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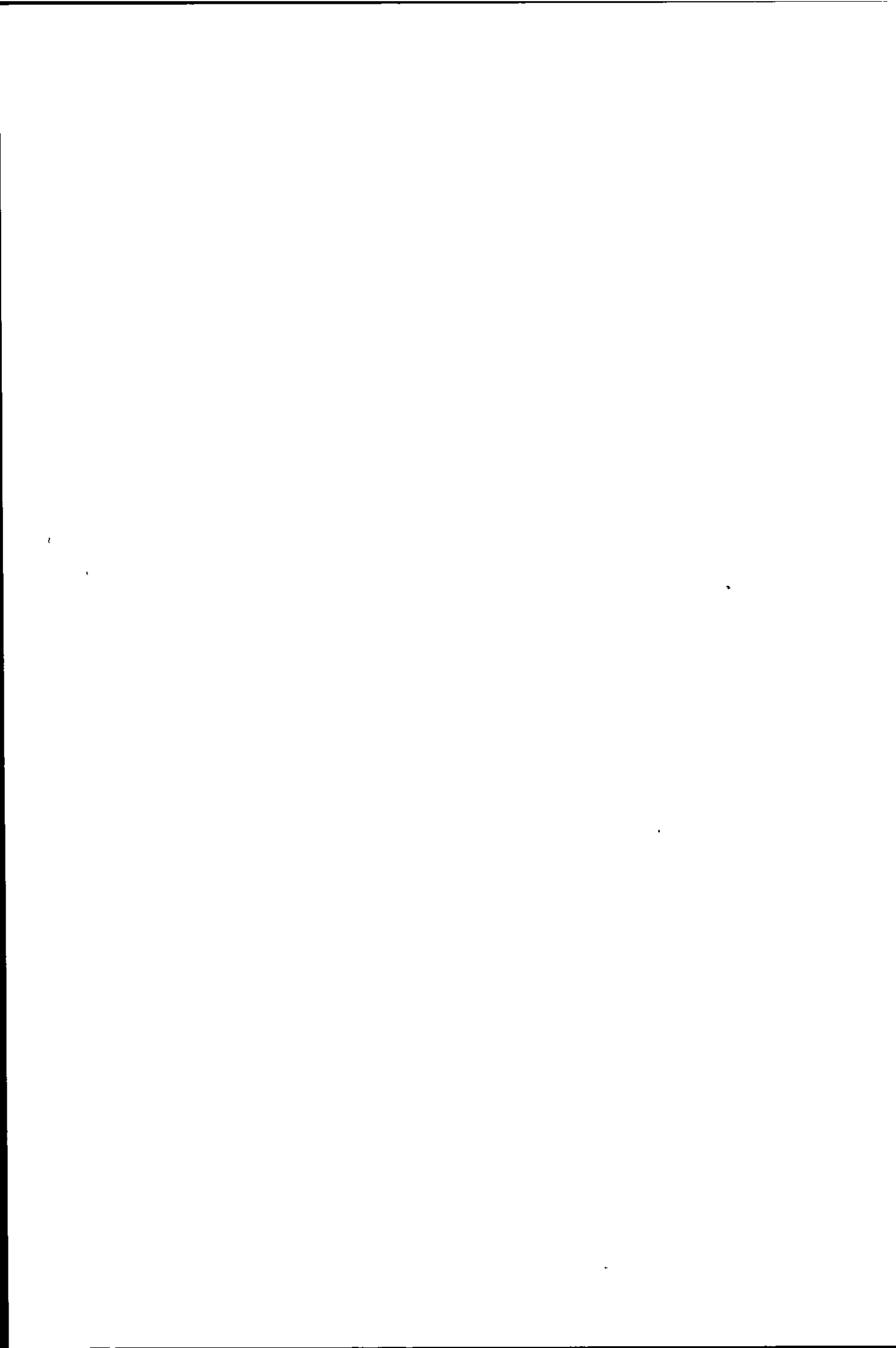
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EQUIVALENCE TRANSFORMATIONS IN LINEAR SYSTEMS

THEORY

by

Paul Fretwell

A Doctoral Thesis

Submitted in partial fulfilment of the requirements
for the award of

Doctor of Philosophy of the Loughborough University of
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PUBLICATIONS

1. Hayton, G.E., Fretwell, P., Pugh, A.C.; 1984, "Fundamental equivalence of generalised state space systems", Proc. I.E.E.E., 23rd C.D.C., Las Vegas, U.S.A., p.289.
2. Hayton, G.E., Fretwell, P., Pugh, A.C.; 1984, "On transformations of generalised state space systems", Institute of Mathematics and its Applications 4th International Conference on Control, Cambridge, proceedings pp.270-278.
3. Hayton, G.E., Fretwell, P., Pugh, A.C.; 1986, "Fundamental equivalence of generalised state space systems", I.E.E.E. transactions on Automatic Control, Vol. AC-31, NO. 5, May, pp431-440.
4. Hayton, Pugh, A.C., G.E., Fretwell, P., 1985, "Finite and infinite frequency preserving transformations in linear systems theory", I.E.E.E., 24th C.D.C., Fort Lauderdale, U.S.A.
5. Pugh, A.C., Fretwell, P., Hayton, G.E.; 1983, "Some transformations of matrix equivalence arising in linear systems theory", Proc. of ACC, 1983, San Francisco, U.S.A., p.633.
6. Pugh, A.C., Hayton, G.E., Fretwell; 1985, "On transformations of matrix pencils and implications in linear systems theory", Accepted for publication in International Journal of Control.

ABSTRACT

There is growing interest in infinite frequency structure of linear systems, and transformations preserving this type of structure. Most work has been centered around Generalised State Space (g-s-s) systems. Two constant equivalence transformations for such systems are Rosenbrock's Restricted System Equivalence (r.s.e.) and Verghese's Strong Equivalence (str.eq.). Both preserve finite and infinite frequency system structure. R.s.e. is over restrictive in that it is constrained to act between systems of the same dimension. While overcoming this basic difficulty str.eq. on the other hand has no closed form description. In this work all these difficulties have been overcome. A constant pencil transformation termed Complete Equivalence (c.e.) is proposed, this preserves finite elementary divisors and non-unity infinite elementary divisors. Applied to g-s-s systems c.e. yields Complete System Equivalence (c.s.e.) which is shown to be a closed form description of str.eq. and is more general than r.s.e. as it relates systems of different dimensions.

Equivalence can be described in terms of mappings of the solution sets of the describing differential equations together with mappings of the constrained initial conditions. This provides a conceptually pleasing definition of equivalence. The new equivalence is termed Fundamental Equivalence (f.e.) and c.s.e. is shown to be a matrix characterisation of it.

A polynomial system matrix transformation termed Full Equivalence (fl1.e.) is proposed. This relates general matrix polynomials of different dimensions while preserving finite and infinite frequency structure. A definition of infinite zeros is also proposed along with a generalisation of the concept of infinite elementary divisors (i.e.d.) from matrix pencils to general polynomial matrices. The i.e.d. provide an additional method of dealing with infinite zeros.

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NOTATION

List of symbols

u.e.	Unimodular Equivalence
s.s.e.	Strict System Equivalence
e.u.e.	Extended Unimodular Equivalence
s.e.	Strict Equivalence
e.s.e.	Extended Strict Equivalence
e.s.s.e.	Extended Strict System Equivalence
c.e.	Complete Equivalence
c.s.e.	Complete System Equivalence
str.eq.	Strong Equivalence
r.s.e.	Rosenbrock's Restricted Equivalence
f.e.	Fundamental Equivalence
fl.e.	Full Equivalence
g.c.d.	Greatest Common Divisor
f.e.d.	Finite Elementary Divisor
i.e.d.	Infinite Elementary Divisor
z.e.d.	Zero Elementary Divisor
n.z.e.d.	Non Zero Elementary Divisor
C	Set of complex numbers
R	Set of reals

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CHAPTER ONE

INTRODUCTION

The first systematic analysis of systems of high-order differential equations from a control theory viewpoint was due to Rosenbrock [43]. He introduced the concept of Strict System Equivalence (s.s.e.) in terms of admissible transformations of the associated system matrix. It transpired that this concept is crucial for the analysis of the relation between general differential operator representations and state space descriptions of time-invariant linear systems

Rosenbrock's definition arose from a desire to systematise all the operations normally used to reduce a given system to state-space form. Its theoretical foundation was not completely clarified and there remained some technical questions (Pernebo [35]). This stimulated a new discussion on s.s.e. which led to a better understanding of the transformation, Fuhrmann [18], Pernebo [35], Pugh and Shelton [41].

In an important paper [18], Fuhrmann presented a canonical state space model for polynomial system matrices and thereby provided the basis for a structural analysis of s.s.e. He extended the class of equivalence transformations to systems having different dimensions. The underlying matrix operation in Fuhrmann's transformation has been investigated by Pugh and Shelton [41]. They developed a polynomial transformation which acts upon matrices of different dimensions while preserving finite zero structure.

Another approach to s.s.e. is due to Pernebo [35]. He characterised s.s.e. by the existence of one-one mappings between the solution sets of the corresponding differential equations (for fixed control functions u). By this method

he was able to answer some questions unresolved in Rosenbrock's exposition [43].

A great deal of recent interest in the literature has been centred on infinite frequency structure in systems theory Rosenbrock [45], Verghese et al [59]. Generalised state-space systems are the simplest linear system to display finite and infinite frequency behaviour. They have in recent times aroused much interest, for example, Rosenbrock[45], Verghese [55], Verghese et al, [59] and Cobb [9]. Rosenbrock proposed a system transformation termed Restricted System Equivalence (r.s.e.) [45]. Two g-s-s systems are r.s.e. if their associated system matrices are related by

$$\begin{bmatrix} M & : & 0 \\ \dots & & \dots \\ 0 & : & I \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & : & B_1 \\ \dots & & \dots \\ -C_1 & : & 0 \end{bmatrix} \begin{bmatrix} N & : & 0 \\ \dots & & \dots \\ 0 & : & I \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & : & B_2 \\ \dots & & \dots \\ -C_2 & : & 0 \end{bmatrix} \quad (1.1.1)$$

where M and N are nonsingular matrices. The matrix operation on which r.s.e. is based, is Gantmacher's Strict Equivalence (s.e.) (Gantmacher [19]). This restricts r.s.e. to operate between systems of the same dimension. In Chapter 4 an attempt is made to generalise the operation of s.e. to pencils of different dimensions. Unfortunately, the proposed constant operation termed Extended Strict Equivalence (e.s.e.), although composed of constant matrices, is shown to affect infinite frequency structure of the pencils.

In Chapter 5, an additional condition is imposed on e.s.e. with respect to the point at infinity. This leads to a new pencil transformation termed Complete Equivalence (c.e.) which has the desired property of maintaining finite and infinite zero structure, while allowing pencils of different dimensions to be related. C.e. is applied to g-s-s systems to form the basis of a system transformation for generalised state space systems. This new operation, termed Complete System Equivalence (c.s.e.), acts upon

systems of different dimension and is shown to preserve all desirable properties at both finite and infinite frequencies.

In an attempt to overcome the restrictions imposed by Rosenbrock's r.s.e., namely the necessity of having the transformed system of the same dimension as the original, Verghese proposed the notion of Strong Equivalence (str.eq.) [55]. The terms under which str.eq. is described make it a useful concept from both a technical and an algorithmic viewpoint since the definition actually catalogues the permitted elementary operations. However strong equivalence itself has certain shortcomings in a mathematical sense, since no closed form description of the transformation was originally given nor was one readily apparent. In Chapter 5, the transformation of complete system equivalence is shown to provide such a description.

Verghese et al [59] show that str.eq. induces a mapping between the solution sets associated with the differential equations. In particular, if P_1 and P_2 are two strongly equivalent systems, then there exists a bijective map of the form

$$x_2(t) = T_1 x_1(t) + V_1 u(t), \quad t > 0 \quad (1.1.2)$$

between the solutions of the systems P_1 and P_2 for any given input $u(t)$, such that the outputs $y(t)$ of the two systems are identical under the mapping. The corresponding initial conditions are related by

$$E_2 x_2(0-) = Q_1 E_1 x_1(0-) \quad (1.1.3)$$

where T_1 , V_1 and Q_1 are constant matrices. The converse is also true, that is, if two systems have isomorphic solutions in the above sense, then they are strongly equivalent.

In Chapter 6, a different approach is made to the problem of defining equivalence in terms of mappings. This provides

an elegant definition of equivalence for g-s-s systems, since the input/solution pairs $(x(t), -u(t))$ determine the controllability characteristics of the system, while the initial condition/output pairs $(Ex(0^-), -y(t))$ describe the observability characteristics. The conditions reflect a certain duality and in particular, it will be shown that taken together they guarantee that the systems P_1 and P_2 have isomorphic pairs $(x(t), -u(t))$ and isomorphic pairs $(Ex(0^-), -y(t))$. A major result of Chapter 6 will demonstrate that this notion of equivalence possesses a neat matrix characterisation which will be shown to be complete system equivalence. It will further be shown that the proposed definition allows a more unified discussion of the equivalence of g-s-s systems than has previously been possible.

In an attempt to extend the above ideas on equivalence to higher order systems, chapter 7 deals with the development of a matrix polynomial transformation which has the property of relating general polynomial matrices while preserving finite and infinite zero structure. A new way of viewing infinite zeros and a new definition of Infinite Elementary Divisors (i.e.d.) for polynomial matrices are proposed. These help in deciding the additional conditions which the transforming matrices need to satisfy if they are to be capable of maintaining finite and infinite zero structure.

A polynomial transformation, termed Full Equivalence (flle.), is considered and is shown to have the desired properties. Full equivalence was proposed by Ratcliffe [42] although there was no clear motivation for certain of the conditions that were applied. The development proposed here is more concise and elegant.

CHAPTER TWO

SURVEY OF LITERATURE RELEVANT TO LINEAR SYSTEM

TRANSFORMATIONS WITH EMPHASIS ON INFINITE

FREQUENCY BEHAVIOUR

2.1 Introduction

This chapter presents a review of the literature relevant to linear system theory with particular emphasis on infinite frequency behaviour.

Matrix algebra plays a large part in this thesis and in current linear systems theory in general. Therefore, a brief list of the major sources of background algebra is given in section 2.2 with the accent on the algebra of matrices.

Transformations of linear systems which preserve finite frequency behaviour are presented in section 2.3.

Structure and behaviour of linear systems at infinite frequency has only been more recently considered in the literature. The fourth and largest section of the survey (section 2.4) is devoted to describing briefly the major publications in the field with particular reference to authors whose work concerns transformations of the system matrix which preserve infinite frequency behaviour.

2.2 Algebraic background theory

A great deal of the background algebra relevant to this thesis may be found in the following publications and the references therein.

Barnett "Matrices in control theory" [3], Barnett [4], Bellman "Matrix analysis" [5], Fuhrmann [17], Gantmacher "Theory of matrices" [19], MacDuffee "The theory of matrices" [32], MacLane and Birkhoff "Algebra" [34], Turnbull and Aitken "An introduction to the theory of canonical matrices" [50]

2.3 Finite frequency transformations

Modern control theory has for a large part considered mathematical models in state space form. A state space system is a first order linear system which is represented by a set of differential equations

$$\begin{aligned}\dot{\bar{x}}(t) &= A\bar{x}(t) + B\bar{u}(t) \\ y(t) &= C\bar{x}(t) + D\bar{u}(t)\end{aligned}\tag{2.3.1}$$

or in the Laplace domain by

$$\begin{aligned}(sI-A)\bar{x}(s) &= B\bar{u}(s) \\ \bar{y}(s) &= C\bar{x}(s) + D\bar{u}(s)\end{aligned}$$

This may be represented by a partitioned first order polynomial matrix

$$\begin{bmatrix} sI-A & \vdots & B \\ \vdots & \ddots & \vdots \\ -C & \vdots & D \end{bmatrix}\tag{2.3.2}$$

This area of linear systems theory is much documented and well understood, Rosenbrock [43], Wolovich [60], Kailath [23]. A more general form of system was proposed by Rosenbrock [43], which is represented in the Laplace domain, by a partitioned polynomial matrix of the form

$$\begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \quad (2.3.3)$$

which is called a polynomial system matrix. Clearly the state space model is a special case of (2.3.3) with $T(s)=sI-A$, $U(s)=B$, $V(s)=C$ and $W(s)=D$. In [43] the problem of determining the relationship between systems that possess the same properties, for example, transfer function and system order, was reduced to finding an equivalence relation between matrix representations of the form (2.3.3). A transformation of polynomial system matrices that meets the above requirements is Rosenbrock's Strict System Equivalence (s.s.e.) It is based on unimodular matrices and embodies all the normally permitted operations upon systems of differential equations. Several problems concerning s.s.e. were left unresolved in [43]. For example, the equivalence classes of s.s.e. were not well defined. This issue was settled by Pernebo who approached the problem by defining an equivalence relation for systems as in (2.3.3), based upon the solution sets of the differential equations [35]. A complete list of invariants of s.s.e. was not included by Rosenbrock in [43]. It was not until 1983 that Hinrichsen and Pratzel-Wolters made some significant advances concerning the invariants of s.s.e. for the special case of reachable systems [20].

One drawback of s.s.e is that it can only relate systems that have the same dimension. Module theory provided Fuhrmann with an elegant generalisation of s.s.e. to systems of different dimensions, [17] [18]. Pugh and Shelton investigated the underlying matrix theory of Fuhrmann's transformation and from this detailed some of the invariants of Fuhrmann's equivalence [41]. For example the finite system zero structure was shown to remain invariant under Fuhrmann's transformation. For a survey of system transformations operating at finite frequency and their unification, see [27]. Although these operations maintain

the invariance of some system properties, information concerning infinite frequency behaviour may not be preserved. Ikeda et al [22] have approached the problem of relating state space systems of different dimensions by proposing two linear constant transformations. The first being between the solutions and the second between the initial conditions.

2.4 Infinite frequency structure

This section is subdivided into two main parts. Section 2.4.1 presents a survey of structures at infinity relevant to linear systems theory. Section 2.4.2 describes the current research on various system characteristics and matrix properties at infinity that play a part in g-s-s theory. It also establishes the main areas of numerical computation for such systems.

2.4.1 Infinite frequency structures in linear systems

Much work has been done in clarifying the definitions of infinite poles and zeros. In linear systems theory one of the first definitions of infinite zeros was proposed by Rosenbrock. He proposed definitions of infinite input and output decoupling zeroes for systems of the form

$$\begin{bmatrix} sE-A & \vdots & B \\ \vdots & \ddots & \vdots \\ \vdots & -C & \vdots \\ & & D \end{bmatrix} \quad (2.4.1)$$

Such systems may occur in hierarchical systems theory Rosenbrock and Pugh [46]. Pugh and Ratcliffe [39] presented a definition of infinite poles and zeros for any rational and polynomial matrix. Their definition is more general than Rosenbrock's and is at present the standard definition.

Bosgra and Van Der Weiden [7] considered non proper and

interconnected linear multivariable systems. They defined poles at infinity, transmission zeros and decoupling zeros at infinity on the basis of the degrees of certain minors of the system matrix. The consistency of some of these definitions with those proposed by Pugh and Ratcliffe [39] was demonstrated in [6]. A Smith Macmillan form for a rational matrix at infinity has been developed [23]. Vardulakis and Karcanias have used this form to investigate the relationship between strict equivalence invariants and pole/zero structure at infinity of matrix pencils, [54]. Pugh and Krishnaswamy made use of this form to develop another characterisation of infinite zeros and poles based upon column degeneracy at infinity, [38].

A great deal of material has been written about matrix pencils, see Gantmacher [19]. These first order matrices have proved very useful in linear systems theory especially for studying g-s-s systems (see Rosenbrock [45]). Several authors have tried to tie in the early work on matrix pencils to that of modern linear systems theory. Karcanias and Hayton [26] examined the relationship between the infinite root locus and certain invariant system zeros for state space systems. They also investigated the relationship between various types of infinite zeros using elementary divisor structure of certain matrix pencils. Karcanias and MacBean [25] have studied the invariants and canonical forms of matrix pencils under strict equivalence from the transformation group point of view.

In addition to defining infinite decoupling zeros Rosenbrock constructed a canonical decomposition for g-s-s systems and proposed a definition of controllability indices [45]. More precisely, he showed that the minimal indices of the input and output pencils of the system were invariant under certain transformations and that they corresponded to the dynamic indices defined by Forney [16].

2.4.2 System characteristics and matrix properties at infinity

The term Generalised State Space system was proposed by Verghese [55], in which he considered the impulsive elements in the solution associated with the system (see also [56] [57] [59].) He related these impulsive modes to the infinite pole and zero structure of the system and proposed a new matrix transformation for g-s-s systems termed Strong Equivalence (Str.eq). This preserves finite and infinite system properties. Cobb [9] approached the problem of justifying the existence of an impulsive component in the response of a g-s-s system by using perturbation theory. He revealed that the impulsive components originate from initial conditions that do not satisfy the constraints imposed by the system. Therefore, he termed such initial conditions unconstrained initial conditions. This point is considered further by Cobb in [11]. Verghese, Van Dooren and Kailath [58] detail many of the properties of the system matrix of a generalised state space system.

Interest in system characteristics and matrix properties at infinite frequency has been growing, see, for example, MacFarlane and Postlethwaite [33] as well as Banks and Abbas-Ghilmansara [2] on infinite root locus, Forney [16], Pugh and Ratcliffe [40] on minimal basis theory. Many aspects of g-s-s systems have been and are being studied. For example, Luenberger, who termed systems of equations such as those described in (2.4.1) Descriptor Systems, concentrated upon the formation of such sets of differential equations [30] [31], while Lewis has investigated solvability and conditionability as well as system inversion [29] [28]. Van Der Weiden and Bosgra have developed a canonical form for the system matrix, based upon Rosenbrock's restricted system similarity [51], while Silverman and Kitapci have developed an algorithm that

computes system structure at infinity [47].

Sincovec has examined some of the numerical problems associated with g-s-s systems [49]. In a more general context Van Dooren has studied the problems associated with the numerical evaluation of generalised eigenvalues [53]. Numerical methods for dealing with generalised eigen structure problems are dealt with by Van Dooren [52]. Optimal state regulation of linear systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad \text{with } E \text{ singular.}$$

has been examined by Cobb [10] where he finds that optimal control can be obtained by solving a reduced order Riccati equation. Fletcher [15] discusses pole assignment at finite frequency for descriptor systems.

Commault and Dion have applied geometric methods to systems as in (2.4.1), in the study of infinite zero structure [12], while Descusse and Dion considered the structure at infinity of square invertible systems (C,A,B) and the conditions under which they are decouplable with static state feedback [13]. This theme is continued in Dion [14] where feedback block decoupling and infinite structure are analysed.

Bosgra and Van Der Weiden have proposed an algorithm for reducing a general system as described by (2.3.3) to a first order system as at (2.4.1) while keeping invariant all desirable system properties at finite and infinite frequencies [7].

2.5 Conclusion

Several key areas in which interesting and important problems lie may be selected from section 2.4. The first area is in g-s-s system theory. Although much effort has been expended in defining system degeneracy at infinity and

standard forms for the system matrix, there has been little activity in developing an elegant matrix transformation which preserves all desirable system properties at finite and infinite frequencies. Verghese developed an algorithm which lists the elementary operations necessary to determine such an equivalence. This method has by its algorithmic nature no closed form which is unfortunate from a mathematical viewpoint. This thesis develops a system matrix transformation that preserves all valuable system properties at finite and infinite frequencies. It is also shown that this transformation is a matrix characterisation of Verghese's str.eq. Pernebo's equivalence relation for linear systems based upon the finite frequency solutions is worthy of generalisation to infinite frequency. For the special case of g-s-s systems such a generalisation is made. Additionally the concept of mappings of the constrained initial conditions is proposed. This is used in the formation of an equivalence relation for g-s-s systems based on mappings of the solution/input pairs and mappings of the constrained initial condition/output pairs. The relationship between this equivalence and previous definitions is shown.

CHAPTER THREE

PRELIMINARY RESULTS AND DEFINITIONS

3.1 Introduction

This chapter contains definitions and background mathematics used in this thesis. The reader will be assumed to be familiar with the fundamental concepts and ideas of linear systems theory.

3.2 Constant matrices

The Jordan form of a constant matrix is needed in later chapters of the thesis. Let A be a constant matrix of dimension $n \times n$.

Definition (3.2.1)

The JORDAN CANONICAL FORM of A under the action of similarity is the following

$$\text{Diag}\{J_1, J_2, \dots, J_k\} \tag{3.2.1}$$

where the J_i are elementary Jordan blocks of the form

$$J_i = \begin{bmatrix} \lambda_i * & 0 & \dots & 0 \\ 0 & \lambda_i * & & \\ & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & 0 \\ & & & \cdot & \cdot & \cdot & 0 \\ & & & & \cdot & \cdot & \cdot & \lambda_i * \\ 0 & \dots & \dots & \dots & 0 & \lambda_i \end{bmatrix} \quad (n_i \times n_i) \tag{3.2.2}$$

where the λ_i , $i=1,2,\dots,n$ are the eigen values of A . The elements on the first super diagonal are either 1 if the

block is not degenerate, or 0 if it is degenerate. The J_i correspond to eigen values of A but the λ_i can appear in more than one Jordan block, (Gantmacher [19]).

3.3 Polynomial and rational matrices

Fundamental ideas concerning polynomial and rational matrices which are relevant to this thesis are presented in this section. Gantmacher [19] is the source of the references unless otherwise stated. The coefficients in the polynomials and rational functions in the following will be taken to be the reals. Greater generality can be achieved but this is not needed here.

Definition (3.3.1)

A square polynomial matrix $P(s)$ is said to be **NONSINGULAR** if and only if $|P(s)| \neq 0$. Otherwise the matrix is said to be **SINGULAR**.

Unimodular matrices are an important subclass of nonsingular matrices. They are used in linear systems theory and are defined as follows.

Definition (3.3.2)

A square polynomial matrix $P(s)$ is said to be **UNIMODULAR** if and only if $|P(s)| = c$, c a constant non zero real.

Thus a unimodular matrix has a constant determinant. A characterisation of unimodular matrices is the following

Lemma (3.3.1)

A polynomial matrix $P(s)$ is unimodular if and only if its inverse $P^{-1}(s)$ is also polynomial.

Proof

See, for instance, Gantmacher [19].

Unimodular matrices can be used to define an equivalence relation on the set of polynomial matrices which is useful in linear systems theory.

Definition (3.3.3)(Gantmacher [19])

Two polynomial matrices $P(s)$ and $Q(s)$, both of dimension $m \times n$, are said to be UNIMODULAR EQUIVALENT (u.e.) if and only if there exist unimodular matrices $M(s)$ and $N(s)$, with dimensions $m \times m$ and $n \times n$ respectively, such that

$$M(s)P(s)N(s) = Q(s) \quad (3.3.1)$$

A matrix transformation is useful if it is an equivalence relation on the set on which it acts. Concerning unimodular equivalence is the following

Lemma (3.3.2)

Unimodular equivalence is reflexive, symmetric and transitive, that is, it is an equivalence relation on the set of $m \times n$ polynomial matrices.

Proof

See, for instance, Gantmacher [19].

Unimodular equivalence may be generated by elementary operations of the following types

- (i) Multiply any row (resp. column) by a non-zero constant
- (ii) Add a multiple, by a polynomial, of any row (resp. column) to any other row (resp. column).
- (iii) Interchange any two among the rows (resp. column).

The concept of greatest common divisor of polynomials may be generalised to the matrix polynomial case.

Definition (3.3.4)

Let $P(s)$ and $Q(s)$ be two polynomial matrices with the same number of columns. A **GREATEST COMMON RIGHT DIVISOR** (g.c.r.d) of $P(s)$ and $Q(s)$ is any square matrix $R(s)$ which has the following properties

- (i) $R(s)$ is a right divisor of $P(s)$ and $Q(s)$ i.e. there exist polynomial matrices $P'(s)$, $Q'(s)$ such that

$$P(s) = P'(s)R(s) \quad Q(s) = Q'(s)R(s) \quad (3.3.2)$$

- (ii) If $R'(s)$ is any other right divisor of $P(s)$ and $Q(s)$ then $R'(s)$ is also a right divisor of $R(s)$.

The **GREATEST COMMON LEFT DIVISOR** (g.c.l.d) of two polynomial matrices having the same number of rows may be defined analogously. Because matrix multiplication is not in general commutative the greatest common left divisor of two matrices will be in general different from their greatest common right divisor.

(Further definitions and results are given for g.c.r.ds but analogous statements may be made for g.c.l.ds). Note that g.c.r.ds are not unique.

Definition (3.3.5)(Rosenbrock [43])

Two polynomial matrices with the same number of columns are **RELATIVELY RIGHT PRIME** (or **RIGHT COPRIME**) if and only if all their g.c.r.ds are unimodular.

Using unimodular equivalence the following standard form for a polynomial matrix may be obtained.

Definition (3.3.6)

The **SMITH FORM** of the $m \times n$ matrix $P(s)$ with rank r is defined as the matrix $S_p(s)$ where

$$S_P(s) = \begin{cases} [E(s), 0] & ; m < n \\ E(s) & ; m = n \\ \begin{bmatrix} E(s) \\ 0 \end{bmatrix} & ; m > n \end{cases} \quad (3.3.3)$$

where $E(s) = \text{diag}\{\lambda_i(s)\}$, the λ_i being the INVARIANT POLYNOMIALS of $P(s)$ given by

$$\lambda_i(s) = \frac{d_i(s)}{d_{i-1}(s)} \quad (3.3.4)$$

where $d_0 = 1$ and $d_i, i=1,2,\dots,\min(m,n)$ is the i 'th DETERMINENTAL DIVISOR, that is the greatest common divisor of all minors of matrix $P(s)$ of order i . $d_i=0$ for $i > r$. The λ_i are monic (i.e. the coefficient of the highest power of s in the polynomial is unity), λ_i divides λ_{i+1} for $i < r$ and $\lambda_i = 0$ for all $i > r$. The monic irreducible factors over C of λ_i are called ELEMENTARY DIVISORS.

The Smith form can be used to give a useful criterion for coprimeness. This test along with others is presented in the following lemma:

Lemma (3.3.3)

Two polynomial matrices $P(s)$ and $Q(s)$ of dimension $n \times n$ and $m \times n$ respectively, are relatively right prime if and only if one of the following equivalent conditions are satisfied

(i) The rank of $\begin{bmatrix} P(s) \\ \dots \\ Q(s) \end{bmatrix}$ is n for all $s \in C$

(ii) Smith form of $\begin{bmatrix} P(s) \\ \dots \\ Q(s) \end{bmatrix}$ is $\begin{bmatrix} I_n \\ \dots \\ 0 \end{bmatrix}$

(iii) There exist polynomial matrices $V(s)$ and $W(s)$ of dimension $n \times m$ and $m \times m$ such that the Smith form of the matrix

$$\begin{bmatrix} P(s) \\ \vdots \\ -Q(s) \\ \vdots \\ V(s) \\ \vdots \\ W(s) \end{bmatrix} \quad (3.3.5)$$

is I_{n+m} . If $P(s)$ and $Q(s)$ have real coefficients then $V(s)$ and $W(s)$ can be chosen with real coefficients.

- (iv) There exist relatively left prime polynomial matrices $X(s)$ and $Y(s)$ of dimension $n \times n$ and $n \times m$ respectively such that

$$X(s)P(s) + Y(s)Q(s) = I_n \quad (3.3.6)$$

If $P(s)$ and $Q(s)$ have real coefficients then $X(s)$ and $Y(s)$ can be chosen with real coefficients.

Proof

See Rosenbrock [43].

Corollary 1

For the special case of $P(s)=N$ a constant matrix and $Q(s)=sI-A$ of dimension $m \times n$ and $n \times n$ respectively are relatively right prime if and only if

$$\text{rank} \begin{bmatrix} N \\ NA \\ NA^2 \\ \cdot \\ \cdot \\ NA^{n-1} \end{bmatrix} = n$$

Proof

See Rosenbrock [43].

Analogous results hold for left primeness.

It follows from Definition (3.3.6) and the interpretation of unimodular equivalence in terms of elementary row and column operations over $R[s]$ that

Lemma (3.3.4)

A polynomial matrix is unimodular equivalent to its Smith form.

Let $P(s)$ be a polynomial matrix. The finite zeros associated with P will be defined and some alternative characterisations given.

Definition (3.3.7)

$s_0 \in \mathbb{C}$ is a FINITE ZERO of degree k of $P(s)$ if and only if $(s-s_0)^k$ is an elementary divisor of $P(s)$. The set of finite zeros of $P(s)$ is the set of all such s_0 , any zero of degree k being included k times.

Some alternative characterisations of finite zeros are:

Lemma (3.3.5)

s_0 is a zero of $P(s)$ if and only if one of the following equivalent conditions holds

- (i) Rank $P(s_0) < \rho(P(s))$ where $\rho(P(s))$ denotes the normal rank of $P(s)$
- (ii) There exist non zero rational vectors $u(s)$ and $v(s)$ with the property that

$$(a) \quad \lim_{s \rightarrow s_0} u(s) \text{ and } \lim_{s \rightarrow s_0} v(s) \text{ exist} \quad (3.3.7)$$

$$(b) \quad \lim_{s \rightarrow s_0} u(s) \neq 0, \quad \lim_{s \rightarrow s_0} v(s) \neq 0 \quad (3.3.8)$$

and

$$(c) \quad P(s)u(s) = (s-s_0)^k v(s) \quad (3.3.9)$$

for some positive integer k .

Proof

See for instance Kailath [23].

Note that the integer k in Lemma (3.3.5) is not in general the degree of the finite zero as defined in Definition (3.3.7).

Now consider those matrices whose elements belong to the field of rational functions $R(s)$.

Definition (3.3.8)

The Smith MacMillan form of a rational matrix $G(s)$ is the diagonal matrix

$$M(s) = \text{Diag}\{(\alpha_i(s)/\beta_i(s))\} \quad (3.3.10)$$

where α_i and β_i are monic and

$$\alpha_i(s) \mid \alpha_{i+1} \quad \text{and} \quad \beta_i \mid \beta_{i-1} \quad (3.3.11)$$

The Smith MacMillan form maybe found by using the elementary operations of unimodular equivalence.

An important quantity associated with a polynomial or rational matrix is the MacMillan degree. In order to define this property the concept of least order is needed.

Definition (3.3.9)

The LEAST ORDER of a rational matrix $G(s)$ is denoted by $\nu(G(s))$ and it is defined as the degree of the least common denominator of minors of all orders of $G(s)$ [43].

This leads to the MACMILLAN DEGREE of $G(s)$.

Definition (3.3.10)

The MACMILLAN DEGREE of $G(s)$ denoted by $\delta(G)$, is defined as follows: if $G(s)$ is written as

$$G(s) = G_S(s) + D(s)$$

where $D(s)$ is a polynomial matrix and

$$\lim_{s \rightarrow \infty} G_S(s) = 0$$

then the MacMillan degree of $G(s)$ is defined as (Kalman [24])

$$\delta(G(s)) = \nu(G_S(s)) + \nu(D(s^{-1}))$$

It follows, since $D(s)$ is a polynomial matrix that

$$\nu(D(s^{-1})) = \delta(D(s)) = \delta(D(s^{-1}))$$

Two alternative characterisations of the MacMillan degree of a polynomial matrix are:

Theorem (3.3.1)

For a polynomial matrix $P(s)$, $\delta(P)$ is equivalent to the following numbers:

- (i) The highest degree for minors of all orders of $P(s)$,

$$(ii) \quad \text{RANK} \begin{bmatrix} P_t & 0 & \dots & 0 & 0 \\ P_{t-1} & P_t & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ P_2 & P_3 & \dots & P_t & 0 \\ P_1 & P_2 & \dots & P_{t-1} & P_t \end{bmatrix} \quad (3.3.12)$$

where

$$P(s) = P_t s^t + P_{t-1} s^{t-1} + \dots + P_1 s + P_0 \quad (3.3.13)$$

and each P_i is a matrix of real constants.

Proof

For (i) see [43] and for (ii) see [3].

To define the finite zeros of an $m \times 1$ rational matrix $G(s)$ a **PRIME MATRIX FRACTION DESCRIPTION** is used i.e.

$$G(s) = T^{-1}(s)N(s) = N_1(s)T_1^{-1}(s) \quad (3.3.14)$$

where $T(s)$ and $N(s)$ are relatively left prime and $N_1(s)$ and $T_1(s)$ are relatively right prime polynomial matrices. In (3.3.14) any $m \times 1$ polynomial matrix $N(s)$ or $N_1(s)$ will be called a **NUMERATOR** of $G(s)$ while any $m \times m$ polynomial matrix such as $T(s)$, or 1×1 matrix such as $T_1(s)$ will be called a **DENOMINATOR** of $G(s)$. (see Pugh and Ratcliffe [39])

Definition (3.3.11) (Pugh and Ratcliffe [39])

$s_0 \in \mathbb{C}$ is a **FINITE ZERO** (respectively **POLE**) of degree k of a rational matrix $G(s)$ if and only if it is a zero of degree k of any numerator (respectively denominator) of $G(s)$.

To extend this definition to include the point at infinity the standard bilinear transformation

$$s = 1/\omega \quad (3.3.15)$$

is made. This takes the point $s=\infty$ to the point $\omega=0$ and the point $s=0$ to the point $\omega=\infty$. All other points in the s -plane are carried onto finite points in the ω -plane in a one to one manner.

Definition (3.3.12)(Pugh and Ratcliffe [39])

$G(s)$ has an INFINITE ZERO (respectively POLE) of degree k in case $G(1/\omega)$ has a finite zero (respectively POLE) of precisely that degree at $\omega=0$.

Let $w(s)$ be a rational vector. $W(s)$ will be called PROPER if $\lim w(s)$, as $s \rightarrow \infty$ exists. If the limit is zero then $w(s)$ will be called STRICTLY PROPER while if this limit is non zero $w(s)$ will be called EXACTLY PROPER [38].

An alternative characterisation of infinite zeros is the following:

Lemma (3.3.6)

The rational matrix $G(s)$ has a zero at infinity if and only if there exists an integer $q > 0$ and exactly proper rational vectors $w(s)$ and $v(s)$ such that

$$G(s)v(s) = s^{-q}w(s) \quad (3.3.16)$$

Proof

See Pugh and Krishnaswamy [38]

The integer q in Lemma (3.3.6) is not in general the degree of the infinite zero as defined in Definition (3.3.11). Although not required in this thesis it is noted that analogous results hold for the case of infinite poles Pugh and Krishnaswamy [38].

A useful condition for determining when a full rank matrix has no infinite zeros is

Theorem (3.3.2)

Let $P(s)$ be an $m \times l$ polynomial matrix of full rank. $P(s)$ has no infinite zeros if and only if there exists a high-order minor ($m \times m$ or $l \times l$ whichever is the less) of $P(s)$ with degree $\delta(P)$, where $\delta(P)$ denotes the MacMillan degree of $P(s)$.

Proof

See Pugh and Ratcliffe [39].

Definition (3.3.13)

A rational or polynomial matrix $G(s)$ is said to have **FULL RANK AT INFINITY** if it has full normal rank and no infinite zeros.

3.4 Matrix pencils

Linear polynomial matrices of the form $sE - A$, where E and A are constant matrices are known as **MATRIX PENCILS**. They are the simplest form of polynomial matrix to display infinite zeros.

Definition (3.4.1)

A matrix pencil $sE - A$ is termed **REGULAR** if and only if it is square and $|sE - A| \neq 0$; otherwise it is termed **SINGULAR**.

Definition (3.4.2)

The **FINITE ELEMENTARY DIVISORS (f.e.d)** of a matrix pencil $sE - A$ are obtained by factoring the invariant polynomials (see Definition (3.3.6)) into powers of polynomials which are irreducible in the field C . A distinction is made between **ZERO ELEMENTARY DIVISORS (z.e.d.)** of the form s^p and **NON-ZERO FINITE ELEMENTARY DIVISORS (n.z.f.e.d.)** i.e. those of the form $(s - s_0)^p$ for some $s_0 \in (C - 0)$.

Another elementary divisor associated with a matrix pencil is

Definition (3.4.3)

The INFINITE ELEMENTARY DIVISORS (i.e.d.) of the matrix pencil $sE-A$ are the z.e.d. of the dual pencil $E-\hat{s}A$. They have the form \hat{s}^q . Non unity i.e.d. have $q > 0$.

The relationship between elementary divisors (finite and infinite) and the zeros of a pencil is described by:

Lemma (3.4.1)

To each finite elementary divisor of degree k of a matrix pencil there corresponds a finite zero of exactly that degree, while to every infinite elementary divisor of degree k , there corresponds an infinite zero of degree $k-1$ and vice versa.

Proof

See Verghese [55].

The conventional transformation for analysing the structure of matrix pencils is

Definition (3.4.4)

Two $m \times n$ matrix pencils are STRICTLY EQUIVALENT if they are related by

$$M(sE_1 - A_1)N = (sE_2 - A_2) \quad (3.4.1)$$

where M and N are constant square nonsingular matrices of respective dimensions $m \times m$ and $n \times n$.

Lemma (3.4.2)

The f.e.d. and i.e.d. together form a complete set of invariants for regular pencils under the operation of strict

equivalence, that is two regular pencils belonging to the same equivalence class have exactly the same sets of elementary divisors and vice versa.

Proof

See Gantmacher [19].

The elementary divisors, finite and infinite, thus characterise the strict equivalence classes of a regular pencil and, in fact define a canonical form for such pencils: the KRONECKER CANONICAL FORM $KCF(sE-A)$, where

$$KCF(sE-A) = \begin{bmatrix} sI-\bar{A} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I-sJ \end{bmatrix} \quad (3.4.2)$$

where the matrix $sI-\bar{A}$ corresponds to the finite elementary divisors of $sE-A$. The matrix \bar{A} is in first natural form, Rosenbrock [43]. The $I-sJ$ correspond to the non unity infinite elementary divisors while the identity block corresponds to the infinite elementary divisors with $q=1$.

Now consider a singular pencil $sE-A$ with dimensions $m \times n$ and rank r . From the definition of singularity it follows that at least one of the inequalities $r < n$ or $r < m$ always holds.

Let $r < n$ then the columns of the pencil are linearly dependent that is there exists some non zero column vector $x(s)$ such that

$$(sE-A)x(s) = 0 \quad (3.4.3)$$

The $x(s)$ satisfying (3.4.3) form a vector space over the rational functions (i.e. the right solution space or right null space of the pencil). A basis for this space will now be constructed. Among all the polynomial vectors $x(s)$ in

the space there will be at least one, $x_1(s)$ say, of some minimum degree d_1 , select this as the first basis vector. Among the polynomial $x(s)$ independent of $x_1(s)$ there will be again at least one, $x_2(s)$ say, of minimum degree d_2 . Select this as the second basis vector. By continuing in this way a polynomial basis for the space may be constructed. Note that there is likely to be a choice of minimum degree vector at each stage so, the basis is not unique.

This leads to

Definition (3.4.5)

The degrees d_i $i=1,2,\dots,p$ of the vectors in any such basis are unique, and are termed **COLUMN MINIMAL INDICES** for the pencil. **ROW MINIMAL INDICES** may be defined analogously by considering the case $r < m$ when the rows of the pencil are linearly dependent.

A singular pencil may have both row and column minimal indices, see Gantmacher [19].

Lemma (3.4.3)

✓ The f.e.d., the i.e.d., the row minimal and the column minimal indices together form a complete set of invariants for pencils under the operation of strict equivalence, that is two pencils belonging to the same equivalence class have exactly the same sets of elementary divisors and minimal indices.

Proof

See Gantmacher [19].

The elementary divisors, finite and infinite, and the row and column minimal indices thus characterise the strict equivalence classes of a singular pencil and may be used to define a canonical form for such pencils, the **KRONECKER**

CANONICAL FORM KCF(sE-A) where

KCF(sE-A)

$$= \text{Diag}\{0_{h,g}; L_{r1}, \dots, L_r; L_{c1}, \dots, L_c; sE_w - A_w\} \quad (3.4.4)$$

where $sE_w - A_w$ is the Kronecker Canonical form associated with the regular part of the pencil; $0_{h,g}$ is a zero block parameterised by the zero column and row indices and $L_{ri}(s)$ and $L_{cj}(s)$ are blocks associated with the non zero column and row indices respectively and are defined by

$$L_{ri} = \begin{bmatrix} s & -1 & 0 & \dots & 0 & 0 \\ 0 & s & -1 & & & 0 \\ & & & & & \\ 0 & 0 & 0 & & -1 & 0 \\ 0 & 0 & 0 & & s & -1 \end{bmatrix} \quad \begin{array}{c} \uparrow \\ d_{ri} \\ \downarrow \end{array} \quad (3.4.5)$$

← $d_{ri}+1$ →

$$L_{cj} = \begin{bmatrix} s & 0 & \dots & 0 & 0 \\ -1 & s & \dots & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ & & & & \\ 0 & 0 & \dots & s & 0 \\ 0 & 0 & \dots & -1 & s \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix} \quad \begin{array}{c} \uparrow \\ d_{cj+1} \\ \downarrow \end{array} \quad (3.4.6)$$

← d_{cj} →

3.5 Systems theory: Definitions and results

Some basic systems theory results and definitions will now be given. This section will be subdivided into three parts. The first will deal with system matrices the second and third with system properties at finite and infinite frequencies respectively.

3.5.1 System matrices

Consider a system of first-order, linear, differential equations of the form

$$E\dot{\chi}(t) = A\chi(t) + B\bar{u}(t) \quad (3.5.1a)$$

$$\bar{y}(t) = C\chi(t) + D\bar{u}(t) \quad (3.5.1b)$$

where $\chi(t)$ is an n -vector, $\bar{u}(t)$ an l -vector, $\bar{y}(t)$ an m -vector and E, A, B, C, D are constant matrices with dimensions $n \times n, n \times n, n \times l, m \times n$ and $m \times l$ respectively. Assuming non-zero initial conditions $\chi(0^-)$, Laplace transformation of (3.5.1a), (3.5.1b) gives

$$\begin{bmatrix} sE-A & B \\ \cdots & \cdots \\ -C & D \end{bmatrix} \begin{bmatrix} x(s) \\ \cdots \\ -u(s) \end{bmatrix} = \begin{bmatrix} Ex(0^-) \\ \cdots \\ -y(s) \end{bmatrix} \quad (3.5.2)$$

Where $x(s), y(s)$ and $u(s)$ are the Laplace transforms of $\chi(t), \bar{y}(t)$ and $\bar{u}(t)$ respectively. Provided that $|sE-A| \neq 0$ (that is provided that $(sE-A)^{-1}$ exists) (3.5.1) describes a system in **GENERALISED STATE SPACE FORM** (Verghese, [55]). The classical state space form for proper systems is a special case of (3.5.2) with $E = I_n$ and clearly if E is nonsingular the equations (3.5.1) may be reduced to the classical state space form by premultiplying (3.5.1a) by E^{-1} . This is no longer so if E is singular and it should be noted that in this case the system equation (3.5.1a) maps only to a subset of R^n . This subset $\{Ex(0^-), x(0^-) \in R^n\}$ will be referred to

as the RESTRICTED INITIAL CONDITION SET. The restricted initial conditions are equivalence classes induced by E on the space of all initial conditions R_n . It is the dimension of this space of restricted initial conditions that represents the dimension of the generalised solution space of (3.5.1a). Thus it is the distinct members of the restricted initial condition set which are distinguishable from consideration of the generalised solutions themselves. The issue of restricted initial conditions has been more fully discussed in [9], [10], [55], [56], [57], [59], [61]. The underlying system of differential equations is said to be singular. See Campbell [8] for examples of systems of singular differential equations.

The $(n+m) \times (n+1)$ matrix pencil

$$P(s) = \begin{bmatrix} sE-A & B \\ \dots & \dots \\ -C & D \end{bmatrix} \quad (3.5.3)$$

which occurs in (3.5.2) contains all the mathematical information about the system which is needed to discuss its behaviour. Accordingly it is called the SYSTEM MATRIX. The advantage of using the system matrix is that all the transformations of the system equations can be expressed as operations on the system matrix. By this means the operations can be systematized and their properties more easily studied.

An important result that will be used extensively in this work is the following due to Rosenbrock [45].

Lemma (3.5.1)

There exist constant nonsingular matrices M and N such that,

$$\begin{bmatrix} M:0 \\ \vdots \\ 0:I \end{bmatrix} \begin{bmatrix} sE-A:B \\ \vdots \\ -C:D \end{bmatrix} \begin{bmatrix} N:0 \\ \vdots \\ 0:I \end{bmatrix} = \begin{bmatrix} sI-\bar{A} & 0 & \vdots & \bar{B} \\ 0 & I-sJ & \vdots & \hat{B} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{C} & -\hat{C} & \vdots & \hat{D} \end{bmatrix} \quad (3.5.4)$$

where

$$\begin{bmatrix} sI-\bar{A} & 0 \\ 0 & I-sJ \end{bmatrix}$$

is the Kronecker form of $sE-A$. The trivial i.e.d. (i.e. $q=0$) are incorporated into the $I-sJ$ matrix.

Proof

See Rosenbrock [45].

A more general higher order system may be represented by the following $(r+m) \times (r+1)$ polynomial system matrix

$$\begin{bmatrix} T(s):U(s) \\ \vdots \\ -V(s):W(s) \end{bmatrix} \quad (3.5.5)$$

See Rosenbrock [43].

The **TRANSFER FUNCTION** matrix associated with the system matrix in (3.5.5) is the $(r+m)$ rational matrix

$$G(s) = V(s)T(s)^{-1}U(s) + W(s) \quad (3.5.6)$$

Define $p = \min(m, 1)$.

3.5.2 System properties at finite frequency

Some system properties occurring at finite frequency will now be given. The references are due to Rosenbrock [43]. The definitions will consider only G-S-S systems as they are the object of interest in this work.

Definition (3.5.2)

The FINITE TRANSMISSION ZEROS of $G(s)$ are the zeros of a numerator of $G(s)$ i.e. they are zeros of $G(s)$. (see definition (3.3.10))

Definition (3.5.3)

The FINITE INVARIANT ZEROS of the system are defined to be the finite zeros of the system matrix taken as a polynomial matrix.

Consider the G-S-S system $P(s)$ in (3.5.3).

Definition (3.5.4)

The FINITE INPUT/(OUTPUT) DECOUPLING ZEROS of $P(s)$ are defined to be the finite zeros of the partitioned matrix $[sE-A|B] \left(\begin{bmatrix} (sE-A)^t \\ -C^t \end{bmatrix} \right)^t$ respectively.

Definition (3.5.5)

The FINITE POLES OF $G(s)$ are the zeros of a denominator of $G(s)$.

Additionally the poles of the system may be described directly in terms of the system matrix

Definition (3.5.6)

The FINITE POLES OF $P(s)$ are the roots of the determinant of pole matrix $T(s)$.

3.5.3 System properties at infinite frequency

Now consider the system structures given in section 3.5.2 at infinite frequency. In particular,

Definition (3.5.7)

The INFINITE TRANSMISSION ZEROS of the system are the infinite zeros of the transfer function matrix $G(s)$.

Definition (3.5.8)

The INFINITE INVARIANT ZEROS of $P(s)$ are the infinite zeros of $P(s)$ considered as a polynomial matrix.

Definition (3.5.9)

The INFINITE INPUT/(OUTPUT) DECOUPLING ZEROS of $P(s)$ are defined to be the infinite zeros of the partitioned matrix $[sE-A|B]$ ($[(sE-A)^t|-C^t]^t$) respectively.

Note that the definition of infinite decoupling zeros given in Bosgra and Van Der Weiden [7] is different to that above. They define decoupling zeros for general high order systems described by (3.5.5) via the following generalised system matrix

$$\begin{bmatrix} T(s) & U(s) & 0 & \vdots & 0 \\ -V(s) & W(s) & I & \vdots & 0 \\ 0 & I & 0 & \vdots & I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I & \vdots & 0 \end{bmatrix} = \begin{bmatrix} T_1(s) & \vdots & U_1(s) \\ -V_1(s) & \vdots & W_1(s) \\ \vdots & & \vdots \end{bmatrix}$$

Their definition of infinite input/(output) decoupling zeros is defined via the partitioned matrices

$$\begin{bmatrix} T_1(s) & \vdots & U_1(s) \end{bmatrix}; \begin{bmatrix} T_1(s) \\ \vdots \\ -V_1(s) \end{bmatrix}$$

However in g-s-s systems the constant matrices B and C in the system matrix mean that the two definitions of

decoupling zero are defined on the same submatrix of the original system matrix. This is not the case for general high order systems. The relation between the definitions of system structure given in [7] is yet not completely resolved.

Definition (3.5.10)

The INFINITE TRANSFER FUNCTION POLES are the infinite poles of the rational matrix $G(s)$.

Definition (3.5.11)

The INFINITE SYSTEM POLES OF $P(s)$ are the infinite zeros of the pole matrix $sE-A$. Another concept of physical importance in linear systems theory is .

Definition (3.5.12)

Consider a state space system at any time t_0 and with initial state $x(t_0)=x_0$ and any given final state x_f . The system is COMPLETELY CONTROLLABLE if there exists a finite time $t_1>t_0$ and a control $u(t)$, $t_0\leq t\leq t_1$ such that $x(t_1)=x_f$. The system is COMPLETELY OBSERVABLE if there exists a finite time $t_1>t_0$ such that knowledge of $u(t)$ and $y(t)$ for $t_0\leq t\leq t_1$ suffices to determine x_0 uniquely.

For state space systems complete controllability(/observability) is equivalent to no input(/output) decoupling zeros. Controllability and observability for g-s-s systems is more complex. For a physical interpretation similar to Definition (3.5.12) see Yip and Sincovec [68]. The result of interest in the context of this work is the following which is analogous to the state space case.

Definition (3.5.13)

A g-s-s system $P(s)$ CONTROLLABLE/(OBSERVABLE) if and only if $P(s)$ has no finite or infinite input(/output) decoupling zeros

CHAPTER FOUR

FINITE FREQUENCY EQUIVALENCE TRANSFORMATIONS AND INFINITE FREQUENCY STRUCTURE

4.1 Introduction

In linear systems theory it is sometimes necessary to be able to relate the structure of matrices of different dimensions. A typical example of this arises from the various matrix fraction descriptions of a given $m \times 1$ rational matrix $G(s)$. These factorisations are essentially of two types, the classification depending on whether the denominator is factored on the left or on the right. If only prime factorisations are considered e.g.

$$G(s) = D_1(s)^{-1}N_1(s) = N_2(s)D_2(s)^{-1} \quad (4.1.1)$$

where $D_1(s)$, $N_1(s)$ (resp. $N_2(s)$, $D_2(s)$) are relatively left (resp. right) prime then the denominator matrices $D_1(s)$ and $D_2(s)$ both describe the finite pole structure of $G(s)$. In some sense, therefore, the matrices $D_1(s)$ and $D_2(s)$ although of different dimensions ($D_1(s)$ is $m \times m$ and $D_2(s)$ is 1×1) must be related.

Such problems have not been covered in the standard matrix theory literature which invariably deals with transformations between matrices of identical dimensions, for example operations of similarity and equivalence upon constant matrices [34]. Although such problems occur frequently in the linear systems area they have been handled on a rather ad hoc basis as typified by the operation of trivially expanding any polynomial system matrix

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ \dots & \dots \\ -V(s) & W(s) \end{bmatrix} \quad (r+m) \times (r+1) \quad (4.1.2)$$

in which it is found that $n = \text{deg}(T(s))$ is larger than r , the dimension of the square matrix $T(s)$. This operation was suggested by Rosenbrock [43] but however, was not formally embodied within the set of elementary operations permitted under the framework of that author's definition of strict system equivalence.

In the following a brief review will be given of work in the linear systems area concerned with such problems as described above. As will be seen these basic problems have been resolved in the sense that if only the system properties at finite frequencies are of interest then the precise connection between the corresponding system matrices $P(s)$ or the denominator matrices $D(s)$ of (4.1.2) may be simply expressed [18], [35], [41]. If however, the infinite frequency behaviour of linear systems is considered of interest then the precise connection between the corresponding system matrices is not known. The main purpose of this chapter is to offer some reflections on this specific problem and to present results relating to the simplest form of system equations capable of exhibiting finite and infinite frequency behaviour, the so called generalised state space descriptions. Many of the results in this chapter can be found in Pugh, Fretwell and Hayton [65]. in

4.2 Extended transformations of equivalence

The standard definition of matrix equivalence in linear multivariable theory is due to Rosenbrock and is termed strict system equivalence [43]. It is constructed on the basis of unimodular matrices so that strict system equivalence is the system theory analogue of the standard

polynomial matrix theory definition which is here called unimodular equivalence. More specifically

Definition (4.2.1)

(i) Two $m \times 1$ polynomial matrices $D_1(s)$, $D_2(s)$ are said to be **UNIMODULAR EQUIVALENT** (u.e.) if there exist unimodular matrices $N_1(s)$, $N_2(s)$ such that

$$D_1(s) = N_1(s)D_2(s)N_2(s)^{-1} \tag{4.2.1}$$

(ii) Two $(r+m) \times (r+1)$ polynomial system matrices $P_1(s)$, $P_2(s)$ of the form (4.1.2) are said to be **STRICTLY SYSTEM EQUIVALENT** (s.s.e) if there exist unimodular matrices $N_1(s)$, $N_2(s)$ and polynomial matrices $X(s)$ and $Y(s)$ such that

$$\begin{bmatrix} T_1(s) : U_1(s) \\ -V_1(s) : W_1(s) \end{bmatrix} = \begin{bmatrix} N_2(s) : 0 \\ X(s) : I \end{bmatrix} \begin{bmatrix} T_2(s) : U_2(s) \\ -V_2(s) : W_2(s) \end{bmatrix} \begin{bmatrix} N_1(s) : Y(s) \\ 0 : I \end{bmatrix}^{-1} \tag{4.2.2}$$

It is noticed within these definitions that there is an immediate restriction to matrices of identical dimensions, and that (ii) is a simple adaptation of (i) to the systems theory context to preserve the underlying input-output structure of the system as described by the transfer function matrix. The required generalisation of these definitions to relate matrices whose dimensions may not be the same is as follows. Let

$$P(m,1) = \left\{ P(s) : P(s) \text{ is an } (r+m) \times (r+1) \text{ polynomial matrix, where } r \text{ is any integer with } r > \max(-m, -1) \right\} \tag{4.2.3}$$

That subset of $P(m,1)$ given by the restriction $r > 0$ will be denoted by $P_0(m,1)$.

Definition (4.2.2)(Pugh and Shelton[41])

(i) Two matrices $D_1(s)$, $D_2(s)$ in $P(m,1)$ are said to be **EXTENDED UNIMODULAR EQUIVALENT (e.u.e)** if there exist polynomial matrices $N_1(s)$, $N_2(s)$ such that

$$D_1(s)N_2(s) = N_1(s)D_2(s)$$

where

$$\begin{aligned} D_1(s), N_1(s) & \text{ are relatively left prime} \\ N_2(s), D_2(s) & \text{ are relatively right prime} \end{aligned} \quad (4.2.4)$$

(ii) Two polynomial system matrices $P_1(s)$, $P_2(s)$ in $P_0(m,1)$ are said to be **EXTENDED STRICT SYSTEM EQUIVALENT (e.s.s.e.)** if there exist polynomial matrices $N_1(s)$, $N_2(s)$, $X(s)$ and $Y(s)$ such that

$$\begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} \begin{bmatrix} N_2(s) & Y(s) \\ 0 & I \end{bmatrix} = \begin{bmatrix} N_1(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \quad (4.2.5)$$

where

$$\begin{aligned} T_1(s), N_1(s) & \text{ are relatively left prime} \\ N_2(s), T_2(s) & \text{ are relatively right prime} \end{aligned} \quad (4.2.6)$$

In respect of these transformations it is noted that it is now possible for matrices of different dimensions to be related. Further it is noted that (ii) appears as a simple adaption of (i) to the systems theory context. The truth of the matter however is that Definition (4.2.2) (ii) was first proposed by Fuhrmann [18] and the Definition (4.2.2) (i) arose only as a consequence of this [41]. From Definition (4.2.2) (i) the following results may be obtained.

Theorem (4.2.1)

- (i) e.u.e is an equivalence transformation on $P(m,1)$.
- (ii) $D_1(s), D_2(s)$ in $P(m,1)$ are e.u.e if and only if their Smith forms are related by a trivial expansion.

Proof

It is not immediately apparent that e.u.e is an equivalence relation but this is in fact the case and a proof may be found in Pugh and Shelton [41]. The result (ii) was also established in [41], but a neater proof has since been provided by Smith [48].

It is the result of (ii) that justifies the use of the term 'extended' in the Definition (4.2.2), it now being apparent that whatever can be accomplished by e.u.e. may equally well be accomplished by u.e and a trivial expansion.

It is further seen from the above results that the essential invariants under e.u.e of a given polynomial matrix $D(s)$ are just its non-unit invariant polynomials or what is equivalent its non-unit finite elementary divisors. Accordingly since these in no way reflect the infinite frequency structure of $D(s)$ the transformation of e.u.e only represents a complete description of the relationship that holds between polynomial matrices whose finite frequency structure is identical. In this respect it is seen that the denominator matrices $D_1(s), D_2(s)$ of the factorisation of $G(s)$ given in (4.1.1) are related exactly as in Definition (4.2.2) (i) and in view of Theorem (4.2.1) (ii) both these matrices reflect the finite pole structure of $G(s)$. It is noted that analogous comments to those above may be made in relation to the transformation of e.s.s.e. since a basic result is that two system matrices are e.s.s.e. if and only if any state-space descriptions arising from them are system similar [43], [18]. Such realisations exhibit no infinite frequency behaviour and so e.s.s.e again only offers a complete description of system

matrices whose finite frequency structure coincides.

4.3 Transformations of matrix pencils.

Consider the generalised state space system described by the system matrix

$$P(s) = \begin{bmatrix} sE-A & : & B \\ \dots & : & \dots \\ -C & : & D \end{bmatrix} \quad (4.3.1)$$

where the constant matrix E is possibly singular but

$$|sE-A| \neq 0 \quad (4.3.2)$$

The representations (4.3.1) are of interest because they are the simplest form of system description which can simultaneously display finite and infinite frequency behaviour.

As a first step in the discussion consider the case of matrix pencils of the form

$$D(s) = sE-A \quad (4.3.3)$$

The conventional transformation by which the infinite frequency properties of such pencils are studied is strict equivalence (see Definition (3.4.4)).

From Lemma (3.4.2) it follows immediately that s.e. preserves the structure of the regular pencil at finite and infinite frequencies in the sense that its finite elementary divisors (which are precisely those of $sI-A$) and infinite elementary divisors as represented by the individual blocks of $I-sJ$ and the trivial Jordan blocks contained within I (see the Kronecker standard form of a regular pencil in (3.4.2)) form a complete set of invariants under s.e. Since

s.e. clearly has the desired properties an attempt is made to generalise it along the lines suggested by e.u.e. and u.e. Consider therefore,

Definition (4.3.1)

Two pencils $D_1(s), D_2(s)$ in $P(m,1)$ are said to be EXTENDED STRICT EQUIVALENT (e.s.e.) if there exist constant matrices N_1, N_2 such that

$$D_1(s)N_2 = N_1D_2(s) \quad (4.3.4)$$

where

$$\begin{aligned} D_1(s), N_1 & \text{ are relatively left prime} & (4.3.5) \\ N_2, D_2(s) & \text{ are relatively right prime} \end{aligned}$$

A first result concerning this transformation which is not immediately seen to be one of equivalence, is

Lemma (4.3.1)

- (i) e.s.e. is a special case of e.u.e.
- (ii) Two pencils $D_1(s), D_2(s)$ in $P(m,1)$ are e.s.e. if and only if any pencils generated from them by s.e. are e.s.e.

Proof

If e.u.e. is restricted to act between matrix pencils in $P(m,1)$ with constant transforming matrices the the result is e.s.e. and (i) is shown. For (ii) suppose $D_1(s), D_2(s)$ are e.s.e. Then for some constant matrices N_1, N_2 the relations (4.3.4) and (4.3.5) hold. Suppose $\bar{D}_1(s), \bar{D}_2(s)$ are obtained from $D_1(s), D_2(s)$ respectively by s.e., i.e.

$$\bar{D}_1 = L_1^{-1}D_1L_2 \quad \bar{D}_2 = M_1^{-1}D_2M_2 \quad (4.3.6)$$

for L_1, L_2, M_1, M_2 square and nonsingular. Then from (4.3.4)

$$\bar{D}_1L_2^{-1}N_2M_2 = L_1^{-1}N_1M_1\bar{D}_2 \quad (4.3.7)$$

which is the required form for e.s.e. To establish the relative primeness conditions consider

$$(\bar{D}_1 :: L_1^{-1} N_1 M_1) = L_1^{-1} (D_1 :: N_1) \begin{bmatrix} L_2 :: 0 \\ 0 :: M_1 \end{bmatrix} \quad (4.3.8)$$

Hence by (4.3.5) and the nonsingularity of L_1 , L_2 , M_1 it follows that D_1 and $L_1^{-1} N_1 M_1$ are relatively left prime. Similarly the relative right primeness of the other matrices in (4.3.7) may be established, while the converse is obtained merely by reversing the roles of D_1 , D_2 and \bar{D}_1 , \bar{D}_2 .

It is noted that the above results apply equally well to regular or singular pencils, but in the case of the former the following holds

Corollary 1

Two regular pencils are e.s.e. if and only if their s.e. Kronecker forms are e.s.e.

The first major result concerning e.s.e. is curious and disappointing.

Theorem (4.3.1)

The transformation of e.s.e. preserves the finite elementary divisors of a pencil but not the infinite elementary divisors.

Proof

e.s.e. is a special case of e.u.e. and Theorem (4.2.1) (ii) applies to show the finite elementary divisors are invariant. For the infinite elementary divisors consider a pencil $D(s)$ in $P(m,1)$. The Kronecker form of $D(s)$ is

$$\begin{bmatrix} sI-A & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I-sJ & 0 & 0 \\ 0 & 0 & 0 & L_C & 0 \\ 0 & 0 & 0 & 0 & L_R \end{bmatrix} = \text{KCF}(D)$$

where the blocks L_C and L_R are block diagonal matrices associated with the minimal column and row indices respectively, see (3.4.4). These blocks only exist when $D(s)$ is singular (Gantmacher [19]). Consider the following transformation of $\text{KCF}(D)$

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \text{KCF}(D) = \begin{bmatrix} sI-A & 0 & 0 \\ 0 & L_C & 0 \\ 0 & 0 & L_R \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (4.3.9)$$

N_1 $\text{KCF}(D)$ $D_1(s)$ N_2

Consider the partitioned matrix

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & \vdots & sI-A & 0 & 0 \\ 0 & 0 & 0 & I & 0 & \vdots & 0 & L_C & 0 \\ 0 & 0 & 0 & 0 & I & \vdots & 0 & 0 & L_R \end{bmatrix} \quad (4.3.10)$$

N_1 $D_1(s)$

This matrix has full rank for all finite s due to the full size identity minor in N_1 . Thus lemma (3.3.3) adapted for left primeness applies and gives that N_1 and $D_1(s)$ are relatively left prime. To show $\text{KCF}(D)$ and N_2 are relatively right prime consider

$$\begin{bmatrix}
 sI-A & 0 & 0 & 0 & 0 \\
 0 & I & 0 & 0 & 0 \\
 0 & 0 & I-sJ & 0 & 0 \\
 0 & 0 & 0 & L_C & 0 \\
 0 & 0 & 0 & 0 & L_R \\
 \hline
 I & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & I & 0 \\
 0 & 0 & 0 & 0 & I
 \end{bmatrix} = \begin{bmatrix} KCF(D) \\ \hline N_2 \end{bmatrix}$$

This can be seen to have full column rank for all finite s due to the identity blocks in N_2 and the identity block and the unimodular matrix $I-sJ$ in $KCF(D)$. Thus Lemma (3.3.3) part (i) applies directly to show that $KCF(D)$ and N_2 are relatively right prime. Therefore it has been show that (4.3.9) is a relation of e.s.e. Note that the transformed matrix on the right has no block matrix $I-sJ$. This block contains the infinite elementary divisors of the matrix. Thus e.s.e. does not preserve infinite elementary structure of pencils in $P(m,1)$.

Example (4.3.1)

Consider the following example which illustrates the above result and which will be used later

$$D(s) = \begin{bmatrix} sI-A' & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.3.11}$$

then

$$\begin{matrix}
 \begin{bmatrix} sI-A' & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} & = & \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} & (sI-A') & (4.3.12) \\
 D_1(s) & N_2 & & N_1 & D_2(s)
 \end{matrix}$$

and it is easily verified that the matrices

$$\begin{bmatrix} sI-A' & 0 & 0 & 0 & \vdots & I \\ 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & s & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} ; \begin{bmatrix} sI-A' \\ \vdots \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4.3.13)$$

have full rank for all finite s . This full rank condition is an alternative characterisation of left and right primeness of the partitioned matrices (see Lemma (3.3.3)). Therefore (4.3.12) is a transformation of e.s.e.

Despite the above results e.s.e. still retains value as an extension of s.e. which leaves invariant all finite frequency structure of the pencil. Further,

Theorem (4.3.2)

e.s.e. is an equivalence relation on the set of regular pencils $sE-A$.

Proof

e.s.e. will be shown to be reflexive, transitive and symmetric on the set of regular pencils. Reflexivity is established by noting that any pencil is e.s.e. to itself in the following way

$$I(sE-A) = (sE-A)I$$

Let $D_i(s) = sE_i - A_i$ $i=1,2,3$ be three regular pencils. Suppose they are related in the following way: $D_1(s)$ is e.s.e. to $D_2(s)$ and $D_2(s)$ is e.s.e. to $D_3(s)$, i.e. there exist constant matrices M_1, N_1, M_2, N_2 such that the following are relations of e.s.e.

$$M_1 D_1(s) = D_2(s) N_1 \quad (4.3.14)$$

$$M_2 D_2(s) = D_3(s) N_2 \quad (4.3.15)$$

(4.3.14) and (4.3.15) give after some simple manipulation

$$M_2 M_1 D_1(s) = D_3(s) N_2 N_1 \quad (4.3.16)$$

The relationship between $D_1(s)$ and $D_3(s)$ expressed in (4.3.16) is one of e.s.e. if $M_2 M_1$ and $D_3(s)$ can be shown to be relatively left prime and $D_1(s)$, $N_2 N_1$ relatively right prime. Because (4.3.14) and (4.3.15) are operations of e.s.e. $D_2(s)$ and M_1 are relatively left prime and $D_3(s)$ and M_2 are also relatively left prime. Using the characterisation (iv) of left primeness in Lemma (3.3.3) there exist four polynomial matrices $Q_i(s)$ $i=1,2,3,4$ (two for each co-prime pair) with the following properties

$$D_2(s) Q_1(s) + M_1 Q_2(s) = I \quad (4.3.17)$$

$$D_3(s) Q_3(s) + M_2 Q_4(s) = I \quad (4.3.18)$$

Pre multiplying (4.3.17) by M_2 and post multiplying the result by $Q_4(s)$ gives

$$M_2 D_2(s) Q_1(s) Q_4(s) + M_2 M_1 Q_2(s) Q_4(s) = M_2 Q_4(s)$$

Substituting for $M_2 D_2$ from (4.3.15)

$$D_3(s)N_2Q_1(s)Q_4(s) + M_2M_1Q_2(s)Q_4(s) = M_2Q_4(s)$$

Substituting for M_2Q_4 from (4.3.18) gives

$$\text{i.e. } D_3(s)(N_2Q_1(s)Q_4(s) + Q_3(s)) + M_2M_1(Q_2(s)Q_4(s)) = I$$

Hence by part (iv) of Lemma (3.3.3) $D_3(s)$ and M_2M_1 are relatively left prime as required. In a similar way $D_1(s)$ and N_2N_1 can be shown to be relatively right prime. Therefore (4.3.16) is a relation of e.s.e. and hence e.s.e. is transitive.

Let sE_1-A_1 and sE_2-A_2 be two regular pencils. Suppose they are e.s.e. Then by Theorem (4.3.1) they possess the same finite elementary divisors. Theorem (4.3.1) also demonstrated that they can both be reduced by e.s.e. to the pencil $sI-A$ which encapsulates the finite zero property of the pencil. Let the following denote the reductions

$$M_1(sI-A) = (sE_1-A_1)N_1 \quad (4.3.19)$$

$$M_2(sI-A) = (sE_2-A_2)N_2 \quad (4.3.20)$$

The proof of Theorem (4.3.1) shows that the particular operations described in (4.3.19) and (4.3.20) are symmetric operations. Therefore, (4.3.20) implies the existence of an inverse relation of e.s.e.

$$N_3(sE_2-A_2) = (sI-A)M_3 \quad (4.3.21)$$

Because e.s.e. is transitive, (4.3.19) together with (4.3.21) gives sE_2-A_2 is e.s.e. to sE_1-A_1 and so symmetry is demonstrated.

It has been shown that e.s.e. is reflexive, symmetric and transitive on the set of regular pencils i.e. e.s.e. is an equivalence relation thus proving the theorem.

4.4 Conclusion

A brief review of transformations acting at finite frequency has been given. Equivalence transformations between matrices of different dimensions were shown to be desirable but only well understood if the restriction to finite frequency was made.

As a first step in developing an operation of equivalence for g-s-s systems which would preserve both finite and infinite frequency structure, a new matrix pencil transformation termed extended strict equivalence e.s.e. was proposed. e.s.e. was shown to be an equivalence relation when restricted to regular pencils. However, it was disappointing to find that infinite frequency structure was not invariant. This meant that when e.s.e. was interpreted in the systems context the resulting transformation would not preserve the structure of the system at infinity which is the main item of interest in the generalised state-space. Thus despite its apparent construction from constant transforming matrices it suprisingly had a dynamic effect on the system description. The reason for this was that essentially there were no conditions on the way the transformation of e.s.e. should act at the point at infinity. The form of the constraints on e.s.e. at finite frequencies given in (4.3.5) suggests however that a similar condition be placed on the transformation in respect of the point at infinity and this will be examined subsequently.

CHAPTER FIVE

EQUIVALENCE TRANSFORMATIONS AT FINITE AND INFINITE FREQUENCY

5.1 Introduction

In this chapter an additional condition for e.s.e. with respect to the point at infinity is proposed. This leads to a new matrix pencil transformation which has the property of maintaining the invariance of the finite and infinite zero structure, although the trivial 1×1 identity blocks corresponding to i.e.d.'s of order 1 (see Lemma (3.4.1)) are not invariant. When applied to g-s-s systems, the transformation is shown to preserve non-trivial internal structure (for example decoupling zeros at finite and infinite frequency). The meaning of the trivial 1×1 blocks in terms of g-s-s systems has been explored by Verghese [55] who shows that they do not have any dynamical significance.

In section 5.2 the new matrix pencil transformation is proposed and is termed complete equivalence (c.e.). It is compared with s.e. and it is shown to preserve both finite and infinite zeros. In section 5.3 the domain of interest is restricted to regular pencils and c.e. is shown to have many desirable properties. Section 5.4 goes on to describe how c.e. is applied directly to g-s-s systems and details the invariants under the resulting system matrix transformation. The transformation is found to be one of equivalence for singular pencils of this type. Many of the results in this chapter can be found in Pugh, Hayton and Fretwell [66] and Taylor and Pugh [67].

5.2 A new matrix pencil transformation

It was illustrated in example (4.3.1) that e.s.e. does not preserve infinite elementary divisors and thus infinite zeros. In seeking an extra condition for e.s.e. which will allow infinite zero structure to remain invariant it is of value to reconsider this example and discover in what way the transformation failed at $s=\infty$.

Consider the partitioned matrix $(D_1(s) \vdots N_1)$ from Example (4.3.1)

$$(sE_1 - A_1 \vdots N_1) = \begin{bmatrix} sI - A & 0 & 0 & 0 & \vdots & I \\ 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & s & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 \end{bmatrix} \quad (5.2.1)$$

Although this has no finite zeros, it does have an infinite zero in other words it is not full rank at $s=\infty$. For definition of full rank at infinity see definition (3.3.12). For

$$((1/\omega)E_1 - A_1 \vdots N_1) = \begin{bmatrix} \omega I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} I - \omega A & 0 & 0 & \omega I \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a relatively prime factorisation and so

$$\begin{bmatrix} I - \omega A & 0 & 0 & \omega I \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.2.2)$$

is a numerator, which clearly has a zero at $\omega = 0$. So the transformation is not constrained at the point at infinity

in the same way as it is at finite frequency. This suggests that e.s.e. should be constrained to satisfy

$$\begin{bmatrix} N_1 & \vdots & sE_1 - A_1 \end{bmatrix} \quad \text{full rank at } s = \infty$$

$$\begin{bmatrix} sE_2 - A_2 \\ \dots \\ -N_2 \end{bmatrix} \quad "$$

This is a natural generalisation of the finite frequency case to infinite frequency.

Thus the following pencil transformation is proposed:

Definition (5.2.1)

Two pencils $sE_1 - A_1$, $sE_2 - A_2$ in $P(m,1)$ (see (4.2.3) for definition of the set $P(m,1)$) are said to be COMPLETELY EQUIVALENT (c.e.) in case there exist constant matrices N_1 , N_2 such that

$$(sE_1 - A_1) N_2 = N_1 (sE_2 - A_2) \quad (5.2.3)$$

where

$$\begin{bmatrix} sE_1 - A_1 & \vdots & N_1 \\ -N_2 & \dots & sE_2 - A_2 \end{bmatrix} \quad \begin{array}{l} \text{full rank for all finite and} \\ \text{infinte } s. \\ " \end{array} \quad (5.2.4)$$

It should be noted that the full rank condition in (5.2.4) is exactly the relative primeness condition found in extended unimodular equivalence (see Definition (4.2.2)). For regular matrix pencils the concept of no finite zeros and full rank for all finite s coincide. This can be seen by considering the determinant of a regular pencil which is by definition non zero. No finite zeros means the determinant is constant, and full rank for all finite s means that the determinant is also constant. But for

singular pencils which have column or row degeneracy this is not the case as the following matrix illustrates:

$$\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has no finite zeros but is clearly rank deficient. Because the partitioned matrices in (5.2.2) are in general singular the more general idea of full rank for all finite s is used.

Additionally the concept of full rank at infinity and no infinite zeros only coincides on pencils which have full normal rank for if a full rank matrix $P(s)$ has no infinite zeros it has full rank at infinity. A matrix may have no infinite zeros, but may be rank deficient.

It is noted here that at present there is no algorithmic method of determining whether two singular pencils are c.e. The case of regular pencils is explored in section 5.3

Some preliminary results concerning c.e. are

Lemma (5.2.1)

- (i) s.e. is a special case of c.e.
- (ii) Two pencils $D_1(s)$, $D_2(s)$ in $P(m,1)$ are c.e. if and only if pencils obtained from them by s.e. are also c.e.

Proof

(i) If D_1 and D_2 are s.e. pencils then by definition there exist nonsingular N_1 and N_2 such that

$$D_1 N_2 = N_1 D_2$$

N_1 and N_2 are nonsingular therefore $(N_1; D_2)$ and $(D_1^t - N_2^t)^t$ have full rank for all finite s and at $s = \infty$ i.e. s.e. fulfils

the conditions of c.e.

(ii) Suppose $D_1(s)$ and $D_2(s)$ are c.e. and $\bar{D}_1(s)$ and $\bar{D}_2(s)$ are obtained from $D_1(s)$, $D_2(s)$ respectively by s.e., i.e.

$$\bar{D}_1 = L_1^{-1} D_1 L_2 \quad \bar{D}_2 = M_1^{-1} D_2 M_2$$

for L_1 , L_2 , M_1 , M_2 square and nonsingular. Thus \bar{D}_1 and \bar{D}_2 are related as

$$\bar{D}_1 L_2^{-1} N_2 M_2 = L_1^{-1} N_1 M_1 \bar{D}_2$$

which is the required form of c.e. To establish the conditions of c.e. consider the partitioned matrices corresponding to the first matrix in (5.2.4) in particular consider

$$(\bar{D}_1 \vdots L_1^{-1} N_1 M_1) = L_1^{-1} (D_1 \vdots N_1) \begin{bmatrix} L_2 \vdots 0 \\ 0 \vdots M_1 \end{bmatrix}$$

By (5.2.4) and the nonsingularity of L_1 , L_2 and M_1 it follows that $(\bar{D}_1 \vdots L_1^{-1} N_1 M_1)$ has full rank for all finite s and no infinite zeros. Similarly the matrix corresponding to the second partitioned matrix in (5.2.4) can be shown to conform to the requirements of c.e. The converse is obtained merely by reversing the roles of D_1 , D_2 and \bar{D}_1 and \bar{D}_2 . Thus the lemma is proved.

Lemma (5.2.2)

Two regular pencils obtained from each other by trivial deflation/inflation are c.e.

Proof

Two pencils

$$sE-A \quad \text{and} \quad \begin{bmatrix} I & 0 \\ 0 & sE-A \end{bmatrix}$$

are related by

$$\text{Deflation: } \begin{bmatrix} I & 0 \\ 0 & sE-A \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} sE-A$$

$$\text{Inflation: } [sE-A] \begin{bmatrix} 0 & I \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & sE-A \end{bmatrix}$$

and the conditions in (5.2.4) are clearly satisfied.

The most pleasing feature of c.e. however is

Theorem (5.2.1)

The transformation of c.e. preserves the finite and infinite zeros of a (regular or singular) pencil $sE-A$.

Proof

Note firstly that the finite zeros are preserved since c.e. is a special case of e.u.e. and so Theorem (4.2.1) applies. For the infinite zeros note that

$$D_1(s)N_2 = N_1D_2(s) \tag{5.2.5}$$

where $D_1(s)=sE_1-A_1$, $D_2(s)=sE_2-A_2$ and the conditions (5.2.4) hold. Let

$$\begin{aligned} \begin{bmatrix} D_1(1/\omega) \\ \vdots \\ N_1 \end{bmatrix} &= T_1(\omega)^{-1} (\bar{D}_1(\omega) \vdots \bar{N}_1(\omega)) \\ \begin{bmatrix} -N_2 \\ \vdots \\ D_2(1/\omega) \end{bmatrix} &= \begin{bmatrix} -\bar{N}_2(\omega) \\ \vdots \\ \bar{D}_2(\omega) \end{bmatrix} T_2(\omega)^{-1} \end{aligned} \tag{5.2.6}$$

be relatively prime factorisations. Thus the following two partitioned matrices are numerator matrices of $[D_1(1/\omega)|N_1]$ and $[-N_2^t|D_2(1/\omega)^t]^t$ respectively.

$$\left[\bar{D}_1(\omega) \begin{array}{c} \vdots \\ \bar{N}_1(\omega) \end{array} \right]; \quad \left[\begin{array}{c} -\bar{N}_2(\omega) \\ \vdots \\ \bar{D}_2(\omega) \end{array} \right] \quad (5.2.7)$$

$D_1(s)$ and $D_2(s)$ are c.e. so (5.2.4) applies and $[D_1(1/\omega)|N_1]$ and $[-N_2^t|D_2(1/\omega)^t]^t$ have full rank for finite and infinite s . This implies that the numerators in (5.2.7) have full rank for all finite ω . Substituting $s=1/\omega$ into (5.2.5) and substituting from (5.2.6) yields

$$\bar{D}_1(\omega)\bar{N}_2(\omega) = \bar{N}_1(\omega)\bar{D}_2(\omega) \quad (5.2.8)$$

which by (5.2.7) is a relation of e.u.e. between $\bar{D}_1(\omega)$ and $\bar{D}_2(\omega)$. Hence $\bar{D}_1(\omega)$ and $\bar{D}_2(\omega)$ have the same structure at $\omega=0$. It remains to prove that $\bar{D}_1(\omega)$ and $\bar{D}_2(\omega)$ are numerators of $D_1(1/\omega)$ and $D_2(1/\omega)$ and as such (when $\omega=0$) represent the infinite zero structure of $D_1(s)$ and $D_2(s)$ respectively.

Now if $\nu(\cdot)$ denotes the least order and $\delta(\cdot)$ denotes the MacMillan degree of the indicated matrix, it follows since $D_1(s)$ is polynomial and N_1 is constant, that

$$\begin{aligned} \nu((1/\omega)E_1 - A_1 \quad N_1) &= \delta(sE_1 - A_1 \quad N_1) \\ &= \delta(sE_1 - A_1) \\ &= \nu((1/\omega)E_1 - A_1) \end{aligned} \quad (5.2.9)$$

The left matrix fraction description

$$(1/\omega)E_1 - A_1 = T_1(\omega)^{-1}\bar{D}_1(\omega)$$

will be left prime if $\text{degree}(|T(s)|) = \nu((1/\omega)E_1 - A_1)$. But

$\text{degree}(|T(s)|) = \nu((1/\omega)E_1 - A_1 \begin{array}{c} \vdots \\ N_1 \end{array})$ since factorisations in (5.2.6) are prime.

$$= \nu((1/\omega)E_1 - A_1) \quad \text{from (5.2.9)}$$

Thus $\bar{D}_1(\omega)$ is a numerator of $D_1(1/\omega)$. In a similar way it follows that $\bar{D}_2(\omega)$ is a numerator of $D_2(1/\omega)$ and the result follows.

Corollary 1

c.e. preserves all the finite elementary divisors and those infinite elementary divisors of degree two and more. c.e. does not preserve the infinite elementary divisors of degree 1.

Proof

From Lemma (3.4.1) it is known, (see [55]), for matrix pencils that to each finite elementary divisor of a certain degree there corresponds a finite zero of exactly that degree, while to every infinite elementary divisor of degree k there corresponds an infinite zero of degree $k-1$, and vice versa.

Example (5.2.1)

Consider the following example,

$$\begin{array}{cccc}
 \begin{bmatrix} sI-A & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & = & \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sI-A & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \\
 D_2(s) & N_1 & N_2 & D_1(s)
 \end{array} \tag{5.2.10}$$

This is an operation of c.e. if

$$\begin{bmatrix} sI-A & 0 & 0 & 0 & \vdots & I & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 1 & s & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 \\ & & & & \vdots & & & \end{bmatrix} = [D_2(s) | N_2] \quad (5.2.11)$$

and

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dots & \dots & \dots \\ sI-A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} N_1 \\ \vdots \\ D_1(s) \end{bmatrix} \quad (5.2.12)$$

have full rank for finite and infinite s . This is the case due to the 4×4 identity minor in (5.2.11) and the 3×3 identity minor in (5.2.12). The $sI-A$ blocks represent the f.e.d. structure of the pencils while the Jordan block

$$I-sJ = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

represents the i.e.d. of degree=2. These are both preserved in (5.2.10). The i.e.d. of degree=1 are represented by the 1 on the diagonal of the left matrix in (5.2.10) and these are not preserved in (5.2.10) thus the example bears out the statements of the Corollary.

5.3 Complete equivalence on regular pencils

In this section the domain of c.e. is restricted to regular pencils

Theorem (5.3.1)

A canonical form for a regular pencil $sE-A$ under c.e. is $KCF(sE-A)$ with the identity blocks removed from the $I-sJ$ matrix, i.e.

$$\begin{bmatrix} sI-A' & 0 \\ 0 & I-sJ \end{bmatrix} \quad (5.3.1)$$

where J has no zero blocks and A' is in first natural form.

Proof

$sE-A$ may be transformed to its Kronecker canonical form $KCD(sE-A)$

$$M(sE-A) = \begin{bmatrix} sI_{n_1}-A' & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I-sJ \end{bmatrix} N \quad (5.3.2)$$

where N and M are nonsingular matrices and $I-sJ$ has no 1×1 blocks. See (3.4.2) and Lemma (3.4.2) for further details on the meaning of the block elements in (5.3.2). Let M and N be partitioned in accordance with the block elements in $KCF(sE-A)$ i.e.

$$N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \quad M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

Simple manipulation gives

$$\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} (sE-A) = \begin{bmatrix} sI-A' & 0 \\ 0 & I-sJ \end{bmatrix} \begin{bmatrix} N_1 \\ N_3 \end{bmatrix} \quad (5.3.3)$$

Because M and N are nonsingular there exist full size nonsingular minors in

$$\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} N_1 \\ N_3 \end{bmatrix}$$

Therefore, the partitioned matrices

$$\begin{bmatrix} M_1 & \vdots & sI-A' & 0 \\ \vdots & \vdots & \vdots & \vdots \\ M_3 & \vdots & 0 & I-sJ \end{bmatrix}, \quad \begin{bmatrix} sE-A \\ \dots\dots\dots \\ N_1 \\ N_3 \end{bmatrix}$$

have full rank for finite s . Also both partitioned matrices possess full sized minors that have degree equal to the MacMillan degree. These minors can be constructed in the following way. Firstly, a minor possessing maximum degree can be constructed from rows and columns of the matrix pencil part of the partition. This minor can be expanded up to full size by using the independent rows and columns from the constant part of the partition. Thus the partition matrices have full rank and have a minor of maximum degree that is full size. Thus Theorem (3.3.2) applies to show that the matrices have no infinite zeros. Thus (5.3.3) is an operation of c.e. as required.

Corollary 1

What may be achieved by c.e. may also be achieved by trivial expansion or deflation and the operation of s.e. and vice versa.

Proof

The proof rests on the fact that two c.e. pencils in Kronecker canonical form are related by operations of s.e.

C.e. generalises s.e. to pencils of different dimension. Theorem (5.2.1) states that it preserves finite and infinite zeros. The following theorem goes further and shows that these zeros completely characterise c.e, that is, they are a set of independent invariants.

Theorem (5.3.2)

A complete set of independent invariants for regular pencils under c.e. are the finite and infinite zeros of the pencils.

Proof

Suppose $D_1(s) = sE_1 - A_1$ and $D_2(s) = sE_2 - A_2$ are two regular pencils which have the same finite and infinite zeros, i.e. they have the same finite elementary divisors and the same infinite elementary divisors of degree 2 or more.

Therefore, $KCF(D_1)$ and $KCF(D_2)$ only differ by the identity blocks in the $I-sJ$ matrix. Let these two blocks have dimension $p_1 \times p_1$ and $p_2 \times p_2$ respectively. It is easily seen that

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_2, p_1} & 0 \\ 0 & 0 & I_q \end{bmatrix} KCF(D_1) = KCF(D_2) \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_2, p_1} & 0 \\ 0 & 0 & I_q \end{bmatrix} \quad (5.3.4)$$

is an operation of c.e. So the Kronecker canonical forms for D_1 and D_2 are c.e. Therefore, by Lemma (5.2.1) (ii) D_1 and D_2 are c.e.

For the converse notice that if D_1 and D_2 are c.e. then Theorem (5.2.1) applies directly, so the theorem follows.

Corollary 1 to Theorem (5.3.1) illustrated the relationship between c.e. and s.e. for regular pencils and it was seen that c.e. consists of operations of s.e. and trivial inflation or deflation. These three operations are clearly symmetric, reflexive and transitive on pencils (regular or singular). Therefore, it would be expected that c.e. is an equivalence relation on the set of regular pencils. This is the case:

Theorem (5.3.3)

c.e. is an equivalence relation on the set of regular pencils.

Proof

Every pencil is related to itself in the following way

$$I(sE-A) = (sE-A)I$$

which is clearly an operation of c.e. by virtue of the identity blocks. Therefore reflexivity is established.

For symmetry let D_1 and D_2 be two c.e. regular pencils as in the proof of Theorem (5.3.2). The relation in (5.3.4) therefore follows. It is clear that the inverse to this relation could equally well have been written down and further that it is also a special case of c.e. So $KCF(D_1)$ and $KCF(D_2)$ are c.e. Therefore, D_2 and D_1 are c.e. (by Lemma (5.2.1)). Therefore, symmetry is proved.

Transitivity: Let D_1 , D_2 and D_3 be three regular pencils related as D_1 c.e. D_2 and D_2 c.e. D_3 . As in the proof of Theorem (5.3.1) a matrix relation between $KCF(D_1)$ and $KCF(D_2)$ and $KCF(D_2)$, and $KCF(D_3)$ may be written. This has the form

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_2, p_1} & 0 \\ 0 & 0 & I_q \end{bmatrix} KCF(D_1) = KCF(D_2) \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_2, p_1} & 0 \\ 0 & 0 & I_q \end{bmatrix}$$

and

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_3, p_2} & 0 \\ 0 & 0 & I_q \end{bmatrix} KCF(D_2) = KCF(D_3) \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_3, p_2} & 0 \\ 0 & 0 & I_q \end{bmatrix}$$

The connection between $KCF(D_1)$ and $KCF(D_3)$ is

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_3, p_1} & 0 \\ 0 & 0 & I_q \end{bmatrix} KCF(D_1) = KCF(D_3) \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0_{p_3, p_1} & 0 \\ 0 & 0 & I_q \end{bmatrix}$$

This can be easily verified to be an operation of c.e. Thus

transitivity is proved completing the proof.

Another important result concerning c.e. of regular pencils in Kronecker form is the following

Theorem (5.3.4)

Let $KCF(D_1)$ and $KCF(D_2)$ be two c.e. regular pencils in Kronecker standard form. If $KCF(D_1)$, $KCF(D_2)$ have the same dimension then the transforming matrices N_1 and N_2 in (5.2.3) are square and non singular.

Proof

Since $KCF(D_1)$ and $KCF(D_2)$ are c.e.

$$\begin{bmatrix} sI_{n_1} - A_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I - sJ_1 \end{bmatrix} \begin{bmatrix} N_{21} & \vdots & N_{22} \\ \vdots & \ddots & \vdots \\ N_{23} & \vdots & N_{24} \end{bmatrix} = \begin{bmatrix} N_{11} & \vdots & N_{12} \\ \vdots & \ddots & \vdots \\ N_{13} & \vdots & N_{14} \end{bmatrix} \begin{bmatrix} sI_{n_2} - A_2 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I - sJ_2 \end{bmatrix} \quad (5.3.5)$$

with conditions in Definition (5.2.1) holding. As the pencils have equal dimension there has been no trivial inflation or deflation and hence the Corollary to Theorem (5.2.1) may be applied to show that

$$J_1 = J_2 = J \quad ; \quad n_1 = n_2 = n \text{ (say)} \quad (5.3.6)$$

Considering block element 1,2 in (5.3.5)

$$(sI - A_1)N_{22} = N_{12}(I - sJ_2) \quad (5.3.7)$$

and considering coefficients of s and constants gives

$$N_{22} = N_{12}J_2 \quad (5.3.8)$$

$$-A_1 N_{22} = N_{12} \quad (5.3.9)$$

Eliminating N_{12} between (5.3.8) and (5.3.9)

$$N_{22} = -A_1 N_{22} J_2 \quad (5.3.10)$$

Let p be the dimension of the first Jordan block in J_2 and let n_i $i=1,2,\dots,p$ denote the first p columns of N_{22} , then the first p columns of (5.3.10) may be written

$$(n_1 \ n_2 \ \dots \ n_p) = -(0 \ A n_1 \ A n_2 \ \dots \ A n_{p-1}) \quad (5.3.11)$$

Thus

$$n_1 = 0$$

and (5.3.12)

$$n_i = A n_{i-1} = 0 \quad i = 2, 3, \dots, p$$

The argument above may be repeated for each Jordan block in J_2 in turn and so

$$N_{22} = 0 \quad (5.3.13)$$

and hence from (5.3.9)

$$N_{12} = 0 \quad (5.3.14)$$

An analogous argument may be used to show

$$N_{23} = N_{13} = 0 \quad (5.3.15)$$

Now consider block element 1,1 in (5.3.5)

$$(sI-A_1)N_{21} = N_{11}(sI-A_2) \quad (5.3.16)$$

with the conditions of Definition (5.2.1) holding so that

N_{11} , $sI-A_1$ are relatively left prime and hence the associated result to Corollary 1 to Lemma (3.3.3) for for right primeness applies to give

$$\text{rank}(N_{11}, A_1 N_{11}, \dots, A_1^{n-1} N_{11}) = n \quad (5.3.17)$$

Equating constants in (5.3.16)

$$A_1 N_{21} = N_{11} A_2 \quad (5.3.18)$$

and equating powers of s

$$N_{21} = N_{11} \quad (5.3.19)$$

from (5.3.17), (5.3.18) and (5.3.19)

$$\text{rank}(N_{11}, N_{11} A_2, \dots, N_{11} A_2^{n-1}) = n \quad (5.3.20)$$

Clearly the matrix

$$(N_{11}, N_{11} A_2, \dots, N_{11} A_2^{n-1})$$

may be reduced by invertible column operations to

$(N_{11}, 0, \dots, 0)$ and hence the condition (5.3.17) becomes

$$\text{rank}(N_{11}) = n$$

Thus, since N_{11} is $n \times n$ it is nonsingular.

Finally consider block position 2,2 in (5.3.5)

$$(I-sJ)N_{24} = N_{14}(I-sJ) \quad (5.3.21)$$

with the conditions in Definition (5.2.1) holding so that

$$\begin{bmatrix} I-sJ \\ \vdots \\ N_{24} \end{bmatrix}$$

has no infinite zeros (the special form $(I-sJ)$ has no finite zeros). A matrix fraction description for

$$\begin{bmatrix} I-1/sJ \\ \vdots \\ N_{24} \end{bmatrix}$$

is

$$\begin{bmatrix} I-1/sJ \\ \vdots \\ N_{24} \end{bmatrix} = \begin{bmatrix} \text{diag}(L_{p_i}) & 0 \\ 0 & \vdots \\ 0 & \vdots \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \text{diag}(T_{p_i}) \\ \vdots \\ N_{24} \end{bmatrix}$$

where L_{p_i} and T_{p_i} are matrices of the form

$$L_p = \begin{bmatrix} s & 0 & & 0 & 0 \\ & \ddots & & & \\ & & \ddots & & \\ & & & s & 0 \\ 0 & & & 0 & 1 \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \downarrow \\ \downarrow \end{matrix} p_i$$

$$T_p = \begin{bmatrix} s-1 & 0 & & 0 \\ & s-1 & & 0 \\ & & \ddots & \\ & & & s-1 \\ 0 & & & 0 & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \downarrow \\ \downarrow \end{matrix} p_i$$

(5.3.22)

respectively with p_i the dimension of the corresponding Jordan block in J . The factorisation is prime if the following matrix has full rank. Then lemma (3.3.3) part (i) will be applicable for the left prime case and the factorisation will be prime. So consider

$$\begin{bmatrix} \text{diag}(L_{p_i}) & 0 & \text{diag}(T_{p_i}) \\ 0 & \vdots & \vdots \\ 0 & I & N_4 \end{bmatrix}$$

This has full rank if all the block rows have full rank. A typical row will be full rank if

$$\begin{bmatrix} s & 0 & & 0 & \vdots & s-1 & 0 & & 0 \\ 0 & \ddots & & & \vdots & 0 & s-1 & & 0 \\ & & \ddots & & \vdots & & & \ddots & \\ & & & s & 0 & \vdots & 0 & & s-1 \\ 0 & & & 0 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has full rank for all finite s . When $s=0$ this is clearly the case due to the matrix in the (1,1) position in the partition being nonsingular. At $s=0$ there is a full size non singular minor consisting of the last column of the (1,1) matrix and every column but the first of the the matrix in the (2,2) position. Thus the matrix fraction description is prime. Let

$$N' = (n'_1, n'_2, \dots, n'_{p_1})$$

be a matrix comprising the first p_1 columns of N_{24} . Since

$$\begin{bmatrix} I-sJ \\ \dots \\ N_{24} \end{bmatrix}$$

has no infinite zeros

$$\begin{bmatrix} \text{diag}(T_{p_1}) \\ \dots \\ N_{24} \end{bmatrix}$$

has full rank at $s=0$ and hence it is clear from the form of T_{p_1} that $n'_1=0$. Equating constants in (5.3.21)

$$N_{24} = N_{14} \tag{5.3.23}$$

and equating powers of s

$$JN_{24} = N_{14}J \tag{5.3.24}$$

Assume N_{24} has less than full rank then there exists a nonzero vector ξ such that

$$\begin{bmatrix} N_{24} \\ J^1 N_{24} \\ J^2 N_{24} \\ \cdot \\ \cdot \\ \cdot \\ J^{p_i-1} N_{24} \end{bmatrix} \xi = 0 \quad (5.3.25)$$

Substituting into (5.3.25) from (5.3.23), (5.3.24) and considering only the first p_1 columns of the N_{24} together with the corresponding elements of ξ

$$\begin{bmatrix} N' \\ N'J \\ \cdot \\ \cdot \\ \cdot \\ N'J^{p-1} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_{p_1} \end{bmatrix} = 0 \quad (5.3.26)$$

that is

$$\begin{bmatrix} n'_1 & n'_2 & \dots & n'_{p_i} \\ & n'_1 & \dots & n'_{p_i-1} \\ & & \cdot & \\ & & & \cdot \\ & & & & n'_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_{p_i} \end{bmatrix} = 0 \quad (5.3.27)$$

Considering the last block row of (5.3.27)

$$n'_1 \xi_{p_i} = 0 \quad (5.3.28)$$

and since $n'_1 = 0$, $\xi_{p_1} = 0$.

Substituting for ξ_p in the penultimate block row of (5.3.27)

$$\begin{aligned} n'_1 \xi_{p_1-1} &= 0 \\ \xi_{p_1-1} &= 0 \end{aligned} \tag{5.3.29}$$

Proceeding in this way and considering each of the Jordan blocks in J in turn it may be shown that $\xi=0$ which contradicts the initial assumption that N_{24} is singular and collecting all the results together (5.3.5) reduces to

$$\begin{bmatrix} sI-A_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & I-sJ \end{bmatrix} \begin{bmatrix} N_{21} & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & N_{24} \end{bmatrix} = \begin{bmatrix} N_{21} & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & N_{24} \end{bmatrix} \begin{bmatrix} sI-A_2 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & I-sJ \end{bmatrix} \tag{5.3.30}$$

with N_{21} , N_{24} nonsingular, thus proving the theorem.

The above theorem has important Corollaries which describe the way in which the transformation of c.e. is a generalisation of another well known matrix pencil transformation.

Corollary 1

If P_1 and P_2 are two regular pencils of the same dimension then P_1 and P_2 are c.e. if and only if they are s.e.

Proof

This follows immediately from Theorem (5.3.4) on applying Lemma (5.2.1) part (ii).

For the special case of pencils in the conventional form $sI-A$ note the following

Corollary 2

Let $sI-A_1$ and $sI-A_2$ be two regular pencils. They are c.e.

if and only if they are similar.

Proof

In this case the relation in (5.3.30) reduces to

$$sI-A_1 = N(sI-A_2)N^{-1}$$

as required.

5.4 Systems theory considerations

Some implications of the transformation discussed previously will now be discussed.

The $(r+m) \times (r+1)$ system matrix $P(s)$ in (3.5.3) is the object of consideration. In setting up an equivalence relation for g-s-s systems there are additional requirements to be satisfied arising from the physical meaning of $P(s)$. Specifically since $P(s)$ represents an internal description of the system, the proposed transformation of equivalence should be seen to act internally and not externally. That is to say, that it in no way should affect the reference input signal nor the corresponding output signals. This restriction necessitates (in the same way as in (4.2.5)) an equivalence transformation of the form

$$\begin{bmatrix} N_1 & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & B_2 \\ \dots & -C_2 & D_2 \end{bmatrix} = \begin{bmatrix} sE_1 - A_1 & B_1 \\ \dots & -C_1 & D_1 \end{bmatrix} \begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix} \quad (5.4.1)$$

So let the following definition be proposed

Definition (5.4.1)

Let $P_1(s), P_2(s)$ in $P_0(m,1)$ be two generalised state space system matrices. $P_1(s), P_2(s)$ are said to be **COMPLETELY SYSTEM EQUIVALENT (c.s.e.)** if there exist constant matrices N_1, N_2, X, Y such that

$$\begin{bmatrix} sE_1 - A_1 & \vdots & B_1 \\ \dots & \dots & \dots \\ -C_1 & \vdots & D_1 \end{bmatrix} \begin{bmatrix} N_2 & \vdots & Y \\ 0 & \dots & \vdots \\ \vdots & \vdots & I \end{bmatrix} = \begin{bmatrix} N_1 & \vdots & 0 \\ \vdots & \vdots & \vdots \\ X & \vdots & I \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & \vdots & B_2 \\ \dots & \dots & \dots \\ -C_2 & \vdots & D_2 \end{bmatrix} \quad (5.4.2)$$

where

$$\begin{bmatrix} sE_1 - A_1 & \vdots & N_1 \\ \dots & \dots & \dots \\ -N_2 & \dots & \dots \\ sE_2 - A_2 & \vdots & \vdots \end{bmatrix} \quad \text{has full rank for all finite } s \text{ and no infinite zeros} \quad (5.4.3)$$

There are at present no methods for determining when two g-s-s systems are c.s.e. Additionally, a complete list of invariants for g-s-s systems under c.s.s. is not known. However, c.s.e. is an equivalence relation on the set of g-s-s system matrices. This important fact is established in Theorem (5.4.1). In order to prove this more results are needed.

In studying generalised state space systems Verghese [55] has used the transformation called "strong equivalence". The definition of this is somewhat lengthy and a closed form expression for it is not given. The definition proceeds as follows.

Definition (5.4.2)

Consider the operations embodied in the transformation

$$\begin{bmatrix} sE_1 - A_1 & \vdots & B_1 \\ \dots & \dots & \dots \\ -C_1 & \vdots & D_1 \end{bmatrix} = \begin{bmatrix} N_1 & \vdots & 0 \\ \dots & \dots & \dots \\ Q & \vdots & I \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & \vdots & B_2 \\ \dots & \dots & \dots \\ -C_2 & \vdots & D_2 \end{bmatrix} \begin{bmatrix} N_2^{-1} & \vdots & R \\ 0 & \dots & \vdots \\ \vdots & \vdots & I \end{bmatrix} \quad (5.4.4)$$

Where

$$\begin{aligned} N_1, N_2 & \text{ square and nonsingular} & (5.4.5) \\ QE_2 = 0 = E_1R & \end{aligned}$$

The operations permitted under this transformation are

termed OPERATIONS OF STRONG EQUIVALENCE. This leads to

Definition (5.4.3)

Two generalised state space system matrices $P_1(s)$, $P_2(s)$ in $P(m,1)$ are said to be STRONGLY EQUIVALENT (str.eq.) if after some sequence of operations of strong equivalence and each has been trivially deflated as far as possible, the resulting system matrices are related by operations of strong equivalence.

The lack of a closed form expression for the equivalence expressed in this definition is clearly a disadvantage.

It will now be shown how this may be remedied. The first result indicates a connection between the definitions of Str.eq and c.s.e.

Lemma (5.4.1)

Two generalised state space system matrices P_1, P_2 in $P_0(m,1)$ are c.s.e. if and only if system matrices derived from them by operations of strong equivalence are also c.s.e.

Proof

Let P_1 and P_2 be two c.s.e. g-s-s systems. Let P_1 and P_2 be constructed from P_1 and P_2 respectively by operations of strong equivalence

$$\begin{bmatrix} sE_i - A_i & B_i \\ \dots & \dots \\ -C_i & D_i \end{bmatrix} = \begin{bmatrix} N_{i1} & 0 \\ \dots & \dots \\ Q_i & I \end{bmatrix} \begin{bmatrix} sE_i - A_i & B_i \\ \dots & \dots \\ -C_i & D_i \end{bmatrix} \begin{bmatrix} N_{i2}^{-1} & R_i \\ 0 & \dots & \dots \\ \dots & \dots & I \end{bmatrix} \quad (5.4.6) \quad i=1,2.$$

as in Definition (5.4.2). Combining (5.4.2) together with (5.4.6) above gives the following relation between P_1 and P_2

$$\begin{bmatrix} N_{11} & 0 \\ \dots & \dots \\ Q_1 & I \end{bmatrix}^{-1} \begin{bmatrix} N_1 & 0 \\ \dots & \dots \\ X & I \end{bmatrix} \begin{bmatrix} N_{21} & 0 \\ \dots & \dots \\ Q_2 & I \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & B_2 \\ \dots & \dots \\ -C_2 & D_2 \end{bmatrix}$$

$$= \begin{bmatrix} sE_1 - A_1 & \vdots & B_1 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & -C_1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & D_1 \end{bmatrix} \begin{bmatrix} N_{12} & \vdots & R_1 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & I \end{bmatrix} \begin{bmatrix} N_2 & \vdots & Y \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & I \end{bmatrix} \begin{bmatrix} N_{22}^{-1} & \vdots & R_2 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & I \end{bmatrix}^{-1}$$

This may easily be verified to be a relationship of c.s.e. by virtue of the non singularity of the operations of strong equivalence. The nonsingularity of the operations involved allows the converse to the proof above to be established in an exactly similar way.

As a first step to establishing the precise connection between the notions of equivalence it will now be formally proved that c.s.e. is an equivalence relation.

Theorem (5.4.1)

C.s.e. is an equivalence relation on the set $P(m,1)$.

Proof

For reflexivity the transforming matrices N_1 and N_2 are taken as unit matrices and X, Y as zero matrices. The conditions (5.4.3) are easily verified.

For transitivity suppose $P_1(s), P_2(s), P_3(s)$ in $P(m,1)$ are related as

$$\begin{bmatrix} sE_1 - A_1 & \vdots & B_1 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & -C_1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & D_1 \end{bmatrix} \begin{bmatrix} N_1 & \vdots & Y_1 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & I \end{bmatrix} = \begin{bmatrix} N_2 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & X_2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & I \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & \vdots & B_2 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & -C_2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & D_2 \end{bmatrix} \quad (5.4.7)$$

$$\begin{bmatrix} sE_2 - A_2 & \vdots & B_2 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & -C_2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & D_2 \end{bmatrix} \begin{bmatrix} N_3 & \vdots & Y_3 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & I \end{bmatrix} = \begin{bmatrix} N_4 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & X_4 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & I \end{bmatrix} \begin{bmatrix} sE_3 - A_2 & \vdots & B_3 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & -C_3 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & D_3 \end{bmatrix} \quad (5.4.8)$$

Combining (5.4.7) and (5.4.8) gives

$$\begin{bmatrix} sE_1 - A_1 & \vdots & B_1 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & -C_1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & D_1 \end{bmatrix} \begin{bmatrix} N_1 N_3 & \vdots & N_1 Y_3 + Y_1 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & I \end{bmatrix} = \begin{bmatrix} N_2 N_4 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & X_2 N_4 + X_4 I \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & I \end{bmatrix} \begin{bmatrix} sE_3 - A_2 & \vdots & B_3 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & -C_3 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & D_3 \end{bmatrix} \quad (5.4.9)$$

This clearly has the form of c.s.e. and the properties in (5.4.3) may be verified from Theorem (5.3.3).

For symmetry assume that $P_1(s), P_2(s)$ in $P_0(m,1)$ are c.s.e. Then by Lemma (5.4.1), $P_i(s)$ may be replaced by $P'_i(s)$ in which the pole pencil is $KCF(sE_i - A_i)$ ($i=1,2$). It is thus only necessary to show that $P'_2(s)$ and $P'_1(s)$ are c.s.e. Let $P''_i(s)$ be the totally deflated form of $P'_i(s)$ ($i=1,2$). The deflation of $P'_2(s)$ to $P''_2(s)$ and the inflation of $P''_1(s)$ to $P'_1(s)$ are clearly both transformations of c.s.e. Hence by the transitivity property of c.s.e. it follows that $P''_1(s)$ and $P''_2(s)$ are c.s.e. That is

$$\begin{bmatrix} sI - \bar{A} & 0 & \vdots & \bar{B}_1 \\ 0 & I - sJ & \vdots & \hat{B}_1 \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{C}_1 & \vdots & -\hat{C}_1 & \hat{D}_1 \end{bmatrix} \begin{bmatrix} N_2 & \vdots & Y \\ 0 & \vdots & I \end{bmatrix} = \begin{bmatrix} N_1 & \vdots & 0 \\ X & \vdots & I \end{bmatrix} \begin{bmatrix} sI - \bar{A} & 0 & \vdots & \bar{B}_2 \\ 0 & I - sJ & \vdots & \hat{B}_2 \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{C}_2 & \vdots & -\hat{C}_2 & \hat{D}_2 \end{bmatrix} \quad (5.4.10)$$

Application of Theorem (5.3.4) shows that N_2 and N_1 and hence

$$\begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix}, \begin{bmatrix} N_1 & 0 \\ X & I \end{bmatrix}$$

are square and invertible so that

$$\begin{bmatrix} N_1^{-1} & \vdots & 0 \\ \vdots & \vdots & \vdots \\ -XN_1^{-1} & \vdots & I \end{bmatrix} \begin{bmatrix} sI - \bar{A} & 0 & \vdots & \bar{B}_1 \\ 0 & I - sJ & \vdots & \hat{B}_1 \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{C}_1 & \vdots & -\hat{C}_1 & \hat{D}_1 \end{bmatrix} = \begin{bmatrix} sI - \bar{A} & 0 & \vdots & \bar{B}_2 \\ 0 & I - sJ & \vdots & \hat{B}_2 \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{C}_2 & \vdots & -\hat{C}_2 & \hat{D}_2 \end{bmatrix} \begin{bmatrix} N_2^{-1} & \vdots & -N_2^{-1}Y \\ 0 & \vdots & I \end{bmatrix} \quad (5.4.11)$$

That is $P''_2(s)$ is c.s.e. to $P''_1(s)$. Further application of transitivity then gives the required result.

The next result achieves the unification between str.eq. and c.s.e.

Theorem (5.4.2)

Two generalised state space system matrices $P_1(s)$, $P_2(s)$ in $P_0(m,1)$ are c.s.e. if and only if they are str.eq.

Proof

From Lemma (5.4.2) and Lemma (5.4.2) respectively it is seen that both trivial inflation/deflation and operations of strong equivalence may be described by transformations of c.s.e. It follows from the transitivity property of c.s.e. that if $P_1(s)$ and $P_2(s)$ are str.eq. they are also c.s.e. Conversely suppose that $P_1(s)$ and $P_2(s)$ are c.s.e. then $P_i(s)$ may be replaced by $P'_i(s)$ as in the proof of symmetry in Theorem (5.4.1) above. $P'_1(s)$ and $P'_2(s)$ are related by trivial inflation/deflation together with the operation of (5.4.10) which are clearly operations of strong equivalence. It thus follows that $P'_1(s)$ and $P'_2(s)$ are str.eq. and hence $P_1(s)$ and $P_2(s)$ are str.eq. completing the proof.

It is seen from the above that the notions of str.eq. and c.s.e. are identical and as such c.s.e. provides a closed form statement of the requirements of str.eq.

The action of c.e. on singular and regular pencils (see theorem (5.2.1)) suggests that c.s.e. will keep invariant certain system structure at infinity. C.s.e. is found to preserve all currently interesting system properties at finite frequencies and at infinite frequency. The following theorem summarises these findings.

Theorem (5.4.3)

c.s.e. leaves invariant the following sets for g-s-s systems. The finite and infinite

- (i) transfer function poles
- (ii) transmission zeros
- (iii) decoupling zeros
- (iv) invariant zeros

(v) system poles

Proof

The finite frequency structure in (i) to (iv) is preserved because c.s.e. is a special case of e.u.e. In [41] it is shown that e.u.e. preserves finite frequency structure. Therefore it is only necessary to consider the infinite system structure.

The individual equations corresponding to (5.4.2) are

$$\begin{matrix} \text{it's not} & \text{necessavly} \\ \text{see} & \text{pugh} \end{matrix} (sE_1 - A_1)N_2 = N_1(sE_2 - A_2) \quad (5.4.12)$$

$$(sE_1 - A_1)Y + B_1 = N_1B_2 \quad (5.4.13)$$

$$-C_1N_2 = X(sE_2 - A_2) - C_2 \quad (5.4.14)$$

$$-C_1Y + D_1 = XB_2 + D_2 \quad (5.4.15)$$

Let G_1 and G_2 be the transfer-function matrices corresponding to P_1 and P_2 respectively, then

$$\begin{aligned} G_2(s) &= C_2(sE_2 - A_2)^{-1}B_2 + D_2 \\ &= (X(sE_2 - A_2) + C_1N_2)(sE_2 - A_2)^{-1}B_2 + D_2 && \text{from (5.4.14)} \\ &= XB_2 + C_1(sE_1 - A_1)^{-1}N_1B_2 + D_2 && \text{from (5.4.12)} \\ &= XB_2 + C_1Y + C_1(sE_1 - A_1)^{-1}B_1 + D_2 && \text{from (5.4.13)} \\ &= C_1(sE_1 - A_1)^{-1}B_1 + D_1 \text{ (from (5.4.15))} \\ &= G_1(s) \end{aligned}$$

The transfer function poles and transmission zeros are both defined via the transfer function (see Definition (3.5.10) and Definition (3.5.7)). P_1 and P_2 have the same transfer function therefore (i) and (ii) are proved.

(iii) The decoupling zeros of $P_1(s)$ are defined as the

finite and infinite zeros of the $(sE-A \ B)$ submatrix of $P(s)$ (see Definition (3.4.9)). Under c.s.e. this is transformed as

$$(sE_1 - A_1 \ \vdots \ B_1) \begin{bmatrix} N_2 \ \vdots \ Y \\ \cdot \ \vdots \ \cdot \\ 0 \ \vdots \ I \end{bmatrix} = N_1 (sE_2 - A_2 \ B_2) \quad (5.4.16)$$

where

$$(sE_1 - A_1 \ \vdots \ N_1) \quad (5.4.17)$$

$$\begin{bmatrix} -N_2 \\ \cdot \ \vdots \ \cdot \\ sE_2 - A_2 \end{bmatrix} \quad \text{have full rank and no finite or infinite zeros}$$

The requirements (5.4.3) then imply that

$$(sE_1 - A_1 \ B_1 \ \vdots \ N_1); \quad \begin{bmatrix} -N_2 & -Y \\ 0 & -I \\ \cdot \ \vdots \ \cdot & \cdot \ \vdots \ \cdot \\ sE_2 - A_2 & B_2 \end{bmatrix}$$

have no finite or infinite zeros. Hence (5.4.4) is a statement of c.e. in respect of the singular pencils $(sE_1 - A_1 \ \vdots \ B_1)$ and $(sE_2 - A_2 \ \vdots \ B_2)$ and the result follows from Theorem (5.2.1).

The infinite invariant zeros (see Definition (3.5.8)) are the infinite zeros of the systems matrices. C.s.e. is a relation of c.e. between the system matrices and thus Theorem (5.2.1) applies and thus the invariant zero are preserved.

The infinite system poles (see Definition (3.5.11)) are the infinite zeros of the poles pencils $sE - A$ of the systems matrices. The pole pencils are related by c.e. and so Theorem (5.2.1) applies to give the infinite system poles are invariant.

Theorem (5.4.3) shows that c.s.e. leaves the pole/zero

structure of the system unchanged. Other parameters of importance to the dynamical behaviour of the system are the input (resp. output) dynamical indices defined as the column (resp. row) minimal indices of the singular pencil

$$\begin{bmatrix} sE-A \\ \vdots \\ B \end{bmatrix} \text{ (resp. } \begin{bmatrix} sE-A \\ \cdots \\ -C \end{bmatrix} \text{)}$$

(see Rosenbrock and Hayton [44]). The following result shows that these are also invariant under c.s.e.

The following result is due to Pugh and Hayton but is included for completeness.

Theorem (5.4.4) (important)

C.s.e. leaves invariant the input and output dynamical indices of a system

Proof

Let $P_1(s), P_2(s)$ in $P(m,1)$ be c.s.e. and let $d_{i1} \ll d_{i2} \ll \dots \ll d_{in}$ be the input dynamical indices of $P_i(s)$ ($i=1,2$). Consider

$$\begin{bmatrix} sE_i - A_i & B_i \end{bmatrix} \begin{bmatrix} \xi(s) \\ u(s) \end{bmatrix} = 0 \quad i=1,2 \quad (5.4.18)$$

and let

$$\begin{bmatrix} \xi_{21}(s) \\ u_{22}(s) \end{bmatrix}$$

be a solution of (5.4.18) for $i=1,2$ of lowest degree then

$$\begin{bmatrix} \xi_{21} \\ u_{21} \end{bmatrix}$$

has degree d_{21} (See Gantmacher [19]). Since $P_1(s), P_2(s)$ are c.s.e. it is readily seen that

$$\begin{bmatrix} \xi_{11}(s) \\ u_{11}(s) \end{bmatrix} = \begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_{21} \\ u_{22} \end{bmatrix} \quad (5.4.19)$$

is a solution of (5.4.18) for $i=1$. The transforming matrix in (5.4.19) is constant so

$$\begin{bmatrix} \xi_{11} \\ u_{11} \end{bmatrix}$$

has degree no greater than the degree of

$$\begin{bmatrix} \xi_{21} \\ u_{21} \end{bmatrix}$$

that is

$$d_{11} \leq d_{21} \quad (5.4.20)$$

By the symmetry property of Theorem (5.4.1), however, $P_2(s)$, $P_1(s)$ are c.s.e. and the argument above may be repeated to show

$$d_{21} \leq d_{11} \quad (5.4.21)$$

it follows from (5.4.20) and (5.4.21) that

$$d_{11} = d_{21} \quad (5.4.22)$$

Now among those solutions of (5.4.18), $i=1,2$, which are independent of

$$\begin{bmatrix} \xi_{21}(s) \\ u_{21}(s) \end{bmatrix}$$

let

$$\begin{bmatrix} \xi_{22}(s) \\ u_{22}(s) \end{bmatrix}$$

be one having lowest degree. Then the degree of

$$\begin{bmatrix} \xi_{22}(s) \\ u_{22}(s) \end{bmatrix}$$

is d_{22} .

$$\begin{bmatrix} \xi_{12}(s) \\ u_{12}(s) \end{bmatrix} = \begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_{22} \\ u_{22} \end{bmatrix} \quad (5.4.23)$$

is certainly a solution of (5.4.18) for $i=1,2$; it will have degree greater or equal to d_{12} if it is independent of

$$\begin{bmatrix} \xi_{11}(s) \\ u_{11}(s) \end{bmatrix}$$

Suppose that

$$\begin{bmatrix} \xi_{1i}(s) \\ u_{1i}(s) \end{bmatrix} \quad i=1,2$$

are linearly independent, that is suppose there exists constant non zero a_1 and a_2 such that

$$a_1 \begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_{21}(s) \\ u_{21}(s) \end{bmatrix} + a_2 \begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_{22}(s) \\ u_{22}(s) \end{bmatrix} = 0 \quad (5.4.24)$$

From consideration of the second block row in (5.4.24)

$$a_1 u_{21}(s) + a_2 u_{22}(s) = 0 \quad (5.4.25)$$

Recalling that

$$\begin{bmatrix} \xi_{2i}(s) \\ u_{2i}(s) \end{bmatrix} \quad i=1,2$$

are linearly independent solutions of (5.4.18), $i=1,2$ it follows from (5.4.25) that there exists a nonzero vector $\xi'(s)=a_1\xi_{21}(s)+a_2\xi_{22}(s)$ such that

$$(sE_2-A_2)\xi'(s) = 0 \quad (5.4.26)$$

(5.4.26) contradicts the regularity of the pole pencil sE_2-A_2 . Hence

$$\begin{bmatrix} \xi_{1i}(s) \\ u_{1i}(s) \end{bmatrix} \quad i=1,2$$

are linearly independent, hence

$$\begin{bmatrix} \xi_{12}(s) \\ u_{12}(s) \end{bmatrix} \quad i=1,2$$

has degree d_{12} and so

$$d_{12} < d_{22} \quad (5.4.27)$$

The symmetry of e.s.e. is again used to show that equality holds in (5.4.27). The argument may be repeated to show that $d_{1i}=d_{2i}$ $i=1,2,\dots,n$ and thus that the input dynamical indices of the system are invariant under c.s.e. Invariance of the output dynamical indices may be shown in the same way by considering solutions of

$$(\xi^t(s), y^t(s)) \begin{bmatrix} sE_i - A_i \\ \dots \\ -C_i \end{bmatrix} = 0 ; \quad i=1,2 \quad (5.4.28)$$

5.5 Conclusion

(e.s.e) has been extended to encompass the point at infinity. The resulting transformation termed complete equivalence has been shown to preserve the finite and infinite zeros of both regular and singular pencils. Additionally, c.e. was shown to be an equivalence relation on the set of regular pencils.

c.e. was then modified by system theoretic requirements to produce a new equivalence transformation called complete system equivalence (c.s.e.). This system transformation has the properties required, namely that of relating systems of different size while preserving finite and infinite frequency structure.

c.s.e. was compared to a previous notion of equivalence for g-s-s systems termed strong equivalence. Str.eq. is somewhat lengthy and algorithmic in nature. (c.s.e) was demonstrated to be a closed form description of this algorithmic transformation.

CHAPTER SIX

FUNDAMENTAL EQUIVALENCE OF GENERALISED

STATE SPACE SYSTEMS

6.1 Introduction

The previous chapter detailed a equivalence transformation based upon a study of the underlying transformations of matrix pencils. This chapter takes a different approach to the problem by defining equivalence of generalised state space systems in terms of mappings of the solution sets of the describing differential equations in the manner suggested by Pernebo [35] together with mappings of the sets of restricted initial conditions. This provides a conceptually pleasing definition of equivalence within which the invariants of the controllability and observability characteristics are implicit.

Section 6.2 summarises a number of existing results and derives the basic properties of the proposed equivalence relation. Section 6.3 contains the major result which demonstrates that this notion of equivalence is an alternative characterisation of complete system equivalence. Many of the results in this chapter can be found in Hayton, Fretwell and Pugh [62], Hayton, Fretwell and Pugh [63] and also Pugh, Hayton and Fretwell [66]. [65].

6.2 Preliminary Results and Definitions

The system to be considered is described by a set of first-order, linear, differential equations of the form

$$E\dot{\bar{\chi}}(t) = A\bar{\chi}(t) + B\bar{u}(t) \quad (6.2.1a)$$

$$\bar{y}(t) = C\bar{\chi}(t) + D\bar{u}(t) \quad (6.2.1b)$$

where $\bar{\chi}(t)$ is an n-vector, $\bar{u}(t)$ an l-vector, $\bar{y}(t)$ an m-vector and E, A, B, C, D are constant matrices with dimensions nxn, nxn, nxl, mxn and mxl respectively. Assuming non-zero initial conditions $\chi(0^-)$, Laplace transformation of (6.2.1a), (6.2.1b) gives

$$\begin{bmatrix} sE-A & : & B \\ \dots & \dots & \dots \\ & -C & : & D \end{bmatrix} \begin{bmatrix} \chi(s) \\ \dots \\ -u(s) \end{bmatrix} = \begin{bmatrix} E\chi(0^-) \\ \dots \\ -y(s) \end{bmatrix} \quad (6.2.2)$$

Where $\chi(s)$, $y(s)$ and $u(s)$ are the Laplace transforms of $\bar{\chi}(t)$, $\bar{y}(t)$ and $\bar{u}(t)$ respectively. This is a GENERALISED STATE SPACE system. See Chapter 3 section 3.5 for more details. A new equivalence relation for g-s-s systems will be developed in this chapter.

The transformation of complete system equivalence retains both input/output behaviour and the (finite and infinite) pole and zero structure of any g-s-s realisation. The transformation thus preserves all the properties fundamental to the dynamic behaviour of the system. Complete system equivalence was originally derived as a transformation of the system matrices, but an alternative and more intuitively attractive way of defining equivalence of two systems is to do so in terms of maps between

(i) the solution/input vector pairs

$$\begin{bmatrix} \chi_1(s) \\ -u(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \chi_2(s) \\ -u(s) \end{bmatrix}$$

and

(ii) the restricted initial condition/output vector pairs

$$\begin{bmatrix} E_1\chi_1(0^-) \\ -y(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_2\chi_2(0^-) \\ -y(s) \end{bmatrix}$$

It is noted from the form of the system equation (6.2.2) that the pairs referred to here are fundamental to the system. The system equations themselves are simply mappings from an appropriate set of solution/input pairs to an appropriate set of initial condition/output pairs. It is the nature of these two sets that characterise a system, for the solution/input pairs completely describe the controllability properties of the system, while the restricted initial condition/output pairs encapsulate the observability properties. These comments are probably best understood in the context of system equations in state space form for then the number of initial condition/output pairs which contain a given output describes the extent to which the system is unobservable. For the case of controllability and again in the state space context, consider the union of all pairs $(x(\tau)^t - u(\tau)^t)^t$ over all admissible control functions. Then controllability demands that the union of all solutions occurring in such pairs should be R^n for each fixed τ .

It would appear that to preserve the structure of the pairs in (i) and (ii) above (and hence to preserve the controllability and observability properties of the system) it would be necessary for there to be two bijections, one relating the solution/input vector pairs and the other relating the restricted initial condition/output vector pairs. However, this is not necessary and a subsequent result shows that the following definition suffices.

Definition (6.2.1)

Let P_1, P_2 , be two generalised state space systems. They are said to be **FUNDAMENTALLY EQUIVALENT** if there exist

- (i) a constant, ²⁻¹ injective map

$$\begin{bmatrix} \chi_2(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \chi_1(s) \\ -u(s) \end{bmatrix} \quad (6.2.3)$$

and

(ii) a constant, surjective map ^{eni}

$$\begin{bmatrix} E_2 \chi_2(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} E_1 \chi_1(0-) \\ -y(s) \end{bmatrix}, \quad XE_1=0 \quad (6.2.4)$$

Note that the special structure of these maps is dictated purely by the requirement that the input/output behaviour of P_1, P_2 be identical. Further, as with system similarity and system matrices in the conventional state space form ([43]), the constancy of the maps (6.2.3) and (6.2.4) is necessary to preserve the generalised state space form. In general, polynomial transformations will result in descriptions of the system by differential equations of higher order than the first. Note however, that in contrast to system similarity, the matrices M and N defining the maps (6.2.3) and (6.2.4) are not necessarily square. Following Pernebo [35], Verghese et al. have proposed a similar definition to (6.2.1) which refers only to the mapping in (6.2.3) with the requirement that it be a bijection [59]. However, to specify two maps as in definition (6.2.1) is a recognition of the relevance from a systems viewpoint of the pairs referred to in (6.2.3) and (6.2.4). Further, the splitting of the bijection requirement in the manner indicated permits the duality of the controllability and observability concepts contained within the pairs to be fully exploited.

The first result formally establishes the bijectivity of the two maps of Definition (6.2.1).

Theorem (6.2.1)

Let P_1, P_2 be two systems in generalised state space form

which are fundamentally equivalent then the maps (6.2.3), (6.2.4) are both bijective.

Proof

The restricted sets of initial conditions $\{E_1\chi_1(0-)\}$, $\{E_2\chi_2(0-)\}$ form vector spaces over R . Let these spaces have dimensions t_1 ($= \text{rank } E_1$) and t_2 ($= \text{rank } E_2$) respectively. From (6.2.4) there exists a map between the two spaces such that

$$E_2\chi_2(0-) = ME_1\chi_1(0-) \quad (6.2.5)$$

Further (6.2.5) is a surjection and hence

$$t_1 \geq t_2 \quad (6.2.6)$$

From (6.2.3) there exists a map between solutions such that

$$\chi_2(s) = N\chi_1(s) - Yu(s) \quad (6.2.7)$$

(6.2.7) is an injection and substituting bijectively for $\chi_1(s)$ and $\chi_2(s)$ from the system equation (6.2.2) gives the following map.

$$\begin{aligned} & (sE_2 - A_2)^{-1}E_2\chi_2(0-) + (sE_2 - A_2)^{-1}B_2u(s) \\ &= N(sE_1 - A_1)^{-1}E_1\chi_1(0-) + N(sE_1 - A_1)^{-1}B_1u(s) - Yu(s) \end{aligned} \quad (6.2.8)$$

(6.2.8) holds for all $u(s)$ and, in particular, for $u(s)=0$, hence

$$E_2\chi_2(0-) = (sE_2 - A_2)N(sE_1 - A_1)^{-1}E_1\chi_1(0-) \quad (6.2.9)$$

is an injection between the restricted initial condition spaces which will map any set of independent elements in $\{E_1\chi_1(0-)\}$ to a set of independent elements in $\{E_2\chi_2(0-)\}$. It follows that

$$t_1 \leq t_2 \quad (6.2.10)$$

From (6.2.6), (6.2.10)

$$t_1 = t_2 = t \quad (6.2.11)$$

and so (6.2.5) and hence (6.2.4) is a bijection.

Now suppose (6.2.7) is not bijective. That is suppose there exists a solution $\chi_2(s)$ such that

$$\chi_2(s) \neq N\chi_1(s) - Yu(s) \quad (6.2.12)$$

again substituting for $\chi_1(s)$, $\chi_2(s)$ from the system equation and setting $u(s)=0$ there exists an $E_2\tilde{\chi}_2(0-)$ such that

$$E_2\tilde{\chi}_2(0-) \neq (sE_2 - A_2)N(sE_1 - A_1)^{-1}E_1\chi_1(0-) \quad (6.2.13)$$

Let $E_1\chi_{1bi}(0-)$ $i=1,2,\dots,t$ be some basis for $\{E_1\chi_1(0-)\}$ then, since $(sE_2 - A_2)N(sE_1 - A_1)^{-1}$ is injective, the vectors $E_2\chi_{2bi}(0-)$ $i=1,2,\dots,t$ where

$$E_2\chi_{2bi}(0-) = (sE_2 - A_2)N(sE_1 - A_1)^{-1}E_1\chi_{1bi}(0-) \quad (6.2.14)$$

is a set of independent vectors in $\{E_2\chi_2(0-)\}$. For (6.2.13) to hold $E_2\tilde{\chi}_2(0-)$ is independent of this set and this implies that $\{E_2\chi_2(0-)\}$ has dimension greater than t_1 which contradicts (6.2.6). Hence the assumption that (6.2.7) is not surjective is false and (6.2.7) is a bijection.

The two pairs in (6.2.3) and (6.2.4) characterise the controllability and observability of the system respectively. Theorem (6.2.1) has demonstrated that two fundamentally equivalent systems have isomorphic solution/input pairs and isomorphic initial condition/output pairs. This implicitly guarantees the invariance of the

observability and controllability characteristics.
equivalence

It is to be emphasised that the solution/input pairs $(\chi_i(s)^t, u(s)^t)^t$ form a subset of the relevant space $R(s)^{n+1}$. This subset is determined by the associated system equations and it is between these subsets (not $R(s)^{n+1}$) that the map (6.2.3) is bijective. Similar comments apply to the restricted initial condition/output pairs.

There are a number of technical results arising from Theorem (6.2.1) and required subsequently which will be presented as Corollaries.

Corollary 1

With the matrices defined as in the theorem

$$(sE_2 - A_2)^{-1} B_2 - N(sE_1 - A_1)^{-1} B_1 + Y = 0 \quad (6.2.15)$$

Proof

Since (6.2.8) holds for all $u(s)$ and the $E_i \chi_i(0-)$ $i=1,2$ are independent of $u(s)$, the matrix coefficient of $u(s)$ in this equation must be identically zero.

Corollary 2

With the matrices defined as in the Lemma there exists a bijective map M' , (not necessarily identical to M) between the restricted initial condition spaces such that

$$((sE_2 - A_2)^{-1} M' - N(sE_1 - A_1)^{-1}) E_1 = 0 \quad (6.2.16)$$

Proof

From the theorem, (6.2.9) is a rational bijection between the restricted initial condition spaces. I_n is a basis for $\chi_1(0-)$ and each element of this basis is mapped (not

necessarily bijectively) to a constant vector $E_2\chi_2(0^-)$ that is

$$(sE_2 - A_2)N(sE_1 - A_1)^{-1}E_1 = Q \quad , Q \text{ constant} \quad (6.2.17)$$

Let

$$(sE_2 - A_2)N(sE_1 - A_1)^{-1} = A_0 + sA_1(s) + A_2(s) \quad (6.2.18)$$

where A_0 is constant, $A_1(s)$ is polynomial and $A_2(s)$ is strictly proper. Substituting into (6.2.17)

$$(A_0 + sA_1(s) + A_2(s))E_1 = Q$$

and equating coefficients of different powers of s :-

$$sA_1(s)E_1 = A_2(s)E_1 = 0 \quad (6.2.19)$$

Hence

$$(sE_2 - A_2)N(sE_1 - A_1)^{-1}E_1 = A_0E_1 \quad (6.2.20)$$

$M' = A_0$ is a bijection between the restricted initial condition spaces and premultiplying both sides of (6.2.20) by $(sE_2 - A_2)^{-1}$ gives (6.2.16).

Note that it is not possible to obtain (6.2.16) directly from (6.2.5) and (6.2.9) since given $E_1\chi_1(0^-)$ may be mapped by (6.2.5) and (6.2.9) to different members of the set $\{E_2\chi_2(0^-)\}$. Thus $M' = M$ as is illustrated by the following example.

Example (6.2.1)

Consider two, identical, undriven, state space systems

$$P_i = \begin{bmatrix} s-1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \chi_1(s) \\ \chi_2(s) \\ \chi_3(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} \chi_1(0-) \\ \chi_2(0-) \\ \chi_3(0-) \\ -y(s) \end{bmatrix} \quad i=a,b \quad (6.2.21)$$

The solution space $\{\chi_a(s)\}$ is mapped bijectively to the solution space $\{\chi_b(s)\}$ by the unit matrix I_3 . Clearly I_3 is also a bijective map between the initial condition sets, but an equally valid choice in terms of Definition (6.2.1) is M where

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (6.2.22)$$

and in this case

$$((sE_2 - A_2)^{-1} M - N(sE_1 - A_1)^{-1}) E_1 =$$

$$\begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$$

$\neq 0$

Definition (6.2.2)

A generalised state space system with system matrix

$$P_K(s) = \begin{bmatrix} sI - \bar{A} & 0 & : & \bar{B} \\ 0 & I - sJ & : & \hat{B} \\ \cdots & -\bar{C} & \cdots & \vdots \\ & -\hat{C} & & \hat{D} \end{bmatrix} \quad (6.2.23)$$

is said to be in KRONECKER FORM. The transformation to this

form is

$$\begin{bmatrix} M & 0 \\ \dots & \dots \\ 0 & I \end{bmatrix} \begin{bmatrix} sE-A & B \\ \dots & \dots \\ -C & D \end{bmatrix} \begin{bmatrix} N & 0 \\ \dots & \dots \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI-\bar{A} & 0 & \dots & \bar{B} \\ \dots & \dots & \dots & \dots \\ 0 & I-sJ & \dots & \hat{B} \\ \dots & \dots & \dots & \dots \\ -\bar{C} & -\hat{C} & \dots & \hat{D} \end{bmatrix} \quad (6.2.24)$$

Where M and N are constant nonsingular matrices. (See Lemma (3.5.1))

In this case the subsystem

$$\begin{bmatrix} sI-\bar{A} & \bar{B} \\ \dots & \dots \\ -\bar{C} & 0 \end{bmatrix} \begin{bmatrix} x(s) \\ \dots \\ -u(s) \end{bmatrix} = \begin{bmatrix} x(0-) \\ \dots \\ -y(s) \end{bmatrix}$$

will be referred to as the state space subsystem and the subsystem

$$\begin{bmatrix} I-sJ & \hat{B} \\ \dots & \dots \\ -\hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \eta(s) \\ \dots \\ -u(s) \end{bmatrix} = \begin{bmatrix} -J\eta(0-) \\ \dots \\ -y(s) \end{bmatrix}$$

will be referred to as the impulsive subsystem.

It should be noted that Definition (6.2.2) is a slight misuse of the Kronecker standard form as it is only the block $sE-A$ which is reduced to the Kronecker form (Gantmacher, [19]). This form is not unique but is useful for proving results and will be used extensively.

Lemma (3.5.1) has shown that the transformation of a system to its Kronecker form is a special case of complete system equivalence. The following shows that this reduction may also be regarded as a transformation of fundamental equivalence.

Lemma (6.2.1)

A generalised state space system is fundamentally

equivalent to its Kronecker form and vice versa.

Proof

Equating powers of s in block position (1,1) of (6.2.24)

$$MEN = \begin{bmatrix} I_n & 0 \\ 0 & -J \end{bmatrix} \quad (6.2.25)$$

Multiplying both sides of (6.2.25) by $N^{-1}\chi_1(0-)$ gives

$$ME\chi(0-) = \begin{bmatrix} I & 0 \\ 0 & -J \end{bmatrix} N^{-1}\chi(0-) \quad (6.2.26)$$

$\begin{bmatrix} I & 0 \\ 0 & -J \end{bmatrix} N^{-1}\chi(0-)$ is certainly an element of $\begin{bmatrix} x(0-) \\ -J\eta(0-) \end{bmatrix}$

and so (6.2.26) may be generalised to

$$\begin{bmatrix} x(0-) \\ -J\eta(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E\chi(0-) \\ -y(s) \end{bmatrix} \quad (6.2.27)$$

M is non singular and hence (6.2.27) is the required map between the restricted initial condition/output pairs.

Equating block elements (1,1) in (6.2.24) and post multiplying by $N^{-1}\chi(s)$

$$M(sE-A)\chi(s) = \begin{bmatrix} sI-\bar{A} & 0 \\ 0 & I-sJ \end{bmatrix} N^{-1}\chi(s)$$

premultiplying by

$$\begin{bmatrix} sI-\bar{A} & 0 \\ 0 & I-sJ \end{bmatrix}^{-1}$$

gives

$$\begin{bmatrix} sI-\bar{A} & 0 \\ 0 & I-sJ \end{bmatrix}^{-1} M(sE-A)\chi(s) = N^{-1}\chi(s)$$

Substituting on the left hand side for $\chi(s)$ in terms of $\chi(0-)$ gives

$$\begin{bmatrix} sI-\bar{A} & 0 \\ 0 & I-sJ \end{bmatrix}^{-1} ME\chi(0-) + \begin{bmatrix} sI-\bar{A} & 0 \\ 0 & I-sJ \end{bmatrix}^{-1} MBu(s) = N^{-1}\chi(s) \quad (6.2.28)$$

From consideration of (block) element (1,2) in (6.2.24)

$$B = M^{-1} \begin{bmatrix} \bar{B} \\ \vdots \\ \hat{B} \end{bmatrix} \quad (6.2.29)$$

and substituting from (6.2.29) and (6.2.27) into (6.2.28) and using the subsystem equations gives

$$\begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix} = N^{-1}\chi(s) \quad (6.2.30)$$

The nonsingularity of N ensures (6.2.30) is an injection and hence the system is fundamentally equivalent to its Kronecker form. The converse result follows immediately from (6.2.30) and (6.2.27) on noting that M and N are both nonsingular.

With the aid of Lemma (3.5.1) and Lemma (6.2.1) it now can be shown that fundamental equivalence acts in a decoupled fashion on the Kronecker standard form.

Theorem (6.2.2)

Let P_1, P_2 be fundamentally equivalent generalised state space systems in Kronecker form then

(i) the two state space sub systems are f.e.

(ii) the two impulsive subsystems are f.e.

Proof

The solution/input map of (6.2.3) may be partitioned and written

$$\begin{bmatrix} x_2(s) \\ \eta_2(s) \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \begin{bmatrix} x_1(s) \\ \eta_1(s) \end{bmatrix} - \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} u(s) \quad (6.2.31)$$

Setting $u(s)=0$ in (6.2.31)

$$x_2(s) = N_1 x_1(s) + N_2 \eta_1(s) \quad (6.2.32)$$

and since $x_2(s) - N_2 \eta_1(s)$ is strictly proper and $N_2 \eta_1(s)$ is polynomial comparing coefficients in (6.2.32) gives, after substitution from the system equations with $u(s)=0$.

$$N_2 J_1 = 0 \quad (6.2.33)$$

An analogous argument concerning the second block equation in (6.2.31) gives

$$N_3 = 0 \quad (6.2.34)$$

(6.2.33), (6.2.34) show that (6.2.31) may be reduced to to the decoupled form

$$\begin{bmatrix} x_2(s) \\ \eta_2(s) \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & N_4 \end{bmatrix} \begin{bmatrix} x_1(s) \\ \eta_1(s) \end{bmatrix} - \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} u(s) \quad (6.2.35)$$

It should be noted that this is not unique, however, since N_2 can, in fact, be any matrix satisfying (6.2.33)

From Corollary 2 to Theorem (6.2.1) there exists a constant bijective map between initial condition/output pairs

$$\begin{bmatrix} x_2(0-) \\ -J_2\eta_2(0-) \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} x_1(0-) \\ -J_1\eta_1(0-) \end{bmatrix} \quad (6.2.36)$$

such that

$$\begin{bmatrix} \left[\begin{array}{cc} (sI-A_2)^{-1} & 0 \\ 0 & (I-J_2)^{-1} \end{array} \right] \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} - \begin{bmatrix} N_1 & 0 \\ 0 & N_4 \end{bmatrix} & \left[\begin{array}{cc} (sI-A_1)^{-1} & 0 \\ 0 & (I-sJ_1)^{-1} \end{array} \right] \begin{bmatrix} I & 0 \\ 0 & -J_1 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \quad (6.2.37)$$

Examining the individual blocks in (6.2.37) gives from the (1,1) position

$$(sI-A_2)^{-1}M_1 = N_1(sI-A_1)^{-1}$$

that is

$$M_1(sI-A_1) = (sI-A_2)N_1$$

and comparing coefficients of s

$$M_1 = N_1 \quad (6.2.38)$$

also from the (1,2) position in (6.2.37)

$$(sI-A_2)^{-1}M_2J_1 = 0$$

that is

$$M_2J_1 = 0 \quad (6.2.39)$$

From the (2,1) position in (6.2.37)

$$(I-sJ_2)^{-1}M_3 = 0$$

that is

$$M_3 = 0 \quad (6.2.40)$$

and from (2,2) position in (6.2.37)

$$((I-sJ_2)^{-1}M_4-N_4(I-sJ_1)^{-1})J_1 = 0$$

that is

$$((I+sJ_2+o(J_2^2))M_4-N_4(I+sJ_1+o(J_1^2)))J_1 = 0$$

and comparing constants

$$M_4J_1 = N_4J_1 \quad (6.2.41)$$

(6.2.38), (6.2.39), (6.2.40), (6.2.41) show that there exists a decoupled bijection between the initial condition pairs of the form

$$\begin{bmatrix} x_2(0-) \\ -J_2\eta_2(0-) \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & N_4 \end{bmatrix} \begin{bmatrix} x_1(0-) \\ -J_1\eta_1(0-) \end{bmatrix} \quad (6.2.42)$$

It should be noted that the structure of (6.2.42) is not unique since the element in (block) position (1,2) is any constant matrix in the kernel of J_1 and the element in (block) position (2,2) is any constant matrix satisfying (6.2.41). Finally

$$y(s) = -X_1x_1(0-)+X_2J_1\eta_1(0-)+y(s) \quad (6.2.43)$$

that is

$$-X_1x_1(0-)+X_2J_1\eta_1(0-) = 0 \quad \forall x_1(0-), \eta_1(0-)$$

and hence

$$X_1 = 0 \quad (6.2.44)$$

$$X_2 J_1 = 0 \quad (6.2.45)$$

Summarising (6.2.35), (6.2.42), (6.2.43) and (6.2.45)

$$\begin{bmatrix} x_2(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} N_1 & Y_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(s) \\ -u(s) \end{bmatrix} \quad \text{is a bijection} \quad (6.2.46)$$

$$\begin{bmatrix} x_2(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(0-) \\ -y(s) \end{bmatrix} \quad \text{is a bijection} \quad (6.2.47)$$

so the state space subsystems are fundamentally equivalent

$$\begin{bmatrix} \eta_2(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} N_4 & Y_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ -u(s) \end{bmatrix} \quad \text{is a bijection} \quad (6.2.48)$$

$$\begin{bmatrix} J_2 \eta_2(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} N_4 & 0 \\ X_2 & I \end{bmatrix} \begin{bmatrix} J_1 \eta_1(0-) \\ -y(s) \end{bmatrix} \quad \text{is a bijection} \quad (6.2.49)$$

$$X_2 J_1 = 0$$

that is the impulsive subsystems are fundamentally equivalent.

The relationship between the state space subsystems may be further specialised to provide the expected connection with the conventional state space transformation.

Corollary 1

If two generalised state space systems in standard form are fundamentally equivalent then their state space subsystems are system similar.

Proof

Recalling that N_1 in (6.2.46) of the Lemma is, in fact, M' of Corollary 2 to Theorem (6.2.1) it is clear that for this special case (6.2.16) may be simplified to

$$(sI - \bar{A}_2)N_1 - N_1(sI - \bar{A}_1) = 0 \quad (6.2.50)$$

Substituting from (6.2.50) into (6.2.15)

$$B_2 = N_1 \bar{B}_1 - (sI - \bar{A}_2) Y_1$$

Equating coefficients of s then shows that

$$Y_1 = 0$$

and hence

$$\bar{B}_2 = N_1 \bar{B}_1 \quad (6.2.51)$$

Further recalling that

$$x_2(0-) = N_1 x_1(0-)$$

is a bijection between initial condition spaces of the same dimension it is clear that the two state space subsystems have the same order and that N_1 is square and nonsingular

Finally the initial condition/output pair map of (6.2.47) ensures that any $x_1(0-)$ and its image $x_2(0-) = N_1 x_1(0-)$ correspond to the same output $y(s)$ and, considering the undriven case, this gives

$$\bar{C}_1 (sI - \bar{A}_1)^{-1} x_1(0-) = \bar{C}_2 (sI - \bar{A}_2)^{-1} N_1 x_1(0-) \quad \forall x_1(0-)$$

replacing $x_1(0-)$ by the basis set I and invoking (6.2.50)

$$\bar{C}_1 = \bar{C}_2 N_1 \quad (6.2.52)$$

so that

$$\begin{bmatrix} sI - \bar{A}_2 & \bar{B}_2 \\ \hline -\bar{C}_2 & 0 \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - \bar{A}_1 & \bar{B}_1 \\ \hline -\bar{C}_1 & 0 \end{bmatrix} \begin{bmatrix} N_1^{-1} & 0 \\ 0 & I \end{bmatrix} \quad (6.2.53)$$

and hence the subsystems are system similar.

It should be noted that the result above actually holds for any pair of systems in the classical state space form which are fundamentally equivalent and is essentially that due to Pernebo [35].

The main difficulty in deriving a matrix characterisation of the notion of fundamental equivalence lies in fact that the maps whose existence is guaranteed by the Definition (6.2.1) may not commute in a diagrammatic sense. Specifically, when considering the impulsive subsystems of fundamentally equivalent, generalised state space systems in Kronecker standard form, the equation (6.2.16), after some manipulation becomes

$$((I-sJ_2)N_4 - N_4(I-sJ_1))J_1 = 0 \quad (6.2.54)$$

but it is not necessarily true that

$$(I-sJ_2)N_4 - N_4(I-sJ_1) = 0$$

as may be seen from the following example

Example (6.2.2)

Consider two identical, undriven systems described by the state equations

$$\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \end{bmatrix} = \begin{bmatrix} -\eta_2(0-) \\ 0 \end{bmatrix}$$

Solutions and initial conditions in both systems then take the form $\{(-\eta_2(0-), 0)\}$ and an appropriate map N_4 is

$$N_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$N_4 J_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$J_2 N_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

that is

$$J_2 N_4 \neq N_4 J_1 \quad \text{although} \quad (J_2 N_4 - N_4 J_1) J_1 = 0$$

The problem of non-commutability highlighted above may always be resolved by the construction of new maps defining the fundamental equivalence which, in addition, possess the required commutability property. The following result gives the details.

Lemma (6.2.3)

If two generalised state space systems in Kronecker standard form are fundamentally equivalent there exists a constant map N'_4 such that

$$\begin{bmatrix} \eta_2(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} N'_4 & Y_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ -u(s) \end{bmatrix} \quad \text{is a bijection}$$

and

$$\begin{bmatrix} -J_2 \eta_2(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} N'_4 & 0 \\ X_2 & I \end{bmatrix} \begin{bmatrix} -J_1 \eta_1(0-) \\ -y(s) \end{bmatrix} \quad \text{is a bijection}$$

and

$$(I-sJ_2)N_4' - N_4'(I-sJ_1) = 0 \quad (6.2.55)$$

Proof

Equating powers of s in (6.2.54)

$$N_4J_1^2 = J_2N_4J_1 \quad (6.2.56)$$

Also rewriting (6.2.9) for the impulsive subsystems

$$J_2\eta_2(0-) = (I-sJ_2)N_4(I+sJ_1+o(J_1^2))J_1\eta_1(0-) \quad (6.2.57)$$

and hence equating constants in (6.2.57) one obtains

$$J_2\eta_2(0-) = N_4J_1\eta_1(0-) \quad (6.2.58)$$

Substituting the unit matrix, I , for $\eta_1(0-)$ in (6.2.58)

$$J_2Q = N_4J_1 \quad Q \text{ constant} \quad (6.2.59)$$

Consider now the matrix defined as

$$Z = (Q-N_4) - (Q-N_4)J_1J_1^t \quad (6.2.60)$$

where J_1^t (the transpose of J_1) is a generalised inverse of J_1 so that $J_1J_1^tJ_1 = J_1$

$$\begin{aligned} ZJ_1 &= (Q-N_4)J_1 - (Q-N_4)J_1J_1^tJ_1 \\ &= (Q-N_4)J_1 - (Q-N_4)J_1 \\ &= 0 \end{aligned} \quad (6.2.61)$$

Also

$$J_2 Z = J_2(Q-N_4) - J_2(Q-N_4)J_1 J_1^t$$

Substituting from (6.2.59) gives

$$J_2 Z = N_4 J_1 - J_2 N_4 - (N_4 J_1 - J_2 N_4) J_1 J_1^t$$

and substituting from (6.2.56)

$$J_2 Z = N_4 J_1 - J_2 N_4 \tag{6.2.62}$$

Define

$$N'_4 = N_4 + Z \tag{6.2.63}$$

(6.2.61) ensures that N_4 in (6.2.48) and (6.2.49) can be replaced by N'_4 , since the image of $J_1 \eta_1(0-)$ is the same for both maps. Also

$$(I - sJ_2)N'_4 - N'_4(I - sJ_1) = s(N_4 J_1 - J_2 N_4 + Z J_1 - J_2 Z)$$

and substituting from (6.2.61), (6.2.62) gives (6.2.55) the required result.

The results of lemmas (3.5.1), (6.2.1) and (6.2.3) together with Theorem (6.2.2) may now be assembled to give the following theorem.

Theorem (6.2.3)

If two generalised state space systems P_1 and P_2 are fundamentally equivalent then there exists maps M' and N' such that:-

$$\begin{bmatrix} \chi_2(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} N' & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \chi_1(s) \\ -u(s) \end{bmatrix} \quad (6.2.64)$$

and

$$\begin{bmatrix} E_2 \chi_2(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} M' & 0 \\ X & I \end{bmatrix} \begin{bmatrix} E_1 \chi_1(0-) \\ -y(s) \end{bmatrix}, \quad XE_1=0 \quad (6.2.65)$$

are bijections and

$$(sE_2 - A_2)N' = M'(sE_1 - A_1) \quad (6.2.66)$$

Proof

The proof follows directly from the earlier results.

6.3 The relationship between fundamental equivalence and complete system equivalence

In order to complete the matrix characterisation of fundamental equivalence described in partial terms by Theorem (6.2.3) it is necessary to consider the effect of the maps (6.2.3) and (6.2.4) on the input matrix B, the output matrix C, and the feedforward term, D, in the impulsive subsystem in definition (6.2.2)

Lemma (6.3.1)

In the notation of Theorem (6.2.2) there exists a matrix relationship between the impulsive subsystems P_1 and P_2 of the form

$$\begin{bmatrix} N_4' & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ C_1 - C_2 N_4' & \dots & \dots & \dots & I \end{bmatrix} \begin{bmatrix} I - sJ_1 & B_1 \\ \dots & D_1 \end{bmatrix} = \begin{bmatrix} I - sJ_2 & B_2 \\ \dots & D_2 \end{bmatrix} \begin{bmatrix} N_4' & Y_2 \\ 0 & I \end{bmatrix} \quad (6.3.1)$$

Proof

Substituting the impulsive subsystems and the map N'_4 into (6.2.15)

$$(I-sJ_2)^{-1}B_2-N'_4(I-sJ_1)^{-1}B_1+Y_2 = 0$$

and using the result of Lemma (6.2.3)

$$B_2 = N'_4B_1-(I-sJ_2)Y_2$$

equating constants

$$B_2 = N'_4B_1-Y_2 \quad (6.3.2)$$

and equating coefficients of s

$$J_2Y_2 = 0 \quad (6.3.3)$$

Finally note that the restricted initial condition/output pair map of (6.2.49) ensures that any $J_1\eta_1(0-)$ and its image $J_2\eta_2(0-)=N'_4J_1\eta_1(0-)$ correspond to the same output $y(s)$ and this gives

$$\begin{aligned} & C_1(I-sJ_1)^{-1}J_1\eta_1(0-)+C_1(I-sJ_1)^{-1}B_1u(s)+D_1u(s) \\ & = C_2(I-sJ_2)^{-1}N'_4J_1\eta_1(0-)+C_2(I-sJ_2)^{-1}B_2u(s)+D_2u(s) \end{aligned} \quad (6.3.4)$$

Considering first the undriven case, replacing $\eta_1(0-)$ by the basis I and invoking the result of Lemma (6.2.3)

$$C_1J_1 = C_2N'_4J_1 \quad (6.3.5)$$

Define

$$X = C_1-C_2N'_4$$

then

$$C_1 = X + C_2 N_4' \quad (6.3.6)$$

and from (6.3.5)

$$XJ_1 = 0 \quad (6.3.7)$$

Since $u(s)$ is independent of $\eta_1(0^-)$ the matrix coefficient of $u(s)$ in (6.3.4) must be zero and hence

$$C_1(I - sJ_1)^{-1}B_1 + D_1 = C_2(I - sJ_2)^{-1}B_2 + D_2$$

substituting from (6.3.2) and (6.3.6) for B_2 , C_2 respectively this may be reduced, after some manipulation to

$$D_2 = XB_1 + C_2Y_2 + D_1 \quad (6.3.8)$$

Lemma (6.2.3) together with equations (6.3.2), (6.3.5) and (6.3.8) may be assembled to give (6.3.1).

Using the result above together with those of the previous section it now becomes possible to develop the main result of the chapter which unifies the ideas of fundamental equivalence and complete system equivalence for generalised state space systems.

Theorem (6.3.1)

Two generalised state space systems, P_1 , P_2 , are fundamentally equivalent if and only if they are completely system equivalent.

Proof

Assume that the systems are fundamentally equivalent then from Lemma (6.2.1) their standard forms are fundamentally

equivalent and using the results of Corollary 1 to Theorem (6.2.2) the two state space subsystems are system similar. Since system similarity is a special case of complete system equivalence it remains only to consider the impulsive subsystems. Lemma (6.3.1) shows that a matrix relationship of the required form exists, but for this to be complete system equivalence it is necessary to show that neither $[N'_4 : I-sJ_2]$ nor $[N'_4{}^t : (I-sJ_1)^t]{}^t$ has infinite zeros (the special form of $I-sJ_1$ does not allow the presence of finite zeros). Consider $[N'_4 : I-sJ_2]$ and replace s by $1/s$ to give a new matrix

$$G^\infty(1/s) = [N'_4 : I-1/sJ_2]$$

A matrix fraction description of this is

$$G^\infty(1/s) = [I-J_2J_2{}^t+sJ_2J_2{}^t]^{-1} [(I-J_2J_2{}^t+sJ_2J_2{}^t)N'_4 : I-J_2J_2{}^t+sJ_2J_2{}^t-J_2] \quad (6.3.9)$$

The (diagonal) denominator of $G^\infty(1/s)$ in (6.3.9) has full rank except at $s=0$. It will now be shown that the numerator, $N(s)$ has full rank at $s=0$ thus proving simultaneously that (6.3.9) is a relatively prime matrix fraction description and that $[N'_4 : I-sJ_2]$ has no infinite zeros.

$$N(0) = [(I-J_2J_2{}^t)N'_4 : I-J_2J_2{}^t-J_2] \quad (6.3.10)$$

subtracting N'_4 times (block) element (1,2) from (block) element (1,1) in (6.3.10) gives the new matrix

$$[J_2N'_4 : I-J_2J_2{}^t-J_2] \quad (6.3.11)$$

$J_2N'_4$ has a nonzero row corresponding to each nonzero row of J_2 , further since N'_4 is a bijection between restricted initial condition pairs $J_2N'_4 = N'_4J_1$ has the same rank as J_2 and so these rows are independent. $I-J_2J_2{}^t-J_2$ certainly has a set of

independent rows corresponding to the zero rows in J_2 (in such rows the only nonzero entry in $I - J_2 J_2^t - J_2$ is a 1 on the diagonal) and so (6.3.11) and hence $N(0)$ has full rank and $[N_4' : I - sJ_2]$ has no infinite zeros. A similar argument may be used to show that $[N_4' : (I - sJ_1)^t]^t$ has no infinite zeros and hence the impulsive subsystems are completely system equivalent and, since any system matrix is completely system equivalent to its Kronecker form (see Lemma (5.4.1)) it follows that systems which are fundamentally equivalent are also completely system equivalent.

Now suppose P_2, P_1 are completely system equivalent then the system matrices are related as in (5.4.2). Equating powers of s in (block) elements (1,1) of (5.4.2)

$$N_1 E_2 = E_1 N_2 \quad (6.3.12)$$

multiplying both sides of (6.3.12) by $\chi_2(0-)$ and noting that $N_2 \chi_2(0-)$ is certainly an element of $\chi_1(0-)$

$$E_1 \chi_1(0-) = N_1 E_2 \chi_2(0-) \quad (6.3.13)$$

Hence N_1 maps $\{E_2 \chi_2(0-)\}$ to members of $\{E_1 \chi_1(0-)\}$. Suppose that it is not a surjection then the rank of $N_1 E_2$ is less than that of E_1 and hence there exists a constant, nonzero vector γ such that

$$\gamma^t N_1 E_2 = 0 \quad (6.3.14)$$

but

$$\gamma^t E_1 = 0 \quad (6.3.15)$$

Considering (block) elements (1,1) in (5.4.2)

$$N_1(sE_2 - A_2) = (sE_1 - A_1)N_2 \quad (6.3.16)$$

(6.3.14), (6.3.15) and (6.3.16) may be manipulated to give

$$\begin{bmatrix} (sE_1 - A_1)^t \\ N_1^t \end{bmatrix} \alpha(s) = \begin{bmatrix} E_1^t \gamma \\ 0 \end{bmatrix} \quad (6.3.17)$$

where $\alpha(s) = ((sE_1 - A_1)^{-1})^t E_1^t \gamma = \alpha_p(s) + \alpha_{sp}(s)$

with $\alpha_p(s)$ polynomial and $\alpha_{sp}(s)$ strictly proper.

$$\begin{bmatrix} (sE_1 - A_1)^t \\ N_1^t \end{bmatrix} \alpha_p(s)$$

may be written as a constant vector plus a polynomial vector- that is it has no strictly proper part and hence from (6.3.17)

$$\begin{bmatrix} (sE_1 - A_1)^t \\ N_1^t \end{bmatrix} \alpha_{sp}(s) = \alpha_0 \quad \alpha_0 \text{ constant} \quad (6.3.18)$$

let $(s-s_0)$ be a factor of the common denominator of the elements of $\alpha_{sp}(s)$ then there exists a constant, k , such that

$$\psi(s) = (s-s_0)^k \alpha_{sp}(s)$$

is a rational vector and

$$\lim_{s \rightarrow s_0} \psi(s) \text{ exists and is non zero}$$

From (6.3.18)

$$\begin{bmatrix} (sE_1 - A_1)^t \\ N_1^t \end{bmatrix} \psi(s) = (s-s_0)^k \alpha_0 \quad (6.3.19)$$

If the right hand side of (6.3.19) is nonzero $[N_1 \ sE_1 - A_1]$ has a finite zero at $s=s_0$ ([43]). This is certainly not so since the systems were assumed to be completely system equivalent and hence $\alpha_0=0$, $\alpha_{sp}(s)=0$ and it follows that

$$\begin{bmatrix} (sE_1 - A_1)^t \\ N_1^t \end{bmatrix} \alpha(s) = \begin{bmatrix} E^t \gamma \\ 0 \end{bmatrix} \quad (6.3.20)$$

Let q be the highest power of s in $\alpha_p(s)$ and assume $q \neq 0$ then

$$\beta(s) = \alpha_p(s)/s^q \quad \text{is exactly proper} \quad (6.3.21)$$

and substituting into (6.3.20) from (6.3.21)

$$\begin{bmatrix} (sE_1 - A_1)^t \\ N_1^t \end{bmatrix} \beta(s) = \frac{1}{s^q} \begin{bmatrix} E^t \gamma \\ 0 \end{bmatrix} \quad (6.3.22)$$

If $E^t \gamma$ in (6.3.22) is nonzero then $[N_1; sE_1 - A_1]$ has an infinite zero ([38]). Hence, as the systems were assumed to be completely system equivalent, either $E^t \gamma = 0$ or $q=0$, that is $\alpha_p(s)$ is a constant vector, β_0 say, and (6.3.20) becomes

$$\begin{bmatrix} (sE_1 - A_1)^t \\ N_1^t \end{bmatrix} \beta_0 = \begin{bmatrix} E^t \gamma \\ 0 \end{bmatrix} \quad (6.3.23)$$

Let

$$1/\omega E_1 - A_1 = N_2(\omega) D^{-1}(\omega) \quad (6.3.24)$$

be a relatively prime factorisation then

$$[N_1; 1/\omega E_1 - A_1] = [N_1; N_2(\omega)] \begin{bmatrix} I & 0 \\ 0 & D(\omega) \end{bmatrix}^{-1} \quad (6.3.25)$$

is also a relatively prime factorisation and thus $[N_1; N_2(\omega)]$ is a numerator of $[N_1; 1/\omega E_1 - A_1]$. Consider

$$\beta_0^t [N_1; N_2(\omega)] = \beta_0^t [N_1; (1/\omega E_1 - A_1)D(\omega)] \quad (6.3.26)$$

substituting into (6.3.26) from (6.3.23)

$$\beta_0^t [N_1; N_1(\omega)] = [0; \gamma^t E_1 D(\omega)] \quad (6.3.27)$$

Rearranging (6.3.24)

$$E_1 D(\omega) = \omega A_1 D(\omega) + \omega N_2(\omega)$$

and thus

$$E_1 D(0) = 0 \quad (6.3.28)$$

Substituting from (6.3.27) into (6.3.28)

$$\beta_0^t [N_1; N_2(0)] = 0 \quad (6.3.29)$$

From (6.3.29) it can be seen that if β_0 is nonzero then $[N_1 \ N_2(\omega)]$ has less than full row rank at $\omega=0$ and thus $[N_1; sE_1 - A_1]$ has an infinite zero. This cannot be so since the systems were assumed to be completely system equivalent and so $\beta_0=0$ and $E_1^t \gamma$ is zero which contradicts the assumption that N_1 is not a surjection. It follows that N_1 is a surjection as required.

Premultiplying both sides of (6.3.16) by $(sE_1 - A_1)^{-1}$ and postmultiplying by $\chi_2(s)$

$$(sE_1 - A_1)^{-1} N_1 (sE_2 - A_2) \chi_2(s) = N_2 \chi_2(s) \quad (6.3.30)$$

Substituting from the system equation into (6.3.30)

$$(sE_1 - A_1)^{-1} N_1 E_2 \chi_2(0^-) + (sE_1 - A_1)^{-1} N_1 B_2 u(s) = N_2 \chi_2(s) \quad (6.3.31)$$

From consideration of (block) elements (1,2) in (5.4.2)

$$N_1 B_2 = (sE_1 - A_1)Y + B_1 \quad (6.3.32)$$

Substituting from (6.3.13) and (6.3.32) into (6.3.31) and invoking the system equation gives

$$\chi_1(s) = N_2 \chi_2(s) - Yu(s) \quad (6.3.33)$$

and thus

$$\begin{bmatrix} \chi_1(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \chi_2(s) \\ -u(s) \end{bmatrix} \quad (6.3.34)$$

is certainly a map between solution/input pairs. Assume (6.3.34) is not an injection. Then there exists some nonzero $\chi_2(s)$ such that

$$\begin{bmatrix} N_2 & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} \chi_2(s) \\ -u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that is $u(s)=0$ so

$$\begin{bmatrix} (sE_2 - A_2) \\ -N_2 \end{bmatrix} \chi_2(s) = \begin{bmatrix} E_2 \chi_2(0^-) \\ 0 \end{bmatrix}$$

Write $\chi_2(s) = \alpha_{sp}(s) + \alpha_p(s)$ the proof then follows that given above for the map N_1 and thus (6.3.34) is an injection between solution/input pairs.

It remains only to complete the initial condition/output pair map and this may be done by considering the output from P_1

$$Y_1(s) = C_1 \chi_1(s) + D_1 u(s) \quad (6.3.35)$$

substituting for $\chi_1(s)$ from (6.3.34)

$$y_1(s) = C_1 N_2 \chi_2(s) - C_1 Y u(s) + D_1(s)$$

substituting for $C_1 N_2$ and D_1 from (5.4.2)

$$\begin{aligned} y_1(s) &= -X(sE_2 - A_2)\chi_2(s) + X B_2 u(s) + C_2 \chi_2(s) + D_2 u(s) \\ &= y_2(s) - X E_2 \chi_2(0-) \end{aligned}$$

Thus

$$\begin{bmatrix} E_1 \chi_1(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ X & I \end{bmatrix} \begin{bmatrix} E_2 \chi_2(0-) \\ -y(s) \end{bmatrix}, \quad X E_2 = 0 \quad (6.3.36)$$

is a surjection and (6.3.34) and (6.3.36) together show that if P_1, P_2 are complete system equivalent then they are fundamentally equivalent.

The above theorem thus unifies the various ideas contained in several definitions of equivalence of generalised state space systems. A point which is discussed below in more detail.

6.4 Conclusion

A new type of equivalence - fundamental equivalence - between two systems in the generalised state space form has been proposed and defined in terms of mappings of the solution sets of the describing differential equations together with mappings of the restricted initial conditions. This definition is intuitively attractive since it is expressed in terms of solution/input pairs and restricted initial condition/output pairs. This implicitly guarantees the invariance of the observability and controllability characteristics as well as the input/output behaviour. The definition of fundamental equivalence in terms of the two

maps (6.2.3) and (6.2.4) also appears natural from another point of view. The solution/input pairs and the restricted initial condition pairs are, as one would expect, duals of each other. This fact emerges clearly and is exploited in the main result (Theorem (6.2.1)). It is noted that Verghese et al., [59] have applied the ideas of Pernebo, [35] to the generalised state space setting and have proposed a mapping interpretation of strong equivalence. However, this definition considers only a single mapping (one between the solution/input pairs) and so does not admit the duality ideas referred to above. There are also subsidiary conditions required of this latter definition which in the approach adopted here appear as consequences of the rather natural Definition (6.2.1).

In summary then, the definition of equivalence of generalised state space systems proposed here gives a precise mapping interpretation of the matrix transformation known as complete system equivalence. It has already been shown in Chapter 5 that the elementary operations which generate this transformation are those specified by Verghese et al. [59] in the definition of strong equivalence. Thus the results of this chapter taken together with those in Chapter 5 and in [59] give a full description of equivalence from a generalised state space view.

CHAPTER SEVEN

FINITE AND INFINITE FREQUENCY CONSIDERATIONS FOR GENERAL POLYNOMIAL MATRICES

7.1 Introduction

Interest in pole and zero structure at infinity leads to an interest in transformations which preserve such structure and a number of definitions of system equivalence for generalised state space systems have been proposed in earlier chapters of this thesis and elsewhere Rosenbrock [45], Verghese [55], Verghese et al [59], Anderson [1]). Extension of these ideas to more general polynomial matrices has proved difficult although a number of partial results have been proposed. One example of these is "full equivalence" (Ratcliffe [42]) which preserves both the finite and infinite poles and zeros of a polynomial system matrix. However the underlying matrix transformation includes a constraint on the MacMillan degree of the matrices concerned which has no obvious significance. A second example is the algorithm due to Bosgra and Van der Weiden [7] which allows a polynomial system matrix to be reduced to generalised state space form while preserving both the finite and infinite dynamic structure. No closed form description of the transformations involved in this reduction has been proposed, however. A rather different approach is offered by Anderson et al [1] who give two separate transformations, one preserving only finite dynamic structure the second only infinite dynamic structure.

This chapter attempts to show the links between a number of these results. A new characterisation of infinite zeros

in terms of a natural extension of the idea of infinite elementary divisors of a matrix pencil to matrix polynomials is presented. This characterisation, which makes transparent the equivalence of the definitions of Pugh and Ratcliffe [39] and Bosgra and Van der Weiden [7] is used to examine the physical significance of the MacMillan degree constraint in full equivalence. Many of the results in this chapter can be found in Hayton, Pugh and Fretwell [64]. [65].

7.2 Infinite elementary divisors and Infinite zeros

In classical matrix theory [19] the infinite elementary divisors of the matrix pencil, $sE-A$, defined as the the finite elementary divisors of the dual pencil $E-sA$ of the form s^q , are invariants of the pencil under transformations of unimodular equivalence and are intimately related to the infinite zeros of the pencil with infinite elementary divisors of multiplicity $q > 1$ corresponding to infinite zeros of multiplicity $q-1$. The concept of infinite elementary divisors may be extended from matrix pencils to more general matrix polynomials as follows

Definition (7.2.1)

Consider an $m \times 1$, $m < 1$ matrix polynomial $P(s)$ where

$$P(s) = P_0 + P_1s + \dots + P_n s^n \quad (7.2.1)$$

where

$$P_i, i=0,1,2,\dots,n \text{ constant matrices and } P_n \neq 0$$

Then the dual polynomial matrix $\hat{P}(\hat{s})$ is defined by

$$\hat{P}(\hat{s}) = P_0 \hat{s}^n + P_1 \hat{s}^{n-1} + \dots + P_n \quad (7.2.2)$$

If $\hat{P}(\hat{s})$ has a finite elementary divisor of the form \hat{s}^q then $P(s)$ is said to have an INFINITE ELEMENTARY DIVISOR of multiplicity q .

The finite elementary divisors of $P(s)$ give the finite zeros of the matrix, but the relationship between the infinite elementary divisors and the infinite frequency structure is more complex as is demonstrated by the following

Theorem (7.2.1)

Let $P(s)$ be a matrix polynomial of n 'th degree with dimensions $m \times 1$, $m < n$ as in (7.2.1) then

1. Infinite elementary divisors of $P(s)$ of multiplicity $q < n$ correspond to infinite poles of $P(s)$ of multiplicity $n - q$.
2. Infinite elementary divisors of $P(s)$ of multiplicity $q > n$ correspond to infinite zeros of $P(s)$ of multiplicity $q - n$.
3. Infinite elementary divisors of multiplicity $q = n$ have no dynamic significance.

Proof

Form $\hat{P}(\hat{s})$ the dual of $P(s)$ as in (7.2.2) and bring it to Smith form so that

$$\hat{P}(\hat{s}) = R(\hat{s}) [\text{Diag}(\hat{s}^p E_1(\hat{s}), \hat{s}^p E_2(\hat{s}), \dots, \hat{s}^p E_m(\hat{s}) \ 0)] L(\hat{s}) \quad (7.2.3)$$

where $R(\hat{s})$, $L(\hat{s})$ are polynomial matrices unimodular in \hat{s} , and $E_i(\hat{s})$, $i=1,2,\dots,m$ are polynomial matrices whose elements contain no factors of the form \hat{s}^q , $q > 0$, $E_i(\hat{s}) \neq 0$ $i > r$ where r

is the normal rank of $P(s)$. Recall that the infinite pole and zero structure of $P(s)$ is the finite pole and zero structure of $P(1/s)$ at $s=0$. Now

$$P(1/\hat{s}) = \frac{1}{\hat{s}^n} P(\hat{s}) \quad (7.2.4)$$

and hence using (7.2.2) the Smith MacMillan form of $P(1/s)$ is

$$P_{SM}(1/\hat{s}) = [\text{Diag}(\hat{s}^{p_1-n} E_1(\hat{s}), \hat{s}^{p_2-n} E_2(\hat{s}), \dots, \hat{s}^{p_m-n} E_m(\hat{s}); 0] \quad (7.2.5)$$

Denominator zeros of $P_{SM}(1/\hat{s})$ correspond to poles of $P(1/\hat{s})$, numerator zeros of $P_{SM}(1/\hat{s})$ correspond to zeros of $P(1/\hat{s})$ and hence the required result follows immediately on inspection of (7.2.5).

Notice that infinite poles of multiplicity n do not correspond to infinite elementary divisors, but to invariant polynomials of $P(s)$ which have no factor of the form s^q , $q > 0$.

The result concerning the relationship between the infinite elementary divisors and the infinite zeros of a pencil is seen to be a special case of Theorem (7.2.1) with $n=1$. Two related results of interest concern the multiplicity of the infinite poles.

Corollary 1

The maximum multiplicity of an infinite pole of a n 'th degree matrix polynomial $P(s)$ is n .

Proof

This follows on noting that $p_1 \geq 0$.

The degree of a matrix pencil is unity and hence

Corollary 2

The maximum multiplicity of an infinite pole of a matrix pencil is unity.

An alternative characterisation of infinite zero structure in terms of the degrees of certain minors of the matrix was proposed by Bosgra and Van der Weiden [7]. The results above allow a connection to be made between this and the more generally used characterisation of Pugh and Ratcliffe [39]. The following definition is essentially that of Bosgra and Van der Weiden but has been extended to include the infinite poles.

Definition (7.2.2)

Let $P(s)$ be an $m \times l$, $m < l$ polynomial matrix with rank r and δ_i the highest degree occurring among minors of order i of $P(s)$. Let δ be the largest of the δ_i and let k_1 be the smallest and k_2 be the largest order among minors with degree δ . Then $P(s)$ is said to have $r - k_2$ INFINITE ZEROS with multiplicities $\delta - \delta_{k_2+1}, \delta_{k_2+1} - \delta_{k_2+2}, \dots, \delta_{r-1} - \delta_r$ respectively and k_1 INFINITE POLES of multiplicities $\delta - \delta_{k_1-1}, \delta_{k_1-1} - \delta_{k_1-2}, \dots, \delta_2 - \delta_1, \delta_1 - 0$.

Theorem (7.2.2)

Let $P(s)$ be a $m \times l$, $m < l$, matrix polynomial of degree n and let $\delta, \delta_i, i=1,2,\dots, r, k_1, k_2$ be defined as in Definition (7.2.2). Then each infinite zero of multiplicity $z_i, i=1,2,\dots, r - k_2$ corresponds to an i.e.d. of degree $n + z_i$ and each infinite pole of multiplicity $q_k, k=1,2,\dots, k_1$ corresponds to an i.e.d. of degree $n - q_k$.

Proof

Since δ_i is the highest degree occurring among minors of $P(s)$ of order i it is clear that the greatest common divisor

of the i 'th order minor of $P(s)$ has the form

$$\hat{s}^{n_i - \delta_i} \prod_j (\hat{s} - \alpha_j), \quad \alpha_j \neq 0, \text{ for all } j$$

and hence the i 'th infinite elementary divisor has degree

$$n_i - \delta_i - (n_{i-1} - \delta_{i-1}) = n - \delta_i - \delta_{i-1} \quad (7.2.6)$$

and the result follows immediately

Theorems (7.2.1) and (7.2.2) taken together show that the matrix fraction description approach of Pugh to infinite frequency behaviour and the matrix minor approach of Definition (7.2.2.) are equivalent.

Theorem (7.2.3)

The definition of infinite zero in Definition (7.2.1) and Definition (3.3.11) coincides.

Proof

The concept of infinite elementary divisor introduced in Definition (7.2.1) is the common link between the two approaches to defining infinite frequency behaviour. Theorem (7.2.1) shows the relationship between the infinite zeros and poles of a polynomial matrix to certain infinite elementary divisors of the matrix. Theorem (7.2.2) shows the relationship between the same infinite elementary divisors and the newly defined infinite zeros and poles. Thus the two concepts correspond to the same infinite elementary divisors and hence are equivalent.

To summarise it has been shown that the method of determining infinite zeros and poles shown in Definition (7.2.2) produces exactly the same results as using the matrix fraction description. Therefore, the two definitions are equivalent and the Theorem is proved.

7.3 Finite and infinite frequency preserving

transformations

The above results may be used to investigate appropriate system matrix transformations. As a first step, transformations of a general matrix polynomial are considered. The following is due to Ratcliffe [42]

Definition (7.3.1)

Let $P_1(s)$ and $P_2(s)$ be two polynomial matrices with dimensions $(m+r_1) \times (l+r_1)$ and $(m+r_2) \times (l+r_2)$ (i.e. they have the same row and column difference) are said to be **FULLY EQUIVALENT** (f.l.l.e.) if they are related by two polynomial matrices $M(s)$ and $N(s)$ of appropriate dimensions such that

$$\begin{bmatrix} M(s) & P_2(s) \end{bmatrix} \begin{bmatrix} P_1(s) \\ \dots \\ -N(s) \end{bmatrix} = 0 \quad (7.3.1)$$

and $\begin{bmatrix} P_1(s) \\ \dots \\ N(s) \end{bmatrix}$ (respectively $\begin{bmatrix} M(s) & P_2(s) \end{bmatrix}$) has full rank and no

finite or infinite zeros and MacMillan degree equal to the MacMillan degree of $P_1(s)$ (respectively $P_2(s)$).

Definition (7.3.1) appears a natural extension of the transformation of complete equivalence proposed for matrix pencils except for the restriction on the MacMillan degrees. The necessity of the restriction is demonstrated by

Example (7.3.1)

Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s-1 & s^2+1 \\ s & s^2 \end{bmatrix} = \begin{bmatrix} s-1 & s+1 \\ s & 0 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \quad (7.3.2)$$

$$D_1(s) \quad N_2(s) \quad N_1(s) \quad D_2(s)$$

The pairs $(D_1(s):N_1(s))$, $\begin{bmatrix} D_2(s) \\ \dots \\ -N_2(s) \end{bmatrix}$ have neither finite or infinite zeros.

This can be seen by considering firstly,

$$\begin{bmatrix} D_1(s):N_1(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & : & s-1 & s+1 \\ 0 & 1 & : & s & 0 \end{bmatrix}$$

This matrix has full rank because $D_1(s)$ is a constant nonsingular identity matrix. This means that $D_1(s) N_1(s)$ has full rank for all finite s . Additionally, the $N_1(s)$ is a minor of largest degree and maximum size. Therefore, Theorem (3.3.2) applies and $D_1(s) N_1(s)$ has no infinite zeros. Similarly consider

$$\begin{bmatrix} D_2(s) \\ \dots \\ -N_2(s) \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \\ \dots & \dots \\ 1-s & -s^2-1 \\ s & 0 \end{bmatrix}$$

$D_2(s)$ is a unimodular full sized minor and therefore the above has full rank for all finite s . $N_2(s)$ is a full sized minor with degree equal to that of the MacMillan degree. Therefore, Theorem (3.3.2) applies and $(D_2(s)^t:N_2(s)^t)^t$ has no infinite zeros.

$$\delta(N_1(s)) = \delta(D_1(s):N_1(s)) = 2 \quad \& \quad \delta(D_1(s)) = 0$$

$$\delta(N_2(s)) = \delta \begin{bmatrix} D_2(s) \\ \dots \\ -N_2(s) \end{bmatrix} = 2 \quad \& \quad \delta(D_2(s)) = 1$$

Thus if $N_i(s)$ are regarded as the transforming matrices the MacMillan degree condition for fll.e. are not satisfied and $D_1(s)$ and $D_2(s)$ do not have the same infinite zero structure. However, if $D_i(s)$ are regarded as the transforming matrices all conditions for fll.e. are satisfied and $N_1(s)$ and $N_2(s)$ have the same finite and infinite zero structure.

Full equivalence preserves finite and infinite zeros. In order to prove this an associated lemma will be proved. This is due to Hayton but is included for completeness

Lemma (7.3.1)

Let $A(s)$ and $B(s)$ be two polynomial matrices with the following properties

- (i) $[A(s) \ B(s)]$ has degree n .
- (ii) $[A(s) \ B(s)]$ has no finite or infinite zeros.
- (iii) $\delta([A(s) \ B(s)]) = \delta(A(s))$

Let $\widehat{[A(s) \ B(s)]} = [A'(\hat{s}) \ B'(\hat{s})]$ be the dual of $[A(s) \ B(s)]$.
Form

$$[A'(\hat{s}) \ B'(\hat{s})] = Q(\hat{s})[A''(\hat{s}) \ B''(\hat{s})]$$

where $Q(\hat{s})$ is a greatest common left divisor of $A'(\hat{s})$ and $B'(\hat{s})$. (i.e. $A''(\hat{s})$ and $B''(\hat{s})$ are relatively left prime.)

Then the zeros at the origin of the matrix $B''(\hat{s})$ have the exactly the same multiplicities as the infinite zeros of $B(s)$.

Proof

Let $B'(\hat{s})$ have invariant polynomials $\hat{s}^{D_i} E_i(s)$ and let

$B'(s)$ have invariant polynomials $s^{z_i} E_i(s)$. Since $[A(s) B(s)]$ has no finite or infinite zeros, the zeros of $[A'(s) B'(s)]$ occur only at the origin and have multiplicities $\leq n$. Hence the invariant polynomials of $[A'(s) B'(s)]$ are of the form s^{q_i} where $0 \leq q_i \leq n$ (from Theorem (7.2.2)). A $Q(s)$ may be constructed as follows. Transform $[A'(s) B'(s)]$ to its Smith form $[\text{Diag}(s^{q_i}) : 0]$ with two unimodular matrices $L(s)$ and $R(s)$. This can be represented as follows with $R(s)$ partitioned appropriately.

$$\begin{aligned}
 [A'(s) : B'(s)] &= L(s) [\text{Diag}(s^{q_i}) : 0] \begin{bmatrix} R_1(s) & R_2(s) \\ R_3(s) & R_4(s) \end{bmatrix} \\
 &= L(s) \text{Diag}(s^{q_i}) [R_1(s) \ R_2(s)]
 \end{aligned}$$

define $Q(s) = L(s) \text{Diag}(s^{q_i})$. Because $R(s)$ is unimodular all rows in $R(s)$ are independent for finite s . Therefore the matrix $[R_1(s) \ R_2(s)]$ is full rank for all finite s . Therefore, simply adapting Lemma (3.3.3) part (i) to the definition of left primeness gives $R_1(s)$ and $R_2(s)$ are relatively leftprime.

Clearly the Smith form of $Q(s)$ is $\text{Diag}(s^{q_i})$. Let k_1 be the order of the smallest minor of $B(s)$ having degree δ , the MacMillan degree of $B(s)$ and k' be the order of the smallest minor of $[A(s) B(s)]$ having degree δ' , the MacMillan degree of $[A(s) B(s)]$. Since

$$\delta' = \delta$$

then

$$k' = k_1$$

From Definition (7.2.2) $[A(s) : B(s)]$ has k' infinite poles and from Theorem (7.2.2) these poles have multiplicities $(n-q_1), (n-q_2), \dots, (n-q_k)$. Because the total number of poles of a polynomial matrix counted according to their degree and

multiplicity is equal to the MacMillan degree of the matrix

$$\sum_{i=1}^{k'} q_i = k'n - \delta$$

It also follows that because $[A(s) \ B(s)]$ has no infinite zeros that for $i=k'+1, k'+2, \dots, m$ (where $m=\min(\text{row dimension}, \text{column dimension})$)

$$q_i = n$$

Consider

$$B'(s) = L(\hat{S}) \text{Diag}(\hat{S}^{q_i}) R_2(\hat{S})$$

The Smith form of $B''(\hat{S})$ (i.e. $R_2(\hat{S})$) is unaffected by multiplication by unimodular $L(\hat{S})$. Therefore, consider the affect multiplication by $\text{Diag}(\hat{S}^{q_i})$ has on the Smith form of $B''(\hat{S})$. Row i of $B''(\hat{S})$ is multiplied by \hat{S}^{q_i} . Inspecting minors of order j shows that invariant polynomials of the form \hat{S}^k are related by the following

$$\sum_{i=1}^j p_i > \sum_{i=1}^j q_i + \sum_{i=1}^j z_i$$

Then letting $j=k_1$ and recalling $k' < k_1$

$$\sum_{i=1}^{k_1} p_i = k_1 n - \delta \geq k_1 n - \delta + \sum_{i=1}^{k_1} z_i$$

This implies that $z_i=0$ for $i=1, 2, \dots, k_1$. Also note the following

$$\begin{aligned}
 p_i &\leq \max(q_i) + z_i \\
 &\leq n + z_i
 \end{aligned}$$

But each p_i for $i > k_1$ must differ from z_i by at least n to keep determinantal totals correct. So

$$p_i \geq n + z_i$$

Hence

$$z_i = p_i - n \text{ for } i > k_1$$

From Theorem (7.2.1) $p_i - n$ is the multiplicity of the i 'th infinite zero of $B(s)$, thus proving the result.

Lemma (7.3.1) can be used to show in a different manner to that of Ratcliffe [42] that full equivalence preserves finite and infinite zeros. This method of proof makes the MacMillan degree constraint clear.

Theorem (7.3.1)

If polynomial matrices $P_1(s)$, $P_2(s)$ are fully equivalent they have the same finite and infinite zeros.

Proof

Fl1.eq. is a special case of e.u.e. which preserves finite zeros. For infinite zeros, since $P_1(s)$, $P_2(s)$ are fl1.eq. there exists two polynomial matrices M and N such that

$$\begin{bmatrix} M(s) & P_2(s) \end{bmatrix} \begin{bmatrix} P_1(s) \\ N(s) \end{bmatrix} = 0$$

with the conditions of Definition (7.3.1) holding. Form the

dual matrices

$$\widehat{[M(s) \vdots P_2(s)]} = [M'(\hat{s}) \vdots P_2'(\hat{s})]$$

and

$$\widehat{\begin{bmatrix} P_1(s) \\ \dots \\ N(s) \end{bmatrix}} = \begin{bmatrix} P_1'(s) \\ \dots \\ N'(s) \end{bmatrix}$$

and consider

$$[M'(\hat{s}) \ P_2'(\hat{s})] = Q_2(\hat{s}) [M''(\hat{s}) \vdots P_2''(\hat{s})]$$

and

$$\begin{bmatrix} P_1'(\hat{s}) \\ N'(\hat{s}) \end{bmatrix} = \begin{bmatrix} P_1''(\hat{s}) \\ N''(\hat{s}) \end{bmatrix} Q_1(\hat{s})$$

where $M''(s), P_2''(s)$ are relatively left prime and $P_1''(s), N''(s)$ are relatively right prime. $Q_1(s)$ and $Q_2(s)$ are both polynomial matrices. Then

$$Q_2(\hat{s}) [M''(\hat{s}) \ P_2''(\hat{s})] \begin{bmatrix} P_1''(\hat{s}) \\ N''(\hat{s}) \end{bmatrix} Q_1(\hat{s}) = 0$$

Premultiplying by inverse of $Q_2(\hat{s})$ and postmultiplying by inverse of $Q_1(\hat{s})$ gives

$$[M''(\hat{s}) \ P_2''(\hat{s})] \begin{bmatrix} P_1''(\hat{s}) \\ N''(\hat{s}) \end{bmatrix} = 0$$

that is $P_2''(\hat{s})$ and $P_1''(\hat{s})$ are extended unimodular equivalent and hence have the same finite zeros which from Lemma (7.3.1) are respectively the infinite zeros of $P_2(s)$ and $P_1(s)$. Thus the theorem is proved.

This transformation should extend to the system case but the exact form of the transformation is at present not clear.

7.4 Conclusion

A new characterisation of infinite zeros has been given. This is in terms of a natural extension of the idea of infinite elementary divisors of matrix pencil to more general matrix polynomials. The characterisation makes transparent the equivalence of the definitions of Pugh and Ratcliffe [39] and Bosgra and Van der Weiden [7]. The physical significance of the MacMillan degree constraint in full equivalence has been examined in the light of the new representation of infinite zeros.

CHAPTER EIGHT

CONCLUSIONS AND FUTURE WORK

Progress has been advanced during the course of this thesis in several areas of linear system transformation theory. In Chapter 4 inadequacies inherent in Rosenbrock's Restricted System Equivalence (r.s.e.) were noted and proposals for improvements made. The main difficulty was found to be in the fact that r.s.e. only allows systems having the same dimension to be related. Because r.s.e. is based on the pencil transformation of Strict Equivalence an attempt was made to generalise s.e. to pencils of different dimensions. However, the proposed transformation did not keep infinite frequency structure invariant. This was due to the lack of constraints acting at the point at infinity. Such an additional condition was proposed in Chapter 5 where the matrix pencil operation of Complete Equivalence (c.e.) was defined. C.e. was found to preserve finite and infinite zero structure while allowing differently dimensioned pencils to be related. In addition to the enlarged equivalence classes which c.e. possesses, c.e. differs from s.e. in that it does not maintain invariant the infinite elementary divisors of degree one. In the field of linear systems, Verghese [55] has shown that for g-s-s systems the i.e.d. of degree one have "no dynamical significance" and can be thought of in terms of differential equations as null equations. This suggests c.e. may be a useful transformation for linear systems. By applying c.e. to g-s-s systems a system transformation termed Complete System Equivalence (c.s.e.) was obtained. This transformation performed all the operations inherent in r.s.e., but was found to have a larger set of equivalence classes due to its ability to relate systems that may not

have the same dimension.

Two seemingly very different definitions of equivalence were unified in Chapter 5. One was an algorithmic equivalence relation termed Strong Equivalence (str.eq.) which has been developed by Verghese[55]. This catalogues the elementary operations necessary to establish equivalence and is based on r.s.e. and trivial inflation and deflation of system matrices. The other was c.s.e. It was demonstrated that c.s.e. is an elegant matrix characterisation of Str.eq.

Chapter 6 took a different approach in the search for equivalence transformations of g-s-s systems. In a similar manner to Pernebo [35] and to a certain extent Hinrichsen and Pratzel-Wolters[21] an equivalence relation termed Fundamental Equivalence (f.e.) was proposed. Although it was noted that Verghese had established a connection between str.eq. and certain bijections of the associated system solutions f.e. develops the idea to a far greater extent and includes many features not considered in Verghese[55], for example non uniqueness of the maps is investigated in Chapter 6. C.s.e was shown to be a matrix characterisation of f.e. and this provided a new approach to the investigation of g-s-s systems and thereby enables a greater degree of understanding of such systems

A new characterisation of infinite poles and infinite zeros in terms of a natural extension of the idea of infinite elementary divisors of a matrix pencil to matrix polynomials has been proposed in Chapter 7. This has been used to show that two earlier separate definitions of infinite frequency structure are, in fact, equivalent.

Future work might include an attempt to generalise Fuhrmann's important result that two systems are Fuhrmann equivalent if and only if their associated state-space forms are system similar to the infinite frequency case. Such a generalisation could be proved by showing that two systems are fl1.e. if and only if their associated g-s-s forms are

c.s.e. This would accurately mirror the work of equivalence of linear systems at finite frequency and would therefore demonstrate that fll.e. and c.s.e. have an important role to play in the infinite frequency behaviour of linear systems.

Important insights about c.s.e. and fll.e. can possibly be obtained by investigating the associated algebraic structure of the systems in a similar way to Fuhrmann's method of developing his system equivalence by using module theory.

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