Stationary Solutions of Stochastic Partial Differential Equations and Infinite Horizon Backward Doubly Stochastic Differential Equations

by

Qi ZHANG

Doctoral Thesis
Submitted in partial fulfillment of the requirements for the award of
Doctor of Philosophy of Loughborough University

16th February, 2008

Supervisor: Prof. Huaizhong ZHAO
Prof. Shige PENG (Shandong University)

Loughborough University

Abstract

In this thesis we study the existence of stationary solutions for stochastic partial differential equations. We establish a new connection between solutions of backward doubly stochastic differential equations (BDSDEs) on infinite horizon and the stationary solutions of the SPDEs. For this, we prove the existence and uniqueness of the $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued solutions of BDSDEs with Lipschitz nonlinear term on both finite and infinite horizons, so obtain the solutions of initial value problems and the stationary weak solutions (independent of any initial value) of SPDEs. Also the $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued BDSDE with non-Lipschitz term is considered. Moreover, we verify the time and space continuity of solutions of real-valued BDSDEs, so obtain the stationary stochastic viscosity solutions of real-valued SPDEs. The connection of the weak solutions of SPDEs and BDSDEs has independent interests in the areas of both SPDEs and BSDEs.

Keywords: stationary solution, stochastic partial differential equations, backward doubly stochastic differential equations, weak solutions, stochastic viscosity solutions, random dynamical systems.

AMS 2000 subject classifications: 60H15, 60H10, 37H10.

ACKNOWLEDGEMENTS

I'd like to give my hearty thanks to my supervisor, Prof. Huaizhong Zhao, whose eminent academic ability and perfect personality deeply affect me. I very appreciate his long-term and warm help in my study and in my life. In November 2004, I was chosen as one of students in Loughborough-Shandong University joint PhD programme to study in Loughborough. To me, the most lucky thing in UK is to be a student of Prof. Zhao. He is very kind and generous to others and impresses everyone by his righteousness and empressement. In the mean time, he is a mathematician with responsible attitude, creative idea and hard-work spirit. Under Prof. Zhao's instruction, I learned little by little about how to do research. Although during this period I was frustrated many many times, Prof. Zhao always discussed the methods with me and helped me solve the difficulties with great patient and sincerity, whether the difficulties came from study or life and whether I was in UK or in China. It was Prof. Zhao's encouragement and help that gave me the confidence to hold on in my goal. Moreover, under his recommendation, the Department offered me a chance to visit Loughborough again this summer and it was very helpful to my thesis.

I'd like to give my hearty thanks to my another supervisor, Prof. Shige Peng. Since I became a student of Prof. Peng at Shandong University in 2001, I have been greatly benefited from his profound knowledge and strict requirement all the time. It was Prof. Peng who lead me to the stochastic analysis world and showed me how interesting this field was. In the past six years, Prof. Peng gave me countless instruction and suggestion in my study which I deeply appreciated. In addition, Prof. Peng always does his endeavor to create the best learning environment for us and gives us the chance to have an access to top lectures, reports and meetings in our field. In November 2004, Prof. Peng chose me as one of students in Loughborough-Shandong University joint PhD programme, which gave me the opportunity to study in Loughborough and broadened my outlook. Although strict in academics, Prof. Peng always shows his sense of humor and kindness to others in life. He often organized activities after class which left us a lot of happy memories. From him, we learn about the great personality charm of a mathematician.

I'd like to give my thanks to Prof. Zdzislaw Brzezniak, Prof. David Elworthy, Prof. Zhiming Ma, Prof. Etienne Pardoux and Prof. Tusheng Zhang for useful conversations in our paper.

I'd like to give my thanks to Head of Department, Prof. Chris Linton, who gives his long-term support for our Stochastic Analysis group. Give my thanks to Jo Corlett and all other secretaries in Department for their help in administration. In fact, the friendliness from all the staff in Department makes me feel warm and homey in Loughborough.

I'd like to thank Dr. Yufeng Shi, Dr. Zuohuan Zheng and my classmates Chunrong Feng, Fajin Wei, Xiaohui Yan and Bo Zhou for beneficial discussion in my study and generous help in my life in UK.

At last, I'd like to give my thanks to my parents and my family for their nurture and support through my life. Their love is my most precious wealth and I will cherish it forever.

Contents

1 Introduction **1**

2 The Correspondence Between Stationary Solutions of SPDEs and BDSDEs **6**

§2.1 General BDSDEs with General Norm 6

§2.2 Stationary Solutions of BDSDEs Derived by Perfection Procedure . . . 10

§2.3 Transferring the Stationarity from BDSDEs to the Corresponding SPDEs 15

§2.3.1 Definition for weak solutions of SPDEs and the corresponding $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)\otimes L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)$ valued BDSDEs 15

§2.3.2 Generalized equivalence of norm principle 18

§2.3.3 Conditions, examples and main results in Chapter 3 20

§2.3.4 Stationary weak solution 22

3 Stationary Weak Solutions of SPDEs **26**

§3.1 Finite Horizon BDSDEs 26

§3.1.1 Conditions, definition and main result 26

§3.1.2 Substitution theorem 28

§3.1.3 The proof of main result 32

§3.2 The Corresponding SPDEs 39

§3.2.1 Weak solutions of SPDEs with finite dimensional noise and introduction of Bally and Matoussi's idea 39

§3.2.2 Existence and uniqueness of solutions of SPDEs with infinite dimensional noise 43

§3.3 Infinite Horizon BDSDEs 46

§3.4 Time Continuity of Solutions of SPDEs 52

4 Non-Lipschitz Condition **57**

§4.1 Conditions, Examples and Main Results 57

§4.2 Finite Horizon BDSDEs and the Corresponding SPDEs 59

§4.2.1 Conditions and main results 59

§4.2.2 Existence and uniqueness of solutions of BDSDEs with finite dimensional noise 60

§4.2.3 Existence and uniqueness of solutions of BDSDEs with infinite dimensional noise 71

§4.2.4 The corresponding SPDEs 74

§4.3 Infinite Horizon BDSDEs 80

5 Stationary Stochastic Viscosity Solutions of SPDEs **83**

§5.1 Doss-Sussmann Transformation and Definition for Stochastic Viscosity Solution of SPDE 83

§5.2 Infinite Horizon BDSDEs 86

§5.2.1 Introduction of Pardoux and Peng's work for finite horizon BDSDEs 86

§5.2.2 Existence and uniqueness of solutions of infinite horizon BDSDEs 88

§5.3 Space and Time Continuity of Solutions of SPDEs and Stationary Stochastic Viscosity Solution 95

§5.3.1 Continuity of solutions of the corresponding BDSDEs 95

§5.3.2 Stationary stochastic viscosity solution of the corresponding SPDE 98

Bibliography **100**

Chapter 1

Introduction

Let $u : [0, \infty) \times U \times \Omega \to U$ be a measurable random dynamical system on a measurable space $(U, \mathcal{B})$ over a metric dynamical system $(\Omega, \mathscr{F}, P, (\theta_t)_{t\geq 0})$, then a stationary solution is a $\mathscr{F}$ measurable random variable $Y : \Omega \to U$ such that (Arnold [1])

$$u(t, Y(\omega), \omega) = Y(\theta_t \omega) \quad \text{for all } t \geq 0 \text{ a.s..} \tag{1.1}$$

This "one-force, one-solution" setting is a natural extension of equilibria or fixed points in deterministic systems to stochastic counterparts. The simplest nontrivial example is the Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$du(t) = -u(t)dt + dB_t.$$

It defines a random dynamical system

$$u(t, u_0) = u_0 e^{-t} + \int_0^t e^{-(t-s)} dB_s$$

and its stationary point is given by $Y(\omega) = \int_{-\infty}^0 e^s dB_s$. Moreover, for any u_0, $u(t, u_0, \theta_{-t}\omega) \to Y(\omega)$ as $t \to \infty$, where θ_t is the shift operator of the Brownian path:

$$(\theta_t B)(s) = B(t+s) - B(s) \text{ for any } s \in (-\infty, +\infty)..$$

A pathwise stationary solution describes the pathwise invariance of the stationary solution over time along the measurable and P-preserving transformation θ_t: $\Omega \longrightarrow \Omega$ and the pathwise limit of the solutions of random dynamical systems. Once $Y(\cdot)$ is known, $Y(\theta_t \cdot)$ is known. Needless to say, it is one of the fundamental questions of basic importance ([1], [12], [13], [20], [38], [50], [51]). For random dynamical systems generated by stochastic partial differential equations (SPDEs), such random fixed points consist of infinitely many random moving invariant surfaces on the configuration space due to the random external force pumped to the system constantly. They are more realistic models than many deterministic models as they demonstrate some complicated phenomena such as turbulence. Their existence and stability are of great interests in both mathematics and physics. However, in contrast to the deterministic dynamical systems, also due to the fact that the external random force exists at all time, the existence of stationary solutions of stochastic dynamical systems generated e.g. by stochastic differential equations (SDEs) or SPDEs, is a difficult and subtle problem. We would like to point out that there have been extensive works on stability and invariant manifolds of random dynamical systems, and researchers usually assume there is an invariant set (or a single point: a stationary solution or a fixed point, often assumed to be 0), then prove invariant manifolds and stability results at a point of the invariant set (Arnold [1] and references therein, Ruelle [48], Duan, Lu and Schaumulfuss [18], [19], Li and Lu [31], Mohammed, Zhang and Zhao [38] to name but a few). But the invariant manifolds theory gives neither the existence results of the invariant set and the stationary solution nor a way to find them. In particular, for the existence of stationary solutions for SPDEs, results are only known in very few cases ([13], [20], [38], [50], [51]). In [50], [51], the stationary strong solution of the stochastic Burgers' equations with periodic or random forcing (C^3 in the space variable) was established by Sinai using the Hopf-Cole transformation. In [38], the stationary solution of the stochastic evolution equations was identified as a solution of the corresponding integral equation up to time $+\infty$ and the existence was established for certain SPDEs by Mohammed, Zhang and Zhao. But the existence of solutions of such a stochastic integral equation in general is far from clear.

The main purpose of this thesis is to find the stationary solution of the following SPDE

$$\begin{aligned} dv(t,x) &= [\mathscr{L}v(t,x) + f\big(x, v(t,x), \sigma^*(x)Dv(t,x)\big)]dt \\ &\quad + g\big(x, v(t,x), \sigma^*(x)Dv(t,x)\big)dB_t. \end{aligned} \tag{1.2}$$

Here B is a two-sided cylindrical Brownian motion on a separable Hilbert space U_0; $\mathscr{L}$ is the infinitesimal generator of a diffusion process $X_s^{t,x}$ (solution of Eq.(2.14)) given by

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} \tag{1.3}$$

with $(a_{ij}(x)) = \sigma\sigma^*(x)$. Eq.(1.2) is very general, especially when we consider the weak solution of Eq.(1.2), the nonlinear functions f and g can include ∇v and the second order differential operator $\mathscr{L}$ is allowed to be degenerate, while in most literature, g is not allowed to depend on ∇v or g only depends on ∇v linearly (Da Prato and Zabczyk [16], Gyöngy and Rovira [23], Krylov [27], Mikulevicius and Rozovskii [37], Pardoux [41]). As an intermediate step, the result of existence and uniqueness of the weak solutions of (1.2), obtained by solving the corresponding backward doubly stochastic differential equations (BDSDEs) under weak Lipschitz condition, appears also new. The existence and uniqueness of such equations when g is independent of ∇v or linearly dependent of ∇v were studied by Da Prato and Zabczyk [16], Krylov [27]. But we don't claim here our results on the existence and uniqueness for the types of SPDEs studied in [16] and [27] have superseded their previous results.

From a pathwise stationary solution we can construct an invariant measure for the skew product of the metric dynamical system and the random dynamical system. In recent years, substantial results on the existence and uniqueness of invariant measures for SPDEs and weak convergence of the law of the solutions as time tends to infinity have been proved for many important SPDEs ([7], [8], [17], [22], [24] to name but a few). The invariant measure describes the invariance of a certain solution in law when time changes, therefore it is a stationary measure of the Markov transition probability. It is well known that an invariant measure gives a stationary solution when it is a random Dirac measure. Although an invariant measure of a random dynamical system on $\mathbb{R}^1$ gives a stationary solution, in general, this is not true unless one considers an extended probability space. However, considering the extended probability space, one essentially regards the random dynamical system as noise as well, so the dynamics is different. See [36] for some examples of SDEs on $\mathbb{R}^1$ and a perfect cocycle on $\mathbb{S}^1$ having an invariant measure, but not a stationary solution. In fact, the stationary solution we study in this thesis gives the support of the corresponding invariant measure, so reveals more detailed information than an invariant measure.

In this thesis, BDSDEs will be used as our tool to study stationary solutions of SPDEs. We will prove that the solutions of the corresponding infinite horizon BDSDEs give the desired stationary solutions of the SPDEs (1.2). Backward stochastic differential equations (BSDEs) have been studied extensively in the last 17 years since the pioneering work of Pardoux and Peng [42]. The connection between BSDEs and quasilinear parabolic partial differential equations (PDEs) was discovered by Pardoux and Peng in [43] and Peng in [45]. The study of the connection of weak solutions of

PDEs and BSDEs began in Barles and Lesigne [4]. The BDSDEs and their connections with the SPDEs were studied by Pardoux and Peng in [44] for the strong solutions, by Bally and Matoussi in [3] for the weak solutions and by Buckdahn and Ma in [9]-[11] for the stochastic viscosity solutions. On the other hand, the infinite horizon BSDE was first studied by Peng in [45] and it was shown that the corresponding PDE is a Poisson equation (elliptic equation). This was studied systematically by Pardoux in [40]. Notice that the solutions of the Poisson equations can be regarded as the stationary solutions of the parabolic PDEs. Deepening this idea, it would not be unreasonable to conjecture that the solutions of infinite horizon BDSDEs (if exists) be the stationary solutions of the corresponding SPDEs. Of course, we cannot write them as solutions of Poisson equations or stochastic Poisson equations like in the deterministic cases. However, it is very natural to describe the stationary solutions of SPDEs by the solutions of infinite horizon BDSDEs. In this sense, BDSDEs (or BSDEs) can be regarded as more general SPDEs (or PDEs).

As far as we know, the connection of the pathwise stationary solutions of the SPDEs and infinite horizon BDSDEs we study in this thesis is new. Enlightened by [1] and [2], we first apply a "perfection procedure" to the solution of BDSDE in Section 2.2 and then transfer the stationary property from BDSDE to the corresponding SPDE in Section 2.3. We believe this new method can be used to many SPDEs such as those with quadratic or polynomial growth nonlinear terms. We don't intend to include all these results in the present thesis, but only study Lipschitz continuous nonlinear term in Chapter 3 and Chapter 5 to initiate this intrinsic method to the study of this basic problem in dynamics of SPDEs and study linear growth non-Lipschitz nonlinear term in Chapter 4. We would like to point out that our BDSDE method depends on neither the continuity of the random dynamical system (continuity means $u(t,\cdot,\omega):U\to U$ is a.s. continuous) nor on the method of the random attractors. The continuity problem for the SPDE (1.2) with the nonlinear noise considered in this thesis still remains open mainly due to the failure of Kolmogorov's continuity theorem in infinite dimensional setting as pointed out by some researchers (e.g. [18], [38]).

In Chapter 3, one of the necessary intermediate steps is to study the BDSDEs on finite horizon and establish their connections with the weak solutions of SPDEs (Sections 3.1 and 3.2). Our method to study the $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued solutions of BDSDEs on finite horizon was inspired by Bally and Matoussi's approach on the existence and uniqueness of solutions of BDSDEs with finite dimensional Brownian motions ([3]). But our results are stronger and our conditions are weaker. We will solve the BDSDEs driven by the cylindrical Brownian motion and nonlinear terms satisfying Lipschitz conditions in the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1) \otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$. We obtain a unique solution $(Y^{t,\cdot}_\cdot, Z^{t,\cdot}_\cdot) \in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. The result $Y^{t,\cdot}_\cdot \in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$, which plays an important role in solving the nonlinear BDSDEs and proving the connection with the weak solutions of SPDEs (also BSDEs and PDEs), was not obtained in [3]. The generalized equivalence of norm principle (proved in Section 2.3), which is a simple extension of the equivalence of norm principle obtained by Kunita ([28]), Barles and Lesigne ([4]), Bally and Matoussi ([3]) to random functions, also plays an important role in the proofs of our results. We believe our results for finite horizon BDSDEs are new even for BSDEs. In Section 3.3, we will solve the BDSDEs on infinite horizon and in Section 3.4, we study continuity of the solution in order to ensure that it gives the perfect stationary weak solutions of the SPDEs.

We further consider $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1) \otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued BDSDE on finite and infinite horizon with linear growth non-Lipschitz nonlinear term (Sections 4.2 and 4.3). The monotone condition plays an important role to weaken the Lipschitz continuous condition in Chapter 4.

In Chapter 5, we recall the Doss-Sussmann transformation and Buckdahn and Ma's idea to define the so-called stochastic viscosity solutions of SPDEs (Section 5.1). Then in Section 5.2, we solve the corresponding real-valued BDSDEs on infinite horizon by a similar method as in Section 3.3. In fact, comparing the stochastic viscosity solution with the weak solution, we need more information for the stochastic viscosity solution. In particular, the space continuity of solutions of SPDEs is considered in Section 5.3 as well as time continuity so that we can perfect the stochastic viscosity solutions of real-valued SPDEs. Finally, we would like to point out that the techniques in Chapter 4 can be similarly applied to studying the stochastic viscosity solutions of SPDEs with linear growth non-Lipschitz term although we don't intend to include the analysis in this thesis.

Chapter 2

The Correspondence Between Stationary Solutions of SPDEs and BDSDEs

§2.1 General BDSDEs with General Norm

On a probability space $(\Omega, \mathscr{F}, P)$, let $(\hat{B}_t)_{t\geq 0}$ be a Q-Wiener process with values in U and let $(W_t)_{t\geq 0}$ be an independent standard Brownian motion with values in $\mathbb{R}^d$. Here U is a separable Hilbert space with countable base $\{e_i\}_{i=1}^{\infty}$; $Q \in L(U)$ is a symmetric nonnegative trace class operator such that $Qe_i = \lambda_i e_i$ and $\sum\limits_{i=1}^{\infty} \lambda_i < \infty$. It is well known that $\hat{B}$ has the following expansion (e.g. [16]): for each t,

$$\hat{B}_t = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\hat{\beta}_j(t)e_j, \tag{2.1}$$

where

$$\hat{\beta}_j(t) = \frac{1}{\sqrt{\lambda_j}} < \hat{B}_t, e_j >_U, \quad j = 1, 2, \cdots$$

are mutually independent real-valued Brownian motions on $(\Omega, \mathscr{F}, P)$ and the series (2.1) is convergent in $L^2(\Omega, \mathscr{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathscr{F}$. We define

$$\mathscr{F}_{t,T} \triangleq \mathscr{F}_{t,T}^{\hat{B}} \bigvee \mathscr{F}_t^W \bigvee \mathcal{N}, \quad \text{for } 0 \leq t \leq T;$$
$$\mathscr{F}_t \triangleq \mathscr{F}_{t,\infty}^{\hat{B}} \bigvee \mathscr{F}_t^W \bigvee \mathcal{N}, \quad \text{for } t \geq 0.$$

Here for any process $(\eta_t)_{t\geq 0}$,

$$\mathscr{F}^{\eta}_{s,t} = \sigma\{\eta_r - \eta_s;\ 0 \leq s \leq r \leq t\},\ \mathscr{F}^{\eta}_t = \mathscr{F}^{\eta}_{0,t},\ \mathscr{F}^{\eta}_{t,\infty} = \bigvee_{T\geq 0} \mathscr{F}^{\eta}_{t,T}.$$

Definition 2.1.1. *Let $\mathbb{S}$ be a Hilbert space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ-field $\mathscr{S}$. For $K \in \mathbb{R}^+$, we denote by $M^{2,-K}([0,\infty);\mathbb{S})$ the set of $\mathscr{B}_{\mathbb{R}^+} \otimes \mathscr{F}/\mathscr{S}$ measurable random processes $\{\phi(s)\}_{s\geq 0}$ with values in $\mathbb{S}$ satisfying*

(i) $\phi(s) : \Omega \to \mathbb{S}$ is $\mathscr{F}_s$ measurable for $s \geq 0$;

(ii) $E[\int_0^\infty e^{-Ks}\|\phi(s)\|^2_{\mathbb{S}}ds] < \infty$.

Also we denote by $S^{2,-K}([0,\infty);\mathbb{S})$ the set of $\mathscr{B}_{\mathbb{R}^+}\otimes\mathscr{F}/\mathscr{S}$ measurable random processes $\{\psi(s)\}_{s\geq 0}$ with values in $\mathbb{S}$ satisfying

(i) $\psi(s) : \Omega \to \mathbb{S}$ is $\mathscr{F}_s$ measurable for $s \geq 0$ and $\psi(\cdot,\omega)$ is continuous P-a.s.;

(ii) $E[\sup_{s\geq 0} e^{-Ks}\|\psi(s)\|^2_{\mathbb{S}}] < \infty$.

Similarly, for $0 \leq t \leq T < \infty$, we define $M^{2,0}([t,T];\mathbb{S})$ and $S^{2,0}([t,T];\mathbb{S})$ on a finite time interval.

Definition 2.1.2. *Let $\mathbb{S}$ be a Hilbert space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ-field $\mathscr{S}$. We denote by $M^{2,0}([t,T];\mathbb{S})$ the set of $\mathscr{B}_{[t,T]} \otimes \mathscr{F}/\mathscr{S}$ measurable random processes $\{\phi(s)\}_{t\leq s\leq T}$ with values in $\mathbb{S}$ satisfying*

(i) $\phi(s) : \Omega \to \mathbb{S}$ is $\mathscr{F}_{s,T} \bigvee \mathscr{F}^{\hat{B}}_{T,\infty}$ measurable for $t \leq s \leq T$;

(ii) $E[\int_t^T \|\phi(s)\|^2_{\mathbb{S}}ds] < \infty$.

Also we denote by $S^{2,0}([t,T];\mathbb{S})$ the set of $\mathscr{B}_{[t,T]} \otimes \mathscr{F}/\mathscr{S}$ measurable random processes $\{\psi(s)\}_{t\leq s\leq T}$ with values in $\mathbb{S}$ satisfying

(i) $\psi(s) : \Omega \to \mathbb{S}$ is $\mathscr{F}_{s,T} \bigvee \mathscr{F}^{\hat{B}}_{T,\infty}$ measurable for $t \leq s \leq T$ and $\psi(\cdot,\omega)$ is continuous P-a.s.;

(ii) $E[\sup_{t\leq s\leq T} \|\psi(s)\|^2_{\mathbb{S}}] < \infty$.

For a positive K, we consider the following infinite horizon BDSDE with the infinite dimensional Brownian motion $\hat{B}$ as noise and Y_t taking values in a separable

Hilbert space H, Z_t taking values in $\mathcal{L}^2_{\mathbb{R}^d}(H)$ (the space of all Hilbert-Schmidt operators from $\mathbb{R}^d$ to H with the Hilbert-Schmidt norm):

$$\begin{aligned} e^{-Kt}Y_t &= \int_t^\infty e^{-Kr}f(r,Y_r,Z_r)dr + \int_t^\infty Ke^{-Kr}Y_r dr \\ &\quad - \int_t^\infty e^{-Kr}g(r,Y_r,Z_r)d^\dagger\hat{B}_r - \int_t^\infty e^{-Kr}Z_r dW_r, \quad t \geq 0. \end{aligned} \tag{2.2}$$

Assume $f : [0,\infty)\times\Omega\times H\times\mathcal{L}^2_{\mathbb{R}^d}(H)\longrightarrow H$, $g : [0,\infty)\times\Omega\times H\times\mathcal{L}^2_{\mathbb{R}^d}(H)\longrightarrow \mathcal{L}^2_{U_0}(H)$ are $\mathscr{B}_{\mathbb{R}^+}\otimes\mathscr{F}\otimes\mathscr{B}_H\otimes\mathscr{B}_{\mathcal{L}^2_{\mathbb{R}^d}(H)}$ measurable such that for any $(t,Y,Z)\in[0,\infty)\times H\times\mathcal{L}^2_{\mathbb{R}^d}(H)$, $f(t,Y,Z)$, $g(t,Y,Z)$ are $\mathscr{F}_t$ measurable, where $U_0 = Q^{\frac{1}{2}}(U) \subset U$ is a separable Hilbert space with the norm

$$< u, v >_{U_0} = < Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v >_U$$

and the complete orthonormal base $\{\sqrt{\lambda_i}e_i\}_{i=1}^\infty$, $\mathcal{L}^2_{U_0}(H)$ is the space of all Hilbert-Schmidt operators from U_0 to H with the Hilbert-Schmidt norm. It is noted that the Q-Wiener process $(\hat{B}_t)_{t\geq 0}$ is a cylindrical Wiener process on U_0, and both $\mathcal{L}^2_{U_0}(H)$ and $\mathcal{L}^2_{\mathbb{R}^d}(H)$ are Hilbert spaces.

Note that the integral w.r.t. $\hat{B}$ is a "backward Itô's integral" and the integral w.r.t. W is a standard forward Itô's integral. The forward integrals in a Hilbert space with respect to Q-Wiener processes were defined in Da Prato and Zabczyk [16]. To see the backward one, let $\{h(s)\}_{s\geq 0}$ be a stochastic process with values in $\mathcal{L}^2_{U_0}(H)$ such that $h(s)$ is $\mathscr{F}_s$ measurable for any $s \geq 0$ and locally square integrable, i.e. for any $0 \leq a \leq b < \infty$,

$$\int_a^b \|h(s)\|^2_{\mathcal{L}^2_{U_0}(H)} ds < \infty \text{ a.s..}$$

Since $\mathscr{F}_s$ is a backward filtration with respect to $\hat{B}$, so from the one-dimensional backward Itô's integral and relation with forward integral, for $0 \leq T \leq T'$, we have

$$\int_t^T \sqrt{\lambda_j} < h(s)e_j, f_k > d^\dagger\hat{\beta}_j(s) = -\int_{T'-T}^{T'-t} \sqrt{\lambda_j} < h(T'-s)e_j, f_k > d\beta_j(s), \ j,k = 1,2,\cdots$$

where $\beta_j(s) = \hat{\beta}_j(T'-s) - \hat{\beta}_j(T')$, $j = 1,2,\cdots$, and so $B_s = \hat{B}_{T'-s} - \hat{B}_{T'}$. Here $\{f_k\}$ is the complete orthonormal basis in H. From approximation theorem of the stochastic integral in a Hilbert space ([16]), we have

$$\int_{T'-T}^{T'-t} h(T'-s)dB_s = \sum_{j,k=1}^\infty \int_{T'-T}^{T'-t} \sqrt{\lambda_j} < h(T'-s)e_j, f_k > d\beta_j(s)f_k.$$

Similarly we also have

$$\int_t^T h(s)d^\dagger\hat{B}_s = \sum_{j,k=1}^{\infty}\int_t^T \sqrt{\lambda_j} < h(s)e_j, f_k > d^\dagger\hat{\beta}_j(s)f_k.$$

It turns out that

$$\int_t^T h(s)d^\dagger\hat{B}_s = -\int_{T'-T}^{T'-t} h(T'-s)dB_s \quad \text{a.s..} \tag{2.3}$$

Later we will consider another Hilbert space $\mathcal{L}^p_{U_0}(H)$ $(p>2)$, a subspace of $\mathcal{L}^2_{U_0}(H)$, consisting of all $h \in \mathcal{L}^2_{U_0}(H)$ which satisfy

$$\|h\|^p_{\mathcal{L}^p_{U_0}(H)} \triangleq \sum_{j,k=1}^{\infty} \lambda_j^{\frac{p}{2}} |\langle he_j, f_k\rangle|^p < \infty.$$

Definition 2.1.3. *Let H_0 be a dense subset of H. If $(Y,Z) \in S^{2,-K}\bigcap M^{2,-K}([0,\infty);H)$ $\bigotimes M^{2,-K}([0,\infty);\mathcal{L}^2_{\mathbb{R}^d}(H))$, and for any $\varphi \in H_0$,*

$$\begin{aligned}\langle e^{-Kt}Y_t,\varphi\rangle &= \langle\int_t^\infty e^{-Kr}f(r,Y_r,Z_r)dr,\varphi\rangle + \langle\int_t^\infty Ke^{-Kr}Y_rdr,\varphi\rangle \\ &-\langle\int_t^\infty e^{-Kr}g(r,Y_r,Z_r)d^\dagger\hat{B}_r,\varphi\rangle - \langle\int_t^\infty e^{-Kr}Z_rdW_r,\varphi\rangle,\ t\geq 0\ P-\text{a.s.},\end{aligned} \tag{2.4}$$

or equivalently

$$\begin{cases}\langle Y_t,\varphi\rangle = \langle Y_T,\varphi\rangle + \langle\int_t^T f(r,Y_r,Z_r)dr,\varphi\rangle - \langle\int_t^T g(r,Y_r,Z_r)d^\dagger\hat{B}_r,\varphi\rangle - \langle\int_t^T Z_rdW_r,\varphi\rangle \\ \lim_{T\to\infty}\langle e^{-KT}Y_T,\varphi\rangle = 0 \quad \text{a.s.,}\end{cases}$$

then we call (Y,Z) a solution of Eq.(2.2) in H.

Remark 2.1.4. **(i)** *Applying Itô's formula in H (see [16]), we have the equivalent form of Eq.(2.2)*

$$\begin{cases}Y_t = Y_T + \int_t^T f(r,Y_r,Z_r)dr - \int_t^T g(r,Y_r,Z_r)d^\dagger\hat{B}_r - \int_t^T Z_rdW_r \\ \lim_{T\to\infty} e^{-KT}Y_T = 0 \quad \text{a.s.;}\end{cases} \tag{2.5}$$

(ii) *One can easily verify that the above definition doesn't depend on the choice of H_0 due to the continuity of the inner product;*

(iii) *The uniqueness of Y in $S^{2,-K}([0,\infty);H)$ implies if (Y',Z') is another solution, then $Y_s = Y'_s$ for $s\geq 0$ a.s.. The uniqueness of Z implies $Z_s = Z'_s$ for a.a. $s\in[0,\infty)$ a.s.. But we can modify the Z at the measure zero exceptional set of s such that $Z_s = Z'_s$ for $s\geq 0$ a.s..*

The first main purpose of this chapter is to study the stationary property of the solution of BDSDE (2.2) on H if the solution exists and is unique. In order to show the main idea, we first assume that there exists a unique solution of Eq.(2.2). The study of the existence and uniqueness of Eq.(2.2) will be deferred to later chapters (Chapters 3-4).

We now construct the measurable metric dynamical system through defining a measurable and measure-preserving shift. Let $\hat{\theta}_t : \Omega \longrightarrow \Omega$, $t \geq 0$, be a measurable mapping on $(\Omega, \mathscr{F}, P)$, defined by

$$\hat{\theta}_t \circ \hat{B}_s = \hat{B}_{s+t} - \hat{B}_t, \quad \hat{\theta}_t \circ W_s = W_{s+t} - W_t.$$

Then for any s, $t \geq 0$,

(i) $P \cdot \hat{\theta}_t^{-1} = P$;

(ii) $\hat{\theta}_0 = I$, where I is the identity transformation on Ω;

(iii) $\hat{\theta}_s \circ \hat{\theta}_t = \hat{\theta}_{s+t}$.

Also for an arbitrary $\mathscr{F}$ measurable $\phi : \Omega \longrightarrow H$, set

$$\hat{\theta} \circ \phi(\omega) = \phi(\hat{\theta}(\omega)).$$

We give the following boundedness and stationarity conditions for f, g w.r.t. $\hat{\theta}$.:

(A.2.1). There exist a constant $M_1 \geq 0$, and functions $\tilde{f}(\cdot) \in M^{2,-K}([0,\infty); R^+)$, $\tilde{g}(\cdot) \in M^{2,-K}([0,\infty); R^+)$ s.t. for any $s \geq 0$, $Y \in H$ and $Z \in \mathcal{L}^2_{\mathbb{R}^d}(H)$,

$$\|f(s,Y,Z)\|_H^2 \leq \tilde{f}^2(s) + M_1\|Y\|_H^2 + M_1\|Z\|^2_{\mathcal{L}^2_{\mathbb{R}^d}(H)},$$

$$\|g(s,Y,Z)\|^2_{\mathcal{L}^2_{U_0}(H)} \leq \tilde{g}^2(s) + M_1\|Y\|_H^2 + M_1\|Z\|^2_{\mathcal{L}^2_{\mathbb{R}^d}(H)};$$

(A.2.2). For any r, $s \geq 0$, $Y \in H$ and $Z \in \mathcal{L}^2_{\mathbb{R}^d}(H)$,

$$\hat{\theta}_r \circ f(s,Y,Z) = f(s+r,Y,Z), \quad \hat{\theta}_r \circ g(s,Y,Z) = g(s+r,Y,Z).$$

§2.2 Stationary Solutions of BDSDEs Derived by Perfection Procedure

We start from the following general result about the stationarity of the solution of infinite horizon BDSDE.

Proposition 2.2.1. *Assume Eq.(2.2) has a unique solution (Y,Z), then under Conditions (A.2.1) and (A.2.2), $(Y_t, Z_t)_{t\geq 0}$ is a "crude" stationary solution, i.e. for any $r \geq 0$,*

$$\hat{\theta}_r \circ Y_t = Y_{t+r}, \quad \hat{\theta}_r \circ Z_t = Z_{t+r} \quad \text{for all } t \geq 0 \text{ a.s..}$$

Proof. Let $B_s = \hat{B}_{T'-s} - \hat{B}_{T'}$ for arbitrary $T' > 0$ and $-\infty < s \leq T'$. Then B_s is a Brownian motion with $B_0 = 0$. For any $r \geq 0$, applying $\hat{\theta}_r$ to B_s, we have

$$\begin{aligned}\hat{\theta}_r \circ B_s &= \hat{\theta}_r \circ (\hat{B}_{T'-s} - \hat{B}_{T'}) = \hat{B}_{T'-s+r} - \hat{B}_{T'+r} \\ &= (\hat{B}_{T'-s+r} - \hat{B}_{T'}) - (\hat{B}_{T'+r} - \hat{B}'_T) = B_{s-r} - B_{-r}.\end{aligned}$$

So for $0 \leq t \leq T \leq T'$ and a locally square integrable process $\{h(s)\}_{s\geq 0}$, by (2.3)

$$\begin{aligned}\hat{\theta}_r \circ \int_t^T h(s) d^\dagger \hat{B}_s &= -\hat{\theta}_r \circ \int_{T'-T}^{T'-t} h(T'-s) dB_s \\ &= -\int_{T'-T}^{T'-t} \hat{\theta}_r \circ h(T'-s) dB_{s-r} \\ &= -\int_{T'-T-r}^{T'-t-r} \hat{\theta}_r \circ h(T'-s-r) dB_s \\ &= \int_{t+r}^{T+r} \hat{\theta}_r \circ h(s-r) d^\dagger \hat{B}_s.\end{aligned}$$

As T' can be chosen arbitrarily, so we can get for arbitrary $T \geq 0$, $0 \leq t \leq T$, $r \geq 0$,

$$\hat{\theta}_r \circ \int_t^T h(s) d^\dagger \hat{B}_s = \int_{t+r}^{T+r} \hat{\theta}_r \circ h(s-r) d^\dagger \hat{B}_s. \tag{2.6}$$

It is easy to see that $g(\cdot, Y_\cdot, Z_\cdot)$ is locally square integrable from Condition (A.2.1), hence by Condition (A.2.2) and (2.6)

$$\hat{\theta}_r \circ \int_t^T g(s, Y_s, Z_s) d^\dagger \hat{B}_s = \int_{t+r}^{T+r} g(s, \hat{\theta}_r \circ Y_{s-r}, \hat{\theta}_r \circ Z_{s-r}) d^\dagger \hat{B}_s. \tag{2.7}$$

We consider the equivalent form Eq.(2.5) instead of Eq.(2.2). Applying the operator $\hat{\theta}_r$ to both sides of Eq.(2.5) and by (2.7), we know that $\hat{\theta}_r \circ Y_t$ satisfies the following equation

$$\begin{cases}\hat{\theta}_r \circ Y_t = \hat{\theta}_r \circ Y_T + \displaystyle\int_{t+r}^{T+r} f(s, \hat{\theta}_r \circ Y_{s-r}, \hat{\theta}_r \circ Z_{s-r}) ds \\ \qquad\qquad - \displaystyle\int_{t+r}^{T+r} g(s, \hat{\theta}_r \circ Y_{s-r}, \hat{\theta}_r \circ Z_{s-r}) d^\dagger \hat{B}_s - \int_{t+r}^{T+r} \hat{\theta}_r \circ Z_{s-r} dW_s \\ \lim_{T\to\infty} \mathrm{e}^{-K(T+r)} (\hat{\theta}_r \circ Y_T) = 0 \quad \text{a.s..}\end{cases} \tag{2.8}$$

On the other hand, from Eq.(2.5), it follows that

$$\begin{cases} Y_{t+r} = Y_{T+r} + \int_{t+r}^{T+r} f(s, Y_s, Z_s)ds - \int_{t+r}^{T+r} g(s, Y_s, Z_s)d^\dagger \hat{B}_s - \int_{t+r}^{T+r} Z_s dW_s \\ \lim_{T\to\infty} e^{-K(T+r)} Y_{T+r} = 0 \quad \text{a.s..} \end{cases} \tag{2.9}$$

Let $\hat{Y}_\cdot = \hat{\theta}_r \circ Y_{\cdot-r}$, $\hat{Z}_\cdot = \hat{\theta}_r \circ Z_{\cdot-r}$. By the uniqueness of solution of Eq.(2.5) and Remark 2.1.4 (iii), it follows from comparing (2.8) with (2.9) that for any $r \geq 0$,

$$\hat{\theta}_r \circ Y_t = \hat{Y}_{t+r} = Y_{t+r}, \quad \hat{\theta}_r \circ Z_t = \hat{Z}_{t+r} = Z_{t+r} \quad \text{for all } t \geq 0 \text{ a.s..}$$

◇

Proposition 2.2.1 gives a "crude" stationary property of Y and Z. We then give the following theorem which makes the "crude", even "very crude" stationary property of Y "perfect". The main idea of the proof is from [1] and [2] perfecting crude cocycles. We include a proof here as our refined proof seems easy to follow.

Theorem 2.2.2. *Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\mathbb{H}$ be a separable Hausdorff topological space with σ-algebra $\mathscr{H}$. Assume $Y\colon [0,\infty) \times \Omega \longrightarrow \mathbb{H}$ is $\mathcal{B}_{\mathbb{R}^+} \otimes \mathscr{F}$ measurable, continuous w.r.t. t a.s. and satisfies for any t, $r \geq 0$,*

$$\hat{\theta}_r \circ Y_t = Y_{t+r} \quad \text{a.s..} \tag{2.10}$$

Then there exists a $\hat{Y}_t$ which is an indistinguishable version of Y_t s.t. $\hat{Y}$ is $\mathcal{B}_{\mathbb{R}^+} \otimes \mathscr{F}$ measurable, continuous w.r.t. t for all ω and satisfies

$$\hat{\theta}_r \circ \hat{Y}_t = \hat{Y}_{t+r} \quad \text{for all } t, r \geq 0 \text{ a.s..}$$

Proof. From the continuity of Y w.r.t. t and using a standard argument, we easily see that for any $r \geq 0$,

$$\hat{\theta}_r \circ Y_t(\omega) = Y_{t+r}(\omega) \quad \text{for all } t \geq 0 \text{ a.s..} \tag{2.11}$$

Define

$$\begin{aligned} M &= \{(r,\omega) : \hat{\theta}_r \circ Y_t(\omega) = Y_{t+r}(\omega) \text{ for all } t\}; \\ \tilde{\Omega} &= \{\omega : (r,\omega) \in M \text{ for a.a. } r\}; \\ \Omega^* &= \{\omega : \hat{\theta}_r\omega \in \tilde{\Omega} \text{ for a.a. } r\}; \\ A(r,t,\omega) &= \hat{\theta}_r \circ Y_t - Y_{t+r}. \end{aligned}$$

Obviously, $A(r,t,\omega)$ is measurable w.r.t. $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{\mathbb{R}^+} \otimes \mathscr{F}$. If we denote by Q the normalized Lebesgue measure on $\mathbb{R}^+$ such that $Q(\mathbb{R}^+) = 1$, then by (2.11),

$$Q \otimes Q \otimes P\big(A^{-1}(0)\big) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\Omega} I_{A^{-1}(0)}(r,t,\omega)dPdQdQ = 1, \qquad (2.12).$$

where I is the indicator function in $(\mathbb{R}^+ \times \mathbb{R}^+ \times \Omega,\ \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{\mathbb{R}^+} \otimes \mathscr{F})$. It is easy to see that

$$M = \{(r,\omega) : \int_{\mathbb{R}^+} I_{A^{-1}(0)}(r,t,\omega)dQ = 1\} \in \mathcal{B}_{\mathbb{R}^+} \otimes \mathscr{F}.$$

And since (2.12)

$$Q \otimes P(M) = Q \otimes P\big(\{(r,\omega) : \int_{\mathbb{R}^+} I_{A^{-1}(0)}(r,t,\omega)dQ = 1\}\big) = 1.$$

Similarly, we have

$$\tilde{\Omega} = \{\omega : \int_{\mathbb{R}^+} I_M(r,\omega)dQ = 1\} \in \mathscr{F}$$

and

$$P(\tilde{\Omega}) = P\big(\{\omega : \int_{\mathbb{R}^+} I_M(r,\omega)dQ = 1\}\big) = 1.$$

Moreover, the measurability of Ω^* can be seen easily as

$$\Omega^* = \{\omega : \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} I_M(r,\hat{\theta}_u\omega)dQdQ = 1\} \in \mathscr{F}.$$

And since $\tilde{\Omega}$ has full measure, so

$$\begin{aligned}
P(\Omega^*) &\geq P\big(\{\omega : Y_{t+r}(\hat{\theta}_u\omega) = Y_t(\hat{\theta}_r \circ \hat{\theta}_u\omega) \text{ for a.a. } r \text{ and } u, \text{ and all } t\} \cap \tilde{\Omega}\big)\\
&= P\big(\{\omega : Y_{t+r+u}(\omega) = Y_t(\hat{\theta}_{r+u}\omega) \text{ for a.a. } r \text{ and } u, \text{ and all } t\} \cap \tilde{\Omega}\big)\\
&= P\big(\{\omega : Y_{t+r'}(\omega) = Y_t(\hat{\theta}_{r'}\omega) \text{ for a.a. } r', \text{ and all } t\} \cap \tilde{\Omega}\big)\\
&= P(\tilde{\Omega})\\
&= 1.
\end{aligned}$$

One can prove $\hat{\theta}_u\Omega^* \subset \Omega^*$ for any $u \geq 0$. To see this for any $\omega \in \hat{\theta}_u\Omega^*$, there exists $\hat{\omega} \in \Omega^*$ s.t. $\omega = \hat{\theta}_u\hat{\omega}$ and $\hat{\theta}_r\hat{\omega} \in \tilde{\Omega}$ for a.e. $r \geq 0$. But $\hat{\theta}_r\omega = \hat{\theta}_{u+r}\hat{\omega} \in \tilde{\Omega}$ for a.e. $r \geq 0$, so $\omega \in \Omega^*$. That is to say $\hat{\theta}_u\Omega^* \subset \Omega^*$. Define

$$\begin{cases} \hat{Y}_t(\omega) = Y_{t-r}(\hat{\theta}_r\omega), & \text{where } r \in [0,t] \text{ with } \hat{\theta}_r\omega \in \tilde{\Omega}, \text{ if } \omega \in \Omega^*,\\ \hat{Y}_t(\omega) = 0, & \text{if } \omega \in \Omega^{*c}. \end{cases}$$

An important fact is that if $\omega \in \Omega^*$, then for an arbitrary $r \in [0,t]$ with $\hat{\theta}_r\omega \in \tilde{\Omega}$, $Y_{t-r}(\hat{\theta}_r\omega)$ is independent of r and

$$Y_{t-r}(\hat{\theta}_r\omega) = Y_t(\omega). \tag{2.13}$$

To see this, as $\hat{\theta}_r\omega \in \tilde{\Omega}$, so there exists $u \geq r$ s.t. $(u, \hat{\theta}_r\omega) \in M$ and $(u-r, \hat{\theta}_r\omega) \in M$. If not, it means for a.e. r there doesn't exist u satisfying $(u, \hat{\theta}_r\omega) \in M$ and $(u-r, \hat{\theta}_r\omega) \in M$. Then one can easily get the measure of $\{u : (u, \hat{\theta}_r\omega) \notin M\}$ is positive. That is a contradiction. So such a u certainly exists and satisfies

$$\hat{\theta}_u Y_{t-r}(\hat{\theta}_r\omega) = Y_{t-r+u}(\hat{\theta}_r\omega) = Y_t(\hat{\theta}_{u-r}\hat{\theta}_r\omega) = Y_t(\hat{\theta}_u\omega).$$

So

$$Y_{t-r}(\hat{\theta}_r\omega) = \hat{\theta}_u^{-1} Y_t(\hat{\theta}_u\omega) = Y_t(\omega).$$

Therefore (2.13) is true and $\hat{Y}_t$ doesn't depend on the choice of r. That is to say $\hat{Y}_t(\omega)$ is well defined. Moreover (2.13) implies that $Y_t = \hat{Y}_t$ for all $t \in \mathbb{R}^+$ on full measure set Ω^*, thus Y_t and $\hat{Y}_t$ are indistinguishable. Define

$$\begin{cases} B(r,t,\omega) = Y_{t-r}(\hat{\theta}_r\omega), & \text{if } r \in [0,t], \ \hat{\theta}_r\omega \in \tilde{\Omega}, \ \text{and } \omega \in \Omega^*, \\ B(r,t,\omega) = 0, & \text{otherwise.} \end{cases}$$

Then $B(r,t,\omega)$ is $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{\mathbb{R}^+} \otimes \mathscr{F}$ measurable. By definition of Ω^*, if $\omega \in \Omega^*$, then for a.e. $0 \leq r \leq t$, $\hat{\theta}_r\omega \in \tilde{\Omega}$. We denote by $L(r)$ the Lebesgue measure on $[0,t]$. Since the countable base of H generates $\mathscr{H}$ and separates points, $(H, \mathscr{H})$ is isomorphic as a measurable space to a subset of $[0,1]$. Consequently,

$$\hat{Y}_t(\omega) = \int_0^t B(r,t,\omega)dL(r)$$

for all t, ω. So by Fubini's theorem, $\hat{Y}_t(\omega)$ is $\mathcal{B}_{\mathbb{R}^+} \otimes \mathscr{F}$ measurable. $\hat{Y}_t(\omega)$ is a.s continuous w.r.t. t due to the a.s continuity of $Y_{t-r}(\omega)$. But there exists null measure set $N \in \mathscr{F}$ s.t. $\{\omega : \hat{Y}_t(\omega)$ is not continuous w.r.t. $t\} \subset N$. Let $\hat{Y}_t(\omega)$ on N equal 0. We still denote this new version of $\hat{Y}_t(\omega)$ by $\hat{Y}_t(\omega)$, then we have $\hat{Y}_t(\omega)$ is continuous for all ω.

The remaining work is to check for $\omega \in \Omega^*$, all $r \geq 0$, $\hat{Y}_t(\omega)$ satisfies stationary property. Since $\omega \in \Omega^*$, for all $r \geq 0$, $\hat{\theta}_r\omega \in \Omega^*$. Pick a u s.t. $\hat{\theta}_u\omega \in \tilde{\Omega}$, $\hat{\theta}_{u+r}\omega \in \tilde{\Omega}$, then by (2.13) we have

$$\hat{Y}_t(\hat{\theta}_r\omega) = Y_{t-u}(\hat{\theta}_{u+r}\omega) = Y_{t+r-u-r}(\hat{\theta}_{u+r}\omega) = Y_{t+r}(\omega) = Y_{t+r-u}(\hat{\theta}_u\omega) = \hat{Y}_{t+r}(\omega).$$

The theorem is proved. ◇

Remark 2.2.3. *From the above perfection argument for "very crude" continuous function, it is easy to see that if the Y in Theorem 2.2.2 is not continuous, but instead it is a "crude" function, then we can also obtain its indistinguishable "perfect" version.*

With Theorem 2.2.2 and Remark 2.2.3, we can deduce directly from Proposition 2.2.1 that

Theorem 2.2.4. *Assume Eq.(2.2) has a unique solution (Y, Z) in the Hilbert space H, then under Conditions (A.2.1) and (A.2.2), $(Y_t, Z_t)_{t\geq 0}$ is a "perfect" stationary solution, i.e.*

$$\hat{\theta}_r \circ Y_t = Y_{t+r}, \quad \hat{\theta}_r \circ Z_t = Z_{t+r} \text{ for all } r,\ t \geq 0 \text{ a.s..}$$

§2.3 Transferring the Stationarity from BDSDEs to the Corresponding SPDEs

An important application of the BDSDEs is to connect their solutions with the solutions of the corresponding SPDEs. If some kind of relationship is established, we can transfer the stationary property from the infinite horizon BDSDEs to SPDEs. In this way, we are able to access stationary solutions of the SPDEs due to the stationary property of solutions of infinite horizon BDSDEs. In this section, we take weak solutions of SPDEs ($L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ valued solutions of SPDEs) as an example to show this method. The main aim of the rest of this chapter is to find the stationary weak solutions of the SPDEs through its corresponding BDSDEs. Some proofs are given in this section, but many detailed proofs are postponed to Chapter 3.

§2.3.1 Definition for weak solutions of SPDEs and the corresponding $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued BDSDEs

From now on, we consider the general BDSDE in the case $H = L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ with the inner product

$$\langle u_1, u_2\rangle = \int_{\mathbb{R}^d} u_1(x)u_2(x)\rho^{-1}(x)dx,$$

i.e. a ρ-weighted L^2 space. Here $\rho(x) = (1 + |x|)^q$, $q > 3$, is a weight function. It is easy to see that $\rho(x) : \mathbb{R}^d \longrightarrow \mathbb{R}^1$ is a continuous positive function satisfying

$\int_{\mathbb{R}^d} |x|^p \rho^{-1}(x)dx < \infty$ for any $p \in (2, q-1)$. Note that we can consider more general ρ which satisfies the above condition and conditions in [3] and all the results of this thesis still hold.

We can write down the solution spaces following Definition 2.1.1: $M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$, $M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$ and $S^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$. Similar to the definition for $M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$, we can also define $M^{p,-K}([0,\infty); L^p_\rho(\mathbb{R}^d; \mathbb{R}^d))$ for $p > 2$.

For $k \geq 0$, we denote by $C^k_{l,b}$ the set of C^k-functions whose partial derivatives of order less than or equal to k are bounded and by H^k_ρ the ρ-weighted Sobolev space (See e.g. [3]). In order to connect BDSDEs with SPDEs, the form of BDSDEs should be a kind of FBDSDEs (forward and backward doubly SDEs). So we first give the following forward SDE.

For $s \geq t$, let $X^{t,x}_s$ be a diffusion process given by the solution of

$$X^{t,x}_s = x + \int_t^s b(X^{t,x}_u)du + \int_t^s \sigma(X^{t,x}_u)dW_u, \tag{2.14}$$

where $b \in C^2_{l,b}(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$, and for $0 \leq s < t$, we regulate $X^{t,x}_s = x$.

For any $r \geq 0$, $s \geq t$, $x \in \mathbb{R}^d$, apply $\hat{\theta}_r$ defined in Section 2.1 to SDE (2.14), then

$$\hat{\theta}_r \circ X^{t,x}_s = x + \int_{t+r}^{s+r} b(\hat{\theta}_r \circ X^{t,x}_{u-r})du + \int_{t+r}^{s+r} \sigma(\hat{\theta}_r \circ X^{t,x}_{u-r})dW_u.$$

So by the uniqueness of the solution and a perfection procedure (c.f. [1]), we have

$$\hat{\theta}_r \circ X^{t,x}_s = X^{t+r,x}_{s+r} \quad \text{for all } r, s, t, x \quad \text{a.s.}. \tag{2.15}$$

Now we consider the following BDSDE with infinite dimensional noise on infinite horizon

$$\begin{aligned} e^{-Ks}Y^{t,x}_s &= \int_s^\infty e^{-Kr} f(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr + \int_s^\infty K e^{-Kr} Y^{t,x}_r dr \\ &\quad - \int_s^\infty e^{-Kr} g(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)d^\dagger \hat{B}_r - \int_s^\infty e^{-Kr} \langle Z^{t,x}_r, dW_r \rangle. \end{aligned} \tag{2.16}$$

Here $\hat{B}_r = \sum_{j=1}^\infty \sqrt{\lambda_j}\hat{\beta}_j(r)e_j$, $\{\hat{\beta}_j(r)\}_{j=1,2,\cdots}$ are mutually independent one-dimensional Brownian motions. Note that we will solve Eq.(2.16) for $Y^{t,\cdot}_r \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$ and $Z^{t,\cdot}_r \in \mathcal{L}^2_{\mathbb{R}^d}(L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)) = L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$.

Set $g_j \triangleq g\sqrt{\lambda_j}e_j : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$, then Eq.(2.16) is equivalent to

$$e^{-Ks}Y^{t,x}_s = \int_s^\infty e^{-Kr} f(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr + \int_s^\infty K e^{-Kr} Y^{t,x}_r dr$$

$$-\sum_{j=1}^{\infty}\int_s^{\infty} e^{-Kr} g_j(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger\hat{\beta}_j(r) - \int_s^{\infty} e^{-Kr}\langle Z_r^{t,x}, dW_r\rangle.$$

Referring to Definition 2.1.3 and noting that $C_c^0(\mathbb{R}^d;\mathbb{R}^1)$ is dense in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$ under the norm $(\int_{\mathbb{R}^d}|\cdot|^2\rho^{-1}(x)dx)^{\frac{1}{2}}$, we can define the solution in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$ as follows:

Definition 2.3.1. *A pair of processes* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2,-K}\bigcap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ *is called a solution of Eq.(2.16) if for an arbitrary* $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-Ks}Y_s^{t,x}\varphi(x)dx &= \int_s^{\infty}\int_{\mathbb{R}^d} e^{-Kr} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxdr \\ &+\int_s^{\infty}\int_{\mathbb{R}^d} Ke^{-Kr}Y_r^{t,x}\varphi(x)dxdr \\ &-\sum_{j=1}^{\infty}\int_s^{\infty}\int_{\mathbb{R}^d} e^{-Kr}g_j(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r) \\ &-\int_s^{\infty}\langle\int_{\mathbb{R}^d} e^{-Kr}Z_r^{t,x}\varphi(x)dx, dW_r\rangle \quad P-\text{a.s.}. \qquad (2.17)\end{aligned}$$

Note that in (2.17) we leave out the weight function ρ in the inner product due to the arbitrariness of φ.

If Eq.(2.16) has a unique solution, then for an arbitrary T, $Y_T^{t,x}$ satisfies

$$Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})d^\dagger\hat{B}_r - \int_s^T\langle Z_r^{t,x}, dW_r\rangle. \qquad (2.18)$$

In Section 3.2, we will prove that the following SPDE can be associated with BDSDE (2.18)

$$\begin{aligned}u(t,x) &= u(T,x) + \int_t^T[\mathscr{L}u(s,x) + f\big(x, u(s,x), (\sigma^*\nabla u)(s,x)\big)]ds \\ &-\int_t^T g\big(x, u(s,x), (\sigma^*\nabla u)(s,x)\big)d^\dagger\hat{B}_s. \qquad (2.19)\end{aligned}$$

Here $\mathscr{L}$ is given by (1.3) and $u(T,x) = Y_T^{T,x}$. But we can normally study general $u(T,x)$ unless we consider the stationary solution.

Now following Definition 2.1.2 we write down the solution spaces needed in this chapter: $M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, $M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and $S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$.

Definition 2.3.2. *A process u is called a weak solution (solution in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$) of Eq.(2.19) if $(u,\sigma^*\nabla u)\in M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and for an arbitrary $\Psi\in C_c^{1,\infty}([0,T]\times\mathbb{R}^d;\mathbb{R}^1)$,*

$$\begin{aligned}
&\int_t^T\int_{\mathbb{R}^d}u(s,x)\partial_s\Psi(s,x)dxds+\int_{\mathbb{R}^d}u(t,x)\Psi(t,x)dx-\int_{\mathbb{R}^d}u(T,x)\Psi(T,x)dx\\
&-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}(\sigma^*\nabla u)(s,x)(\sigma^*\nabla\Psi)(s,x)dxds\\
&-\int_t^T\int_{\mathbb{R}^d}u(s,x)div\big((b-\tilde{A})\Psi\big)(s,x)dxds\\
=\ &\int_t^T\int_{\mathbb{R}^d}f\big(x,u(s,x),(\sigma^*\nabla u)(s,x)\big)\Psi(s,x)dxds\\
&-\sum_{j=1}^{\infty}\int_t^T\int_{\mathbb{R}^d}g_j\big(x,u(s,x),(\sigma^*\nabla u)(s,x)\big)\Psi(s,x)dx d^\dagger\hat{\beta}_j(s)\quad P-\text{a.s.}. \qquad (2.20)
\end{aligned}$$

Here $\tilde{A}_j\triangleq\frac{1}{2}\sum_{i=1}^d\frac{\partial a_{ij}(x)}{\partial x_i}$, and $\tilde{A}=(\tilde{A}_1,\tilde{A}_2,\cdots,\tilde{A}_d)^*$.

This definition can be easily understood if we note the following integration by parts formula: for $\varphi_1,\varphi_2\in C^2(\mathbb{R}^d)$,

$$-\int_{\mathbb{R}^d}\mathscr{L}\varphi_1(x)\varphi_2(x)dx=\frac{1}{2}\int_{\mathbb{R}^d}(\sigma^*\nabla\varphi_1)(x)(\sigma^*\nabla\varphi_2)(x)dx+\int_{\mathbb{R}^d}\varphi_1(x)div\big((b-\tilde{A})\varphi_2\big)(x)dx.$$

§2.3.2 Generalized equivalence of norm principle

We introduce the so-called "generalized equivalence of norm principle" which plays an important role in the analysis not only in this chapter, but also through this thesis.

It is well-known that the solution of Eq.(2.14) defines a stochastic flow of diffeomorphism $X_s^{t,\cdot}:\mathbb{R}^d\to\mathbb{R}^d$ (See e.g. Kunita [28]). We denote by $\hat{X}_s^{t,\cdot}$ the inverse flow and by $J(\hat{X}_s^{t,x})$ the determinant of the Jacobi matrix of $\hat{X}_s^{t,x}$ respectively. For $\varphi\in H_\rho^k(\mathbb{R}^d;\mathbb{R}^1)$, we define a process $\varphi_t:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}^1$ by $\varphi_t(s,x)=\varphi(\hat{X}_s^{t,x})J(\hat{X}_s^{t,x})$. It is proved in [3] that $\varphi_t(s,\cdot)\in H_\rho^k(\mathbb{R}^d;\mathbb{R}^1)$ and for $u\in H_\rho^{k*}(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}
\int_{\mathbb{R}^d}u(x)\varphi(x)dx &\triangleq \sum_{0\leq|\alpha|\leq k}\int_{\mathbb{R}^d}u_\alpha(x)D^\alpha\varphi(x)dx\\
&\leq \sum_{0\leq|\alpha|\leq k}\sqrt{\int_{\mathbb{R}^d}|u_\alpha(x)|^2\rho^{-1}(x)dx\int_{\mathbb{R}^d}|D^\alpha\varphi(x)|^2\rho(x)dx}<\infty
\end{aligned}$$

and

$$\int_{\mathbb{R}^d} u(y)\varphi_t(s,y)dy = \int_{\mathbb{R}^d} u(X_s^{t,x})\varphi(x)dx.$$

The following lemma is an extension of equivalence of norm principle given in [29], [4], [3] to the cases when φ and Ψ are random.

Lemma 2.3.3. *(generalized equivalence of norm principle) Let ρ be the weight function and X be the diffusion process defined in Subsection 2.3.1. If $s \in [t,T]$, $\varphi : \Omega\times\mathbb{R}^d \to \mathbb{R}^1$ is independent of $\mathscr{F}_{t,s}^W$ and $\varphi\rho^{-1} \in L^1(\Omega\otimes\mathbb{R}^d)$, then there exist two constants $c > 0$ and $C > 0$ such that*

$$cE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx] \leq E[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})|\rho^{-1}(x)dx] \leq CE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx].$$

Moreover if $\Psi : \Omega \times [t,T] \times \mathbb{R}^d \to \mathbb{R}^1$, $\Psi(s,\cdot)$ is independent of $\mathscr{F}_{t,s}^W$ and $\Psi\rho^{-1} \in L^1(\Omega\otimes[t,T]\otimes\mathbb{R}^d)$, then

$$\begin{aligned} & cE[\int_t^T \int_{\mathbb{R}^d} |\Psi(s,x)|\rho^{-1}(x)dxds] \leq E[\int_t^T \int_{\mathbb{R}^d} |\Psi(s,X_s^{t,x})|\rho^{-1}(x)dxds] \\ \leq\ & CE[\int_t^T \int_{\mathbb{R}^d} |\Psi(s,x)|\rho^{-1}(x)dxds]. \end{aligned}$$

Proof. Using the conditional expectation w.r.t. $\mathscr{F}_{t,s}^W$ and noting that $\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}$ is $\mathscr{F}_{t,s}^W$ measurable and $|\varphi(y)|\rho^{-1}(y)$ is independent of $\mathscr{F}_{t,s}^W$, we have

$$\begin{aligned} & E[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})|\rho^{-1}(x)dx] \\ =\ & \int_{\mathbb{R}^d} E[\ E[\ |\varphi(y)|\rho^{-1}(y)\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}\ |\mathscr{F}_{t,s}^W]\ ]dy \\ =\ & \int_{\mathbb{R}^d} E[\ |\varphi(y)|\rho^{-1}(y)]E[\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}]dy. \end{aligned}$$

By Lemma 5.1 in [3], for any $y \in \mathbb{R}^d$, $s \in [t,T]$,

$$c \leq E[\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}] \leq C,$$

so the first claim follows. The second claim can be proved similarly. ◇

Remark 2.3.4. *By Lemma 2.3.3 and the fact that $\int_{\mathbb{R}^d} x^p\rho^{-1}(x)dx < \infty$, it is easy to deduce that $X_\cdot^{t,\cdot} \in M^{p,-K}([0,\infty); L_\rho^p(\mathbb{R}^d;\mathbb{R}^d))$ for $K \in \mathbb{R}^+$.*

§2.3.3 Conditions, examples and main results in Chapter 3

The main purpose of this section is to find the stationary weak solution of SPDE (1.2) via the solution of BDSDE (2.16). In this subsection, we give the following conditions, examples and main results in Chapter 3 for BDSDE (2.16) in order to prove the stationarity of SPDE in next subsection, and the proofs of theorems will be given in Chapter 3.

(A.2.1)'. Functions $f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ and $g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathcal{L}^2_{U_0}(\mathbb{R}^1)$ are $\mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable, and there exist constants $M_2, M_{2j}, C, C_j, \alpha_j \geq 0$ with $\sum_{j=1}^{\infty} M_{2j} < \infty$, $\sum_{j=1}^{\infty} C_j < \infty$ and $\sum_{j=1}^{\infty} \alpha_j < \frac{1}{2}$ s.t. for any $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$, $X_1, X_2, Z_1, Z_2 \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$, measurable $U : \mathbb{R}^d \to [0,1]$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} U(x)|f(X_1(x), Y_1(x), Z_1(x)) - f(X_2(x), Y_2(x), Z_2(x))|^2 \rho^{-1}(x)dx \\
\leq\ & \int_{\mathbb{R}^d} U(x)\big(M_2|X_1(x) - X_2(x)|^2 + C|Y_1(x) - Y_2(x)|^2 \\
&\qquad + C|Z_1(x) - Z_2(x)|^2\big)\rho^{-1}(x)dx, \\
&\int_{\mathbb{R}^d} U(x)|g_j(X_1(x), Y_1(x), Z_1(x)) - g_j(X_2(x), Y_2(x), Z_2(x))|^2 \rho^{-1}(x)dx \\
\leq\ & \int_{\mathbb{R}^d} U(x)\big(M_{2j}|X_1(x) - X_2(x)|^2 + C_j|Y_1(x) - Y_2(x)|^2 \\
&\qquad + \alpha_j|Z_1(x) - Z_2(x)|^2\big)\rho^{-1}(x)dx;
\end{aligned}
$$

(A.2.2)'. For $p \in (2, q-1)$,

$$
\int_{\mathbb{R}^d} |f(x,0,0)|^p \rho^{-1}(x)dx < \infty \text{ and } \int_{\mathbb{R}^d} \|g(x,0,0)\|^p_{\mathcal{L}^p_{U_0}(\mathbb{R}^1)} \rho^{-1}(x)dx < \infty;
$$

(A.2.3)'. $b \in C^2_{l,b}(\mathbb{R}^d; \mathbb{R}^1)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^1)$, furthermore, for p is given in (A.2.2)', if L is the global Lipschitz constant for b and σ, L satisfies $K - pL - \frac{p(p-1)}{2}L^2 > 0$;

(A.2.4)'. There exists a constant $\mu > 0$ with $2\mu - pK - pC - \frac{p(p-1)}{2}\sum_{j=1}^{\infty} C_j > 0$ s.t. for any $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$, $X, Z \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$, measurable $U : \mathbb{R}^d \to [0,1]$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} U(x)\big(Y_1(x) - Y_2(x)\big)\big(f(X(x), Y_1(x), Z(x)) - f(X(x), Y_2(x), Z(x))\big)\rho^{-1}(x)dx \\
\leq\ & -\mu \int_{\mathbb{R}^d} U(x)|Y_1(x) - Y_2(x)|^2 \rho^{-1}(x)dx.
\end{aligned}
$$

Remark 2.3.5. *We need monotone condition* (A.2.4)' *in order to solve the infinite horizon BDSDEs. But it does not seem obvious to replace the Lipschitz condition in* *the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ (we call it weak Lipschitz condition) for f in* (A.2.1)$'$ *by a weaker condition on f such as f is continuous in y using the infinite horizon BSDE procedure (e.g. [40]). The difficulty is due to the fact that we consider various conditions in the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ here rather than pointwise ones, therefore we cannot solve the BDSDEs pointwise in x. However, the Lipschitz condition can be relaxed if we strengthen our assumption by changing some conditions in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ to pointwise ones. Later we will study one non-Lipschitz case in Chapter 4. Here we only consider the Lipschitz continuous function f to initiate this intrinsic method to the study of this basic problem.*

Now we give some simple examples. Since the examples for only Lipschitz condition are very familiar, we concentrate ourself on examples for f with $\mathbb{R}^1$ domain. Without losing any generality, assume that p is just a little bit bigger than 2.

Example 2.3.6. $f(y) = -\frac{1}{2}y+2$, $y \in \mathbb{R}^1$. *In this case $\mu = \frac{1}{2}$, $C = \frac{1}{4}$ and $2\mu - pC > 0$. We can construct K, C_j, L and α_j to satisfy Conditions* (A.2.1)$'$–(A.2.4)$'$.

From Example 2.3.6, it is easy to see that f can be any linear function like $f(y) = -Ay + B$, $0 < A < 1$ and $B \in \mathbb{R}^1$.

Example 2.3.7. $f(y) = (-\frac{1}{4}y)I_{\{y<2\}} + (-\frac{1}{5}y - \frac{1}{10})I_{\{y\geq 2\}}$, $y \in \mathbb{R}^1$. *In this case one can verify that $\mu = \frac{1}{4}\bigwedge\frac{1}{5} = \frac{1}{5}$, $C = \frac{1}{16}\bigvee\frac{1}{25} = \frac{1}{16}$ and $2\mu - pC > 0$, then K, C_j, L and α_j can be constructed to satisfy Conditions* (A.2.1)$'$–(A.2.4)$'$.

The Example 2.3.7 provides a method to construct more nonlinear functions.

Example 2.3.8. *If there is a family of linear functions like $f_j(y) = -A_jy + B_j$, $0 < A_j < 1$ and $B_j \in \mathbb{R}^1$, $j = 1, 2, \cdots$, and their coefficients satisfies $\bigwedge_j A_j > \bigvee_j A_j^2$, then from Example 2.3.7, we can see that any new function constructed by an arbitrary combination of these functions satisfies Conditions* (A.2.1)$'$–(A.2.4)$'$.

Remark 2.3.9. *All the above examples are of the pointwise Lipschitz continuous condition. Obviously, the weak Lipschitz continuous condition* (A.2.1)$'$ *is weaker. For instance, we can change the function f in Example 2.3.6 to $f(y) = (-\frac{1}{2}y + 2)I_{\{y\in\mathbb{R}_0^c\}}$, where $\mathbb{R}_0$ is the rational number set in $\mathbb{R}^1$ and $\mathbb{R}_0^c$ is its complementary set.*

We first acknowledge the two theorems below and give their proofs in Section 3.4.

Theorem 2.3.10. *Under Conditions* (A.2.1)$'$–(A.2.4)$'$, *Eq.(2.16) has a unique solution $(Y_s^{t,x}, Z_s^{t,x})$. Moreover $E[\sup_{s\geq 0}\int_{\mathbb{R}^d} \mathrm{e}^{-pKs}|Y_s^{t,x}|^p\rho^{-1}(x)dx] < \infty$.*

Theorem 2.3.11. *Under Conditions* (A.2.1)′–(A.2.4)′, *let* $u(t,\cdot) \triangleq Y_t^{t,\cdot}$, *where* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ *is the solution of Eq.(2.16). Then for arbitrary T and $t \in [0,T]$, $u(t,\cdot)$ is a weak solution for Eq.(2.19). Moreover, $u(t,\cdot)$ is* a.s. *continuous w.r.t. t in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$.*

§2.3.4 Stationary weak solution

Using the conditions and results given in last subsection, we prove the main results in this section.

Theorem 2.3.12. *Under Conditions* (A.2.1)′–(A.2.4)′, *let* $u(t,\cdot) \triangleq Y_t^{t,\cdot}$, *where* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ *is the solution of Eq.(2.16). Then $u(t,\cdot)$ has an indistinguishable version which is a "perfect" stationary weak solution of Eq.(2.19).*

Proof. For $Y \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $Z \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$, let

$$\hat{f}(\mathcal{T},Y,Z) = f(X_s^{t,\cdot},Y,Z), \quad \hat{g}(\mathcal{T},Y,Z) = g(X_s^{t,\cdot},Y,Z).$$

Here we take $\mathcal{T} = (s,t)$ as a dual time variable (t is fixed). By Condition (A.2.1)′, we have

$$\begin{aligned} & \|\hat{f}(\mathcal{T},Y,Z)\|^2_{L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)} \\ = & \int_{\mathbb{R}^d} |f(X_s^{t,x},Y(x),Z(x))|^2\rho^{-1}(x)dx \\ \leq & C_p\int_{\mathbb{R}^d} |f(X_s^{t,x},0,0)|^2\rho^{-1}(x)dx + C_p\int_{\mathbb{R}^d}|Y(x)|^2\rho^{-1}(x)dx + C_p\int_{\mathbb{R}^d}|Z(x)|^2\rho^{-1}(x)dx. \end{aligned}$$

Here and through the thesis, C_p is a generic constant. By Lemma 2.3.3 and Condition (A.2.2)′,

$$\begin{aligned} E[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}|f(X_s^{t,x},0,0)|^2\rho^{-1}(x)dxds] & \leq C_p\int_0^\infty\int_{\mathbb{R}^d}\mathrm{e}^{-Ks}|f(x,0,0)|^2\rho^{-1}(x)dxds \\ & \leq C_p\int_{\mathbb{R}^d}|f(x,0,0)|^p\rho^{-1}(x)dx < \infty. \end{aligned}$$

We take $\bar{f}(\mathcal{T}) = (\int_{\mathbb{R}^d}|f(X_s^{t,x},0,0)|^2\rho^{-1}(x)dx)^{\frac{1}{2}}$, then $\hat{f}(\mathcal{T},Y,Z)$ satisfies Condition (A.2.1). Similarly we can prove $\hat{g}(\mathcal{T},Y,Z)$ also satisfies Condition (A.2.1). On the other hand, applying $\hat{\theta}_r$ to $\hat{f}(\mathcal{T},Y,Z)$, by (2.15), we have for any $Y \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ and $Z \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$,

$$\hat{\theta}_r \circ \hat{f}(\mathcal{T},Y,Z) = f(\hat{\theta}_r \circ X_s^{t,\cdot},Y,Z) = f(X_{s+r}^{t+r,\cdot},Y,Z).$$

Verifying $\hat{g}(\mathcal{T}, Y, Z)$ in the same way, we know that $\hat{f}(\mathcal{T}, Y, Z)$ and $\hat{g}(\mathcal{T}, Y, Z)$ satisfy Condition (A.2.2). Since by Theorem 2.3.10, Eq.(2.16) has a unique solution $(Y_{\mathcal{T}}, Z_{\mathcal{T}})$, this $(Y_{\mathcal{T}}, Z_{\mathcal{T}})$ is a stationary solution as a consequence of Theorem 2.2.4. Note that t is fixed, so for any $t \geq 0$,

$$\hat{\theta}_r \circ Y_{\mathcal{T}} = \hat{\theta}_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \hat{\theta}_r \circ Z_{\mathcal{T}} = \hat{\theta}_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad \text{for all } r \geq 0, \ s \geq t \text{ a.s..}$$

In particular, for any $t \geq 0$,

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for all } r \geq 0 \text{ a.s..} \tag{2.21}$$

By Theorem 2.3.11, we know that $u(t,\cdot) \triangleq Y_t^{t,\cdot}$ is the weak solution of Eq.(2.19), so we get from (2.21) that for any $t \geq 0$,

$$\hat{\theta}_r \circ u(t,\cdot) = u(t+r,\cdot) \quad \text{for all } r \geq 0 \text{ a.s..}$$

Until now, we only know the "crude" stationary property for $u(t,\cdot)$. But by Theorem 2.3.11, $u(t,\cdot)$ is continuous w.r.t. t, thus we can obtain an indistinguishable version of $u(t,\cdot)$, still denoted by $u(t,\cdot)$, s.t.

$$\hat{\theta}_r \circ u(t,\cdot) = u(t+r,\cdot) \quad \text{for all } t, \ r \geq 0 \ \text{ a.s..}$$

So we proved the desired result. ◇

By Definition 2.3.2, Conditions (A.2.1)′ and (A.2.2)′, one can calculate that

$$g\big(\cdot, u(s,\cdot), (\sigma^*\nabla u)(s,\cdot)\big) \in \mathcal{L}^2_{U_0}(L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$$

is locally square integrable in $[0,T]$. Now we consider Eq.(1.2) with cylindrical Brownian motion B on U_0. For arbitrary $T > 0$, let Y be the solution of Eq.(2.16) and $u(t,\cdot) = Y_t^{t,\cdot}$ be the stationary solution of Eq.(2.19) with $\hat{B}$ chosen as the time reversal of B from time T, i.e. $\hat{B}_s = B_{T-s} - B_T$ or $\hat{\beta}_j(s) = \beta_j(T-s) - \beta_j(T)$ for $s \geq 0$. By (2.3) and integral transformation in Eq.(2.19), we can see that $v(t,x) \triangleq u(T-t,x)$ satisfies Eq.(1.2) or its equivalent form

$$\begin{aligned} v(t,x) = v(t,v_0)(x) = v_0(x) + \int_0^t [\mathscr{L}v(s,x) + f\big(x, v(s,x), (\sigma^*\nabla v)(s,x)\big)]ds \\ + \sum_{j=1}^{\infty}\int_0^t g_j\big(x, v(s,x), (\sigma^*\nabla v)(s,x)\big)d\beta_j(s), \ t \geq 0. \end{aligned} \tag{2.22}$$

Here $v_0(x) = v(0,x)$.

In fact, we can prove a claim that $v(t,\cdot)(\omega) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ does not depend on the choice of T. For this, we only need to show that for any $T' \geq T$, $Y_{T-t}^{T-t,\cdot}(\hat{\omega}) = Y_{T'-t}^{T'-t,\cdot}(\hat{\omega}')$ when $0 \leq t \leq T$, where $\hat{\omega}(s) = B_{T-s} - B_T$ and $\hat{\omega}'(s) = B_{T'-s} - B_{T'}$. Let $\hat{\theta}_\cdot$ and $\hat{\theta}'_\cdot$ be the shifts of $\hat{\omega}(\cdot)$ and $\hat{\omega}'(\cdot)$ respectively. Using the stationary property (2.21) for BDSDE (2.16) proved in Theorem 2.3.12, we have

$$Y_{T-t}^{T-t,\cdot}(\hat{\omega}) = \hat{\theta}_{T-t} Y_0^{0,\cdot}(\hat{\omega}) = Y_0^{0,\cdot}(\hat{\theta}_{T-t}\hat{\omega}),$$
$$Y_{T'-t}^{T'-t,\cdot}(\hat{\omega}') = \hat{\theta}'_{T'-t} Y_0^{0,\cdot}(\hat{\omega}') = Y_0^{0,\cdot}(\hat{\theta}'_{T'-t}\hat{\omega}').$$

So we only need to assert that $\hat{\theta}_{T-t}\hat{\omega} = \hat{\theta}'_{T'-t}\hat{\omega}'$. Indeed we have for any $s \geq 0$,

$$\begin{aligned}(\hat{\theta}_{T-t}\hat{\omega})(s) &= \hat{\omega}(T-t+s) - \hat{\omega}(T-t) \\ &= (B_{T-(T-t+s)} - B_T) - (B_{T-(T-t)} - B_T) \\ &= B_{t-s} - B_t.\end{aligned}$$

Note that the right hand side of the above formula does not depend on T, therefore $\hat{\theta}_{T-t}\hat{\omega}(s) = \hat{\theta}'_{T'-t}\hat{\omega}'(s) = B_{t-s} - B_t$.

On the probability space $(\Omega, \mathscr{F}, P)$, we define $\theta_t = (\hat{\theta}_t)^{-1}$, $t \geq 0$. Actually $\hat{B}$ is a two-sided Brownian motion, so $(\hat{\theta}_t)^{-1} = \hat{\theta}_{-t}$ is well defined (see [1]). It is easy to see that θ_t is a shift w.r.t. B satisfying

(i) $P \cdot (\theta_t)^{-1} = P$;

(ii) $\theta_0 = I$;

(iii) $\theta_s \circ \theta_t = \theta_{s+t}$;

(iv) $\theta_t \circ B_s = B_{s+t} - B_t$.

Since $v(t,\cdot)(\omega) = u(T-t,\cdot)(\hat{\omega}) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ a.s., so

$$\begin{aligned}\theta_r v(t,\cdot)(\omega) &= \hat{\theta}_{-r} u(T-t,\cdot)(\hat{\omega}) = \hat{\theta}_{-r}\hat{\theta}_r u(T-t-r,\cdot)(\hat{\omega}) \\ &= u(T-t-r,\cdot)(\hat{\omega}) = v(t+r,\cdot)(\omega),\end{aligned}$$

for all $r \geq 0$ and $T \geq t+r$ a.s.. In particular, let $Y(\omega) = v_0(\omega) = Y_T^{T,\cdot}(\hat{\omega})$. Then the above formula implies (1.1):

$$\theta_t Y(\omega) = Y(\theta_t\omega) = v(t,\omega) = v(t, v_0(\omega), \omega) = v(t, Y(\omega), \omega) \text{ for all } t \geq 0 \text{ a.s..}$$

That is to say $v(t,\cdot)(\omega) = Y(\theta_t\omega)(\cdot) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ is a stationary weak solution of Eq.(1.2) w.r.t. θ. Therefore we proved the following theorem

Theorem 2.3.13. *Under Conditions* (A.2.1)′–(A.2.4)′, *for arbitrary* T *and* $t \in [0,T]$, *let* $v(t,\cdot) \triangleq Y_{T-t}^{T-t,\cdot}$, *where* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ *is the solution of Eq.(2.16) with* $\hat{B}_s = B_{T-s} - B_T$ *for all* $s \geq 0$. *Then* $v(t,\cdot)$ *is a "perfect" stationary weak solution of Eq.(1.2).*

Chapter 3

Stationary Weak Solutions of SPDEs

In this chapter, we will study the weak solutions of SPDEs and the corresponding $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued BDSDEs. Before studying the BDSDEs on infinite horizon and giving the proofs of Theorems 2.3.10 and 2.3.11, we first study the BDSDEs on finite horizon and establish the connection with SPDEs.

§3.1 Finite Horizon BDSDEs

§3.1.1 Conditions, definition and main result

For finite dimensional noise and under Lipschitz condition for a.e. $x\in\mathbb{R}^d$, the problem was studied in Bally and Matoussi [3]. In this section, we consider the following BDSDE with infinite dimensional noise on finite horizon:

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x})+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr \\ &\quad -\int_s^T g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d^\dagger\hat{B}_r-\int_s^T\langle Z_r^{t,x},dW_r\rangle,\quad 0\le s\le T. \end{aligned} \tag{3.1}$$

Here $h:\Omega\times\mathbb{R}^d\longrightarrow\mathbb{R}^1$, $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\longrightarrow\mathbb{R}^1$, $g:[0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\longrightarrow\mathcal{L}^2_{U_0}(\mathbb{R}^1)$. Set $g_j\triangleq g\sqrt{\lambda_j}e_j:[0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\longrightarrow\mathbb{R}^1$, then Eq.(3.1) is equivalent to

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x})+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr \\ &\quad -\sum_{j=1}^\infty\int_s^T g_j(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d^\dagger\hat{\beta}_j(r)-\int_s^T\langle Z_r^{t,x},dW_r\rangle,\quad 0\le s\le T. \end{aligned}$$

We assume

(H.3.1). Function h is $\mathscr{F}^{\hat{B}}_{T,\infty} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable and $E[\int_{\mathbb{R}^d} |h(x)|^2\rho^{-1}(x)dx] < \infty$;

(H.3.2). Functions f and g are $\mathscr{B}_{[0,T]} \otimes \mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable and there exist constants $C, C_j, \alpha_j \geq 0$ with $\sum_{j=1}^{\infty} C_j < \infty$ and $\sum_{j=1}^{\infty} \alpha_j < \frac{1}{2}$ s.t. for any $t \in [0,T]$, $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $X, Z_1, Z_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$,

$$\begin{aligned}
& \int_{\mathbb{R}^d} |f(t,X(x),Y_1(x),Z_1(x)) - f(t,X(x),Y_2(x),Z_2(x))|^2\rho^{-1}(x)dx \\
\leq\ & C\int_{\mathbb{R}^d} (|Y_1(x)-Y_2(x)|^2 + |Z_1(x)-Z_2(x)|^2)\rho^{-1}(x)dx, \\
& \int_{\mathbb{R}^d} |g_j(t,X(x),Y_1(x),Z_1(x)) - g_j(t,X(x),Y_2(x),Z_2(x))|^2\rho^{-1}(x)dx \\
\leq\ & \int_{\mathbb{R}^d} (C_j|Y_1(x)-Y_2(x)|^2 + \alpha_j|Z_1(x)-Z_2(x)|^2)\rho^{-1}(x)dx;
\end{aligned}$$

(H.3.3). $\int_0^T\int_{\mathbb{R}^d} |f(s,x,0,0)|^2\rho^{-1}(x)dxds < \infty$ and $\int_0^T\int_{\mathbb{R}^d} \|g(s,x,0,0)\|^2_{\mathcal{L}^2_{U_0}(\mathbb{R}^1)}\rho^{-1}(x)dxds < \infty$;

(H.3.4). $b \in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^d \times \mathbb{R}^d)$.

Needless to say, the conditions (H.3.1)-(H.3.4) for the existence and uniqueness of solution of Eq.(3.1) are weaker than what are needed for the case of infinite horizon in Subsection 2.3.3. We would like to point out that for the finite horizon problem, our conditions are weaker than those in Bally and Matoussi [3]. In (H.3.1), we allow the terminal function h to depend on the $\mathscr{F}_{t,T}$-independent σ-field $\mathscr{F}^{\hat{B}}_{T,\infty}$. One can easily verify that it doesn't affect the results in [3]. Moreover, here we only need Lipschitz condition in the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ instead of the pointwise Lipschitz condition posed in [3].

Definition 3.1.1. *A pair of processes* $(Y^{t,\cdot}_\cdot, Z^{t,\cdot}_\cdot) \in S^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ *is called a solution of Eq.(3.1) if for any* $\varphi \in C^0_c(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} Y^{t,x}_s\varphi(x)dx =\ & \int_{\mathbb{R}^d} h(X^{t,x}_T)\varphi(x)dx + \int_s^T\int_{\mathbb{R}^d} f(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)\varphi(x)dxdr \\
& - \sum_{j=1}^{\infty}\int_s^T\int_{\mathbb{R}^d} g_j(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)\varphi(x)dx d^\dagger\hat{\beta}_j(r) \\
& - \int_s^T \langle \int_{\mathbb{R}^d} Z^{t,x}_r\varphi(x)dx, dW_r\rangle \quad P-\text{a.s.}. \qquad (3.2)
\end{aligned}$$

The main objective of this section is to prove

Theorem 3.1.2. *Under Conditions* (H.3.1)-(H.3.4), *Eq.(3.1) has a unique solution.*

This theorem is an extension of Theorem 3.1 in [3]. The idea is to start from Bally and Matoussi's results for finite dimensional noise and then take the limit to obtain the solution for the case of infinite dimensional noise. But Bally and Matoussi's results cannot apply immediately here as we have a weaker Lipschitz condition and some of the key claims in the proof of Theorem 3.1 ([3]) are not obvious under their conditions. Moreover, the result $Y_\cdot^{t,\cdot} \in S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ was not obtained in [3].

§3.1.2 Substitution theorem

We study a sequence of BDSDEs

$$\begin{aligned} Y_s^{t,x,n} = \quad & h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})dr \\ & -\sum_{j=1}^n \int_s^T g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})d^\dagger\hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r\rangle. \quad (3.3) \end{aligned}$$

A solution of (3.3) is a pair of processes $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}$ $([0,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ satisfying the spatial integral form of Eq.(3.3), i.e. (3.2) with a finite number of one dimensional backward stochastic integrals.

Lemma 3.1.3. *Under Conditions* (H.3.1)-(H.3.4), *if there exists* $(Y_\cdot(\cdot), Z_\cdot(\cdot)) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ *satisfying the spatial integral form of Eq.(3.3) for* $t \leq s \leq T$, *then* $Y_\cdot(\cdot) \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ *and therefore* $(Y_s(x), Z_s(x))$ *is a solution of Eq.(3.3).*

Proof. Let's first see $Y_s(\cdot)$ is continuous w.r.t. s in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$. Since $(Y_s(x), Z_s(x))$ satisfies the form of Eq.(3.3) for $t \leq s < T$, a.e. $x \in \mathbb{R}^d$, therefore

$$\begin{aligned} & \int_{\mathbb{R}^d} |Y_{s+\Delta s}(x) - Y_s(x)|^2 \rho^{-1}(x)dx \\ \leq \ & C_p \int_{\mathbb{R}^d} \int_s^{s+\Delta s} |f(r, X_r^{t,x}, Y_r(x), Z_r(x))|^2 dr \rho^{-1}(x)dx \\ & + C_p \sum_{j=1}^n \int_{\mathbb{R}^d} |\int_s^{s+\Delta s} g_j(r, X_r^{t,x}, Y_r(x), Z_r(x))d^\dagger\hat{\beta}_j(r)|^2 \rho^{-1}(x)dx \\ & + C_p \int_{\mathbb{R}^d} |\int_s^{s+\Delta s} \langle Z_r(x), dW_r\rangle|^2 \rho^{-1}(x)dx. \end{aligned}$$

For the forward stochastic integral part, it is trivial to see that for $0 \leq \Delta s \leq T - s$,

$$|\int_s^{s+\Delta s} \langle Z_r(x), dW_r\rangle|^2 \leq \sup_{0\leq\Delta s\leq T-s} |\int_s^{s+\Delta s} \langle Z_r(x), dW_r\rangle|^2 \ \text{ a.s..}$$

And by the B-D-G inequality and $Z_\cdot(\cdot) \in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, we can deduce that

$$\int_{\mathbb{R}^d} \sup_{0\le \triangle s\le T-s} |\int_s^{s+\triangle s} \langle Z_r(x), dW_r\rangle|^2 \rho^{-1}(x)dx < \infty \quad \text{a.s.}.$$

So by the dominated convergence theorem,

$$\lim_{\triangle s\to 0^+} \int_{\mathbb{R}^d} |\int_s^{s+\triangle s} \langle Z_r(x), dW_r\rangle|^2 \rho^{-1}(x)dx = 0.$$

Similarly we can prove for $t < s \le T$,

$$\lim_{\triangle s\to 0^-} \int_{\mathbb{R}^d} |\int_{s+\triangle s}^{s} \langle Z_r(x), dW_r\rangle|^2 \rho^{-1}(x)dx = 0.$$

The backward stochastic integral part tends to 0 as $\triangle s \to 0$ can be deduced similarly. So $Y_s(\cdot)$ is continuous w.r.t. s in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$.

From Conditions (H.3.2)–(H.3.4) and $(Y_\cdot(\cdot), Z_\cdot(\cdot)) \in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, it follows that for a.e. $x \in \mathbb{R}^d$,

$$E[\int_t^T |f(r, X_r^{t,x}, Y_r(x), Z_r(x))|^2 dr] < \infty$$

and

$$\sum_{j=1}^n E[\int_t^T |g_j(r, X_r^{t,x}, Y_r(x), Z_r(x))|^2 dr] < \infty.$$

For a.e. $x \in \mathbb{R}^d$, referring to Lemma 1.4 in [44], we apply the generalized Itô's formula (c.f. Elworthy, Truman and Zhao [21]) to $\psi_M\big(Y_r(x)\big)$, where

$$\psi_M(x) = x^2 I_{\{-M\le x<M\}} + M(2x-M)I_{\{x\ge M\}} - M(2x+M)I_{\{x<-M\}}.$$

Then

$$\begin{aligned}
&\psi_M(Y_s(x)) + \int_s^T I_{\{-M\le Y_r(x)<M\}}|Z_r(x)|^2 dr \\
= \;& \psi_M\big(h(X_T^{t,x})\big) + \int_s^T \psi_M'(Y_r(x)) f(r, X_r^{t,x}, Y_r(x), Z_r(x))dr \\
&+\sum_{j=1}^n \int_s^T I_{\{-M\le Y_r(x)<M\}}|g_j(r, X_r^{t,x}, Y_r(x), Z_r(x))|^2 dr \qquad (3.4)\\
&-\sum_{j=1}^n \int_s^T \psi_M'(Y_r(x)) g_j(r, X_r^{t,x}, Y_r(x), Z_r(x)) d^\dagger\hat{\beta}_j(r) - \int_s^T \langle \psi_M'(Y_r(x)) Z_r(x), dW_r\rangle.
\end{aligned}$$

We can use the Fubini theorem to perfect (3.4) so that (3.4) is satisfied for a.e. $x \in \mathbb{R}^d$, on a full measure set that is independent of x. Taking integration over $\mathbb{R}^d$ on both sides and applying the stochastic Fubini theorem ([16]), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \psi_M(Y_s(x))\rho^{-1}(x)dx + \int_s^T \int_{\mathbb{R}^d} I_{\{-M\le Y_r(x)<M\}}|Z_r(x)|^2\rho^{-1}(x)dxdr \\
\le\ & \int_{\mathbb{R}^d} \psi_M\big(h(X_T^{t,x})\big)\rho^{-1}(x)dx + \int_s^T \int_{\mathbb{R}^d} \psi_M'(Y_r(x))f(r, X_r^{t,x}, 0, 0)\rho^{-1}(x)dxdr \\
&+ \int_s^T \int_{\mathbb{R}^d} \psi_M'(Y_r(x))\big(f(r, X_r^{t,x}, Y_r(x), Z_r(x)) - f(r, X_r^{t,x}, 0, 0)\big)\rho^{-1}(x)dxdr \\
&+C_p\sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} |g_j(r, X_r^{t,x}, Y_r(x), Z_r(x)) - g_j(r, X_r^{t,x}, 0, 0)|^2\rho^{-1}(x)dxdr \\
&+C_p\sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} |g_j(r, X_r^{t,x}, 0, 0)|^2\rho^{-1}(x)dxdr \\
&- \int_s^T \langle \int_{\mathbb{R}^d} \psi_M'(Y_r(x))Z_r(x)\rho^{-1}(x)dx, dW_r\rangle \\
&- \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} \psi_M'(Y_r(x))g_j(r, X_r^{t,x}, Y_r(x), Z_r(x))\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r).
\end{aligned}$$

Noting that $\psi_M\big(h(X_T^{t,x})\big) \le |h(X_T^{t,x})|^2$ and $|\psi_M'(Y_r(x))|^2 \le 4|Y_r(x)|^2$, so by Lemma 2.3.3, the B-D-G inequality and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&E[\sup_{t\le s\le T} \int_{\mathbb{R}^d} \psi_M(Y_s(x))\rho^{-1}(x)dx] \\
\le\ & C_pE[\int_{\mathbb{R}^d} |h(x)|^2\rho^{-1}(x)dx] + C_pE[\int_t^T \int_{\mathbb{R}^d} (|Y_r(x)|^2 + |Z_r(x)|^2)\rho^{-1}(x)dxdr] \\
&+C_pE[\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} (|g_j(r, x, 0, 0)|^2 + |f(r, x, 0, 0)|^2)\rho^{-1}(x)dxdr] \\
&+C_pE[\sqrt{\int_t^T \int_{\mathbb{R}^d} |\psi_M'(Y_s(x))|^2\rho^{-1}(x)dx \int_{\mathbb{R}^d} \sum_{j=1}^n |g_j(r, X_r^{t,x}, Y_r(x), Z_r(x))|^2\rho^{-1}(x)dxdr}] \\
&+C_pE[\sqrt{\int_t^T \int_{\mathbb{R}^d} |\psi_M'(Y_s(x))|^2\rho^{-1}(x)dx \int_{\mathbb{R}^d} |Z_r(x)|^2\rho^{-1}(x)dxdr}] \\
\le\ & C_pE[\int_{\mathbb{R}^d} |h(x)|^2\rho^{-1}(x)dx] + C_pE[\int_t^T \int_{\mathbb{R}^d} (|Y_r(x)|^2 + |Z_r(x)|^2)\rho^{-1}(x)dxdr] \\
&+C_pE[\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} (|g_j(r, x, 0, 0)|^2 + |f(r, x, 0, 0)|^2)\rho^{-1}(x)dxdr] \\
&+\frac{1}{5}E[\sup_{t\le s\le T} \int_{\mathbb{R}^d} |\psi_M'(Y_s(x))|^2\rho^{-1}(x)dx]. \qquad (3.5)
\end{aligned}$$

Since $(Y_\cdot(\cdot), Z_\cdot(\cdot)) \in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, taking the limit as $M \to \infty$ and applying the monotone convergence theorem, we have

$$E[\sup_{t\leq s\leq T}\int_{\mathbb{R}^d} |Y_s(x)|^2\rho^{-1}(x)dx] < \infty.$$

So $Y_\cdot(\cdot) \in S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ follows. That is to say $(Y_s(x), Z_s(x))$ is a solution of Eq.(3.3). ◇

For the rest of Chapter 3, we will leave out the similar localization argument as in the proof of Lemma 3.1.3 when applying Itô's formula to save the space of this thesis.

Theorem 3.1.4. (Substitution theorem) *Under Conditions* (H.3.1)–(H.3.4), *assume Eq.(3.3) has a unique solution* $(Y_r^{t,x,n}, Z_r^{t,x,n})$, *then for any* $t \leq s \leq T$,

$$Y_r^{s,X_s^{t,x},n} = Y_r^{t,x,n} \text{ and } Z_r^{s,X_s^{t,x},n} = Z_r^{t,x,n} \text{ for } r\in[s,T], \text{ a.a. } x\in\mathbb{R}^d \text{ a.s..}$$

Proof. For $t\leq s\leq r\leq T$, note that $(Y_r^{s,\cdot,n}, Z_r^{s,\cdot,n})$ is $\mathscr{F}^{\hat{B}}_{r,\infty}\otimes\mathscr{F}^W_{s,r}$ measurable, so is independent of $\mathscr{F}^W_{t,s}$. Thus by Lemma 2.3.3, we have

$$\begin{aligned}&E[\int_s^T\int_{\mathbb{R}^d}(|Y_r^{s,X_s^{t,x},n}|^2 + |Z_r^{s,X_s^{t,x},n}|^2)\rho^{-1}(x)dxdr]\\ \leq\ & C_pE[\int_s^T\int_{\mathbb{R}^d}(|Y_r^{s,x,n}|^2 + |Z_r^{s,x,n}|^2)\rho^{-1}(x)dxdr] < \infty.\end{aligned}$$

Moreover, it is easy to see that $X_r^{s,X_s^{t,x}} = X_r^{t,x}$ and $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ is $\mathscr{F}^{\hat{B}}_{r,\infty}\otimes\mathscr{F}^W_{t,r}$ measurable, so $(Y_\cdot^{s,X_s^{t,\cdot},n}, Z_\cdot^{s,X_s^{t,\cdot},n}) \in M^{2,0}([s,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([s,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ satisfies the spatial integral form of Eq.(3.3) for $s\leq r\leq T$.

Define $Y_r^{s,X_s^{t,x},n} = Y_r^{t,x,n}$, $Z_r^{s,X_s^{t,x},n} = Z_r^{t,x,n}$ when $t\leq r<s$. Then $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ satisfies the spatial integral form of Eq.(3.3) for $t\leq r\leq T$ and $(Y_\cdot^{s,X_s^{t,\cdot},n}, Z_\cdot^{s,X_s^{t,\cdot},n}) \in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. Therefore, by Lemma 3.1.3, $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ is the solution of Eq.(3.3).

By the uniqueness of the solution of Eq.(3.3), we have for any $s\in[t,T]$,

$$(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n}) = (Y_r^{t,x,n}, Z_r^{t,x,n}) \text{ for } r\in[s,T], \text{ a.a. } x\in\mathbb{R}^d \text{ a.s..}$$

◇

§3.1.3 The proof of main result

Before Proving Theorem 3.1.2, the main result in this section, we need first to prove the following theorem which plays an key role in the proof of main result.

Theorem 3.1.5. *Under Conditions* (H.3.1)–(H.3.4), *Eq.(3.3) has a unique solution, i.e. there exists a unique* $(Y^{t,\cdot,n}_\cdot, Z^{t,\cdot,n}_\cdot) \in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ *such that for an arbitrary* $\varphi \in C^0_c(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}\int_{\mathbb{R}^d} Y^{t,x,n}_s\varphi(x)dx &= \int_{\mathbb{R}^d} h(X^{t,x}_T)\varphi(x)dx + \int_s^T\int_{\mathbb{R}^d} f(r,X^{t,x}_r,Y^{t,x,n}_r,Z^{t,x,n}_r)\varphi(x)dxdr \\ &\quad -\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d} g_j(r,X^{t,x}_r,Y^{t,x,n}_r,Z^{t,x,n}_r)\varphi(x)dxd^\dagger\hat\beta_j(r) \\ &\quad -\int_s^T\langle\int_{\mathbb{R}^d} Z^{t,x,n}_r\varphi(x)dx, dW_r\rangle \quad P-\text{a.s.}.\end{aligned} \tag{3.6}$$

Proof. Uniqueness. Assume there exists another $(\hat Y^{t,x,n}_s, \hat Z^{t,x,n}_s) \in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ satisfying (3.6). Define

$$\bar Y^{t,x,n}_s = Y^{t,x,n}_s - \hat Y^{t,x,n}_s \text{ and } \bar Z^{t,x,n}_s = Z^{t,x,n}_s - \hat Z^{t,x,n}_s,\ t\le s\le T.$$

From Condition (H.3.2) and $(Y^{t,\cdot,n}_\cdot, Z^{t,\cdot,n}_\cdot)$, $(\hat Y^{t,\cdot,n}_\cdot, \hat Z^{t,\cdot,n}_\cdot) \in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, it follows that for a.e. $x\in\mathbb{R}^d$,

$$E[\int_t^T |f(r,X^{t,x}_r,Y^{t,x,n}_r,Z^{t,x,n}_r) - f(r,X^{t,x}_r,\hat Y^{t,x,n}_r,\hat Z^{t,x,n}_r)|^2dr] < \infty$$

and

$$\sum_{j=1}^n E[\int_t^T |g_j(r,X^{t,x}_r,Y^{t,x,n}_r,Z^{t,x,n}_r) - g_j(r,X^{t,x}_r,\hat Y^{t,x,n}_r,\hat Z^{t,x,n}_r)|^2dr] < \infty.$$

For a.e. $x\in\mathbb{R}^d$, similar as in (3.4), we apply the generalized Itô's formula to $e^{Kr}\psi_M(\bar Y^{t,x,n}_r)$, where $K\in\mathbb{R}^1$, then take integration over $\Omega\otimes\mathbb{R}^d$ on both sides and apply the stochastic Fubini theorem. Note that the stochastic integrals are martingales, so taking the limit as $M\to\infty$, we have

$$\begin{aligned}&E[e^{Ks}\int_{\mathbb{R}^d}|\bar Y^{t,x,n}_s|^2\rho^{-1}(x)dx] + (\frac{1}{2}-\sum_{j=1}^\infty\alpha_j)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar Z^{t,x,n}_r|^2\rho^{-1}(x)dxdr] \\ &+(K-2C-\sum_{j=1}^\infty C_j-\frac{1}{2})E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar Y^{t,x,n}_r|^2\rho^{-1}(x)dxdr] \le 0.\end{aligned} \tag{3.7}$$

All the terms on the left hand side of (3.7) are positive when K is sufficiently large, so it is easy to see that for each $s \in [t,T]$, $\bar{Y}_s^{t,x} = 0$ for a.a. $x \in \mathbb{R}^d$ a.s.. By a "standard" argument taking s in the rational number space and noting $\int_{\mathbb{R}^d} e^{Ks}|\bar{Y}_s^{t,x,n}|^2\rho^{-1}(x)dx$ is continuous w.r.t. s, we have $\bar{Y}_s^{t,x,n} = 0$ for $s \in [t,T]$, a.a. $x \in \mathbb{R}^d$ a.s.. Also by (3.7), for a.e. $s \in [t,T]$, $\bar{Z}_s^{t,x,n} = 0$ for a.a. $x \in \mathbb{R}^d$ a.s.. We can modify the values of Z at the measure zero exceptional set of s such that $\bar{Z}_s^{t,x,n} = 0$ for $s \in [t,T]$, a.a. $x \in \mathbb{R}^d$ a.s..

Existence. Step 1: We prove for the following equation:

$$\tilde{Y}_s^{t,x,n} = h(X_T^{t,x}) + \int_s^T \tilde{f}(r, X_r^{t,x})dr - \sum_{j=1}^n \int_s^T \tilde{g}_j(r, X_r^{t,x})d^\dagger\hat{\beta}_j(r) - \int_s^T \langle \tilde{Z}_r^{t,x,n}, dW_r\rangle, \tag{3.8}$$

if (H.3.1) and (H.3.4) are satisfied, and $\tilde{f}(\cdot, X_\cdot^{t,\cdot})$, $\tilde{g}_j(\cdot, X_\cdot^{t,\cdot}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, then there exists a unique solution. For this, we can first use a similar method as in the proof of Theorem 2.1 in [3] to prove there exists $(\tilde{Y}_\cdot^{t,\cdot,n}, \tilde{Z}_\cdot^{t,\cdot,n}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ such that for an arbitrary $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}&\int_{\mathbb{R}^d} \tilde{Y}_s^{t,x,n}\varphi(x)dx = \int_{\mathbb{R}^d} h(X_T^{t,x})\varphi(x)dx + \int_s^T\int_{\mathbb{R}^d} \tilde{f}(r, X_r^{t,x})\varphi(x)dxdr \\ &-\sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} \tilde{g}_j(r, X_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r) - \int_s^T \langle \int_{\mathbb{R}^d} \tilde{Z}_r^{t,x,n}\varphi(x)dx, dW_r\rangle \quad P-\text{a.s.}.\end{aligned}$$

By Lemma 3.1.3, $\tilde{Y}_\cdot^{t,\cdot,n} \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Then Step 1 follows.

Step 2: Given $(Y_s^{t,x,n,N-1}, Z_s^{t,x,n,N-1}) \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$, define $(Y_s^{t,x,n,N}, Z_s^{t,x,n,N})$ as follows:

$$\begin{aligned}Y_s^{t,x,n,N} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,n,N-1}, Z_r^{t,x,n,N-1})dr \\ &-\sum_{j=1}^n \int_s^T g_j(r, X_r^{t,x}, Y_r^{t,x,n,N-1}, Z_r^{t,x,n,N-1})d^\dagger\hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n,N}, dW_r\rangle.\end{aligned} \tag{3.9}$$

Let $(Y_r^{t,x,n,0}, Z_r^{t,x,n,0}) = (0,0)$. By Conditions (H.3.1), (H.3.3), (H.3.4) and Lemma 2.3.3, we know h, $f(r, X_r^{t,x}, 0, 0)$ and $g_j(r, X_r^{t,x}, 0, 0)$ satisfy the conditions in Step 1, so Eq.(3.8) has a unique solution $(Y_\cdot^{t,\cdot,n,1}, Z_\cdot^{t,\cdot,n,1}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ when $\tilde{f}(r, X_r^{t,x}) = f(r, X_r^{t,x}, 0, 0)$ and $\tilde{g}(r, X_r^{t,x}) = g(r, X_r^{t,x}, 0, 0)$. From Theorem 3.1.4 and the Fubini theorem, we have

$$Y_r^{t,x,n,1} = Y_r^{r,X_r^{t,x},n,1} \text{ and } Z_r^{t,x,n,1} = Z_r^{r,X_r^{t,x},n,1} \text{ for a.a. } r \in [t,T],\ x \in \mathbb{R}^d \text{ a.s.}.$$

Thus by Conditions (H.3.1)–(H.3.4) and Lemma 2.3.3, we have

$$f(r,X_r^{t,x},Y_r^{t,x,n,1},Z_r^{t,x,n,1})=f(r,X_r^{t,x},Y_r^{r,X_r^{t,x},n,1},Z_r^{r,X_r^{t,x},n,1}),$$

$$g_j(r,X_r^{t,x},Y_r^{t,x,n,1},Z_r^{t,x,n,1})=g_j(r,X_r^{t,x},Y_r^{r,X_r^{t,x},n,1},Z_r^{r,X_r^{t,x},n,1})$$

and h satisfy the conditions in Step 1. Following the same procedure, we obtain $(Y_\cdot^{t,\cdot,n,2},Z_\cdot^{t,\cdot,n,2})\in M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. In general, we see (3.9) is an iterated mapping from $S^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ to itself and obtain a sequence $\{(Y_r^{t,x,n,i},Z_r^{t,x,n,i})\}_{i=0}^\infty$. We will prove that (3.9) is a contraction mapping. For this, define for $t\le s\le T$,

$$\begin{aligned}
&\bar Y_s^{t,x,n,i}=Y_s^{t,x,n,i}-Y_s^{t,x,n,i-1},\quad \bar Z_s^{t,x,n,i}=Z_s^{t,x,n,i}-Z_s^{t,x,n,i-1},\\
&\bar f^i(s,x)=f(s,X_s^{t,x},Y_s^{t,x,n,i},Z_s^{t,x,n,i})-f(s,X_s^{t,x},Y_s^{t,x,n,i-1},Z_s^{t,x,n,i-1}),\\
&\bar g_j^i(s,x)=g_j(s,X_s^{t,x},Y_s^{t,x,n,i},Z_s^{t,x,n,i})-g_j(s,X_s^{t,x},Y_s^{t,x,n,i-1},Z_s^{t,x,n,i-1}),\ i=2,3,\cdots
\end{aligned}$$

Then, for a.e. $x\in\mathbb{R}^d$, $(\bar Y_s^{t,x,n,N},\bar Z_s^{t,x,n,N})$ satisfies

$$\bar Y_s^{t,x,n,N}=\int_s^T\bar f^{N-1}(r,x)dr-\sum_{j=1}^n\int_s^T\bar g_j^{N-1}(r,x)d^\dagger\hat\beta_j(r)-\int_s^T\langle\bar Z_r^{t,x,n,N},dW_r\rangle.$$

Applying the generalized Itô's formula to $e^{Kr}|\bar Y_r^{t,x,n,N}|^2$ for a.e. $x\in\mathbb{R}^d$, by the Young inequality and Condition (H.3.2), we can deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^d}e^{Ks}|\bar Y_s^{t,x,n,N}|^2\rho^{-1}(x)dx+K\int_s^T\int_{\mathbb{R}^d}e^{Kr}|\bar Y_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr\\
&+\int_s^T\int_{\mathbb{R}^d}e^{Kr}|\bar Z_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr\\
\le&\int_s^T\int_{\mathbb{R}^d}e^{Kr}\big(2C|\bar Y_r^{t,x,n,N}|^2+\frac12|\bar Y_r^{t,x,n,N-1}|^2+\frac12|\bar Z_r^{t,x,n,N-1}|^2\big)\rho^{-1}(x)dxdr\\
&+\int_s^T\int_{\mathbb{R}^d}e^{Kr}\big(\sum_{j=1}^\infty C_j|\bar Y_r^{t,x,n,N-1}|^2+\sum_{j=1}^\infty\alpha_j|\bar Z_r^{t,x,n,N-1}|^2\big)\rho^{-1}(x)dxdr\\
&-\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d}e^{Kr}2\bar Y_r^{t,x,n,N}\bar g_j^{N-1}(r,x)\rho^{-1}(x)dxd^\dagger\hat\beta_j(r)\\
&-\int_s^T\langle\int_{\mathbb{R}^d}e^{Kr}2\bar Y_r^{t,x,n,N}\bar Z_r^{t,x,n,N}\rho^{-1}(x)dx,dW_r\rangle.
\end{aligned}\tag{3.10}$$

Then we have

$$(K-2C)E[\int_s^T\int_{\mathbb{R}^d}e^{Kr}|\bar Y_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr]$$

$$
\begin{aligned}
& +E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} |\bar{Z}_r^{t,x,n,N}|^2 \rho^{-1}(x) dx dr] \\
\leq & \ (\frac{1}{2} + \sum_{j=1}^{\infty} \alpha_j) E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big((1 + 2\sum_{j=1}^{\infty} C_j) |\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2 \big) \rho^{-1}(x) dx dr].
\end{aligned}
$$

Letting $K = 1 + 2C + 2\sum_{j=1}^{\infty} C_j$, we have

$$
\begin{aligned}
& E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big((1 + 2\sum_{j=1}^{\infty} C_j) |\bar{Y}_r^{t,x,n,N}|^2 + |\bar{Z}_r^{t,x,n,N}|^2 \big) \rho^{-1}(x) dx dr] \qquad (3.11) \\
\leq & \ (\frac{1}{2} + \sum_{j=1}^{\infty} \alpha_j) E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big((1 + 2\sum_{j=1}^{\infty} C_j) |\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2 \big) \rho^{-1}(x) dx dr].
\end{aligned}
$$

Note that $E[\int_t^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big((1+2\sum_{j=1}^{\infty} C_j)|\cdot|^2 + |\cdot|^2 \big) \rho^{-1}(x) dx dr]$ is equivalent to $E[\int_t^T \int_{\mathbb{R}^d} \big(|\cdot|^2 + |\cdot|^2 \big) \rho^{-1}(x) dx dr]$. From the contraction principle, the mapping (3.9) has a pair of fixed point $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ that is the limit of the Cauchy sequence $\{(Y_\cdot^{t,\cdot,n,N}, Z_\cdot^{t,\cdot,n,N})\}_{N=1}^{\infty}$ in $M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. We then prove that $Y_\cdot^{t,\cdot,n}$ is also the limit of $Y_\cdot^{t,\cdot,n,N}$ in $S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ as $N \to \infty$. For this, we only need to prove that $\{Y_\cdot^{t,\cdot,n,N}\}_{N=1}^{\infty}$ is a Cauchy sequence in $S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Similar as in (3.5), by the B-D-G inequality and the Cauchy-Schwartz inequality, from (3.10), we have

$$
\begin{aligned}
& E[\sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \mathrm{e}^{Ks} |\bar{Y}_s^{t,x,n,N}|^2 \rho^{-1}(x) dx] \qquad (3.12) \\
\leq & \ M_3 E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big(|\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2 + |\bar{Y}_r^{t,x,n,N}|^2 + |\bar{Z}_r^{t,x,n,N}|^2 \big) \rho^{-1}(x) dx dr],
\end{aligned}
$$

where $M_3 > 0$ is independent of n and N. Without losing any generality, assume that $M \geq N$. We can deduce from (3.11) and (3.12) that

$$
\begin{aligned}
& \big(E[\sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x,n,M} - Y_s^{t,x,n,N}|^2 \rho^{-1}(x) dx] \big)^{\frac{1}{2}} \\
\leq & \ \sum_{i=N+1}^{M} \big(E[\sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x,n,i}|^2 \rho^{-1}(x) dx] \big)^{\frac{1}{2}} \\
\leq & \ \sum_{i=N+1}^{M} \big(M_3 E[\int_t^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big(|\bar{Y}_r^{t,x,n,i-1}|^2 + |\bar{Z}_r^{t,x,n,i-1}|^2 \\
& \qquad + |\bar{Y}_r^{t,x,n,i}|^2 + |\bar{Z}_r^{t,x,n,i}|^2 \big) \rho^{-1}(x) dx dr] \big)^{\frac{1}{2}} \\
\leq & \ \sum_{i=N+1}^{M} \big(2M_3 E[\int_t^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big((1 + 2\sum_{j=1}^{\infty} C_j) |\bar{Y}_r^{t,x,n,i-1}|^2 + |\bar{Z}_r^{t,x,n,i-1}|^2 \big) \rho^{-1}(x) dx dr] \big)^{\frac{1}{2}} \\
\leq & \ \sum_{i=N+1}^{\infty} (\frac{1}{2} + \sum_{j=1}^{\infty} \alpha_j)^{\frac{i-2}{2}} \big(2M_3 E[\int_t^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big((1 + 2\sum_{j=1}^{\infty} C_j) |Y_r^{t,x,n,1}|^2
\end{aligned}
$$

$$+|Z_r^{t,x,n,1}|^2)\rho^{-1}(x)dxdr])^{\frac{1}{2}} \longrightarrow 0 \text{ as } M,\ N \longrightarrow \infty.$$

The theorem is proved. ◇

Following a similar procedure as in the proof of Lemma 3.1.3, and applying Itô's formula to $e^{Kr}|Y_r^{t,x,n}|^2$, by the B-D-G inequality, we have the following estimation for the solution of Eq.(3.3):

Proposition 3.1.6. *Under the conditions of Theorem 3.1.2, $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ satisfies*

$$\sup_n E[\sup_{t\le s\le T}\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^2\rho^{-1}(x)dx]+\sup_n E[\int_t^T\int_{\mathbb{R}^d}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr]<\infty.$$

Remark 3.1.7. *For $s\in[0,t]$, Eq.(3.3) is equivalent to the following BDSDE*

$$\begin{aligned} Y_s^{x,n} = \quad & Y_t^{t,x,n}+\int_s^t f(r,x,Y_r^{x,n},Z_r^{x,n})dr \\ & -\sum_{j=1}^n\int_s^t g_j(r,x,Y_r^{x,n},Z_r^{x,n})d^\dagger\hat{\beta}_j(r)-\int_s^t\langle Z_r^{x,n},dW_r\rangle. \end{aligned} \tag{3.13}$$

Note that $Y_t^{t,x,n}$ satisfies Condition (H.3.1). *By a similar method as in the proof of Theorem 3.1.5 and Proposition 3.1.6, we can obtain a $(Y_\cdot^{\cdot,n}, Z_\cdot^{\cdot,n})\in S^{2,0}([0,t];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,t];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$, is the unique solution of Eq.(3.13). Moreover,*

$$\sup_n E[\sup_{0\le s\le t}\int_{\mathbb{R}^d}|Y_s^{x,n}|^2\rho^{-1}(x)dx]+\sup_n E[\int_0^t\int_{\mathbb{R}^d}|Z_r^{x,n}|^2\rho^{-1}(x)dxdr]<\infty.$$

To unify the notation, we define $(Y_s^{t,x,n}, Z_s^{t,x,n})=(Y_s^{x,n},Z_s^{x,n})$ when $s\in[0,t)$. Then $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})\in S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. Furthermore, we have

$$\sup_n E[\sup_{0\le s\le T}\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^2\rho^{-1}(x)dx]+\sup_n E[\int_0^T\int_{\mathbb{R}^d}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr]<\infty. \tag{3.14}$$

Proof of Theorem 3.1.2. The proof of the uniqueness is rather similar to the uniqueness proof in Theorem 3.1.5, so it is omitted.

Existence. By Theorem 3.1.5 and Remark 3.1.7, for each n, there exists a unique solution $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})\in S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ to Eq.(3.3). We will prove $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ is a Cauchy sequence in $S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. Without losing any generality, assume that $m\ge n$, and define

$$\bar{Y}_s^{t,x,m,n}=Y_s^{t,x,m}-Y_s^{t,x,n},\quad \bar{Z}_s^{t,x,m,n}=Z_s^{t,x,m}-Z_s^{t,x,n},$$

$$\bar{f}^{m,n}(s,x) = f(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}) - f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}),$$
$$\bar{g}_j^{m,n}(s,x) = g_j(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}) - g_j(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}), \qquad 0 \leq s \leq T.$$

Then for $0 \leq s \leq T$ and a.e. $x \in \mathbb{R}^d$,

$$\begin{cases} d\bar{Y}_s^{t,x,m,n} = -\bar{f}^{m,n}(s,x)ds + \sum_{j=1}^{n} \bar{g}_j^{m,n}(s,x)d^\dagger\hat{\beta}_j(s) \\ \qquad\qquad + \sum_{j=n+1}^{m} g_j(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m})d^\dagger\hat{\beta}_j(s) + \langle \bar{Z}_s^{t,x,m,n}, dW_s\rangle \\ \bar{Y}_T^{t,x,m,n} = 0 \quad a.s.. \end{cases}$$

Applying Itô's formula to $\mathrm{e}^{Kr}|\bar{Y}_r^{t,x,m,n}|^2$ for a.e. $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho^{-1}(x)dx + (\frac{1}{2} - \sum_{j=1}^{\infty}\alpha_j)\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr \\
&+(K - 2C - \sum_{j=1}^{\infty} C_j - \frac{1}{2})\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr \\
\leq\ & C_p \sum_{j=n+1}^{m}\{(C_j + \alpha_j)(\int_s^T\int_{\mathbb{R}^d}(|Y_r^{t,x,m}|^2 + |Z_r^{t,x,m}|^2)\rho^{-1}(x)dxdr \\
&+ \int_s^T\int_{\mathbb{R}^d}|g_j(r, X_r^{t,x}, 0, 0)|^2\rho^{-1}(x)dxdr)\} \\
&- \sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d} 2\mathrm{e}^{Kr}\bar{Y}_r^{t,x,m,n}\bar{g}_j^{m,n}(r,x)\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r) \\
&- \sum_{j=n+1}^{m}\int_s^T\int_{\mathbb{R}^d} 2\mathrm{e}^{Kr}\bar{Y}_r^{t,x,m,n}g_j(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m})\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r) \\
&- \int_s^T\langle\int_{\mathbb{R}^d} 2\mathrm{e}^{Kr}\bar{Y}_r^{t,x,m,n}\bar{Z}_r^{t,x,m,n}\rho^{-1}(x)dx, dW_r\rangle. \qquad (3.15)
\end{aligned}$$

All the terms on the left hand side of (3.15) are positive when K is sufficiently large. Take expectation on both sides of (3.15), then by Lemma 2.3.3 and (3.14), we have

$$\begin{aligned}
&E[\int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr] + E[\int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr] \\
\leq\ & C_p \sum_{j=n+1}^{m}\{(C_j + \alpha_j)(\sup_n E[\int_0^T\int_{\mathbb{R}^d}(|Y_r^{t,x,n}|^2 + |Z_r^{t,x,n}|^2)\rho^{-1}(x)dxdr] \\
&+ \int_0^T\int_{\mathbb{R}^d}|g_j(r, x, 0, 0)|^2\rho^{-1}(x)dxdr)\} \longrightarrow 0, \qquad \text{as } n,\ m \longrightarrow \infty. \qquad (3.16)
\end{aligned}$$

Also by the B-D-G inequality, from (3.15) we have

$$E[\sup_{0\leq s\leq T}\int_{\mathbb{R}^d} \mathrm{e}^{Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho^{-1}(x)dx]$$

$$
\begin{aligned}
\leq\ & C_pE[\int_0^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}(|\bar{Y}_r^{t,x,m,n}|^2+|\bar{Z}_r^{t,x,m,n}|^2)\rho^{-1}(x)dxdr] \\
&+C_p\sum_{j=n+1}^{m}(C_j+\alpha_j)\big(\sup_n E[\int_0^T\int_{\mathbb{R}^d}(|Y_r^{t,x,n}|^2+|Z_r^{t,x,n}|^2)\rho^{-1}(x)dxdr]\big) \\
&+C_p\sum_{j=n+1}^{m}\int_0^T\int_{\mathbb{R}^d}|g_j(r,x,0,0)|^2\rho^{-1}(x)dxdr.
\end{aligned}
$$

So by (3.14), (3.16) and Condition (H.3.3), we have

$$
E[\sup_{0\leq s\leq T}\int_{\mathbb{R}^d}\mathrm{e}^{Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho^{-1}(x)dx]\longrightarrow 0,\qquad \text{as } n,\ m\longrightarrow\infty.
$$

Therefore $(Y_\cdot^{t,\cdot,n},Z_\cdot^{t,\cdot,n})$ is a Cauchy sequence in $S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ with its limit denoted by $(Y_s^{t,x},Z_s^{t,x})$. We will show that $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})$ is the solution of Eq.(3.1), i.e. $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})$ satisfies (3.2) for an arbitrary $\varphi\in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$. For this, we will prove that Eq.(3.6) converges to Eq.(3.2) in $L^2(\Omega)$ term by term as $n\longrightarrow\infty$. Here we only show the convergence of the third term:

$$
\begin{aligned}
& E[\,|\sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d}g_j(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})\varphi(x)dxd^\dagger\hat{\beta}_j(r) \\
&\qquad -\sum_{j=1}^{\infty}\int_s^T\int_{\mathbb{R}^d}g_j(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2] \\
\leq\ & 2E[\,|\sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d}\big(g_j(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})-g_j(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\big)\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2] \\
&+2E[\,|\sum_{j=n+1}^{\infty}\int_s^T\int_{\mathbb{R}^d}g_j(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2] \\
\leq\ & C_p\sum_{j=1}^{\infty}(C_j+\alpha_j)E[\int_s^T\int_{\mathbb{R}^d}(|Y_r^{t,x,n}-Y_r^{t,x}|^2+|Z_r^{t,x,n}-Z_r^{t,x}|^2)\rho^{-1}(x)dxdr] \\
&+C_pE[\,|\sum_{j=n+1}^{\infty}\int_s^T\int_{\mathbb{R}^d}\big(g_j(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-g_j(r,X_r^{t,x},0,0)\big)\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2] \\
&+C_pE[\,|\sum_{j=n+1}^{\infty}\int_s^T\int_{\mathbb{R}^d}g_j(r,X_r^{t,x},0,0)\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2].
\end{aligned}
$$

Note

$$
\begin{aligned}
& E[\,|\sum_{j=n+1}^{\infty}\int_s^T\int_{\mathbb{R}^d}\big(g_j(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-g_j(r,X_r^{t,x},0,0)\big)\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2] \\
=\ & E[\int_s^T\|\int_{\mathbb{R}^d}\big(g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-g(r,X_r^{t,x},0,0)\big)\varphi(x)dx(\sum_{j=n+1}^{\infty}\lambda_je_j\otimes e_j)^{\frac{1}{2}}\|_{L_U}^2dr]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\int_s^T \sum_{i=1}^{\infty} | \int_{\mathbb{R}^d} (g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - g(r, X_r^{t,x}, 0, 0))\varphi(x)dx \\
&\qquad \times \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \langle e_j, e_i \rangle |^2 dr] \\
&= E[\sum_{j=n+1}^{\infty} \int_s^T | \int_{\mathbb{R}^d} (g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - g_j(r, X_r^{t,x}, 0, 0))\varphi(x)dx|^2 dr] \\
&\leq C_p E[\sum_{j=n+1}^{\infty} \int_s^T \int_{\mathbb{R}^d} |g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - g_j(r, X_r^{t,x}, 0, 0)|^2 \rho^{-1}(x) dx dr] \\
&\leq C_p \sum_{j=n+1}^{\infty} (C_j + \alpha_j) E[\int_s^T \int_{\mathbb{R}^d} (|Y_r^{t,x}|^2 + |Z_r^{t,x}|^2)\rho^{-1}(x) dx dr] \longrightarrow 0. \qquad (3.17)
\end{aligned}
$$

Here we used $(\sum_{j=n+1}^{\infty} \lambda_j e_j \otimes e_j)^{\frac{1}{2}} = \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \otimes e_j$. This can be verified as follows: for an arbitrary $u \in U$, by definition of tensor operator,

$$
\begin{aligned}
(\sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \otimes e_j)(\sum_{i=n+1}^{\infty} \sqrt{\lambda_i} e_i \otimes e_i)u &= \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \langle e_j, \sum_{i=n+1}^{\infty} \sqrt{\lambda_i} e_i \langle e_i, u \rangle \rangle \\
&= \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \langle \sqrt{\lambda_j} e_j, e_j \rangle \langle e_j, u \rangle \\
&= (\sum_{j=n+1}^{\infty} \lambda_j e_j \otimes e_j)u.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
& C_p E[\, | \sum_{j=n+1}^{\infty} \int_s^T \int_{\mathbb{R}^d} g_j(r, X_r^{t,x}, 0, 0)\varphi(x) dx d^{\dagger}\hat{\beta}_j(r)|^2] \\
\leq\ & C_p \int_s^T \int_{\mathbb{R}^d} \sum_{j=n+1}^{\infty} |g_j(r, x, 0, 0)|^2 \rho^{-1}(x) dx dr \longrightarrow 0. \qquad (3.18)
\end{aligned}
$$

That is to say $(Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$ satisfies Eq.(3.2). The proof of Theorem 3.1.2 is completed. ◇

§3.2 The Corresponding SPDEs

§3.2.1 Weak solutions of SPDEs with finite dimensional noise and introduction of Bally and Matoussi's idea

In section 3.1, we proved the existence and uniqueness of solution of BDSDE (3.1) and obtained the solution $(Y_s^{t,x}, Z_s^{t,x})$ by taking the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ of the solutions of Eq.(3.3) in the space $S^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. We still start from Eq.(3.3) in this section. A direct application of Theorem 3.1.4 and Fubini theorem immediately leads to

Proposition 3.2.1. *Under Conditions* (H.3.1)-(H.3.4), *if we define* $u^n(t,x)=Y_t^{t,x,n}$, $v^n(t,x)=Z_t^{t,x,n}$, *then*

$$u^n(s,X_s^{t,x})=Y_s^{t,x,n},\ v^n(s,X_s^{t,x})=Z_s^{t,x,n} \text{ for a.a. } s\in[t,T],\ x\in\mathbb{R}^d \text{ a.s..}$$

Proof. By Theorem 3.1.4, for any $t\leq s\leq T$, we have

$$u^n(s,X_s^{t,x})=Y_s^{t,x,n},\ v^n(s,X_s^{t,x})=Z_s^{t,x,n} \text{ for a.a. } x\in\mathbb{R}^d \text{ a.s..}$$

So

$$E[\int_{\mathbb{R}^d}(|u^n(s,X_s^{t,x})-Y_s^{t,x,n}|+|v^n(s,X_s^{t,x})-Z_s^{t,x,n}|)\rho^{-1}(x)dx]=0.$$

By the Fubini theorem, we have

$$\begin{aligned}
& E[\int_t^T\int_{\mathbb{R}^d}(|u^n(s,X_s^{t,x})-Y_s^{t,x,n}|+|v^n(s,X_s^{t,x})-Z_s^{t,x,n}|)\rho^{-1}(x)dxds]\\
=& \int_t^T E[\int_{\mathbb{R}^d}(|u^n(s,X_s^{t,x})-Y_s^{t,x,n}|+|v^n(s,X_s^{t,x})-Z_s^{t,x,n}|)\rho^{-1}(x)dx]ds\\
=& 0,
\end{aligned}$$

then the conclusion follows. ◇

We use the idea of Bally and Matoussi [3] to establish the connection between the weak solutions of SPDEs and BDSDEs with finite dimensional noise. Consider BDSDE (3.8). Define the mollifier

$$\begin{cases} K^m(x)=mc\exp\{\frac{1}{(mx-1)^2-1}\} & \text{if } 0<x<\frac{2}{m},\\ K^m(x)=0 & \text{otherwise,}\end{cases}$$

where c is chosen such that $\int_{-\infty}^{+\infty}K^m(x)dx=1$. Define

$$\begin{aligned}
h^m(x)&=\int_{\mathbb{R}^d}h(y)K^m(x-y)dy,\\
\tilde{f}^m(r,x)&=\int_{\mathbb{R}^d}\tilde{f}(r,y)K^m(x-y)dy,\\
\tilde{g}_j^m(r,x)&=\int_{\mathbb{R}^d}\tilde{g}_j(r,y)K^m(x-y)dy.
\end{aligned}$$

It is easy to see from standard results in analysis that $h^m(\cdot) \to h(\cdot)$, $\tilde{f}^m(r,\cdot) \to \tilde{f}(r,\cdot)$ and $\tilde{g}_j^m(r,\cdot) \to \tilde{g}_j(r,\cdot)$ in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ respectively. Denote by $(\tilde{Y}^{t,x,n}_{s,m}, \tilde{Z}^{t,x,n}_{s,m})$ the solution of the following BDSDEs:

$$\tilde{Y}^{t,x,n}_{s,m} = h^m(X^{t,x}_T) + \int_s^T \tilde{f}^m(r, X^{t,x}_r)dr - \sum_{j=1}^n \int_s^T \tilde{g}_j^m(r, X^{t,x}_r)d^\dagger\hat{\beta}_j(r) - \int_s^T \langle \tilde{Z}^{t,x,n}_{r,m}, dW_r\rangle.$$

Let $\tilde{u}^n_m(t,x) = \tilde{Y}^{t,x,n}_{t,m}$. Then following classical results of Pardoux and Peng [44], we have $\tilde{Z}^{t,x,n}_{t,m} = \sigma^*\nabla\tilde{u}^n_m(t,x)$, and

$$\tilde{Y}^{t,x,n}_{s,m} = \tilde{u}^n_m(s, X^{t,x}_s) = \tilde{Y}^{s,X^{t,x}_s,n}_{s,m}, \quad \tilde{Z}^{t,x,n}_{s,m} = \sigma^*\nabla\tilde{u}^n_m(s, X^{t,x}_s) = \tilde{Z}^{s,X^{t,x}_s,n}_{s,m}.$$

Moreover $\tilde{u}^n_m(t,x)$ satisfies the smootherized SPDE. In particular, for any smooth test function $\Psi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d;\mathbb{R}^1)$, we still have

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}^d} \tilde{u}^n_m(s,x)\partial_s\Psi(s,x)dxds + \int_{\mathbb{R}^d} \tilde{u}^n_m(t,x)\Psi(t,x)dx - \int_{\mathbb{R}^d} h^m(x)\Psi(T,x)dx \\
&-\frac{1}{2}\int_t^T \int_{\mathbb{R}^d} (\sigma^*\nabla\tilde{u}^n_m)(s,x)(\sigma^*\nabla\Psi)(s,x)dxds \\
&-\int_t^T \int_{\mathbb{R}^d} \tilde{u}^n_m(s,x)\nabla\big((b-\tilde{A})\Psi\big)(s,x)dxds \qquad (3.19)\\
=\ &\int_t^T \int_{\mathbb{R}^d} \tilde{f}^m(s,x)\Psi(s,x)dxds - \sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} \tilde{g}_j^m(s,x)\Psi(s,x)dxd^\dagger\hat{\beta}_j(s) \quad P-\text{a.s..}
\end{aligned}$$

But by standard estimates

$$E[\int_t^T \int_{\mathbb{R}^d} (|\tilde{Y}^{t,x,n}_{s,m} - \tilde{Y}^{t,x,n}_s|^2 + |\tilde{Z}^{t,x,n}_{s,m} - \tilde{Z}^{t,x,n}_s|^2)\rho^{-1}(x)dxds] \longrightarrow 0 \quad \text{as } m\to\infty.$$

And as m_1, $m_2 \to \infty$

$$\begin{aligned}
&E[\int_t^T \int_{\mathbb{R}^d} (|\tilde{u}^n_{m_1}(s, X^{t,x}_s) - \tilde{u}^n_{m_2}(s, X^{t,x}_s)|^2 \\
&\qquad +|\sigma^*\nabla\tilde{u}^n_{m_1}(s, X^{t,x}_s) - \sigma^*\nabla\tilde{u}^n_{m_2}(s, X^{t,x}_s)|^2)\rho^{-1}(x)dxds] \\
=\ &E[\int_t^T \int_{\mathbb{R}^d} (|\tilde{Y}^{t,x,n}_{s,m_1} - \tilde{Y}^{t,x,n}_{s,m_2}|^2 + |\tilde{Z}^{t,x,n}_{s,m_1} - \tilde{Z}^{t,x,n}_{s,m_2}|^2)\rho^{-1}(x)dxds] \longrightarrow 0. \qquad (3.20)
\end{aligned}$$

We define $\mathcal{H}$ to be the set of random fields $\{w(s,x);\ s\in[0,T],\ x\in\mathbb{R}^d\}$ such that $(w, \sigma^*\nabla w) \in M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ with the norm

$$(E[\int_0^T \int_{\mathbb{R}^d} (|w(s,x)|^2 + |(\sigma^*\nabla)w(s,x)|^2)\rho^{-1}(x)dxds)^{\frac{1}{2}}.$$

Following a standard argument as in the proof of the completeness of the Sobolev spaces, we can prove $\mathcal{H}$ is complete. Now by the generalized equivalence of norm principle and (3.20), we can see that $\tilde{u}^n_m$ is a Cauchy sequence in $\mathcal{H}$. So there exists $\tilde{u}^n \in \mathcal{H}$ such that $(\tilde{u}^n_m, \sigma^*\nabla\tilde{u}^n_m) \to (\tilde{u}^n, \sigma^*\nabla\tilde{u}^n)$ in $M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. Moreover

$$\tilde{Y}^{t,x,n}_s = \tilde{u}^n(s, X^{t,x}_s), \quad \tilde{Z}^{t,x,n}_s = \sigma^*\nabla\tilde{u}^n(s, X^{t,x}_s) \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s..}$$

Now it is easy to pass the limit as $m \to \infty$ on (3.19) to conclude that $\tilde{u}^n$ is a weak solution of the corresponding SPDEs. For the nonlinear case, we can take

$$\tilde{f}^n(r,x) = f(r,x,\tilde{u}^n(r,x), \sigma^*\nabla\tilde{u}^n(r,x)), \quad \tilde{g}^n_j(r,x) = g_j(r,x,\tilde{u}^n(r,x),\sigma^*\nabla\tilde{u}^n(r,x)),$$

then $\tilde{f}$ and $\tilde{g}_j$ satisfy the conditions in the above argument.

If we define $u^n(t,x) = Y^{t,x,n}_t$ and $v^n(t,x) = Z^{t,x,n}_t$, using Theorem 3.1.5 and Proposition 3.2.1, we have (since the proof is very similar to the existence part in Theorem 4.2.10, we don't intend to involve a proof here), under Conditions (H.3.1)–(H.3.4), $v^n(t,x) = (\sigma^*\nabla u^n)(t,x)$, moreover, $(u^n, \sigma^*\nabla u^n) \in M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and $u^n(t,x)$ is the weak solution of the following SPDE:

$$\begin{aligned} u^n(t,x) &= h(x) + \int_t^T [\mathscr{L}u^n(s,x) + f\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)]ds \\ &\quad -\sum_{j=1}^n \int_t^T g_j\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)d^\dagger\hat{\beta}_j(s), \qquad 0 \le t \le s \le T. \end{aligned} \tag{3.21}$$

That is to say, for any $\Psi \in C^{1,\infty}_c([0,T]\times\mathbb{R}^d;\mathbb{R}^1)$, we have

$$\begin{aligned} &\int_t^T \int_{\mathbb{R}^d} u^n(s,x)\partial_s\Psi(s,x)dxds + \int_{\mathbb{R}^d} u^n(t,x)\Psi(t,x)dx - \int_{\mathbb{R}^d} h(x)\Psi(T,x)dx \\ &-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}(\sigma^*\nabla u^n)(s,x)(\sigma^*\nabla\Psi)(s,x)dxds \\ &-\int_t^T\int_{\mathbb{R}^d} u^n(s,x)\nabla\big((b-\tilde{A})\Psi\big)(s,x)dxds \\ = &\int_t^T\int_{\mathbb{R}^d} f\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)\Psi(s,x)dxds \\ &-\sum_{j=1}^n\int_t^T\int_{\mathbb{R}^d} g_j\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)\Psi(s,x)dxd^\dagger\hat{\beta}_j(s) \qquad P-\text{a.s..} \end{aligned} \tag{3.22}$$

In this section, we study Eq.(2.19) with f and g allowed to depend on time as discussed in Section 3.1 and this section. By intuition if we define $u(t,x) = Y^{t,x}_t$, it should be a "weak solution" of the Eq.(2.19) with $u(T,x) = h(x)$. We will prove this result in next subsection.

§3.2.2 Existence and uniqueness of solutions of SPDEs with infinite dimensional noise

First we need some necessary preparations.

Proposition 3.2.2. *Under Conditions* (H.3.1)-(H.3.4), *let* $(Y_s^{t,x}, Z_s^{t,x})$ *be the solution of Eq.(3.1). If we define* $u(t,x) = Y_t^{t,x}$, *then* $\sigma^*\nabla u(t,x)$ *exists for a.a.* $t \in [0,T]$, $x \in \mathbb{R}^d$ *a.s., and*

$$u(s, X_s^{t,x}) = Y_s^{t,x},\ (\sigma^*\nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s..}$$

Proof. First we prove u^n is a Cauchy sequence in $\mathcal{H}$. For this, by Lemma 2.3.3 and Proposition 3.2.1, as m, $n \to \infty$, we have

$$\begin{aligned} & E[\int_0^T \int_{\mathbb{R}^d} (|u^m(s,x) - u^n(s,x)|^2 + |(\sigma^*\nabla u^m)(s,x) - (\sigma^*\nabla u^n)(s,x)|^2)\rho^{-1}(x)dxds] \\ \leq\ & C_p E[\int_0^T \int_{\mathbb{R}^d} (|u^m(s, X_s^{0,x}) - u^n(s, X_s^{0,x})|^2 \\ & \qquad + |(\sigma^*\nabla u^m)(s, X_s^{0,x}) - (\sigma^*\nabla u^n)(s, X_s^{0,x})|^2)\rho^{-1}(x)dxds] \\ =\ & C_p E[\int_0^T \int_{\mathbb{R}^d} (|Y_s^{0,x,m} - Y_s^{0,x,n}|^2 + |Z_s^{0,x,m} - Z_s^{0,x,n}|^2)\rho^{-1}(x)dxds] \longrightarrow 0. \end{aligned}$$

So there exists $\tilde{u} \in \mathcal{H}$ as the limit of u^n such that $\nabla\tilde{u}(s,x)$ exists for a.a. $s \in [0,T]$, $x \in \mathbb{R}^d$ a.s. and

$$E[\int_0^T \int_{\mathbb{R}^d} (|u^n(s,x) - \tilde{u}(s,x)|^2 + |(\sigma^*\nabla u^n)(s,x) - (\sigma^*\nabla\tilde{u})(s,x)|^2)\rho^{-1}(x)dxds] \longrightarrow 0.$$

We define $u(t,x) = Y_t^{t,x}$, then similar to the proof as in Proposition 3.2.1, by the uniqueness of solution of Eq.(3.1), we have

$$u(s, X_s^{t,x}) = Y_s^{t,x} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s..}$$

Since

$$\begin{aligned} & E[\int_0^T \int_{\mathbb{R}^d} |u(s,x) - \tilde{u}(s,x)|^2 \rho^{-1}(x)dxds] \\ \leq\ & 2E[\int_0^T \int_{\mathbb{R}^d} (|u(s,x) - u^n(s,x)|^2 + |u^n(s,x) - \tilde{u}(s,x)|^2)\rho^{-1}(x)dxds] \end{aligned}$$

$$\leq C_pE[\int_0^T\int_{\mathbb{R}^d}(|Y_s^{0,x}-Y_s^{0,x,n}|^2+|u^n(s,x)-\tilde{u}(s,x)|^2)\rho^{-1}(x)dxds] \longrightarrow 0,$$

so

$$u(t,x)=\tilde{u}(t,x) \text{ for a.a. } t\in[0,T],\ x\in\mathbb{R}^d \text{ a.s..}$$

Therefore $\sigma^*\nabla u(t,x)$ exists for a.a. $t\in[0,T]$, $x\in\mathbb{R}^d$ a.s.. Using Lemma 2.3.3 again, we have

$$\begin{aligned}
& E[\int_t^T\int_{\mathbb{R}^d}|(\sigma^*\nabla u)(s,X_s^{t,x})-Z_s^{t,x}|^2\rho^{-1}(x)dxds] \\
\leq\ & 2E[\int_t^T\int_{\mathbb{R}^d}(|(\sigma^*\nabla u)(s,X_s^{t,x})-(\sigma^*\nabla u^n)(s,X_s^{t,x})|^2 \\
& \qquad +|(\sigma^*\nabla u^n)(s,X_s^{t,x})-Z_s^{t,x}|^2)\rho^{-1}(x)dxds] \\
\leq\ & C_pE[\int_t^T\int_{\mathbb{R}^d}(|(\sigma^*\nabla u)(s,x)-(\sigma^*\nabla\tilde{u})(s,x)|^2+|(\sigma^*\nabla\tilde{u})(s,x)-(\sigma^*\nabla u^n)(s,x)|^2 \\
& \qquad +|Z_s^{t,x,n}-Z_s^{t,x}|^2)\rho^{-1}(x)dxds] \longrightarrow 0.
\end{aligned}$$

So

$$(\sigma^*\nabla u)(s,X_s^{t,x})=Z_s^{t,x} \text{ for a.a. } s\in[t,T],\ x\in\mathbb{R}^d \text{ a.s..}$$

◇

From Proposition 3.2.2 and Lemma 2.3.3, it is easy to know that

$$\begin{aligned}
& E[\int_0^T\int_{\mathbb{R}^d}|u^n(s,x)-u(s,x)|^2\rho^{-1}(x)dxds] \\
& +E[\int_0^T\int_{\mathbb{R}^d}|(\sigma^*\nabla u^n)(s,x)-(\sigma^*\nabla u)(s,x)|^2\rho^{-1}(x)dxds] \\
\leq\ & C_pE[\int_0^T\int_{\mathbb{R}^d}|u^n(s,X_s^{0,x})-u(s,X_s^{0,x})|^2\rho^{-1}(x)dxds] \\
& +C_pE[\int_0^T\int_{\mathbb{R}^d}|(\sigma^*\nabla u^n)(s,X_s^{0,x})-(\sigma^*\nabla u)(s,X_s^{0,x})|^2\rho^{-1}(x)dxds] \\
=\ & C_pE[\int_0^T\int_{\mathbb{R}^d}|Y_s^{0,x,n}-Y_s^{0,x}|^2\rho^{-1}(x)dxds] \\
& +C_pE[\int_0^T\int_{\mathbb{R}^d}|Z_s^{0,x,n}-Z_s^{0,x}|^2\rho^{-1}(x)dxds] \longrightarrow 0, \quad \text{as } n\to\infty. \qquad (3.23)
\end{aligned}$$

This will be used in the proof of the following theorem.

Theorem 3.2.3. *Under Conditions* (H.3.1)–(H.3.4)*, if we define* $u(t,x) = Y_t^{t,x}$*, where* $(Y_s^{t,x}, Z_s^{t,x})$ *is the solution of Eq.(3.1), then* $u(t,x)$ *is the unique weak solution of Eq.(2.19) with* $u(T,x) = h(x)$*. Moreover,*

$$u(s, X_s^{t,x}) = Y_s^{t,x},\ (\sigma^*\nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s..}$$

Proof. From Proposition 3.2.2, we only need to verify that this u is the unique weak solution of Eq.(2.19) with $u(T,x) = h(x)$. By Lemma 2.3.3, it is easy to see that

$$(\sigma^*\nabla u)(t,x) = Z_t^{t,x} \text{ for a.a. } t \in [0,T],\ x \in \mathbb{R}^d \text{ a.s..}$$

Furthermore, by the generalized equivalence of norm principle again we have

$$\begin{aligned}
& E[\int_0^T \int_{\mathbb{R}^d} (|u(s,x)|^2 + |(\sigma^*\nabla u)(s,x)|^2)\rho^{-1}(x)dxds] \\
\leq\ & C_p E[\int_0^T \int_{\mathbb{R}^d} (|u(s, X_s^{0,x})|^2 + |(\sigma^*\nabla u)(s, X_s^{0,x})|^2)\rho^{-1}(x)dxds] \\
=\ & C_p E[\int_0^T \int_{\mathbb{R}^d} (|Y_s^{0,x}|^2 + |Z_s^{0,x}|^2)\rho^{-1}(x)dxds] < \infty. \qquad (3.24)
\end{aligned}$$

Now we verify that $u(t,x)$ satisfies (2.20) with $u(T,x) = h(x)$ by passing the limit on (3.22) in $L^2(\Omega)$. We only show the convergence of the second and the last terms. For $0 \leq t \leq T$, by Lemma 2.3.3, we have

$$\begin{aligned}
& E[\ |\int_{\mathbb{R}^d} u^n(t,x)\Psi(t,x)dx - \int_{\mathbb{R}^d} u(t,x)\Psi(t,x)dx|^2] \\
\leq\ & E[\int_{\mathbb{R}^d} |u^n(t,x) - u(t,x)|^2\rho^{-1}(x)dx]E[\int_{\mathbb{R}^d} |\Psi(t,x)|^2\rho(x)dx] \\
\leq\ & C_p E[\int_{\mathbb{R}^d} |u^n(t,x) - u(t,x)|^2\rho^{-1}(x)dx] \\
\leq\ & C_p E[\int_{\mathbb{R}^d} |u^n(t, X_t^{0,x}) - u(t, X_t^{0,x})|^2\rho^{-1}(x)dx] \\
=\ & C_p E[\int_{\mathbb{R}^d} |Y_t^{0,x,n} - Y_t^{0,x}|^2\rho^{-1}(x)dx] \\
\leq\ & C_p E[\sup_{t\geq 0} \int_{\mathbb{R}^d} |Y_t^{0,x,n} - Y_t^{0,x}|^2\rho^{-1}(x)dx] \longrightarrow 0, \quad \text{as } n \to \infty.
\end{aligned}$$

The last term includes infinite dimensional integral, but

$$\begin{aligned}
E[\ |\sum_{j=1}^{n} \int_t^T \int_{\mathbb{R}^d} g_j\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)\Psi(s,x)dxd^\dagger\hat{\beta}_j(s) \\
-\sum_{j=1}^{\infty} \int_t^T \int_{\mathbb{R}^d} g_j\big(s,x,u(s,x),(\sigma^*\nabla u)(s,x)\big)\Psi(s,x)dxd^\dagger\hat{\beta}_j(s)|^2]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & 2E[\,|\sum_{j=1}^{n}\int_t^T\int_{\mathbb{R}^d}\big(g_j(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)) \\
&\qquad\qquad -g_j(s,x,u(s,x),(\sigma^*\nabla u)(s,x))\big)\Psi(s,x)dxd^\dagger\hat{\beta}_j(s)|^2] \\
&+2E[\,|\sum_{j=n+1}^{\infty}\int_t^T\int_{\mathbb{R}^d}g_j(s,x,u(s,x),(\sigma^*\nabla u)(s,x))\Psi(s,x)dxd^\dagger\hat{\beta}_j(s)|^2] \\
\leq\ & C_pE[\sum_{j=1}^{\infty}(C_j+\alpha_j)\int_t^T\int_{\mathbb{R}^d}(|u^n(t,x)-u(t,x)|^2 \\
&\qquad\qquad +|(\sigma^*\nabla u^n)(s,x)-(\sigma^*\nabla u)(s,x)|^2)\rho^{-1}(x)dxds] \\
&+C_pE[\,|\sum_{j=n+1}^{\infty}\int_t^T\int_{\mathbb{R}^d}\big(g_j(s,x,u(s,x),(\sigma^*\nabla u)(s,x)) \\
&\qquad\qquad -g_j(s,x,0,0)\big)\Psi(s,x)dxd^\dagger\hat{\beta}_j(s)|^2] \\
&+C_pE[\,|\sum_{j=n+1}^{\infty}\int_t^T\int_{\mathbb{R}^d}g_j(s,x,0,0)\Psi(s,x)dxd^\dagger\hat{\beta}_j(s)|^2].
\end{aligned}
$$

It is obvious that the first term tends to 0 as $n\to\infty$. The last two terms can be treated by a similar method as (3.17) and (3.18) respectively.

Therefore $u(t,x)$ satisfies (2.20), i.e. it is a weak solution of Eq.(2.19) with $u(T,x)=h(x)$. The uniqueness can be proved by a very similar argument as in the uniqueness part in Theorem 4.2.10, so we omit the proof here. ◇

§3.3 Infinite Horizon BDSDEs

We consider the following BDSDE with infinite dimensional noise on infinite horizon

$$
\begin{aligned}
\mathrm{e}^{-Ks}Y_s^{t,x} =\ & \int_s^\infty \mathrm{e}^{-Kr}f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+\int_s^\infty K\mathrm{e}^{-Kr}Y_r^{t,x}dr \\
&-\int_s^\infty \mathrm{e}^{-Kr}g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d^\dagger\hat{B}_r-\int_s^\infty \mathrm{e}^{-Kr}\langle Z_r^{t,x},dW_r\rangle. \quad (3.25)
\end{aligned}
$$

Here $f:[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\longrightarrow\mathbb{R}^1$, $g:[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\longrightarrow\mathcal{L}^2_{U_0}(\mathbb{R}^1)$. Eq.(3.25) is equivalent to

$$
\begin{aligned}
\mathrm{e}^{-Ks}Y_s^{t,x} =\ & \int_s^\infty \mathrm{e}^{-Kr}f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+\int_s^\infty K\mathrm{e}^{-Kr}Y_r^{t,x}dr \\
&-\sum_{j=1}^{\infty}\int_s^\infty \mathrm{e}^{-Kr}g_j(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d^\dagger\hat{\beta}_j(r)-\int_s^\infty \mathrm{e}^{-Kr}\langle Z_r^{t,x},dW_r\rangle.
\end{aligned}
$$

We assume

(H.3.5). Change "$\mathscr{B}_{[0,T]}$" to "$\mathscr{B}_{\mathbb{R}^+}$" and "$t \in [0,T]$" to "$t \geq 0$" in (H.3.2);

(H.3.6). Change "$\int_0^T$" to "$\int_0^\infty e^{-Ks}$" in (H.3.3);

(H.3.7). There exists a constant $\mu > 0$ with $2\mu - K - 2C - \sum_{j=1}^\infty C_j > 0$ s.t. for any $t \geq 0$, $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $X, Z \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$,

$$\begin{aligned}&\int_{\mathbb{R}^d}(Y_1(x) - Y_2(x))\big(f(t,X(x),Y_1(x),Z(x)) - f(t,X(x),Y_2(x),Z(x))\big)\rho^{-1}(x)dx\\ \leq\ & -\mu\int_{\mathbb{R}^d}|Y_1(x) - Y_2(x)|^2\rho^{-1}(x)dx.\end{aligned}$$

Note that the definition for the solution of Eq.(3.25) is similarly given as Definition 2.3.1. The main objective of this section is to prove

Theorem 3.3.1. *Under Conditions* (H.3.4)-(H.3.7), *Eq.(3.25) has a unique solution.*

Proof. Uniqueness. Let $(Y_s^{t,x}, Z_s^{t,x})$ and $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ be two solutions of Eq.(3.25). Define

$$\begin{aligned}&\bar{Y}_s^{t,x} = \hat{Y}_s^{t,x} - Y_s^{t,x}, \quad \bar{Z}_s^{t,x} = \hat{Z}_s^{t,x} - Z_s^{t,x},\\ &\bar{f}(s,x) = f(s,X_s^{t,x},\hat{Y}_s^{t,x},\hat{Z}_s^{t,x}) - f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x}),\\ &\bar{g}(s,x) = g(s,X_s^{t,x},\hat{Y}_s^{t,x},\hat{Z}_s^{t,x}) - g(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x}), \qquad s \geq 0.\end{aligned}$$

Then for $s \geq 0$ and a.e. $x \in \mathbb{R}^d$, $(Y_s^{t,x}, Z_s^{t,x})$ and $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ satisfy

$$\begin{cases} d\bar{Y}_s^{t,x} = -\bar{f}(s,x)ds + \sum_{j=1}^\infty \bar{g}_j(s,x)d^\dagger\hat{\beta}_j(s) + \langle \bar{Z}_s^{t,x}, dW_s\rangle\\ \lim_{T\longrightarrow\infty} e^{-KT}\bar{Y}_T^{t,x} = 0 \quad a.s.. \end{cases}$$

For a.e. $x \in \mathbb{R}^d$, applying Itô's formula for infinite dimensional noise to $e^{-Ks}|\bar{Y}_s^{t,x}|^2$, and by the Young inequality and Conditions (H.3.5), (H.3.7), we obtain

$$\begin{aligned}&E[\int_{\mathbb{R}^d} e^{-Ks}|\bar{Y}_s^{t,x}|^2\rho^{-1}(x)dx] + (\frac{1}{2} - \sum_{j=1}^\infty \alpha_j)E[\int_s^T\int_{\mathbb{R}^d} e^{-Kr}|\bar{Z}_r^{t,x}|^2\rho^{-1}(x)dxdr]\\ &+(2\mu - K - 2C - \sum_{j=1}^\infty C_j)E[\int_s^T\int_{\mathbb{R}^d} e^{-Kr}|\bar{Y}_r^{t,x}|^2\rho^{-1}(x)dxdr]\\ \leq\ & E[\int_{\mathbb{R}^d} e^{-KT}|\bar{Y}_T^{t,x}|^2\rho^{-1}(x)dx]. \end{aligned} \tag{3.26}$$

Taking $K' > K$ s.t. $2\mu - K' - 2C - \sum_{j=1}^{\infty} C_j > 0$ as well, we can see that (3.26) remains true with K replaced by K'. In particular,

$$E[\int_{\mathbb{R}^d} \mathrm{e}^{-K's} |\bar{Y}_s^{t,x}|^2 \rho^{-1}(x) dx] \leq E[\int_{\mathbb{R}^d} \mathrm{e}^{-K'T} |\bar{Y}_T^{t,x}|^2 \rho^{-1}(x) dx].$$

Therefore, we have

$$E[\int_{\mathbb{R}^d} \mathrm{e}^{-K's} |\bar{Y}_s^{t,x}|^2 \rho^{-1}(x) dx] \leq \mathrm{e}^{-(K'-K)T} E[\int_{\mathbb{R}^d} \mathrm{e}^{-KT} |\bar{Y}_T^{t,x}|^2 \rho^{-1}(x) dx]. \tag{3.27}$$

Since $\hat{Y}_s^{t,x}, Y_s^{t,x} \in S^{2,-K} \bigcap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$, so

$$\sup_{T\geq 0} E[\int_{\mathbb{R}^d} \mathrm{e}^{-KT} |\bar{Y}_T^{t,x}|^2 \rho^{-1}(x) dx] \leq E[\sup_{T\geq 0} \int_{\mathbb{R}^d} \mathrm{e}^{-KT} (2|\hat{Y}_T^{t,x}|^2 + 2|Y_T^{t,x}|^2) \rho^{-1}(x) dx] < \infty.$$

Therefore, taking the limit as $T \to \infty$ in (3.27), we have

$$E[\int_{\mathbb{R}^d} \mathrm{e}^{-K's} |\bar{Y}_s^{t,x}|^2 \rho^{-1}(x) dx] = 0.$$

Then the uniqueness is proved.

<u>Existence</u>. For each $n \in \mathbb{N}$, we define a sequence of BDSDEs by setting $h = 0$ and $T = n$ in Eq.(3.1):

$$\begin{aligned} Y_s^{t,x,n} &= \int_s^n f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) dr - \int_s^n g(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) d^\dagger \hat{B}_r \\ &\quad - \int_s^n \langle Z_r^{t,x,n}, dW_r \rangle, \quad 0 \leq s \leq n. \end{aligned} \tag{3.28}$$

It is easy to verify that for each n, these BDSDEs satisfy conditions of Theorem 3.1.2. Therefore, for each n, there exists a $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^{2,0}([0,n]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,0}([0,n]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ which is equivalent to the space $S^{2,-K}([0,n]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,-K}([0,n]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the unique solution of Eq.(3.28). That is to say, for an arbitrary $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$, $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} \mathrm{e}^{-Ks} Y_s^{t,x,n} \varphi(x) dx &= \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx dr \\ &+ \int_s^n \int_{\mathbb{R}^d} K \mathrm{e}^{-Kr} Y_r^{t,x,n} \varphi(x) dx dr - \int_s^n \langle \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} Z_r^{t,x,n} \varphi(x) dx, dW_r \rangle \\ &- \sum_{j=1}^{\infty} \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \varphi(x) dx d^\dagger \hat{\beta}_j(r) \quad P-\text{a.s.}. \end{aligned} \tag{3.29}$$

Let $(Y_t^n, Z_t^n)_{t>n} = (0,0)$, then $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^{2,-K} \bigcap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. We will prove $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence. For this, let

$(Y_s^{t,x,m}, Z_s^{t,x,m})$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ be the solutions of Eq.(3.28) when taking m and n as the terminal time respectively. Without losing any generality, assume that $m \geq n$, and define

$$\begin{aligned}
&\bar{Y}_s^{t,x,m,n} = Y_s^{t,x,m} - Y_s^{t,x,n}, \quad \bar{Z}_s^{t,x,m,n} = Z_s^{t,x,m} - Z_s^{t,x,n},\\
&\bar{f}^{m,n}(s,x) = f(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}) - f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}),\\
&\bar{g}_j^{m,n}(s,x) = g_j(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}) - g_j(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}), \qquad s \geq 0.
\end{aligned}$$

Consider two cases:

(i) When $n \leq s \leq m$, $\bar{Y}_s^{t,x,m,n} = Y_s^{t,x,m}$. Since $(Y_s^{t,x,m}, Z_s^{t,x,m})$ is the solution of Eq.(3.28) with the terminal time m, we have for any $m \in \mathbb{N}$,

$$\begin{cases}
dY_s^{t,x,m} = -f(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m})ds + \sum_{j=1}^{\infty} g_j(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m})d^\dagger\hat{\beta}_j(s)\\
\qquad\qquad +\langle Z_s^{t,x,m}, dW_s\rangle\\
Y_m^{t,x,m} = 0 \quad \text{for } s \in [0,m], \text{ a.a. } x \in \mathbb{R}^d \text{ a.s..}
\end{cases}$$

Noting that $E[\int_0^m \|g(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m})\|^2_{\mathcal{L}^2_{U_0}(\mathbb{R}^1)}dr] < \infty$ for a.e. $x \in \mathbb{R}^d$, we can apply Itô's formula to $\mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2$ for a.e. $x \in \mathbb{R}^d$, then taking integration over $\mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}|Y_s^{t,x,m}|^2\rho^{-1}(x)dx\\
&+(2\mu - K - 2C - \sum_{j=1}^{\infty} C_j - (1+\sum_{j=1}^{\infty} C_j)\varepsilon)\int_s^m\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho^{-1}(x)dxdr\\
&+(\frac{1}{2} - \sum_{j=1}^{\infty}\alpha_j - \sum_{j=1}^{\infty}\alpha_j\varepsilon)\int_s^m\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Z_r^{t,x,m}|^2\rho^{-1}(x)dxdr\\
\leq\ & C_p\int_s^m\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|f(r, X_r^{t,x}, 0, 0)|^2\rho^{-1}(x)dxdr\\
&+C_p\int_s^m\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\sum_{j=1}^{\infty}|g_j(r, X_r^{t,x}, 0, 0)|^2\rho^{-1}(x)dxdr\\
&-\sum_{j=1}^{\infty}\int_s^m\int_{\mathbb{R}^d} 2\mathrm{e}^{-Kr}Y_r^{t,x,m}g_j(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m})\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r)\\
&-\int_s^m\langle\int_{\mathbb{R}^d} 2\mathrm{e}^{-Kr}Y_r^{t,x,m}Z_r^{t,x,m}\rho^{-1}(x)dx, dW_r\rangle.
\end{aligned} \tag{3.30}$$

Note that the constant ε can be chosen to be sufficiently small s.t. all the terms on the left hand side of (3.30) are positive. By (3.30), as n, $m \longrightarrow \infty$ we have

$$E[\int_n^m\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho^{-1}(x)dxdr] + E[\int_n^m\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Z_r^{t,x,m}|^2\rho^{-1}(x)dxdr]$$

$$\leq C_pE[\int_n^m \int_{\mathbb{R}^d} e^{-Kr}(|f(r,X_r^{t,x},0,0)|^2+\sum_{j=1}^{\infty}|g_j(r,X_r^{t,x},0,0)|^2)\rho^{-1}(x)dxdr] \longrightarrow 0. \quad (3.31)$$

Note that the right hand side of (3.31) converges to 0 follows from the generalized equivalence of norm principle. Also using the B-D-G inequality to deal with (3.30) on the interval $[n,m]$, by (3.31), as $n,\ m \longrightarrow \infty$ we have

$$\begin{aligned} & E[\sup_{n\leq s\leq m}\int_{\mathbb{R}^d} e^{-Ks}|Y_s^{t,x,m}|^2\rho^{-1}dx] \\ \leq\ & C_pE[\int_n^m \int_{\mathbb{R}^d} e^{-Kr}(|f(r,X_r^{t,x},0,0)|^2+\sum_{j=1}^{\infty}|g_j(r,X_r^{t,x},0,0)|^2)\rho^{-1}(x)dxdr] \\ & +C_pE[\int_n^m \int_{\mathbb{R}^d} e^{-Kr}(|Y_r^{t,x,m}|^2+|Z_r^{t,x,m}|^2)\rho^{-1}(x)dxdr] \longrightarrow 0. \end{aligned} \quad (3.32)$$

(ii) When $0\leq s\leq n$,

$$\bar{Y}_s^{t,x,m,n} = Y_n^{t,x,m} + \int_s^n \bar{f}^{m,n}(r,x)dr - \sum_{j=1}^{\infty}\int_s^n \bar{g}_j^{m,n}(r,x)d^\dagger\hat{\beta}_j(r) - \int_s^n \langle \bar{Z}_r^{t,x,m,n}, dW_r\rangle.$$

Apply Itô's formula to $e^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2$ for a.e. $x\in\mathbb{R}^d$, then

$$\begin{aligned} & \int_{\mathbb{R}^d} e^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho^{-1}(x)dx + (\frac{1}{2}-\sum_{j=1}^{\infty}\alpha_j)\int_s^n\int_{\mathbb{R}^d} e^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr \\ & +(2\mu-K-2C-\sum_{j=1}^{\infty}C_j)\int_s^n\int_{\mathbb{R}^d} e^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr \\ \leq\ & \int_{\mathbb{R}^d} e^{-Kn}|Y_n^{t,x,m}|^2\rho^{-1}(x)dx - \sum_{j=1}^{\infty}\int_s^n\int_{\mathbb{R}^d} 2e^{-Kr}\bar{Y}_r^{t,x,m,n}\bar{g}_j^{m,n}(r,x)\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r) \\ & -\int_s^n\langle\int_{\mathbb{R}^d} 2e^{-Kr}\bar{Y}_r^{t,x,m,n}\bar{Z}_r^{t,x,m,n}\rho^{-1}(x)dx, dW_r\rangle. \end{aligned} \quad (3.33)$$

Taking expectation on both sides of (3.33), as $n,\ m\longrightarrow\infty$, using (3.32), we have

$$\begin{aligned} & E[\int_s^n\int_{\mathbb{R}^d} e^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr] + E[\int_s^n\int_{\mathbb{R}^d} e^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr] \\ \leq\ & C_pE[\sup_{n\leq s\leq m}\int_{\mathbb{R}^d} e^{-Ks}|Y_s^{t,x,m}|^2\rho^{-1}(x)dx] \longrightarrow 0. \end{aligned} \quad (3.34)$$

Also by the B-D-G inequality, (3.32), (3.33) and (3.34), as $n,\ m\longrightarrow\infty$, we have

$$E[\sup_{0\leq s\leq n}\int_{\mathbb{R}^d} e^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho^{-1}(x)dx] \leq C_pE[\sup_{n\leq s\leq m}\int_{\mathbb{R}^d} e^{-Ks}|Y_s^{t,x,m}|^2\rho^{-1}(x)dx] \longrightarrow 0.$$

Therefore taking a combination of cases (i) and (ii), as $n, m \longrightarrow \infty$, we have

$$E[\sup_{s\geq 0}\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho^{-1}(x)dx] + E[\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr]$$
$$+E[\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr] \longrightarrow 0.$$

That is to say $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence. Take $(Y_s^{t,x}, Z_s^{t,x})$ as the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ in the space $S^{2,-K}\bigcap M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and we will show that $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of Eq.(3.25). We only need to verify that for arbitrary $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$, $(Y_s^{t,x}, Z_s^{t,x})$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}Y_s^{t,x}\varphi(x)dx &= \int_s^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxdr \\ &\quad + \int_s^\infty \int_{\mathbb{R}^d} K\mathrm{e}^{-Kr}Y_r^{t,x}\varphi(x)dxdr \\ &\quad - \sum_{j=1}^\infty \int_s^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r) \\ &\quad - \int_s^\infty \langle \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}Z_r^{t,x}\varphi(x)dx, dW_r\rangle \quad P-\text{a.s.}. \end{aligned} \tag{3.35}$$

Since $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies Eq.(3.29), so we verify that Eq.(3.29) converges to Eq.(3.35) in $L^2(\Omega)$ term by term as $n \longrightarrow \infty$. We only show the infinite dimensional stochastic integral term:

$$\begin{aligned}&E[\,|\sum_{j=1}^\infty \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})\varphi(x)dxd^\dagger\hat{\beta}_j(r) \\ &\qquad - \sum_{j=1}^\infty \int_s^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2] \\ \leq\ & 2E[\,|\sum_{j=1}^\infty \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big(g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) \\ &\qquad\qquad - g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\big)\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2] \\ &+2E[\,|\sum_{j=1}^\infty \int_n^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2].\end{aligned}$$

We see that each term on the right hand side of the above inequality tends to 0 as $n \rightarrow \infty$ since

$$E[\,|\sum_{j=1}^\infty \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big(g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})$$

$$-g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}))\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2]$$
$$\leq C_pE[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}(|Y_r^{t,x,n} - Y_r^{t,x}|^2 + |Z_r^{t,x,n} - Z_r^{t,x}|^2)\rho^{-1}(x)dxdr] \longrightarrow 0, \text{ as } n \to \infty,$$

and

$$E[\,|\sum_{j=1}^\infty\int_n^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}g_j(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r)|^2]$$
$$\leq C_pE[\int_n^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}(|Y_r^{t,x}|^2 + |Z_r^{t,x}|^2)\rho^{-1}(x)dxdr]$$
$$+C_p\int_n^\infty\int_{\mathbb{R}^d}\sum_{j=1}^\infty \mathrm{e}^{-Kr}|g_j(r, x, 0, 0)|^2\rho^{-1}(x)dxdr \longrightarrow 0, \quad \text{as } n \to \infty.$$

That is to say $(Y_s^{t,x}, Z_s^{t,x})_{s\geq 0}$ satisfies Eq.(3.35). The proof of Theorem 3.3.1 is completed. ◇

By a similar method as in the proof of the existence part in case (i) in Theorem 3.3.1, we have the following estimation:

Proposition 3.3.2. *Let $(Y_s^{t,x,n}, Z_s^{t,x,n})$ be the solution of Eq.(3.28), then under the conditions of Theorem 3.3.1,*

$$\sup_n E[\sup_{s\geq 0}\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}|Y_s^{t,x,n}(x)|^2\rho^{-1}(x)dx] + \sup_n E[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Y_r^{t,x,n}(x)|^2\rho^{-1}(x)dxdr]$$
$$+\sup_n E[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Z_r^{t,x,n}(x)|^2\rho^{-1}(x)dxdr] < \infty.$$

§3.4 Time Continuity of Solutions of SPDEs

Now we study BDSDE (2.16), a simpler form of Eq.(3.25).

Proof of Theorem 2.3.10. Since the conditions here are stronger than those in Theorem 3.3.1, so there exists a unique solution $(Y_s^{t,x}, Z_s^{t,x})$ to Eq.(2.16). We only need to prove $E[\sup_{s\geq 0}\int_{\mathbb{R}^d} \mathrm{e}^{-pKs}|Y_s^{t,x}|^p\rho^{-1}(x)dx] < \infty$. Let

$$\varphi_{N,p}(x) = x^{\frac{p}{2}}I_{\{0\leq x<N\}} + N^{\frac{p-2}{2}}(\frac{p}{2}x - \frac{p-2}{2}N)I_{\{x\geq N\}}.$$

We apply the generalized Itô's formula to $\mathrm{e}^{-pKr}\varphi_{N,p}(\psi_M(Y_r^{t,x}))$ for a.e. $x \in \mathbb{R}^d$ to have the following estimation

$$\mathrm{e}^{-pKs}\varphi_{N,p}(\psi_M(Y_s^{t,x})) - pK\int_s^T \mathrm{e}^{-pKr}\varphi_{N,p}(\psi_M(Y_r^{t,x}))dr$$

$$
\begin{aligned}
&+\frac{1}{2}\int_s^T e^{-pKr}\varphi''_{N,p}(\psi_M(Y_r^{t,x}))|\psi'_M(Y_r^{t,x})|^2|Z_r^{t,x}|^2dr\\
&+\int_s^T e^{-pKr}\varphi'_{N,p}(\psi_M(Y_r^{t,x}))I_{\{-M\le Y_r^{t,x}<M\}}|Z_r^{t,x}|^2dr\\
\le\ & e^{-pKT}\varphi_{N,p}(\psi_M(Y_T^{t,x}))+\int_s^T e^{-pKr}\varphi'_{N,p}(\psi_M(Y_r^{t,x}))\psi'_M(Y_r^{t,x})f(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr\\
&+\int_s^T e^{-pKr}\varphi'_{N,p}(\psi_M(Y_r^{t,x}))I_{\{-M\le Y_r^{t,x}<M\}}\sum_{j=1}^\infty|g_j(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2dr\\
&+\frac{1}{2}\int_s^T e^{-pKr}\varphi''_{N,p}(\psi_M(Y_r^{t,x}))|\psi'_M(Y_r^{t,x})|^2\sum_{j=1}^\infty|g_j(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2dr\\
&-\sum_{j=1}^\infty\int_s^T e^{-pKr}\varphi'_{N,p}(\psi_M(Y_r^{t,x}))\psi'_M(Y_r^{t,x})g_j(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d^\dagger\hat{\beta}_j(r)\\
&-\int_s^T\langle e^{-pKr}\varphi'_{N,p}(\psi_M(Y_r^{t,x}))\psi'_M(Y_r^{t,x})Z_r^{t,x},dW_r\rangle. \qquad (3.36)
\end{aligned}
$$

Note that $\lim_{T\to\infty}e^{-pKT}\varphi_{N,p}(\psi_M(Y_T^{t,x}))=0$, so after taking limit as $T\to\infty$, we take the integration over $\Omega\otimes\mathbb{R}^d$. As $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})\in S^{2,-K}\bigcap M^{2,-K}([0,\infty);L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,-K}([0,\infty);L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and $\varphi'_{N,p}(\psi_M(Y_r^{t,x}))\psi'_M(Y_r^{t,x})$ is bounded, we can use the stochastic Fubini theorem and all the stochastic integrals have zero expectation. Using Conditions (A.2.1)′–(A.2.4)′, and taking the limit as $M\to\infty$ first, then the limit as $N\to\infty$, by the monotone convergence theorem, we have

$$
\begin{aligned}
&\Big(p\mu-pK-pC-\frac{p(p-1)}{2}\sum_{j=1}^\infty C_j-(3+\frac{p(p-1)}{2}\sum_{j=1}^\infty C_j)\varepsilon\Big)\\
&\times E\Big[\int_s^\infty\int_{\mathbb{R}^d}e^{-pKr}|Y_r^{t,x}|^p\rho^{-1}(x)dxdr\Big]\\
&+\frac{p}{4}\Big(2p-3-(2p-2)\sum_{j=1}^\infty\alpha_j-(2p-2)\sum_{j=1}^\infty\alpha_j\varepsilon\Big)\\
&\times E\Big[\int_s^\infty\int_{\mathbb{R}^d}e^{-pKr}|Y_r^{t,x}|^{p-2}|Z_r^{t,x}|^2\rho^{-1}(x)dxdr\Big]\\
\le\ & C_p\int_{\mathbb{R}^d}|f(x,0,0)|^p\rho^{-1}(x)dx+C_p\int_{\mathbb{R}^d}\sum_{j=1}^\infty|g_j(x,0,0)|^p\rho^{-1}(x)dx]<\infty. \qquad (3.37)
\end{aligned}
$$

Note that the constant ε can be chosen to be sufficiently small s.t. all the terms on the left hand side of (3.37) are positive. Also by the B-D-G inequality, the Cauchy-Schwartz inequality and the Young inequality, from (3.36) we have

$$
E[\sup_{s\ge0}\int_{\mathbb{R}^d}e^{-pKs}|Y_s^{t,x}|^p\rho^{-1}(x)dx]
$$

$$\leq C_p\int_{\mathbb{R}^d}|f(x,0,0)|^p\rho^{-1}(x)dx + C_p\int_{\mathbb{R}^d}\sum_{j=1}^{\infty}|g_j(x,0,0)|^p\rho^{-1}(x)dx$$
$$+C_pE[\int_0^{\infty}\int_{\mathbb{R}^d}e^{-pKr}|Y_r^{t,x}|^{p-2}|Z_r^{t,x}|^2\rho^{-1}(x)dxdr]$$
$$+C_pE[\int_0^{\infty}\int_{\mathbb{R}^d}e^{-pKr}|Y_r^{t,x}|^p\rho^{-1}(x)dxdr]. \quad (3.38)$$

So by (3.37), Theorem 2.3.10 is proved. ◇

We need to prove two lemmas before giving a proof of Theorem 2.3.11.

Lemma 3.4.1. *Under Condition* (A.2.3)′*, for arbitrary* $T > 0$*,* $t, t' \in [0,T]$*,*

$$E[\int_0^{\infty}\int_{\mathbb{R}^d}e^{-Kr}|X_r^{t',x} - X_r^{t,x}|^p\rho^{-1}(x)dxdr] \leq C_p|t'-t|^{\frac{p}{2}} \quad \text{a.s..}$$

Proof. It is not difficult to deduce from Lemma 4.5.6 in [28], and also it is a simpler case of Lemma 5.3.1, so we don't intend to involve the proof here. ◇

Lemma 3.4.2. *Under Conditions* (A.2.1)′*–*(A.2.4)′*, for arbitrary* $T > 0$*,* $t, t' \in [0,T]$*, let* $(Y_s^{t',x})_{s\geq 0}$*,* $(Y_s^{t,x})_{s\geq 0}$ *be the solutions of Eq.(2.16), then*

$$E[\sup_{s\geq 0}\int_{\mathbb{R}^d}e^{-pKs}|Y_s^{t',x} - Y_s^{t,x}|^p\rho^{-1}(x)dx] \leq C_p|t'-t|^{\frac{p}{2}}.$$

Proof. Let

$$\bar{Y}_s = Y_s^{t',x} - Y_s^{t,x}, \quad \bar{Z}_s = Z_s^{t',x} - Z_s^{t,x},$$
$$\bar{f}(s) = f(X_s^{t',x}, Y_s^{t',x}, Z_s^{t',x}) - f(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}),$$
$$\bar{g}_j(s) = g_j(X_s^{t',x}, Y_s^{t',x}, Z_s^{t',x}) - g_j(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}), \qquad s \geq 0.$$

Then

$$\begin{cases} d\bar{Y}_s = -\bar{f}(s)ds + \sum_{j=1}^{\infty}\bar{g}_j(s)d^{\dagger}\hat{\beta}_j(s) + \langle \bar{Z}_s, dW_s\rangle \\ \lim_{T\to\infty} e^{-KT}\bar{Y}_T = 0 \quad \text{for a.a. } x \in \mathbb{R}^d \text{ a.s..} \end{cases}$$

First note that from Theorem 2.3.10, we know $E[\sup_{s\geq 0}\int_{\mathbb{R}^d}e^{-pKs}|\bar{Y}_s|^p\rho^{-1}(x)dx] < \infty$. Applying Itô's formula to $e^{-pKr}|\bar{Y}_r|^p$ for a.e. $x \in \mathbb{R}^d$ (we leave out procedure of localization as in (3.36) for simplicity) and taking integration over $\mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d}e^{-pKs}|\bar{Y}_s|^p\rho^{-1}(x)dx$$

$$
\begin{aligned}
&+\big(p\mu - pK - pC - \frac{p(p-1)}{2}\sum_{j=1}^{\infty}C_j - 3\varepsilon\big)\int_s^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^p\rho^{-1}(x)dxdr \\
&+\frac{p}{4}\big(2p-3-(2p-2)\sum_{j=1}^{\infty}\alpha_j\big)\int_s^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2\rho^{-1}(x)dxdr \\
\leq\ & C_p\int_s^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{X}_r|^p\rho^{-1}(x)dxdr - p\int_s^{\infty}\langle\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^{p-2}\bar{Y}_r\bar{Z}_r\rho^{-1}(x)dx, dW_r\rangle \\
&-p\sum_{j=1}^{\infty}\int_s^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^{p-2}\bar{Y}_r\bar{g}_j(r)\rho^{-1}(x)dxd^{\dagger}\hat{\beta}_j(r). \qquad (3.39)
\end{aligned}
$$

Note that the constant ε can be chosen to be sufficiently small s.t. all the terms on the left hand side of (3.39) are positive. Taking integration over Ω on both sides of (3.39), by Lemma 3.4.1 we have

$$
\begin{aligned}
&E[\int_s^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^p\rho^{-1}(x)dxdr] + E[\int_s^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2\rho^{-1}(x)dxdr] \\
\leq\ & C_pE[\int_s^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{X}_r|^p\rho^{-1}(x)dxdr] \\
\leq\ & C_p|t'-t|^{\frac{p}{2}}. \qquad (3.40)
\end{aligned}
$$

Also by the B-D-G inequality, from (3.39) and (3.40), we have

$$
\begin{aligned}
&E[\sup_{s\geq 0}\int_{\mathbb{R}^d} e^{-pKs}|\bar{Y}_s|^p\rho^{-1}(x)dx] \\
\leq\ & C_pE[\int_0^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{X}_r|^p\rho^{-1}(x)dxdr] + C_pE[\int_0^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^p\rho^{-1}(x)dxdr] \\
&+C_pE[\int_0^{\infty}\int_{\mathbb{R}^d} e^{-pKr}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2\rho^{-1}(x)dxdr] \\
\leq\ & C_p|t'-t|^{\frac{p}{2}}.
\end{aligned}
$$

◇

Proof of Theorem 2.3.11. By Lemma 3.4.2, we have

$$
\begin{aligned}
&E([\sup_{s\geq 0}\int_{\mathbb{R}^d} e^{-2Ks}|Y_s^{t',x} - Y_s^{t,x}|^2\rho^{-1}(x)dx])^{\frac{p}{2}} \\
\leq\ & C_pE[\sup_{s\geq 0}\int_{\mathbb{R}^d} e^{-pKr}|Y_s^{t',x} - Y_s^{t,x}|^p\rho^{-1}(x)dx]\big(\int_{\mathbb{R}^d}\rho^{-1}(x)dx\big)^{\frac{p-2}{2}} \\
\leq\ & C_p|t'-t|^{\frac{p}{2}}.
\end{aligned}
$$

Noting $p > 2$, by the Kolmogorov continuity theorem (see [28]), we have $t \longrightarrow Y_s^{t,x}$ is a.s. continuous for $t \in [0, T]$ under the norm

$$
(\sup_{s\geq 0}\int_{\mathbb{R}^d} e^{-2Ks}|\cdot|^2\rho^{-1}(x)dx)^{\frac{1}{2}}.
$$

Without losing any generality, assume that $t' \geq t$. Then we can see

$$\begin{aligned}\lim_{t'\to t}\Big(\int_{\mathbb{R}^d} \mathrm{e}^{-2Kt'}|Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2\rho^{-1}(x)dx\Big)^{\frac{1}{2}} &\leq \lim_{t'\to t}\Big(\sup_{s\geq 0}\int_{\mathbb{R}^d} \mathrm{e}^{-2Ks}|Y_s^{t',x} - Y_s^{t,x}|^2\rho^{-1}(x)dx\Big)^{\frac{1}{2}} \\ &= 0 \quad \text{a.s.}.\end{aligned}$$

Notice $t' \in [0, T]$, so

$$\lim_{t'\to t}\Big(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2\rho^{-1}(x)dx\Big)^{\frac{1}{2}} = 0 \quad \text{a.s.}. \tag{3.41}$$

Since $Y_\cdot^{t,\cdot} \in S^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, $Y_{t'}^{t,\cdot}$ is continuous w.r.t. t' in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$. That is to say for each t,

$$\lim_{t'\to t}\Big(\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_{t}^{t,x}|^2\rho^{-1}(x)dx\Big)^{\frac{1}{2}} = 0 \quad \text{a.s.}. \tag{3.42}$$

Now by (3.41) and (3.42)

$$\begin{aligned}&\lim_{t'\to t}\Big(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t}^{t,x}|^2\rho^{-1}(x)dx\Big)^{\frac{1}{2}} \\ \leq\ &\lim_{t'\to t}\Big(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2\rho^{-1}(x)dx\Big)^{\frac{1}{2}} + \lim_{t'\to t}\Big(\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_{t}^{t,x}|^2\rho^{-1}(x)dx\Big)^{\frac{1}{2}} \\ =\ &0 \quad \text{a.s.}.\end{aligned}$$

For arbitrary $T > 0$, $0 \leq t \leq T$, define $u(t,\cdot) = Y_t^{t,\cdot}$, then $u(t,\cdot)$ is a.s. continuous w.r.t. t in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$. Since $Y_\cdot^{t,\cdot} \in S^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, $Y_T^{T,x}$ is $\mathscr{F}_{T,\infty}^{\hat{B}} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable and $E[\int_{\mathbb{R}^d} |Y_T^{T,x}|^2\rho^{-1}(x)dx] < \infty$. It follows that Condition (H.3.1) is satisfied. Moreover, Conditions (A.2.1)′–(A.2.3)′ are stronger than Conditions (H.3.2)–(H.3.4), so by Theorem 3.2.3, $u(t,x)$ is a weak solution of Eq.(2.19). Theorem 2.3.11 is proved. ◇

Now all the theorems listed in Subsection 2.3.3 are proved. We can use the method as shown in Subsection 2.3.4 to transfer the stationarity from BDSDE to the corresponding SPDE and achieve the stationary weak solution of SPDE (1.2) in Theorem 2.3.13.

Chapter 4

Non-Lipschitz Condition

§4.1 Conditions, Examples and Main Results

The main purpose of this chapter is to find the stationary weak solution of SPDE (1.2) via the solution of BDSDE (2.16) under non-Lipschitz conditions:

(A.4.1)′. Functions $f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ and $g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathcal{L}^2_{U_0}(\mathbb{R}^1)$ are $\mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable, and there exist constants $M_4, M_{4j}, C, C_j, \alpha_j \geq 0$ with $\sum_{j=1}^{\infty} M_{4j} < \infty$, $\sum_{j=1}^{\infty} C_j < \infty$ and $\sum_{j=1}^{\infty} \alpha_j < \frac{1}{2}$ s.t. for any $Y \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$, $X_1, X_2, Z_1, Z_2 \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$, measurable $U : \mathbb{R}^d \to [0,1]$,

$$\begin{aligned}
& \int_{\mathbb{R}^d} U(x)|f(X_1(x), Y(x), Z_1(x)) - f(X_2(x), Y(x), Z_2(x))|^2 \rho^{-1}(x)dx \\
\leq & \int_{\mathbb{R}^d} U(x)\big(M_4|X_1(x) - X_2(x)|^2 + C|Z_1(x) - Z_2(x)|^2\big)\rho^{-1}(x)dx, \\
& \int_{\mathbb{R}^d} U(x)|g_j(X_1(x), Y_1(x), Z_1(x)) - g_j(X_2(x), Y_2(x), Z_2(x))|^2 \rho^{-1}(x)dx \\
\leq & \int_{\mathbb{R}^d} U(x)\big(M_{4j}|X_1(x) - X_2(x)|^2 + C_j|Y_1(x) - Y_2(x)|^2 \\
& \qquad + \alpha_j|Z_1(x) - Z_2(x)|^2\big)\rho^{-1}(x)dx;
\end{aligned}$$

(A.4.2)′. For $p \in (2, q-1)$, $\int_{\mathbb{R}^d} \|g(x,0,0)\|^p_{\mathcal{L}^p_{U_0}(\mathbb{R}^1)} \rho^{-1}(x)dx < \infty$;

(A.4.3)′. There exists a constant $M_5 \geq 0$ s.t. for any $t \geq 0$, $x, z \in \mathbb{R}^d$, $y \in \mathbb{R}^1$,

$$|f(x,y,z)| \leq M_5(1 + |y|);$$

(A.4.4)′. There exists a constant $\mu > 0$ with $2\mu - pK - pC - \frac{p(p-1)}{2}\sum_{j=1}^{\infty} C_j > 0$ s.t.

for any $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $X, Z \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$, measurable $U : \mathbb{R}^d \to [0,1]$,

$$\begin{aligned}&\int_{\mathbb{R}^d} U(x)\big(Y_1(x) - Y_2(x)\big)\big(f(X(x), Y_1(x), Z(x)) - f(X(x), Y_2(x), Z(x))\big)\rho^{-1}(x)dx \\ \leq\ & -\mu\int_{\mathbb{R}^d} U(x)|Y_1(x) - Y_2(x)|^2\rho^{-1}(x)dx;\end{aligned}$$

(A.4.5)$'$. For any $x, z \in \mathbb{R}^d$, $y \to f(x,y,z)$ is continuous;

(A.4.6)$'$. $b \in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^1)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^1)$, furthermore, for p is given in (A.4.2)$'$, if L is the global Lipschitz constant for b and σ, L satisfies $K - pL - \frac{p(p-1)}{2}L^2 > 0$.

Although in this chapter, the continuity condition and linear growth condition for f are the pointwise condition rather than the weak condition as in Chapter 3, we can extend our results to SPDEs with a large number of functions which are not included in Chapter 3. Here we give some examples which do not satisfy the conditions in last chapter.

Example 4.1.1. $f(y) = -Ay + B$, $A > 0$ *and* $B \in \mathbb{R}^1$. *In this case* $\mu = A$ *and we don't need to worry about the choices of* p, C, K, C_j, L *and* α_j *since all of them can be independent of* A. *We can take* A *as large as we like, so the relationship of all the coefficients in* (A.4.1)$'$-(A.4.6)$'$ *is much easier to be satisfied. Then the choice of* f *and* g *is also much more than before.*

We then give an example of nonlinear function:

Example 4.1.2. $f(y) = (-4y)I_{\{y<-2\}} + 2y^2 I_{\{-2\leq y<-1\}} + (-2y)I_{\{y\geq -1\}}$, $y \in \mathbb{R}^1$. *In this case one can verify that* $\mu = 2$, *then* p, C, K, C_j, L *and* α_j *can be constructed to satisfy Conditions* (A.4.1)$'$-(A.4.6)$'$.

In fact, with the help of the linear function in Example 4.1.1, we can construct countless nonlinear functions satisfying our new conditions by a similar method as shown in Example 2.3.8:

Example 4.1.3. *If there is a family of finite linear functions like* $f_j(y) = -A_j y + B_j$, $A_j \geq a > 0$ *and* $B_j \in \mathbb{R}^1$, $j = 1, 2, \cdots$, *then from Example 4.1.1, we can see that any new function constructed by an arbitrary combination of the functions in this family satisfies Conditions* (A.4.1)$'$-(A.4.6)$'$ *with* $\mu \geq \bigwedge_j A_j$.

In the latter sections, we will follow a similar procedure as in Chapter 3, but different methods, to obtain Theorem 2.3.10 and 2.3.11 under the above non-Lipschitz conditions. As for Theorem 2.3.12, it is easy to prove it with the help of Theorem 2.3.10, 2.3.11 and non-Lipschitz conditions. After that, we can use the method as shown in Subsection 2.3.4 to transfer the stationarity from BDSDE to the corresponding SPDE and achieve the stationary weak solution of SPDE:

Theorem 4.1.4. *Under Conditions* (A.4.1)′–(A.4.6)′, *for arbitrary* T *and* $t \in [0,T]$, *let* $v(t,\cdot) \triangleq Y_{T-t}^{T-t,\cdot}$, *where* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ *is the solution of Eq.(2.16) with* $\hat{B}_s = B_{T-s} - B_T$ *for all* $s \geq 0$. *Then* $v(t,\cdot)$ *is a "perfect" stationary weak solution of Eq.(1.2).*

§4.2 Finite Horizon BDSDEs and the Corresponding SPDEs

§4.2.1 Conditions and main results

Following the procedure of Chapter 3, we first study the BDSDEs on finite horizon and establish the connection with SPDEs. In this section, we consider Eq.(3.1) and assume

(H.4.1). Function h is $\mathscr{F}_{T,\infty}^{\hat{B}} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable and $E[\int_{\mathbb{R}^d} |h(x)|^2 \rho^{-1}(x)dx] < \infty$;

(H.4.2). Functions f and g are $\mathscr{B}_{[0,T]} \otimes \mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable and there exist constants $C, C_j, \alpha_j \geq 0$ with $\sum_{j=1}^{\infty} C_j < \infty$ and $\sum_{j=1}^{\infty} \alpha_j < \frac{1}{2}$ s.t. for any $r \in [0,T]$, $Y_1, Y_2 \in L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$, $X, Z_1, Z_2 \in L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |f(r,X(x),Y_1(x),Z_1(x)) - f(r,X(x),Y_2(x),Z_2(x))|^2 \rho^{-1}(x)dx \\
\leq\ & C\int_{\mathbb{R}^d} |Z_1(x) - Z_2(x)|^2 \rho^{-1}(x)dx, \\
&\int_{\mathbb{R}^d} |g_j(r,X(x),Y_1(x),Z_1(x)) - g_j(r,X(x),Y_2(x),Z_2(x))|^2 \rho^{-1}(x)dx \\
\leq\ & \int_{\mathbb{R}^d} (C_j|Y_1(x) - Y_2(x)|^2 + \alpha_j|Z_1(x) - Z_2(x)|^2)\rho^{-1}(x)dx;
\end{aligned}$$

(H.4.3). $\int_0^T \int_{\mathbb{R}^d} \|g(r,x,0,0)\|^2_{\mathcal{L}^2_{U_0}(\mathbb{R}^1)} \rho^{-1}(x)dxdr < \infty$;

(H.4.4). There exists a constant $M_6 \geq 0$ s.t. for any $r \in [0,T]$, $x,z \in \mathbb{R}^d$, $y \in \mathbb{R}^1$,

$$|f(r,x,y,z)| \leq M_6(1 + |y|);$$

(H.4.5). There exists a constant $\mu \in \mathbb{R}^1$ s.t. for any $r \in [0,T]$, $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $X, Z \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$, measurable $U : \mathbb{R}^d \to [0,1]$,

$$\begin{aligned}&\int_{\mathbb{R}^d} U(x)\big(Y_1(x) - Y_2(x)\big)\big(f(r, X(x), Y_1(x), Z(x))\\ &\qquad\qquad -f(r, X(x), Y_2(x), Z(x))\big)\rho^{-1}(x)dx\\ \leq\ & \mu \int_{\mathbb{R}^d} U(x)|Y_1(x) - Y_2(x)|^2\rho^{-1}(x)dx;\end{aligned}$$

(H.4.6). For any $r \in [0,T]$, $x, z \in \mathbb{R}^d$, $y \to f(r,x,y,z)$ is continuous;

(H.4.7). $b \in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^d \times \mathbb{R}^d)$.

The first objective of this section is to prove

Theorem 4.2.1. *Under Conditions* (H.4.1)-(H.4.7), *Eq.(3.1) has a unique solution.*

Then we will make the connection between the solutions of BDSDE (3.1) and SPDE (2.19).

Theorem 4.2.2. *Under Conditions* (H.4.1)-(H.4.7), *if we define* $u(t,x) = Y_t^{t,x}$, *where* $(Y_s^{t,x}, Z_s^{t,x})$ *is the solution of Eq.(3.1), then* $u(t,x)$ *is the unique weak solution of Eq.(2.19) with* $u(T,x) = h(x)$. *Moreover,* $u(s, X_s^{t,x}) = Y_s^{t,x}$, $(\sigma^*\nabla u)(s, X_s^{t,x}) = Z_s^{t,x}$ *for a.a.* $s \in [t,T]$, $x \in \mathbb{R}^d$ *a.s..*

§4.2.2 Existence and uniqueness of solutions of BDSDEs with finite dimensional noise

In their pioneering work [42], Pardoux and Peng solved the following BSDE with Lipschitz conditions on the coefficient:

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T \langle Z_r, dW_r\rangle. \tag{4.1}$$

After that, many researchers studied how to weaken the Lipschitz conditions so that the BSDE system can include more equations. To name but a few, in [45], [30], [40], [26], [5] and [6], researchers made their significant contributions to this subject. In this chapter, we use the method in [30] to tackle the non-Lipschitz condition problem. In [30], Lepeltier and San Martin assumed that the $\mathbb{R}^1$-valued function $f(r,y,z)$ satisfies the measurable condition, the y, z linear growth condition and the y, z continuous condition, then they proved the existence of solution of Eq.(4.1). But the uniqueness of solution failed to be proved since the comparison theorem cannot be used under non-Lipschitz condition.

In [49], after proving the comparison theorem of BDSDE with Lipschitz condition, the authors followed the same procedure as in [30] and proved the corresponding result for the following $\mathbb{R}^1$-valued BDSDE:

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T \langle g(r, Y_r, Z_r), d^\dagger \hat{B}_r\rangle - \int_s^T \langle Z_r, dW_r\rangle. \qquad (4.2)$$

For the nonlinear term $f(r, y, z)$, they assumed the same measurable condition, the y, z linear growth condition and the y, z continuous condition as Lepeltier and San Martin did. In the term $g(r, y, z)$, besides the standard measurable condition, they assumed Lipschitz condition w.r.t. y and z. Then in Theorem 4.1 in [49], they proved the existence of solution of Eq.(4.2).

First we study the BDSDE with finite dimensional noise, Eq.(3.3), under non-Lipschitz conditions. Note here the existing results, such as in [49] and [30] in the case of BSDE, only dealt with the solution of Eq.(3.3) for a fixed x. But our solution is in the space $S^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$. The main task in this subsection is to prove

Theorem 4.2.3. *Under Conditions* (H.4.1)-(H.4.7), *Eq.(3.3) has a unique solution*

$$(Y^{t,\cdot,n}_\cdot, Z^{t,\cdot,n}_\cdot) \in S^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)).$$

We will first acknowledge that the following Proposition 4.2.4 is true at this moment, then we prove Theorem 4.2.3 with the help of Proposition 4.2.4. Note that in the proof of Theorem 4.2.3 and Proposition 4.2.4, we can only consider the solution in $S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$ due to the arguments in Remark 3.1.7.

Proposition 4.2.4. *Given* $(U_\cdot(\cdot), V_\cdot(\cdot)) \in S^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$, *then under Conditions* (H.4.1)-(H.4.7), *the equation*

$$\begin{aligned} Y^{t,x,n}_s = \quad & h(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x,n}_r, V_r(x))dr \\ & - \sum_{j=1}^n \int_s^T g_j(r, X^{t,x}_r, U_r(x), V_r(x))d^\dagger \hat{\beta}_j(r) - \int_s^T \langle Z^{t,x,n}_r, dW_r\rangle \qquad (4.3) \end{aligned}$$

has a unique solution.

Proof of Theorem 4.2.3. Uniqueness. Assume there exists another $(\hat{Y}^{t,\cdot,n}_\cdot, \hat{Z}^{t,\cdot,n}_\cdot) \in S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ satisfying (3.3). Define

$$\bar{Y}^{t,x,n}_s = Y^{t,x,n}_s - \hat{Y}^{t,x,n}_s \text{ and } \bar{Z}^{t,x,n}_s = Z^{t,x,n}_s - \hat{Z}^{t,x,n}_s,\ t \le s \le T.$$

Then for a.e. $x \in \mathbb{R}^d$, $(\bar{Y}^{t,x,n}_s, \bar{Z}^{t,x,n}_s)$ satisfies

$$\begin{aligned} \bar{Y}^{t,x,n}_s = & \int_s^T \big(f(r, X^{t,x}_r, Y^{t,x,n}_r, Z^{t,x,n}_r) - f(r, X^{t,x}_r, \hat{Y}^{t,x,n}_s, \hat{Z}^{t,x,n}_s)\big)dr \\ & -\sum_{j=1}^n \int_s^T \big(g_j(r, X^{t,x}_r, Y^{t,x,n}_r, Z^{t,x,n}_r) - g_j(r, X^{t,x}_r, \hat{Y}^{t,x,n}_s, \hat{Z}^{t,x,n}_s)\big)d^\dagger\hat{\beta}_j(r) \\ & - \int_s^T \langle \bar{Z}^{t,x,n}_r, dW_r\rangle. \end{aligned}$$

From Conditions (H.4.4) and $(\hat{Y}^{t,\cdot,n}_\cdot, \hat{Z}^{t,\cdot,n}_\cdot), (Y^{t,\cdot,n}_\cdot, Z^{t,\cdot,n}_\cdot) \in S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}$ $([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, it follows that

$$\begin{aligned} & E[\int_t^T \int_{\mathbb{R}^d} |f(r, X^{t,x}_r, Y^{t,x,n}_r, Z^{t,x,n}_r) - f(r, X^{t,x}_r, \hat{Y}^{t,x,n}_s, \hat{Z}^{t,x,n}_s)|^2 \rho^{-1}(x)dxdr] \\ \le\ & 2E[\int_t^T \int_{\mathbb{R}^d} (|f(r, X^{t,x}_r, Y^{t,x,n}_r, Z^{t,x,n}_r)|^2 + |f(r, X^{t,x}_r, \hat{Y}^{t,x,n}_s, \hat{Z}^{t,x,n}_s)|^2)\rho^{-1}(x)dxdr] \\ \le\ & C_pE[\int_t^T \int_{\mathbb{R}^d} (1 + |Y^{t,x,n}_r|^2 + |\hat{Y}^{t,x,n}_r|^2)\rho^{-1}(x)dxdr] < \infty. \end{aligned}$$

So we have for a.e. $x \in \mathbb{R}^d$,

$$E[\int_t^T |f(r, X^{t,x}_r, Y^{t,x,n}_r, Z^{t,x,n}_r) - f(r, X^{t,x}_r, \hat{Y}^{t,x,n}_r, \hat{Z}^{t,x,n}_r)|^2 dr] < \infty.$$

Similarly, with Condition (H.4.2), we have for a.e. $x \in \mathbb{R}^d$,

$$\sum_{j=1}^n E[\int_t^T |g_j(r, X^{t,x}_r, Y^{t,x,n}_r, Z^{t,x,n}_r) - g_j(r, X^{t,x}_r, \hat{Y}^{t,x,n}_r, \hat{Z}^{t,x,n}_r)|^2 dr] < \infty.$$

For a.e. $x \in \mathbb{R}^d$, we apply the generalized Itô's formula to $\mathrm{e}^{Ks}\psi_M\big(\bar{Y}^{t,x,n}_s\big)$, where $K \in \mathbb{R}^1$ and

$$\psi_M(x) = x^2 I_{\{-M \le x < M\}} + M(2x - M)I_{\{x \ge M\}} - M(2x + M)I_{\{x < -M\}}.$$

Then

$$\mathrm{e}^{Ks}\psi_M(\bar{Y}^{t,x,n}_s) + K\int_s^T \mathrm{e}^{Kr}\psi_M(\bar{Y}^{t,x,n}_r)dr + \int_s^T \mathrm{e}^{Kr} I_{\{-M \le \bar{Y}^{t,x,n}_r < M\}} |\bar{Z}^{t,x,n}_r|^2 dr$$

$$
\begin{aligned}
= & \int_s^T \mathrm{e}^{Kr}\psi_M'(\bar{Y}_r^{t,x,n})\big(f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) - f(r,X_r^{t,x},\hat{Y}_r^{t,x,n},\hat{Z}_r^{t,x,n})\big)dr \\
& + \sum_{j=1}^{n}\int_s^T \mathrm{e}^{Kr} I_{\{-M\le \bar{Y}_r^{t,x,n}<M\}}|g_j(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) - g_j(r,X_r^{t,x},\hat{Y}_r^{t,x,n},\hat{Z}_r^{t,x,n})|^2 dr \\
& - \sum_{j=1}^{n}\int_s^T \mathrm{e}^{Kr}\psi_M'(\bar{Y}_r^{t,x,n})\big(g_j(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) - g_j(r,X_r^{t,x},\hat{Y}_r^{t,x,n},\hat{Z}_r^{t,x,n})\big)d^\dagger\hat{\beta}_j(r) \\
& - \int_s^T \langle \mathrm{e}^{Kr}\psi_M'(\bar{Y}_r^{t,x,n})\bar{Z}_r^{t,x,n}, dW_r\rangle. \qquad (4.4)
\end{aligned}
$$

We can use the Fubini theorem to perfect (4.4) so that (4.4) is satisfied for a.e. $x \in \mathbb{R}^d$ on a full measure set that is independent of x. If we define $\frac{\psi_M'(x)}{x} = 2$ when $x = 0$, then $0 \le \frac{\psi_M'(\bar{Y}_r^{t,x,n})}{\bar{Y}_r^{t,x,n}} \le 2$. Taking integration over $\mathbb{R}^d$ on both sides and applying the stochastic Fubini theorem ([16]), we have

$$
\begin{aligned}
& \int_{\mathbb{R}^d} \mathrm{e}^{Ks}\psi_M(\bar{Y}_s^{t,x,n})\rho^{-1}(x)dx + K\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\psi_M(\bar{Y}_r^{t,x,n})\rho^{-1}(x)dxdr \\
& + \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr} I_{\{-M\le \bar{Y}_r^{t,x,n}<M\}}|\bar{Z}_r^{t,x,n}|^2\rho^{-1}(x)dxdr \\
= & \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\frac{\psi_M'(\bar{Y}_r^{t,x,n})}{\bar{Y}_r^{t,x,n}}\bar{Y}_r^{t,x,n}\big(f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) \\
& \qquad\qquad - f(r,X_r^{t,x},\hat{Y}_r^{t,x,n},Z_r^{t,x,n})\big)\rho^{-1}(x)dxdr \\
& + \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\psi_M'(\bar{Y}_r^{t,x,n})\big(f(r,X_r^{t,x},\hat{Y}_r^{t,x,n},Z_r^{t,x,n}) \\
& \qquad\qquad - f(r,X_r^{t,x},\hat{Y}_r^{t,x,n},\hat{Z}_r^{t,x,n})\big)\rho^{-1}(x)dxdr \\
& + \sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr} I_{\{-M\le \bar{Y}_r^{t,x,n}<M\}}|g_j(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) \\
& \qquad\qquad - g_j(r,X_r^{t,x},\hat{Y}_r^{t,x,n},\hat{Z}_r^{t,x,n})|^2\rho^{-1}(x)dxdr \\
& - \sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\psi_M'(\bar{Y}_r^{t,x,n})\big(g_j(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) \\
& \qquad\qquad - g_j(r,X_r^{t,x},\hat{Y}_r^{t,x,n},\hat{Z}_r^{t,x,n})\big)\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r) \\
& - \int_s^T \langle \int_{\mathbb{R}^d} \mathrm{e}^{Kr}\psi_M'(\bar{Y}_s^{t,x,n})\bar{Z}_r^{t,x,n}\rho^{-1}(x)dx, dW_r\rangle \\
\le & \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\frac{\psi_M'(\bar{Y}_r^{t,x,n})}{\bar{Y}_r^{t,x,n}}\mu|\bar{Y}_r^{t,x,n}|^2\rho^{-1}(x)dxdr \\
& + \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}(2C|\bar{Y}_r^{t,x,n}|^2 + \frac{1}{2}|\bar{Z}_r^{t,x,n}|^2)\rho^{-1}(x)dxdr \\
& + \sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}(\sum_{j=1}^{\infty} C_j|\bar{Y}_r^{t,x,n}|^2 + \sum_{j=1}^{\infty}\alpha_j|\bar{Z}_r^{t,x,n}|^2)\rho^{-1}(x)dxdr
\end{aligned}
$$

$$-\sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d} e^{Kr}\psi_M'(\bar{Y}_r^{t,x,n})\big(g_j(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})$$
$$-g_j(r,X_r^{t,x},\hat{Y}_r^{t,x,n},\hat{Z}_r^{t,x,n})\big)\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r)$$
$$-\int_s^T\langle\int_{\mathbb{R}^d} e^{Kr}\psi_M'(\bar{Y}_r^{t,x,n})\bar{Z}_r^{t,x,n}\rho^{-1}(x)dx, dW_r\rangle. \qquad (4.5)$$

Note that the stochastic integrals are martingales, so taking the expectation, we have

$$\begin{aligned}
&E[\int_{\mathbb{R}^d} e^{Ks}\psi_M(\bar{Y}_s^{t,x,n})\rho^{-1}(x)dx] + KE[\int_s^T\int_{\mathbb{R}^d} e^{Kr}\psi_M(\bar{Y}_r^{t,x,n})\rho^{-1}(x)dxdr]\\
&+E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}I_{\{-M\leq\bar{Y}_r^{t,x,n}<M\}}|\bar{Z}_r^{t,x,n}|^2\rho^{-1}(x)dxdr]\\
\leq\ & E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}\frac{\psi_M'(\bar{Y}_r^{t,x,n})}{\bar{Y}_r^{t,x,n}}\mu|\bar{Y}_r^{t,x,n}|^2\rho^{-1}(x)dxdr]\\
&+(2C+\sum_{j=1}^{\infty}C_j)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Y}_r^{t,x,n}|^2\rho^{-1}(x)dxdr]\\
&+(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Z}_r^{t,x,n}|^2\rho^{-1}(x)dxdr].
\end{aligned}$$

Taking the limit as $M\to\infty$ and applying the monotone convergence theorem, we have

$$\begin{aligned}
&E[\int_{\mathbb{R}^d} e^{Ks}|\bar{Y}_s^{t,x,n}|^2\rho^{-1}(x)dx] + (\frac{1}{2}-\sum_{j=1}^{\infty}\alpha_j)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Z}_r^{t,x,n}|^2\rho^{-1}(x)dxdr]\\
&+(K-2\mu-2C-\sum_{j=1}^{\infty}C_j)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Y}_r^{t,x,n}|^2\rho^{-1}(x)dxdr]\leq 0. \qquad (4.6)
\end{aligned}$$

Note that all the terms on the left hand side of (4.6) are positive when K is sufficiently large. So by a "standard" argument, we have $\bar{Y}_s^{t,x,n}=0$ for $s\in[t,T]$, a.a. $x\in\mathbb{R}^d$ a.s.. Also by (4.6), for a.e. $s\in[t,T]$, $\bar{Z}_s^{t,x,n}=0$ for a.a. $x\in\mathbb{R}^d$ a.s.. We can modify the values of Z at the measure zero exceptional set of s such that $\bar{Z}_s^{t,x,n}=0$ for $s\in[t,T]$, a.a. $x\in\mathbb{R}^d$ a.s..

Existence. If we regard Eq.(4.3) as a mapping, then by Proposition 4.2.4, (4.3) is an iterated mapping from $S^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ to itself and we can obtain a sequence $\{(Y_r^{t,x,n,i},Z_r^{t,x,n,i})\}_{i=1}^{\infty}$ from this mapping. We will prove that (4.3) is a contraction mapping. For this, define for $t\leq s\leq T$,

$$\bar{Y}_s^{t,x,n,i}=Y_s^{t,x,n,i}-Y_s^{t,x,n,i-1},\quad \bar{Z}_s^{t,x,n,i}=Z_s^{t,x,n,i}-Z_s^{t,x,n,i-1},$$
$$\bar{g}_j^i(s,x)=g_j(s,X_s^{t,x},Y_s^{t,x,n,i},Z_s^{t,x,n,i})-g_j(s,X_s^{t,x},Y_s^{t,x,n,i-1},Z_s^{t,x,n,i-1}),\ i=2,3,\cdots$$

Then for a.e. $x \in \mathbb{R}^d$, $(\bar{Y}_s^{t,x,n,N}, \bar{Z}_s^{t,x,n,N})$ satisfies

$$\begin{aligned}\bar{Y}_s^{t,x,n,N} &= \int_s^T \big(f(r, X_r^{t,x}, Y_r^{t,x,n,N}, Z_r^{t,x,n,N-1}) - f(r, X_r^{t,x}, Y_s^{t,x,n,N-1}, Z_s^{t,x,n,N-2})\big)dr \\ &\quad -\sum_{j=1}^{n}\int_s^T \bar{g}_j^{N-1}(r,x)d^\dagger\hat{\beta}_j(r) - \int_s^T \langle \bar{Z}_r^{t,x,n,N}, dW_r\rangle.\end{aligned}$$

Similar as in (4.5), applying generalized the Itô's formula to $e^{Kr}\psi_M(\bar{Y}_r^{t,x,n,N})$ for a.e. $x \in \mathbb{R}^d$, by the Young inequality, Condition (H.4.2) and (H.4.5), we can deduce that

$$\begin{aligned}&\int_{\mathbb{R}^d} e^{Ks}\psi_M(\bar{Y}_s^{t,x,n,N})\rho^{-1}(x)dx + K\int_s^T\int_{\mathbb{R}^d} e^{Kr}\psi_M(\bar{Y}_r^{t,x,n,N})\rho^{-1}(x)dxdr \\ &+\int_s^T\int_{\mathbb{R}^d} e^{Kr}I_{\{-M\le \bar{Y}_r^{t,x,n,N}<M\}}|\bar{Z}_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr \\ \le\ &\int_s^T\int_{\mathbb{R}^d} e^{Kr}\frac{\psi_M'(\bar{Y}_r^{t,x,n,N})}{\bar{Y}_r^{t,x,n,N}}\mu|\bar{Y}_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr \\ &+2C\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Y}_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr + \sum_{j=1}^{\infty}C_j\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Y}_r^{t,x,n,N-1}|^2\rho^{-1}(x)dxdr \\ &+(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j)\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Z}_r^{t,x,n,N-1}|^2\rho^{-1}(x)dxdr \\ &-\sum_{j=1}^{n}\int_s^T\int_{\mathbb{R}^d} e^{Kr}\psi_M'(\bar{Y}_r^{t,x,n,N})\bar{g}_j^{N-1}(r,x)\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r) \\ &-\int_s^T\langle\int_{\mathbb{R}^d} e^{Kr}\psi_M'(\bar{Y}_r^{t,x,n,N})\bar{Z}_r^{t,x,n,N}\rho^{-1}(x)dx, dW_r\rangle. \end{aligned} \tag{4.7}$$

Then we have

$$\begin{aligned}&(K-2\mu-2C)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Y}_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr] \\ &+E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}|\bar{Z}_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr] \\ \le\ &(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}(2\sum_{j=1}^{\infty}C_j|\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2)\rho^{-1}(x)dxdr].\end{aligned}$$

Letting $K = 2\mu + 2C + 2\sum_{j=1}^{\infty}C_j$, we have

$$\begin{aligned}&E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}(2\sum_{j=1}^{\infty}C_j|\bar{Y}_r^{t,x,n,N}|^2 + |\bar{Z}_r^{t,x,n,N}|^2)\rho^{-1}(x)dxdr] \\ \le\ &(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j)E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}(2\sum_{j=1}^{\infty}C_j|\bar{Y}_r^{t,x,n,N-1}|^2 + |\bar{Z}_r^{t,x,n,N-1}|^2)\rho^{-1}(x)dxdr].\end{aligned} \tag{4.8}$$

Note that $E[\int_t^T \int_{\mathbb{R}^d} e^{Kr}(2\sum_{j=1}^{\infty} C_j|\cdot|^2+|\cdot|^2)\rho^{-1}(x)dxdr]$ is equivalent to $E[\int_t^T \int_{\mathbb{R}^d} (|\cdot|^2+|\cdot|^2)\rho^{-1}(x)dxdr]$. From the contraction principle, the mapping (4.3) has a pair of fixed point $(Y_\cdot^{t,\cdot,n,\infty}, Z_\cdot^{t,\cdot,n,\infty})$ that is the limit of the Cauchy sequence $\{(Y_\cdot^{t,\cdot,n,N}, Z_\cdot^{t,\cdot,n,N})\}_{N=1}^{\infty}$ in $M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. We then prove that $Y_\cdot^{t,\cdot,n,\infty}$ is also the limit of $Y_\cdot^{t,\cdot,n,N}$ in $S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ as $N\to\infty$. For this, we only need to prove that $\{Y_\cdot^{t,\cdot,n,N}\}_{N=1}^{\infty}$ is a Cauchy sequence in $S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$. From (4.7), by the B-D-G inequality and the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
& E[\sup_{t\le s\le T}\int_{\mathbb{R}^d} e^{Ks}\psi_M(\bar{Y}_s^{t,x,n,N})\rho^{-1}(x)dx] \\
\le\ & C_pE[\int_t^T\int_{\mathbb{R}^d}(|\bar{Y}_r^{t,x,n,N}|^2+|\bar{Y}_r^{t,x,n,N-1}|^2+|\bar{Z}_r^{t,x,n,N-1}|^2)\rho^{-1}(x)dxdr] \\
& +C_pE[\sqrt{\int_t^T\int_{\mathbb{R}^d}|\psi_M'(\bar{Y}_r^{t,x,n,N})|^2\rho^{-1}(x)dx\int_{\mathbb{R}^d}\sum_{j=1}^{n}|\bar{g}_j^{N-1}(r,x)|^2\rho^{-1}(x)dxdr]} \\
& +C_pE[\sqrt{\int_t^T\int_{\mathbb{R}^d}|\psi_M'(\bar{Y}_r^{t,x,n,N})|^2\rho^{-1}(x)dx\int_{\mathbb{R}^d}|\bar{Z}_r^{t,x,n,N}|^2\rho^{-1}(x)dxdr]} \\
\le\ & C_pE[\int_t^T\int_{\mathbb{R}^d}(|\bar{Y}_r^{t,x,n,N}|^2+|\bar{Z}_r^{t,x,n,N}|^2+|\bar{Y}_r^{t,x,n,N-1}|^2+|\bar{Z}_r^{t,x,n,N-1}|^2)\rho^{-1}(x)dxdr] \\
& +\frac{1}{5}E[\sup_{t\le s\le T}\int_{\mathbb{R}^d}|\psi_M'(Y_s(x))|^2\rho^{-1}(x)dx],
\end{aligned}
\tag{4.9}
$$

where C_p depends on $|\mu|$, C, $\sum_{j=1}^{\infty}\alpha_j$, $\sum_{j=1}^{\infty}C_j$ and the fixed B-D-G inequality constant. Taking the limit as $M\to\infty$ and applying the monotone convergence theorem, we have

$$
\begin{aligned}
& E[\sup_{t\le s\le T}\int_{\mathbb{R}^d} e^{Ks}|\bar{Y}_s^{t,x,n,N}|^2\rho^{-1}(x)dx] \qquad (4.10)\\
\le\ & M_7E[\int_s^T\int_{\mathbb{R}^d} e^{Kr}(|\bar{Y}_r^{t,x,n,N-1}|^2+|\bar{Z}_r^{t,x,n,N-1}|^2+|\bar{Y}_r^{t,x,n,N}|^2+|\bar{Z}_r^{t,x,n,N}|^2)\rho^{-1}(x)dxdr],
\end{aligned}
$$

where $M_7>0$ is independent of n and N. Without losing any generality, assume that $M\ge N$. We can deduce from (4.8) and (4.10) that

$$
\begin{aligned}
& (E[\sup_{t\le s\le T}\int_{\mathbb{R}^d}|Y_s^{t,x,n,M}-Y_s^{t,x,n,N}|^2\rho^{-1}(x)dx])^{\frac{1}{2}} \\
\le\ & \sum_{i=N+1}^{M}(E[\sup_{t\le s\le T}\int_{\mathbb{R}^d}|\bar{Y}_s^{t,x,n,i}|^2\rho^{-1}(x)dx])^{\frac{1}{2}} \\
\le\ & \sum_{i=N+1}^{M}(M_7E[\int_t^T\int_{\mathbb{R}^d} e^{Kr}(|\bar{Y}_r^{t,x,n,i-1}|^2+|\bar{Z}_r^{t,x,n,i-1}|^2 \\
& \qquad +|\bar{Y}_r^{t,x,n,i}|^2+|\bar{Z}_r^{t,x,n,i}|^2)\rho^{-1}(x)dxdr])^{\frac{1}{2}}
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{i=N+1}^{M}\big((1+\frac{1}{2\sum_{j=1}^{\infty}C_j})M_7E[\int_t^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}(2\sum_{j=1}^{\infty}C_j|\bar{Y}_r^{t,x,n,i-1}|^2+|\bar{Z}_r^{t,x,n,i-1}|^2\\
&\qquad +2\sum_{j=1}^{\infty}C_j|\bar{Y}_r^{t,x,n,i}|^2+|\bar{Z}_r^{t,x,n,i}|^2)\rho^{-1}(x)dxdr]\big)^{\frac{1}{2}}\\
&\leq \sum_{i=N+1}^{M}\big((2+\frac{1}{\sum_{j=1}^{\infty}C_j})M_7E[\int_t^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}(2\sum_{j=1}^{\infty}C_j|\bar{Y}_r^{t,x,n,i-1}|^2\\
&\qquad +|\bar{Z}_r^{t,x,n,i-1}|^2)\rho^{-1}(x)dxdr]\big)^{\frac{1}{2}}\\
&\leq \sum_{i=N+1}^{\infty}(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j)^{\frac{i-2}{2}}\big((2+\frac{1}{\sum_{j=1}^{\infty}C_j})M_7\\
&\qquad \times E[\int_t^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}(2\sum_{j=1}^{\infty}C_j|Y_r^{t,x,n,1}|^2+|Z_r^{t,x,n,1}|^2)\rho^{-1}(x)dxdr]\big)^{\frac{1}{2}}\\
&\longrightarrow 0 \text{ as } M,\ N\longrightarrow\infty.
\end{aligned}$$

The theorem is proved. ◇

The remaining work in this subsection is to prove Proposition 4.2.4. First we do some preparations.

Lemma 4.2.5. *Under Conditions* (H.4.1)–(H.4.7), *if there exists* $(Y_\cdot(\cdot),Z_\cdot(\cdot))\in M^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ *satisfying the spatial integral form of Eq.(3.3) for* $t\leq s\leq T$, *then* $Y_\cdot(\cdot)\in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ *and therefore* $(Y_s(x),Z_s(x))$ *is a solution of Eq.(3.3).*

Proof. Referring to Lemma 3.1.3, we can prove $Y_s(\cdot)$ is continuous w.r.t. s in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ under the conditions in this chapter. We only mention that we can use Condition (H.4.4) to deal with the term $f(r,X_r^{t,x},Y_r(x),Z_r(x))$ although there is no weak Lipschitz condition for $Y_r(x)$.

Then we prove $E[\sup_{t\leq s\leq T}\int_{\mathbb{R}^d}|Y_s(x)|^2\rho^{-1}(x)dx]<\infty$. For a.e. $x\in\mathbb{R}^d$, applying the generalized Itô's formula to $\psi_M\big(Y_r(x)\big)$, by Lemma 2.3.3, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\psi_M(Y_s(x))\rho^{-1}(x)dx+\int_s^T\int_{\mathbb{R}^d}I_{\{-M\leq Y_r(x)<M\}}|Z_r(x)|^2\rho^{-1}(x)dxdr\\
\leq&\int_{\mathbb{R}^d}\psi_M\big(h(X_T^{t,x})\big)\rho^{-1}(x)dx+\int_s^T\int_{\mathbb{R}^d}\psi'_M(Y_r(x))f(r,X_r^{t,x},0,0)\rho^{-1}(x)dxdr\\
&+\int_s^T\int_{\mathbb{R}^d}\psi'_M(Y_r(x))\big(f(r,X_r^{t,x},Y_r(x),Z_r(x))-f(r,X_r^{t,x},Y_r(x),0)\big)\rho^{-1}(x)dxdr\\
&+\int_s^T\int_{\mathbb{R}^d}\frac{\psi'_M(Y_r(x))}{Y_r(x)}Y_r(x)\big(f(r,X_r^{t,x},Y_r(x),0)-f(r,X_r^{t,x},0,0)\big)\rho^{-1}(x)dxdr
\end{aligned}$$

$$
\begin{aligned}
&+C_p\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d}|g_j(r,X_r^{t,x},Y_r(x),Z_r(x))-g_j(r,X_r^{t,x},0,0)|^2\rho^{-1}(x)dxdr\\
&+C_p\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d}|g_j(r,X_r^{t,x},0,0)|^2\rho^{-1}(x)dxdr\\
&-\int_s^T\langle\int_{\mathbb{R}^d}\psi_M'(Y_r(x))Z_r(x)\rho^{-1}(x)dx,dW_r\rangle\\
&-\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d}\psi_M'(Y_r(x))g_j(r,X_r^{t,x},Y_r(x),Z_r(x))\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r)\\
\leq\ & C_p\int_{\mathbb{R}^d}|h(x)|^2\rho^{-1}(x)dx+C_p\int_s^T\int_{\mathbb{R}^d}(|Y_r(x)|^2+|Z_r(x)|^2)\rho^{-1}(x)dxdr\\
&+C_p\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d}(1+|g_j(r,x,0,0)|^2)\rho^{-1}(x)dxdr\\
&-\int_s^T\langle\int_{\mathbb{R}^d}\psi_M'(Y_r(x))Z_r(x)\rho^{-1}(x)dx,dW_r\rangle\\
&-\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d}\psi_M'(Y_r(x))g_j(r,X_r^{t,x},Y_r(x),Z_r(x))\rho^{-1}(x)dxd^\dagger\hat{\beta}_j(r).
\end{aligned}
\tag{4.11}
$$

Similar as in (4.9), by the B-D-G inequality and the Cauchy-Schwartz inequality, from (4.11), we have

$$
\begin{aligned}
&E[\sup_{t\leq s\leq T}\int_{\mathbb{R}^d}\psi_M(Y_s(x))\rho^{-1}(x)dx]\\
\leq\ & C_pE[\int_{\mathbb{R}^d}|h(x)|^2\rho^{-1}(x)dx]+C_pE[\int_t^T\int_{\mathbb{R}^d}(|Y_r(x)|^2+|Z_r(x)|^2)\rho^{-1}(x)dxdr]\\
&+C_p\sum_{j=1}^n\int_t^T\int_{\mathbb{R}^d}(1+|g_j(r,x,0,0)|^2)\rho^{-1}(x)dxdr<\infty.
\end{aligned}
$$

So taking the limit as $M\to\infty$ and applying the monotone convergence theorem, we have $Y_\cdot(\cdot)\in S^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Recall that a solution of Eq.(3.3) is a pair of processes in $S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ satisfying the spatial integral form of Eq.(3.3), therefore $(Y_s(x),Z_s(x))$ is a solution of Eq.(3.3). ◇

From the proof of Lemma 4.2.5, one can similarly deduce that

Corollary 4.2.6. *Under Conditions* (H.4.1)-(H.4.7), *if there exists* $(Y_\cdot(\cdot),Z_\cdot(\cdot))\in M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ *satisfying the spatial integral form of Eq.(4.3) for* $t\leq s\leq T$, *then* $Y_\cdot(\cdot)\in S^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ *and therefore* $(Y_s(x),Z_s(x))$ *is a solution of Eq.(4.3).*

For the rest of Chapter 4, we will leave out the similar localization argument as in the proof of Theorem 4.2.3 and Lemma 4.2.5 when applying Itô's formula to save the space of this thesis.

Proof of Proposition 4.2.4. The proof of the uniqueness is rather similar to the uniqueness proof in Theorem 4.2.3, so it is omitted.

Existence. Define

$$\tilde{f}^x(r,y) = f(r, X_r^{t,x}, y, V_r(x)) \text{ and } \tilde{g}_j^x(r) = g_j(r, X_r^{t,x}, U_r(x), V_r(x)),$$

then for a.e. $x \in \mathbb{R}^d$, (4.3) becomes

$$Y_s^{t,x,n} = h(X_T^{t,x}) + \int_s^T \tilde{f}^x(r, Y_r^{t,x,n})dr - \sum_{j=1}^n \int_s^T \tilde{g}_j^x(r)d^\dagger\hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r\rangle. \tag{4.12}$$

Then it is easy to see that for a.e. $x \in \mathbb{R}^d$, $\tilde{f}^x$ and $\tilde{g}_j^x$ satisfy

(H.4.1)$'$. $\tilde{f}^x : [t,T] \times \Omega \times \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ is $\mathscr{B}_{[t,T]} \otimes \mathscr{F}_{s,T} \bigvee \mathscr{F}_{T,\infty}^{\hat{B}} \otimes \mathscr{B}_{\mathbb{R}^1}$ measurable and $\tilde{g}_j^x : [t,T] \times \Omega \longrightarrow \mathbb{R}^1$ is $\mathscr{B}_{[t,T]} \otimes \mathscr{F}_{s,T} \bigvee \mathscr{F}_{T,\infty}^{\hat{B}}$ measurable;

(H.4.2)$'$. For any $r \in [t,T]$, $y \in \mathbb{R}^1$, $|\tilde{f}^x(r,y)| \leq M_6(1+|y|)$;

(H.4.3)$'$. For any $r \in [t,T]$, $y \to \tilde{f}^x(r,y)$ is continuous.

By Theorem 4.1 in [49], for a.e. $x \in \mathbb{R}^d$, Eq.(4.12), as well as Eq.(4.3), has a solution $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in M^{2,0}([t,T];\mathbb{R}^1) \bigotimes M^{2,0}([t,T];\mathbb{R}^d)$. Actually, we can further prove $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ under the conditions of Proposition 4.2.4.

First by Condition (H.4.4) or Condition (H.4.2)$'$, for a.e. $x \in \mathbb{R}^d$, we have

$$E[\int_t^T |f(r, X_r^{t,x}, Y_r^{t,x,n}, V_r(x))|^2 dr] < \infty,$$

also by Conditions (H.4.2), (H.4.3) and (H.4.7), for a.e. $x \in \mathbb{R}^d$, we have

$$\sum_{j=1}^n E[\int_t^T |g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2 dr] < \infty.$$

Then for a.e. $x \in \mathbb{R}^d$, applying the generalized Itô's formula to $\mathrm{e}^{Kr}|Y_r^{t,x,n}|^2$, we have

$$E[\mathrm{e}^{Ks}|Y_s^{t,x,n}|^2] + KE[\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n}|^2 dr] + E[\int_s^T \mathrm{e}^{Kr}|Z_r^{t,x,n}|^2 dr]$$

$$
\begin{aligned}
&\leq E[\mathrm{e}^{KT}|h(X_T^{t,x})|^2] + E[\int_s^T \mathrm{e}^{Kr}(|Y_r^{t,x,n}|^2 + |f(r, X_r^{t,x}, Y_r^{t,x,n}, V_r(x))|^2)dr] \\
&\quad + \sum_{j=1}^n E[\int_s^T \mathrm{e}^{Kr}|g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2 dr] \\
&\leq E[\mathrm{e}^{KT}|h(X_T^{t,x})|^2] + E[\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n}|^2 dr] + E[\int_s^T \mathrm{e}^{Kr} 2M_6^2(1 + |Y_r^{t,x,n}|^2)dr] \\
&\quad + \sum_{j=1}^n E[\int_s^T \mathrm{e}^{Kr}|g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2 dr].
\end{aligned}
$$

It turns out that

$$
\begin{aligned}
&E[\mathrm{e}^{Ks}|Y_s^{t,x,n}|^2] + (K - 2M_6^2 - 1)E[\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n}|^2 dr] + E[\int_s^T \mathrm{e}^{Kr}|Z_r^{t,x,n}|^2 dr] \\
\leq\; & E[\mathrm{e}^{KT}|h(X_T^{t,x})|^2] + 2M_6^2\mathrm{e}^{KT}T + \sum_{j=1}^n E[\int_s^T \mathrm{e}^{Kr}|g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2 dr].
\end{aligned}
$$

Taking the integration over $\mathbb{R}^d$ and by Conditions (H.4.2), (H.4.3), (H.4.7) and Lemma 2.3.3, we have

$$
\begin{aligned}
&E[\int_{\mathbb{R}^d} \mathrm{e}^{Ks}|Y_s^{t,x,n}|^2\rho^{-1}(x)dx] + (K - 2M_6^2 - 1)E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr}|Y_r^{t,x,n}|^2\rho^{-1}(x)dxdr] \\
&+E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr] \\
\leq\; & E[\int_{\mathbb{R}^d} \mathrm{e}^{KT}|h(X_T^{t,x})|^2\rho^{-1}(x)dx] + \int_{\mathbb{R}^d} 2M_6^2\mathrm{e}^{KT}T\rho^{-1}(x)dx \\
&+\sum_{j=1}^n E[\int_s^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr}|g_j(r, X_r^{t,x}, U_r(x), V_r(x))|^2\rho^{-1}(x)dxdr] \\
\leq\; & C_pE[\int_{\mathbb{R}^d} |h(x)|^2\rho^{-1}(x)dx] + C_p + C_pE[\int_s^T \int_{\mathbb{R}^d}(|U_r(x)|^2 + |V_r(x)|^2)\rho^{-1}(x)dxdr] \\
&+C_p\sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} |g_j(r, x, 0, 0)|^2\rho^{-1}(x)dxdr \\
<\; & \infty. \qquad (4.13)
\end{aligned}
$$

Taking K sufficiently large, we have $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ and for a.e. $x \in \mathbb{R}^d$, $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies Eq.(4.3) on a full set Ω^x dependent on x. But we can use the Fubini theorem to perfect Eq.(4.3) so that $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies (4.3) for a.e. $x \in \mathbb{R}^d$ on a full measure set $\tilde{\Omega}$ independent of x. In the following we give more details.

From (4.13), we have for any $s \in [t,T]$, $E[\int_{\mathbb{R}^d} \mathrm{e}^{Ks}|Y_s^{t,x,n}|^2\rho^{-1}(x)dx] < \infty$, so there exist a full measure set $\Omega' \subset \Omega$ independent of x and a full set $\mathcal{E}' \subset \mathbb{R}^d$ probably dependent on ω s.t. $Y_s^{t,x,n} < \infty$ on $\Omega' \otimes \mathcal{E}'$. Let

$$\begin{aligned} F(s,x) &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,n}, V_r(x))dr \\ &\quad - \sum_{j=1}^n \int_s^T g_j(r, X_r^{t,x}, U_r(x), V_r(x))d^\dagger\hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r\rangle. \end{aligned}$$

Then by Eq.(4.3), for $x \in \mathcal{E}$, where $\mathcal{E}$ is also a full measure set in $\mathbb{R}^d$, $Y_s^{t,x,n} = F(s,x)$ on Ω^x. Then for $(\omega, x) \in \Omega^x \bigcap \Omega' \otimes \mathcal{E} \bigcap \mathcal{E}'$, we have $Y_s^{t,x,n} = F(s,x)$. Since now $Y_s^{t,x,n} < \infty$, so $F(s,x) < \infty$ and we can move $F(s,x)$ to the other side of the equality to have $Y_s^{t,x,n} - F(s,x) = 0$ on the full measure set $\Omega^x \bigcap \Omega' \otimes \mathcal{E} \bigcap \mathcal{E}'$ in the product space $\Omega \otimes \mathbb{R}^d$. Thus

$$\int_{\Omega\otimes\mathbb{R}^d} |Y_s^{t,x,n} - F(s,x)|(dP \otimes dx) = 0.$$

By the Fubini theorem, we have

$$E[\int_{\mathbb{R}^d} |Y_s^{t,x,n} - F(s,x)|dx] = 0.$$

This means there exists a full set $\ddot{\Omega}$ independent of x s.t. on $\ddot{\Omega}$, $Y_s^{t,x,n} - F(s,x) = 0$ for $x \in \ddot{\mathcal{E}}^\omega$, where $\ddot{\mathcal{E}}^\omega$ is a full set in $\mathbb{R}^d$ and depend on ω. Take $\tilde{\Omega} = \ddot{\Omega} \bigcap \Omega'$ and $\tilde{\mathcal{E}}^\omega = \ddot{\mathcal{E}}^\omega \bigcap \mathcal{E}'$, then both are still a full measure set and on $\tilde{\Omega} \otimes \tilde{\mathcal{E}}^\omega$, $Y_s^{t,x,n} < \infty$, furthermore $F(s,x) < \infty$. We can move items in the equality $Y_s^{t,x,n} - F(s,x) = 0$ to have $Y_s^{t,x,n} = F(s,x)$ for $x \in \tilde{\mathcal{E}}^\omega$ on a full measure set $\tilde{\Omega}$ independent of x.

Now we have $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ and for $t \le s \le T$, $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies (4.3) for a.e. $x \in \mathbb{R}^d$ on a full measure set $\tilde{\Omega}$ independent of x. Then for any $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$, multiplying by φ on both sides of Eq.(4.3) and taking the integration over $\mathbb{R}^d$, we have $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies the spatial integral form of Eq.(4.3) for $t \le s \le T$. By Corollary 4.2.6, $Y_\cdot^{t,\cdot,n} \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a solution of Eq.(4.3). ◇

§4.2.3 Existence and uniqueness of solutions of BDSDEs with infinite dimensional noise

Following a similar procedure as in the proof of Lemma 4.2.5, and applying Itô's formula to $e^{Kr}|Y_r^{t,x,n}|^2$, by the B-D-G inequality we have the following estimation for the solution of Eq.(3.3):

Proposition 4.2.7. *Under the conditions of Theorem 4.2.1,* $(Y_s^{t,x,n}, Z_s^{t,x,n})$ *satisfies*

$$\sup_n E[\sup_{0\le s\le T}\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^2\rho^{-1}(x)dx] + \sup_n E[\int_0^T\int_{\mathbb{R}^d}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr] < \infty.$$

Now we turn to the proof of the first main theorem of this section.

Proof of Theorem 4.2.1. The proof of the uniqueness is rather similar to the uniqueness proof in Theorem 4.2.3, so it is omitted.

Existence. By Theorem 4.2.3, for each n, there exists a unique solution $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ $\in S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ to Eq.(3.3). We will prove $(Y_\cdot^{t,\cdot,n}$, $Z_\cdot^{t,\cdot,n})$ is a Cauchy sequence in $S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. We use the same notation as in Theorem 3.1.2, then by Lemma 2.3.3 and Proposition 4.2.7, as n, $m\longrightarrow\infty$ we have

$$E[\int_0^T\int_{\mathbb{R}^d}e^{Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr] + E[\int_0^T\int_{\mathbb{R}^d}e^{Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho^{-1}(x)dxdr]\longrightarrow 0 \tag{4.14}$$

and

$$E[\sup_{0\le s\le T}\int_{\mathbb{R}^d}e^{Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho^{-1}(x)dx]\longrightarrow 0.$$

Therefore $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ is a Cauchy sequence in $S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2$ $(\mathbb{R}^d;\mathbb{R}^d))$ with its limit denoted by $(Y_s^{t,x}, Z_s^{t,x})$.

We will show that $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (3.2) for an arbitrary $\varphi\in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$. For this, we prove that along a subsequence (3.6), the spatial integral form of Eq.(3.3), converges to Eq.(3.2) in $L^2(\Omega)$ term by term as $n\longrightarrow\infty$. Here we only show that along a subsequence

$$E[\,|\int_s^T\int_{\mathbb{R}^d}\big(f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) - f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\big)\varphi(x)dxdr|^2]\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Other items are under the same conditions as in Section 3.1, therefore the convergence can be dealt with similarly. Notice

$$\begin{aligned}
&E[\,|\int_s^T\int_{\mathbb{R}^d}\big(f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) - f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\big)\varphi(x)dxdr|^2]\\
\le\ &E[\int_s^T\int_{\mathbb{R}^d}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}) - f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2\rho^{-1}(x)dxdr\\
&\times\int_s^T\int_{\mathbb{R}^d}|\varphi(x)|^2\rho(x)dxdr]
\end{aligned}$$

$$
\begin{aligned}
&\le C_pE[\int_s^T\int_{\mathbb{R}^d}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2\rho^{-1}(x)dxdr]\\
&\le C_pE[\int_s^T\int_{\mathbb{R}^d}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})-f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x})|^2\rho^{-1}(x)dxdr]\\
&\quad +C_pE[\int_s^T\int_{\mathbb{R}^d}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x})-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2\rho^{-1}(x)dxdr]\\
&\le C_pE[\int_s^T\int_{\mathbb{R}^d}|Z_r^{t,x,n}-Z_r^{t,x}|^2\rho^{-1}(x)dxdr] \qquad (4.15)\\
&\quad +C_pE[\int_s^T\int_{\mathbb{R}^d}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x})-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2\rho^{-1}(x)dxdr].
\end{aligned}
$$

We only need to prove that along a subsequence

$$
E[\int_s^T\int_{\mathbb{R}^d}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x})-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2\rho^{-1}(x)dxdr]\longrightarrow 0 \text{ as } n\longrightarrow\infty.
$$

First we will find a subsequence of $\{Y_r^{t,x,n}\}_{n=1}^\infty$ still denoted by $\{Y_r^{t,x,n}\}_{n=1}^\infty$ s.t. $Y_r^{t,x,n}\longrightarrow Y_r^{t,x}$ for a.e. $r\in[0,T]$, $x\in\mathbb{R}^d$, a.s. ω and $E[\int_0^T\int_{\mathbb{R}^d}\sup_n|Y_r^{t,x,n}|^2\rho^{-1}(x)dxdr]<\infty$. From (4.14), we know that $E[\int_0^T\int_{\mathbb{R}^d}|Y_r^{t,x,n}-Y_r^{t,x}|^2\rho^{-1}(x)dxdr]\longrightarrow 0$, therefore we may assume without losing any generality that $Y_r^{t,x,n}\longrightarrow Y_r^{t,x}$ for a.e. $r\in[0,T]$, $x\in\mathbb{R}^d$, a.s. ω and extract a subsequence of $\{Y_r^{t,x,n}\}_{n=1}^\infty$ still denoted by $\{Y_r^{t,x,n}\}_{n=1}^\infty$ s.t.

$$
\sqrt{E[\int_0^T\int_{\mathbb{R}^d}|Y_r^{t,x,n+1}-Y_r^{t,x,n}|^2\rho^{-1}(x)dxdr]}\le\frac{1}{2^n}.
$$

For any n,

$$
|Y_r^{t,x,n}|\le|Y_r^{t,x,1}|+\sum_{i=1}^{n-1}|Y_r^{t,x,i+1}-Y_r^{t,x,i}|\le|Y_r^{t,x,1}|+\sum_{i=1}^{\infty}|Y_r^{t,x,i+1}-Y_r^{t,x,i}|.
$$

Then by the property of norm, we have

$$
\begin{aligned}
&\sqrt{E[\int_0^T\int_{\mathbb{R}^d}\sup_n|Y_r^{t,x,n}|^2\rho^{-1}(x)dxdr]}\\
\le&\sqrt{E[\int_0^T\int_{\mathbb{R}^d}(|Y_r^{t,x,1}|+\sum_{i=1}^{\infty}|Y_r^{t,x,i+1}-Y_r^{t,x,i}|)^2\rho^{-1}(x)dxdr]}\\
\le&\sqrt{E[\int_0^T\int_{\mathbb{R}^d}|Y_r^{t,x,1}|^2\rho^{-1}(x)dxdr]}+\sum_{i=1}^{\infty}\sqrt{E[\int_0^T\int_{\mathbb{R}^d}|Y_r^{t,x,i+1}-Y_r^{t,x,i}|^2\rho^{-1}(x)dxdr]}\\
\le&\sqrt{E[\int_0^T\int_{\mathbb{R}^d}|Y_r^{t,x,1}|^2\rho^{-1}(x)dxdr]}+\sum_{i=1}^{\infty}\frac{1}{2^i}
\end{aligned}
$$

$< \infty.$

On the other hand, for this subsequence $\{Y_r^{t,x,n}\}_{n=1}^{\infty}$, by Condition (H.4.4), we have

$$
\begin{aligned}
& E[\int_0^T \int_{\mathbb{R}^d} \sup_n |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
\leq\ & C_p E[\int_0^T \int_{\mathbb{R}^d} \sup_n (1 + |Y_r^{t,x,n}|^2 + |Y_r^{t,x}|^2) \rho^{-1}(x) dx dr] \\
=\ & C_p E[\int_0^T \int_{\mathbb{R}^d} (1 + \sup_n |Y_r^{t,x,n}|^2 + |Y_r^{t,x}|^2) \rho^{-1}(x) dx dr] \\
<\ & \infty.
\end{aligned}
$$

Then, by the Lebesgue's dominated convergence theorem and Condition (H.4.6), we have

$$
\begin{aligned}
& \lim_{n\to\infty} E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
=\ & E[\int_s^T \int_{\mathbb{R}^d} \lim_{n\to\infty} |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr] \\
=\ & 0.
\end{aligned}
$$

That is to say $(Y_s^{t,x}, Z_s^{t,x})_{0\leq s\leq T}$ satisfies Eq.(3.2). The proof of Theorem 4.2.1 is completed. ◇

§4.2.4 The corresponding SPDEs

In the previous subsection, we proved the existence and uniqueness of solution of BDSDE (3.1) and obtained the solution $(Y_s^{t,x}, Z_s^{t,x})$ by taking the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ of the solutions of Eq.(3.3) in the space $S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \bigotimes M^{2,0}$ $([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ along a subsequence. We still start from Eq.(3.3) in this subsection.

Proposition 4.2.8. *Under Conditions* (H.4.1)-(H.4.7), *assume Eq.(3.3) has a unique solution* $(Y_r^{t,x,n}, Z_r^{t,x,n})$, *then for any* $t \leq s \leq T$,

$$Y_r^{s,X_s^{t,x},n} = Y_r^{t,x,n} \text{ and } Z_r^{s,X_s^{t,x},n} = Z_r^{t,x,n} \text{ for } r \in [s,T], \text{ a.a. } x \in \mathbb{R}^d \text{ a.s..}$$

Proof. The proof is similar to the proof of Theorem 3.1.4. Here Lemma 4.2.5 plays the same role as Lemma 3.1.3 in the proof of Theorem 3.1.4. ◇

A direct application of Proposition 4.2.8 and Fubini theorem immediately leads to

Proposition 4.2.9. *Under Conditions* (H.4.1)–(H.4.7), *if we define* $u^n(t,x) = Y_t^{t,x,n}$, $v^n(t,x) = Z_t^{t,x,n}$, *then*

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n},\ v^n(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s..}$$

Now we consider the SPDE (3.21).

Theorem 4.2.10. *Under Conditions* (H.4.1)–(H.4.7), *if we define* $u^n(t,x) = Y_t^{t,x,n}$, *where* $(Y_s^{t,x,n}, Z_s^{t,x,n})$ *is the solution of Eq.(3.3), then* $u^n(t,x)$ *is the unique weak solution of Eq.(3.21). Moreover,*

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n},\ (\sigma^*\nabla u^n)(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s..}$$

Proof. Uniqueness. Let u be a solution of Eq.(3.21). Define

$$\begin{aligned} F^n(s,x) &= f\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big),\\ G_j^n(s,x) &= g_j\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big). \end{aligned}$$

Since u is the solution, so $E[\int_0^T\int_{\mathbb{R}^d}\big(|u^n(s,x)|^2+|(\sigma^*\nabla u^n)(s,x)|^2\big)\rho^{-1}(x)dxds] < \infty$ and

$$\begin{aligned} & E[\int_0^T\int_{\mathbb{R}^d}(|F^n(s,x)|^2+\sum_{j=1}^n|G_j^n(s,x)|^2)\rho^{-1}(x)dxds]\\ = \ & E[\int_0^T\int_{\mathbb{R}^d}\big(|f\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)|^2\\ & \qquad +\sum_{j=1}^n|g_j\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)|^2\big)\rho^{-1}(x)dxds]\\ \leq \ & E[\int_0^T\int_{\mathbb{R}^d}M_6^2(1+|u^n(s,x)|)^2\rho^{-1}(x)dxds]\\ & +E[\int_0^T\int_{\mathbb{R}^d}(2\sum_{j=1}^n|g_j\big(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x)\big)-g(s,x,0,0)|^2\\ & \qquad +2|g(s,x,0,0)|^2)\rho^{-1}(x)dxds]\\ \leq \ & C_pE[\int_0^T\int_{\mathbb{R}^d}(1+|u^n(s,x)|^2+|(\sigma^*\nabla u^n)(s,x)|^2+\sum_{j=1}^n|g_j(s,x,0,0)|^2)\rho^{-1}(x)dxds]\\ < \ & \infty. \end{aligned} \tag{4.16}$$

If we define $Y_s^{t,x,n} = u^n(s, X_s^{t,x})$ and $Z_s^{t,x,n} = (\sigma^*\nabla u^n)(s, X_s^{t,x})$, then by Lemma 2.3.3,

$$E[\int_t^T\int_{\mathbb{R}^d}(|Y_s^{t,x,n}|^2+|Z_s^{t,x,n}|^2)\rho^{-1}(x)dxds]$$

$$
\begin{aligned}
&= E[\int_t^T \int_{\mathbb{R}^d} (|u^n(s, X_s^{t,x})|^2 + |(\sigma^*\nabla u^n)(s, X_s^{t,x})|^2)\rho^{-1}(x)dxds] \\
&\leq C_p E[\int_t^T \int_{\mathbb{R}^d} |u^n(s,x)|^2 + |(\sigma^*\nabla u^n)(s,x)|^2\rho^{-1}(x)dxds] \\
&< \infty.
\end{aligned}
$$

Using some ideas of Theorem 2.1 in [3], similar to the argument as in Subsection 3.2.1, we have for $t \leq s \leq T$, $(Y_s^{t,x,n}, Z_s^{t,x,n})$ solves the following linear BDSDE:

$$
\begin{aligned}
Y_s^{t,x,n} &= h(X_T^{t,x}) + \int_s^T F^n(r, X_r^{t,x})dr \\
&\quad -\sum_{j=1}^n \int_s^T G_j^n(r, X_r^{t,x})d^\dagger\hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,n}, dW_r\rangle. \qquad (4.17)
\end{aligned}
$$

Then for any $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} Y_s^{t,x,n}\varphi(x)dx &= \int_{\mathbb{R}^d} h(X_T^{t,x})\varphi(x)dx + \int_s^T \int_{\mathbb{R}^d} F^n(r, X_r^{t,x})\varphi(x)dxdr \\
&\quad -\sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} G_j^n(r, X_r^{t,x})\varphi(x)dxd^\dagger\hat{\beta}_j(r) \\
&\quad -\int_s^T \langle \int_{\mathbb{R}^d} Z_r^{t,x,n}\varphi(x)dx, dW_r\rangle \quad P-\text{a.s.}. \qquad (4.18)
\end{aligned}
$$

Noting the definition of $F^n(s,x)$, $G_j^n(s,x)$, $Y_s^{t,x,n}$ and $Z_s^{t,x,n}$, from (4.18), we have that $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies the spatial integration form of Eq.(3.3). By Corollary 4.2.6, $Y_\cdot^{t,\cdot,n} \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ and therefore $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a solution of Eq.(3.3). If there is another solution $\hat{u}$ to Eq.(3.21), then by the same procedure, we can find another solution $(\hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n})$ to Eq.(3.3), where

$$
\hat{Y}_s^{t,x,n} = \hat{u}^n(s, X_s^{t,x}) \text{ and } \hat{Z}_s^{t,x,n} = (\sigma^*\nabla\hat{u}^n)(s, X_s^{t,x}).
$$

By Theorem 4.2.3, the solution of Eq.(3.3) is unique, therefore

$$
Y_s^{t,x,n} = \hat{Y}_s^{t,x,n} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s..}
$$

Especially for $t = 0$,

$$
Y_s^{0,x,n} = \hat{Y}_s^{0,x,n} \text{ for a.a. } s \in [0,T],\ x \in \mathbb{R}^d \text{ a.s..}
$$

By Lemma 2.3.3 again,

$$
E[\int_0^T \int_{\mathbb{R}^d} |u^n(s,x) - \hat{u}^n(s,x)|^2\rho^{-1}(x)dxds]
$$

$$
\begin{aligned}
&\leq C_pE[\int_0^T\int_{\mathbb{R}^d}|u^n(s,X_s^{0,x})-\hat{u}^n(s,X_s^{0,x})|^2\rho^{-1}(x)dxds] \\
&= C_pE[\int_0^T\int_{\mathbb{R}^d}|Y_s^{0,x,n}-\hat{Y}_s^{0,x,n}|^2)\rho^{-1}(x)dxds] \\
&= 0.
\end{aligned}
$$

So $u^n(s,x)=\hat{u}^n(s,x)$ for a.a. $s\in[0,T]$, $x\in\mathbb{R}^d$ a.s.. The uniqueness is proved.

Existence. For each $(t,x)\in[0,T]\otimes\mathbb{R}^d$, define $u^n(t,x)=Y_t^{t,x,n}$ and $v^n(t,x)=Z_t^{t,x,n}$, where $(Y_\cdot^{t,\cdot,n},Z_\cdot^{t,\cdot,n})\in S^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^{2,0}([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ is the solution of Eq.(3.3). Then by Proposition 4.2.9,

$$
u^n(s,X_s^{t,x})=Y_s^{t,x,n},\ v^n(s,X_s^{t,x})=Z_s^{t,x,n} \text{ for a.a. } s\in[t,T],\ x\in\mathbb{R}^d \text{ a.s..}
$$

Set

$$
\begin{aligned}
F^n(s,x)&=f\big(s,x,u^n(s,x),v^n(s,x)\big),\\
G_j^n(s,x)&=g_j\big(s,x,u^n(s,x),v^n(s,x)\big).
\end{aligned}
$$

Then it is easy to see that $(Y_s^{t,x,n},Z_s^{t,x,n})$ is a solution of Eq.(4.17). By Lemma 2.3.3,

$$
\begin{aligned}
&E[\int_0^T\int_{\mathbb{R}^d}|u^n(s,x)|^2+|v^n(s,x)|^2\rho^{-1}(x)dxds] \\
&\leq C_pE[\int_0^T\int_{\mathbb{R}^d}(|u^n(s,X_s^{0,x})|^2+|v^n(s,X_s^{0,x})|^2)\rho^{-1}(x)dxds] \\
&= C_pE[\int_0^T\int_{\mathbb{R}^d}(|Y_s^{0,x,n}|^2+|Z_s^{0,x,n}|^2)\rho^{-1}(x)dxds] \\
&< \infty.
\end{aligned}
$$

Then from a similar computation as in (4.16) we have

$$
\begin{aligned}
&E[\int_0^T\int_{\mathbb{R}^d}(|F^n(s,x)|^2+\sum_{j=1}^n|G_j^n(s,x)|^2)\rho^{-1}(x)dxds] \\
&\leq C_pE[\int_0^T\int_{\mathbb{R}^d}(1+|u^n(s,x)|^2+|v^n(s,x)|^2+\sum_{j=1}^n|g_j(s,x,0,0)|^2)\rho^{-1}(x)dxds] \\
&< \infty.
\end{aligned}
$$

Now using some ideas of Theorem 2.1 in [3], similar to the argument as in Subsection 3.2.1, we know that $v^n(s,x)=(\sigma^*\nabla u^n)(s,x)$ and u^n is the weak solution of the following linear SPDE:

$$
u^n(t,x) = h(x)+\int_t^T[\mathscr{L}u^n(s,x)+F^n(s,x)]ds
$$

$$-\sum_{j=1}^{n}\int_t^T G_j^n(s,x)d^\dagger\hat{\beta}_j(s), \qquad 0\le t\le s\le T. \tag{4.19}$$

Noting the definition of $F^n(s,x)$ and $G_j^n(s,x)$ and the fact that $v^n(s,x)=(\sigma^*\nabla u^n)(s,x)$, from (4.19), we have that u^n is the weak solution of Eq.(3.21). ◇

In this subsection, we study Eq.(2.19) with f and g allowed to depend on time. If $(Y_s^{t,x},Z_s^{t,x})$ is the solution of Eq.(3.1) and we define $u(t,x)=Y_t^{t,x}$, then by Proposition 3.2.2, we have $\sigma^*\nabla u(t,x)$ exists for a.a. $t\in[0,T]$, $x\in\mathbb{R}^d$ a.s., and

$$u(s,X_s^{t,x})=Y_s^{t,x},\ (\sigma^*\nabla u)(s,X_s^{t,x})=Z_s^{t,x} \text{ for a.a. } s\in[t,T],\ x\in\mathbb{R}^d \text{ a.s..}$$

Also by Proposition 4.2.9 and Lemma 2.3.3, we have (3.23). With (3.23), we prove the other main theorem in this section.

Proof of Theorem 4.2.2. We only need to verify that this u defined through $Y_t^{t,x}$ is the unique weak solution of Eq.(2.19). By Lemma 2.3.3, it is easy to see that

$$(\sigma^*\nabla u)(t,x)=Z_t^{t,x} \text{ for a.a. } t\in[0,T],\ x\in\mathbb{R}^d \text{ a.s..}$$

Furthermore, using the generalized equivalence norm principle again we have (3.24). Then we will verify that $u(t,x)$ satisfies (2.20). Since $u^n(t,x)$ is the weak solution of SPDE (3.21), so for any $\Psi\in C_c^{1,\infty}([0,T]\times\mathbb{R}^d;\mathbb{R}^1)$, $u^n(t,x)$ satisfies (3.22). By proving that along a subsequence (3.22) converges to (2.20) in $L^2(\Omega)$, we have that $u(t,x)$ satisfies (2.20). We only show that along a sequence

$$\begin{aligned}E[\,|\int_t^T\int_{\mathbb{R}^d}\big(&f(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x))\\&-f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))\big)\Psi(s,x)dxds|^2]\longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned}$$

First note

$$\begin{aligned}&E[\,|\int_t^T\int_{\mathbb{R}^d}\big(f(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x))\\&\qquad\qquad -f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))\big)\Psi(s,x)dxds|^2]\\ \le\ &C_pE[\int_t^T\int_{\mathbb{R}^d}|f(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x))\\&\qquad\qquad -f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds]\\ \le\ &C_pE[\int_t^T\int_{\mathbb{R}^d}|f(s,x,u^n(s,x),(\sigma^*\nabla u^n)(s,x))\end{aligned}$$

$$
\begin{aligned}
&\qquad -f(s,x,u^n(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds] \\
&+C_pE[\int_t^T\int_{\mathbb{R}^d}|f(s,x,u^n(s,x),(\sigma^*\nabla u)(s,x)) \\
&\qquad -f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds] \\
\leq\ & C_pE[\int_t^T\int_{\mathbb{R}^d}|(\sigma^*\nabla u^n)(s,x)-(\sigma^*\nabla u)(s,x)|^2\rho^{-1}(x)dxds] \\
&+C_pE[\int_t^T\int_{\mathbb{R}^d}|f(s,x,u^n(s,x),(\sigma^*\nabla u)(s,x)) \\
&\qquad -f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds].
\end{aligned}
$$

We face a similar situation as in (4.15) and only need to prove that along a subsequence

$$
\begin{aligned}
E[\int_t^T\int_{\mathbb{R}^d}&|f(s,x,u^n(s,x),(\sigma^*\nabla u)(s,x)) \\
&-f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds]\longrightarrow 0\ \text{ as } n\longrightarrow\infty.
\end{aligned}
$$

For this, note that we have (3.23) which plays the same role as (4.14) in the proof of Theorem 4.2.1. Thus we can find a subsequence of $\{u^n(s,x)\}_{n=1}^{\infty}$ still denoted by $\{u^n(s,x)\}_{n=1}^{\infty}$ s.t. $u^n(s,x)\longrightarrow u(s,x)$ for a.e. $s\in[0,T]$, $x\in\mathbb{R}^d$, a.s. ω and $E[\int_0^T\int_{\mathbb{R}^d}\sup_n|u^n(s,x)|^2\rho^{-1}(x)dxds]<\infty$. On the other hand, for this subsequence $\{u^n(s,x)\}_{n=1}^{\infty}$, by Condition (H.4.4), we have

$$
\begin{aligned}
&E[\int_0^T\int_{\mathbb{R}^d}\sup_n|f(s,x,u^n(s,x),(\sigma^*\nabla u)(s,x)) \\
&\qquad -f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds] \\
\leq\ & C_pE[\int_0^T\int_{\mathbb{R}^d}\sup_n(1+|u^n(s,x)|^2+|u(s,x)|^2)\rho^{-1}(x)dxds] \\
=\ & C_pE[\int_0^T\int_{\mathbb{R}^d}(1+\sup_n|u^n(s,x)|^2+|u(s,x)|^2)\rho^{-1}(x)dxds] \\
<\ & \infty.
\end{aligned}
$$

Then, by the Lebesgue's dominated convergence theorem and Condition (H.4.6), we have

$$
\begin{aligned}
&\lim_{n\to\infty}E[\int_t^T\int_{\mathbb{R}^d}|f(s,x,u^n(s,x),(\sigma^*\nabla u)(s,x)) \\
&\qquad -f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds] \\
=\ & E[\int_t^T\int_{\mathbb{R}^d}\lim_{n\to\infty}|f(s,x,u^n(s,x),(\sigma^*\nabla u)(s,x)) \\
&\qquad -f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))|^2\rho^{-1}(x)dxds] \\
=\ & 0.
\end{aligned}
$$

Therefore $u(t,x)$ satisfies (2.20), i.e. it is a weak solution of Eq.(2.19) with $u(T,x)=h(x)$. We can prove the uniqueness following a similar argument in Theorem 4.2.10 or Theorem 3.1 in Bally and Matoussi [3]. $\diamond$

§4.3 Infinite Horizon BDSDEs

We consider infinite horizon BDSDE (3.25) in this section and assume

(H.4.8). Change "$\mathscr{B}_{[0,T]}$" to "$\mathscr{B}_{\mathbb{R}^+}$" and "$r \in [0,T]$" to "$r \geq 0$" in (H.4.2);

(H.4.9). Change "$\int_0^T$" to "$\int_0^\infty \mathrm{e}^{-Kr}$" in (H.4.3);

(H.4.10). Change "$r \in [0,T]$" to "$r \geq 0$" in (H.4.4);

(H.4.11). Change "$\mu \in \mathbb{R}^1$" to "$\mu > 0$ with $2\mu - K - 2C - \sum_{j=1}^\infty C_j > 0$", "$r \in [0,T]$" to "$r \geq 0$" and "$\leq \mu \int_{\mathbb{R}^d} U(x)|Y_1(x)-Y_2(x)|^2 \rho^{-1}(x)dx$" to "$\leq -\mu \int_{\mathbb{R}^d} U(x)|Y_1(x)-Y_2(x)|^2 \rho^{-1}(x)dx$" in (H.4.5);

(H.4.12). Change "$r \in [0,T]$" to "$r \geq 0$" in (H.4.6).

The main result of this section is

Theorem 4.3.1. *Under Conditions* (H.4.7)-(H.4.12), *Eq.(3.25) has a unique solution.*

Proof. We use the same notation as in the proof of Theorem 3.3.1. Here we only prove the existence of solution. It is easy to verify that BDSDE (3.28) satisfies conditions of Theorem 4.2.1. Therefore, for each n, there exists $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,-K}([0,n]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,-K}([0,n]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the unique solution of Eq.(3.28). Let $(Y_t^n, Z_t^n)_{t>n} = (0,0)$, then $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,-K} \bigcap M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. Using a similar argument as in the proof of Theorem 3.3.1, we can prove that $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence. Take $(Y_s^{t,x}, Z_s^{t,x})$ as the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ in the space $S^{2,-K} \bigcap M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes M^{2,-K}([0,\infty); L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and we will show that $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of Eq.(3.25). We only need to prove that for an arbitrary $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$, $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (3.35). Noting that $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies Eq.(3.29), we can prove that $(Y_s^{t,x}, Z_s^{t,x})$ satisfies Eq.(3.35) by verifying that along a subsequence Eq.(3.29) converges to Eq.(3.35) in $L^2(\Omega)$ term by term as $n \longrightarrow \infty$. Here we only show that along a subsequence

$$E[\,|\int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})\varphi(x)dxdr$$

$$-\int_s^\infty\int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxdr|^2] \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

For this, note

$$\begin{aligned}
& E[\,|\int_s^n\int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})\varphi(x)dxdr \\
& \quad -\int_s^\infty\int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxdr|^2] \\
\leq\ & 2E[\,|\int_s^n\int_{\mathbb{R}^d} e^{-Kr}\big(f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\big)\varphi(x)dxdr|^2] \\
& +2E[\,|\int_n^\infty\int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxdr|^2] \\
\leq\ & C_pE[\int_s^\infty\int_{\mathbb{R}^d} e^{-Kr}|f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x})|^2\rho^{-1}(x)dxdr] \\
& +C_pE[\int_s^n\int_{\mathbb{R}^d} e^{-Kr}|f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2\rho^{-1}(x)dxdr] \\
& +C_pE[\int_n^\infty\int_{\mathbb{R}^d} e^{-Kr}|f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2\rho^{-1}(x)dxdr] \\
\leq\ & C_pE[\int_s^\infty\int_{\mathbb{R}^d} e^{-Kr}|Z_r^{t,x,n} - Z_r^{t,x}|^2\rho^{-1}(x)dxdr] \\
& +C_pE[\int_n^\infty\int_{\mathbb{R}^d} e^{-Kr}(1+|Y_r^{t,x}|^2)\rho^{-1}(x)dxdr] \\
& +C_pE[\int_s^\infty\int_{\mathbb{R}^d} e^{-Kr}|f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2\rho^{-1}(x)dxdr].
\end{aligned}$$

Similar to (4.15), we only need to prove that along a subsequence

$$\begin{aligned}
& E[\int_s^\infty\int_{\mathbb{R}^d} e^{-Kr}|f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) \\
& \qquad -f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2\rho^{-1}(x)dxdr] \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Since $\{Y_s^{t,x,n}\}_{n=1}^\infty$ is a Cauchy sequence in the space $M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ with the limit $Y_s^{t,x}$, as $n \to 0$, we have

$$E[\int_0^\infty\int_{\mathbb{R}^d} e^{-Kr}|Y_r^{t,x,n} - Y_r^{t,x}|^2\rho^{-1}(x)dxdr] \longrightarrow 0. \tag{4.20}$$

Then from (4.20) we can find a subsequence of $\{Y_r^{t,x,n}\}_{n=1}^\infty$ still denoted by $\{Y_r^{t,x,n}\}_{n=1}^\infty$ s.t. $Y_r^{t,x,n} \longrightarrow Y_r^{t,x}$ for a.e. $r \geq 0$, $x \in \mathbb{R}^d$, a.s. ω and $E[\int_0^\infty\int_{\mathbb{R}^d} e^{-Kr}\sup_n|Y_r^{t,x,n}|^2\rho^{-1}(x)dxdr]$ $< \infty$. On the other hand, for this subsequence $\{Y_r^{t,x,n}\}_{n=1}^\infty$, by Condition (H.4.10), we have

$$E[\int_0^\infty\int_{\mathbb{R}^d} e^{-Kr}\sup_n|f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2\rho^{-1}(x)dxdr]$$

$$\begin{aligned}
&\leq C_pE[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\sup_n(1+|Y_r^{t,x,n}|^2+|Y_r^{t,x}|^2)\rho^{-1}(x)dxdr]\\
&= C_pE[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}(1+\sup_n|Y_r^{t,x,n}|^2+|Y_r^{t,x}|^2)\rho^{-1}(x)dxdr]\\
&< \infty.
\end{aligned}$$

Then, by the Lebesgue's dominated convergence theorem and Condition (H.4.12), we have

$$\begin{aligned}
&\lim_{n\to\infty} E[\int_s^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x})-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2\rho^{-1}(x)dxdr]\\
&= E[\int_s^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\lim_{n\to\infty}|f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x})-f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^2\rho^{-1}(x)dxdr]\\
&= 0.
\end{aligned}$$

That is to say $(Y_s^{t,x},Z_s^{t,x})_{s\geq0}$ satisfies Eq.(3.35). The proof of Theorem 4.3.1 is completed. ◇

By a similar method as in the proof of the existence part in case (i) in Theorem 3.3.1, we have the following estimation:

Proposition 4.3.2. *Let $(Y_s^{t,x,n},Z_s^{t,x,n})$ be the solution of Eq.(3.28), then under the conditions of Theorem 4.3.1,*

$$\begin{aligned}
&\sup_n E[\sup_{s\geq0}\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}|Y_s^{t,x,n}(x)|^2\rho^{-1}(x)dx]+\sup_n E[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Y_r^{t,x,n}(x)|^2\rho^{-1}(x)dxdr]\\
&+\sup_n E[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}|Z_r^{t,x,n}(x)|^2\rho^{-1}(x)dxdr]<\infty.
\end{aligned}$$

All the proofs until now in this chapter have shown us how to deal with the non-Lipschitz term. So as for BDSDE (2.16), we don't intend to give details. Indeed, we can follow the procedure in Section 3.4 to prove Theorem 2.3.10 and 2.3.11 under the non-Lipschitz conditions. We leave it to readers.

Chapter 5

Stationary Stochastic Viscosity Solutions of SPDEs

In this chapter, we discuss the stationary stochastic viscosity solutions of SPDEs. Comparing the stochastic viscosity solution with the weak solution, we need more information for the stochastic viscosity solution. In particular, the space continuity of solutions of SPDEs as well as time continuity is considered in this chapter. With them, we can perfect the stochastic viscosity solutions of real-valued SPDEs and achieving the stationary solution. In this chapter, we only deal with the Lipschitz condition, but we would like to point out that the techniques in Chapter 4 can be similarly applied to studying the stochastic viscosity solutions of SPDEs with linear growth non-Lipschitz term although we don't intend to include the analysis in this chapter.

§5.1 Doss-Sussmann Transformation and Definition for Stochastic Viscosity Solution of SPDE

The main purpose of this chapter is to find the stationary solution of the following SPDE

$$
\begin{aligned}
v(t,x) &= v(0,x) + \int_0^t [\mathscr{L}v(s,x) + f\big(x, v(s,x), \sigma^*(x)Dv(s,x)\big)]ds \\
&\quad + \int_0^t \langle g\big(x, v(s,x)\big), dB_s\rangle. \qquad (5.1)
\end{aligned}
$$

In this chapter B_s is a two-sided Brownian motion in $\mathbb{R}^l$, $\mathcal{L}$ is the infinitesimal generator of a diffusion process as in (1.3).

As shown in Subsection 2.3.4, under appropriate conditions, for $T \geq t$, defining $u(t,x) \triangleq v(T-t,x)$, we can obtain the time reverse version of Eq.(5.1):

$$\begin{aligned} u(t,x) &= u(T,x) + \int_t^T [\mathscr{L}u(s,x) + f\big(x, u(s,x), (\sigma^*\nabla u)(s,x)\big)]ds \\ &\quad - \int_t^T \langle g\big(x, u(s,x)\big), d^\dagger \hat{B}_s\rangle. \end{aligned} \tag{5.2}$$

Here $\hat{B}_s = B_{T-s} - B_T$ is also a two-sided Brownian motion in $\mathbb{R}^l$ and the integral w.r.t. $\hat{B}$ is a "backward Itô's integral".

The BDSDE associated with SPDE (5.2) has the following form

$$Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T \langle g(X_r^{t,x}, Y_r^{t,x}), d^\dagger \hat{B}_r\rangle - \int_s^T \langle Z_r^{t,x}, dW_r\rangle. \tag{5.3}$$

Here W_s is a two-sided Brownian motion in $\mathbb{R}^d$ and the integral w.r.t. W is a standard forward Itô's integral.

We assume

(A.5.1). Functions $f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ and $g : \mathbb{R}^d \times \mathbb{R}^1 \longrightarrow \mathbb{R}^l$ are $\mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ and $\mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1}$ measurable respectively, and there exist constants C_0, C_1, $C \geq 0$ s.t. for any (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d$,

$$|f(x_1,y_1,z_1) - f(x_2,y_2,z_2)|^2 \leq C_1|x_1-x_2|^2 + C_0|y_1-y_2|^2 + C|z_1-z_2|^2,$$
$$|g(x_1,y_1) - g(x_2,y_2)|^2 \leq C_1|x_1-x_2|^2 + C|y_1-y_2|^2;$$

(A.5.2). $g(\cdot,\cdot) \in C_b^{2,3}(\mathbb{R}^d, \mathbb{R}^1)$;

(A.5.3). There exist $K \in \mathbb{R}^+$, $p > d+2$, $K < K' < 2K$ and a constant $\mu > 0$ with $2\mu - \frac{p}{2}K' - \frac{p(p+1)}{2}C > 0$ s.t. for any $y_1, y_2 \in \mathbb{R}^1$, x, $z \in \mathbb{R}^d$,

$$(y_1 - y_2)(f(x,y_1,z) - f(x,y_2,z)) \leq -\mu|y_1-y_2|^2;$$

(A.5.4). $b(\cdot) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma(\cdot) : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ are globally Lipschitz continuous with Lipschitz constant L and for p, K in (A.5.3), $K - pL - \frac{p(p-1)}{2}L^2 > 0$.

The stochastic viscosity solution of SPDE was studied by Lions and Souganidis in [32]-[35] and by Buckdahn and Ma in [9]-[11]. In this chapter, we adopt Buckdahn and Ma's definition for stochastic viscosity solution of SPDE (5.1) which is done through the Doss-Sussmann transformation and the viscosity solution of a random PDE. Denote the set of C^0-functions with linear growth by C_l^0. Buckdahn and Ma proved that if $u(T,\cdot) \in C_l^0(\mathbb{R}^d)$ is given, the solution $Y_t^{t,x}$ of Eq.(5.3), $(t,x) \in [0,T] \times \mathbb{R}^d$, is a stochastic viscosity solution of Eq.(5.2) under Conditions (A.5.1), (A.5.2) and (A.5.4). Therefore it gives the stochastic viscosity solution of Eq.(5.1) through the time reversal argument. To benefit the reader, we include briefly Buckdahn and Ma's main idea of definition for stochastic viscosity solution of SPDE (5.1) in [9]-[11] through the Doss-Sussmann transformation and the viscosity solution of a random PDE. Define $\lambda(t,x,y)$ as the solution of the following SDE

$$\lambda(t,x,y) = y + \frac{1}{2}\int_0^t \langle g, D_y g\rangle(x, \lambda(s,x,y))ds - \int_0^t \langle g(x,\lambda(s,x,y)), dB_s\rangle.$$

Under the Condition (A.5.2), $\lambda(t,x,y)$ is a stochastic flow, i.e. for fixed x, the random field $\lambda(\cdot,x,\cdot)$ is continuously differentiable in the variable y, and the mapping $y \longrightarrow \lambda(t,x,y)$ defines a diffeomorphism for all (t,x), P-a.s.. Denote its inverse by $\zeta(t,x,y) = (\lambda(t,x,\cdot))^{-1}(y)$. Let $\tilde{v}(t,x) = \zeta(t,x,v(t,x))$, the so-called Doss-Sussmann transformation, then $\tilde{v}(t,x)$ satisfies the following random PDE

$$\tilde{v}(t,x) = \tilde{v}(0,x) + \int_0^t [\mathcal{L}\tilde{v}(s,x) + \tilde{f}(s,x,\tilde{v}(s,x),\sigma^*(x)D\tilde{v}(s,x))]ds. \tag{5.4}$$

Here

$$\begin{aligned}\tilde{f}(t,x,y,z) = \frac{1}{D_y\lambda(t,x,y)}\big(&f(x,\lambda(t,x,y),\sigma^*(x)D_x\lambda(t,x,y) + D_y\lambda(t,x,y)z) \\ &+\mathscr{L}_x\lambda(t,x,y) + \langle\sigma^*(x)D_{xy}\lambda(t,x,y),z\rangle + \frac{1}{2}D_{yy}\lambda(t,x,y)|z|^2\big).\end{aligned}$$

Then the stochastic viscosity solution of Eq.(5.1) $v(t,x)$ was defined in [9]-[11] through the viscosity solution of "deterministic" PDE (5.4) $\tilde{v}(t,x)$ for a.s. ω via the relation $\tilde{v}(t,x) = \zeta(t,x,v(t,x))$. The notion of viscosity solutions of partial differential equations was first introduced by Crandall and Lions in [15]. Here for fixed ω, we give the definition for the viscosity solution of "deterministic" PDE (5.4):

Definition 5.1.1. (e.g. [43]) For fixed ω, $\hat{v} \in C([0,\infty)\times\mathbb{R}^d;\mathbb{R}^1)$ is called a viscosity subsolution (resp. supersolution) of (5.4) if $\hat{v}(0,x) \le \tilde{v}(0,x,\omega)$ (resp. $\hat{v}(0,x) \ge \tilde{v}(0,x,\omega)$), $x \in \mathbb{R}^d$, and moreover for any $\varphi \in C^{1,2}([0,\infty)\times\mathbb{R}^d;\mathbb{R}^1)$ and $(t,x) \in (0,\infty)\times\mathbb{R}^d$ which is a local maximum of $\hat{v}-\varphi$ (resp. minimum of $\hat{v}-\varphi$),

$$\begin{aligned}&\frac{\partial\varphi}{\partial t}(t,x) - \mathcal{L}\varphi(t,x) - \tilde{f}\big(t,x,\varphi(t,x),\sigma^*(x)D\varphi(t,x)\big) \le 0 \\ (\text{resp.}\quad &\frac{\partial\varphi}{\partial t}(t,x) - \mathcal{L}\varphi(t,x) - \tilde{f}\big(t,x,\varphi(t,x),\sigma^*(x)D\varphi(t,x)\big) \ge 0\ );\end{aligned}$$

$\hat{v} \in C([0,\infty) \times \mathbb{R}^d; \mathbb{R}^1)$ is called a viscosity solution of (5.4) if it is both a viscosity subsolution and supersolution.

Remark 5.1.2. *The definition for the stochastic viscosity solution of SPDE in [9]-[11] was not directly defined by the Doss-Sussmann transformation and the viscosity solution of a random PDE, instead they used "test function language" as Crandall and Lions did in [15] and gave a very complicated definition. But Buckdahn and Ma proved that their definition is equivalent to the definition through the Doss-Sussmann transformation and the viscosity solution of a random PDE. Since the definition through the Doss-Sussmann transformation and the viscosity solution of a random PDE is easy to understand, here we would rather use it to introduce their definition for stochastic viscosity solution of SPDE than the complicated definition through "test function language".*

Then according to the Definition 2.1.1 and 2.1.2, for K, $p \in \mathbb{R}^+$ and $0 \leq t \leq T < \infty$, we can write down the notation adopted in this chapter: $M^{2,-K}([0,\infty); \mathbb{R}^d)$, $S^{2,-K}([0,\infty); \mathbb{R}^1)$, $M^{2,0}([t,T]; \mathbb{R}^d)$, $S^{2,0}([t,T]; \mathbb{R}^1)$ and $S^{p,-K}([0,\infty); \mathbb{R}^1)$.

Buckdahn and Ma established the connection between the solution of BDSDE (5.3) and the stochastic viscosity solution of SPDE (5.1). The following Buckdahn and Ma's result will be used in this chapter.

Theorem 5.1.3. ([9]) *Assume Conditions* (A.5.1), (A.5.2), (A.5.4) *and that the function* $v(0,\cdot) \in C_l^0(\mathbb{R}^d)$ *is given. Then* $v(t,x) = u(T-t,x) = Y_{T-t}^{T-t,x}$, *where* $Y_\cdot^{t,x} \in S^{2,0}([0,T]; \mathbb{R}^1)$ *is the solution of Eq.(5.3), is a stochastic viscosity solution of Eq.(5.1).*

Remark 5.1.4. *From the argument of Buckdahn and Ma, if we assume that the given* $v(0,x)$ *is continuous w.r.t.* x *and* $E[|v(0,X_T^{t,x})|^2] < \infty$, *the condition* $v(0,\cdot) \in C_l^0(\mathbb{R}^d)$ *can be replaced by the above conditions in Theorem 5.1.3, but the conclusion of Theorem 5.1.3 remains true since* $E[|v(0,X_T^{t,x})|^2] = E[|Y_T^{T,X_T^{t,x}}|^2] = E[|Y_T^{t,x}|^2] < \infty$ *guarantees the corresponding BDSDE has a square-integrable terminal condition.*

§5.2 Infinite Horizon BDSDEs

§5.2.1 Introduction of Pardoux and Peng's work for finite horizon BDSDEs

In this subsection, we will briefly introduce the pioneering work by Pardoux and Peng in [44] for the following finite horizon BDSDE:

$$Y_s = Y_T + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T \langle g(r, Y_r, Z_r), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r, dW_r \rangle. \qquad (5.5)$$

Here we only consider $\mathbb{R}^1$-valued BDSDE for our purpose. One can also refer to [44] for multi-dimensional BDSDE if interested. We assume

(A.5.1)′. Functions $f : \Omega \times [0,T] \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ and $g : \Omega \times [0,T] \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^l$ are $\mathscr{F} \otimes \mathscr{B}_{[0,T]} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable, and for any $(y,z) \in \mathbb{R}^1 \times \mathbb{R}^d$, $f(\cdot, y, z) \in M^{2,0}([0,T];\mathbb{R}^1)$ and $g(\cdot, y, z) \in M^{2,0}([0,T];\mathbb{R}^d)$, moreover there exist constants $C \geq 0$ and $0 \leq \alpha < 1$ s.t. for any $r \in [0,T]$, (y_1, z_1), $(y_2, z_2) \in \mathbb{R}^1 \times \mathbb{R}^d$,

$$
\begin{aligned}
|f(r,y_1,z_1) - f(r,y_2,z_2)|^2 &\leq C|y_1 - y_2|^2 + C|z_1 - z_2|^2, \\
|g(r,y_1,z_1) - g(r,y_2,z_2)|^2 &\leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2.
\end{aligned}
$$

Theorem 5.2.1. ([44]) *Under Condition* (A.5.1)′, *for any given* $\mathcal{F}_T \bigvee \mathscr{F}^{\hat{B}}_{T,\infty}$ *measurable* $Y_T \in L^2(\Omega)$, *Eq.(5.5) has a unique solution*

$$
(Y_\cdot, Z_\cdot) \in S^{2,0}([0,T];\mathbb{R}^1) \bigotimes M^{2,0}([0,T];\mathbb{R}^d).
$$

In [44], Pardoux and Peng also discussed a type of FBDSDE, a special case of Eq.(5.5),

$$
\begin{aligned}
Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\
&\quad - \int_s^T \langle g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle, \qquad (5.6)
\end{aligned}
$$

where $(X_s^{t,x})_{t \leq s \leq T}$ is the solution of Eq.(2.14). We assume

(A.5.2)′. Functions $f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ and $g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^l$ are $\mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable, and there exist constants $C \geq 0$ and $0 \leq \alpha < 1$ s.t. for any (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d$,

$$
\begin{aligned}
|f(x_1,y_1,z_1) - f(x_2,y_2,z_2)|^2 &\leq C|x_1 - x_2|^2 + C|y_1 - y_2|^2 + C|z_1 - z_2|^2, \\
|g(x_1,y_1,z_1) - g(x_2,y_2,z_2)|^2 &\leq C|x_1 - x_2|^2 + C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2.
\end{aligned}
$$

Theorem 5.2.2. ([44]) *Under Condition* (A.5.2)′, *for any given* $\mathcal{F}_T \bigvee \mathscr{F}^{\hat{B}}_{T,\infty}$ *measurable* h *of linear growth, and for each* $x \in \mathbb{R}^d$, *Eq.(5.6) has a unique solution*

$$
(Y_\cdot, Z_\cdot) \in S^{2,0}([0,T];\mathbb{R}^1) \bigotimes M^{2,0}([0,T];\mathbb{R}^d).
$$

In [44], for the first time, Pardoux and Peng associated the classical solution of SPDE, if any, with the solution of BDSDE (5.6). They proved that under some strong smoothness conditions of h, b, σ, f and g (for details see [44]), $u(t,x) = Y_t^{t,x}$, where Y

is the unique solution of Eq.(5.6), $(t,x)\in[0,T]\times\mathbb{R}^d$, is independent of $\mathcal{F}_T^W$ and is the unique classical solution of the following backward SPDE

$$\begin{aligned} u(t,x) &= h(x)+\int_t^T[\mathcal{L}u(s,x)+f\big(x,u(s,x),\sigma^*(x)Du(s,x)\big)]ds \\ &\quad -\int_t^T\langle g\big(x,u(s,x),\sigma^*(x)Du(s,x)\big),d^\dagger\hat{B}_s\rangle,\quad 0\le t\le T. \end{aligned}$$

§5.2.2 Existence and uniqueness of solutions of infinite horizon BDSDEs

The main purpose of this subsection is to prove the existence and uniqueness of solution of the following BDSDE on infinite horizon:

$$\begin{aligned} \mathrm{e}^{-\frac{K'}{2}t}Y_t &= \int_t^\infty \mathrm{e}^{-\frac{K'}{2}s}f(s,Y_s,Z_s)ds+\int_t^\infty\frac{K'}{2}\mathrm{e}^{-\frac{K'}{2}s}Y_sds \\ &\quad -\int_t^\infty \mathrm{e}^{-\frac{K'}{2}s}\langle g(s,Y_s,Z_s),d^\dagger\hat{B}_s\rangle-\int_t^\infty \mathrm{e}^{-\frac{K'}{2}s}\langle Z_s,dW_s\rangle, \end{aligned} \tag{5.7}$$

or equivalently, for arbitrary $T>0$ and $0\le t\le T$,

$$\begin{cases} dY_t=-f(t,Y_t,Z_t)dt+\langle g(t,Y_t,Z_t),d^\dagger\hat{B}_t\rangle+\langle Z_t,dW_t\rangle, \\ \lim\limits_{T\to\infty}\mathrm{e}^{-\frac{K'}{2}T}Y_T=0\quad \text{a.s..} \end{cases}$$

We assume that

(H.5.1). Functions $f:\Omega\times[0,\infty)\times\mathbb{R}^1\times\mathbb{R}^d\longrightarrow\mathbb{R}^1$ and $g:\Omega\times[0,\infty)\times\mathbb{R}^1\times\mathbb{R}^d\longrightarrow\mathbb{R}^l$ are $\mathscr{F}\otimes\mathscr{B}_{[0,\infty)}\otimes\mathscr{B}_{\mathbb{R}^1}\otimes\mathscr{B}_{\mathbb{R}^d}$ measurable, and there exist constants C_0, $C\ge0$ and $0\le\alpha<\frac{1}{2}$ s.t. for any $(\omega,t)\in\Omega\times[0,\infty)$, (y_1,z_1), $(y_2,z_2)\in\mathbb{R}^1\times\mathbb{R}^d$,

$$\begin{aligned} |f(t,y_1,z_1)-f(t,y_2,z_2)|^2&\le C_0|y_1-y_2|^2+C|z_1-z_2|^2, \\ |g(t,y_1,z_1)-g(t,y_2,z_2)|^2&\le C|y_1-y_2|^2+\alpha|z_1-z_2|^2; \end{aligned}$$

(H.5.2). There exist $K\in\mathbb{R}^+$, $p>d+2$, $K<K'<2K$ and a constant $\mu>0$ with $2\mu-K'-\frac{p(p+1)}{2}C>0$ s.t. for any $(\omega,t)\in\Omega\times[0,\infty)$, y_1, $y_2\in\mathbb{R}^1$, $z\in\mathbb{R}^d$,

$$(y_1-y_2)(f(t,y_1,z)-f(t,y_2,z))\le-\mu|y_1-y_2|^2;$$

(H.5.3). For p, K in (H.5.2), $f(\cdot,0,0)\in M^{p,-K}([0,\infty);\mathbb{R}^1)$, $g(\cdot,0,0)\in M^{p,-K}([0,\infty);\mathbb{R}^l)$.

Remark 5.2.3. *In this chapter, we use the exponentially decay function* $\mathrm{e}^{-\frac{K'}{2}s}$ *in infinite horizon BDSDEs (e.g. Eq.(5.7)) rather than* e^{-Ks} *as in the previous chapters. One can easily see that* $\mathrm{e}^{-\frac{K'}{2}s}$ *is a weaker condition, but adequate for us, although there is no essential change. In order to avoid heavy notation, we only use exponentially decay function* $\mathrm{e}^{-\frac{K'}{2}s}$ *in this chapter.*

Theorem 5.2.4. *Under Conditions* (H.5.1)–(H.5.3), *Eq.(5.7) has a unique solution*

$$(Y_\cdot, Z_\cdot) \in S^{p,-K}([0,\infty);\mathbb{R}^1) \cap M^{2,-K}([0,\infty);\mathbb{R}^1) \bigotimes M^{2,-K}([0,\infty);\mathbb{R}^d).$$

Remark 5.2.5. *Since here we consider the BDSDE on $\mathbb{R}^1$ space which is simpler than the BDSDE on the Hilbert space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ we considered in Chapters 3 and 4, therefore we only give the details of the proofs which are different from Chapter 3. We also leave out the localization argument when applying Itô's formula to save the space.*

Proof of Theorem 5.2.4. The uniqueness can be done by the generalized Itô's formula as in the corresponding part in Theorem 3.3.1, so it is omitted.

Existence. For each $n \in \mathbb{N}$, we define a sequence of BDSDEs as follows

$$Y_t^n = \int_t^n f(s, Y_s^n, Z_s^n)ds - \int_t^n \langle g(s, Y_s^n, Z_s^n), d^\dagger \hat{B}_s\rangle - \int_t^n \langle Z_s^n, dW_s\rangle. \tag{5.8}$$

Let $(Y_t^n, Z_t^n)_{t\geq n} = (0,0)$, and according to [44] (or Theorem 5.2.1), Eq.(5.8) has a unique solution $(Y_\cdot^n, Z_\cdot^n) \in S^{2,-K}([0,\infty);\mathbb{R}^1)\cap M^{2,-K}([0,\infty);\mathbb{R}^1) \bigotimes M^{2,-K}([0,\infty);\mathbb{R}^n)$. Also under Conditions (H.5.1)–(H.5.3), we can prove $Y_\cdot^n \in S^{p,-K}([0,\infty);\mathbb{R}^1)$.

Lemma 5.2.6. *Let $(Y_t^n)_{t\geq 0}$ be the solution of Eq.(5.8), then under Conditions* (H.5.1)–(H.5.3), $Y_\cdot^n \in S^{p,-K}([0,\infty);\mathbb{R}^1)$.

Proof. Applying the generalized Itô's formula to $\mathrm{e}^{-Kr}|Y_r^n|^p$ and following a similar calculation as in (3.36)–(3.38), we have

$$\begin{aligned} &E[\int_0^\infty \mathrm{e}^{-Kr}|Y_r^n|^{p-2}|Z_r^n|^2dr] + E[\int_0^\infty \mathrm{e}^{-Kr}|Y_r^n|^p dr] \\ \leq\ & C_pE[\int_0^\infty \mathrm{e}^{-Kr}|f(r,0,0)|^p dr] + C_pE[\int_0^\infty \mathrm{e}^{-Kr}|g(r,0,0)|^p dr] < \infty \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} E[\sup_{t\geq 0} \mathrm{e}^{-Kt}|Y_t^n|^p] \leq\ & C_pE[\int_0^\infty \mathrm{e}^{-Kr}(|f(r,0,0)|^p + |g(r,0,0)|^p)dr] \\ &+C_pE[\int_0^\infty \mathrm{e}^{-Kr}|Y_r^n|^{p-2}|Z_r^n|^2dr] + C_pE[\int_0^\infty \mathrm{e}^{-Kr}|Y_r^n|^p dr]. \end{aligned} \tag{5.10}$$

By (H.5.3) and (5.9), $Y_\cdot^n \in S^{p,-K}([0,\infty);\mathbb{R}^1)$. Lemma 5.2.6 is proved. ◇

Remark 5.2.7. *The proof of Lemma 5.2.6 also works with p replaced by 2. Note that if $f(\cdot,0,0) \in M^{p,-K}([0,\infty);\mathbb{R}^1)$, then by the Hölder inequality*

$$E[\int_0^\infty \mathrm{e}^{-Ks}|f(s,0,0)|^2ds] \leq (\int_0^\infty \mathrm{e}^{-Ks}ds)^{\frac{p-2}{p}}(E[\int_0^\infty \mathrm{e}^{-Ks}|f(s,0,0)|^p ds])^{\frac{2}{p}} < \infty,$$

therefore $f(\cdot,0,0)\in M^{2,-K}([0,\infty);\mathbb{R}^1)$. *Similarly we have* $g(\cdot,0,0)\in M^{2,-K}([0,\infty);\mathbb{R}^l)$ *as well. So it is easy to see in (5.9) with p replaced by* 2 *that*

$$(Y_\cdot^n, Z_\cdot^n)\in M^{2,-K}([0,\infty);\mathbb{R}^1)\bigotimes M^{2,-K}([0,\infty);\mathbb{R}^d).$$

We return to the proof of Theorem 5.2.4. We will show that $(Y_\cdot^n, Z_\cdot^n)$ is a Cauchy sequence in the space of $S^{p,-K}([0,\infty);\mathbb{R}^1)\cap M^{2,-K}([0,\infty);\mathbb{R}^1)\bigotimes M^{2,-K}([0,\infty);\mathbb{R}^d)$ with the norm

$$\Big((E[\sup_{t\geq 0}\mathrm{e}^{-Kt}|\cdot|^p])^{\frac{2}{p}}+E[\int_0^\infty \mathrm{e}^{-Kr}|\cdot|^2dr]+E[\int_0^\infty \mathrm{e}^{-Kr}|\cdot|^2dr]\Big)^{\frac{1}{2}},$$

as in Pardoux [40]. Firstly we show that, for $m,n\in\mathbb{N}$ and $m\geq n$,

$$\lim_{n,m\to\infty}E[\sup_{t\geq 0}\mathrm{e}^{-Kt}|Y_t^m-Y_t^n|^p]=0.$$

Define $\bar{Y}_t^{m,n}=Y_t^m-Y_t^n$, $\bar{Z}_t^{m,n}=Z_t^m-Z_t^n$.

(i) When $n\leq t\leq m$,

$$\bar{Y}_t^{m,n}=Y_t^m=\int_t^m f(s,Y_s^m,Z_s^m)ds-\int_t^m\langle g(s,Y_s^m,Z_s^m),d^\dagger\hat{B}_s\rangle-\int_t^m\langle Z_s^m,dW_s\rangle.$$

Some similar calculations as in (5.9) and (5.10) lead to

$$\begin{aligned}
&E[\sup_{n\leq t\leq m}\mathrm{e}^{-Kt}|Y_t^m|^p]\\
\leq\;& C_pE[\int_n^m \mathrm{e}^{-Kr}|Y_r^m|^{p-2}|Z_r^m|^2dr]+C_pE[\int_n^m \mathrm{e}^{-Kr}|Y_r^m|^pdr]\\
&+C_pE[\int_n^m \mathrm{e}^{-Kr}(|f(r,0,0)|^p+|g(r,0,0)|^p)dr]\\
\leq\;& C_pE[\int_n^m \mathrm{e}^{-Kr}(|f(r,0,0)|^p+|g(r,0,0)|^p)dr]\longrightarrow 0,\ \text{as } n,\ m\longrightarrow\infty. \qquad (5.11)
\end{aligned}$$

(ii) When $0\leq t\leq n$,

$$\bar{Y}_t^{m,n}=Y_n^m+\int_t^n\bar{f}_rdr-\int_t^n\langle\bar{g}_r,d^\dagger\hat{B}_r\rangle-\int_t^n\langle\bar{Z}_r^{m,n},dW_r\rangle.$$

Here

$$\bar{f}_r=f(r,Y_r^m,Z_r^m)-f(r,Y_r^n,Z_r^n),\quad \bar{g}_r=g(r,Y_r^m,Z_r^m)-g(r,Y_r^n,Z_r^n).$$

Applying Itô's formula to $\mathrm{e}^{-Kr}|\bar{Y}_r^{m,n}|^p$ and following a similar calculation as in (3.39), we have for $s\leq n$,

$$\mathrm{e}^{-Ks}|\bar{Y}_s^{m,n}|^p+\big(p\mu-K-pC-\frac{p(p-1)}{2}C\big)\int_s^n\mathrm{e}^{-Kr}|\bar{Y}_r^{m,n}|^pdr$$

$$\begin{aligned}&+\frac{p}{4}(2p-3-(2p-2)\alpha)\int_s^n e^{-Kr}|\bar{Y}_r^{m,n}|^{p-2}|\bar{Z}_r^{m,n}|^2dr\\ \leq\ & e^{-Kn}|Y_n^m|^p-p\int_s^n e^{-Kr}|\bar{Y}_r^{m,n}|^{p-2}\bar{Y}_r^{m,n}\langle\bar{g}_r,d^\dagger\hat{B}_r\rangle\\ &-p\int_s^n e^{-Kr}|\bar{Y}_r^{m,n}|^{p-2}\bar{Y}_r^{m,n}\langle\bar{Z}_r^{m,n},dW_r\rangle.\end{aligned}$$

And some similar calculations as in (5.9) lead to

$$E[\int_0^n e^{-Kr}|\bar{Y}_r^{m,n}|^{p-2}|\bar{Z}_r^{m,n}|^2dr]+E[\int_0^n e^{-Kr}|\bar{Y}_r^{m,n}|^pdr]\leq C_pE[e^{-Kn}|Y_n^m|^p].\quad(5.12)$$

From (i), the right hand side of the above inequality converges to 0, as n, $m\longrightarrow\infty$.

By some similar calculations as in (5.10), we have

$$E[\sup_{0\leq t\leq n}e^{-Kt}|\bar{Y}_t^{m,n}|^p]\leq C_pE[e^{-Kn}|Y_n^m|^p]\longrightarrow 0,\quad \text{as } n,m\longrightarrow\infty.$$

From (i) (ii), we have for $m,n\in\mathbb{N}$,

$$\lim_{n,m\to\infty}E[\sup_{t\geq0}e^{-Kt}|Y_t^m-Y_t^n|^p]=0.$$

Furthermore the above arguments also hold for $p=2$ in (5.11) and (5.12). Noting from Remark 5.2.7, we have as n, $m\longrightarrow\infty$

$$E[\int_0^\infty e^{-Kr}|\bar{Y}_r^{m,n}|^2dr]+E[\int_0^\infty e^{-Kr}|\bar{Z}_r^{m,n}|^2dr]\longrightarrow 0.$$

Therefore, (Y^n,Z^n) is a Cauchy sequence in the Banach space

$$S^{p,-K}([0,\infty);\mathbb{R}^1)\cap M^{2,-K}([0,\infty);\mathbb{R}^1)\bigotimes M^{2,-K}([0,\infty);\mathbb{R}^d).$$

We take $(Y_t,Z_t)_{t\geq0}$ as the limit of $(Y_t^n,Z_t^n)_{t\geq0}$ in the above space and we will show that $(Y_t,Z_t)_{t\geq0}$ is the solution of Eq.(5.7). Since for $t\leq n$,

$$Y_t^n=\int_t^n f(s,Y_s^n,Z_s^n)ds-\int_t^n\langle g(s,Y_s^n,Z_s^n),d^\dagger\hat{B}_s\rangle-\int_t^n\langle Z_s^n,dW_s\rangle,$$

it turns out that for $t\leq n$,

$$\begin{aligned}e^{-\frac{K'}{2}t}Y_t^n\ =\ &\int_t^n e^{-\frac{K'}{2}s}f(s,Y_s^n,Z_s^n)ds+\int_t^n\frac{K'}{2}e^{-\frac{K'}{2}s}Y_s^nds\\ &-\int_t^n e^{-\frac{K'}{2}s}\langle g(s,Y_s^n,Z_s^n),d^\dagger\hat{B}_s\rangle-\int_t^n e^{-\frac{K'}{2}s}\langle Z_s^n,dW_s\rangle.\quad(5.13)\end{aligned}$$

We will show that Eq.(5.13) converges to Eq.(5.7) as $n\longrightarrow\infty$. For this, we verify the convergence in $L^2(\Omega)$ term by term. For the first term,

$$E[\,|e^{-\frac{K'}{2}t}Y_t^n-e^{-\frac{K'}{2}t}Y_t|^2]\leq E[\sup_{t\geq0}e^{-Kt}|Y_t^n-Y_t|^2]\longrightarrow 0.$$

For the second term, by the Hölder inequality,

$$\begin{aligned}
& E[\,|\int_t^n e^{-\frac{K'}{2}s} f(s,Y_s^n,Z_s^n)ds - \int_t^\infty e^{-\frac{K'}{2}s} f(s,Y_s,Z_s)ds|^2] \\
\leq\ & 2E[\,|\int_t^n e^{-\frac{K'}{2}s}\big(f(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)\big)ds|^2] + 2E[\,|\int_n^\infty e^{-\frac{K'}{2}s} f(s,Y_s,Z_s)ds|^2] \\
\leq\ & 2E[\int_t^n e^{-(K'-K)s}ds \int_t^n e^{-Ks}|f(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)|^2 ds] \\
& +2E[\int_n^\infty e^{-(K'-K)s}ds \int_n^\infty e^{-Ks}|f(s,Y_s,Z_s)|^2 ds] \\
\leq\ & C_pE[\int_0^\infty e^{-Ks}|Y_s^n - Y_s|^2 ds] + C_pE[\int_0^\infty e^{-Ks}|Z_s^n - Z_s|^2 ds] + C_pE[\int_n^\infty e^{-Ks}|Y_s|^2 ds] \\
& +C_pE[\int_n^\infty e^{-Ks}|Z_s|^2 ds] + C_pE[\int_n^\infty e^{-Ks}|f(s,0,0)|^2 ds] \longrightarrow 0.
\end{aligned}$$

For the fourth term, noting $K' > K$ and by Itô's isometry, we have

$$\begin{aligned}
& E[\,|\int_t^n e^{-\frac{K'}{2}s}\langle g(s,Y_s^n,Z_s^n), d^\dagger \hat{B}_s\rangle - \int_t^\infty e^{-\frac{K'}{2}s}\langle g(s,Y_s,Z_s), d^\dagger \hat{B}_s\rangle|^2] \\
\leq\ & 2E[\,|\int_t^n e^{-\frac{K'}{2}s}\langle g(s,Y_s^n,Z_s^n) - g(s,Y_s,Z_s), d^\dagger \hat{B}_s\rangle|^2] \\
& +2E[\,|\int_n^\infty e^{-\frac{K'}{2}s}\langle g(s,Y_s,Z_s), d^\dagger \hat{B}_s\rangle|^2] \\
=\ & 2E[\int_t^n e^{-K's}|g(s,Y_s^n,Z_s^n) - g(s,Y_s,Z_s)|^2 ds] + 2E[\int_n^\infty e^{-K's}|g(s,Y_s,Z_s)|^2 ds] \\
\leq\ & C_pE[\int_0^\infty e^{-Ks}|Y_s^n - Y_s|^2 ds] + C_pE[\int_0^\infty e^{-Ks}|Z_s^n - Z_s|^2 ds] + C_pE[\int_n^\infty e^{-Ks}|Y_s|^2 ds] \\
& +C_pE[\int_n^\infty e^{-Ks}|Z_s|^2 ds] + C_pE[\int_n^\infty e^{-Ks}|g(s,0,0)|^2 ds] \longrightarrow 0.
\end{aligned}$$

We can deal with the third term and the last term using the same arguments as the second term and the fourth term respectively. So $(Y_t, Z_t)_{t\geq 0}$ is the solution of Eq.(5.7). The proof of Theorem 5.2.4 is completed. ◇

Now let's turn to the proof for the existence and uniqueness of solution of FBDSDE on infinite horizon:

$$\begin{aligned}
e^{-\frac{K'}{2}s}Y_s^{t,x} =\ & \int_s^\infty e^{-\frac{K'}{2}r} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^\infty \frac{K'}{2} e^{-\frac{K'}{2}r} Y_r^{t,x} dr \\
& - \int_s^\infty e^{-\frac{K'}{2}r}\langle g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), d^\dagger \hat{B}_r\rangle \\
& - \int_s^\infty e^{-\frac{K'}{2}r}\langle Z_r^{t,x}, dW_r\rangle, \qquad s \geq 0,
\end{aligned} \tag{5.14}$$

equivalently, for arbitrary $T > 0$ and $0 \le s \le T$,

$$\begin{cases} Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T \langle g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), d^\dagger \hat{B}_r\rangle - \int_s^T \langle Z_r^{t,x}, dW_r\rangle, \\ \lim_{T\to\infty} \mathrm{e}^{-\frac{K'}{2}T} Y_T^{t,x} = 0 \quad \text{a.s..} \end{cases}$$

We replace Condition (A.5.1) by

(A.5.1)*. Functions $f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ and $g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^l$ are $\mathscr{B}_{\mathbb{R}^d} \otimes \mathscr{B}_{\mathbb{R}^1} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable, and there exist constants C_0, C_1, $C \ge 0$ and $0 \le \alpha < \frac{1}{2}$ s.t. for any (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d$,

$$|f(x_1, y_1, z_1) - f(x_2, y_2, z_2)|^2 \le C_1|x_1 - x_2|^2 + C_0|y_1 - y_2|^2 + C|z_1 - z_2|^2,$$
$$|g(x_1, y_1, z_1) - g(x_2, y_2, z_2)|^2 \le C_1|x_1 - x_2|^2 + C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2.$$

Proposition 5.2.8. *Under Conditions* (A.5.1)*, (A.5.3), (A.5.4), *Eq.(5.14) has a unique solution*

$$(Y_\cdot^{t,x}, Z_\cdot^{t,x}) \in S^{p,-K}([0,\infty); \mathbb{R}^1) \cap M^{2,-K}([0,\infty); \mathbb{R}^1) \bigotimes M^{2,-K}([0,\infty); \mathbb{R}^d).$$

Proof. Let

$$\hat{f}(s, y, z) = f(X_s^{t,x}, y, z), \quad \hat{g}(s, y, z) = g(X_s^{t,x}, y, z).$$

We need to verify that $\hat{f}$, $\hat{g}$ satisfy Conditions (H.5.1)–(H.5.3) in Theorem 5.2.4. It is obvious that $\hat{f}$, $\hat{g}$ satisfy (H.5.1) and (H.5.2), so we only need to show that $\hat{f}$, $\hat{g}$ satisfy (H.5.3) as well, i.e.

$$E[\int_0^\infty \mathrm{e}^{-Ks}|\hat{f}(s,0,0)|^p ds] < \infty \text{ and } E[\int_0^\infty \mathrm{e}^{-Ks}|\hat{g}(s,0,0)|^p ds] < \infty.$$

Since

$$\begin{aligned} & E[\int_0^\infty \mathrm{e}^{-Ks}|\hat{f}(s,0,0)|^p ds] \\ = \ & E[\int_0^\infty \mathrm{e}^{-Ks}|f(X_s^{t,x},0,0)|^p ds] \\ \le \ & C_p E[\int_0^\infty \mathrm{e}^{-Ks}|f(X_s^{t,x},0,0) - f(0,0,0)|^p ds] + C_p E[\int_0^\infty \mathrm{e}^{-Ks}|f(0,0,0)|^p ds] \\ \le \ & C_p E[\int_0^\infty \mathrm{e}^{-Ks} C_1^p |X_s^{t,x}|^p ds] + C_p E[\int_0^\infty \mathrm{e}^{-Ks}|f(0,0,0)|^p ds], \end{aligned}$$

we only need to prove $E[\int_0^\infty e^{-Ks}|X_s^{t,x}|^p ds] < \infty$. Now applying Itô's formula to $e^{-Kr}|X_r^{t,x}|^p$, we have

$$\begin{aligned}
& d\big(e^{-Kr}|X_r^{t,x}|^p\big) \\
= \; & -Ke^{-Kr}|X_r^{t,x}|^p dr + \frac{p}{2}e^{-Kr}|X_r^{t,x}|^{p-2}\big(2\langle X_r^{t,x}, b(X_s^{t,x})\rangle dr + \|\sigma(X_r^{t,x})\|^2 dr \\
& +2\langle X_s^{t,x}, \sigma(X_r^{t,x})dW_r\rangle\big) + \frac{p(p-2)}{2}e^{-Kr}|X_r^{t,x}|^{p-4}\langle\sigma(X_r^{t,x})\sigma^*(X_r^{t,x})X_r^{t,x}, X_r^{t,x}\rangle dr.
\end{aligned}$$

Then by the Lipschitz condition and the Young inequality, we have for $0 \leq t \leq s$,

$$\begin{aligned}
& e^{-Ks}|X_s^{t,x}|^p + \int_t^s Ke^{-Kr}|X_r^{t,x}|^p dr \\
\leq \; & e^{-Kt}|x|^p + \int_t^s pe^{-Kr}|X_r^{t,x}|^{p-1}(L|X_r^{t,x}| + |b(0)|)dr \\
& + \int_t^s \frac{p(p-1)}{2}e^{-Kr}|X_r^{t,x}|^{p-2}(L|X_r^{t,x}| + \|\sigma(0)\|)^2 dr \\
& + \int_t^s pe^{-Kr}|X_r^{t,x}|^{p-2}\langle X_r^{t,x}, \sigma(X_r^{t,x})dW_r\rangle \\
\leq \; & e^{-Kt}|x|^p + pL\int_t^s e^{-Kr}|X_r^{t,x}|^p dr + p\int_t^s e^{-Kr}|X_r^{t,x}|^{p-1}|b(0)|dr \\
& + \int_t^s e^{-Kr}\frac{p(p-1)}{2}|X_r^{t,x}|^{p-2}\big((1+\varepsilon)L^2|X_r^{t,x}|^2 + C_p\|\sigma(0)\|^2\big)dr \\
& + \int_t^s pe^{-Kr}|X_r^{t,x}|^{p-2}\langle X_r^{t,x}, \sigma(X_r^{t,x})dW_r\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& e^{-Ks}|X_s^{t,x}|^p + \big(K - pL - \frac{p(p-1)}{2}L^2 - (2 + \frac{p(p-1)}{2}L^2)\varepsilon\big)\int_t^s e^{-Kr}|X_r^{t,x}|^p dr \\
\leq \; & e^{-Kt}|x|^p + C_p\int_t^s e^{-Kr}(|b(0)|^p + \|\sigma(0)\|^p)dr \\
& +p\int_t^s e^{-Kr}|X_r^{t,x}|^{p-2}\langle X_r^{t,x}, \sigma(X_r^{t,x})dW_r\rangle.
\end{aligned}$$

Due to the arbitrariness of ε and Condition (A.5.4), we have

$$E[\int_t^s e^{-Kr}|X_r^{t,x}|^p dr] \leq e^{-Kt}|x|^p + C_pE[\int_t^s e^{-Kr}(|b(0)|^p + \|\sigma(0)\|^p)dr] < \infty.$$

Taking the limit of s and noting that $(X_s^{t,x})_{s<t} = x$, we have

$$E[\int_0^\infty e^{-Kr}|X_r^{t,x}|^p dr] < \infty. \tag{5.15}$$

So $E[\int_0^\infty e^{-Ks}|\hat{f}(s,0,0)|^p ds] < \infty$. Similarly, $E[\int_0^\infty e^{-Ks}|\hat{g}(s,0,0)|^p ds] < \infty$. ◇

§5.3 Space and Time Continuity of Solutions of SPDEs and Stationary Stochastic Viscosity Solution

§5.3.1 Continuity of solutions of the corresponding BDSDEs

An simple application of the stochastic flow property proved in [28] leads to

Lemma 5.3.1. *Under Condition* (A.5.4), *let* $(X_s^{t,x})_{s\geq 0}$ *be the solution of Eq.(2.14), then for arbitrary* T *and* t, $t'\in[0,T]$, x, x' *belonging to an arbitrary bounded set in* $\mathbb{R}^d$,

$$E[\int_0^\infty \mathrm{e}^{-Kr}|X_r^{t',x'}-X_r^{t,x}|^p dr] \ \leq \ C_p(|x'-x|^p+|t'-t|^{\frac{p}{2}}) \quad \text{a.s..}$$

Proof. First as shown in the argument in Proposition 5.2.8, we have (5.15) by Condition (A.5.4). Without losing any generality, assume that $t'\geq t\geq 0$. Referring to Lemma 4.5.6 in [28] and noting that t, t', x, x' are bounded, we have

$$\begin{aligned}
& E[\int_0^\infty \mathrm{e}^{-Kr}|X_r^{t',x'}-X_r^{t,x}|^p dr] \\
= & \ E[\int_0^t \mathrm{e}^{-Kr}|x'-x|^p dr]+E[\int_t^{t'} \mathrm{e}^{-Kr}|x'-X_r^{t,x}|^p dr]+E[\int_{t'}^\infty \mathrm{e}^{-Kr}|X_r^{t',x'}-X_r^{t,x}|^p dr] \\
= & \int_0^t \mathrm{e}^{-Kr}|x'-x|^p dr+\int_t^{t'} \mathrm{e}^{-Kr}E[|X_{t'}^{t',x'}-X_r^{t,x}|^p]dr+\int_{t'}^\infty \mathrm{e}^{-Kr}E[|X_r^{t',x'}-X_r^{t,x}|^p]dr \\
\leq & \int_0^t \mathrm{e}^{-Kr}|x'-x|^p dr+C_p\int_t^{t'} \mathrm{e}^{-Kr}(|x'-x|^p+|t'-t|^{\frac{p}{2}}+|t'-r|^{\frac{p}{2}})dr \\
& +C_p\int_{t'}^\infty \mathrm{e}^{-Kr}(|x'-x|^p+|t'-t|^{\frac{p}{2}})dr \\
\leq & \ C_p(|x'-x|^p+|t'-t|^{\frac{p}{2}}+|t'-t|^{\frac{p}{2}+1}) \\
\leq & \ C_p(|x'-x|^p+|t'-t|^{\frac{p}{2}}).
\end{aligned}$$

◇

Now consider the following BDSDE on infinite horizon:

$$\begin{aligned}
\mathrm{e}^{-\frac{K'}{2}s}Y_s^{t,x} \ = & \int_s^\infty \mathrm{e}^{-\frac{K'}{2}r}f(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+\int_s^\infty \frac{K'}{2}\mathrm{e}^{-\frac{K'}{2}r}Y_r^{t,x}dr \\
& -\int_s^\infty \mathrm{e}^{-\frac{K'}{2}r}\langle g(X_r^{t,x},Y_r^{t,x}),d^\dagger \hat{B}_r\rangle-\int_s^\infty \mathrm{e}^{-\frac{K'}{2}r}\langle Z_r^{t,x},dW_r\rangle. \quad (5.16)
\end{aligned}$$

It is easy to see that Eq.(5.16) is a simpler form of Eq.(5.14), and for arbitrary given terminal time T, Eq.(5.16) has the same form on $[s,T]$ as the finite horizon BDSDE (5.3).

Proposition 5.3.2. *Under Conditions* (A.5.1)-(A.5.4)*, let* $(Y_s^{t,x})_{s\geq 0}$ *be the solution of Eq.(5.16), then for arbitrary* T *and* $t \in [0,T]$, $x \in \mathbb{R}^d$, $(t,x) \longrightarrow Y_t^{t,x}$ *is a.s. continuous.*

Proof. For t, t', $r \geq 0$, let

$$\begin{aligned}
&\bar{Y}_r = Y_r^{t',x'} - Y_r^{t,x}, \quad \bar{Z}_r = Z_r^{t',x'} - Z_r^{t,x},\\
&\bar{f}_r = f(X_r^{t',x'}, Y_r^{t',x'}, Z_r^{t',x'}) - f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}),\\
&\bar{g}_r = g(X_r^{t',x'}, Y_r^{t',x'}) - g(X_r^{t,x}, Y_r^{t,x}).
\end{aligned}$$

Then

$$\begin{cases}
d\bar{Y}_r = -\bar{f}_r dr + \langle \bar{g}_r, d^\dagger \hat{B}_r\rangle + \langle \bar{Z}_r, dW_r\rangle,\\
\lim\limits_{T\to\infty} \mathrm{e}^{-\frac{K'}{2}T}\bar{Y}_T = 0 \quad \text{a.s..}
\end{cases}$$

Applying Itô's formula to $\mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^p$ and following a similar calculation as in (3.39), we have for $0 \leq s \leq T$,

$$\begin{aligned}
&\mathrm{e}^{-\frac{pK'}{2}s}|\bar{Y}_s|^p + (p\mu - \frac{pK'}{2} - \frac{p(p+1)}{2}C - \varepsilon)\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^p dr\\
&+\frac{p(2p-3)}{4}\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2 dr\\
\leq\ & \mathrm{e}^{-\frac{pK'}{2}T}|\bar{Y}_T|^p + C_p\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{X}_r|^p dr - p\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^{p-2}\bar{Y}_r\langle \bar{g}_r, d^\dagger\hat{B}_r\rangle\\
&-p\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^{p-2}\bar{Y}_r\langle \bar{Z}_r, dW_r\rangle.
\end{aligned} \tag{5.17}$$

Then, for $0 \leq s \leq T$, we have

$$\begin{aligned}
&E[\mathrm{e}^{-\frac{pK'}{2}s}|\bar{Y}_s|^p] + (p\mu - \frac{pK'}{2} - \frac{p(p+1)}{2}C - \varepsilon)E[\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^p dr]\\
&+\frac{p(2p-3)}{4}E[\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2 dr]\\
\leq\ & E[\mathrm{e}^{-\frac{pK'}{2}T}|\bar{Y}_T|^p] + C_p E[\int_s^T \mathrm{e}^{-\frac{pK'}{2}r}|\bar{X}_r|^p dr].
\end{aligned} \tag{5.18}$$

Since $(Y_\cdot^{t,x}) \in S^{p,-K}([0,\infty);\mathbb{R}^1)$, for arbitrary $T \geq 0$, it turns out that

$$E[\mathrm{e}^{-\frac{pK'}{2}T}|\bar{Y}_T|^p] \leq E[\sup_{s\geq 0} \mathrm{e}^{-Ks}|\bar{Y}_s|^p] < \infty.$$

Therefore by the Lebesgue's dominated convergence theorem, we have

$$\lim_{T\to\infty} E[\mathrm{e}^{\frac{pK'}{2}T}|\bar{Y}_T|^p] = E[(\lim_{T\to\infty} \mathrm{e}^{-\frac{K'}{2}T}|\bar{Y}_T|)^p] = 0. \tag{5.19}$$

So taking the limit of T in (5.18), by Lemma 5.3.1 and the monotone convergence theorem, we have

$$\begin{aligned} & E[\mathrm{e}^{-\frac{pK'}{2}s}|\bar{Y}_s|^p] + (p\mu - \frac{pK'}{2} - \frac{p(p+1)}{2}C - \varepsilon)E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^p dr] \\ & + \frac{p(2p-3)}{4}E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2 dr] \\ \leq\ & C_p E[\int_0^\infty \mathrm{e}^{-Kr}|\bar{X}_r|^p dr] \\ \leq\ & C_p(|x'-x|^p + |t'-t|^{\frac{p}{2}}). \end{aligned}$$

Due to the arbitrariness of ε, it follows that

$$E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2 dr] + E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^p dr] \leq C_p(|x'-x|^p + |t'-t|^{\frac{p}{2}}). \quad (5.20)$$

From (5.17), by B-D-G inequality, we have

$$\begin{aligned} & E[\sup_{0\leq s\leq T} \mathrm{e}^{-\frac{pK'}{2}s}|\bar{Y}_s|^p] \\ \leq\ & E[\mathrm{e}^{-\frac{pK'}{2}T}|\bar{Y}_T|^p] + C_p E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{X}_r|^p dr] + pE[\sqrt{\int_0^\infty \mathrm{e}^{-pK'r}|\bar{Y}_r|^{2p-2}|\bar{g}_r|^2 dr}] \\ & + pE[\sqrt{\int_0^\infty \mathrm{e}^{-pK'r}|\bar{Y}_r|^{2p-2}|\bar{Z}_r|^2 dr}]. \end{aligned} \quad (5.21)$$

Taking the limit of T on both sides of (5.21), by (5.19), the monotone convergence theorem and the Young inequality, we have

$$\begin{aligned} & E[\sup_{s\geq 0} \mathrm{e}^{-\frac{pK'}{2}s}|\bar{Y}_s|^p] \\ \leq\ & C_p E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{X}_r|^p dr] + \frac{1}{3}E[\sup_{s\geq 0} \mathrm{e}^{-\frac{pK'}{2}s}|\bar{Y}_s|^p] + C_p E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^p dr] \\ & + \frac{1}{3}E[\sup_{s\geq 0} \mathrm{e}^{-\frac{pK'}{2}s}|\bar{Y}_s|^p] + C_p E[\int_0^\infty \mathrm{e}^{-\frac{pK'}{2}r}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2 dr]. \end{aligned}$$

From this inequality, by Lemma 5.3.1 and (5.20), for arbitrary $T > 0$, t, $t' \in [0, T]$, x, x' belonging to an arbitrary bounded set in $\mathbb{R}^d$, we have

$$E[\sup_{s\geq 0} \mathrm{e}^{-pKs}|\bar{Y}_s|^p] \leq C_p(|x'-x|^p + |t'-t|^{\frac{p}{2}}). \quad (5.22)$$

Noting $p > d+2$ in (5.22), by Kolmogorov Lemma (see [28]), we have that $Y_s^{(\cdot,\cdot)}$ has a continuous modification for $t \in [0, T]$ and x belonging to an arbitrary bounded set in $\mathbb{R}^d$ under the norm $\sup_{s\geq 0} \mathrm{e}^{-Ks}|Y_s^{(\cdot,\cdot)}|$. In particular,

$$\lim_{\substack{t'\to t \\ x'\to x}} \mathrm{e}^{-Kt'}|Y_{t'}^{t',x'} - Y_{t'}^{t,x}| = 0.$$

Then we have a.s.

$$\begin{aligned}
&\lim_{\substack{t'\to t\\ x'\to x}} |e^{-Kt'}Y_{t'}^{t',x'} - e^{-Kt}Y_t^{t,x}| \\
\leq\ &\lim_{\substack{t'\to t\\ x'\to x}} (|e^{-Kt'}Y_{t'}^{t',x'} - e^{-Kt'}Y_{t'}^{t,x}| + |e^{-Kt'}Y_{t'}^{t,x} - e^{-Kt}Y_t^{t,x}|) = 0.
\end{aligned}$$

The convergence of the second term follows from the continuity of $Y_s^{t,x}$ in s. That is to say $e^{-Kt}Y_t^{t,x}$ is a.s. continuous, therefore $Y_t^{t,x}$ is a.s. continuous w.r.t. $t \in [0,T]$ and x belonging to an arbitrary bounded set in $\mathbb{R}^d$.

Denote by $\bar{B}(0,N)$ the closed ball in $\mathbb{R}^d$ of radius N centered at 0. It is obvious that $\bigcup_{N=1}^{\infty} \bar{B}(0,N) = \mathbb{R}^d$. $Y_t^{t,x}$ is continuous w.r.t $t \in [0,T]$ and $x \in \bar{B}(0,N)$ on Ω^N. Take $\tilde{\Omega} = \bigcap_{N=1}^{\infty} \Omega^N$, then $P(\tilde{\Omega}) = 1$. Now for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, there exists an N s.t. $x \in \bar{B}(0,N)$. On the other hand, for all $\omega \in \tilde{\Omega}$, it is obvious that $\omega \in \Omega^N$. So $Y_t^{t,x}$ is continuous w.r.t. $t \in [0,T]$ and $x \in \mathbb{R}^d$ on $\tilde{\Omega}$. Proposition 5.3.2 is proved. ◇

§5.3.2 Stationary stochastic viscosity solution of the corresponding SPDE

Theorem 5.3.3. *Under Conditions* (A.5.1)–(A.5.4), *for arbitrary* T *and* $t \in [0,T]$, $x \in \mathbb{R}^d$, *let* $v(t,x) \triangleq Y_{T-t}^{T-t,x}$, *where* $(Y_s^{t,x}, Z_s^{t,x})$ *is the solution of Eq.(5.16) with* $\hat{B}_s = B_{T-s} - B_T$ *for all* $s \geq 0$. *Then* $v(t,x)$ *is continuous w.r.t.* t *and* x *and is a stochastic viscosity solution of Eq.(5.1).*

Proof. By Proposition 5.3.2, $v(t,x,\omega)$ is a.s. continuous w.r.t. $t \in [0,T]$ and $x \in \mathbb{R}^d$. Since $Y_\cdot^{t,x} \in S^{2,-K}([0,\infty);\mathbb{R}^1)$, $Y_T^{T,x}$ is $\mathscr{F}_{T,\infty}^{\hat{B}} \otimes \mathscr{B}_{\mathbb{R}^d}$ measurable and $E[\ |Y_T^{T,x}|^2] < \infty$. Moreover, Condition (A.5.1) is stronger than Condition (A.5.2)$'$. So by Theorem 5.2.2, $Y_\cdot^{t,x} \in S^{2,0}([0,T];\mathbb{R}^1)$ is the solution of Eq.(5.3). On the other hand, $E[\ |Y_T^{T,X_T^{t,x}}|^2] = E[\ |Y_T^{t,x}|^2] < \infty$. By Theorem 5.1.3 and Remark 5.1.4, $v(t,x)$ is a stochastic viscosity solution of Eq.(5.1). Theorem 5.3.3 is proved. ◇

In the following, we will prove the $v(t,x)$ derived through Theorem 5.3.3 is a stationary solution of Eq.(5.1). Similar as in Subsection 2.3.4, we do it through the "perfection procedure" of Y. Define $\hat{\theta}$ and θ as in Section 2.1 and Subsection 2.3.4 respectively. By the same method as in Proposition 2.2.1, we can get a "crude" version of $Y_\cdot^{t,x}$, i.e. for any r, $t \geq 0$, $x \in \mathbb{R}^d$,

$$\hat{\theta}_r \circ Y_s^{t,x} = Y_{s+r}^{t+r,x} \quad \text{for all } s \geq 0 \text{ a.s..}$$

In particular, for any $r,\ t \geq 0,\ x \in \mathbb{R}^d$,

$$\hat{\theta}_r \circ Y_t^{t,x} = Y_{t+r}^{t+r,x} \quad \text{a.s..}$$

Then noticing the continuity of $Y_t^{t,x}$ w.r.t. t, by Theorem 2.2.2, we have an indistinguishable version of $Y_t^{t,x}$, still denoted by $Y_t^{t,x}$, s.t. for any $x \in \mathbb{R}^d$,

$$\hat{\theta}_r \circ Y_t^{t,x} = Y_{t+r}^{t+r,x} \quad \text{for all } r,\ t \geq 0 \text{ a.s..}$$

In fact, following a "standard" argument, we can obtain from the continuity of $Y_t^{t,x}$ w.r.t. x that

$$\hat{\theta}_r \circ Y_t^{t,x} = Y_{t+r}^{t+r,x} \quad \text{for all } r,\ t \geq 0,\ x \in \mathbb{R}^d \text{ a.s..} \tag{5.23}$$

Note that we have proved $v(t,x) = Y_t^{t,x}$ is a stochastic viscosity solution of Eq.(5.1) in Theorem 5.3.3, then let's see it is a stationary stochastic viscosity solution under shift θ. By (5.23) and the relationship between θ and $\hat{\theta}$, we have

$$\theta_r v(t,x)(\omega) = \hat{\theta}_{-r} Y_{T-t}^{T-t,x}(\hat{\omega}) = \hat{\theta}_{-r}\hat{\theta}_r Y_{T-t-r}^{T-t-r,x}(\hat{\omega}) = Y_{T-t-r}^{T-t-r,x}(\hat{\omega}) = v(t+r,x)(\omega)$$

for all $r \geq 0$ and $T \geq t+r$, $x \in \mathbb{R}^d$ a.s.. In particular, let $Y(x,\omega) = v_0(x,\omega) = Y_T^{T,x}(\hat{\omega})$, then the above formula implies (1.1):

$$\theta_t Y(x,\omega) = Y(x,\theta_t\omega) = v(t,x,\omega) = v(t,v_0(x,\omega),x,\omega) = v(t,Y(x,\omega),x,\omega)$$

for all $t \geq 0$, $x \in \mathbb{R}^d$ a.s.. That is to say $v(t,x)(\omega) = Y(x,\theta_t\omega) = Y_{T-t}^{T-t,x}(\hat{\omega})$ is a stationary solution of Eq.(5.1) w.r.t. θ. As the arguments in Subsection 2.3.4, we can prove a claim that $v(t,\cdot)(\omega) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ does not depend on the choice of T. Therefore we have the following conclusion

Theorem 5.3.4. *Under Conditions* (A.5.1)–(A.5.4), *for arbitrary T and $t \in [0,T]$, let $v(t,x) \triangleq Y_{T-t}^{T-t,x}$, where $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of Eq.(5.16) with $\hat{B}_s = B_{T-s} - B_T$ for all $s \geq 0$. Then $v(t,x)$ is a "perfect" stationary stochastic viscosity solution of Eq.(5.1).*

Bibliography

[1] L. Arnold, *Random dynamical systems.* Springer-Verlag Berlin Heidelberg (1998).

[2] L. Arnold, M. Scheutzow, *Perfect cocycles through stochastic differential equations.* Probab. Theory Relat. Fields, Vol.101 (1995), 65-88.

[3] V. Bally, A. Matoussi, *Weak solutions for SPDEs and backward doubly stochastic differential equations.* J. Theoret. Probab., Vol.14 (2001), 125-164.

[4] G. Barles, E. Lesigne, *SDE, BSDE and PDE.* In: Backward stochastic differential equations. Pitman Res. Notes Math., Ser.364, Longman, Harlow, (1997), 47-80.

[5] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux, L. Stoica, *L^p solutions of backward stochastic differential equations.* Stochastic Process. Appl., Vol.108 (2003), 109-129.

[6] Ph. Briand, Y. Hu, *BSDE with quadratic growth and unbounded terminal value.* Probab. Theory Relat. Fields, Vol.136 (2006), 604-618.

[7] Z. Brzezniak, D. Gatarek, *Martingale solutions and invariant measures for stochastic evolution in Banach spaces.* Stochastic Process. Appl., Vol.84 (1999), 187-226.

[8] Z. Brzezniak, Y. Li, *Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains.* Trans. AMS, Vol.358 (2006), 5587-5629.

[9] R. Buckdahn, J. Ma, *Stochastic viscosity solutions for non-linear stochastic partial differtial equations (I).* Stochastic Process. Appl., Vol.93 (2001), 181-204.

[10] R. Buckdahn, J. Ma, *Stochastic viscosity solutions for non-linear stochastic partial differtial equations (II).* Stochastic Process. Appl., Vol.93 (2001), 205-228.

[11] R. Buckdahn, J. Ma, *Pathwise stochastic Taylor expansions and stochstic viscosity solutions for full nonlinear stochastic PDEs.* Ann. Probab., Vol.30 (2002), 1131-1171.

[12] T. Caraballo, P. E. Kloeden, J. Real, *Discretization of asymptotically stable stationary solutions of delay differential equations with a random stationary delay.* J. Dynam. Differential Equations, Vol.18, No.4 (2006), 863-880.

[13] T. Caraballo, P. E. Kloeden, B. Schmalfuss, *Exponentially stable stationary solutions for stochastic evolution equations and their perturbation.* Appl. Math. Optim., Vol.50 (2004), 183-207.

[14] M. G. Crandall, H. Ishii, P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc., Vol.27 (1992), 1-67.

[15] M. G. Crandall, P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations.* Trans. Amer. Math. Soc., Vol.277 (1983), 1-42.

[16] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions.* Cambridge University Press (1992).

[17] G. Da Prato, J. Zabczyk, *Ergodicity for infinite dimensional systems (London Mathematical Society Lecture Note Series).* Cambridge University Press (1996).

[18] J. Duan, K. Lu, B. Schmalfuss, *Invariant manifolds for stochastic partial differential equations.* Ann. Probab., Vol.31 (2003), 2109-2135.

[19] J. Duan, K. Lu, B. Schmalfuss, *Smooth stable and unstable manifolds for stochastic evolutionary equations.* J. Dynam. Differential Equations, Vol.16 (2004), 949-972.

[20] W. E, K. Khanin, A. Mazel, Ya. Sinai, *Invariant measures for Burgers equation with stochastic forcing.* Ann. of Math., Vol.151 (2000), 877-960.

[21] K. D. Elworthy, A. Truman, H. Z. Zhao, *A generalized Itô formula and space-time Lebesgue-Stieltjes integrals of local times.* Sèmin. Probab., Vol.40, (2007), 117-136.

[22] F. Flandoli, *Dissipativity and invariant measures for stochastic Navier-Stokes equations.* NoDEA, Vol.1 (1994), 403-426.

[23] I. Gyöngy, C. Rovira, *On L^p-solutions of semilinear stochastic partial differential equations.* Stochastic Process. Appl., Vol.90 (2000), 83-108.

[24] M. Hairer, J. Mattingly, *Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing.* Ann. of Math., Vol.164 (2006), 993-1032.

[25] I. Karatzas, S. E. Shreve, *Brownian Motion and Stochastic Calculus.* Springer-Verlag New York (1998).

[26] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth.* Ann. Probab., Vol.28, No.2 (2000), 558-602.

[27] N. V. Krylov, *An analytic approach to SPDEs.* In: Stochastic partial equations: six perspectives, edited by R.A. Carmona and B. Rozovskii, Mathematical Surveys and Monographs, Amer. Math. Soc., Vol.64 (1999), 183-242.

[28] H. Kunita, *Stochastic flows and stochastic differential equations.* Cambridge University Press (1990).

[29] H. Kunita, *Stochastic flow acting on Schwartz distributions.* J. Theoret. Prob., Vol.7(2) (1994), 247-278.

[30] J. P. Lepeltier, J. San Martin, *Backward stochastic differential equations with continuous coefficient.* Statistics and Probability Letters, Vol.32 (1997), 425-430.

[31] W. Li, K. Lu, *Sternberg theorems for random dynamical systems.* Communications on Pure and Applied Mathematics, Vol.LVIII (2005), 941-988.

[32] P.-L. Lions, P. E. Souganidis, *Fully nonlinear stochastic partial differential equations.* C. R. Acad. Sci. Paris, Ser.I, Vol.326 (1998), 1085-1092.

[33] P.-L. Lions, P. E. Souganidis, *Fully nonlinear stochastic partial differential equations: non-smooth equations and applications.* C. R. Acad. Sci. Paris, Ser.I, Vol.327 (1998), 735-741.

[34] P.-L. Lions, P. E. Souganidis, *Fully nonlinear stochastic pde with semilinear stochastic dependence.* C. R. Acad. Sci. Paris, Ser.I, Vol.331 (2000), 617-624.

[35] P.-L. Lions, P. E. Souganidis, *Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations.* C. R. Acad. Sci. Paris, Ser.I, Vol.331 (2000), 783-790.

[36] Y. Liu, H. Z. Zhao, *Representation of pathwise stationary solutions of stochastic Burgers equations.* Preprint.

[37] R. Mikulevicius, B. L. Rozovskii, *A note on Krylov's L_p-theory for systems of SPDEs.* Electronic Journal of Probability, Vol.6, Paper no.12 (2001), 1-35.

[38] S.-E. A. Mohammed, T. Zhang, H. Z. Zhao, *The stable manifold theorem for semi-linear stochastic evolution equations and stochastic partial differential equations.* Mem. Amer. Math. Soc., Vol.196 (2008). (to appear).

[39] D. Nualart, E. Pardoux, *Stochastic calculus with anticipating integrands.* Probab. Theory Relat. Fields, Vol.78 (1988), 535-581.

[40] E. Pardoux, *Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order.* Stochastic Analysis and Related Topics: The Geilo Workshop, 1996, L. Decreusefond, J. Gjerde, B. Oksendal, A. S. Üstüunel eds., Birkhäuser (1998), 79-127.

[41] E. Pardoux, *Stochastic partial differential equations.* Fudan lecture notes (2007).

[42] E. Pardoux, S. Peng, *Adapted solution of a backward stochastic differential equation.* Syst. Control Lett., Vol.14 (1990), 55-61.

[43] E. Pardoux, S. Peng, *Backward stochastic differential equations and quasilinear parabolic partial differential equations.* Stochastic Partial Differential Equations, B. L. Rozuvskii, R. B. Sowers eds., Lect. Notes Control Inf. Sci., Berlin Heidelberg New York: Springer, Vol.176 (1992), 200-217.

[44] E. Pardoux, S. Peng, *Backward doubly stochastic differential equations and systems of quasilinear SPDEs.* Probab. Theory Relat. Fields, Vol.98 (1994), 209-227.

[45] S. Peng, *Probabilistic interpretation for systems of quasilinear parabolic partial differential equations.* Stochastics, Vol.37 (1991), 61-74.

[46] S. Peng, *BSDE and stochastic optimizations.* In: Topics in stochastic analysis, edited by J. Yan, S. Peng, S. Fang and L. M. Wu, Ch.2, (Chinese vers.), Science Publication (1997), 16-39.

[47] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion.* Springer-Verlag Berlin Heidelberg (1994).

[48] D. Ruelle, *Characteristic exponents and invariant manifolds in Hilbert space.* Ann. of Math., Vol.115 (1982), 243-290.

[49] Y. F. Shi, Y. L. Gu, K. Liu, *Comparison theorems of backward doubly stochastic differential equations and applications.* Stochastic Analysis and Applications, Vol.23 (2005), 97-110.

[50] Ya. Sinai, *Two results concerning asymptotic behaviour of solutions of Burgers equation with force.* J. Statist. Phys., Vol.64 (1991), 1-12.

[51] Ya. Sinai, *Burgers system driven by a periodic stochastic flows.* In: Itô's stochastic calculus and probability theory, Springer, Tokyo (1996), 347-353.

[52] Q. Zhang, H. Z. Zhao, *Stationary solutions of SPDEs and infinite horizon BDSDEs.* Journal of Functional Analysis, Vol.252 (2007), 171-219.

CURRICULUM VITAE

Name : Qi ZHANG **Gender :** Male **Nationality :** Chinese
Date of Birth : 30/Sep/1979 **Place of Birth :** Jining, Shandong Province, China
Supervisor : Prof. Huaizhong ZHAO and Prof. Shige PENG (Shandong University)
Department : Mathematical Sciences **Speciality :** Stochastic Analysis
Research Fields : BSDE, SPDE, Stochastic Dynamical Systems

Education Experience:

- Sep/1998 ~ Jun/2002, School of Mathematics and Systems Sciences, Shandong University, Bachelor of Science;
- Sep/2002 ~ present, Department of Mathematical Sciences, Loughborough University & School of Mathematics and Systems Sciences, Shandong University.

Address & Email:

Qi Zhang, Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK
School of Mathematics and System Sciences, Shandong University, Jinan 250100, China
Q.Zhang3@lboro.ac.uk

Teaching Experience:

October 2005 ~ May 2006, Tutorials and Example Classes on Stochastic Calculus (MAP104) and Mathematical Finance (MAP204) at Loughborough University

Publication:

Q. Zhang, H. Z. Zhao, *Stationary solutions of SPDEs and infinite horizon BDSDEs.* Journal of Functional Analysis, Vol.252 (2007), 171-219.