} \\ I_l \end{bmatrix} \bar{u}(s) \tag{4.2}
$$

$$
\bar{y}(s) = \begin{bmatrix} O_{mr} & O_{ml} & I_m \end{bmatrix} \begin{bmatrix} \bar{z}(s) \\ -\bar{u}(s) \\ \bar{y}(s) \end{bmatrix} \tag{4.3}
$$

Now define

$$
\tau(s) = \begin{bmatrix} T(s) & U(s) & O_{rm} \\ -V(s) & W(s) & I_m \\ O_{lr} & -I_l & O_{lm} \end{bmatrix} \tag{4.4}
$$

The following shows how a generalised state space form may be obtained from a normalised system:

Vardulakis's reduction algorithm

(a) Form $\tau(s)$ as in (4.4) above and then $\tau^{-1}(s)$.

(b) Compute $x_{pol}(s)$ and $x_{spr}(s)$ as follows:

$$
\tau^{-1}(s) = x_{spr}(s) + x_{pol}(s) \tag{4.5}
$$

where $x_{spr}(s) \in \mathbb{R}_{pr}^{\rho\text{x}\rho}(s)$, ($\mathbb{R}_{pr}(s)$ denotes the ring of proper rational functions, $\rho = r + l + m$), is strictly proper and $x_{pol}(s) \in \mathbb{R}[s]^{\rho\text{x}\rho}$.

(c) Compute a minimal realisation ($\bar{C} \in \mathbb{R}^{\rho\text{x}n}$, $\bar{J} \in \mathbb{R}^{n\text{x}n}$, $\bar{B} \in \mathbb{R}^{n\text{x}\rho}$ where $n = \deg|\tau(s)|$) of $x_{spr}(s)$ i.e. find matrices $\bar{C}$, $\bar{J}$, $\bar{B}$ such that

$$
\bar{C}\left(sI - \bar{J}\right)^{-1}\bar{B} = x_{spr} \tag{4.6}
$$

Compute a minimal realisation ($\bar{C}_\infty \in \mathbb{R}^{\rho\times\bar{\mu}}$, $\bar{J}_\infty \in \mathbb{R}^{\bar{\mu}\times\bar{\mu}}$, $\bar{B}_\infty \in \mathbb{R}^{\bar{\mu}\times\rho}$) of $x_{pol}(s)$ i.e. find matrices $\bar{C}_\infty$, $\bar{J}_\infty$, $\bar{B}_\infty$ such that

$$\bar{C}_\infty\left(I - s\bar{J}_\infty\right)^{-1}\bar{B}_\infty = x_{pol} \tag{4.7}$$

(d) Compute the matrices

$$\begin{aligned}
C &= [\,O_{mr} \quad O_{ml} \quad I_m\,][\,\bar{C} \quad \bar{C}_\infty\,] \\
E &= \begin{bmatrix} I_n & O_{n\bar{\mu}} \\ O_{\bar{\mu}n} & -\bar{J}_\infty \end{bmatrix} \\
A &= \begin{bmatrix} \bar{J} & O_{n\bar{\mu}} \\ O_{\bar{\mu}n} & -I_{\bar{\mu}} \end{bmatrix} \\
B &= \begin{bmatrix} \bar{B} \\ \bar{B}_\infty \end{bmatrix}\begin{bmatrix} O_{rl} \\ O_{ml} \\ I_l \end{bmatrix}
\end{aligned} \tag{4.8}$$

and form the generalised state space system

$$\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned} \tag{4.9}$$

Hence, the system matrix has been reduced to an equivalent generalised state space system, preserving the transfer function matrix, system poles in $\mathbb{C} \cup \{\infty\}$, decoupling zeros in $\mathbb{C} \cup \{\infty\}$, and the generalised order.

Example 1

$$\begin{aligned}
P(s) &= \left[\begin{array}{c|c} T(s) & U(s) \\ \hline -V(s) & W(s) \end{array}\right] \\
&= \left[\begin{array}{c|c} s^2(s+1) & s \\ \hline -1 & 0 \end{array}\right]
\end{aligned}$$

This system matrix will now be reduced to generalised state space form by Vardulakis's method:

$$\begin{aligned}
\tau(s) &= \begin{bmatrix} T(s) & U(s) & O_{rm} \\ -V(s) & W(s) & I_m \\ O_{lr} & -I_l & O_{lm} \end{bmatrix} \\
&= \begin{bmatrix} s^2(s+1) & s & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\end{aligned}$$

(Here, $m = r = l = 1$.)

Therefore,

$$\tau^{-1}(s) = \frac{1}{s^2(s+1)}\begin{bmatrix} 1 & 0 & s \\ 0 & 0 & -s^2(s+1) \\ 1 & s^2(s+1) & s \end{bmatrix}$$

$$= x_{spr}(s) + x_{pol}(s)$$

where

$$x_{spr}(s) = \frac{1}{s^2(s+1)}\begin{bmatrix} 1 & 0 & s \\ 0 & 0 & 0 \\ 1 & 0 & s \end{bmatrix}$$

($x_{spr}(s)$ is strictly proper), and

$$x_{pol}(s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Now compute a minimal realisation of $x_{spr}(s)$:

$$x_{spr}(s) = \bar{C}\left(sI - \bar{J}\right)^{-1}\bar{B}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} s & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Now compute a minimal realisation of $x_{pol}(s)$:

$$x_{pol}(s) = \bar{C}_\infty\left(I - s\bar{J}_\infty\right)^{-1}\bar{B}_\infty$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$C = [0 \quad 0 \quad 1][\bar{C} \quad \bar{C}_\infty]$$

$$= [0 \quad 0 \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= [1 \quad 0 \quad 0 \quad 1 \quad 0]$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} \bar{B} \\ \bar{B}_\infty \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the generalised state space system matrix is

$$\left[\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} s & -1 & 0 & 0 & 0 & 0 \\ 0 & s+1 & -1 & 0 & 0 & 1 \\ 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline -1 & 0 & 0 & -1 & 0 & 0 \end{array}\right]$$

There is also another method of transforming an arbitrary normalised system to generalised state space form which has been noted by Verghese (1978).

Consider the normalised system

$$T(D)x(t) = Bu(t)$$
$$y(t) = Cx(t) \tag{4.10}$$

where $T(D)$ is a non-singular polynomial matrix with matrix polynomial

$$T(D) = T_0 D^k + T_1 D^{k-1} + \ldots + T_k \tag{4.11}$$

(D denotes the differential operator $\frac{d}{dt}$), and B, C are constant matrices.

Equations (4.10) may be written as

$$\begin{bmatrix} T(D) & -B \\ C & O \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} O \\ y(t) \end{bmatrix} \tag{4.12}$$

The following procedure transforms the above system of high order differential equations in the variable x to a system of first order differential equations:

Form a strongly irreducible generalised state space realisation $\{I - s\bar{J},\ \bar{B},\ \bar{C}\}$ of the polynomial matrix $T(s)$, so that

$$T(s) = \bar{C}(I - s\bar{J})^{-1}\bar{B} \tag{4.13}$$

Now define the variable z:

$$(I - D\bar{J})z(t) = \bar{B}x(t) \tag{4.14}$$

for all t. Then, from (4.12), (4.13) and (4.14),

$$\bar{C}z(t) = Bu(t), \qquad t \geq 0 \tag{4.15}$$

Hence, (4.12) may be written in the form of the generalised state space system:

$$\left[\begin{array}{cc|c} I - D\bar{J} & -\bar{B} & 0 \\ \bar{C} & 0 & -B \\ \hline 0 & C & 0 \end{array}\right] \begin{bmatrix} z(t) \\ x(t) \\ \hline u(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hline y(t) \end{bmatrix} \tag{4.16}$$

Hence, a generalised state space system may be formed, preserving the finite modes, finite decoupling zero structure, and the infinite frequency free response modes, unobservable modes and uncontrollable modes.

VI.5 System matrix linearisation by Zhang

In applications it is of interest to develop strongly irreducible realisations for singular systems (i.e systems with no input decoupling zeros), and which are controllable and observable in the regular sense and at infinity. This may be done using Bosgra and Van der Weiden's algorithm to produce the system matrix linearisation and then removing the decoupling zeros. Alternatively this may be accomplished directly by Zhang's method which is described as follows:

Considered is the realisation of the system that is left coprimely fractioned as.

$$G(s) = T^{-1}(s)\ U(s) \tag{5.1}$$

The realisation is performed through the polynomial matrix linearisation of $[T(s) \quad U(s)]$.

This realisation is always observable, both in the regular sense and at infinity. The controllability at infinity depends, however, on the row reducedness of $[T(s) \quad U(s)]$. To achieve this row reducedness, a class of unimodular matrix operations is used.

There are computer packages which can produce the transfer function matrix of a system matrix and then produce the left coprimely fractioned form $T^{-1}U$. Zhang's method now may be used to give the irreducible singular system. Hence, a linearisation via Zhang's method can be accomplished directly.

However, producing an irreducible singular system via Bosgra and Van der Weiden's algorithm is not so favourable. A system matrix may be linearised to form an equivalent singular system using Bosgra and Van der Weiden's algorithm. Finite decoupling zeros now may be removed using computer packages. However, as yet, there are no computer packages to remove the infinite decoupling zeros. Hence, to produce strongly irreducible realisations for singular systems which are controllable and observable in the regular sense and at infinity, Zhang's method is preferable and will be used here.

Linearisation of polynomial matrices

Consider, as before, the mxl polynomial matrix with corresponding matrix polynomial

$$P(s) = P_0 + P_1 s + P_2 s^2 + \ldots + P_q s^q \tag{5.2}$$

Denote the row degrees of $P(s)$ by $\alpha_i,\ i = 1, 2, \ldots, m$.

$P(s)$ may be written in the form

$$P(s) = \mathrm{diag}\left(s^{\alpha_i}\right) P_0 + \mathrm{diag}\left(s^{\alpha_i - 1}\right) P_1 + \ldots + \mathrm{diag}\left(s^{\alpha_i - \alpha}\right) P_\alpha \tag{5.3}$$

where

$$\alpha = \max\,(\alpha_i) \tag{5.4}$$

For the negative s-power bases, the rows in the coefficient matrices are all zero.

A matrix $P_L(s)$ is constructed as follows:

$$P_L(s) = \begin{bmatrix} sI & 0 & \ldots & 0 & P_\alpha \\ -I & sI & \ldots & 0 & P_{\alpha-1} \\ 0 & -I & \ddots & \vdots & \vdots \\ \vdots & \vdots & & sI & P_1 \\ 0 & 0 & \ldots & -I & P_0 \end{bmatrix} \tag{5.5}$$

The linearisation of $P(s)$ is formed by deleting all rows and the same number of columns in $P_L(s)$ that correspond to negative s-power bases.

Realisation for singular systems

Using the linearisation of polynomial matrices given above, a realisation for singular systems is proposed. Consider the mxl singular system (5.1). Let

$$P(s) = [T(s) \quad U(s)] \tag{5.6}$$

The realisation of the singular system (5.1) is

$$\left[\begin{array}{ccccc|c} sI & 0 & \ldots & 0 & T_\alpha & U_\alpha \\ -I & sI & \ldots & 0 & T_{\alpha-1} & U_{\alpha-1} \\ 0 & -I & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & sI & T_1 & U_1 \\ 0 & 0 & \ldots & -I & T_0 & U_0 \\ \hline 0 & 0 & \ldots & 0 & -I & 0 \end{array}\right] \tag{5.7}$$

where the coefficients T_i, U_i, $i = 1, \ldots, \alpha$, are from $T(s)$, $U(s)$ respectively in a similar way to (5.3).

In Zhang's paper, the '$-I$' in the last row of matrix (5.7) is quoted as 'I'. In this case, '$-I$' is used because considered throughout this thesis is the conventional Rosenbrock system matrix

$$\left[\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right]$$

instead of the matrix

$$\left[\begin{array}{c|c} sE - A & B \\ \hline C & 0 \end{array}\right]$$

Example 1

$$P(s) = \left[\begin{array}{c|c} T(s) & U(s) \\ \hline -V(s) & W(s) \end{array}\right] = \left[\begin{array}{c|c} s(s+1) & 1 \\ \hline -1 & 0 \end{array}\right]$$

The system is left coprimely fractioned as

$$G(s) = T^{-1}U = \frac{1}{s(s+1)}$$

Hence, considered now is the polynomial matrix linearisation of $[\, s(s+1) \quad 1\,]$.

$$\begin{aligned} P(s) &= [\, s(s+1) \quad 1\,] \\ &= [1 \quad 0]\, s^2 + [1 \quad 0]\, s + [0 \quad 1] \\ &= [T_0 \quad U_0]\, s^2 + [T_1 \quad U_1]\, s + [T_2 \quad U_2] \end{aligned}$$

Hence, the realisation of system $G(s)$ is

$$\left[\begin{array}{ccc|c} s & 0 & 0 & 1 \\ -1 & s & 1 & 0 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

Example 2

$$P(s) = \left[\begin{array}{c|c} T(s) & U(s) \\ \hline -V(s) & W(s) \end{array}\right]$$

$$= \left[\begin{array}{c|c} s^2(s+1) & s \\ \hline -1 & 0 \end{array}\right]$$

This system has an input decoupling zero $s = 0$. Zhang's method of linearisation removes this decoupling zero and considers the polynomial matrix linearisation of $[\,s(s+1) \quad 1\,]$ (example 1 above). However, consider the system without the input decoupling zero removed. (In the next section, the linearisations by Zhang, Vardulakis and that based on Bosgra and Van der Weiden's algorithm are compared for systems with and without decoupling zeros.)

Hence, consider now the polynomial matrix linearisation of $[\,s^2(s+1) \quad s\,]$.

$$\begin{aligned} P(s) &= [\,s^2(s+1) \quad s\,] \\ &= [1 \quad 0]\,s^3 + [1 \quad 0]\,s^2 + [0 \quad 1]\,s + [0 \quad 0] \\ &= [\,T_0 \quad U_0\,]\,s^3 + [\,T_1 \quad U_1\,]\,s^2 + [\,T_2 \quad U_2\,]\,s + [\,T_3 \quad U_3\,] \end{aligned}$$

The linearisation is

$$\left[\begin{array}{cccc|c} s & 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 1 \\ 0 & -1 & s & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

VI.6 Comparison of the three types of linearisation

The computerised algorithm which permits the reduction of a general polynomial matrix to an equivalent matrix pencil form does not include the case of system matrices, and hence needs to be modified as such (see VI.3) so that it can be compared with the linearisations by Zhang and Vardulakis, which are for system matrices.

Example 1

Consider the system

$$P(s) = \left[\begin{array}{c|c} T(s) & U(s) \\ \hline -V(s) & W(s) \end{array}\right] = \left[\begin{array}{c|c} s(s+1) & 1 \\ \hline -1 & 0 \end{array}\right]$$

Zhang's method of linearisation (see example 1 section VI.5) gives

$$\left[\begin{array}{ccc|c} s & 0 & 0 & 1 \\ -1 & s & 1 & 0 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

The linearisation of the system matrix based on Bosgra and Van der Weiden's algorithm (see example 1 section VI.3) is

$$\left[\begin{array}{cccc|c} s & 1 & s & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ s & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

It can be seen that this can be deflated to

$$\left[\begin{array}{ccc|c} s & s & 0 & 1 \\ -1 & 0 & 1 & 0 \\ s & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

Firstly, it will be seen whether the two system matrices, one formed by Zhang's method of linearisation, the other using Bosgra and Van der Weiden's algorithm, are complete system equivalent.

The following relation holds:

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
\hline
0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{ccc|c}
s & 0 & 0 & 1 \\
-1 & s & 1 & 0 \\
0 & -1 & 1 & 0 \\
\hline
0 & 0 & -1 & 0
\end{array}\right]
$$

$$
= \left[\begin{array}{ccc|c}
s & s & 0 & 1 \\
-1 & 0 & 1 & 0 \\
s & -1 & 0 & 0 \\
\hline
0 & 0 & -1 & 0
\end{array}\right]
\left[\begin{array}{ccc|c}
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 1
\end{array}\right]
$$

which shows that the two state space systems are, in fact, complete system equivalent.

Example 2

Now consider the system

$$
P(s) = \left[\begin{array}{c|c}
T(s) & U(s) \\
\hline
-V(s) & W(s)
\end{array}\right]
= \left[\begin{array}{c|c}
s^2(s+1) & s \\
\hline
-1 & 0
\end{array}\right]
$$

This system has an input decoupling zero $s = 0$.

Zhang's method of linearisation (see example 2 section VI.5) gives

$$
\left[\begin{array}{cccc|c}
s & 0 & 0 & 0 & 0 \\
-1 & s & 0 & 0 & 1 \\
0 & -1 & s & 1 & 0 \\
0 & 0 & -1 & 1 & 0 \\
\hline
0 & 0 & 0 & -1 & 0
\end{array}\right]
$$

The system matrix linearisation based on Bosgra and Van der Weiden's algorithm (see example 2 section VI.3) is

$$
\left[\begin{array}{ccccc|c}
0 & s & s & s & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
s & 0 & s-1 & -1 & 0 & 0 \\
s & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & -1 & 0
\end{array}\right]
$$

which can be deflated to

$$\left[\begin{array}{cccc|c} 0 & s & s & s & 0 \\ s & 0 & s-1 & -1 & 0 \\ s & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

It will now be seen whether the two system matrices, one formed by Zhang's method of linearisation, the other using Bosgra and Van der Weiden's algorithm, both formed from a system with a decoupling zero, are complete system equivalent.

The following relation holds:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} s & 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 1 \\ 0 & -1 & s & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 0 & s & s & s & 0 \\ s & 0 & s-1 & -1 & 0 \\ s & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

which shows that the two state space systems are, in fact, complete system equivalent.

Hence, it may be concluded that a relationship exists between Zhang's method of linearisation and the linearisation based on Bosgra and Van der Weiden's algorithm. This relationship exists for systems with and without decoupling zeros.

Example 3

Consider again the system

$$P(s) = \left[\begin{array}{c|c} s^2(s+1) & s \\ \hline -1 & 0 \end{array}\right]$$

Zhang's method of linearisation (see example 2 section VI.5) gives

$$\left[\begin{array}{cccc|c} s & 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 1 \\ 0 & -1 & s & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Vardulakis's method of linearisation (see example 1 section VI.4) gives

$$\left[\begin{array}{ccccc|c} s & -1 & 0 & 0 & 0 & 0 \\ 0 & s+1 & -1 & 0 & 0 & 1 \\ 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline -1 & 0 & 0 & -1 & 0 & 0 \end{array}\right]$$

It can be seen that this can be deflated to

$$\left[\begin{array}{cccc|c} s & -1 & 0 & 0 & 0 \\ 0 & s+1 & -1 & 0 & 1 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

It will now be seen whether this system matrix and the one formed by Zhang's method are complete system equivalent.

The following relation holds:

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \end{array}\right]\left[\begin{array}{cccc|c} s & 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 1 \\ 0 & -1 & s & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} s & -1 & 0 & 0 & 0 \\ 0 & s+1 & -1 & 0 & 1 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

which shows that the two state space systems are, in fact, complete system equivalent.

Hence, it may be concluded that a relationship exists between Zhang's method of linearisation, the linearisation by Vardulakis and also the linearisation based on Bosgra and Van der Weiden's algorithm. This relationship exists for systems with and without decoupling zeros.

In this chapter, it has been seen how the three types of system matrix reduction to singular system form are related. However, although an example has been used to illustrate the relation of complete system equivalence, further work needs to be done in the future to prove this result on a theoretical basis.

It has been noted that to produce strongly irreducible realisations for singular systems it is preferable to use Zhang's method of linearisation. To extend on the work in this thesis, a computerised version of Zhang's method of linearisation could be provided.

CHAPTER VII

ON COMPUTING AN ALGORITHM WHICH PERMITS THE REDUCTION OF A GENERAL POLYNOMIAL MATRIX TO AN EQUIVALENT MATRIX PENCIL FORM

CHAPTER VII

ON COMPUTING AN ALGORITHM WHICH PERMITS THE REDUCTION OF A GENERAL POLYNOMIAL MATRIX TO AN EQUIVALENT MATRIX PENCIL FORM

VII.1 Introduction

Bosgra and Van der Weiden (1981) have given a procedure whereby a general polynomial system matrix may be reduced to an equivalent generalised state space form. The sense in which this is equivalent to the original system matrix is that the reduced system exhibits identical system properties both at finite and infinite frequencies.

In this chapter a computerised version of this algorithm is provided which permits the reduction of a general polynomial matrix to a similarly equivalent matrix pencil form (i.e. one which exhibits identical finite and infinite zero structure). The key to this reduction involves an efficient method of selecting a set of linearly independent rows and columns from a block Toeplitz matrix $\Pi(E)$. It will be seen that a set of linearly independent rows is chosen using the "natural row selection". However, the "natural column selection" will not be used to choose a set of linearly independent columns.

A number of examples are used to illustrate the formation of the equivalent matrix pencil. By using the program by Demianczuk (1985), which computes the infinite frequency structure of a given rational matrix from its Laurent expansion, the equivalent infinite zero property of the matrix pencil and the polynomial matrix can be verified directly.

VII.2 The natural row and column selection

Let $P(s)$ be an mxl polynomial matrix which corresponds to the matrix polynomial

$$P(s) \equiv P_0 + P_1 s + P_2 s^2 + \ldots + P_q s^q \qquad (2.1)$$

Recall that the basis of the reduction method proposed by Bosgra and Van der Weiden (1981) is the selection of a maximum number of linearly independent rows and columns from the matrix $\Pi(E)$, defined by

$$\Pi(E) \triangleq \begin{bmatrix} P_2 & P_3 & \ldots & P_q \\ P_3 & P_4 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ P_q & 0 & \ldots & 0 \end{bmatrix} \qquad (2.2)$$

In view of the probable large dimensions of $\Pi(E)$, an efficient method is needed for the selection of the linearly independent rows and columns from the matrix $\Pi(E)$. Since $\Pi(E)$ is composed of smaller block matrices (and is block Toeplitz), it is easier to consider each of these smaller matrices rather than the large matrix itself when selecting required rows and columns. The reason for this is to save on computer storage space. It is not necessary to store the matrix $\Pi(E)$, but instead store each of the submatrices $P_2, P_3, \ldots, P_q$.

Consider starting from the last row in matrix P_q. Select the largest element from this row. This will be the pivot element. If all elements in the row are zero, move up to the next row, etc. The reason for choosing the largest element is for stability reasons since dividing all elements in a row by the largest element causes a smaller percentage error in the calculations than by dividing by, say, the smallest element.

A matrix F is now formed such that when pre-multiplied by P_q will set a "1" in the pivot position and zero all other elements in the row. In the program, a matrix P is that formed by multiplying P_q by F.

Example 1

Consider a polynomial matrix $P(s)$ with matrix polynomial

$$P(s) = P_0 + P_1 s + P_2 s^2 + P_3 s^3$$

where

$$P_2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

and

$$P_3 = \begin{bmatrix} 2 & 6 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

Looking at the last row in matrix P_3, the pivot element is "2". Hence the matrix F is

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e $P_3 \mathrm{x} F$ produces a "1" in the pivot position and zeros all other elements in the row:

$$P = P_3 \mathrm{x} F = \begin{bmatrix} -1 & 3 & -1 & -1 \\ 0 & 2 & 0 & 0 \\ -1/2 & 3/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The "1" formed in the pivot position represents linear independence i.e. row 4 and column 2 are linearly independent. It will be noted that the method of row selection described here is the "natural row selection". At the same time this produces a linearly independent column. Hence, linearly independent columns may be selected in this way. This method is better than performing the "natural row selection", transposing the matrix $\Pi(E)$ and then performing another selection ("natural column selection"). The method described here is more efficient in that both a set of linearly independent rows and columns from a matrix $\Pi(E)$ may be selected at the same time, instead of being selected by performing the selection process twice.

Now select from matrix P the largest element (in modulus) from the next row up. If this pivot element lies in the same column as that already selected for linear independence, then it may be ignored and the next largest element in the row is chosen as the pivot element. The reason for this can be seen by considering example 1 again.

Looking at row 3 of matrix P. the pivot element is "3/2" which is in the same column as the "1" from the last row of P. Forming the matrix F as above using "3/2" as the pivot element and pre-multiplying by P above will change the last row in P and hence will no longer give the linearly independent row 4. Therefore, although it is important to select the largest element as a pivot (for stability reasons described earlier), this does not apply when the largest element lies in a column which has already been selected for linear independence. Instead, the next largest element in the row is chosen as the pivot element. This does not change the stability of the matrix since, using example 1, the linearly independent column 2 has already been selected and now may be ignored.

Again a matrix F is formed which when pre-multiplied by P will set a "1" in the pivot position and zero all other elements in the row. This process of selecting linearly independent rows (and columns) continues until all rows in P_q have been examined for linear independence. The row and column numbers corresponding to linear independence in the submatrix P_q are now stored.

In the program, once a linearly independent column has been selected it is ignored by setting other elements in the column to zero. Consider example 1 again. Continuing the "natural row selection", the matrix F now formed is

$$F = \begin{bmatrix} -2 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

remembering that elements in column 2, rows 1, 2, 3 of matrix P are set to zero because column 2 is linearly independent, giving

$$PxF = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Row 3 and column 1 are now linearly independent and so the final matrix P formed is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

All linearly independent rows and columns have now been selected from the submatrix P_3. The row and column numbers corresponding to linear independence in the submatrix P_3 (i.e. rows 3,4 and columns 1,2) are now stored.

In the program, a matrix $F1$ is formed (from the F matrices) which when pre-multiplied by P_q will produce all linearly independent rows and columns in P_q. Considering example 1, the matrix $F1$ is

$$F1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -1 & -1 \\ 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying P_3 by $F1$ gives

$$P_3xF1 = \begin{bmatrix} 2 & 6 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 & -1 \\ 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Hence. P_3 has linearly independent rows $3, 4$ and linearly independent columns $1, 2$ (remember that a "1" represents linear independence).

Consider now the next block up in $\Pi(E)$ i.e. submatrix P_{q-1} which in the program is called $P1$. $P1$ is multiplied by $F1$ (found previously) and the resulting matrix is called P. P now has elements in certain rows and columns set to zero. These rows and columns are those which correspond to linear independence in the submatrix P_q. These are set to zero because if, say, rows $h_1, h_2, \ldots, h_k$ are selected from P_q, then these rows will always be selected when considering $P_{q-1}, \ldots, P_2$. Hence, such rows and columns may be ignored by setting them to zero. (It will be seen later that it is important to multiply $P1$ by $F1$ firstly and then zero the elements from particular rows and columns (i.e. the rows and columns that correspond to linear independence in P_q), rather than performing the operations the other way round.) With the resulting matrix, the whole process of selecting a pivot element starting from the bottom row continues until all rows in P_{q-1} have been examined for linear independence. The row and column numbers corresponding to any linearly independent rows and columns from this resulting matrix are now stored. Consider again example 1. The matrix $\Pi(E)$ (defined by (2.2)) is

$$\Pi(E) = \left[\begin{array}{cccc|cccc} 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 \\ 4 & 2 & 1 & 4 & 2 & 4 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ \hline 2 & 6 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \tag{2.3}$$

Consider now the submatrix

$$P_2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

This is now multiplied by $F1$ to give

$$P_2 \mathrm{x} F1 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 & -1 \\ 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 0 & 0 \\ -6 & 1 & -3 & 0 \\ -1 & 1/2 & 0 & 0 \\ -3 & 1/2 & -1 & 0 \end{bmatrix}$$

P_3 had linearly independent rows $3, 4$ and linearly independent columns $1, 2$, so setting elements from those rows and columns to zero in the above matrix gives

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With this resulting matrix, the process of selecting linearly independent rows and columns continues giving the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row 2 and column 3 are now stored since they have now been selected for linear independence.

Hence, the matrix $\Pi(E)$ has 5 linearly independent rows i.e. rows $2, 3, 4, 7, 8$. Also $\Pi(E)$ has 5 linearly independent columns i.e. columns $1, 2, 3, 5, 6$.

It will now be seen that it is important to multiply $P1$ by $F1$ firstly and then zero the elements from particular rows and columns (i.e. the rows and columns that correspond to linear independence in P_q), rather than performing the operations the other way round. Consider example 1. From P_3, rows $3, 4$ and columns $1, 2$ are linearly independent. Now suppose that in matrix P_2 the elements in these rows and columns are set to zero firstly i.e. suppose P_2 becomes

$$P_2 = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiplying now by $F1$ gives

$$P_2 \text{x} F1 = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 & -1 \\ 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose now that the process of selecting linearly independent rows and columns continues. It can be seen that 2 further linearly independent rows (and columns) i.e. rows 1, 2 will be selected from the matrix above. However, this is incorrect. It can be seen from $\Pi(E)$ (see (2.3)) that rows 1 and 3 are linearly dependent and row 3 has already been selected. Hence, $P1$ needs to be multiplied by $F1$ firstly before elements from the rows and columns are zeroed.

The selection process now continues until all submatrices P_q to P_2 have been examined for linear independence.

VII.3 Forming the matrix pencil

A method is needed such that only linearly independent rows and columns from $\Pi(E)$ are printed in the program. All row and column numbers of linearly independent rows and columns in each submatrix have already been noted. Consider the submatrix P_2. All linearly independent rows and columns in P_2 are those found in P_q, plus additional linearly independent rows and columns found in $P_{q-1}, \ldots, P_2$.

Consider example 1 section VII.2.

$$\Pi(E) = \begin{bmatrix} P_2 & P_3 \\ P_3 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc} 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 \\ 4 & 2 & 1 & 4 & 2 & 4 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ \hline 2 & 6 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \qquad (3.1)$$

Recall that in P_3 rows 3 and 4 are linearly independent and in P_2 an additional row, row 2, is linearly independent. All the linearly independent rows in P_2 are those found in P_3 i.e. rows $3, 4$ and the additional row in P_2 i.e. row 2. Hence, rows $2, 3, 4$ are stored. Consider the submatrix P_3. Rows 3 and 4 are linearly independent. To correspond to the appropriate row numbers from $\Pi(E)$, to each row number is added $m = 4$ where m is the number of rows in each submatrix P_2, P_3 i.e. rows 7 and 8 are now stored. Hence, the linearly independent rows in $\Pi(E)$ are rows $2, 3, 4, 7, 8$.

In the program, there is a section of code which converts the row (and column) numbers of the linearly independent rows (and columns) in each submatrix to the actual row (and column) numbers that correspond to $\Pi(E)$. These row (resp. column) numbers are now stored in a matrix RR (resp. CC) whose dimension depends on the number of linearly independent rows (resp. columns). Hence, all required rows and columns can be printed.

Let the positive sets of integers $I \triangleq \{i_1, i_2, \ldots, i_{\rho(E)}\}$ (resp. $J \triangleq \{j_1, j_2, \ldots, j_{\rho(E)}\}$), $\rho(E) \triangleq \operatorname{rank} \Pi(E)$, define a row (resp. column) selection also denoted I (resp. J) from $\Pi(E)$ of $\rho(E)$ linearly independent rows (resp. columns). Let P_E (resp. P_A) be that submatrix of $\Pi(E)$ (resp. $\Pi(A)$) formed from rows of the selection I and columns of the selection J. Let P_B be the submatrix of $\Pi(B)$ formed from the rows of the selection I, and P_C be the submatrix of $\Pi(C)$ formed from the columns of the selection J. P_E, P_A, P_B, P_C may now be assembled in the following form to give the required matrix pencil $P_F(s)$:

$$P_F(s) = \begin{bmatrix} P_E - sP_A & P_B s \\ -P_C s & P_1 s + P_0 \end{bmatrix}$$

or

$$P_F(s) = \begin{bmatrix} P_E & 0 \\ 0 & P_0 \end{bmatrix} + \begin{bmatrix} -P_A & P_B \\ -P_C & P_1 \end{bmatrix} s \tag{3.2}$$

which is how the program gives the matrix pencil.

VII.4 Examples using the algorithm

This section illustrates a number of examples where it can be seen that a polynomial matrix $P(s)$ and its associated matrix pencil form $P_F(s)$ have identical finite and infinite zeros. Firstly, the program described in this chapter is used to produce the matrix pencil equivalent of the general polynomial matrix. Then by using the program by Demianczuk (1985), which computes the infinite frequency structure of a given rational matrix from its Laurent expansion, the zeros at infinity of both forms are produced, thus confirming that the infinite zero structures are identical. Also, an example is performed by hand to see that the polynomial matrix $P(s)$ and the matrix pencil $P_F(s)$ constructed from $P(s)$ do have identical finite and infinite zero structures.

Example 1

$$P(s) = \begin{bmatrix} s^2+1 & s^4 \\ 0 & s \end{bmatrix}$$

The program which reduces a general polynomial matrix to an equivalent matrix pencil form gives the following:

Enter q, the highest power of s

4

Enter number of rows of P

2

Enter number of columns of P

2

Enter P .0

1 0
0 0

Enter P 1.0

0 0
0 1

Enter P 2.0

1 0
0 0

Enter P 3.0

$$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

Enter $\quad P \quad 4.0$

$$\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

$$\begin{matrix} .00 & 1.00 \\ .00 & .00 \end{matrix}$$

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

1

Enter column numbers of l.i. columns

2

$$\begin{matrix} .00 & .00 \\ .00 & .00 \end{matrix}$$

Enter number of l.i. rows

0

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The s coefficient is :

$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, the matrix pencil has been formed. Now, Demianczuk's program gives, for both $P(s)$ and its linearisation, a zero at infinity of degree 1 i.e. both the general polynomial matrix and its associated matrix pencil form have identical infinite zero structure.

The example is now performed by hand to see that the polynomial matrix $P(s)$ and its associated matrix pencil form $P_F(s)$ have identical finite zero structure.

It is necessary to produce the Smith forms of $P(s)$ and $P_F(s)$. Firstly consider the matrix

$$P(s) = \begin{bmatrix} s^2+1 & s^4 \\ 0 & s \end{bmatrix}$$

Now perform elementary row and column operations on $P(s)$:

$$\begin{bmatrix} s^2+1 & s^4 \\ 0 & s \end{bmatrix} \xrightarrow[(s^2 \text{ x } col.\ 1)\ -\ (col.\ 2)]{new\ col.\ 2\ =} \begin{bmatrix} s^2+1 & s^2 \\ 0 & -s \end{bmatrix}$$

$$\xrightarrow[(col.\ 1)\ -\ (col.\ 2)]{new\ col.\ 2\ =} \begin{bmatrix} s^2+1 & 1 \\ 0 & s \end{bmatrix}$$

$$\xrightarrow[col.\ 1\ and\ col.\ 2]{interchange} \begin{bmatrix} 1 & s^2+1 \\ s & 0 \end{bmatrix}$$

$$\xrightarrow[(s \text{ x } row\ 1)\ -\ (row\ 2)]{new\ row\ 2\ =} \begin{bmatrix} 1 & s^2+1 \\ 0 & s(s^2+1) \end{bmatrix}$$

$$\xrightarrow[(col.\ 2)\ -\ ((s^2+1) \text{ x } col.\ 1)]{new\ col.\ 2\ =} \begin{bmatrix} 1 & 0 \\ 0 & s(s^2+1) \end{bmatrix}$$

Hence, $P(s)$ has Smith form

$$S(P) = \begin{bmatrix} 1 & 0 \\ 0 & s(s^2+1) \end{bmatrix}$$

Now consider the matrix

$$P_F(s) = \begin{bmatrix} 0 & -s & 1 & s & 0 \\ -s & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & s \\ 0 & 0 & -s & 1 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

which is the matrix pencil constructed from $P(s)$. Perform elementary row and column operations on $P_F(s)$:

$$P_F(s) \xrightarrow[col.\ 1 \text{ and } col.\ 3]{interchange} \begin{bmatrix} 1 & -s & 0 & s & 0 \\ 0 & 1 & -s & 0 & 0 \\ 0 & 0 & 1 & 0 & s \\ -s & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

$$\xrightarrow[\substack{\textit{new col. } 4 = \\ (\textit{col. } 4) \;-\; (s \text{ x } \textit{col. } 1)}]{\substack{\textit{new col. } 2 = \\ (\textit{col. } 2) + (\textit{col. } 4)}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -s & 0 & 0 \\ 0 & 0 & 1 & 0 & s \\ -s & 1 & 0 & 1+s^2 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

$$\xrightarrow[(\textit{col. } 3) \;+\; (s \text{ x } \textit{col. } 2)]{\textit{new col. } 3 =} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & s \\ -s & 1 & s & 1+s^2 & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

$$\xrightarrow[(\textit{col. } 5) \;-\; (s \text{ x } \textit{col. } 3)]{\textit{new col. } 5 =} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -s & 1 & s & 1+s^2 & -s^2 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

$$\xrightarrow[\substack{\textit{to zero using} \\ \textit{rows } 1,2,3}]{\substack{\textit{elements in row } 4, \\ \textit{cols. } 1,2,3 \textit{ are set}}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1+s^2 & -s^2 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

$$\xrightarrow[(\textit{col. } 4) \;+\; (\textit{col. } 5)]{\textit{new col. } 5 =} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1+s^2 & 1 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

$$\xrightarrow[\textit{col. } 4 \text{ and } \textit{col. } 5]{\textit{interchange}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1+s^2 \\ 0 & 0 & 0 & s & 0 \end{bmatrix}$$

$$\xrightarrow[(s \text{ x } \textit{row } 4) \;-\; (\textit{row } 5)]{\textit{new row } 5 =} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1+s^2 \\ 0 & 0 & 0 & 0 & s(1+s^2) \end{bmatrix}$$

$$\xrightarrow[(\textit{col. } 5) \;-\; ((1+s^2) \text{ x } \textit{col. } 4)]{\textit{new col. } 5 =} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s(1+s^2) \end{bmatrix}.$$

Hence, $P_F(s)$ has Smith form

$$S(P_F) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s(s^2+1) \end{bmatrix}$$

$$= \begin{bmatrix} I_3 & 0 \\ 0 & S(P) \end{bmatrix}$$

i.e. the polynomial matrix $P(s)$ and its associated matrix pencil form $P_F(s)$ have identical finite zero structure.

It may also be shown that $P(s)$ has Smith McMillan form at infinity

$$S^\infty(P) = \begin{bmatrix} s^4 & 0 \\ 0 & 1/s \end{bmatrix}$$

i.e. $P(s)$ has an infinite zero of degree 1 and an infinite pole of degree 4. Also it may be shown that $P_F(s)$ has Smith McMillan form at infinity

$$S^\infty(P_F) = \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 1/s \end{bmatrix}$$

i.e. $P_F(s)$ has an infinite zero of degree 1, and 4 infinite poles of degree 1. Hence it may be seen that the polynomial matrix $P(s)$ and its associated matrix pencil form $P_F(s)$ have identical infinite zero structure.

It is interesting to note that the Toeplitz matrix at infinity is built up by including terms corresponding to decreasing powers in s. Therefore, when using the program by Demianczuk, it is important to enter the lowest s power as a number low enough to include all infinite zeros. For the examples considered, in the first instance, the lowest power "0" is entered, producing all poles at infinity. Then the McMillan degree may be calculated from this and the lowest power is then entered as the negative of the McMillan degree, hence producing all zeros at infinity. The following illustrates the importance of entering in

Demianczuk's program an s power lower than the lowest s power shown in the example. Consider example 1 section II.5.

$$T(s) = \begin{bmatrix} s^3 & s^2 & 1 \\ -1 & 0 & 0 \\ -s & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^3 + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= G_3 s^3 + G_2 s^2 + G_1 s^1 + G_0 s^0 + G_{-1} s^{-1} + G_{-2} s^{-2}$$

where

$$G_{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$G_{-2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The lowest s power here is "0". It can be seen from example 1 section II.5 that the Toeplitz matrix T_1^∞ has been constructed using the matrix G_{-1} and the Toeplitz matrix T_2^∞ has been constructed using the matrices G_{-1} and G_{-2} i.e. to produce both the infinite zeros the lowest s power needs to be entered as "-2" and not "0". Hence, this example illustrates the importance of entering in Demianczuk's program the lowest s power as a number low enough to give the infinite zeros.

Example 2

$$P(s) = \begin{bmatrix} 1 & 0 \\ s^2+1 & 1 \end{bmatrix}$$

The program which reduces a general polynomial matrix to an equivalent matrix pencil form gives the following:

Enter q, the highest power of s

2

Enter number of rows of P

2

Enter number of columns of P

2

Enter P .0

1 0
1 1

Enter P 1.0

0 0
0 0

Enter P 2.0

0 0
1 0

.00 .00
1.00 .00

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

2

Enter column numbers of l.i. columns

1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The s coefficient is :

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Hence, the matrix pencil has been formed. Now, Demianczuk's program gives, for both $P(s)$ and its linearisation, a zero at infinity of degree 2 i.e. both the general polynomial matrix and its associated matrix pencil form have identical infinite zero structure.

Example 3

$$P(s) = \begin{bmatrix} s^3+1 & 0 & s^2 \\ 0 & s & s^2 \\ s^2 & 0 & s \end{bmatrix}$$

The program which reduces a general polynomial matrix to an equivalent matrix pencil form gives the following:

Enter q, the highest power of s

3

Enter number of rows of P

3

Enter number of columns of P

3

Enter P .0

```
1 0 0
0 0 0
0 0 0
```

Enter P 1.0

```
0 0 0
0 1 0
0 0 1
```

Enter P 2.0

```
0 0 1
0 0 1
1 0 0
```

Enter P 3.0

```
1 0 0
0 0 0
0 0 0
```

```
1.00  .00  .00
 .00  .00  .00
 .00  .00  .00
```

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

1

Enter column numbers of l.i. columns

1

$$\begin{matrix} .00 & .00 & .00 \\ .00 & .00 & 1.00 \\ .00 & .00 & .00 \end{matrix}$$

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

2

Enter column numbers of l.i. columns

3

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The s coefficient is :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, the matrix pencil has been formed. Now. Demianczuk's program gives, for both $P(s)$ and its linearisation, a zero at infinity of degree 3 i.e. both the general polynomial matrix and its associated matrix pencil form have identical infinite zero structure.

Example 4

$$P(s) = \begin{bmatrix} s^3 & s^2 & 1 \\ -1 & 0 & 0 \\ -s & -1 & 0 \end{bmatrix}$$

The program which reduces a general polynomial matrix to an equivalent matrix pencil form gives the following:

Enter q, the highest power of s

3

Enter number of rows of P

3

Enter number of columns of P

3

Enter P .0

$$\begin{matrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix}$$

Enter P 1.0

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix}$$

Enter P 2.0

$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Enter P 3.0

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1.00 & .00 & .00 \\ .00 & .00 & .00 \\ .00 & .00 & .00 \end{matrix}$$

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

1

Enter column numbers of l.i. columns

1

$$\begin{matrix} .00 & .00 & .00 \\ .00 & .00 & .00 \\ .00 & .00 & .00 \end{matrix}$$

Enter number of l.i. rows

0

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The s coefficient is :

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Now, Demianczuk's program gives, for both $P(s)$ and its linearisation, a zero at infinity of degree 1 and a zero at infinity of degree 2 i.e. both forms have identical infinite zero structure. (It was seen earlier in example 2 section II.4 and example 1 section II.5 that $P(s)$ had one infinite zero of degree 1 and one infinite zero of degree 2.)

Example 5

$$P(s) = \begin{bmatrix} s^5 & s^2 & 1 \\ 1 & 0 & 0 \\ s^3 & 1 & 0 \end{bmatrix}$$

The program which reduces a general polynomial matrix to an equivalent matrix pencil form gives the following:

Enter q, the highest power of s

5

Enter number of rows of P

3

Enter number of columns of P

3

Enter P .0

0	0	1
1	0	0
0	1	0

Enter P 1.0

0	0	0
0	0	0
0	0	0

Enter P 2.0

0	1	0
0	0	0
0	0	0

Enter P 3.0

0	0	0
0	0	0
1	0	0

Enter P 4.0

0	0	0
0	0	0
0	0	0

Enter P 5.0

1	0	0
0	0	0
0	0	0

1.00	.00	.00
.00	.00	.00
.00	.00	.00

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

1

Enter column numbers of l.i. columns

1

$$\begin{matrix} .00 & .00 & .00 \\ .00 & .00 & .00 \\ .00 & .00 & .00 \end{matrix}$$

Enter number of l.i. rows

0

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The s coefficient is :

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, Demianczuk's program gives, for both $P(s)$ and its linearisation, a zero at infinity of degree 2 and a zero at infinity of degree 3 i.e. both forms have identical infinite zero structure.

Example 6

$$P(s) = \begin{bmatrix} s^8 & 0 & s^4 & s \\ s^7 & 1 & s^3 & 1 \\ 1 & 0 & 0 & 0 \\ s^4 & 0 & 1 & 0 \end{bmatrix}$$

The program which reduces a general polynomial matrix to an equivalent matrix pencil form gives the following:

Enter q, the highest power of s

8

Enter number of rows of P

4

Enter number of columns of P

4

Enter P .0

0	0	0	0
0	1	0	1
1	0	0	0
0	0	1	0

Enter P 1.0

0	0	0	1
0	0	0	0
0	0	0	0
0	0	0	0

Enter P 2.0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Enter P 3.0

0	0	0	0
0	0	1	0
0	0	0	0
0	0	0	0

Enter P 4.0

0	0	1	0
0	0	0	0
0	0	0	0
1	0	0	0

Enter P 5.0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Enter P 6.0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Enter P 7.0

0	0	0	0
1	0	0	0
0	0	0	0
0	0	0	0

Enter P 8.0

1	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

1.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

1

Enter column numbers of l.i. columns

1

.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00

Enter number of l.i. rows

0

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The s coefficient is :

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Now, Demianczuk's program gives, for both $P(s)$ and its linearisation, a zero at infinity of degree 3 and a zero at infinity of degree 4 i.e. both forms have identical infinite zero structure.

Example 7

$$
P(s) = \begin{bmatrix}
s^6 + s^3 & s^5 & s^3 & 1 \\
s^3 & s^2 & 1 & 0 \\
s & 1 & 0 & 0 \\
s^3 + 1 & s^2 & 1 & 0
\end{bmatrix}
$$

The program which reduces a general polynomial matrix to an equivalent matrix pencil form gives the following:

Enter q, the highest power of s

6

Enter number of rows of P

4

Enter number of columns of P

4

Enter P .0

$$
\begin{matrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{matrix}
$$

Enter P 1.0

0	0	0	0
0	0	0	0
1	0	0	0
0	0	0	0

Enter P 2.0

0	0	0	0
0	1	0	0
0	0	0	0
0	1	0	0

Enter P 3.0

1	0	1	0
1	0	0	0
0	0	0	0
1	0	0	0

Enter P 4.0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Enter P 5.0

0	1	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Enter P 6.0

1	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

1.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00

Enter number of l.i. rows

1

Enter row numbers of l.i. rows

1

Enter column numbers of l.i. columns

1

.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00
.00	.00	.00	.00

Enter number of l.i. rows

0

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The s coefficient is :

$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, Demianczuk's program gives, for both $P(s)$ and its linearisation, a zero at infinity with multiplicity 3 and degrees $1, 2, 3$ i.e. both forms ~~have~~ identical infinite zero structure.

In this section a number of examples have been used to illustrate the formation of the equivalent matrix pencil of a general polynomial matrix. Then, by using Demianczuk's program (1985), which computes the infinite frequency structure of a given rational matrix from its Laurent expansion, the equivalent infinite zero property of the matrix pencil and the polynomial matrix has been verified directly.

CHAPTER VIII

CONCLUSIONS

VIII

CONCLUSIONS

In this thesis, the Hayton et al. (1989) algorithm which produces a matrix pencil equivalent of a given polynomial matrix has been computerised. The sense in which this is equivalent has been described from two points of view. Firstly, it has been seen that the reduction algorithm is a full system equivalence transformation. Secondly, the polynomial matrix and the associated matrix pencil have identical finite and infinite zero structures.

The computerised algorithm has been tested by a number of examples to see that the linearised form produced does have the same finite and infinite zero structure as the given polynomial matrix. Here, Demianczuk's program (1985), which computes the infinite frequency structure of a given rational matrix from its Laurent expansion, has been used to produce the infinite zeros of both the original polynomial matrix and its associated matrix pencil form, and it has been seen that the infinite frequency property has been preserved. An example has been performed by hand to see that the finite frequency property is preserved.

Three methods of system matrix reduction to linear polynomial form have been described. Firstly discussed is the Hayton et al. (1989) algorithm, and it has been seen how this is based on the Bosgra and Van der Weiden (1981) reduction procedure whereby a general polynomial system matrix may be reduced to an equivalent generalised state space form. Another method discussed is the reduction of a polynomial matrix of a linear multivariable system to generalised state space form proposed by Vardulakis (1991). The final reduction is the linearisation described by Zhang (1989) which produces a strongly irreducible realisation for singular systems. These three types of linearisations have been compared via an example. It has been seen using the example that all three types of linearisation are, in fact, related by complete system equivalence.

However, in the future, further work needs to be done to prove this result on a theoretical basis. Also, it has been seen that to produce an irreducible singular system Zhang's method of linearisation is preferable to the other two methods. To extend on the work in this thesis, a computerised version of Zhang's method of linearisation could be provided.

REFERENCES

REFERENCES

BOSGRA, O.H. and VAN DER WEIDEN, A.J.J. (1981),

Realisations in generalised state space form for polynomial system matrices, and the definitions of poles, zeros and decoupling zeros at infinity. *International Journal of Control*, vol. 33, no. 3, pp. 393-411.

DEMIANCZUK, O. (1985),

On computing the infinite frequency structure of a rational matrix. M.Sc. dissertation, Loughborough University of Technology, U.K.

DESOER, C.A. and SCHULMAN, J.D. (1974),

Zeros and poles of matrix transfer functions and their dynamical interpretation. *I.E.E.E. Transactions on Circuits and Systems*, vol. 21, no. 1, pp. 3-7.

FERREIRA, PEDRO M.G. (1980),

Infinite system zeros. *International Journal of Control*, vol. 32, no. 4, pp. 731-735.

HAYTON, G.E. , PUGH, A.C. and FRETWELL, P. (1988),

Infinite elementary divisors of a matrix polynomial and implications. *International Journal of Control*, vol. 47, no. 1, pp. 53-64.

HAYTON, G.E. , WALKER, A.B. and PUGH, A.C. (1988),

On a transformation for general polynomial matrices with applications. *Linear Circuits, Systems and Signal Processing:* Theory and Application, pp. 101-108.

HAYTON, G.E. , WALKER, A.B. and PUGH, A.C. (1989),

Matrix pencil equivalents of a general polynomial matrix. *International Journal of Control*, vol. 49, no. 6, pp. 1979-1987.

HAYTON, G.E. , WALKER, A.B. and PUGH, A.C. (1990),

Infinite frequency structure-preserving transformations for general polynomial system matrices. *International Journal of Control*, vol. 52, no. 1, pp. 1-14.

KAILATH, T. (1980),

Linear systems, (Prentice-Hall).

KARAMPETAKIS, N.P. and VARDULAKIS, A.I. (1990),

Generalised state space system matrix equivalent of a Rosenbrock system matrix. Internal report of Department of Mathematics, University of Thessaloniki, Greece.

PUGH, A.C. and HAYTON, G.E. (1990),

Transformation issues in linear systems theory. *Progress in Systems and Control Theory 3*, vol. 1, pp. 107-115.

PUGH, A.C. , HAYTON, G.E. and FRETWELL, P. (1987),

Transformation of matrix pencils and implications in linear systems theory. *International Journal of Control*, vol. 45, no. 2, pp. 529-548.

PUGH, A.C. , HAYTON, G.E. and WALKER, A.B. (1990),

System matrix characterisation of input-output equivalence. *International Journal of Control*, vol. 51, no. 6, pp. 1319-1326.

PUGH, A.C. , JONES, E.R.L. , DEMIANCZUK, O. and HAYTON, G.E. (1989),

Infinite frequency structure and a certain matrix Laurent expansion. *International Journal of Control*, vol. 50, no. 5, pp. 1793-1805.

PUGH, A.C. and KRISHNASWAMY, V. (1985),

Algebraic and dynamic characterisations of poles and zeros at infinity. *International Journal of Control*, vol. 42, no. 5, pp. 1145-1153.

PUGH, A.C. and RATCLIFFE, P.A. (1979),

On the zeros and poles of a rational matrix. *International Journal of Control*, vol. 30, no. 2, pp. 213-226.

PUGH, A.C. and SHELTON, A.K. (1978),

On a new definition of strict system equivalence. *International Journal of Control*, vol. 27, no. 5, pp. 657-672.

ROSENBROCK, H.H. (1970),

State space and multivariable theory, (Nelson).

ROSENBROCK, H.H. (1973),

The zeros of a system. *International Journal of Control*, vol. 18, no. 2, pp. 297-299.

ROSENBROCK, H.H. (1974),

Correction to 'The zeros of a system'. *International Journal of Control*, vol. 20, no. 3, pp. 525-527.

VARDULAKIS, A.I.G. (1991),

On the transformation of a polynomial matrix model of a linear multivariable system to generalised state space form. Internal report of Department of Mathematics, University of Thessaloniki, Greece.

VARDULAKIS, A.I.G. (1991),

Linear multivariable control, (Wiley).

VERGHESE, G.C. (1978),

Infinite frequency behaviour in generalised dynamical systems. Ph. D. dissertation, Stanford University, U.S.A.

WALKER, A.B. (1988),

Equivalence transformations for linear systems. Ph. D. dissertation, Hull University, U.K.

ZHANG, S.Y. (1989),

Polynomial matrix linearisation and strongly irreducible realisation for singular systems. *International Journal of Control*, vol. 49, no. 2, pp. 471-479.

BARNETT, S. (1971),

Matrices in control theory, (Krieger).

MACMILLAN, B. (1952),

Introduction to formal realisability theory - II. Bell System Tech., vol. 31, pp. 541 - 600.

PUGH, A.C. (1976),

The McMillan degree of a polynomial system matrix. International Journal of Control, vol. 24, no. 1, pp. 129 - 135.

VAN DOOREN, P.M., DEWILDE, P. and VANDEWALLE, J. (1979),

On the determination of the Smith Mc Millan form of a rational matrix. I.E.E.E. Transactions on Circuits and Systems, vol. 26, pp. 180 - 189.

VARDULAKIS, A.I.G., LIMEBEER, D.N.J. and KARCANIAS, N. (1982),

Structure and Smith McMillan form of a rational matrix at infinity. International Journal of Control, vol. 35, no. 4, pp. 701 - 725.

WOLOVICH, W.A. (1974),

Linear multivariable systems. Springer-Verlag, New York, Inc.

APPENDIX

Program which reduces a general polynomial matrix to an equivalent matrix pencil form.

```
double precision P(10,10),P0(10,10),P1(10,10),P2(10,10)
double precision PP(0:10,10,10)
double precision E(50,50),A(50,50),B(50,50),C(50,50)
double precision F(10,10),F1(10,10),F2(10,10)
double precision FF1(30,30),FF2(30,30)
double precision RR(20),CC(20)
real v,s
integer m,l,q,k,r(2:10),aa(2:10,10),bb(2:10,10)
integer w,x,y
w=10
x=10
y=10
print*,' Enter q,the highest power of s '
read*,q
print*,' Enter number of rows of P '
read*,m
print*,' Enter number of columns of P '
read*,l
do 1 h=0,q
print*,' '
print*,' Enter P ',h
read*,(( PP(h,i,j),j=1,l),i=1,m)
continue
do 2 h=q+1,10
   do 3 i=1,m
      do 4 j=1,l
          PP(h,i,j)=0
      continue
   continue
continue
do 5 k1=0,q-2
   do 6 i=(k1*m)+1,(k1+1)*m
      do 7 n=0,q-2
         do 8 j=(n*l)+1,(n+1)*l
             E(i,j)=PP(n+k1+2,i-(k1*m),j-(n*l))
         continue
      continue
   continue
continue
do 9 k1=0,q-3
   do 10 i=(k1*m)+1,(k1+1)*m
      do 11 n=0,q-3
         do 12 j=(n*l)+1,(n+1)*l
              A(i,j)=PP(n+k1+3,i-(k1*m),j-(n*l))
2        continue
1     continue
0  continue
continue
do 13 i=((q-2)*m)+1,(q-1)*m
   do 14 j=1,(q-1)*l
        A(i,j)=0
4  continue
3 continue
do 15 i=1,(q-1)*m
   do 16 j=((q-2)*l)+1,(q-1)*l
        A(i,j)=0
6  continue
5 continue
do 17 k1=0,q-2
   do 18 i=(k1*m)+1,(k1+1)*m
      do 19 j=1,l
          B(i,j)=PP(k1+2,i-(k1*m),j)
```

```
19          continue
18       continue
17    continue
      do 20 i=1,m
         do 21 n=0,q-2
            do 22 j=(n*l)+1,(n+1)*l
                 C(i,j)=PP(n+2,i,j-(n*l))
22          continue
21       continue
20    continue
      do 23 h=q,2,-1
         do 24 i=1,m
            do 25 j=1,l
                 P(i,j)=PP(h,i,j)
25          continue
24       continue
100      do 26 i1=m,1,-1
            do 27 i2=1,l
                 if ( P(i1,i2).ne.0 )  then
                 v=abs(P(i1,i2))
                 s=i2
                 do 28 i3=i2+1,l
                      if ( abs(P(i1,i3)).gt.v ) then
                      v=abs(P(i1,i3))
                      s=i3
                      endif
28               continue
                 F(s,s)=1/P(i1,s)
                 if ( s.ge.2 ) then
                 do 29 j=1,s-1
                      F(s,j)=-P(i1,j)/P(i1,s)
29               continue
                 do 30 i=1,s-1
                      F(i,i)=1
30               continue
                 do 31 i=1,s-1
                    do 32 j=i+1,l
                         F(i,j)=0
32                  continue
31               continue
                 endif
                 if ( s.ge.3 ) then
                 do 33 j=1,s-2
                    do 34 i=j+1,s-1
                         F(i,j)=0
34                  continue
33               continue
                 endif
                 if ( s.le.(l-1)) then
                 do 35 j=s+1,l
                      F(s,j)=-P(i1,j)/P(i1,s)
35               continue
                 do 36 i=s+1,l
                      F(i,i)=1
36               continue
                 do 37 i=s+1,l
                    do 38 j=1,i-1
                         F(i,j)=0
38                  continue
37               continue
                 endif
                 if ( s.le.(l-2)) then
```

```
          do 39 i=s+1,l-1
             do 40 j=i+1,l
                  F(i,j)=0
40           continue
39        continue
          endif
          if ( i1.lt.k ) then
          ifail=0
          call f01ckf(f2,f1,f,w,x,y,z,l,l,ifail)
          do 41 i=1,m
             do 42 j=1,l
                  F1(i,j)=F2(i,j)
42           continue
41        continue
          goto 200
          endif
          do 43 i=1,m
             do 44 j=1,l
                  F1(i,j)=F(i,j)
44           continue
43        continue
          k=i1
          ifail=0
          call f01ckf(p0,p,f1,w,x,y,z,l,l,ifail)
          goto 201
200       ifail=0
          call f01ckf(p0,p,f,w,x,y,z,l,l,ifail)
201       do 45 i=1,m
             do 46 j=1,l
                  P(i,j)=P0(i,j)
46           continue
45        continue
          else
          goto 27
          endif
          do 47 i=1,i1-1
               P(i,s)=0
47        continue
     goto 26
27   continue
26   continue
     print*,' '
     do 48 i=1,m
     print'(20F9.2)',(P(i,j),j=1,l)
48   continue
     print*,' '
     print*,' Enter number of l.i. rows '
     read*,r(h)
     if ( r(h).eq.0 ) then
     goto 300
     endif
     print*,' Enter row numbers of l.i. rows '
     read*,( aa(h,i),i=1,r(h))
     print*,' Enter column numbers of l.i. columns '
     read*,( bb(h,i),i=1,r(h))
     if ( h.eq.2 ) then
300  ii1=0
     do 49 n=0,q-2
        do 50 h1=n+2,q
           do 51 i=1,r(h1)
                ii1=ii1+1
                RR(ii1)=aa(h1,i)+(n*m)
```

```fortran
51          continue
50       continue
49    continue
      ii2=0
      do 52 n1=0,q-2
         do 53 h2=n1+2,q
            do 54 i1=1,r(h2)
                ii2=ii2+1
                CC(ii2)=bb(h2,i1)+(n1*l)
54          continue
53       continue
52    continue
      do 55 i=1,ii1
         do 56 j=1,ii2
              FF1(i,j)=E(RR(i),CC(j))
56       continue
55    continue
      do 57 i=ii1+1,ii1+m
         do 58 j=ii2+1,ii2+l
              FF1(i,j)=PP(0,i-ii1,j-ii2)
58       continue
57    continue
      do 59 i=1,ii1
         do 60 j=ii2+1,ii2+l
              FF1(i,j)=0
60       continue
59    continue
      do 61 i=ii1+1,ii1+m
         do 62 j=1,ii2
              FF1(i,j)=0
62       continue
61    continue
      do 63 i=1,ii1
         do 64 j=1,ii2
              FF2(i,j)=-A(RR(i),CC(j))
64       continue
63    continue
      do 65 i=ii1+1,ii1+m
         do 66 j=ii2+1,ii2+l
              FF2(i,j)=PP(1,i-ii1,j-ii2)
66       continue
65    continue
      do 67 i=1,ii1
         do 68 j=ii2+1,ii2+l
              FF2(i,j)=B(RR(i),j-ii2)
68       continue
67    continue
      do 69 i=ii1+1,ii1+m
         do 70 j=1,ii2
              FF2(i,j)=-C(i-ii1,CC(j))
70       continue
69    continue
      print*,' '
      do 71 i=1,ii1+m
           print'(20I3)',( FF1(i,j),j=1,ii2+l)
71    continue
      print*,' '
      print*,' The s coefficient is : '
      print*,' '
      do 72 i=1,ii1+m
           print'(20I3)',( FF2(i,j),j=1,ii2+l)
72    continue
```

```
      stop
      elseif ( h.eq.q ) then
      goto 400
      else
      goto 73
      endif
23    continue
400   do 73 h=q-1,2,-1
         do 74 i=1,m
            do 75 j=1,l
               P1(i,j)=PP(h,i,j)
75          continue
74       continue
         ifail=0
         call f01ckf(p2,p1,f1,w,x,y,z,1,1,ifail)
         do 76 i=1,m
            do 77 j=1,l
               P(i,j)=P2(i,j)
77          continue
76       continue
         do 78 h1=h+1,q
            do 79 i=1,r(h1)
               do 80 j=1,l
                  P(aa(h1,i),j)=0
80             continue
               do 81 i1=1,m
                  P(i1,bb(h1,i))=0
81             continue
79          continue
78       continue
         goto 100
73    continue
      stop
      end
```

Demianczuk's program which computes the infinite frequency structure of a given rational matrix from its Laurent expansion.

```
      integer row,col,hi,lo,ro(-30:30)
      double precision d(30,30,-30:30),dsq(30,30,-30:30),dum(30,30)
      double precision pt(30,30),q(30,30),qt(30,30),aa(30,30)
      double precision sv(30),work(130)
      print*,'The transfer function matrix G(s) has an expansion '
      print*,'at infinity of the form:'
      print*,'   G(s)=D(l)s**l + D(l-1)s**l-1 + .........'
      print*,'                 .....+ D(0)  + D(-1)s**-1 + .......'
      print*,'Enter dimensions of the matrix G(s)'
      print*,'-no of rows followed by no. of columns'
      read*,row,col
      print*,'What is the normal rank of G(s)?'
      read *,norank
1     print*,'What is the highest power?'
      read*,hi
      print*,'What is the lowest power?'
      read*,lo
      if(lo.gt.hi)then
        print*,'The lowest power you have given is greater than'
        print*,'the highest power.Please re-enter:'
        goto 1
      endif
      if(row.gt.col)then
        min=col
      else
        min=row
      endif
      lwork=(3*min)+(min**2)
      call one(row,col,hi,lo,norank,lwork,min,ro,d,dsq,dum,sv,pt,q,qt,
     + work,aa,-30,30,30,130)
      stop
      end
      subroutine one(r,c,u,l,norank,lwork,min,ro,d,dsq,dum,sv,pt,q,qt,
     + work,aa,minl,maxu,maxdim,maxlw)
      integer r,c,u,diff,rank,rp,rdif,ro(-30:30)
      double precision d(r,c,l:u),dsq(r,c,-30:u),dum(r,c),sv(min)
      double precision pt(min,c),q(r,min),qt(min,r),work(lwork)
      double precision z(1),dmin(30,30,-30:30),dplus(30,30,-30:30)
      double precision aa(min,c)
      j=u
      print*,'Enter the elements of the matrix which corresponds to'
      print*,'the highest power,row by row'
      read*,((d(i,k,j),k=1,c),i=1,r)
      if(u.eq.l)goto 41
0     jprev=j
1     print*,'Enter the next power (descending) in the Laurent series'
      read*,j
      if(j.ge.jprev)then
        print*,'This power is higher than the previous one.'
        print*,'Please re-enter:'
        goto 31
      endif
      if(j.lt.l)then
        print*,'This power is lower than the lowest power'
        print*,'Please re-enter:'
        goto 31
      endif
      diff=jprev-j
      if(diff.gt.1)then
        jmin=jprev-1
        jpls=j+1
        do 40 n=jmin,jpls,-1
```

```
          do 50 i=1,r
             do 60 k=1,c
                  d(i,k,n)=0.0
60           continue
50        continue
40      continue
      endif
      print*,'Enter the elements of the matrix which corresponds'
      print*,'to this power,row by row'
      read*,((d(i,k,j),k=1,c),i=1,r)
      if(j.gt.l)goto 30
      ll=l-1
41    do 70 i=1,r
         do 80 k=1,c
            do 85 n=u,l,-1
                 dsq(i,k,n)=d(i,k,n)
85          continue
            do 87 m=ll,-30,-1
                 dsq(i,k,m)=0.0
87          continue
80       continue
70    continue
      j=u
100   do 90 i=1,r
         do 95 k=1,c
             dum(i,k)=dsq(i,k,j)
95       continue
90    continue
      ifail=0
      call f02wcf(r,c,min,dum,r,q,r,sv,pt,min,work,lwork,ifail)
      rank=0
      do 25 i=1,min
          if(sv(i).gt.0.0000001)rank=rank+1
25    continue
      ro(j)=rank
      rp=rank+1
      rdif=r-rank
      mindif=min-rank
      do 110 i=1,min
         do 120 k=1,r
              qt(i,k)=q(k,i)
120      continue
110   continue
      do 130 k=j,-30,-1
         do 140 i=1,r
            do 150 n=1,c
                  dum(i,n)=dsq(i,n,k)
150         continue
140      continue
      ifail=0
      call f01ckf(aa,qt,dum,min,c,r,z,1,1,ifail)
      call next(rank,rp,aa,min,c,rdif,dsq,r,l,u,j,k,mindif,dmin,dplus,
    + 30,-30,30)
130   continue
      if(j.eq.-30)then
        print*,'More terms are required in the expansion'
        return
      endif
      if((j.gt.l).or.((j.ge.l).and.(ro(j).lt.norank)))then
        j=j-1
        goto 100
      endif
```

```
      print*,' '
      ipol=0
      if(u.le.0)then
        print*,'There is no pole at infinity'
        goto 108
      endif
      j=u
107   if(ro(j).gt.0)then
        print*,'There is a pole at infinity of order',j
        ipol=1
      endif
404   last=ro(j)
      j=j-1
      if((j.eq.0).and.(ipol.eq.1))goto 108
      if((j.eq.0).and.(ipol.eq.0))then
        print*,'There is no pole at infinity'
        print*,' '
        goto 108
      endif
      new=ro(j)-last
      if(new.gt.0)then
        do 766 jk=1,new
              print*,'There is a pole at infinity of order',j
766     continue
        ipol=1
      else
        goto 404
      endif
108   if(l.gt.0)goto 499
      j=u
      new=ro(j)
      goto 408
407   if(j.eq.l1)goto 499
      last=ro(j)
      j=j-1
      new=ro(j)-last
408   if((ro(j).eq.norank).and.(j.ge.0))goto 499
      if((ro(j).eq.norank).and.(j.lt.0))then
        ij=-j
        do 409 ii=1,new
              print*,'There is a zero at infinity of order',ij
409     continue
        goto 129
      endif
      if((new.gt.0).and.(j.lt.0))then
        ij=-j
        do 498 ii=1,new
              print*,'There is a zero at infinity of order',ij
498     continue
      endif
      goto 407
499   print*,'There is no zero at infinity'
      if(l.gt.0)goto 128
129   if(ro(j).lt.norank)then
        print*,' '
        print*,'LESS THAN NORANK'
        print*,' '
      endif
      return
128   end
      subroutine next(rank,rp,aa,min,c,rdif,dsq,r,l,u,j,k,mindif,
     1 dmin,dplus,maxdim,minl,maxu)
```

```
      integer rank,rp,rdif,u,r,c
      double precision aa(min,c),dmin(rank,c,-30:j)
      double precision dplus(min,c,-30:j),dsq(r,c,-30:u)
      do 210 i=1,rank
         do 220 n=1,c
               dmin(i,n,k)=aa(i,n)
220      continue
210   continue
      do 230 i=rp,min
         do 240 n=1,c
               dplus(i,n,k)=aa(i,n)
240      continue
230   continue
      if(k.eq.j)return
        kplus=k+1
        do 260 n=1,c
           do 270 i=1,rank
                 dsq(i,n,k)=dmin(i,n,kplus)
270        continue
           do 280 i=rp,min
                 dsq(i,n,k)=dplus(i,n,k)
280        continue
260     continue
      return
      end
```