Growth and Integrability in Multi-Valued Dynamics

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A Doctoral Thesis

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Certificate of Originality

I certify that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgements or footnotes, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a degree.

Signed: ....

Date: ....

Abstract

This thesis is focused on the problem of growth and integrability in multi-valued dynamics generated by $SL_2(\mathbb{Z})$ actions. An important example is given by Markov dynamics on the cubic surface

$$x^2 + y^2 + z^2 = 3xyz,$$

generating all the integer solutions of this celebrated Diophantine equation, known as Markov triples.

To study the growth problem of Markov numbers we use the binary tree representation. This allows us to define the Lyapunov exponents $\Lambda(x)$ as the function of the paths on this tree, labelled by $x \in \mathbb{R}P^1$. We prove that $\Lambda(x)$ is a $PGL_2(\mathbb{Z})$-invariant function, which is zero almost everywhere but takes all values in $[0, \ln \varphi]$ (where φ denotes the golden ratio). We also show that this function is monotonic, and that its restriction to the Markov-Hurwitz set of most irrational numbers is convex in the Farey parametrisation.

We also study the growth problem for integer binary quadratic forms using Conway's topograph representation. It is proven that the corresponding Lyapunov exponent $\Lambda_Q(x) = 2\Lambda(x)$ except for the paths along the Conway river.

Finally, we study the tropical version of the Markov dynamics on the tropical version of the Cayley cubic proposed by Adler and Veselov, and show that it is semi-conjugated to the standard action of $SL_2(\mathbb{Z})$ on a torus. This implies the dynamics is ergodic, with the Lyapunov exponent and entropy given by the logarithm of the spectral radius of the corresponding matrix.

Keywords: *Lyapunov exponents, modular group, continued fractions, Markov numbers, binary forms*

Acknowledgements

One of the first things we learn in mathematics is that results often depend on a combination of sufficient and necessary conditions.

With that in mind, I'd like to thank my supervisor, Prof. A. P. Veselov, for his unending patience and enthusiasm. Without his support over the past three years, it goes without saying, this PhD would not have been possible.

Martyn, you've put up with a lot, and never given up on me. Without you, nothing would have been.

Contents

Chapter 1

Introduction

The study of the orbits and invariant properties of maps is one of the central problems in the theory of dynamical systems. The important particular case of rational maps of the complex projective line $\mathbb{C}P^1 = \bar{\mathbb{C}}$ goes back to Julia and Fatou and became the subject of substantial interest in the 1980s (see, for example, the review by Blanchard [8]). In addition, the seminal works by Sullivan, Thurston, Douady and Hubbard uncovered deep connections between rational complex dynamics and the theory of Kleinian groups and Teichmüller spaces.

The notion of integrability for such maps is not obvious and was first discussed by Veselov [78, 79], who proposed an approach to the problem based on the existence of a commuting map. This was motivated by the remarkable works of Julia, Fatou and Ritt in the 1920s, who classified the commuting pairs of polynomial and rational maps [79]. In particular, the map $f : z \to z^2$, $z \in \bar{\mathbb{C}}$ is integrable in this sense since it commutes with $g : z \to z^3$, although it has the Julia set $|z| = 1$, $z = e^{i\varphi}$ with chaotic dynamics $\varphi \to 2\varphi$ on it and positive Lyapunov exponent $\lambda = \ln 2$.

Another approach to integrability, partly motivated by the work of Arnold [4] and Moser [60], is based on growth properties: the numerical characteristics of integrable maps should have polynomial growth, while generically one has exponential growth (see Veselov [80]). This worked well for two-dimensional complex dynamics defined by invertible polynomial maps (forming the so-called affine Cremona group) where the polynomial growth of the degrees of the iteration indeed implies the integrability [80]. Important further steps in this direction came in particular from Bellon and Viallet, who studied the algebraic entropy of the rational 2D Cremona maps [7], and by Halburd, who proposed the Diophantine version of this approach [38].

However, the example of commuting multi-valued maps given by the classical modular correspondences showed that integrability does not necessarily exclude exponential growth, and raises the question of how exceptional such cases are (see discussion in [79]).

This problem is probably out of reach in full generality, so in [82] it was proposed to study in more detail a particular case of multi-valued dynamics given by the action of the group $SL_2(\mathbb{Z})$ (or its close relative, the modular group $PSL_2(\mathbb{Z})$). The motivation came from important work by Dubrovin [28], Fock [31], Cantat et al [13, 14], Iwasaki et al [43, 44], where examples of such dynamics appeared in relation with quantum cohomology, Teichmüller spaces and Painlevé equations.

The main aim of the thesis is to study growth in $SL_2(\mathbb{Z})$-dynamics in more detail using ideas coming from geometry and the theory of integrable systems. We will make particular use of the geometric representation of $SL_2(\mathbb{Z})$-dynamics using the binary tree embedded in the plane, which Conway [23] called the *topograph* and used to represent the values of binary quadratic forms.

The thesis consists of three parts.

The first studies the growth of a particular case of $SL_2(\mathbb{Z})$-dynamics coming from the theory of the classical Markov equation.

The *Markov equation* is the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz,$$

originally described by Markov in his 1880 thesis on quadratic forms [55]. There are an infinite number of solution triples, known as *Markov triples*, connected via involutions and permutations [12]. The famous Unicity Conjecture of Frobenius [33] states that the Markov triple is uniquely defined by its largest member.

Frobenius [33] and Zagier [84], and more recently Bourgain, Gamburd and Sarnak [11] have studied various aspects of the arithmetic and behaviour of the Markov numbers m_n. However, it is more natural to study the growth of Markov numbers using the tree representation. It has been traditional to consider the Markov triples as vertices in a binary tree [2, 9, 12, 15, 25, 84], but we will be using the binary tree embedded in the upper half-plane as dual to the Farey tesselation (see Figure 1.1).

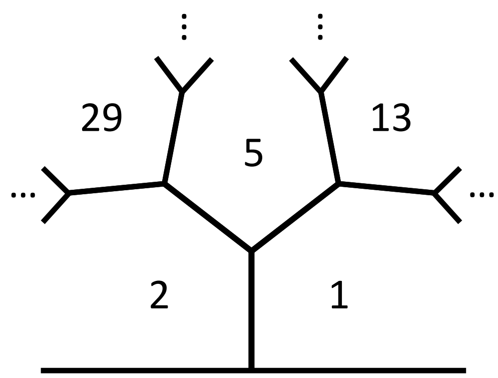


Figure 1.1: The embedded planar Markov tree.

The connection between the Markov equation and the Farey tessellation of the upper half-plane goes back to Frobenius [33], who introduced the Farey parametrisation of the Markov numbers. We can therefore measure growth in the tree along infinite paths γ_ξ, defined by the limit $\xi \in \mathbb{R}P^1$ of the corresponding Farey fractions.

Cohn [20] used the Fricke identities [32] to derive the Markov forms studied by Frobenius, and later [21] considered the *Markov matrices* via a binary tree representation. The resulting *Cohn tree* stands as a link between the combinatorial theory of words, in the tradition of Christoffel [18, 19] and Morse-Hedlund [59], and Markov dynamics [2]. An embedded planar Cohn tree is identified with the Markov tree by a simple trace map. One of the incidental results of this thesis is an apparently new connection, via the Minkowski question mark function, between Cohn's representation of the Markov numbers and the continued fraction representations of certain quadratic irrationals which arise in the context of Diophantine analysis.

To measure the growth along a path γ_ξ in this binary tree, we define the Lyapunov exponent $\Lambda(\xi)$ as

$$\Lambda(\xi) := \limsup_{n\to\infty} \frac{\ln \ln m_n(\xi)}{n}, \tag{1.1}$$

where $m_n(\xi)$ is the n-th Markov number along the path γ_ξ.

Equivalently, $\Lambda(\xi)$ can be defined as

$$\Lambda(\xi) := \limsup_{n\to\infty} \frac{\ln w_n(\xi)}{n}, \tag{1.2}$$

where $w_n(\xi)$ is the largest number in the n-th triple along the path γ_ξ in the Euclid tree described by the equation $a + b = c$ (see Figure 1.2). The equivalence is based on the observation going back to Mordell [58], and explicitly used by Zagier [84], that the modified Markov equation

$$X^2 + Y^2 + Z^2 = 3XYZ + \frac{4}{9} \tag{1.3}$$

after the change of variables

$$X = \frac{2}{3}\cosh a,\ Y = \frac{2}{3}\cosh b,\ Z = \frac{2}{3}\cosh c \tag{1.4}$$

reduces to the Euclid equation $a + b = c$ (and thus can be considered to be "integrable").

The set $Spec_\Lambda = \{\Lambda(x),\ x \in \mathbb{R}P^1\}$ of all possible values of $\Lambda(x)$ is called the *Lyapunov spectrum* of Markov and Euclid trees.

We have proved the following results about the Lyapunov spectrum [69]:

Theorem 1. *The Lyapunov exponent exists for all paths and can be naturally extended to the function $\Lambda(x)$, $x \in \mathbb{R}P^1$, which is $GL_2(\mathbb{Z})$-invariant and almost everywhere vanishing.*

Theorem 2. *The Lyapunov spectrum of Markov and Euclid trees is*

$$Spec_\Lambda = [0, \ln \varphi],$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

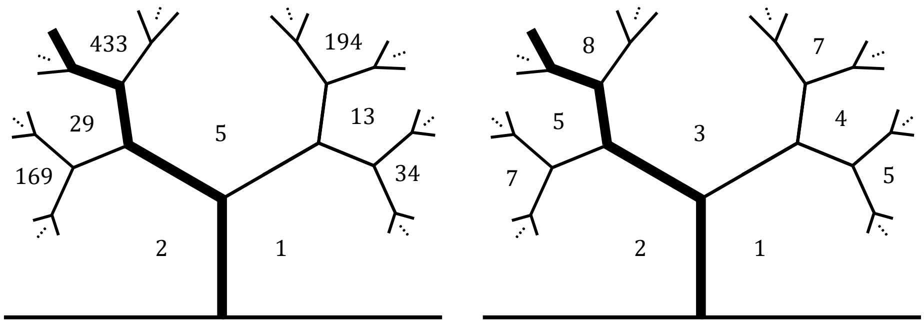


Figure 1.2: The Markov and Euclid trees with 'golden' path marked.

In particular, for all $x \in \mathbb{R}P^1$, $\Lambda(x) \leq \Lambda(\varphi) = \ln \varphi$, so the "golden path" has the maximal Lyapunov exponent.

The celebrated *Markov theorem* says that the Markov numbers are in one-to-one correspondence with the equivalence classes of the *most irrational numbers* (where irrationality is measured by the so-called *Markov constant*). Let $\mathbb{X}$ be the set of the Markov-Hurwitz numbers, which are certain representatives of these equivalence classes.

Theorem 3. *The restriction $\Lambda_{\mathbb{X}}$ of the Lyapunov function on the set $\mathbb{X}$ of Markov-Hurwitz numbers is monotonically increasing from*

$$\Lambda\left(x_2\right) = \frac{1}{2}\ln\left(1+\sqrt{2}\right)$$

to

$$\Lambda\left(x_1\right) = \ln\left(\frac{1+\sqrt{5}}{2}\right).$$

In the Farey parametrization, $\Lambda\left(x\left(\frac{p}{q}\right)\right)$ is convex as a function of $\frac{p}{q}$.

The proof makes use of the link between Markov numbers and the lengths of simple closed geodesics on a special once-punctured torus [35, 20] and a result of Fock [31].

We also have a generalisation of this result to a version of the Markov equation considered by Mordell [58]

$$X^2 + Y^2 + Z^2 = XYZ + 4 - 4a^6, \quad a \in \mathbb{N},$$

related to the one-hole hyperbolic tori [69].

The second part of the thesis is devoted to the growth of the values of binary quadratic forms.

Conway [23] used the notion of the topograph to represent the values of binary quadratic forms geometrically. Hatcher [39] developed this further by explicitly describing the connection between these topographs and the Farey tessellation and continued fractions, which is important for our work.

Theorem 4 (Conway). *Let $Q(x, y)$ be an indefinite binary quadratic form not representing zero. Then the topograph of Q contains a unique periodic path, called the 'river', separating all positive values of Q from all negative values.*

By considering the growth of the values of the binary quadratic quadratic forms along paths γ_ξ in the topograph in the same way as before, we obtain the following result [70]:

Theorem 5. *For definite binary quadratic forms Q the Lyapunov exponent*

$$\Lambda_Q(\xi) = 2\Lambda(\xi),$$

where $\Lambda(x)$ is the function defined above. For indefinite binary quadratic forms Q not representing 0 we have

$$\Lambda_Q(\xi) = \begin{cases} 2\Lambda(\xi) & \text{if } \xi \neq \alpha_\pm \\ 0 & \text{if } \xi = \alpha_\pm \end{cases},$$

where $\alpha_\pm$ are the roots of the quadratic equation $Q(\alpha_\pm, 1) = 0$.

Importantly, the paths $\gamma_{\alpha_\pm}$ are exactly those which follow the Conway river. These paths are described explicitly using the connection to continued fraction representations, which reveals some results regarding continued fraction representations of quadratic irrational numbers going back to Galois [52]. Moreover, we use these results to reconcile the Conway river with a geometric interpretation of continued fractions proposed by Klein [50] and developed further by Arnold, who introduced the notion of the sail (see [45]).

The third part of the thesis is concerned with the tropical version of Markov dynamics.

Cantat [13] considered the dynamics of the modular group on the cubic surfaces S_D defined by the Markov-type equation

$$x^2 + y^2 + z^2 = xyz + D \tag{1.5}$$

and computed the corresponding entropy. The calculation is based on the same important observation [58, 20, 84] that the case $D = 4$ (equivalent to the case (1.3)) is integrable in the sense that the dynamics can be linearised. Geometrically S_4 is the unique cubic surface with four singularities, known as the "Cayley cubic".

Adler and Veselov [1] proposed a tropical version of the bounded part of the Cayley cubic as a certain tetrahedron T, from which we see the corresponding tropical Markov dynamics. We show that this dynamics is semi-conjugated to the standard action of $SL_2(\mathbb{Z})$ on a torus, which allows us to compute the corresponding Lyapunov exponent and entropy [71].

Theorem 6. *The tropical Cayley-Markov action of a hyperbolic element $A \in SL_2(\mathbb{Z})$ on T is ergodic, with the Lyapunov exponent and entropy given by the logarithm of the spectral radius of A. Their growth along the path γ_ξ on the planar binary tree is given by $\Lambda(\xi)$.*

The structure of the thesis is as follows:

Chapter 2 provides a background to the main themes of the thesis; in particular, some historical results concerning Diophantine approximation are discussed, as well as a brief explanation of continued fractions.

In chapter 3, following [69], we discuss the Markov equation and investigate Markov dynamics as a function of the paths in the binary planar tree. We introduce the Lyapunov exponent $\Lambda(\xi)$ and the corresponding Lyapunov spectrum, and discuss some properties of this function. Particular attention is paid to the restriction $\Lambda_{\mathbb{X}_a}$ of Λ to generalised Markov-Hurwitz sets, for which specific results are found following work by Fock [31].

Chapter 4, based on [70], concerns the growth in $SL_2(\mathbb{Z})$-dynamics on binary quadratic forms following the topographic approach of Conway [23]. We also discuss the relation between the Conway river and the Arnold sail from the geometric theory of continued fractions, following [72].

In chapter 5 we study the tropical version of Cayley-Markov dynamics, following [71].

We conclude with a brief summary of the results and some open problems.

Chapter 2

Background

2.1 Diophantine Approximation

Diophantine approximation is the study of how well one can approximate the reals by rational numbers, and is one of the oldest areas in number theory; the name directly refers to the ancient Greek mathematician Diophantus of Alexandria. There are many good introductions to this topic; we recommend for example [2, 12], from which we have taken the following results.

The first results were formalised in the nineteenth century with the classical result of Dirichlet:

Theorem 7 (Dirichlet's Approximation Theorem). *Let $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$. Then there exists $\frac{p}{q} \in \mathbb{Q}$ with $q \leq N$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}.$$

An immediate consequence of this theorem is the following:

Corollary 1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists infinitely many $\frac{p}{q} \in \mathbb{Q}$ with*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Various attempts were made to sharpen this estimate; a famous result of Hurwitz [42] improves on Dirichlet's result:

Theorem 8 (Hurwitz). *For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

This result leads to the *Lagrange spectrum* [2]: let $\alpha \in \mathbb{R}, L \in \mathbb{R}, L > 0$. Then

$$L(\alpha) = \sup \left\{ L : \left| \alpha - \frac{p}{q} \right| < \frac{1}{Lq^2} \text{ holds for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}$$

is called the *Lagrange number* of α. The Lagrange spectrum is then defined as

$$\mathcal{L} = \{L(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q}\}.$$

2.2 The Markov Theorem

In 1880 Andrei A. Markov discovered in his master's thesis [55] a remarkable connection between Diophantine analysis and the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz,$$

known today as the *Markov equation.*

Definition 1 (Singular solutions)**.** *The positive integer solution triples of the Markov equation are known as* Markov triples*. There are exactly two solutions containing at least one repeated element:*

$$(1, 1, 1) \quad \textit{and} \quad (1, 1, 2).$$

These are called the singular solutions*.*

All Markov triples can be found from the initial singular solution $(1, 1, 1)$ via Vieta involutions:

Lemma 1 ([55])**.** *Let (k, l, m) be a Markov triple, i.e.*

$$k^2 + l^2 + m^2 = 3klm, \quad k, l, m \in \mathbb{N}.$$

Then

$$(k, l, 3kl - m), \quad (k, 3km - l, m), \quad (3lm - k, l, m)$$

are also Markov triples.

A number that occurs as a member of a Markov triple is known as a *Markov number.* The famous *Unicity Conjecture* of Frobenius stands as an open problem regarding the uniqueness of the Markov triples:

Conjecture 1 (Frobenius Unicity Conjecture [33])**.** *Each Markov number occurs as the largest member of a Markov triple exactly once.*

The set of Markov numbers we will denote as $\mathcal{M}$; it can be shown that every $m \in \mathcal{M}$ can be found in the way described above.

In particular, Markov proved his celebrated theorem concerning the irrationality of a certain subset of irrational numbers:

Theorem 9 (Markov [55], taken from [2])**.** *Let $\mathcal{M}$ be the set of Markov numbers. The Lagrange spectrum below 3 is given by*

$$\mathcal{L}_{<3} = \left\{ \frac{\sqrt{9m^2-4}}{m} : m \in \mathcal{M} \right\}.$$

More precisely, there is a sequence of quadratic irrationals

$$\beta_m = \frac{a_m + \sqrt{9m^2-4}}{b_m}, \quad m \in \mathcal{M}$$

with a_m, $b_m \in \mathbb{Z}$ whose Lagrange numbers are

$$L(\beta_m) = \frac{\sqrt{9m^2-4}}{m}.$$

Conversely, every $L(\alpha) < 3$ with $\alpha \notin \mathbb{Q}$ is of this form.

The *Markov constant* of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is defined as

$$\mu(\alpha) = L(\alpha)^{-1} \tag{2.1}$$

and can be interpreted as a measure of irrationality. Specifically: a larger Markov constant describes a 'more irrational' number. The set of all Markov constants is called the *Markov spectrum* [12].

For further details, as well as historical notes, the reader is recommended the very nicely-written book by Aigner [2].

2.3 Continued Fractions

A good understanding of continued fractions is necessary for the study of Diophantine approximation, and central to certain results in this thesis. The following can be extracted from any good introduction to the topic, in particular [2, 12, 25, 26, 62].

Definition 2. *A finite continued fraction is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}, \tag{2.2}$$

with $a_0 \in \mathbb{Z}$ and positive $a_1, a_2, \ldots, a_n \in \mathbb{Z}$.

We use the notation $[a_0, a_1, a_2, \ldots, a_n]$ to represent the continued fraction (2.2)*.*

An infinite continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \tag{2.3}$$

with $a_0 \in \mathbb{Z}$ *and positive* $a_1, a_2, \ldots \in \mathbb{Z}$. *We use the notation* $[a_0, a_1, a_2, \ldots]$ *to represent the continued fraction* (2.3).

Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction representation of $\alpha \in \mathbb{R}$. Then we have the following theorem (see e.g. [2]):

Theorem 10. *1. A infinite continued fraction* $[a_0, a_1, a_2, \ldots]$ *is always an irrational number;*

2. *If* $[a_0, a_1, a_2, \ldots] = [b_0, b_1, b_2, \ldots]$ *holds for two infinite continued fractions, then* $a_i = b_i$ *for all* $i \geq 0$*;*

3. *Every irrational number* α *can be uniquely expanded into a infinite continued fraction* $\alpha = [a_0, a_1, a_2, \ldots]$.

Conversely, every finite continued fraction $[a_0, a_1, a_2, \ldots, a_k]$ is equal to a rational number $\frac{p}{q}$ (and vice versa), and this expansion is not unique. Indeed, it is easy to see that

$$[a_0, a_1, a_2, \ldots, a_k] = [a_0, a_1, a_2, \ldots, a_k - 1, 1] .$$

If the infinite continued fraction of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is terminated at a_k, we get the *kth convergent*

$$[a_0, a_1, a_2, \ldots, a_k] = C_k.$$

Theorem 11 (See e.g. [62])**.** *Let* $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$ *be positive. Define the sequences* $p_0, p_1, \ldots, p_n$ *and* $q_0, q_1, \ldots, q_n$ *recursively by*

$$\begin{aligned} p_0 &= a_0 & q_0 &= 1 \\ p_1 &= a_0 a_1 + 1 & q_1 &= a_1 \end{aligned}$$

and

$$p_k = a_k p_{k-1} + p_{k-1} \qquad q_k = a_k q_{k-1} + q_{k-2}$$

for $k = 2, 3, \ldots, n$. *Then the kth convergent* $C_k = [a_0, a_1, a_2, \ldots, a_k]$ *is given by*

$$C_k = \frac{p_k}{q_k}.$$

These convergents are the best rational approximations for the irrational α, in the sense that p_k/q_k is closer to α than any other rational number with denominator less than q_k [62].

It can be shown that

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^{k+1},$$

so that

$$\left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \frac{1}{q_k q_{k+1}}$$

and hence

$$\left|\alpha - \frac{p_k}{q_k}\right| < \left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \frac{1}{q_k q_{k+1}} < \frac{1}{q_k^2}.$$

We have the following classical result:

Theorem 12 (Lagrange)**.** *Let* $\alpha = [a_0, a_1, a_2, \ldots]$*. Then* α *is a quadratic irrational if and only if* $[a_0, a_1, a_2, \ldots]$ *is eventually periodic, i.e.*

$$\alpha = [a_0, a_1, \ldots, a_k, \overline{a_{k+1}, a_{k+2}, \ldots, a_{k+n}}]\,.$$

We will also need the notion of the *continuant* of a continued fraction [25]:

Definition 3 (Continuant)**.** *The continuant* $K(a_0, a_1, \ldots, a_n)$ *is the numerator of the continued fraction* $[a_0, a_1, \ldots, a_n]$*.*

Continuants can be recursively defined by

$$\begin{aligned} K(a_0) &= a_0, \\ K(a_0, a_1) &= 1 + a_0 a_1, \\ K(a_0, a_1, \ldots, a_n) &= a_n K(a_0, a_1, \ldots, a_{n-1}) + K(a_0, a_1, \ldots, a_{n-2}). \end{aligned}$$

2.4 Farey Sequences

The following classical construction is due to the British geologist John Farey Sr. and can be found in [12].

The *Farey sequence of order* N, denoted $\mathcal{F}_N$, is the ordered list of all reduced fractions $\frac{p}{q} \in [0, 1]$ such that $q \leq N$. For example,

$$\begin{aligned} \mathcal{F}_1 &= \left\{\frac{0}{1}, \frac{1}{1}\right\}, \\ \mathcal{F}_2 &= \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}, \\ \mathcal{F}_3 &= \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}, \end{aligned}$$

and so on.

Two fractions next to each other in a Farey sequence are called *Farey neighbours*; a Farey fraction that belongs to $\mathcal{F}_{N+1}$ but not $\mathcal{F}_N$ is the *Farey mediant* of two Farey neighbours of $\mathcal{F}_N$. The mediant of $\frac{p}{q}$ and $\frac{r}{s}$ is defined as

$$\frac{p}{q} * \frac{r}{s} := \frac{p+r}{q+s},$$

and is a new neighbour of both.

For any Farey neighbours $\frac{p}{q}$ and $\frac{r}{s}$, we have that

$$ps - qr = \pm 1.$$

We will show that the Farey fractions can be naturally represented using a binary tree, which we will call the Farey tree.

Chapter 3

Lyapunov spectrum of Markov and Euclid trees

The *Markov numbers* are numbers which occur as positive integer solutions to the *Markov equation*

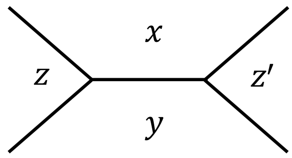
$$x^2 + y^2 + z^2 = 3xyz. \tag{3.1}$$

The growth of the Markov numbers

$$m = 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, \ldots$$

was studied by Zagier [84] and later McShane and Rivin [56]. However, successive Markov numbers are found by the series of Vieta involutions described in Lemma 1, and therefore grow naturally in a binary tree (see e.g. [9]). We therefore study the growth of the Markov numbers along prescribed paths in this tree, rather than following Zagier.

More precisely, we will be using the tree representation with Markov numbers living in the connected components of the complement to a planar binary tree, using the graphical representation of Vieta involution shown in Figure 3.1.



$$z + z' = 3xy$$

Figure 3.1: Graphical representation of Vieta involution

The corresponding *Markov tree* is shown in Figure 3.2 next to the *Farey tree*, for which at each vertex we have fractions $\frac{a}{b}$, $\frac{c}{d}$ and their *Farey mediant*

$$\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d}.$$

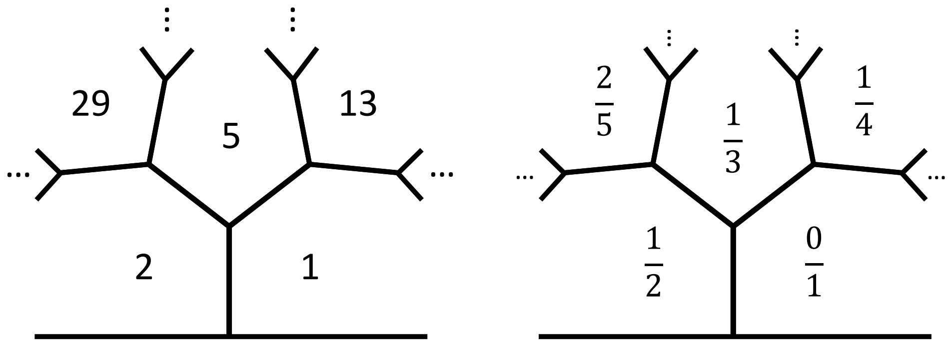


Figure 3.2: Correspondence between Markov numbers and Farey fractions

This defines the *Farey parametrisation* of the Markov numbers

$$m = m\left(\frac{p}{q}\right), \quad \frac{p}{q} \in \left[0, \frac{1}{2}\right],$$

which goes back to Frobenius [33] and will be crucial for the purposes of this thesis.

Using the Farey tree we can assign to every infinite path γ on a rooted planar binary tree a point $x \in \left[0, \frac{1}{2}\right]$ by considering the limit of the Farey fractions along the path: for instance, in Figure 3.3 the path $\gamma_{\overline{\varphi}^2}$, where $\overline{\varphi} = \frac{\sqrt{5}-1}{2}$, has been highlighted.

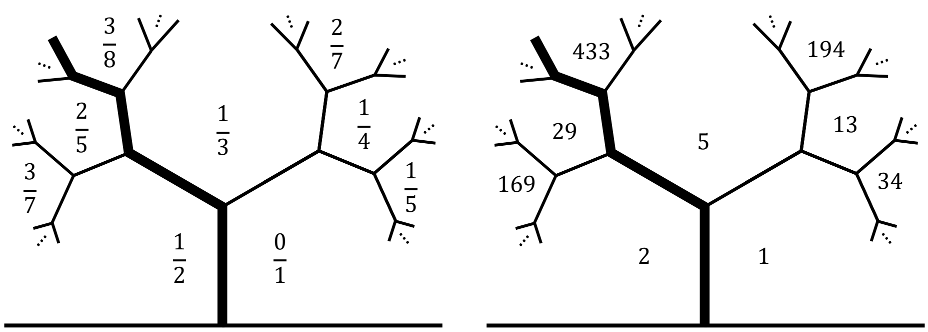


Figure 3.3: Farey and Markov trees with $\gamma_{\overline{\varphi}^2}$, equivalent to the “golden” path

Let $m_n(x)$ be the n-th Markov number along the path $\gamma(x)$ and define the corresponding *Lyapunov exponent* $\Lambda(x)$ as

$$\Lambda(x) = \limsup_{n\to\infty} \frac{\ln(\ln m_n(x))}{n}. \tag{3.2}$$

Equivalently, following [22, 84], we can consider the 'tropical version' of the Markov tree: the Euclid tree, describing the Euclidean algorithm with integer triples (u, v, w) satisfying the relation

$$u + v = w,$$

and define the Lyapunov exponent as

$$\Lambda(x) = \limsup_{n\to\infty} \frac{\ln w_n(x)}{n},$$

where $w_n(x)$ is the last (largest) number in the n-th triple along path $\gamma(x)$.

We prove that the Lyapunov exponent exists for all paths and can be naturally extended to the function $\Lambda(x)$, $x \in \mathbb{R}P^1$, which is $GL_2(\mathbb{Z})$-*invariant*:

$$\Lambda\left(\frac{ax+b}{cx+d}\right) = \Lambda(x), \quad x \in \mathbb{R}P^1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}),$$

and almost everywhere vanishing. This interesting function is the main object of our study.

Theorem 13. *The Lyapunov spectrum of Markov and Euclid trees is*

$$Spec_\Lambda = [0, \ln \varphi],$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio.*

In particular, for all $x \in \mathbb{R}P^1$

$$\Lambda(x) \leq \Lambda(\varphi) = \ln \varphi,$$

so the 'golden path' has the maximal Lyapunov exponent.

To state our second main result we need to introduce the following set of the 'most irrational numbers' $\mathbb{X} \subset \mathbb{R}$.

Recall from Chapter 2 that Hurwitz [42] proved that the golden ratio and its equivalents have the maximal possible Markov constant, which can be considered a measure of irrationality (see [12]).

Markov's theorem states that Markov constants μ larger than 1/3 have the form

$$\mu = \frac{m}{\sqrt{9m^2 - 4}},$$

where m is a Markov number (see details in Delone [27] and Bombieri [9]).

Corresponding equivalence classes of these most irrational numbers are naturally labelled by the Markov numbers $m \in \mathcal{M}$. They have special representatives x_m (which we call Markov-Hurwitz numbers) with pure periodic continued fractions with period consisting of 1 and 2:

$$x_1 = [\overline{1}] = \frac{\sqrt{5}-1}{2},$$
$$x_2 = [\overline{2}] = \sqrt{2} - 1,$$
$$x_5 = [\overline{2, 2, 1, 1}] = \frac{\sqrt{221}-9}{14}, \ldots$$

where

$$[a_1, a_2, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

Note that we use a version of continued fractions with $a_0 = 0$, which will allow us to avoid zeros in continued fractions (cf. [48]).

The set $\mathbb{X}$ of all Markov-Hurwitz numbers is countable and has only one isolated point: $x_1 = \frac{\sqrt{5}-1}{2} \approx 0.6180$, which is also the maximal number in $\mathbb{X}$. The minimal number is $x_2 = \sqrt{2} - 1 \approx 0.4142$, and the maximal limiting point of $\mathbb{X}$ is

$$x_* = [2, 2, \overline{1}] = \frac{7 + \sqrt{5}}{22} \approx 0.4198.$$

Using the Farey parametrization of Markov numbers $m = m\left(\frac{p}{q}\right)$, we can denote the corresponding number x_m as $x\left(\frac{p}{q}\right)$.

Theorem 14. *The restriction $\Lambda_{\mathbb{X}}$ of the Lyapunov function on the set of Markov-Hurwitz numbers is monotonically increasing from*

$$\Lambda(x_2) = \frac{1}{2}\ln(1 + \sqrt{2}) \quad \textit{to} \quad \Lambda(x_1) = \ln\left(\frac{1 + \sqrt{5}}{2}\right).$$

In the Farey parametrization, $\Lambda\left(x\left(\frac{p}{q}\right)\right)$ is convex as a function of $\frac{p}{q}$.

The proof is based on the relation between Markov numbers and geodesics on the punctured torus with hyperbolic metric, which was found by Gorshkov [35] in his thesis in 1953 and, independently, by Cohn [20]. Since then this relation has been very much in use, see in particular Goldman [34], Bowditch [10] and a nice exposition by Series [66].

Our general approach is close to Chekhov and Penner [17], who discussed similar questions in the quantum theory of Teichmüller spaces. The key result for us is due to Fock [31], who proved using Thurston's laminations that a certain function defined in terms of Markov numbers can be extended to a convex function on a real interval.

We also present a generalisation of these results to the countable sets $\mathbb{X}_a$ of quadratic irrationals depending on a natural number a. They are related to the solutions of the Diophantine equation

$$X^2 + Y^2 + Z^2 = XYZ + 4 - 4a^6, \quad a \in \mathbb{N},$$

studied by Mordell [58], and geometrically to the geodesics on one-holed hyperbolic tori. For $a = 1$ we have the scaled Markov equation and Markov-Hurwitz set $\mathbb{X}_1 = \mathbb{X}$.

3.1 The Farey tree, monoid $SL_2(\mathbb{N})$ and Lyapunov exponent

Let $\mathcal{T}$ be a binary tree embedded in the hyperbolic plane $\mathbb{H}$ as the dual graph to the Farey tessellation of $\mathbb{H}$ into ideal triangles (see Figure 3.4).

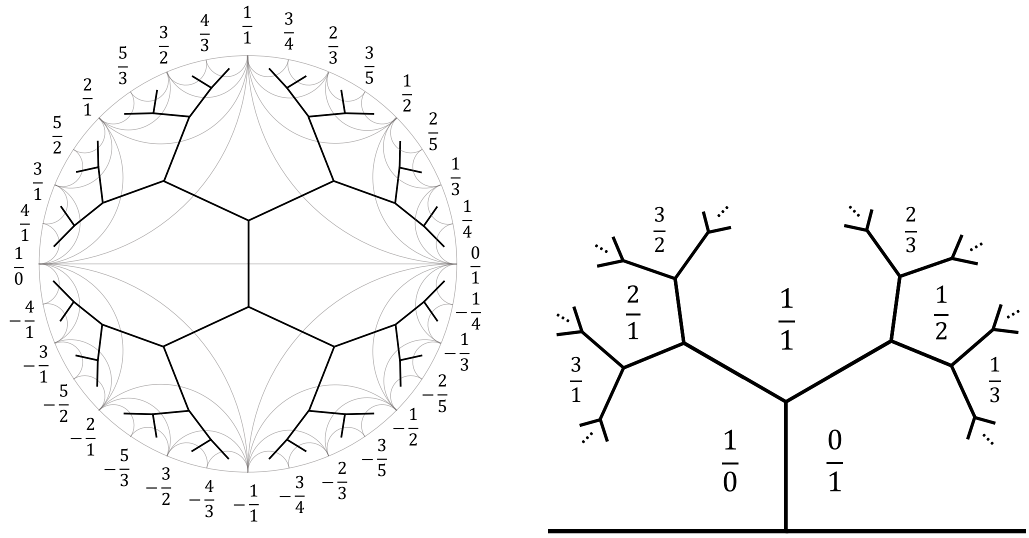


Figure 3.4: Dual tree for Farey tessellation and positive Farey tree

It is enough to consider only the upper half of the tree, which can be considered as the Farey tree T_F of all positive fractions (see Figure 3.4). The Farey tree shown in Figure 3.3 is the branch of this tree corresponding to the fractions lying between 0 and $\frac{1}{2}$.

Let $SL_2(\mathbb{N}) \subset SL_2(\mathbb{Z})$ be the set of matrices with non-negative entries. Such matrices are closed under multiplication and contain the identity, and thus form a monoid.

The positive Farey tree gives a nice parametrisation of this monoid. Indeed, every (naturally oriented) edge E of T_F is adjacent to two neighbouring Farey fractions $\frac{a}{c}$, $\frac{b}{d}$, so we can consider the matrix

$$A_E = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which belongs to $SL_2(\mathbb{N})$. It follows from the construction of the Farey sequences that every matrix $A \in SL_2(\mathbb{N})$ appears in this way exactly once.

Remark. Frobenius [33] introduced the so-called *Frobenius coordinates* to parametrise the Markov numbers. These are pairs of coprime numbers (p, q) such that

$$m(1, 0) = 1, \quad m(0, 1) = 2, \quad m(1, 1) = 5$$

and Markov triples take the form

$$(m(p_1, q_1), m(p_2, q_2), m(p_1 + p_2, q_1 + q_2)).$$

It is easy to see that this construction is exactly analogous to our Farey parametrisation.

Let us define the notion of a path in the positive Farey tree.

Definition 4 (Path)**.** *Let E_I be the edge in the Farey tree separating the Farey fractions $\frac{a}{c}$ and $\frac{b}{d}$, with V_I the vertex adjacent to both and their Farey mediant $\frac{a+b}{c+d}$.*

We define a left turn as moving from E_I to E_L, where E_L is the edge between $\frac{a}{c}$ and $\frac{a+b}{c+d}$. In matrix form, this is equivalent to multiplication from the right by

$$L := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We define similarly a right turn as moving from E_I to E_R, where E_R is the edge between $\frac{a+b}{c+d}$ and $\frac{b}{d}$. In matrix form, this is equivalent to multiplication from the right by

$$R := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The path $\gamma(x)$ is the series of left and right turns from the initial edge separating $\frac{1}{0}$ and $\frac{0}{1}$ to the limit $x \in [0, \infty]$.

Remark. The path $\gamma(x)$ is unique and infinite if and only if x is irrational.

Recall now that the *spectral radius* $\rho(A)$ of a matrix A is defined as the maximum of the modulus of its eigenvalues. For a non-triangular (hyperbolic) matrix A from $SL_2(\mathbb{N})$ the eigenvalues are λ and λ^{-1}, where $\lambda = \lambda(A) > 1$ and

$$\rho(A) = \lambda(A)$$

(for triangular (parabolic) matrices $\rho(A) = 1$).

Consider now the path $\gamma(x)$ in the Farey tree.

Proposition 1. *The Lyapunov exponent can be equivalently defined as*

$$\Lambda(x) = \limsup_{n\to\infty} \frac{\ln \rho(A_n(x))}{n},$$

where $A_n(x) \in SL_2(\mathbb{N})$ is attached to n-th edge along path $\gamma(x)$ and $\rho(A)$ is the spectral radius of matrix A.

Proof. Let us assume for convenience that $x \in [0, 1]$, which corresponds to the right half of the positive Farey tree shown in Figure 3.4. The left half of the tree is related by $x \to 1/x$ and the change of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} = S^{-1}AS, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We see that the denominators of the fractions form the Euclid tree shown on the right of Figure 3.5. Let

$$A_n(x) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

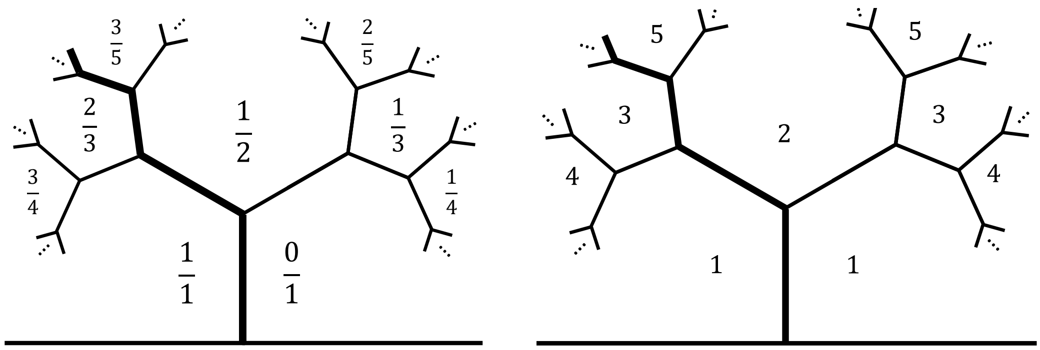


Figure 3.5: Farey and Euclid rooted trees with a path

be the matrix assigned to n-th edge of $\gamma(x)$. Then

$$w_n(x) = \max(c_n(x), d_n(x))$$

is the corresponding sequence from the Euclid tree.

Let λ_n be the maximal eigenvalue of $A_n(x)$. Since $\lambda_n + \lambda_n^{-1} = a_n + d_n$ with $\lambda_n^{-1} \leq 1$ we have

$$\lambda_n \leq a_n + d_n < c_n + d_n \leq 2\max(c_n, d_n) = 2w_n,$$

where we have used that $a_n < c_n$, which is valid on this half of the tree.

To have the estimate of λ_n from below we need to consider the cases $x = 0$ and $x > 0$ separately. For $x = 0$

$$A_n(0) = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

with $\lambda_n = 1$ and $w_n = n$, so

$$\limsup_{n\to\infty} \frac{\ln \lambda_n}{n} = 0 = \limsup_{n\to\infty} \frac{\ln w_n}{n} = \limsup_{n\to\infty} \frac{\ln n}{n}$$

in this case.

If $x > 0$, since $a_n/c_n \to x$ as $n \to \infty$, we have for large n the inequality $a_n > \frac{x}{2}c_n$. This means that

$$\lambda_n \geq \frac{1}{2}(a_n + d_n) > \frac{1}{2}\left(\frac{x}{2}c_n + d_n\right) > \frac{x}{4}\max(c_n, d_n) = \frac{x}{4}w_n.$$

Thus we have for $x > 0$ and large n that $\frac{x}{4}w_n(x) < \lambda_n(x) < 2w_n(x)$, which implies that

$$\limsup_{n\to\infty} \frac{\ln \lambda_n(x)}{n} = \limsup_{n\to\infty} \frac{\ln w_n(x)}{n}$$

provided any of these limits exist, which we will now show. □

Note that under our assumptions $w = \max(c, d) = ||A||_\infty$, where the norm $||A||_\infty$ of a real matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is defined as

$$||A||_\infty := \max(|a|, |b|, |c|, |d|),$$

and so from relation (3.1) it follows that

$$\Lambda(x) = \limsup_{n\to\infty} \frac{\ln ||A_n(x)||_\infty}{n}.$$

Theorem 15. *The Lyapunov exponent $\Lambda(x)$ exists for all real $x \geq 0$ and satisfies*

$$0 \leq \Lambda(x) \leq \ln \varphi,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Moreover, every value in $[0, \ln \varphi]$ is attained.

Proof. Recall that the *operator norm* of a matrix A acting on a Euclidean space is defined as

$$||A|| := \max_{|x|=1} |Ax|.$$

The norm is related to the spectral radius by the formula

$$||A||^2 = \rho(A^*A) \tag{3.3}$$

and satisfies the inequalities (see e.g. [51])

$$\rho(A) \leq ||A||$$

and

$$||AB|| \leq ||A|| \cdot ||B||.$$

Now note that the matrices $A_n(x)$ along a path $\gamma(x)$ have the product form

$$A_n(x) = X_1 \dots X_n,$$

where X_i are either L or R defined as

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

depending on whether we turn left or right on the tree.

Since

$$RL = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

has maximal eigenvalue

$$\lambda(RL) = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2,$$

the norms

$$||L|| = ||R|| = \frac{1+\sqrt{5}}{2} = ||X_i||,$$

by (3.3). Therefore

$$\rho(A_n) \le ||A_n|| \le ||X_1 \dots X_n|| \le ||X_1|| \dots ||X_n|| = \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

which implies that the sequence

$$\frac{\ln \rho(A_n)}{n} \le \ln \frac{1+\sqrt{5}}{2}$$

is bounded. In particular,

$$\Lambda(x) = \limsup_{n\to\infty} \frac{\ln \rho(A_n(x))}{n}$$

exists and satisfies the inequality

$$\Lambda(x) \le \ln \frac{1+\sqrt{5}}{2}.$$

The equality is attained at $x = \frac{\sqrt{5}-1}{2}$ since the corresponding

$$A_{2n} = (RL)^n = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n.$$

Similarly, for the path γ_0 consisting of only right-turns, we have

$$A_n = R^n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

and thus $\Lambda(0) = 0$.

To show that every value $\Lambda_0 \in (0, \ln \varphi)$ is attained we will use the following lemma.

Let (X_n), $n \in \mathbb{N}$ be a sequence of matrices such that $X_i \in \{L, R\}$ and let B be any matrix from $SL_2(\mathbb{N})$. Consider the products

$$A_n = X_1 \dots X_n, \quad B_n = BX_1 \dots X_n, \quad n \in \mathbb{N}.$$

Lemma 2. *For every matrix $B \in SL_2(\mathbb{N})$*

$$\limsup_{n\to\infty} \frac{\ln ||B_n||}{n} = \limsup_{n\to\infty} \frac{\ln ||A_n||}{n}.$$

In particular, for every such B

$$\lim_{n\to\infty} \frac{\ln ||BR^n||}{n} = 0, \quad \lim_{n\to\infty} \frac{\ln ||B(RL)^n||}{2n} = \ln \varphi.$$

Indeed, $B_n = BA_n, A_n = B^{-1}B_n$, so

$$||A_n||/||B^{-1}|| \le ||B_n|| \le ||B||||A_n||,$$

from which (2) follows.

The second part follows from the equalities

$$\lim_{n\to\infty} \frac{\ln \rho(R^n)}{n} = 0, \quad \lim_{n\to\infty} \frac{\ln \rho((RL)^n)}{2n} = \ln \varphi,$$

which are easy to check.

Now the strategy is the following: we start with the matrix $A_0 = I$ and consider

$$A_2 = RL = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

with

$$\frac{\ln ||A_2||}{2} = \ln \varphi.$$

Apply multiplication by R from the right several times until we get to the matrix A_n with

$$\frac{\ln ||A_n||}{n} < \Lambda_0.$$

As soon as this happens we start multiplying from the right by matrix RL until we have matrix A_m with

$$\frac{\ln ||A_m||}{m} > \Lambda_0$$

and then repeat all this. It is easy to see that this process leads to a sequence of matrices A_n such that

$$\lim_{n\to\infty} \frac{\ln ||A_n||}{n} = \Lambda_0.$$

Indeed, say for $A_{n+1} = A_n R$, we have

$$||A_n||/||R^{-1}|| \le ||A_{n+1}|| \le ||R||||A_n||,$$

which implies that

$$\frac{\ln ||A_n|| - \ln ||R^{-1}||}{n+1} \le \frac{\ln ||A_{n+1}||}{n+1} \le \frac{\ln ||R|| + \ln ||A_n||}{n+1},$$

so that the steps

$$\frac{\ln ||A_{n+1}||}{n+1} - \frac{\ln ||A_n||}{n}$$

turn to zero as n goes to infinity.

To complete the proof we use the fact that any two norms on a finite-dimensional vector space are equivalent. In particular, we have

$$c_1||A||_\infty \le ||A|| \le c_2||A||_\infty$$

for some positive constants c_1 and c_2. This implies that

$$\lim_{n\to\infty} \frac{\ln ||A_n||_\infty}{n} = \lim_{n\to\infty} \frac{\ln ||A_n||}{n} = \Lambda_0,$$

and due to (3.1) for the corresponding path $\gamma = \gamma(x)$ we have $\Lambda(x) = \Lambda_0$. □

Bearing in mind the Farey tesselation in Figure 3.4, we extend Λ to negative x by $\Lambda(-x) = \Lambda(x)$ and define $\Lambda(\infty) := \Lambda(\frac{1}{0}) = 0$.

Corollary 2. *The function $\Lambda(x),\ x \in \mathbb{R}P^1$ is $GL_2(\mathbb{Z})$-invariant:*

$$\Lambda\left(\frac{ax+b}{cx+d}\right) = \Lambda(x), \quad x \in \mathbb{R}P^1$$

for all integer $a, b, c, d,$ satisfying $ad - bc = \pm 1$.

Indeed, it is well-known (see e.g. [52]) that two irrational numbers $x, y \in \mathbb{R}$ are $GL_2(\mathbb{Z})$-equivalent, which means that

$$y = \frac{ax+b}{cx+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}),$$

if and only if x and y have continued fraction expansions which eventually coincide. This implies that the corresponding paths γ_x and γ_y have eventually the same sequence of left and right turns (see sections 3.4, 3.5 below), and by Lemma 2, relation (3.1) and equivalence of norms, have the same Lyapunov exponents.

In particular, $\Lambda(x+1) = \Lambda(x)$ is periodic, so it is sufficient to consider only the segment $[0, 1]$.

For the quadratic irrationals the values of Λ can be described explicitly. Let

$$x = [a_1, \dots, a_k, \overline{b_1, b_2, \dots, b_{2n}}]$$

be the continued fraction expansion of a quadratic irrational x, which is known to be periodic by the Lagrange theorem. We can assume that the length of the period is even by doubling it if necessary.

Define the matrix $B(x) \in SL_2(\mathbb{N})$ as the product

$$B = R^{b_1} L^{b_2} \dots R^{b_{2n-1}} L^{b_{2n}} = \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ b_{2n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & b_{2n} \\ 0 & 1 \end{pmatrix}.$$

Let $\tau = tr\, B(x)$ be the trace and

$$\lambda(x) = \frac{\tau + \sqrt{\tau^2 - 4}}{2}$$

be the largest eigenvalue (or spectral radius) of $B(x)$.

Proposition 2. *The Lyapunov exponent of the quadratic irrational*

$$x = [a_1, \dots, a_k, \overline{b_1, b_2, \dots, b_{2n}}]$$

can be described explicitly as

$$\Lambda(x) = \frac{\ln \lambda(x)}{s(x)}, \tag{3.4}$$

where $s(x) = b_1 + \cdots + b_{2n}$.

The proof follows easily from the results of this section.

In particular, we have for $x = \sqrt{2}, \sqrt{3}, \sqrt{5}$ the periods $\overline{2,2}$, $\overline{1,2}$, $\overline{4,4}$ respectively, so

$$\Lambda(\sqrt{2}) = \frac{1}{4}\ln(3+2\sqrt{2}),$$
$$\Lambda(\sqrt{3}) = \frac{1}{3}\ln(2+\sqrt{3}),$$
$$\Lambda(\sqrt{5}) = \frac{1}{8}\ln(9+4\sqrt{5}).$$

Theorem 16. *The Lyapunov exponent $\Lambda(x) = 0$ for almost every $x \in [0,1]$. In particular, for almost every x the limsup in the definition of $\Lambda(x)$ can be replaced by the usual limit.*

Proof. For rational x we have $\Lambda(x) = 0$, so assume that x is irrational. Let

$$x = [a_1, a_2, \ldots]$$

be its expansion as a continued fraction,

$$\frac{p_n(x)}{q_n(x)} = [a_1, a_2, \ldots, a_n]$$

be the n-th convergent and $s_n(x) = a_1 + \cdots + a_n$. Then by equation (3.4) and the Farey tree interpretation of the continued fraction expansions we have

$$\Lambda(x) = \limsup_{n\to\infty} \frac{\ln q_n(x)}{s_n(x)} = \limsup_{n\to\infty} \frac{\ln q_n(x)}{n}\frac{n}{s_n(x)}.$$

But the classical result of Lévy [53] says that for almost all x

$$\lim_{n\to\infty} \frac{\ln q_n(x)}{n} = \frac{\pi^2}{12\ln 2}.$$

Now the result follows from the known fact (see [24], Th. 4 in Ch. 7, Section 4) that for almost every x

$$\lim_{n\to\infty} \frac{s_n(x)}{n} = \infty.$$

□

3.2 Markov forms and the Cohn tree

Before we proceed to the most irrational numbers let us introduce the notion of the Markov binary quadratic form [55].

Definition 5 (Markov form)**.** *Let (k,l,m) be a Markov triple:*

$$k^2 + l^2 + m^2 = 3klm$$

with m being the largest number. The Markov form $f_m(x,y)$ associated to this Markov triple has the form

$$f_m(x,y) = mx^2 + (3m-2p)xy + (q-3p)y^2, \tag{3.5}$$

where

$$p := \min\{x : lx \equiv \pm k \ (mod\ m)\}, \quad q := \frac{1}{m}(p^2+1).$$

A Markov form is an indefinite binary quadratic form with the discriminant

$$\Delta(f_m) = 9m^2 - 4$$

and with $m(f_m) = m$, where by definition

$$m(f) := \min_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|.$$

Markov studied the possible values of the ratio

$$M_f = \frac{m(f)}{\sqrt{\Delta(f)}}$$

for the indefinite integral binary forms and showed that all possible values $M = M_f$ larger than $1/3$ are given by

$$M = \frac{m}{\sqrt{9m^2-4}},$$

where m is a Markov number, and realised by the Markov forms (see [27]).

The corresponding positive roots $x = \alpha_m$ of $f_m(x,1) = 0$ give the most irrational numbers, which we will discuss in the next section. They have the continued fraction expansion

$$\alpha_m = [\overline{a_1,\dots,a_{2n}}]$$

with the following properties (Markov [55], Frobenius [33]; see also Cusick and Flahive [25], Ch. 2 Th.3) for $m > 2$:

$$\begin{aligned} m &= K(a_1,\dots,a_{2n-1}), \\ p &= K(a_2,\dots,a_{2n-1}), \\ q &= K(a_2,\dots,a_{2n-3}), \end{aligned}$$

where $K(s_1,\dots,s_n)$ is the continuant. We also have

$$\begin{aligned} a_1 = a_{2n} &= 2, \\ a_{2n-2} = a_{2n-1} &= 1, \end{aligned}$$

and the sequence $a_2,\dots,a_{2n-3}$ is palindromic.

Motivated by the theory of modular functions, Cohn [20] proposed a new derivation of Markov's original results using multiplication of matrices in $SL_2(\mathbb{N})$, so we can construct the analogous Cohn tree with triples (A,B,C), $C = AB$. His approach was based on arithmetic restrictions on Fricke's identity, from which he found the initial matrices

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \tag{3.6}$$

(see Figure 3.6). The relation between the Cohn and Markov trees is given by the trace map

$$C \to m = \frac{1}{3}\mathrm{tr}C.$$

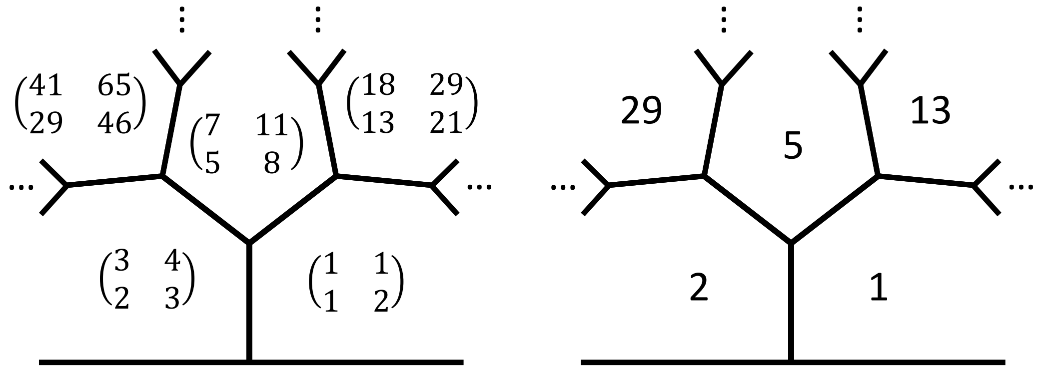


Figure 3.6: Cohn and Markov trees related by trace map

To state the relation with Markov forms we need to recall a standard relation between matrices from $SL_2(\mathbb{Z})$ and integral binary quadratic forms (see e.g. [52]).

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

be hyperbolic. Consider A as the automorphism of the lattice $\mathcal{L} = \mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$ by choosing some basis e_1, e_2 in this lattice. Then we can define the following integral binary quadratic form Q_A by the formula

$$\mathbf{v} \wedge A\mathbf{v} = Q_A(\mathbf{v}) e_1 \wedge e_2, \tag{3.7}$$

where $\mathbf{v}$ is a vector from $\mathbb{R}^2$. Explicitly if $\mathbf{v} = xe_1 + ye_2$ then

$$Q_A(x, y) = \det \begin{pmatrix} x & ax + by \\ y & cx + dy \end{pmatrix} = cx^2 + (d - a)xy - by^2. \tag{3.8}$$

The main property of this form (easily seen from the definition) is that this form is invariant under the action of A:

$$Q_A(A\mathbf{v}) = Q_A(\mathbf{v}).$$

Note that the discriminant of Q_A is

$$D = (d - a)^2 + 4bc = (a + d)^2 - 4(ad - bc) = (a + d)^2 - 4,$$

which is exactly the discriminant of the characteristic equation of A:

$$\lambda^2 - (a + d)\lambda + 1 = 0.$$

In particular, since A is hyperbolic the form Q_A is indefinite.

The following theorem, which seems to be new, gives a direct link between Cohn matrices and Markov forms.

Theorem 17. *Let A_m be the matrix from Cohn tree corresponding to Markov number m. Then Markov form $f_m(x, y)$ can be written as*

$$f_m(x, y) = Q_m(x + y, y)$$

where $Q_m = Q_{A_m}$ is the binary form (3.8) corresponding to A_m.

Proof. Aigner [2] showed that the Cohn matrix A_m has the form

$$A_m = \begin{pmatrix} m+p & 2m+p-q \\ m & 2m-p \end{pmatrix},$$

where p and q are the same as in the definition of Markov form (see Thm. 4.13 in [2], bearing in mind that Aigner's version of Cohn matrices is transposed to ours).

Using (3.8) we have

$$\begin{aligned} Q_m(x,y) &= mx^2 + (m-2p)xy - (2m+p-q)y^2, \\ Q_m(x+y,y) &= m(x+y)^2 + (m-2p)(x+y)y - (2m+p-q)y^2 \\ &= mx^2 + (3m-2p)xy + (q-3p)y^2, \end{aligned}$$

which is exactly the Markov form (3.5). □

Note that we see a very deep relation between the Cohn tree and combinatorial group theory and the automorphisms of free group $\mathbb{F}_2$. This is based on the well-known fact that the mapping class groups of a torus and a punctured torus are both isomorphic to $GL_2(\mathbb{Z})$.

3.3 The Markov-Hurwitz most irrational numbers

Recall that the Markov constant can be considered (after Markov and Hurwitz) as a measure of the irrationality of a number.

The Markov constant $\mu(\alpha)$ of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is defined as the minimal number c such that the inequality

$$\left| \alpha - \frac{p}{q} \right| \le \frac{c}{q^2} \tag{3.9}$$

holds for infinitely many $\frac{p}{q}$.

It can be shown [12] that for $\alpha = [a_1, a_2, \ldots]$ the Markov constant can be computed as

$$\mu(\alpha) = \liminf_{N\to\infty} ([0, a_{N+1}, a_{N+2} \ldots] + [a_N, a_{N-1}, \ldots, a_1])^{-1}.$$

A well-known result of Hurwitz [42] claims that for all irrational α we have

$$\mu(\alpha) \le \frac{1}{\sqrt{5}},$$

and $\mu(\alpha) = \frac{1}{\sqrt{5}}$ if and only if α is equivalent to $\frac{1+\sqrt{5}}{2}$. In other words, the golden ratio (and its equivalents) are the most irrational numbers.

It is natural to consider what happens if we exclude values of α equivalent to $\frac{1+\sqrt{5}}{2}$. The answer is that for the remaining numbers $\mu(\alpha) \le 1/\sqrt{8}$ (see e.g. [12]), and $\mu(\alpha) = 1/\sqrt{8}$ if and only if α is equivalent to $1 + \sqrt{2} = [\overline{2}]$ (the 'silver ratio').

This can be continued to derive the 'bronze ratio'

$$\alpha = [\overline{2,2,1,1}] = \frac{9+\sqrt{221}}{10}, \quad \mu(\alpha) = \frac{5}{\sqrt{221}},$$

a perhaps surprising result.

The *Markov spectrum* of Markov constants $\mu(\alpha) > \frac{1}{3}$ was completely described by the famous Markov theorem:

Theorem 18 (Markov [55]). *All Markov constants $\mu(\alpha) > \frac{1}{3}$ have the form*

$$\mu = \frac{m}{\sqrt{9m^2-4}},$$

where $m \in \mathcal{M}$ is a Markov number.

Note that this is equivalent to the formulation of M_f given in the previous section.

Markov's original result was stated in terms of binary quadratic forms, considered in the previous section. For modern proofs we refer to Bombieri [9] and Cusick and Flahive [25].

It is well-known that the most irrational numbers have periodic continued fractions with even periods consisting of 1's and 2's only (see e.g. [12, 25]).

Let us define the *conjunction* operation of two periods as

$$[\overline{s_1,\dots,s_n}] \odot [\overline{t_1,\dots,t_m}] = [\overline{s_1,\dots,s_n,t_1,\dots,t_m}] \tag{3.10}$$

and construct the new tree using this operation and starting with $A = 2_2$ and $B = 1_2$, where by k_n we mean the sequence $k,\dots,k$ of numbers k taken n times.

As a result we have the *Markov-Hurwitz tree* shown in Figure 3.7.

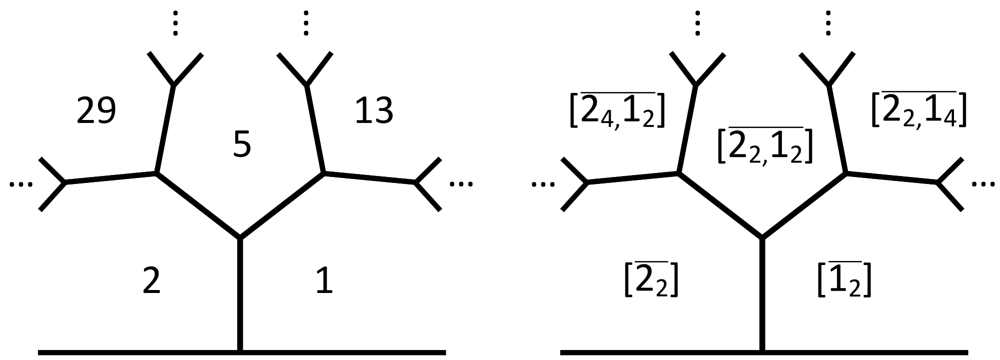


Figure 3.7: Markov and Markov-Hurwitz trees

Let y_m be the number on Markov-Hurwitz tree corresponding to the Markov number m.

The following result can be extracted from Cusick and Flahive [25] (see Lemma 4 in Chapter 2 of [25]), who made a detailed analysis of the roots α_m of $f_m(x,1) = 0$ for Markov forms $f_m(x,y)$.

Theorem 19 ([25]). *The Markov constant*

$$\mu(y_m) = \frac{m}{\sqrt{9m^2 - 4}},$$

so y_m are representatives of the most irrational numbers.

Remark. It follows from the results of [25] that for $m > 1$

$$y_m = \mu_m + 1 = \frac{5c - 2d + \sqrt{9c^2 - 4}}{2c}, \tag{3.11}$$

where $v_m = (\mu_m, 1)$ is the eigenvector with the largest eigenvalue of the corresponding matrix

$$A_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

from the Cohn tree.

3.4 Paths in Farey tree and Minkowski's Function

To describe the most irrational paths in the Farey tree we will need the *question mark function* introduced by Minkowski [57] and denoted by $?(x)$. It was studied later by Denjoy and by Salem (for history and references see [76]) and can be uniquely defined by the following properties:

- $?(0) = 0, \quad ?(1) = 1;$
- For Farey neighbours $\frac{a}{b}$, $\frac{c}{d}$, the value of the ?-function on their mediant is the arithmetic mean of corresponding values:

$$?\left(\frac{a+c}{b+d}\right) = \frac{1}{2}\left(?\left(\frac{a}{b}\right) + ?\left(\frac{c}{d}\right)\right); \tag{3.12}$$

- The ?-function is continuous on $[0, 1]$.

Definition 6 (Dyadic rational). *A reduced fraction $\frac{p}{q} \in \mathbb{Q}$ is called* dyadic rational *when $q = 2^n, n \in \mathbb{Z}$.*

Equivalently, let $\left\{\frac{p}{q}\right\}$ be the fractional part of $\frac{p}{q}$, with binary expansion

$$[0.\epsilon_1\epsilon_2\epsilon_3\ldots]_2, \quad \epsilon_i \in \{0, 1\}.$$

Then $\frac{p}{q}$ is dyadic rational if and only if this binary expansion is finite.

It can be shown that the Minkowski function has the following properties (see e.g. [76]):

- x is rational iff $?(x)$ has finite binary representation (i.e. iff $?(x)$ is dyadic rational);
- x is a quadratic irrational iff $?(x)$ is rational, but not dyadic rational;
- $?(x)$ is strictly increasing and defines a homeomorphism of $[0, 1]$ to itself;
- $?'(x) = 0$ almost everywhere.

Salem [65] gave a very convenient definition of $?(x)$ in terms of continued fractions. Namely, if x is given as a continued fraction

$$x = [a_1, a_2, \dots, a_n, \dots],$$

then

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \dots. \tag{3.13}$$

We claim that Minkowski's function $?(x)$ encodes the path γ leading to x on the Farey tree. More precisely, let γ_x be such a path for $x \in [0,1]$ and define the *path function* using binary representation as

$$\pi(x) = \pi(\gamma_x) := [0.\epsilon_1\epsilon_2 \dots \epsilon_j \dots]_2, \tag{3.14}$$

where

$$\epsilon_j = \begin{cases} 0 & \text{if the } j\text{th step of } \gamma_x \text{ is a right-turn;} \\ 1 & \text{if the } j\text{th step of } \gamma_x \text{ is a left-turn.} \end{cases} \tag{3.15}$$

For example, for the path γ in Figure 3.5 we have $\pi(\gamma) = [0.1010\dots]_2$.

Theorem 20. *The path function $\pi(x)$ is nothing other than Minkowski's question mark function.*

Proof. We simply check that $\pi(x)$ satisfies the defining properties of Minkowski's function. First, we have by definition that

$$\pi(0) = [0.0000\dots]_2 = 0, \quad \pi(1) = [0.1111\dots]_2 = 1,$$

and

$$\pi\left(\frac{1}{2}\right) = [0.1000\dots]_2 = \frac{1}{2} = \frac{0+1}{2} = \frac{\pi(0)+\pi(1)}{2}.$$

Now let us check that $\pi(x)$ satisfies the main property (3.12):

$$\pi\left(\frac{a+c}{b+d}\right) = \frac{1}{2}\left(\pi\left(\frac{a}{b}\right) + \pi\left(\frac{c}{d}\right)\right).$$

Let $\frac{a}{c}$ and $\frac{b}{d}$ be two Farey neighbours, and assume $\frac{a}{c} < \frac{b}{d}$. At every point on the Farey tree apart from $x = \frac{1}{2}$, either $\frac{a}{c}$ or $\frac{b}{d}$ is 'higher up' the Farey tree: the binary expansions will be of different lengths. There are two cases to consider.

Case 1: Assume that $\frac{b}{d}$ is 'higher up' the Farey tree than $\frac{a}{c}$ - i.e. $\frac{b}{d}$ has the longer binary expansion. Since $\frac{a}{c}$ and $\frac{b}{d}$ are neighbours, we know that

$$\pi\left(\frac{a}{c}\right) = [0.b_1b_2\dots b_n1]_2, \quad \pi\left(\frac{b}{d}\right) = [0.b_1b_2\dots b_n\beta_1\beta_2\dots\beta_k1]_2,$$

where $\beta_1\beta_2\dots\beta_k = [10\dots0]$. Then from the definition (3.15) of π we have

$$\pi\left(\frac{a+b}{c+d}\right) = [0.b_1b_2\dots b_n\beta_1\beta_2\dots\beta_k01]_2 = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_n}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+k+2}}$$

$$= \frac{1}{2}\left[\frac{b_1}{2}+\frac{b_2}{2^2}+\cdots+\frac{b_n}{2^n}+\frac{1}{2^{n+1}}\right]+\frac{1}{2}\left[\frac{b_1}{2}+\frac{b_2}{2^2}+\cdots+\frac{b_n}{2^n}+\frac{1}{2^{n+1}}+\frac{1}{2^{n+k+1}}\right]$$
$$= \frac{1}{2}\left(\pi\left(\frac{a}{c}\right)+\pi\left(\frac{b}{d}\right)\right).$$

Case 2: Now assume that $\frac{a}{c}$ has the longer binary expansion, so that

$$\pi\left(\frac{a}{c}\right)=[0.b_1b_2\dots b_n\beta_1\beta_2\dots\beta_k 1]_2,\quad \pi\left(\frac{b}{d}\right)=[0.b_1b_2\dots b_n 1]_2,$$

where $\beta_1\beta_2\dots\beta_k=[01\dots1]$. Again from (3.15) we have

$$\pi\left(\frac{a+b}{c+d}\right)=[0.b_1b_2\dots b_n\beta_1\beta_2\dots\beta_k 11]_2$$
$$=\frac{b_1}{2}+\frac{b_2}{2^2}+\cdots+\frac{b_n}{2^n}+\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+\cdots+\frac{1}{2^{n+k}}+\frac{1}{2^{n+k+1}}+\frac{1}{2^{n+k+2}}$$
$$=\frac{1}{2}\left[\frac{b_1}{2}+\frac{b_2}{2^2}+\cdots+\frac{b_n}{2^n}+\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+\cdots+\frac{1}{2^{n+k}}+\frac{1}{2^{n+k+1}}\right]$$
$$+\frac{1}{2}\left[\frac{b_1}{2}+\frac{b_2}{2^2}+\cdots+\frac{b_n}{2^n}+\frac{1}{2^{n+1}}\right]=\frac{1}{2}\left(\pi\left(\frac{a}{c}\right)+\pi\left(\frac{b}{d}\right)\right).$$

So in either case, we have

$$\pi\left(\frac{a+b}{c+d}\right)=\frac{1}{2}\left(\pi\left(\frac{a}{c}\right)+\pi\left(\frac{b}{d}\right)\right),$$

which means that π coincides with Minkowski function on all rational numbers. Since $\pi(x)$ is monotonic it must coincide with $?(x)$ for all $x\in[0,1]$. □

3.5 The Most Irrational Paths and Minkowski tree

Let $x=\alpha$ be quadratic irrational and assume that α has the pure periodic continued fraction expansion $\alpha=[\overline{a}]$ with even period $a=a_1,\dots,a_{2n}$ (if the period is odd we will double it to make even).

It follows from Salem's formula (3.13) that the value of Minkowski's function $?(\alpha)$ has a pure periodic binary representation

$$?(\alpha)=[0.\overline{A}]_2$$

with period A of length $a_1+\cdots+a_{2n}$ consisting of (a_1-1) 0's followed by a_2 1's, then followed by a_3 0's etc until we have a_{2n} 1's followed by one final 0.

For convenience we will drop the initial zero and write simply $[\overline{A}]_2$ instead of $[0.\overline{A}]_2$.

In particular, we have

$$?([\overline{1,1}])=[\overline{10}]_2,\ \ ?([\overline{2,2}])=[\overline{0110}]_2,\ \ ?([\overline{2,2,1,1}])=[\overline{011010}]_2.$$

Using Salem's representation we can prove the following conjunction property of Minkowski's function.

Proposition 3. *Let* $[\bar{a}] = [\overline{a_1, \ldots, a_{2n}}]$, $[\bar{b}] = [\overline{b_1, \ldots, b_{2m}}]$ *be two continued fractions of even periods and*

$$?([\bar{a}]) = [\overline{A}]_2, \quad ?([\bar{b}]) = [\overline{B}]_2.$$

Then

$$?([\overline{ab}]) = [\overline{AB}]_2. \tag{3.16}$$

Proof. Observe that

$$\begin{aligned}
?([\overline{ab}]) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \cdots - \frac{1}{2^{a_1+\ldots+a_{2n}-1}} \\
+\frac{1}{2^{a_1+\ldots+a_{2n}+b_1-1}} - \frac{1}{2^{a_1+\ldots+a_{2n}+b_1+b_2-1}} + \ldots - \frac{1}{2^{a_1+\ldots+a_{2n}+b_1+\ldots+b_{2m}-1}} \\
+\frac{1}{2^{a_1+\ldots+a_{2n}+b_1+\ldots+b_{2m}+a_1-1}} - \frac{1}{2^{a_1+\ldots+a_{2n}+b_1+\ldots+b_{2m}+a_1+a_2-1}} + \ldots \\
= [A]_2 + [\underbrace{0\ldots0}_{\sum a_i} B]_2 + [\underbrace{0\ldots0}_{\sum a_i+b_i} A]_2 + \ldots = [\overline{AB}]_2.
\end{aligned}$$

□

Thus Minkowski's $?(x)$ function maps the most irrational numbers to particular binary expansions, specifically those which mirror the continued fraction expansion of the most irrational numbers with "1,1" replaced by "10" and "2,2" replaced by "0110".

Applying Minkowski's function to the Markov-Hurwitz tree gives the *Minkowski tree*, encoding the paths to the most irrational numbers (see Figure 3.8, where i_k means i repeated k times).

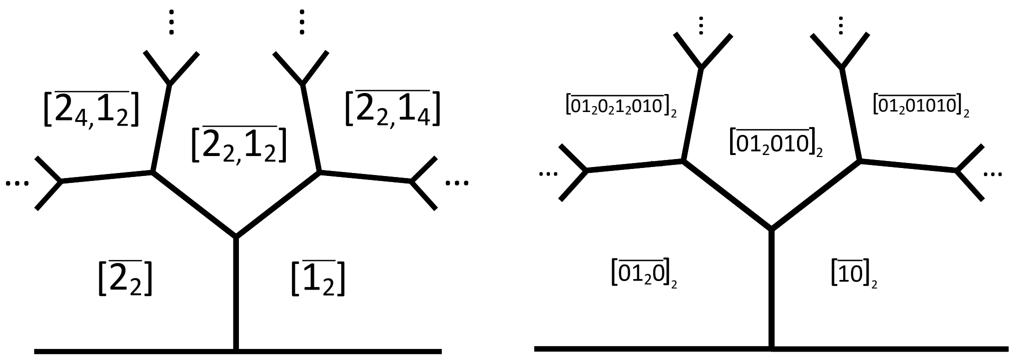


Figure 3.8: Markov-Hurwitz and Minkowski trees related by ?-function

3.6 Lyapunov exponents of most irrational paths.

Let $m\left(\frac{p}{q}\right) \in \mathcal{M}$ be the Markov number corresponding to the Farey fraction $\frac{p}{q} \in \frac{1}{2}$ (see Figure 3.2), and $x\left(\frac{p}{q}\right)$ be a representative of the corresponding class of the most irrational numbers.

It is convenient for us to choose such representatives to be the inverse of the corresponding number y_m from Markov-Hurwitz tree: $x_m = y_m^{-1} \in [0,1]$. We call these representatives *Markov-Hurwitz numbers* and denote by $\mathbb{X}$ the set of all these numbers

$$\mathbb{X} = \{x_m = y_m^{-1} : m \in \mathcal{M}\}.$$

Theorem 21. *The function $\Lambda\left(x\left(\frac{p}{q}\right)\right)$ is convex as a function of $\frac{p}{q} \in \mathbb{Q}$.*

The restriction $\Lambda_{\mathbb{X}}$ of $\Lambda(x)$ on the set of Markov-Hurwitz numbers $\mathbb{X}$ is monotonically increasing from

$$\Lambda(x_2) = \frac{1}{2}\ln(1+\sqrt{2}) \quad \textit{to} \quad \Lambda(x_1) = \ln\left(\frac{1+\sqrt{5}}{2}\right).$$

Proof. Following Fock [31], consider the following function $\psi(\xi)$, $\xi \in [0, \frac{1}{2}]$. First, define it for rational $\xi = \frac{p}{q} \in [0, \frac{1}{2}] \cap \mathbb{Q}$ as follows:

$$\psi\left(\frac{p}{q}\right) = \frac{1}{q}\operatorname{arcosh}\left(\frac{3}{2}m\left(\frac{p}{q}\right)\right). \tag{3.17}$$

Fock proved the following, crucial for us, result (see item 6 in Section 7.3 of [31]).

Theorem 22 (V. Fock [31])**.** *The function ψ can be extended to a continuous convex function of all $\xi \in \mathbb{R}$ with the property*

$$\psi(1-\xi) = \psi(\xi). \tag{3.18}$$

We claim that our function

$$\Lambda\left(x\left(\frac{p}{q}\right)\right) = \frac{1}{2}\psi\left(\frac{p}{q}\right)$$

is simply half of Fock's function.

Indeed, let x_m be a Markov-Hurwitz number and $?(x_m) = [\overline{a}]_2$, with $a = \epsilon_1, \ldots, \epsilon_{2q}$, be its image under Minkowski's function. It is a simple task to prove by induction that the length of the period $2q$ is exactly twice the denominator of the Farey fraction $\frac{p}{q}$ corresponding to m (see Figure 3.8).

Remark. It follows from McShane and Rivin [56] that Fock's function is differentiable at every irrational point and non-differential at every rational point (see [68]).

Let $A_m \in SL_2(\mathbb{N})$ be the matrix in the Farey tree obtained by following the path defined by a. The second key observation is the following:

Remark. A_m is nothing other than the Cohn matrix corresponding to m.

Indeed, for $x_1 = \frac{\sqrt{5}-1}{2} = [\overline{11}]$ we have $?(x_1) = [\overline{10}]_2$ and the corresponding matrix

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Similarly, for $x_2 = \sqrt{2} - 1 = [\overline{22}]$ we have $?(x_2) = [\overline{0110}]_2$ and

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}.$$

The general case follows from the conjunction rule for the Minkowski tree and the product rule for the Cohn tree.

This means that the Lyapunov exponent

$$\Lambda(x_m) = \frac{\ln \lambda(m)}{2q},$$

where $\lambda(m)$ is the largest eigenvalue of A_m.

But we know that the Cohn matrix A_m has the trace $3m$ and thus the characteristic equation

$$\lambda^2 - 3m\lambda + 1 = 0.$$

Thus the Lyapunov exponent is

$$\Lambda(x_m) = \frac{1}{2q} \ln\left(\frac{3m + \sqrt{9m^2 - 4}}{2}\right) = \frac{1}{2q} \operatorname{arcosh}\left(\frac{3m}{2}\right), \tag{3.19}$$

which is exactly half of the Fock function. This proves the convexity of $\Lambda\left(x\left(\frac{p}{q}\right)\right)$.

To prove the monotonicity we note first that the function $x\left(\frac{p}{q}\right)$ is monotonically decreasing, which follows from the conjunction construction of Markov-Hurwitz tree. Since Fock's function ψ is convex and satisfies (3.18) it has the minimum at $\xi = \frac{1}{2}$. This means that $\Lambda\left(x\left(\frac{p}{q}\right)\right)$ is monotonically decreasing when $\frac{p}{q} \in \left[0, \frac{1}{2}\right]$, and thus $\Lambda(x)$ is strictly increasing on $\mathbb{X}$. $\square$

3.7 Generalised Markov-Hurwitz sets

Part of Theorem 21 can be generalised to the following sets.

Let $a \in \mathbb{Z}_{>0}$ be an integer parameter and consider a version of the Hurwitz tree starting with the continued fractions $[\overline{2a_2}] = [\overline{2a, 2a}]$, $[\overline{a_2}] = [\overline{a, a}]$, and the corresponding version of the Minkowski tree growing from $[\overline{0_{2a-1}1_{2a}0}]_2 = [\overline{\underbrace{0\dots0}_{2a-1}\underbrace{1\dots1}_{2a}0}]_2$ and $[\overline{0_{a-1}1_a0}]_2 = [\overline{\underbrace{0\dots0}_{a-1}\underbrace{1\dots1}_{a}0}]_2$, where we continue to drop the initial zero as before (see Figure 3.9).

Let us denote by $\mathbb{X}_a$ the set of the inverses of the corresponding quadratic irrationals from this version of the Hurwitz tree. When $a = 1$ we have the set $\mathbb{X}_1 = \mathbb{X}$ considered before.

The corresponding version of the Cohn tree starts with the generalisation of Cohn matrices (3.6)

$$M_a = \begin{pmatrix} 1 - a + a^2 & a^2 \\ a & a+1 \end{pmatrix}, \quad M_{2a} = \begin{pmatrix} 1 - 2a + 4a^2 & 4a^2 \\ 2a & 2a+1 \end{pmatrix}. \tag{3.20}$$

Indeed, it is easy to check that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{2a-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{2a} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = M_a.$$

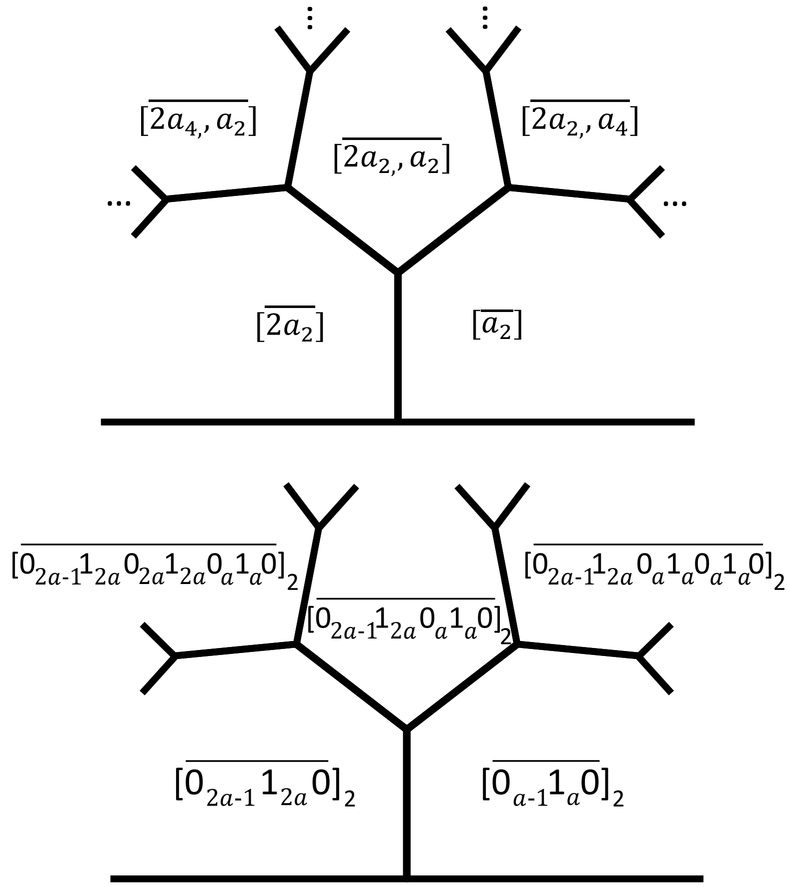


Figure 3.9: Generalised Markov-Hurwitz and Minkowski trees

The trace map $A \to \mathrm{tr}A$ produces the a-generalisation of Markov tree shown in Figure 3.10.

The corresponding triples are the integer solutions of the following version of the Markov equation studied by Mordell [58]

$$X^2 + Y^2 + Z^2 = XYZ + 4 - 4a^6. \tag{3.21}$$

Note that when $a = 1$ we have the equation

$$X^2 + Y^2 + Z^2 = XYZ, \tag{3.22}$$

which is simply a rescaled version of the Markov equation (3.1) and has integer solutions being Markov triples multiplied by 3:

$$X = 3x, \quad Y = 3y, \quad Z = 3z.$$

The modified equation (3.21) has no fully symmetric solutions, but has a solution with $X = Y$:

$$X = Y = a^2 + 2, \quad Z = 4a^2 + 2.$$

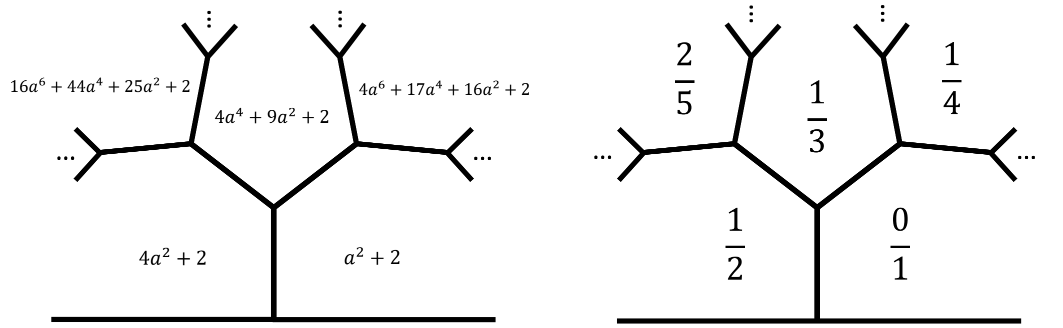


Figure 3.10: The a-generalisation of Markov tree and corresponding Farey fractions.

Applying to this solution the Vieta involution $(X, Y, Z) \to (X, Y, XY - Z)$ and permutations we have the generalised Markov tree above.

Let $A\left(a, \frac{p}{q}\right)$ be the matrix from the a-Cohn tree corresponding to the fraction $\frac{p}{q}$ from the Farey tree, $m\left(a, \frac{p}{q}\right) = \mathrm{tr} A\left(a, \frac{p}{q}\right)$ be the corresponding a-Markov number:

$$m\left(a, \frac{0}{1}\right) = a^2 + 2,$$
$$m\left(a, \frac{1}{2}\right) = 4a^2 + 2,$$
$$m\left(a, \frac{1}{3}\right) = 4a^4 + 9a^2 + 2, \quad \dots,$$

and let $y\left(a, \frac{p}{q}\right)$ be the corresponding quadratic irrational from the a-version of Markov-Hurwitz tree, $x\left(a, \frac{p}{q}\right) = y\left(a, \frac{p}{q}\right)^{-1}$. Note that, as in the previous case (see Remark at the end of Section 3.3), we have

$$y\left(a, \frac{p}{q}\right) = \mu\left(a, \frac{p}{q}\right) + 1,$$

where $v = \left(\mu\left(a, \frac{p}{q}\right), 1\right)^T$ is the eigenvector with the largest eigenvalue of the matrix $A\left(a, \frac{p}{q}\right)$.

The key observation is that on our set $\mathbb{X}_a$ the values of the Lyapunov function have the form

$$\Lambda\left(x\left(a, \frac{p}{q}\right)\right) = \frac{\ln \lambda\left(a, \frac{p}{q}\right)}{2aq}, \tag{3.23}$$

where

$$\lambda\left(a, \frac{p}{q}\right) = \frac{m + \sqrt{m^2 - 4}}{2}, \quad m = m\left(a, \frac{p}{q}\right)$$

is the largest eigenvalue of the Cohn matrix $A\left(a, \frac{p}{q}\right)$. The proof is a straightforward generalisation of the arguments from the previous section.

Geometrically the equation (3.21) describes the lengths of the simple closed geodesics on the equianharmonic hyperbolic torus with a hole (see e.g. [20, 31]) of length

$$l = \mathrm{arcosh}(2a^6 - 1).$$

This follows from the Fricke identities [32] : for any $A, B \in SL_2(\mathbb{R})$, $C = AB$ we have

$$tr\,AB + tr\,AB^{-1} = tr\,A\,tr\,B,$$

$$(tr\,A)^2 + (tr\,B)^2 + (tr\,C)^2 = tr\,A\,tr\,B\,tr\,C + tr\,(ABA^{-1}B^{-1}) + 2,$$

from which we see that $X = tr\,A$, $Y = tr\,B$, $Z = tr\,C$ satisfy (3.21) with

$$tr\,ABA^{-1}B^{-1} = 2 - 4a^6.$$

The matrices $A = M_a$ and $B = M_{2a}$ freely generate the Fuchsian subgroup G_a of $SL_2(\mathbb{R})$. The corresponding quotient of the upper half-plane is a hyperbolic torus with a hole. The length of the hole satisfies

$$2\cosh l = |tr\,ABA^{-1}B^{-1}| = 4a^6 - 2,$$

giving $l = \operatorname{arcosh}(2a^6 - 1)$. When $a = 1$ we have the punctured torus with $l = 0$ and the scaled version of the Markov equation.

Repeating the proof of Fock's theorem for the stable norm of the one-holed torus we have the following result:

Theorem 23. *The function* $\Lambda\left(x\left(a, \frac{p}{q}\right)\right)$ *is convex as a function of* $\frac{p}{q}$ *for all* $a \in \mathbb{N}$.

The restriction of Λ *to the set* $\mathbb{X}_a$ *is monotonically increasing.*

Chapter 4

Growth of Values of Binary Forms and Conway Rivers

In his book "The Sensual (Quadratic) Form" Conway [23] described a "topographic" way to visualise the values of a binary quadratic form

$$Q(x, y) = ax^2 + hxy + by^2, \quad (x, y) \in \mathbb{Z}^2. \tag{4.1}$$

Following Conway we define the *superbase* of the integer lattice $\mathbb{Z}^2$ as a triple $(\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3)$ such that $(\mathbf{e}_1, \mathbf{e}_2)$ is a basis of the lattice and

$$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \mathbf{0}.$$

We construct a binary planar tree starting from this superbase, such that each basis is identified with an edge and each superbase is identified with a vertex. It is easy to see that the edges and vertices which have some vector in common form a path, and we can therefore label each face by the common vector in the path which bounds it (see Figure 4.1). Note that all *primitive* lattice vectors, i.e. those which are not multiples of any other lattice vectors, appear on this tree.

By taking values of the form Q on the vectors of the superbase, we get what Conway called the *topograph* of Q containing the values of Q on all primitive lattice vectors. In particular, if $\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1), \mathbf{e}_3 = -(1, 1)$ we have the values

$$Q(\mathbf{e}_1) = a, \ Q(\mathbf{e}_2) = b, \ Q(\mathbf{e}_3) = c := a + b + h.$$

We can construct the topograph of Q starting from this triple using the following property, which Conway called the *arithmetic progression rule* and is known in geometry as the *parallelogram rule*:

$$Q(\mathbf{u} + \mathbf{v}) + Q(\mathbf{u} - \mathbf{v}) = 2(Q(\mathbf{u}) + Q(\mathbf{v})), \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^2. \tag{4.2}$$

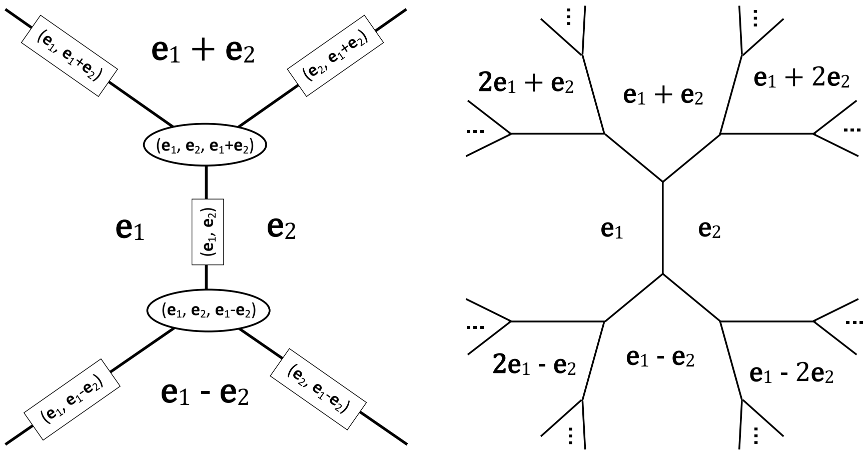


Figure 4.1: The superbase tree and its version with only faces labelled.

Note that quadratic forms (in any dimension) can be characterised as the degree 2 homogeneous functions satisfying relation (4.2). This leads to the following relation on the topograph of Q (see Figure 4.2), which can be used to compute the primitive values of Q recursively.

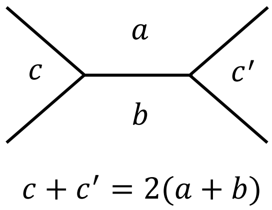


Figure 4.2: Arithmetic progression rule for values of quadratic forms.

For the standard quadratic form $Q(x,y) = x^2 + y^2$ we have the tree shown on the left of Figure 4.3. On the right side of Figure 4.3 we show the corresponding *Farey tree*, where at each vertex we have the fractions $\frac{p}{r}$, $\frac{q}{s}$ and its *Farey mediant* $\frac{p+q}{r+s}$ (see e.g. [39, 69]). As in the previous chapter, we use the Farey tree to identify the infinite paths γ on a binary tree with real numbers $\xi \in [0, \infty]$. For example, for the golden ratio $\xi = \varphi := \frac{\sqrt{5}+1}{2}$ we have the Fibonacci path shown in bold on both trees of Figure 4.3.

We would like to study the growth of the values of Q along the path γ_ξ. Modifying the arguments from the previous chapter, we define

$$\Lambda_Q(\xi) = \limsup_{n\to\infty} \frac{\ln |Q_n(\xi)|}{n}, \tag{4.3}$$

where

$$|Q_n(\xi)| = \max(|a_n(\xi)|, |b_n(\xi)|, |c_n(\xi)|)$$

with $(a_n(\xi), b_n(\xi), c_n(\xi))$ being the n-th triple on the path γ_ξ.

For example, for the Fibonacci path with $\xi = \varphi$ we have $q_n = F_{2n}$ being every second Fibonacci number with the growth

$$\limsup_{n\to\infty} \frac{\ln |Q_n(\xi)|}{n} = \lim_{n\to\infty} \frac{\ln F_{2n}}{n} = \ln \varphi^2 = 2\ln\varphi.$$

We will show that a similar result is true for any positive binary quadratic form Q, namely that

$$\Lambda_Q(\xi) = 2\Lambda(\xi), \tag{4.4}$$

where $\Lambda(\xi)$ is the function introduced in Chapter 3 describing the growth of the Markov numbers, or, equivalently, the growth of the monoid $SL_2(\mathbb{N})$.

Recall that the function $\Lambda(\xi)$ can be extended to $\xi \in \mathbb{R}P^1$ and has very peculiar properties: it is discontinuous everywhere, $GL_2(\mathbb{Z})$-invariant and takes all real values from $[0, \ln\varphi]$.

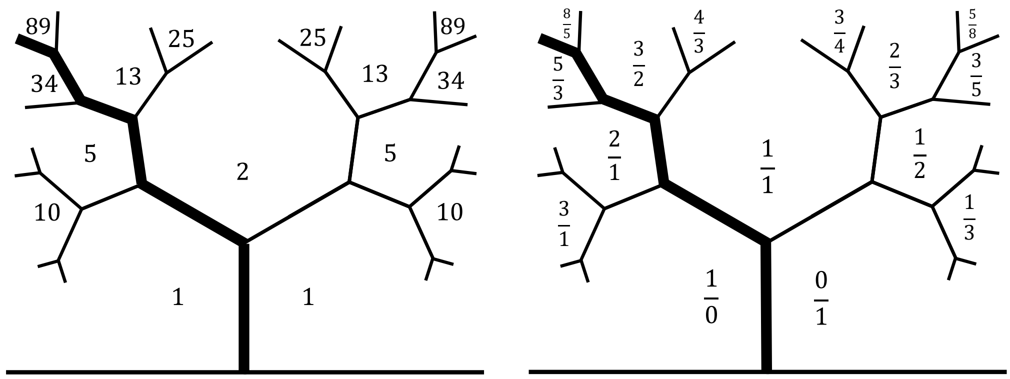


Figure 4.3: Topograph of $Q = x^2 + y^2$ and the corresponding Farey tree with marked “golden” Fibonacci path.

The situation is different for indefinite binary quadratic forms, due to the existence of what Conway [23] called the *river* (see Figure 4.4).

Definition 7 (Conway river)**.** *Let $Q(x, y) = ax^2 + hxy + by^2$, $a, h, b \in \mathbb{Z}$, be an indefinite binary quadratic form such that $Q(x, y) \neq 0$ for all $(x, y) \in \mathbb{Z}^2$. Then the topograph contains a unique infinite path separating positive and negative values in the tree. This path is called the Conway river.*

When a, b, h are integers, this river is periodic both in the sense of its left-right sequence as a path, and the values on either side of it - although it should be noted that the two periods on either side of the river need not be the same.

Let $\alpha_\pm$ be the two real roots of the corresponding quadratic equation

$$Q(\alpha, 1) = 0.$$

The main result of this chapter says that for an indefinite form Q not representing zero

$$\Lambda_Q(\xi) = 2\Lambda(\xi), \quad \xi \neq \alpha_\pm \tag{4.5}$$

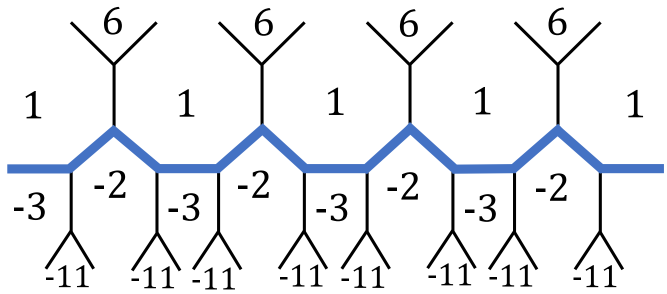


Figure 4.4: Conway river for the indefinite binary quadratic form $Q = x^2 - 2xy - 2y^2$.

with $\Lambda_Q(\alpha_\pm) = 0 \neq 2\Lambda(\alpha_\pm)$. We will show that the two exceptional paths with zero growth are exactly those leading to the two ends of the Conway river.

We will also discuss in more detail the geometry of the corresponding exceptional paths in relation to the continued fraction expansions of the quadratic irrationals $\alpha_\pm$. The Galois result about pure periodic continued fractions naturally appears in this way.

In the case when the indefinite form Q does represent zero (which means that its discriminant is total square) the roots $\alpha_\pm$ are rational, so $\Lambda(\alpha_\pm) = 0$ and $\Lambda_Q(\xi) = 2\Lambda(\xi)$ for all ξ in this case.

Finally, in the case of semidefinite forms Q, the growth $\Lambda_Q(\xi) \equiv 0$.

4.1 Paths in binary trees and the Galois theorem

Let us first describe in more detail the correspondence between $\xi \in [0, \infty]$ and paths γ in the planar binary rooted tree shown on the right of Figure 4.3. Indeed, the corresponding Farey tree has a unique path γ_ξ for any irrational positive ξ such that the limit of the adjacent fractions is ξ (for rational ξ such a path is finite and leads to the corresponding fraction).

In terms of the corresponding continued fraction expansion

$$\xi = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ddots}}} = [c_0, c_1, c_2, c_3 \ldots], \quad c_0 \geq 0, c_i > 0,$$

the path γ_ξ starts from the root and can be described as c_0 left-turns on the tree, followed by c_1 right-turns, followed by c_2 left-turns, and so on.

For every oriented edge E adjacent to two Farey fractions $\frac{a}{c}$, $\frac{b}{d}$ we assign the unimodular matrix

$$A_E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the corresponding matrix $A_n(\gamma)$, $\gamma = \gamma_\xi$ is the product of the first n matrices along the path γ:

$$A_n(\gamma) = L^{c_0} R^{c_1} L^{c_2} R^{c_3} \dots \tag{4.6}$$

where

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The matrices L and R freely generate the monoid $SL_2(\mathbb{N})$, which is the positive part of the group $SL_2(\mathbb{Z})$, acting on the planar binary rooted tree by left and right turns respectively.

We would like to extend this correspondence to the full binary planar tree, as in the left side of Figure 4.5.

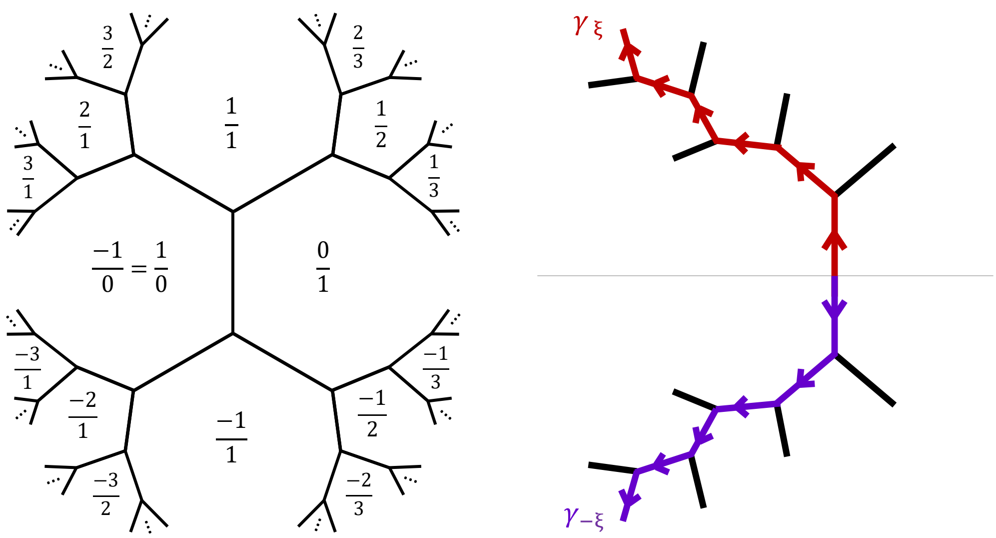


Figure 4.5: Full Farey tree and reflected path $\gamma_{-\xi}$

In particular, we see from this figure that for the reflected path $\bar{\gamma} = \gamma_{-\xi}$ the corresponding matrix is

$$A_n(\bar{\gamma}) = L^{-c_0} R^{-c_1} L^{-c_2} R^{-c_3} \dots,$$

where $A_n(\gamma)$ is given by (4.6) and for an edge E in the bottom half of the Farey tree with two adjacent Farey fractions $\frac{a}{c}$, $\frac{b}{d}$ we assign the matrix

$$A_E = \begin{pmatrix} -a & b \\ -c & d \end{pmatrix}.$$

To be consistent with this it is natural to consider for negative real numbers

$$\eta = -\xi < 0, \quad \xi = [c_0, c_1, c_2, c_3 \dots], \quad c_0 \geq 0, c_i > 0$$

the *negative continued fraction expansions*

$$\eta = -[c_0, c_1, c_2, c_3 \dots] = [-c_0, -c_1, -c_2, -c_3 \dots]. \tag{4.7}$$

Note that it is always possible to avoid negative c_i with $i > 0$ because of the identity

$$-[c_0, c_1, c_2, c_3, \ldots] = [-c_0 - 1, 1, c_1 - 1, c_2, c_3, \ldots], \tag{4.8}$$

but we are not going to do this.

The famous theorem of Lagrange [52] says that any quadratic irrational has a periodic continued fraction expansion

$$\alpha = [a_0, \ldots, a_k, \overline{b_1, \ldots, b_l}],$$

and conversely, any eventually periodic continued fraction represents a quadratic irrational.

Less well-known is the following addition due to Galois (see e.g. [52]) characterising *pure periodic continued fraction expansions*

$$\alpha = [\overline{b_1, \ldots, b_l}].$$

Theorem 24 (Galois)**.** *A quadratic irrational* $\alpha = \frac{A \pm \sqrt{D}}{B}$ *has a pure periodic continued fraction expansion*

$$\alpha = [\overline{b_1, \ldots, b_l}]$$

if and only if its conjugate $\bar{\alpha} = \frac{A \mp \sqrt{D}}{B}$ *satisfies the inequality*

$$-1 < \bar{\alpha} < 0.$$

Moreover, in that case

$$\bar{\alpha} = -[0, \overline{b_l, \ldots, b_1}].$$

As we will see later, geometrically the conjugate $\bar{\alpha}$ determines the path going backwards along the corresponding Conway river.

4.2 Quadratic irrational and conjugate expansions

To describe our main result we need an answer to the following natural question: assume that we know the continued fraction expansion of a quadratic irrational

$$\alpha = [a_0, \ldots, a_k, \overline{b_1, \ldots, b_l}]. \tag{4.9}$$

What is the continued fraction expansion of its conjugate $\bar{\alpha}$?

For example, we have

$$\alpha = \frac{6 + \sqrt{2}}{17} = [0, 2, 3, \overline{2}], \quad \bar{\alpha} = \frac{6 - \sqrt{2}}{17} = [0, 3, 1, \overline{2}]. \tag{4.10}$$

What is the general rule here?

We could not find the answer to this question in the literature, except in the Galois case, so we present it in this section.

Note first that we can assume that in (4.9) $a_k \neq b_l$, because otherwise α can be rewritten with the period $[\overline{b_l, b_1, \ldots, b_{l-1}}]$.

Proposition 4. *Let $\alpha = [a_0, a_1, \dots, a_k, \overline{b_1, \dots, b_l}]$ be the continued fraction expansion of a quadratic irrational with $a_k < b_l$, and $k \geq 1$. Then the continued fraction expansion of its conjugate is*

$$\overline{\alpha} = [a_0, \dots, a_{k-1} - 1, 1, b_l - a_k - 1, \overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}]. \tag{4.11}$$

If $a_k > b_l$, $k \geq 1$, then

$$\overline{\alpha} = \left[a_0, \dots, a_{k-1}, a_k - b_l - 1, 1, b_{l-1} - 1, \overline{b_{l-2}, b_{l-3}, \dots, b_1, b_l, b_{l-1}}\right]. \tag{4.12}$$

Here it will be convenient for us to allow zeros in the continued fraction expansion, which can be always avoided using the following identities:

$$[c_0, \dots, c_i, 0, c_{i+1}, c_{i+2}, c_{i+3} \dots] = [c_0, \dots, c_i + c_{i+1}, c_{i+2}, c_{i+3} \dots], \tag{4.13}$$

$$[c_0, \dots, c_i, 0, 0, c_{i+1}, c_{i+2}, c_{i+3} \dots] = [c_0, \dots, c_i, c_{i+1}, c_{i+2}, c_{i+3} \dots]. \tag{4.14}$$

In particular, in example (4.10) the formula (4.12) gives for $\alpha = [0, 2, 3, \overline{2}]$

$$\bar{\alpha} = [0, 2, 0, 1, 1, \bar{2}] = [0, 3, 1, \overline{2}].$$

A more sophisticated example:

$$\alpha = \frac{11523 + \sqrt{15006}}{9222} = [1, 3, 1, 4, \overline{7, 2, 3, 9}],$$

$$\bar{\alpha} = \frac{11523 - \sqrt{15006}}{9222} = [1, 3, 0, 1, 4, \overline{3, 2, 7, 9}] = [1, 4, 4, \overline{3, 2, 7, 9}].$$

Proof. It is enough to consider the case $k = 1$. Let us first define

$$\beta = \left[\overline{b_1, \dots, b_{l-1}, b_l}\right], \quad \tilde{\beta} = \left[\overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}\right].$$

Then, by definition,

$$\alpha = \left[a_0, a_1, \overline{b_1, \dots, b_l}\right] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\beta}},$$

so that

$$\overline{\alpha} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\overline{\beta}}} := [a_0, a_1, \overline{\beta}].$$

Using the Galois result and identity (4.8) we have

$$\overline{\beta} = -\left[0, \overline{b_l, \dots, b_1}\right] = \left[-1, 1, b_l - 1, \tilde{\beta}\right].$$

where $\tilde{\beta} = [\overline{b_{l-1}, \dots, b_1, b_l}]$.

We can now directly compute

$$
\begin{aligned}
\overline{\alpha} = [a_0, a_1, \overline{\beta}] &= \left[a_0, a_1, -1, 1, b_l - 1, \tilde{\beta}\right] \\
&= \left[a_0, a_1, -1, 1 + \frac{\tilde{\beta}}{(b_l - 1)\,\tilde{\beta} + 1}\right] \\
&= \left[a_0, a_1, -1 + \frac{(b_l - 1)\,\tilde{\beta} + 1}{b_l\tilde{\beta} + 1}\right] \\
&= \left[a_0, a_1 + \frac{b_l\tilde{\beta} + 1}{-\tilde{\beta}}\right] = a_0 - \frac{\tilde{\beta}}{(b_l - a_1)\,\tilde{\beta} + 1} \\
&= a_0 - 1 + \cfrac{1}{\cfrac{(b_l - a_1)\,\tilde{\beta} + 1}{\tilde{\beta}\,(b_l - a_1 - 1) + 1}} \\
&= a_0 - 1 + \cfrac{1}{1 + \cfrac{\tilde{\beta}}{\tilde{\beta}\,(b_l - a_1 - 1) + 1}} \\
&= a_0 - 1 + \cfrac{1}{1 + \cfrac{1}{(b_l - a_1 - 1) + \cfrac{1}{\tilde{\beta}}}} \\
&= \left[a_0 - 1, 1, b_l - a_1 - 1, \overline{b_{l-1}, \ldots, b_0, b_l}\right].
\end{aligned}
$$

This completes the case $a_k < b_l$. For the case $a_k > b_l$ we have by (4.13),

$$
\alpha = \left[a_0, \ldots, a_k - b_l, 0, \overline{b_l, b_1, \ldots, b_{l-1}}\right],
$$

and the result then immediately follows from the previous case with a_k replaced by 0 and a_{k-1} replaced by $a_k - b_l$. $\square$

Note that formulae (4.11), (4.12) determine involutions since for $a_k < b_l$ we have

$$
\begin{aligned}
\overline{\overline{\alpha}} &= \left[a_0, \ldots, a_{k-1} - 1, 1 - 1, 1, b_l - (b_l - a_k - 1) - 1, \overline{b_1, \ldots, b_{l-1}, b_l}\right] \\
&= \left[a_0, \ldots, a_{k-1} - 1, 0, 1, a_k, \overline{b_1, \ldots, b_{l-1}, b_l}\right] \\
&= \left[a_0, \ldots, a_{k-1}, a_k, \overline{b_1, \ldots, b_{l-1}, b_l}\right] = \alpha,
\end{aligned}
$$

while otherwise

$$
\begin{aligned}
\overline{\overline{\alpha}} &= \left[a_0, \ldots, a_{k-1}, a_k - b_l - 1, 1 - 1, 1, b_{l-1} - 1 - (b_{l-1} - 1), \overline{b_l, b_1, \ldots, b_{l-1}}\right] \\
&= \left[a_0, \ldots, a_{k-1}, a_k - b_l - 1, 0, 1, 0, \overline{b_l, b_1, \ldots, b_{l-1}}\right] \\
&= \left[a_0, \ldots, a_{k-1}, a_k - b_l, 0, \overline{b_l, b_1, \ldots, b_{l-1}}\right] = \alpha.
\end{aligned}
$$

Let us consider now the special case $k = 0$.

Proposition 5. *Let $\alpha = [a_0, \overline{b_1, \ldots, b_l}]$ with $a_0 < b_l$. Then the conjugate $\overline{\alpha}$ can be given as the negative continued fraction expansion*

$$\overline{\alpha} = -\left[b_l - a_0, \overline{b_l, \ldots, b_1}\right]. \tag{4.15}$$

When $a_0 > b_l$ we have

$$\overline{\alpha} = [a_0 - b_l - 1, 1, b_l - 1, \overline{b_{l-1}, b_{l-2}, \ldots, b_1, b_l}]. \tag{4.16}$$

Proof. We have

$$\left[a_0, \overline{b_1, \ldots, b_n}\right] = a_0 + \left[0, \overline{b_0, \ldots, b_n}\right],$$

so by the Galois result

$$\begin{aligned}\overline{a_0 + \left[0, \overline{b_1, \ldots, b_l}\right]} &= a_0 - \left[\overline{b_l, \ldots, b_1}\right] \\ &= -\left[b_l - a_0, \overline{b_{l-1}, \ldots, b_1, b_l}\right].\end{aligned}$$

In the case when $a_0 > b_l$, we can rewrite

$$\left[a_0, \overline{b_1, \ldots, b_l}\right] = \left[a_0 - b_l, 0, \overline{b_l, b_1 \ldots, b_{l-1}}\right]$$

and proceed as in the case when $k \geq 1$ to get the formula (4.16). □

Note that only in the Galois pure periodic case and in the case with $k = 0, a_0 < b_l$ do we have to use negative continued fractions. We will see now that these are the only cases when the initial position is on the Conway river.

4.3 Paths to Conway river

Let

$$Q(x, y) = ax^2 + hxy + by^2$$

be an indefinite binary quadratic form not representing zero, by which we mean that $Q(x, y) \neq 0$ for all $(x, y) \in \mathbb{Z}^2 \setminus (0, 0)$. Equivalently, this means that the discriminant of the form

$$D = h^2 - 4ab$$

is positive, but not a total square.

In this case Conway [23] showed that on the topograph of Q positive and negative values are separated by a unique periodic river, as in Figure 4.4.

Let V_Q be the initial vertex on the topograph with $a = Q(1, 0)$, $b = Q(0, 1)$, $c = Q(1, 1) = a + b + h$. We will now explain how the continued fractions determine the path from V_Q to the Conway river, and the relation with the Galois result.

Let us assume for simplicity that all the values a, b, c are of the same sign (say, positive), otherwise V_Q is already on the river. Let us assume also that $h < 0$.

Let $\alpha, \bar{\alpha}$ be the real roots of the quadratic equation

$$Q(\alpha, 1) = a\alpha^2 + h\alpha + b = 0. \tag{4.17}$$

Again for simplicity let us assume that

$$\alpha = \frac{-h + \sqrt{D}}{2a}, \quad D = h^2 - 4ab$$

is the dominant (maximal modulus) root. Let

$$\alpha = [a_0, a_1, \dots, a_k, \overline{b_1, \dots, b_l}] \tag{4.18}$$

be the continued fraction expansion of α.

From the general theory of the Farey tree and continued fractions (see, for example, [39]) we have the following result.

Proposition 6. *The continued fraction expansion (4.18) describes the unique path γ_α going from V_Q to the corresponding Conway river, and then along the river towards α. A similar path $\gamma_{\bar{\alpha}}$ leading to $\bar{\alpha}$ is described by Propositions 4 and 5 above.*

The Conway river of Q is an infinite in both directions path from $\overline{\alpha}$ to α described by the periodic part $\boxed{b_1, \dots, b_l}$ of the expansion.

The finite path π from V_Q to the Conway river can be given for $k \geq 1$ by

$$\pi = \begin{cases} [a_0, \dots, a_k - b_l - 1] & \text{if } a_k > b_l \\ [a_0, \dots, a_{k-1} - 1] & \text{if } a_k < b_l \end{cases}$$

and for $k = 0$ by

$$\pi = \begin{cases} a_0 - b_l & \text{if } a_0 > b_l \\ \emptyset & \text{if } a_0 < b_l \end{cases}.$$

This also gives a geometric interpretation of our formulas from the previous section, which is illustrated in Figure 4.6 in the example of $Q = 17x^2 - 12xy + 2y^2$ with

$$\alpha = \frac{6 + \sqrt{2}}{17} = [0, 2, 3, \overline{2}], \quad \bar{\alpha} = \frac{6 - \sqrt{2}}{17} = [0, 3, 1, \overline{2}].$$

Let us now describe the special Galois case when α is a pure periodic continued fraction.

Let us call the quadratic form $Q(x, y) = ax^2 + hxy + by^2$ a *Galois form* if the continued fraction expansion of the dominant root of $Q(\alpha, 1) = 0$ is pure periodic. By the Galois theorem this is equivalent to the conditions $\alpha > 1$, $-1 < \bar{\alpha} < 0$.

Proposition 7. *A binary quadratic form Q is Galois if and only if $Q(1, 0) = a$, $Q(0, 1) = b$, $Q(1, 1) = c$ satisfy*

$$\begin{aligned} &ab < 0, \\ &ac < 0, \\ &ac' = a(2a + 2b - c) > 0. \end{aligned}$$

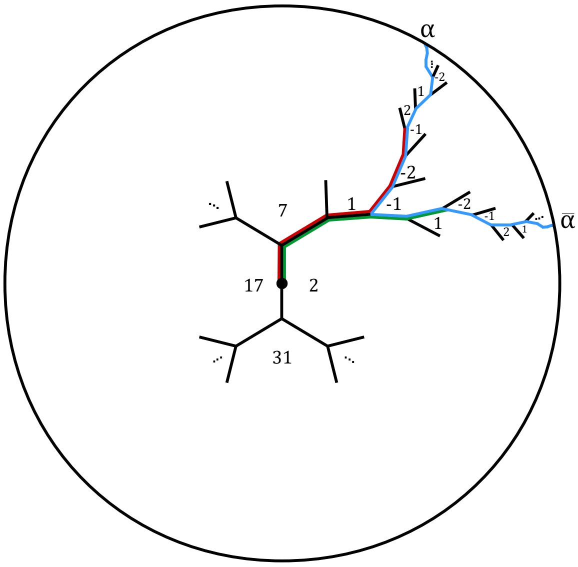


Figure 4.6: Paths to α and $\bar{\alpha}$ and Conway river for $Q = 17x^2 - 12xy + 2y^2$

The proof is obvious geometrically: the conditions $\alpha > 1$ and $-1 < \overline{\alpha} < 0$ are true if and only if the Conway river travels along the edges separating a and c' from b and c, as in Figure 4.7.

An example of a Galois form is the "golden" form $Q = x^2 - xy - y^2$ corresponding to $a = c' = 1, b = c = -1$ with

$$\alpha = \varphi = \frac{1+\sqrt{5}}{2} = [\bar{1}],\ \bar{\alpha} = \frac{1-\sqrt{5}}{2} = -[0, \bar{1}].$$

4.4 Lyapunov exponents for values of binary forms

Now we are ready to state and prove our main results.

Let $Q(x, y) = ax^2 + hxy + by^2$ be a binary quadratic form (definite or indefinite).

If the form Q is indefinite, then we first assume that Q does not represent zero in the sense that $Q(x, y) \neq 0$ for all $(x, y) \in \mathbb{Z}^2 \setminus (0, 0)$. In that case the two roots $\alpha_{\pm} = \alpha, \bar{\alpha}$ of the quadratic equation

$$Q(\xi, 1) = 0$$

are real quadratic irrationals, corresponding to the limits of the Conway river.

To study the growth of the values of Q along the path γ_ξ we define the corresponding Lyapunov exponent as

$$\Lambda_Q(\xi) = \limsup_{n\to\infty} \frac{\ln |Q_n(\xi)|}{n}, \tag{4.19}$$

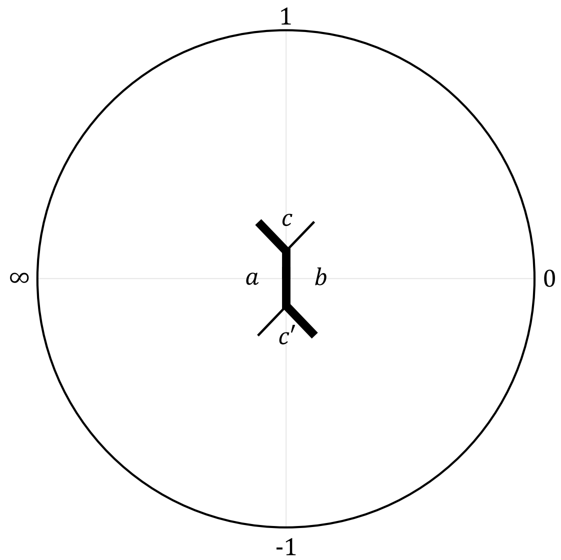


Figure 4.7: Galois form Q on the Conway river

where

$$|Q_n(\xi)| = \max(|a_n(\xi)|, |b_n(\xi)|, |c_n(\xi)|)$$

and $a_n(\xi), b_n(\xi), c_n(\xi)$ are the values of Q at the n-th superbase on the path γ_ξ.

Let $\Lambda(\xi)$ be the Lyapunov exponent of the monoid $SL_2(\mathbb{N})$ introduced in [69]:

$$\Lambda(\xi) = \limsup_{n\to\infty} \frac{\ln \rho(A_n(\xi))}{n}, \tag{4.20}$$

where $A_n(\xi) = A_n(\gamma_\xi)$ are the matrices (4.6) and $\rho(A)$ is the spectral radius of the matrix A.

Theorem 25. *For the definite binary quadratic form Q the Lyapunov exponent*

$$\Lambda_Q(\xi) = 2\Lambda(\xi).$$

For the indefinite binary quadratic form Q not representing 0 we have

$$\Lambda_Q(\xi) = \begin{cases} 2\Lambda(\xi) & \text{if } \xi \neq \alpha_\pm \\ 0 & \text{if } \xi = \alpha_\pm \end{cases}$$

In other words, the only two exceptional paths with zero growth are those leading to the two ends of the Conway river of Q.

Proof. Let us introduce first for the form $Q(x, y) = ax^2 + hxy + by^2$ its matrix defined by

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{h}{2} \\ \frac{h}{2} & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which, slightly abusing the notation, we also denote by Q:

$$Q = \begin{pmatrix} a & \frac{h}{2} \\ \frac{h}{2} & b \end{pmatrix}.$$

Its determinant

$$\det Q = ab - \frac{h^2}{4} = -\frac{D}{4}$$

is minus a quarter of the discriminant $D = h^2 - 4ab$ of the quadratic equation $Q(\xi, 1) = 0$.

The action of the group $SL_2(\mathbb{Z})$ on Q is defined in the matrix form by

$$Q \to Q' = A^t Q A,$$

where

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{4.21}$$

This determines a 3-dimensional representation $A \to \hat{A}$ of $SL_2(\mathbb{Z})$, when A acts on the coefficients (a, h, b) of the form Q by

$$\hat{A} = \begin{pmatrix} p^2 & 2pr & r^2 \\ pq & pr + qs & rs \\ q^2 & 2qs & s^2 \end{pmatrix}.$$

If the eigenvalues of the matrix A are λ and λ^{-1}, then the eigenvalues of $\hat{A}$ are λ^2, λ^{-2} and 1.

Consider first the case when the coefficients of the form a, b, h are all positive. In that case Conway's Climbing Lemma [23] guarantees the permanent growth whichever upward path we choose (see Figure 4.8, in which arrows show the direction of growth in the topograph).

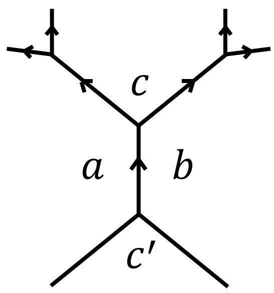


Figure 4.8: Climbing Lemma

Our claim is that the corresponding growth is given by $\Lambda_Q(\xi) = 2\Lambda(\xi)$.

To prove this, we introduce two norms on binary forms $Q = ax^2 + hxy + by^2$. The first norm is the one used in the definition of Λ_Q:

$$|Q| = \max(|a|, |b|, |c|), \; c = a + b + h. \tag{4.22}$$

The second norm is

$$|Q|_h = \max(|a|, |b|, |h|). \tag{4.23}$$

Since any two norms in a finite-dimensional space are equivalent, we have for all Q

$$c_1 |Q|_h \le |Q| \le c_2 |Q|_h$$

for some positive constants c_1, c_2. This means that we can replace $|Q|$ by $|Q|_h$ in the definition

$$\Lambda_Q(\xi) = \limsup_{n\to\infty} \frac{\ln|Q_n(\xi)|}{n} = \limsup_{n\to\infty} \frac{\ln|Q_n(\xi)|_h}{n}.$$

Now let γ_ξ be a path and

$$A_n(\xi) = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} \in SL_2(\mathbb{Z})$$

be the corresponding matrices (4.6), so that the matrices of the forms along the path are

$$Q_n(\xi) = A_n(\xi)^T Q A_n(\xi),$$

or, explicitly

$$\begin{pmatrix} a_n \\ h_n \\ b_n \end{pmatrix} = \begin{pmatrix} p_n^2 & 2p_n r_n & r_n^2 \\ p_n q_n & p_n r_n + q_n s_n & r_n s_n \\ q_n^2 & 2q_n s_n & s_n^2 \end{pmatrix} \begin{pmatrix} a \\ h \\ b \end{pmatrix}. \tag{4.24}$$

Recall also that the function $\Lambda(\xi)$ can be alternatively defined as

$$\Lambda(\xi) = \limsup_{n\to\infty} \frac{\ln w_n(x)}{n},$$

where $w_n = r_n + s_n$ (see the proof of Proposition 1).

As before, we will assume without loss of generality that $\xi \in [0,1]$ and consider the cases $\xi = 0$ and $\xi > 0$ separately. When $\xi = 0$ we have

$$A_n(0) = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

which determines the quadratic growth of $|Q_n|_h$ in n, and as a result

$$\Lambda_Q(0) = 0 = 2\Lambda(0).$$

If $0 < \xi \le 1$ we have $p_n \le r_n, q_n \le s_n$ and thus from (4.24)

$$\begin{aligned} |Q_n(\xi)|_h &\le \max((p_n + r_n)^2, (p_n + s_n)(q_n + r_n), (q_n + s_n)^2)|Q|_h \\ &\le \max(4r_n^2, (r_n + s_n)^2, 4s_n^2)|Q|_h \\ &\le 4(r_n + s_n)^2|Q|_h \\ &= 4w_n^2|Q|_h. \end{aligned}$$

On the other hand, by assumption the initial a, b, h are all positive integers, so $a, b, h \ge 1$. From (4.24) then it follows that

$$\begin{aligned} |Q_n(\xi)|_h &\ge (p_n + r_n)^2 + (p_n + s_n)(q_n + r_n) + (q_n + s_n)^2 \\ &> r_n^2 + s_n^2 \\ &\ge \frac{1}{2}(r_n + s_n)^2 \\ &= \frac{1}{2}w_n^2. \end{aligned}$$

Thus we have

$$\frac{1}{2}w_n^2 \leq |Q_n(\xi)|_h \leq 4w_n^2|Q|_h,$$

which implies that

$$\Lambda_Q(\xi) = \limsup_{n\to\infty} \frac{\ln|Q_n(x)|}{n} = \limsup_{n\to\infty} \frac{\ln w_n^2(x)}{n} = 2\Lambda(\xi).$$

This completes the proof in the growing case. Assume now that the values a, b, c may decrease as we move along the path.

Consider first the case when the form Q is positive definite. Then all the values are positive, so it is clear that for any path at some point the growth will start again and we can repeat our arguments to get the claim.

However, if Q is indefinite this is no longer true, since Q can take negative values as well. By Conway's result [23] positive and negative values of Q are separated by the infinite periodic river.

There are three possibilities: either the path does not cross the river, it crosses the river (or starts on the river and then leaves it), or it is after some point stuck on the river forever.

If the path does not cross the river the values of Q will remain positive and thus bounded from below, so at some point we will have growth and we repeat the arguments to prove the claim in this case as well.

If the path crosses the river then at some point all the values of a_n, b_n, c_n will become negative, and we can repeat the arguments for $-Q$ to get the claim. If we start on the river, then after the point of departure we can use the same arguments as before.

Finally, there are exactly two paths which are stuck on the river, corresponding to $\xi = \alpha_\pm$. In that case we have no growth because of the periodicity of the Conway river [23] and thus

$$\Lambda_Q(\alpha_\pm) = 0.$$

Note that the corresponding $\Lambda(\alpha_\pm) \neq 0$, so $\Lambda_Q(\xi) \neq 2\Lambda(\xi)$ in that case. □

Let us now discuss the remaining case: indefinite forms representing zero and semidefinite forms with zero discriminant.

Proposition 8. *For indefinite forms Q representing zero*

$$\Lambda_Q(\xi) = 2\Lambda(\xi), \ \xi \in \mathbb{R}P^1,$$

while for the semidefinite forms

$$\Lambda_Q(\xi) \equiv 0.$$

Proof. The topograph of the indefinite forms representing zero (with discriminant D being total square) is described in Conway [23]. In that case we have two "lakes" corresponding to zero values connected by a finite river separating positive and negative values of Q. In exceptional cases the river disappears and the two lakes are adjacent (see the First Lecture in [23]).

This picture agrees with the fact that the roots $\alpha_\pm$ of the corresponding equation

$$Q(\alpha, 1) = 0$$

are rational. A simple analysis shows that the previous arguments in this case work as well, with the only difference being that now both $\Lambda_Q(\alpha_\pm)$ and $\Lambda(\alpha_\pm)$ are zero, and thus the equality $\Lambda_Q(\xi) = 2\Lambda(\xi)$ holds in this case as well.

In the semidefinite case equivalent to the case $Q = ax^2$ we have zero growth, which simply follows from the Conway description [23]. $\square$

4.5 Conway river and the Arnold sail

We will now explain a simple connection between the Conway river and the Arnold sail from the geometric theory of continued fractions, following [72].

In 1895, Felix Klein proposed the following geometric representation of continued fractions: draw the ray $\frac{x}{y} = \omega$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$, in the two-dimensional integer point lattice. By marking the integer points whose coordinates are the convergents for ω, we see that the rays to these points approximate the ω-ray increasingly well. Klein then invites us to:

> Imagine pegs or needles affixed at all the integral points, and wrap a tightly drawn string about the sets of pegs to the right and to the left of the ω-ray, then the vertices of the two convex strong-polygons which bound our two point sets will be precisely the points (p_ν, q_ν) whose coordinates are the numerators and denominators of the successive convergents to ω, the left polygon having the even convergents, the right one the odd.

This perspective was later revitalised by Arnold [5] with an emphasis on multi-dimensional generalisations. In particular, for a polyhedral cone he introduced the notion of the *sail* as the boundary of the convex hull of the integer points inside it. In two dimensions, the sail of the angle formed by the ω-ray and x-axis is precisely Klein's construction of the continued fraction expansion of ω (see Figure 4.9).

Karpenkov [45] developed this work in more detail by introducing the *lattice length sine* or *LLS* sequence: let the sail for some arbitrary angle ω be a broken line with sequence of vertices (A_i). Define for all admissible indices

$$\begin{aligned} a_{2k} &= \mathrm{l}l A_k A_{k+1}, \\ a_{2k-1} &= \mathrm{l}\sin\left(\angle A_{k-1} A_k A_{k+1}\right), \end{aligned}$$

where $\mathrm{l}l$ denotes the integer length and $\mathrm{l}\sin$ denotes the *integer sine* of the angle $\angle ABC$, defined as

$$\mathrm{l}\sin\left(\angle ABC\right) = \frac{\mathrm{l}S(\triangle ABC)}{\mathrm{l}l(AB)\mathrm{l}l(BC)}, \tag{4.25}$$

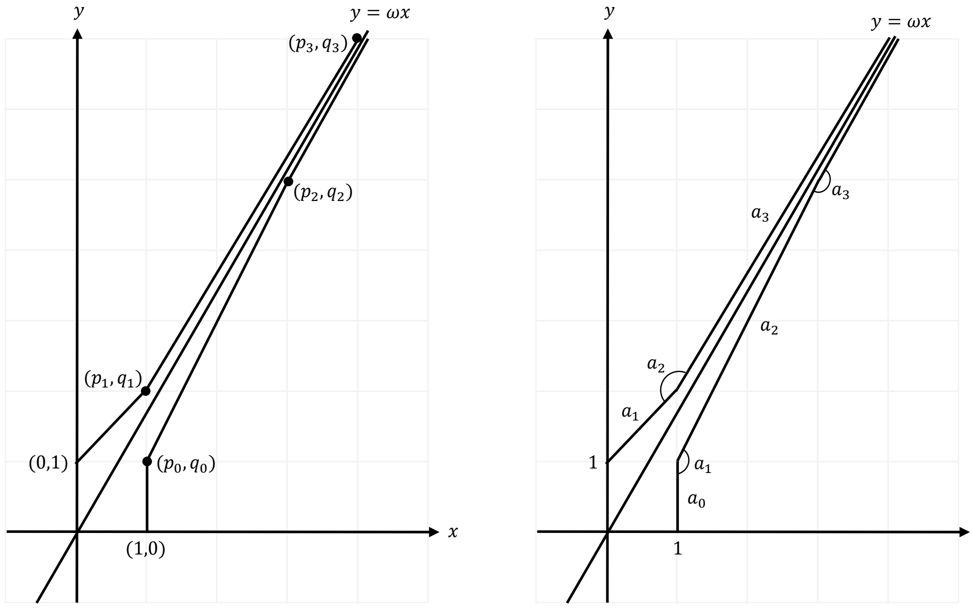


Figure 4.9: Klein's construction and the corresponding LLS sequence

where $\mathrm{l}S(\triangle ABC)$ of the triangle $\triangle ABC$ as the index of the sublattice generated by the integer vectors AB and BC in the integer lattice.

Then (a_n) is called the lattice length sine sequence, and

$$\omega = [a_0, a_1, a_2, \ldots] .$$

He also proved a remarkable *edge-angle duality* between sails of adjacent angles: let $A_0A_1A_2\ldots$ describe the sail underneath the ray (where $A_i,\ i \in \mathbb{Z}$ are the convergent coordinate points approximating ω) and $B_0B_1B_2\ldots$ describe the sail above. Then

$$\mathrm{l}\sin\left(\angle A_iA_{i+1}A_{i+2}\right) = \mathrm{l}l\left(B_iB_{i+1}\right) \tag{4.26}$$

and

$$\mathrm{l}\sin\left(\angle B_iB_{i+1}B_{i+2}\right) = \mathrm{l}l\left(A_{i+1}A_{i+2}\right). \tag{4.27}$$

Karpenkov proved that the LLS sequence determines the angle uniquely up to an integer affine transformation (note that the sequence is defined up to a shift of indices and depends on the orientation of the sail). We can thus explicitly describe the continued fraction expansion of ω using only one sail of the ω-ray (see Figure 4.9).

Karpenkov linked this realisation with the theory of indefinite binary quadratic forms: the zero set of such a form is a pair of lines, forming four angles with four sails, which are either isomorphic or dual to each other (see Figure 4.10).

We consider, therefore, a pair of lines given by $y = \alpha x$ and $y = \beta x$ and one of the angles $\angle \mathcal{AOB}$ formed by them. Let us assume that α and β are irrational, so that the origin O is the

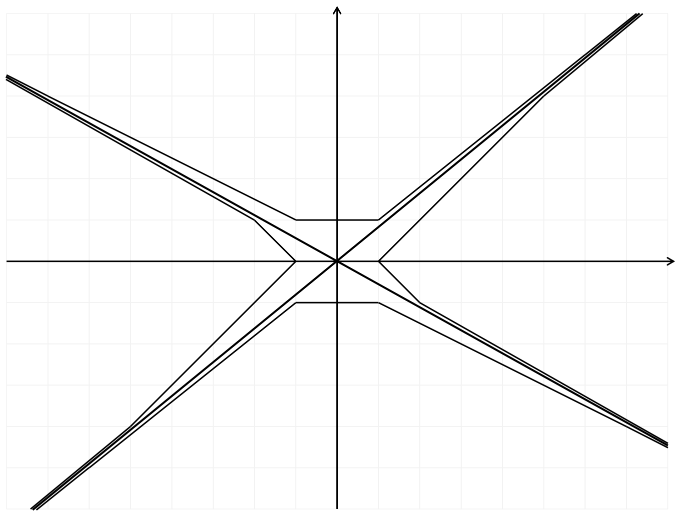


Figure 4.10: Arnold sails for a pair of lines

only integer point on them, and consider the convex hull of the integer points inside $\angle \mathcal{AOB}$ (excluding O). Its boundary is an infinite broken line called the *Arnold sail* of the angle $\angle \mathcal{AOB}$. The corresponding (infinite in both directions) LLS sequence can be considered as *a joint continued fraction expansion* of the numbers α and β, and is related to the rational approximation of the arrangement of these two lines (or equivalently, to the corresponding quadratic form $Q = (y - \alpha x)(y - \beta x)$ [45]).

The main result is the following simple relation between the Conway river and corresponding Arnold sail of an indefinite binary quadratic form:

Theorem 26. *Let $Q(x, y)$ be a real indefinite binary quadratic form and consider the Arnold sail of the pair of lines given by $Q(x, y) = 0$, assuming that the origin is the only integer point on them.*

Then the corresponding LLS sequence (a_i), $i \in \mathbb{Z}$ coincides with the sequence of the left- and right-turns of the Conway river on the topograph of Q. This determines the river uniquely up to the action of the group $PGL(2, \mathbb{Z})$ on the topograph and a change of direction.

Proof. Consider an indefinite quadratic form $Q(x, y) = ax^2 + hxy + by^2$ and factorise it as a product of linear forms

$$Q(x, y) = b(y - \alpha x)(y - \beta x)$$

with irrational α, β, assuming without loss of generality that $\alpha > 0$ and $\beta < 0$.

Let $P = A_0$ be a corner of the Arnold sail of the corresponding pair of lines $y = \alpha x$ and $y = \beta x$. Choose a new basis in the lattice with $e_1 = OA_0$ and e_2 being a primitive vector along the edge A_0A_1 of the Arnold sail. From Klein's construction it follows that this indeed a basis.

In the new coordinate system the corresponding $\alpha > 1$ and $0 > \beta > -1$, and we have the situation shown in Figure 4.11 justified by the following lemma (see also formulas 7 and 8 in Markov [55] and Definition 1.1 in [46]).

Lemma 3. *The LLS sequence of the Arnold sail of a pair of lines $y = \alpha x$ and $y = \beta x$ with $\alpha > 1$ and $0 > \beta > -1$ is*

$$\ldots, b_4, b_3, b_2, b_1, a_0, a_1, a_2, a_3, \ldots, \tag{4.28}$$

where a_i and b_j are given by the continued fraction expansions

$$\alpha = [a_0, a_1, a_2, a_3, \ldots], \quad -\beta = [0, b_1, b_2, b_3, b_4, \ldots]. \tag{4.29}$$

The proof follows directly from Klein's construction and Karpenkov's results [45], Ch. 3 and 7 (see in particular, Prop. 7.5).

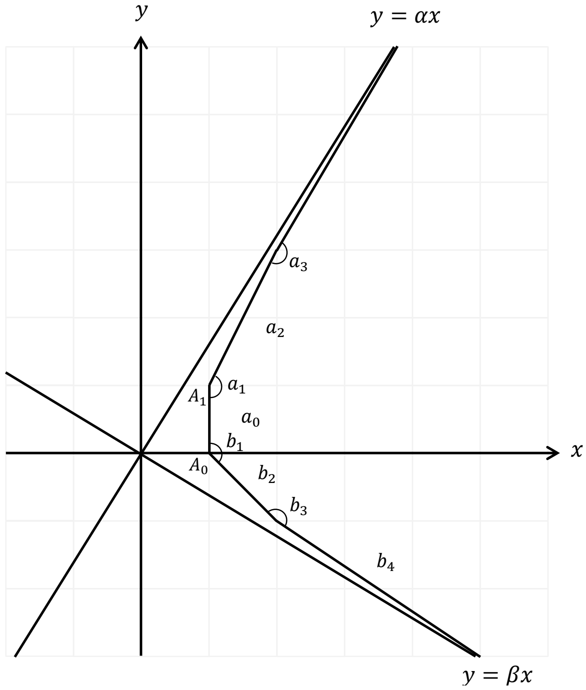


Figure 4.11: Arnold sail in a special basis

Let us now look at the corresponding Conway river. Since $\alpha\beta < 0$ this means that $Q(1, 0) = b\alpha\beta$ and $Q(0, 1) = b$ have different signs, so our initial position is already on the Conway river.

We know that the Conway river is the unique path on the Farey tree connecting the points α and β on the boundary, and thus is the union of two paths γ_α and γ_β. Combining this with the description of the Farey paths in terms of continued fractions described in Section 4.3, we conclude that the sequence (4.28) determines the sequence of the river's left and right turns.

We now claim that this determines the river uniquely modulo the action of $PGL(2, \mathbb{Z})$, which is the symmetry group of the binary tree embedded in the plane. Indeed, we have a well-known isomorphism (see e.g. [23])

$$PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3.$$

This allows us to define the action of $PSL(2, \mathbb{Z})$ on the tree with generators of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ acting as rotations by π about the edge centre and by $2\pi/3$ about the vertex. The element $diag\,(-1, 1) \in GL(2, \mathbb{Z})$ acts by the natural reflection and changes the orientation.

Using this action one can transform any directed edge to any other. After that the sequence of left- and right-turns determines the river uniquely. The left-right symmetry corresponds to the reflection. This proves our theorem. $\square$

Chapter 5

Tropical Markov Dynamics and the Cayley Cubic

In this chapter we consider the tropical version of the integrable case of Markov dynamics

$$x^2 + y^2 + z^2 = 3xyz + \frac{4}{9}$$

or, equivalently after scaling by 3:

$$x^2 + y^2 + z^2 = xyz + 4. \tag{5.1}$$

In this case the reparametrization

$$x = 2\cosh a,\ y = 2\cosh b,\ z = 2\cosh c$$

reduces this equation with positive x, y, z to the same Euclid relation $c = a + b$. This observation is well known (see e.g. [84]) and probably goes back to Mordell [58].

From the algebro-geometric point of view the relation (5.1) determines the classical surface known as the *Cayley cubic.* It was studied by Arthur Cayley in [16] and can be characterised as the cubic surface with 4 (which is the maximum possible) conical singularities (see Figure 5.1 borrowed from [83]).

The central part of the Cayley cubic is the boundary of what is sometimes called a *spectrahedron*, and has the trigonometric parametrization

$$x = 2\cos a,\ y = 2\cos b,\ z = 2\cos c \tag{5.2}$$

with $c = a + b$. Is there a tropical analogue of this part?

Adler and Veselov [1] came up with a natural candidate, replacing this part by the surface of the regular tetrahedron T with vertices at the singular points $(2, 2, 2)$, $(2, -2, -2)$, $(-2, 2, -2)$, $(-2, -2, 2)$, which are determined by the 'tropical' Cayley equation

$$\max\{-u - v - w, -u + v + w, u - v + w, u + v - w\} = 2. \tag{5.3}$$

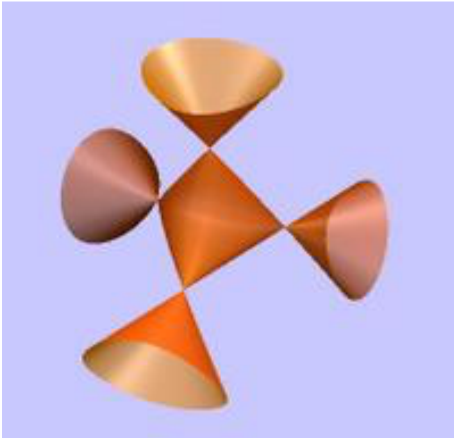


Figure 5.1: Cayley cubic

We have the corresponding action of the modular group $PSL_2(\mathbb{Z})$ generated by cyclic permutation of u, v, w and the tropical Vieta involution

$$(u, v, w) \to (u, v, -w + 2f(u, v)) \tag{5.4}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a piecewise linear function defined by

$$f(u, v) = \begin{cases} v & \text{if} \quad u \geq |v|, \\ u & \text{if} \quad v \geq |u|, \\ -v & \text{if} \quad -u \geq |v|, \\ -u & \text{if} \quad -v \geq |u|. \end{cases} \tag{5.5}$$

The plot of the function f is shown in Figure 5.2.

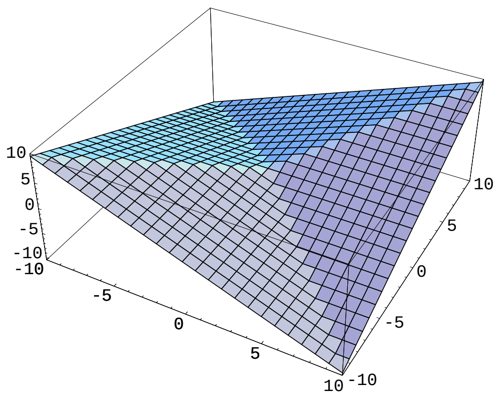


Figure 5.2: The function $f(x, y)$

This chapter will study the properties of this action, which we will call *tropical Cayley-Markov dynamics.* The main result is the following:

Theorem 27. *The tropical Cayley-Markov action of a hyperbolic element $A \in SL_2(\mathbb{Z})$ on T is ergodic, with the Lyapunov exponent and entropy given by the logarithm of the spectral radius of A. Their growth along the path γ_ξ on the planar binary tree is given by $\Lambda(\xi)$.*

The proof is by constructing the semi-conjugation of this action with the standard action of $SL_2(\mathbb{Z})$ on a torus, using a natural tropical analogue of the cosine function.

The same idea was used by Cantat [13] to prove similar results about the entropy of the (generalised) Markov dynamics (see also the important work of Iwasaki and Uehara [43, 44] in this direction).

5.1 Tropicalisation of Markov dynamics & Cayley cubic

Tropicalisation (or ultra-discretisation) can be applied to any dynamical system which can be written in algebraic form without a minus sign (i.e. subtraction-free). Recall that we tropicalise a system by replacing the operation of addition and multiplication by

$$X \oplus Y = \max(X, Y)$$

and

$$X \otimes Y = X + Y$$

respectively. It is clear that this does not work directly for the classical Markov dynamics

$$(x, y, z) \mapsto (x, y, 3xy - z)$$

because of the minus sign.

However, we can consider another Vieta version (cf. Hone [41])

$$(x, y, z) \to (x, y, (x^2 + y^2)/z), \tag{5.6}$$

which can be naturally tropicalised as

$$(X, Y, Z) \to (X, Y, \max(2X, 2Y) - Z). \tag{5.7}$$

Together with cyclic permutations of X, Y, Z, this generates the action of the modular group $PSL_2(\mathbb{Z})$, which is known to be isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

It has an invariant

$$\Phi = \max(2X, 2Y, 2Z) - (X + Y + Z),$$

or, equivalently,

$$\Phi = \max(X - Y - Z, Y - X - Z, Z - X - Y), \tag{5.8}$$

which is the tropical version of the integral

$$F = \frac{x^2 + y^2 + z^2}{xyz},$$

invariant under the Vieta involution (5.6).

It is easy to verify that the tropical equation $\Phi = 0$ for positive integers X, Y, Z defines the Euclidean algorithm and describes the asymptotic growth of the Markov triples in the logarithmic scale.

Let us now turn to the Cayley cubic case

$$x^2 + y^2 + z^2 = xyz + 4.$$

Adding 4 to the right hand side of the equation does not change the asymptotic behaviour at infinity, and thus the tropicalisation is the same as in the Markov case. However, it changes the shape of the surface near the origin by adding the part bounded by 4 singular points (see Figure 5.1).

Adler and Veselov [1] suggested replacing this part by the surface T of the tetrahedron with the same vertices.

The projection of T to the (u, v)-coordinate plane is a 2-to-1 map to the corresponding square (see Figure 5.3).

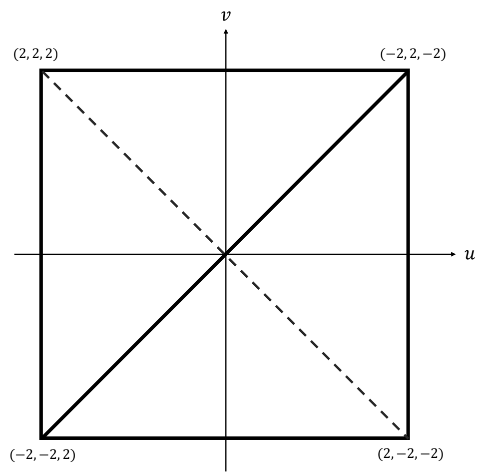


Figure 5.3: Projection of tropical Cayley surface T

It can be checked directly that the piecewise linear involution (5.4), (5.5) swaps the branches of this double cover similarly to the Markov involution (5.6), which was the motivation for introducing the tropical Cayley-Markov dynamics in [1].

Proposition 9. *The function*

$$\Psi = \max\{-u + v + w, u - v + w, u + v - w, -u - v - w\} \tag{5.9}$$

is invariant under these dynamics.

The level set $\Psi = c$, $c > 0$ is the surface of the regular tetrahedron with the vertices (c, c, c), $(c, -c, -c)$, $(-c, c, -c)$, $(-c, -c, c)$.

The proof is again by direct check. Note in particular the difference with the usual case (5.8). We are not aware of any straightforward 'tropicalisation' procedure leading from the Cayley cubic to the equation $\Psi = 2$, which was motivated only by geometric reasons.

5.2 Properties of tropical Cayley-Markov dynamics

We would like to study the dynamical properties of the tropical Cayley-Markov action $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$, where the action of $\mathbb{Z}_2$ is given by by (5.4) and (5.5).

It is easy to see that this action preserves the usual Lebesgue measure on the surface of T.

Numerical calculations [1] showed the ergodic behaviour of the orbits of tropical Cayley-Markov dynamics at the level set $\Psi = c$ (see Figure 5.4).

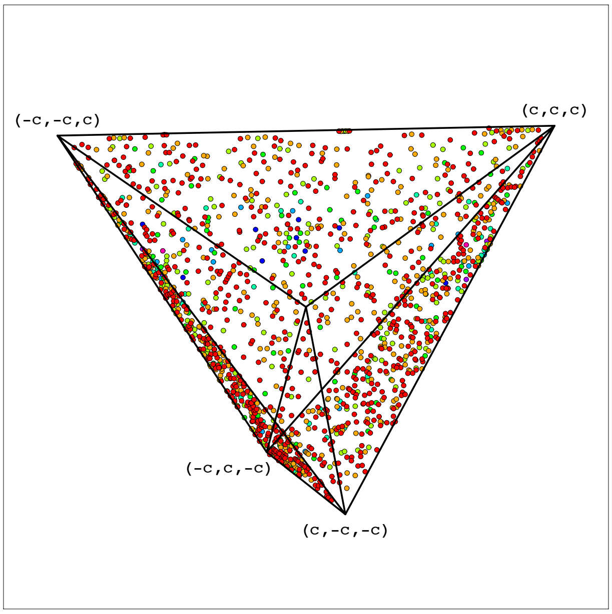


Figure 5.4: The level set of tropical Markov dynamics

Now we are ready to prove this and our main Theorem 27.

Let us introduce the tropical version of the cosine function, costrop x, as the period-2 piecewise linear function given on the period by

$$\operatorname{costrop} x = 1 - 2|x|, \quad x \in [-1, 1],$$

(see Figure 5.5), and define the tropical parametrisation of T by the following tropical analogue of (5.2):

$$u = 2costrop\, \phi,\ v = 2costrop\, \psi,\ w = 2costrop\, \chi, \tag{5.10}$$

where $\chi = \phi + \psi$ and $(\phi, \psi) \in T^2 = \mathbb{R}^2/(2\mathbb{Z})^2$.

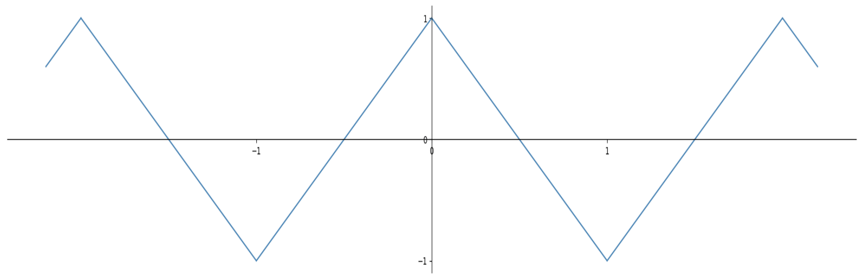


Figure 5.5: Tropical cosine function *costrop* ϕ.

The corresponding map determines the 2-to-1 folding of the torus T^2 into T (see Figure 5.6).

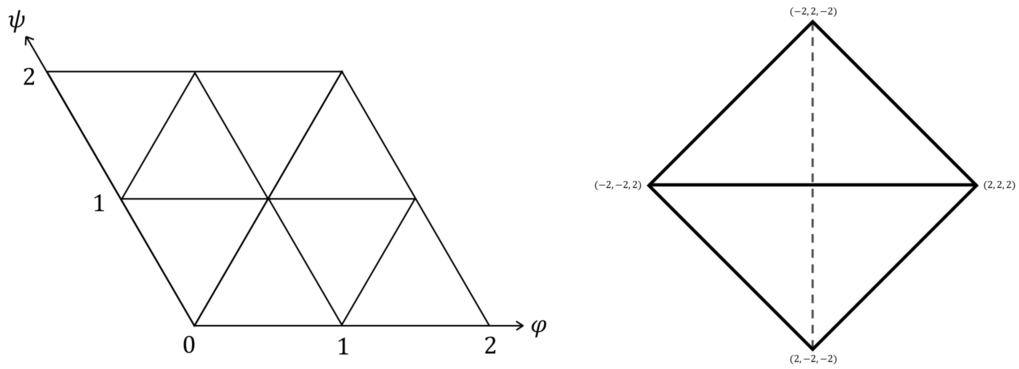


Figure 5.6: Folding of the torus to T

The key observation is the following:

Lemma 4. *The parametrisation (5.10) semi-conjugates the tropical Cayley-Markov action of A with the standard action of A on the torus T^2.*

Indeed, u and v determine ϕ and ψ by (5.10) uniquely up to a sign, which means that the two values of the corresponding coordinate w are $w = 2costrop\,(\phi \pm \psi)$. Thus the tropical Cayley-Markov involution corresponds to the linear maps $(\pm\phi, \pm\psi, \pm(\phi+\psi)) \to (\pm\phi, \pm\psi, \pm(\phi-\psi))$, describing the action of $PSL_2(\mathbb{Z})$ on the so-called lax superbases (see [23]).

Note that the surface of the tetrahedron T is the quotient of the torus T^2 by the central symmetry group $\mathbb{Z}_2$, with fixed points corresponding to the vertices of the tetrahedron, so we have the following commutative diagram of the group actions

$$\begin{array}{lcl} T^2 & \stackrel{SL_2(\mathbb{Z})}{\longrightarrow} & T^2 \\ \downarrow \mathbb{Z}_2 & & \downarrow \mathbb{Z}_2 \\ T & \stackrel{PSL_2(\mathbb{Z})}{\longrightarrow} & T. \end{array} \tag{5.11}$$

Since the action of a hyperbolic A on a torus is known to be ergodic with the Lyapunov exponent and entropy given by the natural logarithm of the spectral radius of A (see e.g. [47]), this completes the proof of the theorem.

Chapter 6

Conclusion

We have studied the problem of growth in some concrete examples of $SL_2(\mathbb{Z})$-dynamics, including the generalised Markov action on the cubics

$$x^2 + y^2 + z^2 = 3xyz + D.$$

When $D = 4/9$ the dynamics can be linearised by the change of variables (1.4), and thus can be considered "integrable". This important observation probably comes from Mordell [58] and allows us to reduce the problem to the study of the Euclid equation. This observation was used earlier by Zagier [84] to study the growth problem for Markov numbers and, more recently, by Cantat and Loray [13, 14] to compute the topological entropy of the corresponding $SL_2(\mathbb{Z})$-dynamics. The underlying idea is that the hyperbolic dynamics is rigid, so its numerical characteristics remain the same under the change of parameters (and thus the solvable cases can be used to compute them).

Let us summarise the main results of the thesis.

In Chapter 3 we considered the growth problem for Markov and Euclid dynamics. Crucially, we introduced the idea of measuring the growth along paths in the Markov and Euclid trees, defined by their limit in $\mathbb{R}P^1$. We defined the Lyapunov exponent along these paths and proved that it exists for all $x \in \mathbb{R}P^1$. In particular, we described the Lyapunov spectrum for the Markov and Euclid trees, and proved that Λ is convex in the Farey parametrisation and monotonically increasing when restricted to the set of most irrational numbers.

In Chapter 4 we applied the techniques used to study the Markov dynamics to binary quadratic forms $Q(x, y)$, using the topograph representation introduced by Conway [23]. The main result is that the corresponding Lyapunov exponent $\Lambda_Q(x) = 2\Lambda(x)$ for all paths in the indefinite binary quadratic topograph, with the exception of the paths along the Conway river.

We also showed that the Conway river gives an explicit description of the continued fraction representation of these roots of Q. From this, we proved a nice connection to the work of Klein, Arnold and Karpenkov in the geometric theory of continued fractions.

It would be interesting to generalise the results for binary cubic forms and higher-dimensional quadratic forms. The lack of local rules means that this is not straightforward in the topographical representation (although some partial results have been found); however, Arnold sails can be defined in much more general situations, in particular, for cubic binary forms and multidimensional simplicial cones. It would be interesting to investigate whether there is an analogue of the Conway river in these cases.

Finally, in Chapter 5 we considered the tropicalised, or ultra-discretised, version of the Markov dynamics on the tropical version of the Cayley cubic proposed by Adler and Veselov. By introducing the tropical version of the parametrisation (1.4), we proved that the tropical action of a hyperbolic element $A \in SL_2(\mathbb{Z})$ on T is ergodic, and has growth which is once again given by the same function $\Lambda(\xi)$.

There are several directions for further study.

One natural extension would be to study the set

$$supp(\Lambda) = \{x \in \mathbb{R} : \Lambda(x) \neq 0\},$$

further and, in particular, to find its Hausdorff dimension (cf. e.g. [40]).

The most intriguing question about the Lyapunov function Λ is whether it is already known in some parts of mathematics. The invariance under the modular group suggests that $\Lambda(x)$ might be interpreted as the limit values of some modular function on the real boundary of the hyperbolic plane (see e.g. [3] for Riemann's approach to this problem).

Another interesting direction would be to study the properties of the related generalised Fock functions, which have been shown to be stable norms in the sense of Federer and Gromov [68]. They are known to be convex, thus having both left and right derivatives at every point. Can these derivatives be computed at some special points explicitly? This would reveal a hidden "integrability" of these functions.

Finally, the Markov equation is known to be closely related to the theory of cluster algebras, and the corresponding dynamics can be interpreted in terms of cluster mutations (see e.g. [6, 61]). It would be natural to explore this relation further.

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