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STABILITY OVER A FINITE-TIME INTERVAL

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SUMMARY

This thesis will be concerned with the study of the new concept of final-stability which is a direct generalisation of the theory of stability over a finite time interval.

The introduction deals with an intuitive historical development which leads to the concept under consideration.

The main part, chapters I and II, is devoted to the study of differential systems. Precise definitions of the concept are stated, and a corresponding theory is established. Use has been extensively made of Liapunov-like functions. The possible relationship with the theory of controllability is indicated.

Dynamical and discrete systems are then studied in some detail. For dynamical systems, a somewhat different approach is considered.

Finally, possible future topics of research are suggested and discussed.

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INTRODUCTION

This thesis will be concerned with the theory of a new concept, i.e., the concept of final-stability (over a finite interval of time). But before introducing the reader to this notion, one needs to give a short history of its development. We do not claim to be exhaustive, but our purpose is merely to give an intuitive reasoning which leads us to the concept of final-stability, which is the main object of the thesis. We also add that although the theory of final-stability, established hereafter, has many applications, the most obvious potential is perhaps that it offers a unified approach to both stability and controllability by the use of Liapunov-like functions.

Although the theory of final-stability can be extended (and is extended) to more general systems, we shall limit our discussion in this introduction to differential systems of the form

$$\dot{x} = f(x,t) \tag{1}$$

and

$$\dot{x} = g(x,u,t), \quad u \text{ control} \tag{EC}$$

where x is n-vector, u an m-vector, f and g are n-vector functions of their arguments.

We know that, under certain conditions [5,11,13,23], given an $\varepsilon > 0$, and an interval of time $I = [t_o, t_o+T)$ there exists a $\delta = \delta(\varepsilon,T,t_o) > 0$ such that

$$||x(t; x_o,t_o) - x(t; x_1,t_o)|| < \varepsilon, \ t \in I$$

provided

$$||x_o - x_1|| < \delta$$

where $x(t;x_o,t_o)$ is the solution through (x_o,t_o) and $x(t; x_1,t_o)$ the solution through (x_1, t_o). ($||\cdot||$ is the euclidean norm or any other appropriate norm.)

This general proposition has a physical sense. Indeed, practically, the initial conditions are determined by means of measurements and any measurement can be only approximate. Thus, the continuity with respect to initial conditions expresses the fact that these errors of measurement do not affect the solution too seriously. In other words, if the admissible error ε is given for a solution, for a given interval I, there exists a $\delta = \delta(\varepsilon,I)$ such that if, when establishing the initial conditions, the error is smaller than δ, then the error in the solution does not exceed the given ε.

We emphasise here the fact that δ depends not only on ε but also on the size of the interval and actually decreases when T increases. It follows that a solution will have a physical meaning in reality only if for a sufficiently large interval δ remains sufficiently large. This can be achieved by requiring that δ does not depend upon the size of the interval. We thus reach the notion of stability in the sense of Liapunov which was and still is extensively investigated [1,9,10,26, 27,38,50].

Although the concept of Liapunov stability concerns itself with the stability of a fixed solution $\tilde{x}(t)$, say, of (1), we can assume that the solution under consideration is the equilibrium state $x \equiv 0$. The trivial solution of (1) is, then, said to be <u>stable in the sense of Liapunov</u> if for any $\varepsilon > 0$, and any $t_o \in R^+$, there exists a $\delta=\delta(\varepsilon,t_o) > 0$ such that $||x_o|| < \delta$, implies that $||x(t; x_o,t_o)|| < \varepsilon$, for all $t \geq t_o$. If δ depends only on ε, the stability is said to be <u>uniform</u>. If, in addition, there exists a $\delta_o = \delta_o(t_o)$ such that for each $\varepsilon > o$, there exists a $T = T(\varepsilon, x_o, t_o)$ such that $||x_o|| < \delta_o$, implies that $||x(t;\, x_o, t_o)|| < \varepsilon$, for all $t \geq t_o + T$, then the trivial solution is said to be asymptotically stable.

In applications, it is obvious that asymptotic stability is more desirable than mere stability. If one wishes to maintain, say, a certain temperature K in a system, it is, clearly, desirable that small deviations actually cancel out, and not desirable to maintain merely some temperature not too far from K.

Another practical consideration: suppose that an electrical system has been designed to operate at N volts. The system is so arranged that small deviations are cancelled out. But, how large are the deviations that cancel out? The system may be asymptotically stable and yet not operate properly if deviations in excess of some millivolts occur. Thus, the system, while asymptotically stable in theory, is actually unstable in practice. To have true asymptotic stability one should allow for deviations of several volts, for example. The main trouble with both stability and symptotic stability is that δ depends on ε. So, the logical step is to require that δ does not depend on ε.

These and other considerations have led LaSalle and Lefschetz [19] to introduce the concept of practical stability which is defined in the following manner:

Consider system (1), with $f(0,t) \equiv 0$, and the perturbed system

$$\dot{x} = f(x,t) + p(x,t), \quad t \geq 0 \qquad \text{(EP)}$$

We are given a positive number δ and two sets Q and Q_o. Q is a closed bounded set containing the origin and Q_o a subset of Q. Let $x^*(t;\, x_o, t_o)$

be the solution of (EP) satisfying $x^*(t_o;x_o,t_o) = x_o$. Let P be the set of all perturbations p(x,t) satisfying $||p(x,t)|| \leq \delta$, for all $t \geq 0$, $x \in R^n$. If for each $p \in P$, each $x_o \in Q_o$, and each $t_o \geq 0$, $x^*(t; x_o,t_o) \in Q$, for all $t \geq t_o$, then the origin is said to be practically stable.

The concept of practical stability is relative to the number δ and the sets Q and Q_o. Q is the set of acceptable responses. Q_o is the set of acceptable initial states.

The common factor in both Liapunov's stability and practical stability is that the time interval is infinite. Consequently, the stability property can never be verified in practice, since a physical system can be observed only during a finite time interval. Therefore, several attempts have been made to modify the different types of stability such that the motion has to be considered only during finite time intervals.

The idea of stability over a finite interval if time is not new [6,7,14, 15,20,21,22,24,25,33] but the first to introduce it in the following form were Weiss and Infante [36]. Further work was carried out by them and many others [8,16,17,18,35,37]: Let

$$\alpha : ||x|| < a, \ \beta : ||x|| < b, \ \gamma : ||x|| < c$$

$$c < a \leq b, \ I = [t_o, t_o+T)$$

(where $||x||$ is the euclidean norm).

Then, system (1) is stable with respect to the sets (α,β,I) if, for any trajectory x(t), the condition $x(t_o) \in \alpha$ implies that $x(t) \in \beta$, all $t \in I$. If, in addition, any trajectory x(t), $x(t_o) \in \alpha$, is such that $x(t) \in \gamma$, all $t \in [t_1, t_o+T)$, for some $t_1 \in I$ (which may depend on the particular trajectory), then the system is said to be contractively stable with respect to the sets (α,β,γ,I).

Notice that the condition $f(0,t) \equiv 0$ is no longer required and no uniqueness condition need be imposed.

One also notes the similarity between asymptotic stability and contractive stability. In fact, we shall see that it would be possible to connect the stability in the sense of Liapunov to the theory of final-stability by means of transformations such as $t = \frac{s}{T - s} + t_o$.

Now, some important questions arise: is it possible to generalise the idea of finite-time stability? That is, take α and β to be any sets in R^n and find out what happens to any trajectory $x(t)$ emanating from α at $t = t_o$. Does it enter β and remain there? Does it enter β but leave it before the end of the interval? Or does it possibly never enter β within the given interval? On the other hand, what happens to a trajectory starting outside α?

The theory of final-stability which we establish in this thesis answers most of these questions.

The first chapter is devoted to the study of differential systems of the form (1). However, to make the concept more acceptable and connect it more directly to the concept of controllability, we consider also system (EC); this will be done in chapter II. System (1) is, of course, a special case of the above system.

Although most of the thesis will be devoted to the study of system (EC), extensions to discrete and dynamical systems are considered in some detail. Finally, the concluding chapter is devoted to the discussion of some ideas worthwhile for future research.

CHAPTER I

Ordinary Differential Equations without the influence of Perturbing Forces "Unforced Systems"

§1. INTRODUCTION AND DEFINITIONS

1.1 Introduction

We shall consider systems of the form

$$\dot{x} = f(x,t) \tag{1.1.1}$$

where x is a real n-vector which represents the state of the system at times $t \in I$, I being the interval $[t_o, t_o+T)$. It will be assumed that the n-vector function $f(x,t)$ possesses the necessary properties, so that there is no difficulty with questions of existence, uniqueness and continuity of solutions with respect to initial conditions. Moreover, it is not required that $f(0,t) = 0$, for all $t \in I$, so that stability with respect to a set rather than a point can be discussed without having to resort to complicated transformations.

Since the theory established in this chapter will be a special case of a more general theory which we will establish in Chapter II, we shall omit most of the proofs. We also note that a more detailed discussion of this theory will be found in $[R_1]$.

1.2 Notation

R^n: the real euclidean space of n-dimensions

R^+: the set of non-negative real numbers

R : the set of all real numbers

I : $[t_o, t_o+T)$, $t_o \in R^+$, $T \in R^+$, $T > 0$.

In the sequel, when not otherwise stated, small greek letters will denote connected sets in R^n. If α, β are two such sets, we define

$$\alpha/\beta = \alpha - \alpha \cap \beta \tag{1.2.1}$$

The closure, boundary, complement and interior of any set $\alpha \subset R^n$ are denoted respectively by $\bar{\alpha}$, Fr.α, α^c and I(α).

Let $V[x,t]$ denote a mapping

$$V : R^n \times I \rightarrow R^1 \tag{1.2.2}$$

Accordingly, we define the following functions of t:

$$V_M^\alpha(t) = \sup_{x\epsilon\alpha}.V[x,t], \quad V_m^\alpha(t) = \inf_{x\epsilon\alpha}.V[x,t] \tag{1.2.3}$$

Finally, we use the notation $V[x,t] \;\epsilon\; C^1[\alpha \times I^*]$, where $\alpha \;\epsilon\; R^n$ and $I^* \subseteq I$, to indicate that the function $V[x,t]$ and its first partial derivatives $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial x_1}$, ..., $\frac{\partial V}{\partial x_n}$ are continuous in both $(x,t) \;\epsilon\; \alpha \times I^*$. We shall also use the notation $\phi(t) \;\epsilon\; R[I^*]$ to indicate that the function $\phi(t)$ is Riemann-integrable over I^*.

1.3 Definitions of semi-final stability:

Definition 1.3.1:

System (1.1.1) is semi-finally stable with respect to the sets (α,β,I) if, for any trajectory x(t), the condition

$$x(t_o) \;\epsilon\; \alpha \tag{1.3.1}$$

implies the existence of a $t_1 \;\epsilon\; I$, such that

$$x(t_1) \;\epsilon\; \beta \tag{1.3.2}$$

where t_1 may depend on the particular trajectory.

Definition 1.3.2:

System (1.1.1) is semi-finally stable with respect to the sets $(\alpha,\ \beta,\ t_1 \varepsilon I)$ if, for any trajectory $x(t)$, the condition

$$x(t_o)\ \varepsilon\ \alpha \tag{1.3.1}$$

implies that

$$x(t_1)\ \varepsilon\ \beta \tag{1.3.2}$$

Definition 1.3.3:

System (1.1.1) is uniformly semi-finally stable with respect to the sets $(\alpha,\ \beta,\ I)$, if there exists a $t_1\ \varepsilon\ I$, such that the system is semi-finally stable with respect to the sets $(\alpha,\ \beta,\ t_1 \varepsilon I)$.

Definition 1.3.4:

System (1.1.1) is strongly semi-finally stable with respect to the sets $(\alpha,\ \beta,\ I)$, if

(i) it is semi-finally stable with respect to the sets $(\alpha,\ \beta,\ I)$, and

(ii) for any trajectory $x(t)$, the condition

$$x(t_o)\ \varepsilon\ \alpha^c/\beta \tag{1.3.3}$$

implies that

$$x(t)\ \varepsilon\ \beta^c, \quad \text{for all } t\ \varepsilon\ I \tag{1.3.4}$$

In this definition, the restriction to the set α^c/β is essential; for any trajectory $x(t)$, with $x(t_o)\ \varepsilon\ \beta$, is such that $x(t^*)\ \varepsilon\ \beta$, for some $t^*\ \varepsilon\ I$ (certainly for $t^* = t_o$). However, in the following definition, we do not need this restriction.

Definition 1.3.5:

System (1.1.1) is strongly semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \varepsilon I)$, if

(i) it is semi-finally stable with respect to the sets (α, β, I), and

(ii) for any trajectory $x(t)$, the condition $x(t_o) \varepsilon \alpha^c$, implies that

$$x(t_1) \varepsilon \beta^c \tag{1.3.5}$$

Definition 1.3.6:

System (1.1.1) is not semi-finally stable with respect to the sets (α, β, I) if there exists a trajectory $x(t)$, with initial condition $x(t_o) \varepsilon \alpha$, and satisfying $x(t) \varepsilon \beta^c$, all $t \varepsilon I$.

It is not semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \varepsilon I)$, if there exists a trajectory $x(t)$, $x(t_o) \varepsilon \alpha$, satisfying $x(t_1) \varepsilon \beta^c$.

1.4 Definitions of final-stability:

Definition 1.4.1:

System (1.1.1) is finally-stable with respect to the sets (α, β, I) if, for any trajectory $x(t)$, the condition

$$x(t_o) \varepsilon \alpha \tag{1.4.1}$$

implies the existence of $t_1 \varepsilon I$, such that

$$x(t) \varepsilon \beta, \quad \text{for all } t \varepsilon [t_1, t_o+T) \tag{1.4.2}$$

where t_1 may depend on the particular trajectory.

Definition 1.4.2:

System (1.1.1) is finally-stable with respect to the sets (α, β, $t_1 \varepsilon I$) if, for any trajectory x(t), the condition

$$x(t_o) \ \varepsilon \ \alpha \qquad (1.4.1)$$

implies that

$$x(t) \ \varepsilon \ \beta, \quad \text{for all } t \ \varepsilon \ [t_1, t_o+T) \qquad (1.4.3)$$

Definition 1.4.3:

System (1.1.1) is uniformly finally-stable with respect to the sets (α, β, I), if there exists a $t_1 \ \varepsilon$ I such that the system is finally-stable with respect to the sets (α, β, $t_1 \ \varepsilon$ I).

Definition 1.4.4:

System (1.1.1) is strongly finally-stable with respect to the sets (α, β, I), if

(i) it is finally-stable with respect to the sets (α, β, I), and

(ii) for any trajectory x(t), the condition

$$x(t_o) \ \varepsilon \ \alpha^c \qquad (1.4.4)$$

implies the existence of $t^* \ \varepsilon$ I, such that

$$x(t) \ \varepsilon \ \beta^c, \quad \text{for all } t \ \varepsilon \ [t^*, t_o+T) \qquad (1.4.5)$$

where t^* may depend on the particular trajectory.

Definition 1.4.5:

System (1.1.1) is strongly finally-stable with respect to the sets (α, β, $t_1 \varepsilon I$), if

(i) it is finally-stable with respect to the sets (α, β, I), and

(ii) for any trajectory $x(t)$, the condition $x(t_o) \in \alpha^c$ implies that $x(t) \in \beta^c$, for all $t \in [t_1, t_o+T)$.

Definition 1.4.6:

System (1.1.1) is not finally-stable with respect to the sets (α, β, I), if there exists a trajectory $x(t)$, with $x(t_o) \in \alpha$, and satisfying

$$x(t) \in \beta^c, \quad \text{for all } t \in [t^*, t_o+T) \tag{1.4.6}$$

for some $t^* \in I$.

It is not finally-stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, if there exists a trajectory $x(t)$, with $x(t_o) \in \alpha$, and satisfying

$$x(t_2) \in \beta^c \tag{1.4.7}$$

for some $t_2 \in [t_1, t_o+T)$.

1.5 Discussion

(1) In the case where $\alpha \subseteq \beta$, any system (1.1.1) is necessarily uniformly semi-finally stable with respect to the sets (α, β, I).

(2) We note that, in general, a system (1.1.1) being (semi-)finally stable with respect to the sets $(\alpha, \beta, t_1 \in I)$ is necessarily uniformly (semi-)finally stable with respect to the sets (α, β, I); but the converse is not necessarily true, i.e. it is possible that a system (1.1.1) is uniformly (semi-)finally stable with respect to the sets (α, β, I), but not (semi-)finally-stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, for some given $t_1 \in I$.

(3) It is to be noted that definition 1.3.4 implies definition 1.3.5 provided the extra condition

(iii) for any trajectory $x(t)$, the condition

$$x(t_o) \in \beta/\alpha$$

implies that

$$x(t_1) \in \beta^c$$

But the converse is not necessarily true, i.e. a system may be strongly semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, without being strongly semi-finally stable with respect to the sets (α, β, I).

(4) In definition 1.3.6, the first part implies the second part, i.e., if a system is not semi-finally stable with respect to the sets (α, β, I), then it is necessarily not semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, for any $t_1 \in I$. On the other hand, it may happen that a system is semi-finally stable with respect to the sets (α, β, I), and not semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, for some $t_1 \in I$.

(5) One can define final-stability with respect to the sets $(\alpha, \beta, t_1 \in I)$ in terms of semi-final stability, i.e., definition 1.4.2 is equivalent to the following definition:

System (1.1.1) is finally-stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, if it is semi-finally stable with respect to the sets $(\alpha, \beta, t_2 \in I)$, for all $t_2 \in [t_1, t_o+T)$.

(6) It is evident that a system (1.1.1) may be strongly finally-stable with respect to the sets (α, β, I) without being strongly-finally

stable with respect to the sets (α, β, $t_1 \varepsilon I$), for some $t_1 \varepsilon I$. This is an essential difference between definition 1.4.4 and definition 1.4.5.

(7) Now, we redefine the concept of finite-time stability introduced by Weiss and Infante [35,36,37] in the following manner:

Definition 1.5.1: [36,18]

System (1.1.1) is stable with respect to the sets (α, β, I), $\alpha \subseteq \beta$, if it is finally-stable with respect to the sets (α, β, $t_o \varepsilon I$).

Definition 1.5.2: [36]

System (1.1.1) is quasi-contractively stable with respect to the sets (α, γ, I), $\alpha \supset \gamma$, if it is finally-stable with respect to the sets (α, γ, I).

Definition 1.5.3: [36]

System (1.1.1) is contractively-stable with respect to the sets (α, β, γ, I), $\gamma \subset \alpha \subseteq \beta$, if

(i) the system is finally-stable with respect to the sets (α, β, $t_o \varepsilon I$), and

(ii) it is finally-stable with respect to the sets (α, γ, I).

Definition 1.5.4: [18]

System (1.1.1) is quasi-expansively stable with respect to the sets (α, γ, I), $\alpha \subseteq \gamma$, if it is finally-stable with respect to the sets (α, γ, I).

Definition 1.5.5: [18]

System (1.1.1) is expansively-stable with respect to the sets (α, β, γ, I), $\alpha \subseteq \gamma \subset \beta$, if

(i) the system (1.1.1) is finally-stable with respect to the sets (α, β, $t_0 \in I$), and

(ii) it is finally-stable with respect to the sets (α, γ, I).

These definitions show that the concept of finite-time stability introduced by Weiss and Infante [36] and developed by them and several other authors [8,16,17,18,35,36,37] is included in this new concept of final-stability.

8. We terminate this discussion by mentioning that a treatment of the case of a scalar differential equation can be found in [R_1].

§2. GENERAL THEOREMS ON THE DIFFERENT TYPES OF SEMI-FINAL STABILITY

2.1 Semi-final stability

The case $\alpha \subseteq \beta$ is trivial, i.e., any system (1.1.1) is semi-finally stable with respect to the sets (α, β, I), $\alpha \subseteq \beta$. Hence, the following theorem is concerned only with the case $\alpha \not\subseteq \beta$.

Theorem 2.1.1:

System (1.1.1) is semi-finally stable with respect to the sets (α, β, I) $\alpha \not\subseteq \beta$, with β an open set, if there exist two functions $V[x,t] \in C^1[\beta^c \times I]$, and $\phi(t) \in R[I]$ such that

(i) $V_m^{\beta^c}(t_o+T)$ and $V_M^{\alpha/\beta}(t_o)$ are finite.

(ii) $\dot{V}[x(t),t] \leq \phi(t)$, $t \in I$, along any trajectory $x(t)$, with initial condition $x(t_o) \in \alpha/\beta$, as long as $x(t) \in \beta^c$.

(iii) $\int_{t_o}^{t_o+T} \phi(t)dt < V_m^{\beta^c}(t_o+T) - V_M^{\alpha/\beta}(t_o)$

We note that the theorem is valid for any connected set α (not necessarily open). This has the advantage that the theorem can be applied to the case where α is a point $x_o \in \beta^c$.

2.2 Semi-final stability with respect to the sets $(\alpha, \beta, t_1 \in I)$:

1. The case $\bar{\alpha} \subset I(\beta)$.

Theorem 2.2.1:

System (1.1.1) is semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, $\bar{\alpha} \subset I(\beta)$, if there exist a set γ, $\gamma \supseteq \alpha$, $\bar{\gamma} \subset I(\beta)$, two functions $V[x,t] \in C^1[I^c(\gamma) \times I^*]$, and $\phi(t) \in R[I^*]$, where

$I^* = [t_o, t_1]$, such that

(i) $V_m^{\beta^c}(t_1)$ and $V_M^{Fr.\gamma}(t_2)$, all $t_2 \in [t_o, t_1)$, are finite.

(ii) $\dot{V}[x(t),t] < \phi(t)$, $t \in I^*$, along any trajectory $x(t)$, with $x(t_o) \in \alpha$, as long as $x(t) \in I^c(\gamma)$.

(iii) $\int_{t_2}^{t_1} \phi(t)dt \leqq V_m^{\beta^c}(t_1) - V_M^{Fr.\gamma}(t_2)$ for all $t_2 \in [t_o, t_1)$

The conditions of the theorem imply that $t_1 > t_o$. The case $t_1 = t_o$ is trivial, since any system (1.1.1) is semi-finally stable with respect to the sets (α, β, $t_o \in I$), $\alpha \subseteq \beta$. (For the proof of the above theorem refer to 2.3.1 (1) Chapter II).

2. <u>The general case</u>:

Theorem 2.2.2:

System (1.1.1) is semi-finally stable with respect to the sets (α, β, $t_1 \in I$), $t_1 > t_o$, if there exist an open set γ, $\bar{\gamma} \subset I(\beta)$, six functions $V_i[x,t] \in C^1[\gamma^c \times I]$, $\phi_i(t) \in R[I]$, $i = 1,2,3$; such that

(i) $V_{1m}^{\gamma^c}(t_o+T)$ and $V_{1M}^{\alpha/\gamma}(t_o)$ are finite.

(ii) $\dot{V}_1[x(t),t] \leqq \phi_1(t)$, $t \in I$, along any trajectory $x(t)$, $x(t_o) \in \alpha/\gamma$, as long as $x(t) \in \gamma^c$.

(iii) $\int_{t_o}^{t_o+T} \phi_1(t)dt < V_{1m}^{\gamma^c}(t_o+T) - V_{1M}^{\alpha/\gamma}(t_o)$

(iv) $V_{2M}^{\beta^c}(t_1)$ and $V_{2m}^{Fr.\gamma}(t_2)$ for all $t_2 \in (t_1, t_o+T)$, are finite.

(v) $\dot{V}_2[x(t),t] < \phi_2(t)$, $t \in [t_1, t_o+T)$, along any trajectory

$x(t)$, with $x(t_o) \varepsilon \alpha$, as long as $x(t) \varepsilon \gamma^c$.

(vi) $\int_{t_1}^{t_2} \phi_2(t)dt \leqq V_{2m}^{Fr.\gamma}(t_2) - V_{2M}^{\beta^c}(t_1)$, all $t_2 \varepsilon (t_1, t_o+T)$

(vii) $V_{3m}^{\beta^c}(t_1)$ and $V_{3M}^{Fr.\gamma}(t_3)$, all $t_3 \varepsilon [t_o,t_1)$, are finite.

(viii) $\dot{V}_3[x(t),t] < \phi_3(t)$, $t \varepsilon [t_o,t_1)$, along any trajectory $x(t)$, $x(t_o) \varepsilon \alpha$, as long as $x(t) \varepsilon \gamma^c$.

(ix) $\int_{t_3}^{t_1} \phi_3(t)dt \leqq V_{3m}^{\beta^c}(t_1) - V_{3M}^{Fr.\gamma}(t_3)$, all $t_3 \varepsilon [t_o,t_1)$.

Proof of theorem 2.2.2:

Conditions(i) - (iii) ensure that the system is semi-finally stable with respect to the sets (α, γ, I), by Theorem 2.1.1

Let $x(t)$ be an arbitrarily chosen trajectory with $x(t_o) \varepsilon \alpha$, and suppose, contrary to the expected conclusion that

$$x(t_1) \varepsilon \beta^c \tag{2.2.1}$$

Since the system is semi-finally stable with respect to the sets (α, γ, I), one of the following situations has to occur:

(a) There exists a $t_2 \varepsilon (t_1, t_o+T)$ such that

$$x(t_2) \varepsilon Fr.\gamma, \quad x(t) \varepsilon \gamma^c, \quad t \varepsilon [t_1,t_2) \tag{2.2.2}$$

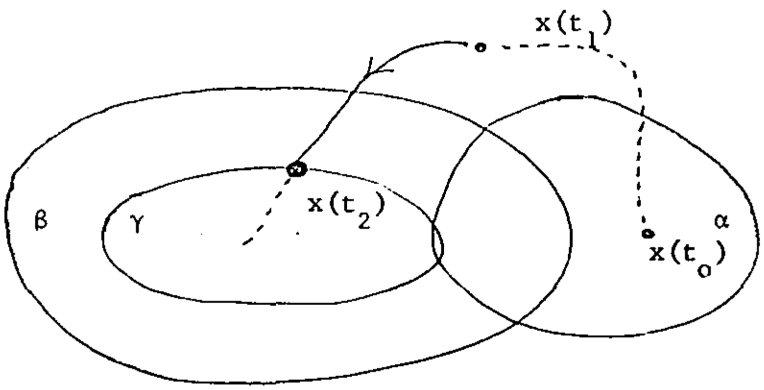


Fig. 2.2.1

In this case,

$$V_2[x(t_2),t_2] = V_2[x(t_1),t_1] + \int_{t_1}^{t_2} \dot{V}_2[x(t),t]dt$$

By (v) - (vi), we get

$$V_2[x(t_2),t_2] < V_{2m}^{Fr.\gamma}(t_2) \tag{2.2.3}$$

which contradicts the assumption (2.2.2). Thus, there is no $t_2 \in (t_1, t_o+T)$ satisfying (2.2.2).

(b) there exists, then, a $t_3 \in [t_o,t_1)$, such that

$$x(t_3) \in Fr.\gamma,\ x(t) \in \gamma^c, \text{ all } t \in (t_3,t_1] \tag{2.2.4}$$

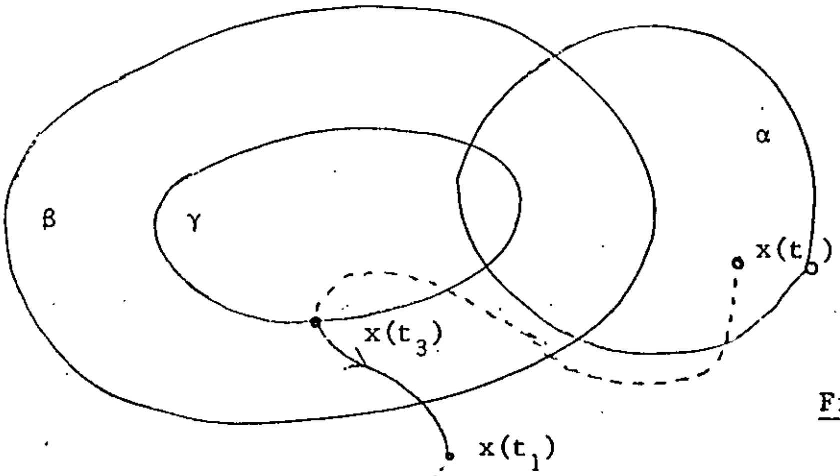


Fig. 2.2.2

In this case, conditions (2.2.4) and (viii) - (ix) yield

$$V_3[x(t_1),t_1] < V_{3m}^{\beta^c}(t_1) \tag{2.2.5}$$

which constitutes an obvious contradiction to the original assumption (2.2.1).

Hence, the assumption that $x(t_1) \in \beta^c$ is false, and we must conclude that

$$x(t_1) \in \beta \tag{2.2.6}$$

Since the above argument is independent of the choice of the trajectory $x(t)$, $x(t_o) \in \alpha$, it holds for all trajectories emanating from α at $t = t_o$. <u>This completes the proof of Theorem 2.2.2.</u>

The assumption $t_1 > t_o$ in the above theorem does not constitute a restriction, since either: (a) $\alpha \subseteq \beta$, and hence any system (1.1.1) is semi-finally stable with respect to $(\alpha, \beta, t_o \in I)$, or (b) $\alpha \not\subseteq \beta$, and then any system (1.1.1) is not semi-finally stable with respect to the sets $(\alpha, \beta, t_o \in I)$.

The assumption that γ is an open set, is essential to the application of theorem 2.1.1. This restriction may be avoided by replacing the conditions (i) - (iii) by the more general statement that system (1.1.1) is semi-finally stable with respect to the sets (α, γ, I), provided of course, the possibility of proving this fact by a theorem other than Theorem 2.1.1, if γ is not open. The requirement that γ exists constitutes, however, a restriction in the sense that Theorem 2.2.2 is not applicable in the case where β is a point $x_1 \in R^n$.

Corollary 2.2.1:

If there exists a $t_1 \in (t_o, t_o+T)$ such that all the conditions of Theorem 2.2.2 hold, then system (1.1.1) is uniformly semi-finally stable with respect to the sets (α, β, I).

2.3 Strong semi-final stability:

Theorem 2.3.1:

System (1.1.1) is strongly semi-finally stable with respect to the sets (α, β, I), $\alpha^c/\beta \neq \emptyset$, if there exist two functions $V[x,t] \in C^1[R^n \times \underline{I}]$,

$\phi(t) \in R[I]$, such that

(i) system (1.1.1) is semi-finally stable with respect to the sets (α, β, I),

(ii) $V_m^\beta(t)$, $t \in (t_o, t_o+T)$, and $V_M^{\alpha^c/\beta}(t_o)$ are finite,

(iii) $\dot{V}[x(t),t] < \phi(t)$, $t \in I$, along any trajectory $x(t)$, $x(t_o) \in \alpha^c/\beta$,

(iv) $\int_{t_o}^{t_1} \phi(t)dt \leqq V_m^\beta(t_1) - V_M^{\alpha^c/\beta}(t_o)$, all $t_1 \in (t_o, t_o+T)$

This theorem will be proved in Chapter II. We note that condition (i) is automatically satisfied in the case $\alpha \subseteq \beta$, and $\alpha^c/\beta = \beta^c$. In the case where $\alpha \not\subseteq \beta$, one may apply Theorem 2.1.1. Moreover, the condition $\alpha^c/\beta \neq \emptyset$ is not restrictive, since the case $\alpha^c/\beta = \emptyset$ is trivial, i.e., any system (1.1.1) is strongly semi-finally stable with respect to the sets (α, β, I), provided it is semi-finally stable with respect to the same sets.

In the above theorem $V[x,t]$ is required to be defined for all $x \in R^n$, and $V_m^\beta(t)$ must be finite for all $t \in (t_o, t_o+T)$. The following theorem is less restrictive in some respects; nevertheless, we require that $\beta = \bar{\beta}$, i.e. β is a closed set in R^n.

Theorem 2.3.2:

System (1.1.1) is strongly semi-finally stable with respect to the sets (α, β, I), $\alpha^c/\beta \neq \emptyset$, if there exist two functions $V[x,t] \in C^1[I^c(\beta) \times \bar{I}]$, $\phi(t) \in R[I]$, such that

(i) system (1.1.1) is semi-finally stable with respect to the sets (α, β, I),

(ii) $V_M^{\alpha^c/\beta}(t_o)$ and $V_m^{Fr.\beta}(t)$, $t \in (t_o, t_o+T)$, are finite

(iii) $\dot{V}[x(t),t] < \phi(t)$, $t \in I$, along any trajectory $x(t)$, $x(t_o) \in \alpha^c/\beta$, as long as $x(t) \in I^c(\beta)$,

(iv) $\int_{t_o}^{t_1} \phi(t)dt \leqq V_m^{Fr.\beta}(t_1) - V_M^{\alpha^c/\beta}(t_o)$, all $t_1 \in (t_o, t_o+T)$

Proof of Theorem 2.3.2:

Let $x(t)$ be an arbitrarily chosen trajectory of system (1.1.1) with initial condition $x(t_o) \in \alpha^c/\beta$.

Suppose, contrary to the expected conclusion, that

$$x(t_2) \in \beta, \quad \text{for some } t_2 \in (t_o, t_o+T) \tag{2.3.1}$$

Since β is closed, then there exists a $t_1 \in (t_o, t_2]$ such that

$$x(t_1) \in Fr.\beta, \quad x(t) \in \beta^c, \quad t \in [t_o, t_1) \tag{2.3.2}$$

but then

$$V[x(t_1),t_1] = V[x(t_o),t_o] + \int_{t_o}^{t_1} \dot{V}[x(t),t]dt$$

$$\leqq V_M^{\alpha^c/\beta}(t_o) + \int_{t_o}^{t_1} \dot{V}[x(t),t]dt$$

Using (iii) - (iv), we get

$$V[x(t_1)t_1] < V_m^{Fr.\beta}(t_1) \tag{2.3.3}$$

which constitutes the required contradiction. Hence, $x(t) \in \beta^c$, for all $t \in I$.

Using the fact that the above argument holds for any trajectory $x(t)$

emanating from α^c/β at $t = t_o$, and condition (i), we conclude that the system is strongly semi-finally stable with respect to the sets (α, β, I). This completes the proof of Theorem 2.3.2.

Remark 2.3.1:

A similar theory may be established concerning strong semi-final stable systems with respect to the sets $(\alpha, \beta, t_1 \epsilon I)$. The negation of these results is also possible. This is done, in more detail, in the next chapter.

2.4 Instability theorems:

(The following theorems are special cases of more general theorems established in Chapter II, so we omit the proofs here.)

1. We know that any system (1.1.1) is semi-finally stable with respect to the sets (α, β, I), provided $\alpha \subseteq \beta$. So the following theorem is restricted to the case $\alpha \nsubseteq \beta$.

Theorem 2.4.1:

System (1.1.1) is not semi-finally stable with respect to the sets (α, β, I), $\alpha/\bar{\beta} \neq \emptyset$, if there exist a point $x_o \ \epsilon\ \alpha/\bar{\beta}$, two functions $V[x,t] \ \epsilon\ C^1[I^c(\beta) \times I]$, and $\phi(t) \ \epsilon\ R[I]$, such that

(i) $V_M^{Fr.\beta}(t)$, all $t \ \epsilon\ (t_o, t_o+T)$, is finite,

(ii) $\dot{V}[x(t),t] > \phi(t)$, $t \ \epsilon\ I$, along the trajectory $x(t)$, $x(t_o) = x_o$, as long as $x(t) \ \epsilon\ I^c(\beta)$,

(iii) $\int_{t_o}^{t_1} \phi(t)dt \geq V_M^{Fr.\beta}(t_1) - V[x_o,t_o]$, all $t_1 \ \epsilon\ (t_o, t_o+T)$.

The condition $\alpha/\bar{\beta} \neq \emptyset$, guarantees the existence of at least one point $x_o \in \alpha/\bar{\beta}$. We note that β is not required to be open as in the stability theorem 2.1.1 However, the restriction, $\alpha/\bar{\beta} \neq \emptyset$ can be avoided by assuming that $V[x,t] \in C^1[R^n \times I]$, and condition (iii) replaced by

$$\text{(iii)}^* \quad \int_{t_o}^{t} \phi(t)dt \geqq V_M^{\beta}(t_1) - V[x_o,t_o], \text{ all } t_1 \in (t_o, t_o+T).$$

This implies, of course, that $V_M^{\beta}(t)$ must be finite (instead of the condition on $V_M^{Fr.\beta}(t)$). It is to be noted, moreover, that $x_o \in \alpha/\beta$.

2. If $\alpha \subseteq \beta$, then any system (1.1.1) is certainly semi-finally stable with respect to the sets (α, β, $t_o \in I$). On the other hand, if $\alpha \not\subseteq \beta$, then any system (1.1.1) is certainly not semi-finally stable with respect to the sets (α, β, $t_o \in I$). Bearing this in mind, we can prove the following theorem for the case $t_1 > t_o$.

Theorem 2.4.2:

System (1.1.1) is not semi-finally stable with respect to the sets (α, β, $t_1 \in I$), $t_1 > t_o$, if there exist two functions $V[x,t] \in C^1[R^n \times I^*]$, $\phi(t) \in R[I^*]$, where $I^* = [t_o,t_1]$, such that

(i) $V_M^{\beta}(t_1)$ is finite

(ii) there exists a point $x_o \in \alpha$, such that

$$\dot{V}[x(t),t] > \phi(t), \; t \in I^*$$

along the trajectory $x(t)$, $x(t_o) = x_o$; and

$$\int_{t_o}^{t_1} \phi(t)dt \geqq V_M^{\beta}(t_1) - V[x_o,t_o].$$

§3. GENERAL THEOREMS ON THE DIFFERENT TYPES OF FINAL-STABILITY

3.1 Final-stability

Theorem 3.1.1:

System (1.1.1) is finally-stable with respect to the sets (α, β, I), if there exist two functions $V[x,t] \in C^1[R^n \times I]$, $\phi(t) \in R[I]$, such that

(i) $V_m^{\beta^c}(t_o+T)$ and $V_M^{\alpha}(t_o)$ are finite.

(ii) $\dot{V}[x(t),t] \leq \phi(t)$, $t \in I$, along any trajectory $x(t)$ with $x(t_o) \in \alpha$.

(iii) $\int_{t_o}^{t_o+T} \phi(t)dt < V_m^{\beta^c}(t_o+T) - V_M^{\alpha}(t_o)$

Here, we allow α to be any connected set. β, however, must be an <u>open</u> connected set. As before, this means that α may be a point $x_o \in R^n$. We note finally that, if the conditions of Theorem 3.1.1 hold, then the conditions of Theorem 2.1.1 are also satisfied and system (1.1.1) is semi-finally stable with respect to the sets (α, β, I), which fact is in our favour, since final-stability implies semi-final-stability with respect to the same sets.

This theorem will be proved in Chapter II where we give more detailed results concerning this type of final-stability.

3.2 Final-stability with respect to the sets $(\alpha, \beta, t_1 \in I)$

Theorem 3.2.1:

System (1.1.1) is finally-stable with respect to the sets $(\alpha, \beta, t_1 \in I)$

if there exists a connected set γ, $\bar{\gamma} \subset \beta$ such that

(a) system (1.1.1) is semi-finally stable with respect to the sets $(\alpha, \gamma, t_1 \in I)$,

and if

(b) there exist two functions $V[x,t] \in C^1[\bar{\beta} - I(\gamma)]$, $\phi(t) \in R[I]$, such that

(i) $V_m^{Fr.\beta}(t_3)$ and $V_M^{Fr.\gamma}(t_2)$ are finite for all $t_3 \in (t_1, t_o+T)$, and all $t_2 \in [t_1, t_o+T)$.

(ii) $\dot{V}[x(t),t] < \phi(t)$, $t \in [t_1, t_o+T)$, along any trajectory $x(t)$, $x(t_o) \in \alpha$, as long as $x(t) \in \bar{\beta} - I(\gamma)$

(iii) $\displaystyle\int_{t_2}^{t_3} \phi(t)dt \leqq V_m^{Fr.\beta}(t_3) - V_M^{Fr.\gamma}(t_2)$, for all $t_2 \in [t_1, t_o+T)$, and $t_3 \in (t_2, t_o+T)$.

Here all the sets under consideration are connected; moreover, β is an open set.

Corollary 3.2.1:

If there exists a $t_1 \in I$ such that the conditions of Theorem 3.2.1 hold, then system (1.1.1) is uniformly finally-stable with respect to the sets (α, β, I).

Remark 3.2.1:

If $\alpha \subset I(\beta)$, α a connected set, then system (1.1.1) is certainly semi-finally stable with respect to the sets $(\alpha, \beta, t_o \in I)$. This means that, in Theorem 3.2.1, the condition (a) is satisfied for $\gamma = \alpha$, $t_1 = t_o$; in this case, Theorem 3.2.1 reduces to a theorem on stability with respect to the sets (α, β, I) given by Weiss and Infante [36]. We note, however, that in Weiss and Infante the sets α and β are less general. In fact, they are of the form

$$\alpha : \ ||x|| < a, \quad \beta : \ ||x|| < b, \quad a < b$$

where $||x||$ is the euclidean norm.

3.3 Strong final-stability:

The following theorem can be proved in the usual manner. Note that condition (i) ensures that the system is finally stable with respect to the sets (α, β, I).

Theorem 3.3.1:

System (1.1.1) is strongly finally-stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, if there exist four functions $V_i[x,t] \in C^1[R^n \times I]$, $\phi_i(t) \in R[I]$, $i = 1,2$, such that

(i) $V_1[x,t]$ satisfies all the conditions of theorem 3.1.1.

(ii) $V^{\beta}_{2m}(t)$, $t \in [t_1, t_o+T)$, $V^{\alpha^c}_{2M}(t_o)$ are finite.

(iii) $\dot{V}_2[x(t),t] < \phi_2(t)$, $t \in I$, along any trajectory $x(t)$, with $x(t_o) \in \alpha^c$.

(iv) $\displaystyle\int_{t_o}^{t_2} \phi_2(t)dt \leqq V^{\beta}_{2m}(t_2) - V^{\alpha^c}_{2M}(t_o)$, for all $t_2 \in [t_1, t_o+T)$.

(β must be open by Theorem 3.1.1.)

Corollary 3.3.1:

System (1.1.1) is strongly finally-stable with respect to the sets (α, β, I), if, in Theorem 3.3.1, conditions (ii) and (iv) are replaced by (ii)* $V^{\beta}_{2m}(t)$, $t \in (t_o, t_o+T)$, and $V^{\alpha^c}_{2M}(t_o)$ are finite.

(iii)* for each $x_o \in \alpha^c$, these exists a $t^* \in (t_o, t_o+T)$ such that

$$\int_{t_o}^{t_2} \phi_2(t)dt \leqq V^{\beta}_{2m}(t_2) - V[x_o,t_o], \text{ all } t_2 \in [t^*, t_o+T).$$

3.4 Instability Theorems:

One can state and prove many theorems on non-final-stability. We, however, limit ourselves to some of them (which can be proved in the usual way). A more detailed account concerning this aspect of the theory is given in Chapter II.

Theorem 3.4.1:

System (1.1.1) is not finally-stable with respect to the sets (α, β, I), $\alpha \cap \beta \neq \emptyset$, if there exist two functions $V[x,t] \in C^1[\beta \times I]$, $\phi(t) \in R[I]$, a point $x_o \in \alpha \cap \beta$, and a $t^* \in I$ such that

(i) $V^{\beta}_{M}(t)$ and $V^{\beta}_{m}(t)$ are finite for all $t \in I$, $t \geqq t^*$.

(ii) $\dot{V}[x(t),t] > \phi(t)$, $t \in I$, along the trajectory $x(t)$, $x(t_o) = x_o$, as long as $x(t) \in \beta$,

(iii) $\int_{t_1}^{t_2} \phi(t)dt \geqq V^{\beta}_{M}(t_2) - V^{\beta}_{m}(t_1)$, all $t_1, t_2 \in I$, $t_2 > t_1 \geqq t^*$.

We note that, if β is an open set, then it is possible that system (1.1.1) is finally-stable with respect to the sets (α, β, I) and, at the same time, there exists a trajectory $x(t)$, $x(t_o) \in \alpha$, such that $x(t_o+T) \in Fr.\beta$. On the other hand, if $x(t_o+T) \in \beta$, for each trajectory $x(t)$ emanating from α at $t = t_o$, then the system is certainly finally-stable with respect to the sets (α, β, I), provided β is open.

Theorem 3.4.2:

System (1.1.1) is not finally-stable with respect to the sets (α, β, I), $\alpha/\beta \neq \emptyset$, if there exist two functions $V[x,t] \in C^1[\bar{\beta} \times I]$, $\phi(t) \in R[I]$, and a point $x_o \in \alpha/\beta$, such that

(i) $V_m^{Fr.\beta}(t)$ and $V_M^{\beta}(t)$ are finite for all $t \in I$.

(ii) $\dot{V}[x(t),t] > \phi(t)$, $t \in I$, along the trajectory $x(t)$, $x(t_o) = x_o$, as long as $x(t) \in \bar{\beta}$.

(iii) $\int_{t_1}^{t_2} \phi(t)dt \geqq V_M^{\beta}(t_2) - V_m^{Fr.\beta}(t_1)$, all $t_1, t_2 \in I$, $t_2 > t_1$.

Theorem 3.4.3:

System (1.1.1) is not finally-stable with respect to the sets $(\alpha, \beta, t_1 \in I)$, if there exist a $t^* \in (t_1, t_o+T)$, two functions $V[x,t] \in C^1[\beta \times I^*]$, and $\phi(t) \in R[I^*]$, where $I^* = [t_1, t^*]$, such that

(i) $V_M^{\beta}(t^*)$ and $V_m^{\beta}(t_1)$ are finite.

(ii) $\dot{V}[x(t),t] > \phi(t)$, $t \in [t_1, t^*]$ along the trajectory $x(t)$, $x(t_o) = x_o$ for some $x_o \in \alpha$, as long as $x(t) \in \beta$.

(iii) $\int_{t_1}^{t^*} \phi(t)dt \geqq V_M^{\beta}(t^*) - V_m^{\beta}(t_1)$.

We note that since condition (ii) in the above theorems is difficult to verify for the given trajectory only, it has to be verified for the whole system in general. This remark indicates that the conditions of the above theorems are too restrictive (in the sense that they are independent of the set α, as far as practical applications are concerned).

§4. CONCLUSION

As expected, the theory of final-stability established in this chapter is more general than the theory of finite-time stability [36], in the sense that, in most cases, stronger results have been proved.

The advantage of α being, in general, only a connected set and not necessarily open is obvious, since this allows us to consider cases where α is a point in the state space R^n. This becomes more obvious if we consider one of the simplest definitions of controllability, i.e., there exists a control $u(\cdot)$ such that the trajectory $x(t)$ of the system $\dot{x} = g(x,u,t)$, emanating from $x_o \in R^n$ at $t = t_o$, reaches the neighbourhood of another point $x_1 \in R^n$ at time t_f. Of course, whether this belongs to the domain of semi-final stability or final-stability depends on what we want to happen to the trajectory $x(t)$ after time t_f. This will be considered in more detail in the following chapter.

CHAPTER II

Ordinary Differential Equations Under the Influence of Perturbing Forces "Forced Systems"

§1. INTRODUCTION, LEMMAS AND DEFINITIONS

1.1 Introduction

In Chapter I the theory of final stability of systems of the form

$$\dot{x} = f(x,t) \tag{1.1.1}$$

where x is an n-vector (the state vector) was discussed. This chapter is concerned with the final-stability of systems under the influence of perturbing forces.

Let R^n be the n-dimensional euclidean space and let

$$S^m \subseteq \{u(x,t) \,|\, u\colon R^n \times I \to R^m\} \tag{1.1.2}$$

$$\left(I = [t_o, t_o+T),\ t_o \in R^+,\ T \in R^+,\ T > 0\right)$$

be a given set of functions which we call the set of admissible controls. This is done in order to indicate the possible connections between final-stability and the known concept of controllability.

Let $g : I \times R^n \times S^m \to R^n$ be smooth enough so that there is no difficulty with the existence of solutions of the differential system

$$\dot{x} = g(x,t,u) \tag{EC}$$

i.e., for each $u(\cdot) \in S^m$, and each $x_o \in R^n$, there exists at least one function $x\left(t;\ x_o,t_o,u(\cdot)\right)$ with the properties

$$\dot{x}\left[t;\ x_o,t_o,u(\cdot)\right] = g\left[t,x\left(t;\ x_o,t_o,u(\cdot)\right),u(\cdot)\right],\ t \in I$$

and

$$x\left[t_o;\ x_o,t_o,u(\cdot)\right] = x_o$$

We call $u(\cdot)$ a <u>control</u>, and $x\left(t;\ x_o,t_o,u(\cdot)\right)$ a <u>response</u> from x_o.

We note that for a given control $u(\cdot) \in S^m$ and $x_o \in R^n$, there may exist many responses from x_o, i.e., many functions of the form mentioned above. To each one of these functions we associate a trajectory (or motion)

$$x(t) = x\big(t;\ x_o, t_o, u(\cdot)\big)$$

where the time t plays the role of a parameter.

1.2 Definitions

We are now in a position to define the different types of final-stability. Let U be a subset of S^m, then

Definition 1.2.1:

System (EC) is semi-finally stable with respect to the sets (α, β, U, I), if, for any trajectory $x(t) = x\big(t;\ x_o, t_o, u(\cdot)\big)$, $u(\cdot) \in U$, the condition $x_o \in \alpha$ implies the existence of a $t_1 \in I$ such that

$$x(t_1) = x\big(t_1;\ x_o, t_o, u(\cdot)\big) \in \beta$$

where t_1 may depend on the particular trajectory.

Definition 1.2.2:

System (EC) is not semi-finally stable with respect to the sets (α, β, U, I), if there exists a trajectory $x^*(t) = x^*\big(t;\ x_o, t_o, u^*(\cdot)\big)$, for some $x_o \in \alpha/\beta$, and $u^*(\cdot) \in U$, such that $x^*(t) \in \beta^c$, all $t \in I$.

From these definitions, we conclude that if $\alpha \subseteq \beta$, then system (EC) is certainly semi-finally stable with respect to the sets (α, β, U, I).

Definition 1.2.3:

System (EC) is semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$ if, for any trajectory $x(t) = x(t; x_o, t_o, u(\cdot))$, $u(\cdot) \in U$, the condition $x_o \in \alpha$ implies that $x(t_1) \in \beta$.

Definition 1.2.4:

System (EC) is not semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$, if there exists a trajectory $x^*(t) = x^*(t; x_o, t_o, u^*(\cdot))$, for some $u^*(\cdot) \in U$, and some $x_o \in \alpha$, such that $x^*(t_1) \in \beta^c$.

It is to be noted immediately that system (EC) may be semi-finally stable with respect to the sets (α, β, U, I) without being semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$, for some $t_1 \in I$.

Definition 1.2.5:

System (EC) is uniformly semi-finally stable with respect to the sets (α, β, U, I), if there exists a $t_1 \in I$ such that the system is semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$.

Definition 1.2.6:

System (EC) is strongly semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$, if

(i) it is semi-finally stable with respect to the sets (α, β, U, I), and

(ii) any trajectory $x(t) = x(t; x_o, t_o, u(\cdot))$, $x_o \in \alpha^c$, $u(\cdot) \in U$, is such that $x(t_1) \in \beta^c$.

If condition (ii) holds for all $t_1 \varepsilon I$, and $x_o \varepsilon \alpha^c/\beta$, then system (EC) is said to be strongly semi-finally stable with respect to the sets (α, β, U, I).

Definition 1.2.7:

System (EC) is finally-stable with respect to the sets (α, β, U, I) if, for any trajectory $x(t) = x\big(t;\, x_o,t_o,u(\cdot)\big)$, $u(\cdot) \varepsilon U$, the condition $x_o \varepsilon \alpha$ implies the existence of a $t_1 \varepsilon I$, such that $x(t) \varepsilon \beta$, for all $t \varepsilon I$, $t \geqq t_1$; where t_1 may depend on the particular trajectory.

We note that this does not imply that $x(t_o+T) \varepsilon \beta$, if β is an open set.

Definition 1.2.8:

System (EC) is not finally-stable with respect to the sets (α, β, U, I), if there exists a trajectory $x^*(t) = x^*\big(t;\, x_o,t_o,u^*(\cdot)\big)$, for some $x_o \varepsilon \alpha$, and some $u^*(\cdot) \varepsilon U$, such that

$$x^*(t) \varepsilon \beta^c$$

for all $t \varepsilon I$, $t \geqq t^*$, for some $t^* \varepsilon I$.

Definition 1.2.9:

System (EC) is finally-stable with respect to the sets $(\alpha, \beta, U, t_1 \varepsilon I)$, if, for any trajectory $x(t) = x\big(t;\, x_o,t_o,u(\cdot)\big)$, $u(\cdot) \varepsilon U$, the condition $x_o \varepsilon \alpha$ implies that $x(t) \varepsilon \beta$, for all $t \varepsilon I$, $t \geqq t_1$.

Definition 1.2.10:

System (EC) is not finally-stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$, if there exists a trajectory $x^*(t) = x^*(t; x_o, t_o, u^*(\cdot))$, for some $x_o \in \alpha$, and some $u^*(\cdot) \in U$, such that $x^*(t_2) \in \beta^c$, for some $t_2 \in I$, $t_2 \geqq t_1$.

Here again, we note that system (EC) may be finally-stable with respect to the sets (α, β, U, I) without being finally-stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$, for some $t_1 \in I$.

Definition 1.2.11:

System (EC) is uniformly finally-stable with respect to the sets (α, β, U, I) if there exists a $t_1 \in I$ such that the system is finally-stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$.

Definition 1.2.12:

System (EC) is strongly finally-stable with respect to the sets (α, β, U, I), if

(i) it is finally-stable with respect to the sets (α, β, U, I),

and

(ii) for any trajectory $x(t) = x(t; x_o, t_o, u(\cdot))$, $u(\cdot) \in U$, the condition $x_o \in \alpha^c$ implies the existence of a $t^* \in I$, such that

$$x(t) \in \beta^c, \quad \text{for all } t \in I,\ t \geqq t^*$$

where t^* may depend on the particular trajectory.

Definition 1.2.13:

System (EC) is strongly finally-stable with respect to the sets $(\alpha, \beta, U, t_1 \in I)$, if

(i) it is finally-stable with respect to the sets (α, β, U, I),

and

(ii) for any trajectory $x(t) = x(t; x_o, t_o, u(\cdot))$, $u(\cdot) \in U$, the condition $x_o \in \alpha^c$ implies that $x(t) \in \beta^c$, for all $t \in I$, $t \geqq t_1$.

Remark 1.2.1:

A theory concerning system (1.1.1) can be established by setting $U \equiv \{u(\cdot) \equiv 0\}$ in the system

$$\dot{x} = f(x,t) + G(x,t)u \tag{EG}$$

1.3 Comparison Principles

In order to establish the intended theory for system (EC), we need to state some essential definitions and lemmas [4,13,18,18.1,18.2]. Let $V[x,t]$ denote a mapping

$$V: R^n \times I \to R^1 \tag{1.3.1}$$

Definition 1.3.1:

$V[x,t]$ is said to be of class $L[\gamma, I^*]$, $\gamma \subseteq R^n$, $I^* \subseteq I$, if it is continuous in both $(x,t) \in \gamma \times I^*$ and satisfies a local lipschitzian condition in x, for each $t \in I^*$. We write

$$V^*[x,t] = \lim_{h \to 0^+} \sup \frac{V[x + hg(x,t,u), t + h] - V[x,t]}{h} \tag{1.3.2}$$

$$V_M^\alpha(t) = \sup_{x \in \alpha} . V[x,t], \quad V_m^\alpha(t) = \inf_{x \in \alpha} . V[x,t] \tag{1.3.3}$$

where α is a connected set in R^n.

Definition 1.3.2:

A function $\omega(t,r) : I \times R^1 \to R^1$ is said to be of class Ω if it is smooth enough to ensure the existence of the maximal solutions of

$$\dot{r} = \omega(t,r) \tag{C}$$

over I. A function $\omega \in \Omega$ is said to be of class Ω^* if, in addition, $\omega(t,r)$ is monotonic increasing in r for each $t \in I$.

Lemma 1.3.1: [4,13,18·1,18·2]

Let $V[x,t] \in L[\gamma, I_1]$, $\gamma \subseteq R^n$, $I_1 \subseteq I$, and $\omega(t,r) \in \Omega$. Suppose that

$$V^*[x,t] \leqq \omega(t, V[x,t]) \tag{1.3.4}$$

for all $(x,t) \in \gamma \times I_1$, and all $u(\cdot) \in U$, where U is a given subset of S^m. Let $r(t)$ be the maximal solution of (C) with initial condition $r(t_1) = r_1$, $t_1 \in I_1$. Then, for any trajectory $x(t) = x\big(t; x_o, t_o, u(\cdot)\big)$, $u(\cdot) \in U$, $x_o \in \gamma$, the condition

$$V[x(t_1), t_1] \leqq r_1 \tag{1.3.5}$$

implies that

$$V[x(t), t] \leqq r(t) \tag{1.3.6}$$

for all $t \in I_1$, $t \geqq t_1$, as long as $x(t) \in \gamma$.

Lemma 1.3.2: [18]

If, in the previous lemma, $\omega \in \Omega^*$, and

$$V^*[x,t] < \omega(t,V[x,t]) \tag{1.3.7}$$

for all $t \in I_1$, $x \in \gamma$, $u(\cdot) \in U$, then (1.3.5) implies that

$$V[x(t),t] < r(t) \tag{1.3.8}$$

for all $t \in I_1$, $t > t_1$, as long as $x(t) \in \gamma$.

Remark 1.3.1:

Let γ be a closed set in R^n, and $I_1 = [a,b]$, then: (a) inequality (1.3.4) need only hold for all $x \in I(\gamma)$, $t \in [a,b)$. The conclusion (1.3.5) still holds at $t = b$ even if $x(b) \in Fr.\gamma$, provided of course that $x(b) = \lim_{t\to b} x(t)$ exists (this condition is usually implied by the conditions of the theorem under consideration.); but (b) for conclusion (1.3.8) to hold for all $x \in \gamma$, $t \in I_1$, $t > t_1$, we need to assume that condition (1.3.7) holds for all $x \in S(\gamma)$, $t \in I_{1\varepsilon}$, where $S(\gamma)$ is some open set containing γ and $I_{1\varepsilon} = [a,b+\varepsilon)$, $\varepsilon > 0$ being arbitrarily small.

(This remark enables us to avoid mentioning these facts throughout the remainder of the thesis.)

1.4 Discussion

In order to be able to extend our results to control theory, i.e., to controllability, we extend the previous definition of the different types of semi-final and final-stability stated in 1.2 to the closed interval $\bar{I} = [t_o,t_o+T]$. In the sequel, when not stated otherwise, J will indicate either I or $\bar{I}$. We note, immediately, that the main difference lies in the fact that t_1 may coincide with t_o+T, in the case where J = $\bar{I}$.

Now, bearing in mind that the control subset U may be considered as a single function u(·) ε S^m, i.e., U = {u(·)}, and assuming that the sets α and β are not necessarily open but only connected sets, we give below the definitions of different types of controllability.

Definition 1.4.1:

System (EC) is (α, β, $\bar{I}$) semi-controllable if there exists a subset $U \subseteq S^m$ such that the system is semi-finally stable with respect to the sets (α, β, U, $\bar{I}$).

Definition 1.4.2:

System (EC) is (α, β, t_1 ε $\bar{I}$) semi-controllable if there exists a subset $U \subseteq S^m$, such that the system is semi-finally stable with respect to the sets (α, β, U, t_1 ε $\bar{I}$).

Definition 1.4.3:

System (EC) is said to be uniformly (α, β, $\bar{I}$) semi-controllable if there exists a t_1 ε $\bar{I}$, such that the system is (α, β, t_1 ε $\bar{I}$) semi-controllable.

Definition 1.4.4:

System (EC) is (α, β, $\bar{I}$)-controllable if there exists a subset $U \subseteq S^m$ such that system (EC) is finally-stable with respect to the sets (α, β, U, $\bar{I}$).

Definition 1.4.5:

System (EC) is $(\alpha, \beta, t_1 \varepsilon \bar{I})$-controllable if there exists a subset $U \subseteq S^m$, such that the system is finally-stable with respect to the sets $(\alpha, \beta, U, t_1 \varepsilon \bar{I})$.

It is said to be uniformly $(\alpha, \beta, \bar{I})$-controllable if there exists a $t_1 \varepsilon \bar{I}$ such that the system is $(\alpha, \beta, t_1 \varepsilon \bar{I})$- controllable.

Remark 1.4.1:

Bearing Remark 1.3.1 in mind, the lemmas of section 1.3 can be extended to cover the case $J = \bar{I}$. However, whenever the case limit $t = t_o+T$ is considered, we shall avoid using Lemma 1.3.2

§2. GENERAL THEOREMS ON THE DIFFERENT TYPES OF SEMI-FINAL STABILITY.

2.1 Semi-final stability

(1) Any system (EC) is semi-final stable with respect to the sets (α, β, U, J), $U \subseteq S^m$, $\alpha \subseteq \beta$. So, we assume that $\alpha \not\subseteq \beta$. If $J = I$, we assume, furthermore, that β is an open set. Let

$$W = W(\alpha, \beta, J, U) = \{x | x = x(t; x_o, t_o, u(\cdot)), \text{ all } t \in J, x_o \in \alpha/\beta, \text{ and all } u(\cdot) \in U\} \tag{2.1.1}$$

Theorem 2.1.1:

System (EC) is semi-finally stable with respect to the sets (α, β, U, J), $\alpha \not\subseteq \beta$, if there exist two functions $V[x,t] \in L[\bar{W}/\beta, J]$, and $\omega(t,r) \in \Omega$ such that

(i) $V_M^{\alpha/\beta}(t_o)$ and $V_m^{\bar{W}/\beta}(t_o+T)$ are finite.

(ii) $V^*[x,t] \leq \omega(t, V[x,t])$, for all $t \in I$, $x \in \bar{W}/\beta$, $u(\cdot) \in U$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha/\beta}(t_o)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t_o+T) < V_m^{\bar{W}/\beta}(t_o+T) \tag{2.1.2}$$

To show that Theorem 2.1.1-I is a special case of Theorem 2.1.1 it will be sufficient to take $\omega(t,r) \equiv \phi(t)$ and note that the solution $r(t)$, $r(t_o) = V_M^{\alpha/\beta}(t_o)$, is given by

$$r(t) = V_M^{\alpha/\beta}(t_o) + \int_{t_o}^{t} \phi(s)ds \tag{2.1.3}$$

Remark 2.1.1:

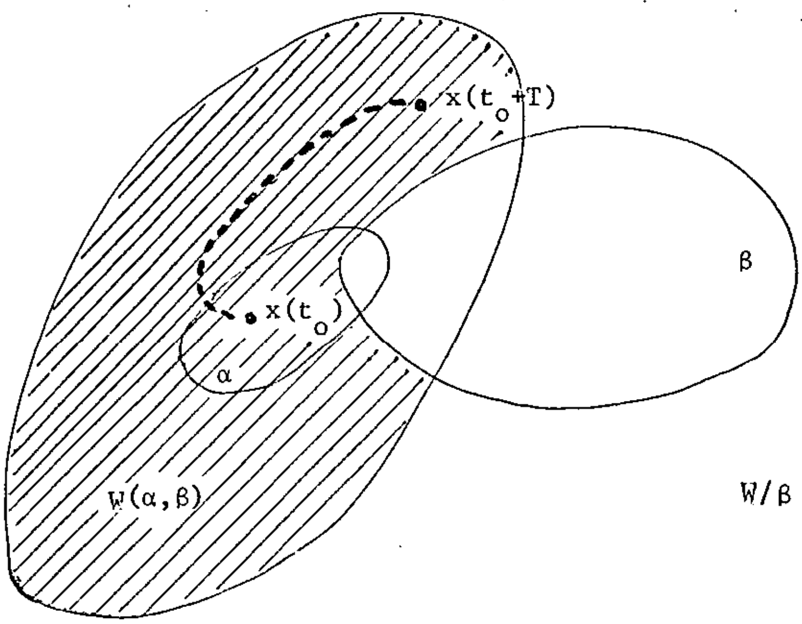
In practical applications, it is usually difficult to determine the set W defined by (2.1.1). This does not constitute a restriction to the application of the theorem, because the conclusion will still be valid if we replace W by a larger set W_+, containing W, bearing in mind that in this case

$$V_m^{\bar{W}_+/\beta}(t_o+T) \leq V_m^{\bar{W}/\beta}(t_o+T)$$

and hence condition (2.1.2) will be satisfied. We note, moreover, that the set W_+ may either be (a) the space R^n, or (b) a given set determined by some aspects of the problem under consideration.

This remark applies to all the results established in the thesis, providing the appropriate modifications are made. Hence, we need not repeat it again.

Proof of Theorem 2.1.1:



W/β = shaded area

Figure 2.1.1

(a) Any trajectory $x(t) = x\big(t; x_o, t_o, u(\cdot)\big)$, $u(\cdot) \in U$, with $x_o \in \alpha \cap \beta$, is such that $x(t^*) \in \beta$, for some $t^* \in J$ (certainly for $t^* = t_o$).

(b) Let $x(t) = x\big(t; x_o, t_o, u(\cdot)\big)$, $u(\cdot) \in U$, be any trajectory of (EC), with $x_o \in \alpha/\beta$, and suppose contrary to the expected conclusion, that

$$x\big(t; x_o, t_o, u(\cdot)\big) \in \beta^c \tag{2.1.4}$$

for all $t \in J$. Since $x(t) \in W$, by (2.1.1), then

$$x\big(t; x_o, t_o, u(\cdot)\big) \in W/\beta, \text{ all } t \in J \tag{2.1.5}$$

and

$$x\big(t_o+T; x_o, t_o, u(\cdot)\big) \in \bar{W}/\beta \tag{2.1.6}$$

(for if $J = I$, then we must assume β to be open; in this case (2.1.4) ensures that $x(t_o+T) \in \beta^c$. On the other hand, if $J = \bar{I}$ then (2.1.4) implies (2.1.6) immediately.)

But since $V[x_o, t_o] \leqq V_M^{\alpha/\beta}(t_o)$, the application of (i),(ii) and Lemma 1.3.1 gives

$$V[x(t), t] \leqq r_M(t), \text{ all } t \in J \tag{2.1.7}$$

where $r_M(t)$ is the maximal solution, $r_M(t_o) = V_M^{\alpha/\beta}(t_o)$, of equation (C). Using (2.1.7) and (2.1.2) we get the inequality

$$V[x(t_o+T), t_o+T] < V_m^{\bar{W}/\beta}(t_o+T) \tag{2.1.8}$$

which constitutes an obvious contradiction to (2.1.6) in view of the definition of $V_m^{\bar{W}/\beta}(t_o+T)$. Thus, the original assumption (2.1.4) is false, and there must exist a $t_1 \in J$, such that

$x(t_1) \in \beta$.

Since the above argument is independent of the exact value of x_o and the particular trajectory chosen, provided $x_o \in \alpha/\beta$, and $u(\cdot) \in U$, it holds for all trajectories emanating from α/β at $t = t_o$. This completes the proof of Theorem 2.1.1.

(2) the following theorem was suggested to us by a similar theorem on controllability [12]. But, in addition to the fact that we apply this theorem to sets rather than points, the conditions are much milder. It is to be noted, also, that, in the case of controllability our theorem will yield another type of controllability (Definition 1.4.1 - II) different from the type of controllability of [12] which, in fact, corresponds to definition 1.4.2 - II with $t_1 = t_o+T$. This is to say, that the following theorem will not reduce completely to the corresponding theorem in [12], as far as controllability is concerned.

Theorem 2.1.2:

System (EC) is semi-finally stable with respect to the sets (α, β, U, J), $\alpha \not\trianglelefteq \beta$, ($\beta$ open if $J = I$), if there exist two functions $V[x,t] \in L[\bar{W}/\beta, I]$, and $\omega(t,r) \in \Omega$ such that

(i) $V_M^{\alpha/\beta}(t_o)$ is finite

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \in I$, $x \in \bar{W}/\beta$, $u(\cdot) \in U$.

(iii) for any continuous n-vector function $c(t)$, the conditions:

(a) $c(t_o) \in \alpha/\beta$,

(b) $c(t_o+T) = \lim_{t \to t_o+T} c(t) \in \bar{W}/\beta$

imply that

$$\lim_{t \to t_o + T} V[c(t),t] = +\infty \qquad (2.1.9)$$

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha/\beta}(t_o)$, of equation (C) is such that $r_M(t_o+T) = \lim_{t \to t_o+T} r_M(t)$ exists and is bounded above.

Proof of Theorem 2.1.2: *(refer to Figure 2.1.1)*

Let $x(t) = x(t;\, x_o, t_o, u(\cdot))$, $x_o \in \alpha/\beta$, $u(\cdot) \in U$, be any trajectory of (EC), then

$$x(t) \in W, \text{ all } t \in J \qquad (2.1.10)$$

Suppose, contrary to the expected conclusion, that

$$x(t) \in W/\beta, \text{ all } t \in J \qquad (2.1.11)$$

Since β is open in the case $J = I$, (2.1.10) and (2.1.11) give

$$x(t_o+T) \in \bar{W}/\beta, \qquad (2.1.12)$$

But then, by (iii) we get

$$\lim_{t \to t_o + T} V[x(t),t] = +\infty \qquad (2.1.13)$$

On the other hand, using (i), (ii), (2.1.11) and Lemma 1.3.1, we get

$$V[x(t_o+T), t_o+T] \leqq r_M(t_o+T) \qquad (2.1.14)$$

where $r_M(t)$ is given by (iv). Since $r_M(t_o+T)$ is bounded above, by (iv), then (2.1.14) constitutes a contradiction to (2.1.13). Hence, the original assumption (2.1.11) is false; the <u>theorem follows</u>.

Remark 2.1.2:

To avoid the question of the existence of $x(t_o+T) = \lim_{t \to t_o+T} x(t)$, one can modify the conditions of the above theorems in an appropriate way. For example, conditions (i) and (iii) of Theorem 2.1.1 may be replaced by

(i)* $V[x_o,t_o]$, all $x_o \in \alpha/\beta$, $V_m^{W/\beta}(t)$, all $t \in J$, are finite

(iii)* for each $r_o \in B$, there corresponds a $t(r_o) \in J$ such that the maximal solution $r_M(t;r_o)$, $r_M(t_o;r_o) = r_o$, of equation (C) is such that

$$r_M(t;r_o) < V_m^{W/\beta}(t), \quad t = t(r_o)$$

where $B \in R$ is such that

$$V[x_o,t_o] \in B, \text{ all } x_o \in \alpha/\beta.$$

For, in this case, assumption (2.1.4) will lead to the conclusion

$V[x(t)t] \leq r_M(t;V[x_o,t_o])$, all $t \in J$ and, by (iii)*, the above inequality gives

$$V[x(t),t] < V_m^{W/\beta}(t), \text{ for } t = t(V[x_o,t_o])$$

which is the required contradiction.

2.2 Non semi-final stability

(1) Let, for any $x_o \in R^n$ and $u_o(\cdot) \in S^m$,

$$W_o = W_o(x_o,u_o,J) \supseteq \{x \mid x = x(t;\ x_o,t_o,u_o(\cdot)),\ \text{all } t \in J\} \tag{2.2.1}$$

then

Theorem 2.2.1:

System (EC) is not semi-finally stable with respect to the sets (α, β, U, J), $\alpha/\bar{\beta} \neq \emptyset$, $U \subseteq S^m$, if there exist a control $u_o(\cdot) \in U$, a point $x_o \in \alpha/\bar{\beta}$, and two functions $V[x,t] \in L[W_o/I(\beta),J]$, $\omega(t,r) \in \Omega^*$ such that

(i) $V_m^{W_o \cap Fr.\beta}(t)$ is finite, for all $t \in J$, $t > t_o$.

(ii) $V^*[x,t] < \omega(t,V[x,t])$ all $t \in I$, $x \in W_o/I(\beta)$, and $u(\cdot) = u_o(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that

$$r_M(t) \leqq V_m^{W_o \cap Fr.\beta}(t), \text{ all } t \in (t_o, t_o+T) \tag{2.2.2}$$

and

$$r_M(t_o+T) < V_m^{W_o \cap Fr.\beta}(t_o+T), \text{ if } J = \bar{I} \tag{2.2.3}$$

Remark 2.2.1:

Condition (2.2.3) is dictated by the fact that inequality (1.3.8 - II) need not be strict at $t = t_o+T$. (see Remark 1.4.1 - Chapter II)

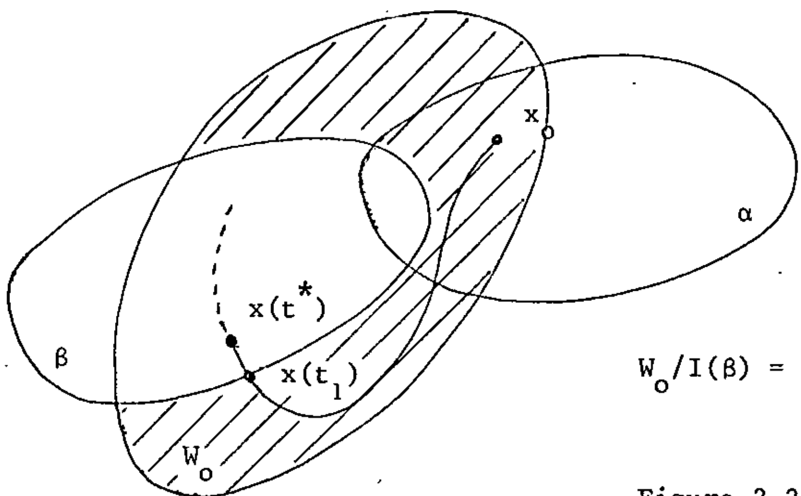
Remark 2.2.2:

An interesting special case of the above theorem can be deduced by setting $\omega(t,r) = \phi(t) \in R[I]$, and assuming that $\dot{V}$ [along trajectories of (EC)] exists ($V^* = \dot{V}$). In this case, the solution $r(t)$, $r(t_o) = V[x_o,t_o]$, of (C) is given by

$$r(t) = V[x_o,t_o] + \int_{t_o}^{t} \phi(s)ds \tag{2.2.4}$$

Furthermore, setting $u_o(\cdot) \equiv 0$ in system (EG) one obtains Theorem 2.4.1 - I, provided the function V of Theorem 2.2.1 - II is replaced by a function $V_1 \equiv - V$.

Proof of Theorem 2.2.1:



$W_o/I(\beta)$ = shaded area

Figure 2.2.1

(a) Suppose, contrary to the expected conclusion, that system (EC) is semi-finally stable with respect to the sets (α, β, U, J).

(b) Consider any trajectory $x(t) = x\big(t;\ x_o,t_o,u_o(\cdot)\big)$, where $x_o \ \varepsilon\ \alpha/\bar{\beta}$, and $u_o(\cdot)\ \varepsilon$ U are given by the conditions of the theorem, then by (2.2.1)

$$x(t)\ \varepsilon\ W_o,\ \text{all } t\ \varepsilon\ J \tag{2.2.5}$$

By (a), there exists a $t^*\ \varepsilon$ J, $t^* > t_o$, such that

$$x(t^*)\ \varepsilon\ \beta \tag{2.2.6}$$

Using the continuity of the trajectory, (2.2.5) and (2.2.6), we get

$$x(t_1)\ \varepsilon\ W_o \cap \text{Fr.}\beta,\ \text{for some } t_1\ \varepsilon\ (t_o,t^*] \tag{2.2.7}$$

and

$$x(t) \in W_o/\beta, \text{ all } t \in [t_o, t_1) \tag{2.2.8}$$

(Refer to Figure 2.2.1).

But then, by (2.2.7), (2.2.8), (i), (ii), and Lemma 1.3.2 we get

$$V[x(t),t] < r_M(t), \text{ all } t \in (t_o, t_1] \tag{2.2.9}$$

where $r_M(t)$ is given by (iii). If $J = \bar{I}$, it may happen that $t_1 = t_o+T$; in this case, we get

$$V[x(t),t] < r_M(t), \text{ all } t \in (t_o, t_o+T),\ V[x(t_o+T),\ t_o+T] \leqq r_M(t_o+T) \tag{2.2.10}$$

(for Lemma 1.3.2 need not be true at $t = t_o+T$.)

In both cases, (2.2.2) and (2.2.3) yield the required contradiction to (2.2.7), i.e.

$$V[x(t_1),t_1] < V_m^{W_o \cap Fr.\beta}(t_1)$$

Thus the assumption (a) is false. <u>This completes the proof of Theorem 2.2.1.</u>

(2) If β is open, then $\alpha/\beta \neq \alpha/\bar{\beta}$, and it may happen that α and β are such that $\alpha/\bar{\beta} = \emptyset$, but $\alpha/\beta \neq \emptyset$. In such cases, the above theorem is not valid. The following theorem considers such a situation, and is stated without proof.

Theorem 2.2.2:

System (EC) is not semi-finally stable with respect to the sets (α, β, U, J), $\alpha/\beta \neq \emptyset$, if there exist a point $x_o \in \alpha/\beta$, a control $u_o(\cdot) \in U$, and two functions $V[x,t] \in L[W_o,J]$, $\omega(t,r) \in \Omega^*$, such that

(i) $V_m^{W_o \cap \beta}(t)$, all $t \in J$, $t > t_o$, is finite

(ii) $V^*[x,t] < \omega(t,V[x,t])$ for all $x \in W_o$, $t \in I$, and $u(\cdot) = u_o(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that

$$r_M(t) \leqq V_m^{W_o \cap \beta}(t), \text{ all } t \in (t_o, t_o+T) \tag{2.2.11}$$

and

$$r_M(t_o+T) < V_m^{W_o \cap \beta}(t_o+T), \text{ if } J = \bar{I} \tag{2.2.12}$$

(3) Using the following corollary, and the theorems stated above, one may deduce some interesting results concerning the non-controllability of system (EC).

Corollary 2.2.1:

System (EC) is not (α, β, J) semi-controllable if it is not semi-finally stable with respect to the sets $(\alpha, \beta, \{u(\cdot)\}, J)$, for all $u(\cdot) \in S^m$.

Finally, we note that the above theorems are meaningful in the case $\alpha \not\subseteq \beta$ only; so the assumption $\alpha/\beta \neq \emptyset$ is not a restriction.

2.3 Semi-final stability with respect to the sets $(\alpha, \beta, U, t_1 \in J)$:

(1) Let $J^* = [t_o, t_1)$, and define the set $Z = Z(\alpha, J^*, U)$, $\alpha \subseteq R^n$, $U \subseteq S^m$, as follows

$$Z = Z(\alpha, J^*, U) = \{x \mid x = x(t; x_o, t_o, u(\cdot)), \text{ for all } x_o \in \alpha,\ t \in J^*,\ u(\cdot) \in U\} \tag{2.3.1}$$

Theorem 2.3.1:

System (EC) is semi-finally stable with respect to the sets (α, β, U, $t_1 \varepsilon J$), $t_1 < t_o+T$, if there exist a set γ, $\bar{\gamma} \subset I(\beta)$, two functions $V[x,t] \varepsilon L[\bar{Z}/I(\gamma), \overline{J^*}]$, and $\omega(t,r) \varepsilon \Omega^*$, such that

(i) the system is semi-finally stable with respect to the sets (α, γ, U, J^*).

(ii) $V_M^{Z \cap Fr.\gamma}(t)$, $t \varepsilon [t_o, t_1)$, and $V_m^{\bar{Z}/\beta}(t_1)$ are finite.

(iii) $V^*[x,t] < \omega(t, V[x,t])$, for all $t \varepsilon \overline{J^*}$, $x \varepsilon \bar{Z}/I(\gamma)$, $u(\cdot) \varepsilon U$.

(iv) for each $t_2 \varepsilon [t_o, t_1)$, the maximal solution $r_M(t)$, $r_M(t_2) = V_M^{Z \cap Fr.\gamma}(t_2)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t_1) \leqq V_m^{\bar{Z}/\beta}(t_1) \tag{2.3.2}$$

Obviously Theorem 2.2.1 - I is a special case of this theorem. To see this, it will be sufficient to take $\omega(t,r) \equiv \phi(t)$ and $\gamma \supseteq \alpha$.

Remark 2.3.1:

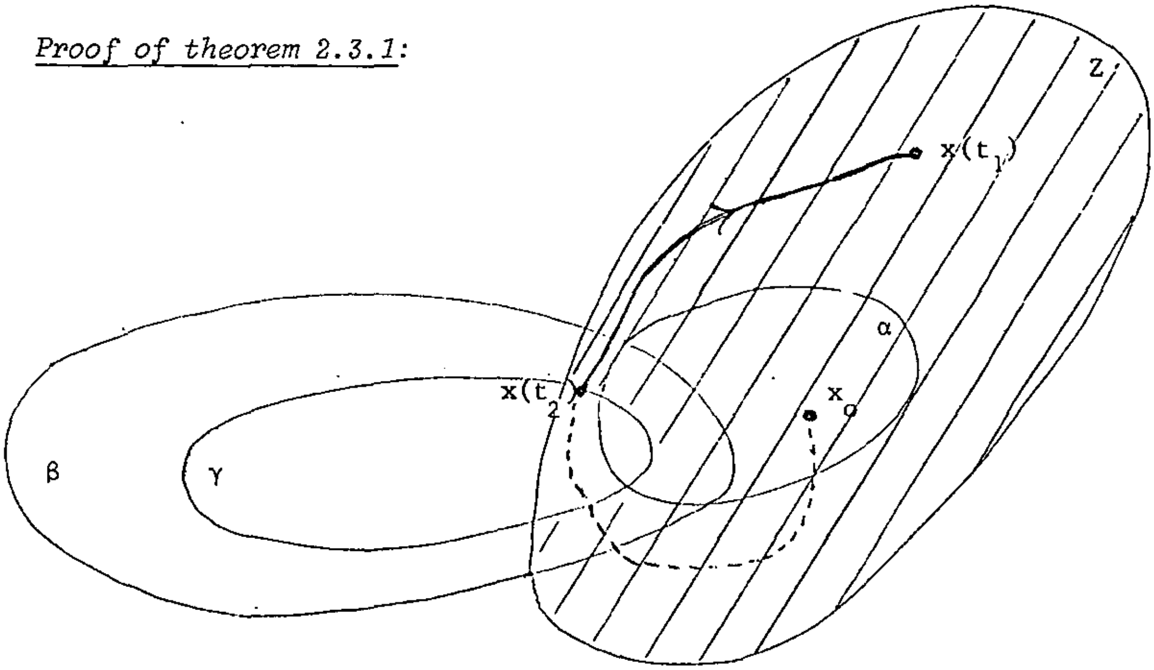
To include the case where $t_1 = t_o+T$, one can modify the assumptions of the theorem in the following manner: we assume that $\omega \varepsilon \Omega$ and

(iii)* $V^*[x,t] \leqq \omega(t, V[x,t])$, all $t \varepsilon J^*$, $x \varepsilon Z/I(\gamma)$, $u(\cdot) \varepsilon U$.

(iv)* $r_M(t_1) < V_m^{\bar{Z}/\beta}(t_1)$

We note, however, that if $t_1 = t_o+T$, then the conclusion of the above theorem is, in fact, final-stability with respect to the sets $(\alpha, \beta, U, \bar{I})$.

Proof of theorem 2.3.1:



$\bar{Z}/I(\gamma)$ = shaded area

Figure 2.3.1

Let $x(t) = x(t; x_o, t_o, u(\cdot))$, $x_o \in \alpha$, $u(\cdot) \in U$, be any trajectory of (EC). Then, by (i), there exists a $t^* \in J^*$ such that

$$x(t^*) \in \gamma \cap \bar{Z} \tag{2.3.3}$$

(where Z is given by (2.3.1)).

Suppose, contrary to the expected conclusion that

$$x(t_1) \in \beta^c \cap \bar{Z} = \bar{Z}/\beta \tag{2.3.4}$$

Since $\bar{\gamma} \subset I(\beta)$, and using (2.3.3), (2.3.4) and (2.3.1) we conclude that there exists a $t_2 \in [t^*, t_1)$ such that

$$x(t_2) \in Z \cap \mathrm{Fr}.\gamma,\ x(t) \in \bar{Z}/\bar{\gamma},\ \text{all } t \in (t_2,t_1] \tag{2.3.5}$$

(refer to Figure 2.3.1).

This relation, together with (ii), (iii) and Lemma 1.3.2, gives

$$V[x(t),t] < r_M(t),\ \text{all } t \in (t_2,t_1] \tag{2.3.6}$$

where $r_M(t)$ is given by (iv). Using (iv) again, we conclude that

$$V[x(t_1),t_1] < V_m^{\bar{Z}/\beta}(t_1) \tag{2.3.7}$$

which is an obvious contradiction to (2.3.4). Thus, $x(t_1) \in \beta$, the theorem follows.

(2) The assumption, in the above theorem, that there exists a set γ, $\bar{\gamma} \subset I(\beta)$, makes it impossible to apply this theorem to the case where β is a point in R^n. Hence, it is not possible to conclude anything about the $(\{x_o\}, \{x_1\}, t_1 \in J)$ semi-controllability. Note also that we have to verify condition (i) by means of the theory established in Section 2.1 - II or any other appropriate method. (This applies in the case $\alpha \not\subset I(\beta)$ only.) The following theorems avoid these difficulties.

Theorem 2.3.2:

System (EC) is semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \in J)$, if there exist two functions $V[x,t] \in L[\bar{Z}, \bar{J^*}]$, and $\omega(t,r) \in \Omega$ such that

(i) $V_M^{\alpha}(t_o)$ and $V_m^{\bar{Z}/\beta}(t_1)$ are finite.

(ii) $V^*[x,t] \leq \omega(t,V[x,t])$, all $t \in J^*$, $x \in Z$, $u(\cdot) \in U$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha}(t_o)$, of equation (C) is such that

$$r_M(t_1) < V_m^{\bar{Z}/\beta}(t_1) \tag{2.3.8}$$

Remark 2.3.2:

The conditions of the above theorem may be modified, provided $t_1 \neq t_o+T$, to relax the inequality (2.3.8). This can be done as follows:
$\omega \in \Omega^*$, and

(ii)* $V^*[x,t] < \omega(t,V[x,t])$, $x \in \bar{Z}$, $t \in \overline{J^*}$, $u(\cdot) \in U$.

(iii) $r_M(t_1) \leqq V_m^{\bar{Z}/\beta}(t_1)$

Remark 2.3.3:

Using Lemma 1.3.1, and the conditions of the theorem, one can show that

$$r_M(t_1) \geqq V[x(t_1),t_1] \tag{2.3.9}$$

for any trajectory emanating from α at $t = t_o$. (2.3.9) then gives

$$r_M(t_1) \geqq V[x(t_1),t_1] \geqq V_m^{\bar{Z}}(t_1) \tag{2.3.10}$$

This inequality and (2.3.8) yield

$$V_m^{\bar{Z}}(t_1) < V_m^{\bar{Z}/\beta}(t_1) \tag{2.3.11}$$

which makes it impossible to reduce β to a point $x_1 \in R^n$.

The following theorem avoids the restriction (2.3.11) implied by the conditions of Theorem 2.3.2.

Theorem 2.3.3:

System (EC) is semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \varepsilon J)$, if there exist two functions $V[x,t] \varepsilon L[\bar{Z}, J^*]$, and $\omega(t,r) \varepsilon \Omega$, such that

(i) $V_M^{\alpha}(t_o)$ is finite,

(ii) for any continuous n-vector function $c(t)$, the conditions:

(a) $c(t_o) \varepsilon \alpha$,

(b) $c(t_1) \varepsilon \bar{Z}/\beta$

imply that

$$\lim_{t \to t_1} V[c(t),t] = +\infty$$

(ii) $V^*[x,t] \leqq \phi(t,V[x,t])$, all $t \varepsilon J^*$, $x \varepsilon Z$, $u(\cdot) \varepsilon U$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha}(t_o)$, of equation (C) is such that $r_M(t_1)$ is bounded above.

Corollary 2.3.1:

System (EC) is $(\{x_o\}, \{x_1\}, t_1 \varepsilon \bar{I})$ semi-controllable, if there exist a control $u^*(\cdot) \varepsilon S^m$, two functions $V[x,t] \varepsilon L[\overline{W_o^*}, J^*]$, and $\omega(t,r) \varepsilon \Omega$, where

$$W_o^* = W_o(x_o, u^*, J^*). \quad [\text{Refer to (2.2.1)}] \qquad (2.3.12)$$

such that

(i) $V[x_o,t_o]$ is finite,

(ii) for any continuous n-vector function $c(t)$, the conditions

(a) $c(t_o) = x_o$,

(b) $c(t_1) \neq x_1$

imply that

$$\lim_{t \to t_1} V[c(t),t] = +\infty$$

(iii) $V^*[x,t] \leq \omega(t,V[x,t])$, all $t \varepsilon J^*$, $x \varepsilon W_o^*$, $u(\cdot) = u^*(\cdot)$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that $r_M(t_1)$ is bounded above.

This result is similar in many respects to a controllability theorem established in [12]. We note, however, that the assumptions are much milder.

(3) We note, finally, that a theorem generalising theorem 2.2.2 - I can be stated and established in a similar manner.

2.4 Non semi-final stability with respect to the sets (α, β, U, t_1 ε J)

Let $J^* = [t_o,t_1)$, and W_o^* be given by (2.3.12), i.e.

$$W_o^* \supseteq \{x | x = x(t; x_o,t_o,u^*(\cdot)), \text{ all } t \varepsilon J^*\}, \text{ for any } x_o \varepsilon R^n,\ u^*(\cdot) \varepsilon S^m. \quad (2.3.12)$$

Theorem 2.4.1: $(t_1 < t_o+T)$.

System (EC) is not semi-finally stable with respect to the sets (α, β, U, t_1 ε J), if there exist a point $x_o \varepsilon \alpha$, $u^*(\cdot) \varepsilon U$, and two functions $V[x,t] \varepsilon L[\overline{W_o^*}, \overline{J^*}]$, $\omega(t,r) \varepsilon \Omega^*$, such that

(i) $V_m^{\overline{W_o^*}\cap\beta}(t_1)$ is finite.

(ii) $V^*[x,t] < \omega(t,V[x,t])$ all $t \varepsilon \overline{J^*}$, $x \varepsilon \overline{W_o^*}$, and $u(\cdot) = u^*(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation

$$\dot{r} = \omega(t,r) \quad (C)$$

is such that

$$r_M(t_1) \leqq V_m^{\overline{W_o^*}\cap\beta}(t_1) \tag{2.4.1}$$

Remark 2.4.1:

The case $t_1 = t_o+T$ is treated, in a similar way, by assuming that $\omega \in \Omega$, and (replacing $\overline{J_o^*}$ by J_o^*)

(ii)* $V^*[x,t] \leqq \omega(t, V[x,t])$, all $t \in J_o^*$, $x \in W_o^*$, and $u(\cdot) = u^*(\cdot)$,

and

$$r_M(t_1) < V_m^{\overline{W_o^*}\cap\beta}(t_1), \tag{2.4.1*}$$

Theorem 2.4.2:

System (EC) is not semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \in J)$, if there exist a point $x_o \in \alpha$, a control $u^*(\cdot) \in U$, two functions $V[x,t] \in L[\overline{W_o^*}, J^*]$, $\omega(t,r) \in \Omega$, such that

(i) for any continuous n-vector function $c(t)$, the conditions:

(a) $c(t_o) = x_o$

(b) $c(t_1) \in \overline{W_o^*}\cap \beta$,

imply that

$$\lim_{t\to t_1} V[c(t),t] = +\infty \tag{2.4.2}$$

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, for all $t \in J^*$, $x \in W_o^*$, $u(\cdot) = u^*(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that $r_M(t_1)$ is bounded above.

Proof of Theorem 2.4.2:

(Theorem 2.4.1 can be proved in a similar manner)

Let $x(t) = x(t;x_o,t_o,u^*(\cdot))$, x_o and $u^*(\cdot)$ given by the conditions of the theorem, be any trajectory of (EC). Assume, contrary to the expected conclusion, that

$$x(t_1) \in \overline{W_o^*} \cap \beta \tag{2.4.3}$$

then (2.4.2), (2.4.3) and (iii) give

$$\lim_{t \to t_1} V[x(t),t] = +\infty$$

On the other hand, (ii), (iii) and Lemma 1.3.1 yield the required contradiction, i.e., $V[x(t_1),t_1]$ is bounded above. The theorem follows.

Comments:

(a) A theorem, similar to theorem 2.4.2 - I, can be deduced by introducing the function $\phi(t)$ and bearing in mind that the solution $r(t)$, $r(t^*) = r^*$, is given by

$$r(t) = r^* + \int_{t^*}^{t} \phi(s)ds$$

(b) Using Theorems 2.4.1 and 2.4.2, and the following corollary, one can deduce some results concerning the non $(\alpha, \beta, t_1 \in J)$ semi-controllability.

Corollary 2.4.1:

System (EC) is not $(\alpha, \beta, t_1 \in J)$ semi-controllable, if it is not semi-finally stable with respect to the sets $(\alpha, \beta, \{u(\cdot)\}, t_1 \in J)$, all $u(\cdot) \in S^m$.

Thus, using Theorem 2.4.2, we get

Corollary 2.4.2:

System (EC) is not $(\{x_o\}, \{x_1\}, t_1 \in J)$ semi-controllable, if there exist two functions $V[x,t] \in L[\overline{W_o^*}, J^*]$, $\omega(t,r) \in \Omega$, such that

(i) for any continuous n-vector function c(t), the conditions:

(a) $c(t_o) = x_o$,

(b) $c(t_1) = x_1$

imply that

$$\lim_{t \to t_1} V[c(t),t] = +\infty$$

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \in J^*$, $x \in W_o^*$, and all $u(\cdot) \in S^m$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that $r_M(t_1)$ is bounded above.

2.5 Strong semi-final stability:

(1) The following theorem is a direct generalisation of Theorem 2.3.1.- I. A similar theorem to Theorem 2.3.2 - I can be stated and proved in a similar manner.

Theorem 2.5.1:

System (EC) is strongly semi-finally stable with respect to the sets (α, β, U, J), $\alpha^c/\beta \neq \emptyset$, if

(a) it is semi-finally stable with respect to the sets (α, β, U, J), and

(b) there exist two functions $V[x,t] \in L[N,J]$, and $\omega(t,r) \in \Omega^*$, where

$$N = N(\alpha, \beta, U, J) = \{x | x = x(t; x_o,t_o,u(\cdot)), \text{ for all } x_o \in \alpha^c/\beta,\ t \in J,\ u(\cdot) \in U\} \quad (2.5.1)$$

such that

(i) $V_M^{\alpha^c/\beta}(t_o)$ and $V_m^{N\cap\beta}(t)$, all $t \in J$, $t > t_o$, are finite.

(ii) $V^*[x,t] < \omega(t,V[x,t])$, for all $t \in I$, $x \in N$, $u(\cdot) \in U$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha^c/\beta}(t_o)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t) \leqq V_m^{N\cap\beta}(t), \text{ all } t \in (t_o,t_o+T) \tag{2.5.2}$$

and

$$r_M(t_o+T) < V_m^{N\cap\beta}(t_o+T), \text{ if } J = \bar{I} \tag{2.5.3}$$

Remark 2.5.1:

The additional condition (2.5.3) is due to the fact that inequality 1.3.8 - II need no longer be strict at $t = t_o+T$. (Ref. Remark 1.4.1 - II).

The condition (2.5.3) is not necessary in the case where β is an open set; for it will be sufficient to show that any trajectory $x(t) = x(t; x_o,t_o,u(\cdot))$, $x_o \in \alpha^c/\beta$, $u(\cdot) \in U$, is such that $x(t) \in \beta^c$, all $t \in (t_o, t_o+T)$, and hence $x(t_o + T) \in \beta^c$.

We can, however, avoid this restriction by changing the conditions of the theorem as follows: $\omega \in \Omega$, and

(ii)* $V^*[x,t] \leqq \omega(t,V[x,t])$, and $r_M(t) < V_m^{N\cap\beta}(t)$, all $t\in J$, $t > t_o$.

Proof of Theorem 2.5.1:

Let $x(t) = x(t; x_o,t_o,u(\cdot))$, $x_o \in \alpha^c/\beta$, $u(\cdot) \in U$, be any trajectory of (EC). Suppose, contrary to the expected conclusion, that

$$x(t_2) \in \beta, \text{ for some } t_2 \in J,\ t_2 > t_o \tag{2.5.4}$$

By (2.5.1), (2.5.4) gives

$$x(t_2) \in N \cap \beta \tag{2.5.5}$$

On the other hand, application of (i), (ii) and Lemma 1.3.2 gives

$$V[x(t),t] < r_M(t), \text{ all } t \in (t_o, t_o+T) \tag{2.5.6}$$

$$\left(V[x(t_o+T),\ t_o+T] \leqq r_M(t_o+T)\right).$$

which, either by (2.5.2) or (2.5.3), yields

$$V[x(t_2),t_2] < V_m^{N\cap\beta}(t_2) \tag{2.5.7}$$

(the application of (2.5.3) is needed if $t_2 = t_o+T$). This is the required contradiction. <u>The theorem follows</u>.

(2) One can show that a system (EC) is not strongly semi-finally stable by the negation of either parts (i) or (ii) of Definition 1.2.6 (adapted to the case under consideration). The negation of part (i) has been investigated in section 2.2 - II. We concentrate then on the negation of part (ii), i.e., we have to show that there exists a trajectory $x(t) = x(t;\ x_o, t_o, u^*(\cdot))$, some $x_o \in \alpha^c/\beta$, $u^*(\cdot) \in U$, such that $x(t_2) \in \beta$, for some $t_2 \in J$. This will be done via the following theorems.

<u>Theorem 2.5.2:</u>

System (EC) is not strongly semi-finally stable with respect to the sets (α, β, U, J), $\alpha^c/\beta \neq \emptyset$, if there exist $x_o \in \alpha^c/\beta$, $u^*(\cdot) \in U$, $t_1 \in J (t_1 > t_o)$, two functions $V[x,t] \in L[\overline{W_o^*}, \overline{J^*}]$, and $\omega(t,r) \in \Omega$, where $J^* = [t_o, t_1)$, and

$$W_o^* \supseteq \{x | x = x(t;\ x_o, t_o, u^*(\cdot)),\ \text{all } t \in J^*\} \qquad (2.3.12)$$

such that

(i) $V_m^{\overline{W_o^*/\beta}}(t_1)$ is finite.

(ii) $\dot{V}^*[x,t] \leqq \omega(t, V[x,t])$, all $t \in J^*$, $x \in W_o^*$, and $u(\cdot) = u^*(\cdot)$.

(iii) the maximal solution $r_M(t_o)$, $r_M(t_o) = V[x_o, t_o]$, of equation (C) is such that

$$r_M(t_1) < V_m^{\overline{W_o^*/\beta}}(t_1) \qquad (2.5.8)$$

Theorem 2.5.3:

System (EC) is not strongly semi-finally stable with respect to the sets $(\alpha,\ \beta,\ U,\ J)$, $\alpha^c/\beta \neq \emptyset$, if there exist $x_o \in \alpha^c/\beta$, $u^*(\cdot) \in U$, $t_1 \in J$, two functions $V[x,t] \in L[\overline{W_o^*},\ J^*]$, and $\omega(t,r) \in \Omega$, such that

(i) for any continuous n-vector function $c(t)$, the condition

(a) $c(t_o) = x_o$,

(b) $c(t_1) \in W_o^*/\beta$,

imply that

$$\lim_{t \to t_1} V[c(t), t] = +\infty \qquad (2.5.9)$$

(ii) $\dot{V}^*[x,t] \leqq \omega(t, V[x,t])$, all $t \in J^*$, $x \in W_o^*$, and $u(\cdot) = u^*(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o, t_o]$, of equation (C) is such that $r_M(t_1)$ is bounded above.

(3) A similar theory, concerning the strong semi-final stability of system (EC) with respect to the sets $(\alpha,\ \beta,\ U,\ t_1 \in J)$, can be established in the same way.

§3. GENERAL THEOREMS ON THE DIFFERENT TYPES OF FINAL-STABILITY

3.1 Final-Stability:

(1) As in Section 3.1 - I, we shall assume that β is an open set in the case $J = I$. Of course, if $J = \bar{I}$, then β need not necessarily be open.

Let $K = K(\alpha, U, J)$ be the set

$$K = K(\alpha, U, J) = \{x \mid x = x\big(t;\, x_o, t_o, u(\cdot)\big),\ \text{all}\ x_o \in \alpha,\ t \in J,\ u(\cdot) \in U\} \tag{3.1.1}$$

Theorem 3.1.1:

System (EC) is finally-stable with respect to the sets (α, β, U, J), if there exist two functions $V[x,t] \in L[\bar{K},J]$, and $\omega(t,r) \in \Omega$, such that:

(i) $V_M^{\alpha}(t_o)$ and $V_m^{\bar{K}/\beta}(t_o+T)$ are finite.

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \in I$, $x \in \bar{K}$, $u(\cdot) \in U$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha}(t_o)$, of equation (C) is such that

$$r_M(t_o+T) < V_m^{\bar{K}/\beta}(t_o+T) \tag{3.1.2}$$

Theorem 3.1.2:

System (EC) is finally-stable with respect to the sets (α, β, U, J), if there exist two functions $V[x,t] \in L[\bar{K},I]$, and $\omega(t,r) \in \Omega$, such that

(i) $V_M^{\alpha}(t_o)$ is finite.

(ii) for any continuous n-vector function $c(t)$ the conditions:

(a) $c(t_o)$

(b) $c(t_o+T) = \lim_{t\to t_o+T} c(t) \varepsilon \bar{K}/\beta$

imply that

$$\lim_{t\to t_o+T} V[c(t),t] = +\infty \tag{3.1.3}$$

(iii) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \varepsilon I$, $x \varepsilon K$, $u(\cdot) \varepsilon U$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^\alpha(t_o)$, of equation (C) is such that $r_M(t_o+T)$ is bounded above.

Proof of Theorem 3.1.1:

(Theorem 3.1.2 can be proved in a similar manner.)

Assuming that any trajectory $x(t) = x(t; x_o, t_o, u(\cdot))$, $x_o \varepsilon \alpha$, $u(\cdot) \varepsilon U$, satisfies the relation

$$x(t) \varepsilon K/\beta \quad \text{all } t \varepsilon J,\ t \geqq t^* \text{ (for some } t^* \varepsilon J) \tag{3.1.4}$$

will lead to the conclusion that

$$x(t_o+T) \varepsilon \bar{K}/\beta \tag{3.1.5}$$

But using (ii), (iii) and Lemma 1.3.1 leads to the required contradiction, i.e.

$$V[x(t_o+T),t_o+T] < V_m^{\bar{K}/\beta}(t_o+T).$$

This completes the proof of Theorem 3.1.1.

(2) In all the above theorems, the assumption has been made that $\lim_{t\to t_o+T} x(t)$ exists. This need not be true in the case where $J = I$.

The following theorem avoids such an assumption.

Theorem 3.1.3:

System (EC) is finally stable with respect to the sets (α, β, U, J), if there exist two functions $V[x,t] \in L[K,J]$, $\omega(t,r) \in \Omega$, such that

(i) $V_m^{K/\beta}(t)$ is finite, for all $t \in J$, $t > t_o$.

(ii) $V^*[x,t] \leq \omega(t,V[x,t])$, all $t \in I$, $x \in K$, $u(\cdot) \in U$.

(iii) for each $r_o \in B$, there corresponds a $t(r_o) \in J$, such that the maximal solution $r_M(t;\ r_o,t_o)$ of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t;\ r_o,t_o) < V_m^{K/\beta}(t),\ \text{all } t \in J,\ t \geq t(r_o), \tag{3.1.6}$$

where B is a subset of R^1 with the following property

$$V[x_o,t_o] \in B,\ \text{all } x_o \in \alpha \tag{3.1.7}$$

(3) Now, let $\alpha \equiv \{x_o\}$, $B \equiv \{x_1\}$, $J = \bar{I}$, then Theorem 3.1.2 gives the following interesting corollary:

Corollary 3.1.1:

System (EC) is $(\{x_o\}, \{x_1\}, J)$-controllable, if there exist a control $u^*(\cdot) \in S^m$, two functions $V[x,t] \in L[R^n,I]$, $\omega(t,r) \in \Omega$, such that

(i) for any continuous n-vector function c(t), the condition

(a) $c(t_o) = x_o$,

(b) $c(t_o+T) = \lim_{t\to t_o+T} c(t) \neq x_1$

imply that

$$\lim_{t \to t_o + T} V[c(t),t] = +\infty$$

(ii) $V^*[x,t] \leq \omega(t,V[x,t]$, all $t \in I$, $x \in R^n$, $u(\cdot) = u^*(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that $r_M(t_o+T)$ is bounded above.

This corollary includes the controllability theorem established in [12].

3.2 Non final-stability

(1) *Theorem 3.2.1:*

System (EC) is not finally-stable with respect to the sets (α, β, U, J), if there exist $x_o \in \alpha$, $u_o(\cdot) \in U$, $t_1 \in J$, and two functions $V[x,t] \in L[W_o,J]$, $\omega(t,r) \in \Omega^*$, where

$$W_o = W_o(x_o,u_o,J). \quad \text{[Ref. to (2.2.1) - II]} \tag{3.2.1}$$

such that

(i) $V_m^{W_o \cap \beta}(t)$, $t \in J$, $t \geq t_1$, is finite

(ii) $V^*[x,t] < \omega(t,V[x,t])$, all $t \in I$, $x \in W_o$, and $u(\cdot) = u_o(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C)

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t) \leq V_m^{W_o \cap \beta}(t), \text{ all } t \in (t_1,t_o+T) \tag{3.2.2}$$

and

$$r_M(t_o+T) < V_m^{W_o \cap \beta}(t_o+T), \text{ if } J = \bar{I} \tag{3.2.3}$$

The last inequality (3.2.3) is needed, for the conclusion of Lemma 1.3.2 need not be true at $t = t_o+T$. As before, one can avoid this distinction by replacing (3.2.2) and (3.2.3) by

$$r_M(t) < V_m^{W_o \cap \beta}(t), \text{ all } t \in J, \; t > t_1 \qquad (3.2.2)^*$$

In this case (ii) is no longer required to be a strict inequality, and it will be sufficient to assume that $\omega \in \Omega$ only.

The following theorem avoids the above situation. However, we must assume either:

(a) $J = \bar{I}$

or

(b) β is closed, and any trajectory $x(t) = x(t; x_o, t_o, u(\cdot))$, $x_o \in \alpha$, is such that $\lim_{t \to t_o+T} x(t)$ exists.

Theorem 3.2.2:

System (EC) is not finally-stable with respect to the sets (α, β, U, J), if there exist $x_o \in \alpha$, $u_o(\cdot) \in U$, two functions $V[x,t] \in L[\bar{W}_o, I]$, and $\omega(t,r) \in \Omega$, such that

(i) for any continuous n-vector function $c(t)$, the conditions:

(a) $c(t_o) = x_o$

(b) $c(t_o+T) = \lim_{t \to t_o+T} c(t) \in \bar{W}_o \cap \beta$

imply that

$$\lim_{t \to t_o+T} V[c(t),t] = +\infty \qquad (3.2.4)$$

(ii) $V^*[x,t] \leqq \omega(t, V[x,t])$, all $t \in I$, $x \in W_o$, $u(\cdot) = u_o(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that $r_M(t_o+T) = \lim_{t\to t_o+T} r_M(t)$ is bounded above.

The above theorems are proved in the usual manner. Note that if, in the proof of Theorem 3.2.2, we assume that for any trajectory $x(t) = x(t; x_o, t_o, u_o(\cdot))$, $x_o \varepsilon \alpha$, we have the relation

$$x(t) \varepsilon \beta, \text{ all } t \varepsilon J, t \geqq t^*, \text{ for some } t^* \varepsilon J$$

then, by the assumption that either β is closed, or $J = \bar{I}$, we conclude that $x(t_o+T) \varepsilon \beta$.

(2) The theorem which follows is different, in many aspects, from the above results. We shall assume, however, that β is closed.

Theorem 3.2.3:

System (EC) is not finally-stable with respect to the sets (α, β, U, J), if there exist sets $\alpha^* \subseteq \alpha$, $\beta^* \subseteq W_o/\beta$, for some $x_o \varepsilon \alpha^*$, $u_o(\cdot) \varepsilon U$, and two functions $V[x,t] \varepsilon L[W_o/I(\beta), J]$, $\omega(t,r) \varepsilon \Omega$, such that

(i) system (EC) is semi-finally stable with respect to the sets (α^*, β^*, U^*, J), for some $U^* \subseteq U$, such that $u_o(\cdot) \varepsilon U^*$.

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \varepsilon I$, $x \varepsilon W_o/I(\beta)$, $u(\cdot) = u_o(\cdot)$.

(iii) for each $t_1 \varepsilon J$, the maximal solution $r_M(t)$, $r_M(t_1) = V_M^{W_o \cap \beta^*}(t_1)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t) < V_m^{W_o \cap Fr.\beta}(t), \text{ all } t > t_1, t \varepsilon J \tag{3.2.5}$$

provided, of course, $V_M^{W_o \cap \beta^*}(t)$, all $t \in J$, and $V_m^{W_o \cap Fr.\beta}(t)$, all $t \in J$, $t > t_o$, are finite.

To avoid the restriction that β is closed, we require that $\beta^* \cap \bar{\beta} = \emptyset$. Moreover, if $\alpha/\bar{\beta} \neq \emptyset$, then we may take $\alpha^* = \beta^* = \{x_o\}$, $x_o \in \alpha/\bar{\beta}$, so that condition (i) of the above theorem is automatically satisfied. Thus, we arrive at the following corollary:

Corollary 3.2.1:

System (EC) is not finally-stable with respect to the sets (α, β, U, J), $\alpha/\bar{\beta} \neq \emptyset$, if there exist a point $x_o \in \alpha/\bar{\beta}$, a control $u_o(\cdot) \in U$, and two functions $V[x,t] \in L[W_o/I(\beta), J]$, $\omega(t,r) \in \Omega$, such that

(i) $V_m^{W_o \cap Fr.\beta}(t)$, $t \in J$, $t > t_o$, is finite.

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \in I$, $x \in W_o/I(\beta)$, $u(\cdot) = u_o(\cdot)$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that

$$r_M(t) < V_m^{W_o \cap Fr.\beta}(t), \text{ all } t \in J, \ t > t_o, \tag{3.2.6}$$

Proof of Theorem 3.2.3:

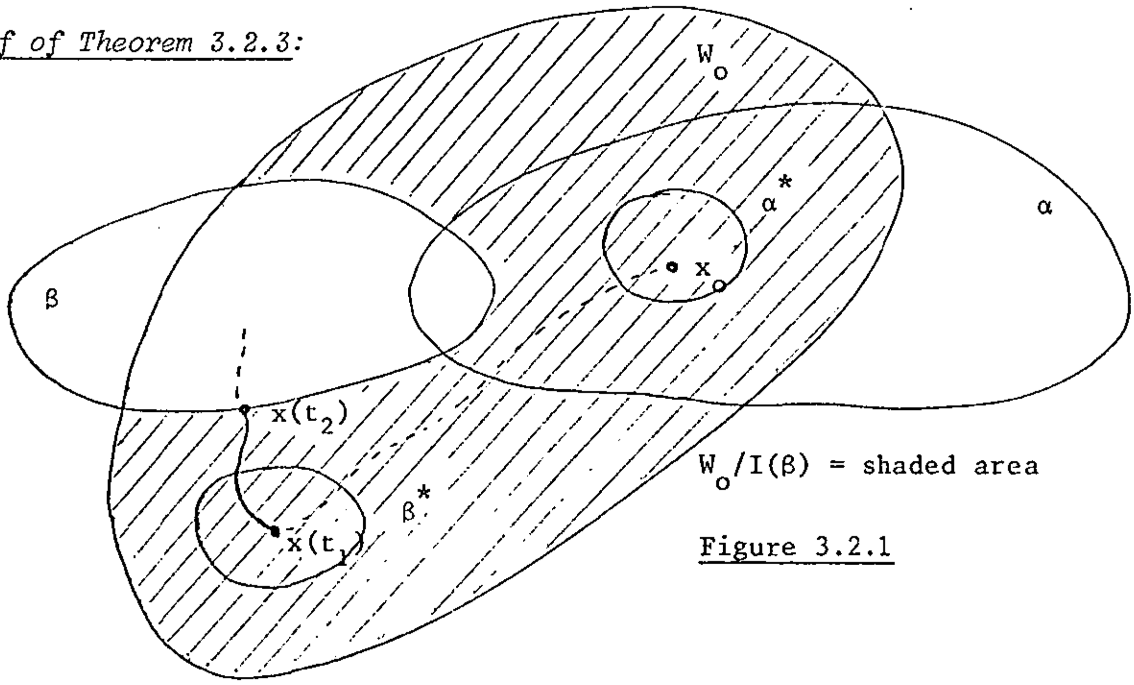


Figure 3.2.1

(a) Suppose, contrary to the expected conclusion, that system (EC) is finally stable with respect to the sets (α, β, U, J).

(b) For the given $x_o \in \alpha^*$, and $u_o(\cdot) \in U^* \subseteq U$, let $x(t) = x(t; x_o, t_o, u_o(\cdot))$ be any trajectory of (EC). By (a), there exist a $t^* \in J$, such that

$$x(t) \in \beta \cap W_o \quad , \quad \text{all } t \in J,\ t \geqq t^* \qquad (3.2.7)$$

By (i), there exists a $t_1 \in J$, $t_1 < t^*$, such that

$$x(t_1) \in \beta^* \qquad (3.2.8)$$

Using (3.2.7) and (3.2.8), we conclude that there exists a $t_2 \in (t_1, t^*]$ such that

$$x(t_2) \in W_o \cap \text{Fr}.\beta,\ x(t) \in W_o/\beta,\ \text{all } t \in (t_1, t_2) \qquad (3.2.9)$$

But, then, applying (ii), (iii) and Lemma 1.3.1, we conclude that

$$V[x(t), t] \leqq r_M(t),\ \text{all } t \in [t_1, t_2] \qquad (3.2.10)$$

which, by (3.2.5), gives the contradictory inequality

$$V[x(t_2), t_2] < V_m^{W_o \cap \text{Fr}.\beta}(t_2) \qquad (3.2.11)$$

Hence, the original assumption (a) is false. <u>This completes the proof of Theorem 3.2.3</u>.

(3) Now, we apply the above results to give some corollaries concerning non-controllability. This will be done by means of the following corollary.

Corollary 3.2.2:

System (EC) is not $(\alpha,\ \beta,\ \bar{I})$ controllable, if it is not finally-stable with respect to the sets $(\alpha,\ \beta,\ \{u(\cdot)\},\ \bar{I})$ for all $u(\cdot) \in S^m$.

Thus, Theorem 3.2.1 gives, by setting $t_1 = t_o+T$,

Corollary 3.2.3:

System (EC) is not $(\{x_o\},\ \{x_1\},\ \bar{I})$ controllable, if there exist two functions $V[x,t] \in L[R^n,\ \bar{I}]$, and $\omega(t,r) \in \Omega$, such that

(i) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \in I$, $x \in R^n$, and all $u(\cdot) \in S^m$.

(ii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t_o+T) < V[x_1,\ t_o+T] \tag{3.2.12}$$

It is to be noted that application of Lemma 1.3.1 gives

$$r_M(t) \geqq V[x(t),t] \tag{3.2.13}$$

for any trajectory $x(t) = x(t;\ x_o,t_o,u(\cdot))$, which means, by (3.2.12), that

$$V_m^{R^n}(t_o+T) < V[x_1,\ t_o+T] \tag{3.2.14}$$

That is to say, the function V has to be chosen so that $V[x,t_o+T]$ does not reach its infimum at $x = x_1$.

Corollary 3.2.4:

[Refer to Theorem 3.2.2. - II]

System (EC) is not $(\{x_o\}, \{x_1\}, \bar{I})$-controllable, if there exist two functions $V[x,t] \in L[R^n,I]$, and $\omega(t,r) \in \Omega$, such that

(i) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \in I$, $x \in R^n$, $u(\cdot) \in S^m$.

(ii) the maximal solution $r_M(t)$, $r_M(t_o) = V[x_o,t_o]$, of equation (C) is such that $r_M(t_o+T)$ is bounded above.

(iii) for any continuous function $c(t)$ (n-vector), the conditions:

$$c(t_o) = x_o, \qquad c(t_o+T) = \lim_{t \to t_o+T} c(t) = x_1$$

imply that

$$\lim_{t \to t_o+T} V[c(t),t] = +\infty \tag{3.2.15}$$

Remark 3.2.1:

R^n, in the above results, may be replaced by the set K, given by (3.1.1), provided $\alpha \equiv \{x_o\}$, $U \equiv S^m$.

Remark 3.2.2:

We note that, setting $t_1 = t_o+T$ in Corollary 2.4.2 - II, we can deduce Corollary 3.2.4 - II. The difference between the two corollaries lies in the fact that Corollary 2.4.2 does not tell us what happens to the system (EC) after t_1, while the other corollary ensures that any trajectory, emanating from x_o, does not arrive at x_1 at some time t^* and remain there for all $t \geqq t^*$, $t \in J$.

3.3 Final-stability with respect to the sets (a, b, U, t_1 ε J):

(1) In the following theorem, we shall assume that β is an open set. This is due to the fact that the conditions of the theorem will imply, as seen from the proof, that any trajectory x(t), emanating from α can never reach Fr.β; so, there is no advantage in considering the case where β is closed. On the other hand, β can not reduce to a point, by the assumed existence of a set γ, $\bar{\gamma} \subset I(\beta)$.

Theorem 3.3.1:

System (EC) is finally-stable with respect to the sets (α, β, U, $t_1 \in J$), β open, if there exist a set γ, $\bar{\gamma} \subset \beta$, two functions $V[x,t] \in L[K \cap (\bar{\beta} - I(\gamma)), J_1]$, $\omega(t,r) \in \Omega^*$, where $J_1 : t \in J, t \geqq t_1$, and K given by (3.1.1) - II; such that

(i) system (EC) is semi-finally stable with respect to the sets (α, γ, U, $t_1 \in J$)

(ii) $V_M^{K\cap Fr.\gamma}(t)$, $t \in [t_1, t_o+T)$, $V_m^{K\cap Fr.\beta}(t)$, $t \in J$, $t > t_1$, are finite.

(iii) $V^*[x,t] < \omega(t,V[x,t])$, all $t \in [t_1, t_o+T)$, $x \in K \cap (\bar{\beta} - I(\gamma))$, and $u(\cdot) \in U$.

(iv) for each $t_2 \in [t_1, t_o+T)$, the maximal solution $r_M(t)$, $r_M(t_2) = V_M^{K\cap Fr.\gamma}(t_2)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t) \leqq V_m^{K\cap Fr.\beta}(t), \text{ all } t \in (t_2, t_o+T) \tag{3.3.1}$$

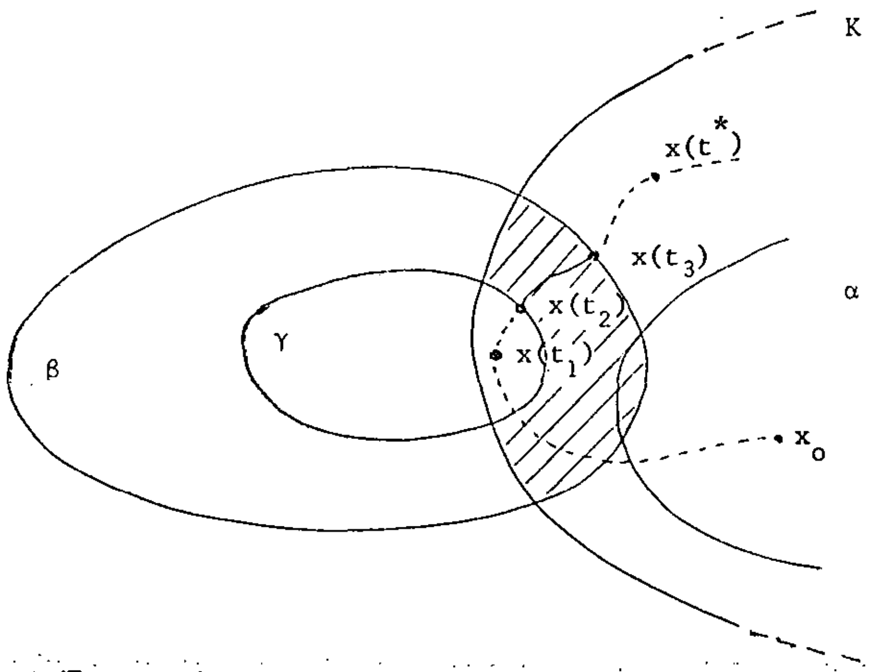
and

$$r_M(t_o+T) < V_m^{K\cap Fr.\beta}(t_o+T), \text{ if } J = \bar{I} \tag{3.3.2}$$

Proof of Theorem 3.3.1:

Let $x(t) = x\big(t; x_o, t_o, u(\cdot)\big)$, $x_o \in \alpha$, $u(\cdot) \in U$, be any trajectory of (EC), then

$$x(t) \in K, \; t \in J, \text{ by (3.1.1) - II} \tag{3.3.3}$$



$K \cap \big(\bar{\beta} - I(\gamma)\big)$ = shaded area

Figure 3.3.1

By (i), we get

$$x(t_1) \in K \cap \gamma \tag{3.3.4}$$

Now, suppose, contrary to the expected conclusion, that there exists a $t^* \in J$, $t^* > t_1$, such that

$$x(t^*) \in K/\beta \tag{3.3.5}$$

then, by (3.3.4), there exist $t_3 \in (t_1, t^*]$, and $t_2 \in [t_1, t_3)$, such that

$$x(t_2) \in K \cap Fr.\gamma,\ x(t_3) \in K \cap Fr.\beta, \text{ and}$$
$$x(t) \in \beta - \bar{\gamma}, \text{ all } t \in (t_2, t_3). \tag{3.3.6}$$

But, then, application of (3.3.6), (iii), (iv) and Lemma 1.3.2 gives

$$V[x(t),t] < r_M(t), \text{ all } t \in (t_2,t_3] \tag{3.3.7}$$

where $r_M(t)$ is given by (iv). By (3.3.1) (and (3.3.2), if $J = \bar{I}$), we conclude that

$$V[x(t_3),t_3] < V_m^{K\cap Fr.\beta}(t_3) \tag{3.3.8}$$

which means that $x(t_3) \notin K \cap Fr.\beta$. Hence, the last inequality (3.3.8) constitutes a contradiction to (3.3.6), in view of the definition of $V_m^{K\cap Fr.\beta}(t)$. Thus, the original assumption (3.3.5) is false and there exists no $t^* \in J$, $t^* > t_1$, such that (3.3.5) holds.

Since the above argument is independent of the choice of the trajectory $x(t) = x(t; x_o, t_o, u(\cdot))$, $x_o \in \alpha$, $u(\cdot) \in U$, it holds for all trajectories emanating from α at $t = t_o$. <u>This completes the proof of the theorem.</u>

(2) In order to show that the theory of final-stability includes, as a special case, the theory of stability over a finite-time interval [18,37], we shall give here the different definitions of stability in terms of the definitions of final-stability; then give the appropriate conclusions. Some of these conclusions include the known results established in [18,37]. The others are new results.

Definition 3.3.1:

System (EC) is stable with respect to the sets (α, β, U, J), $\alpha \subseteq \beta$, if it is finally-stable with respect to the sets $(\alpha, \beta, U, t_o \in J)$.

Definition 3.3.2:

System (EC) is unstable with respect to the sets (α, β, U, J), $\alpha \subseteq \beta$, if it is not finally-stable with respect to the sets $(\alpha, \beta, U, t_o \in J)$.

Definition 3.3.3:

System (EC) is quasi-contractively stable with respect to the sets (α, γ, U, J), $\gamma \subset \alpha$, if it is finally stable with respect to the sets (α, γ, U, J).

Definition 3.3.3 is a generalisation of the corresponding definition given in [18,36], but does not correspond to the definition of quasi-contractive stability given in [37]. The same remark is valid for the following definition.

Definition 3.3.4:

System (EC) is contractively-stable with respect to the sets $(\alpha, \beta, \gamma, U, J)$, $\gamma \subset \alpha \subseteq \beta$, if

(i) it is finally-stable with respect to the sets $(\alpha, \beta, U, t_o \in J)$, and

(ii) it is finally-stable with respect to the sets (α, β, U, J).

Definition 3.3.5:

System (EC) is quasi-expansively stable with respect to the sets (α, β, U, J), $\alpha \subseteq \gamma$, if it is finally-stable with respect to the sets (α, γ, U, J).

Definition 3.3.6:

System (EC) is expansively stable with respect to the sets (α, β, γ, U, J), $\alpha \subseteq \gamma \subset \beta$, if

(i) it is finally-stable with respect to the sets (α, β, U, $t_o \in$ J), and

(ii) it is finally-stable with respect to the sets (α, β, U, J).

We note, furthermore, that these definitions are more general than the ones given in [18,36,37], in the sense that the sets α, β, and γ are more general.

We are now in a position to state some results in connection with the definitions stated above.

Theorem 3.3.2:

System (EC) is stable with respect to the sets (α, β, U, J), $\bar{\alpha} \subset \beta$, β open, if there exist two functions $V[x,t] \in L[K \cap (\bar{\beta} - I(\alpha)), \bar{J}]$, $\omega(t,r) \in \Omega^*$, such that

(i) $V_M^{K\cap Fr.\alpha}(t)$, $t \in I$, and $V_m^{K\cap Fr.\beta}(t)$, $t \in J$, $t > t_o$, are finite.

(ii) $V^*[x,t] < \omega(t,V[x,t])$, all $t \in I$, $x \in K \cap (\bar{\beta} - I(\alpha))$, $u(\cdot) \in U$.

(iii) for each $t_2 \in I$, the maximal solution $r_M(t)$, $r_M(t_2) = V_M^{K\cap Fr.\alpha}(t_2)$, of equation (C) is such that

$$r_M(t) \leqq V_m^{K\cap Fr.\beta}(t),\ \text{all } t \in (t_2,\ t_o+T)$$

and

$$r_M(t_o+T) < V_m^{K\cap Fr.\beta}(t_o+T),\ \text{if } J = \bar{I}$$

This theorem is an immediate consequence of Theorem 3.3.1. It shows clearly that the corresponding theorems established in [18,36,37] are special cases of the theory of final-stability.

Theorem 3.3.3:

System (EC) is contractively stable with respect to the sets $(\alpha, \beta, \gamma, U, J)$, $\gamma \subset \alpha$, $\bar{\alpha} \subset \beta$, β open, if there exist three functions $V[x,t] \in L[\bar{\beta},\bar{I}]$, $\omega_1(t,r) \in \Omega$, and $\omega_2(t,r) \in \Omega^*$, such that

(i) $V_M^{\alpha}(t_o)$, $V_m^{\bar{\beta}-\gamma}(t_o+T)$, $V_M^{Fr.\alpha}(t)$, $t \in J$, $t < t_o+T$, and $V_m^{Fr.\beta}(t)$, $t \in J$, $t > t_o$, are finite.

(ii) $V^*[x,t] \leqq \omega_1(t,V[x,t])$, all $t \in I$, $x \in \bar{\beta}$, $u(\cdot) \in U$.

(iii) $V^*[x,t] < \omega_2(t,V[x,t])$, all $t \in I$, $x \in \bar{\beta} - I(\alpha)$, $u(\cdot) \in U$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha}(t_o)$ of equation

$$\dot{r} = \omega_1(t,r) \tag{C_1}$$

is such that

$$r_M(t_o+T) < V_m^{\bar{\beta}-\gamma}(t_o+T) \tag{3.3.9}$$

(v) for each $t_2 \in [t_o, t_o+T)$, the maximal solution $s_M(t)$, $s_M(t_2) = V_M^{Fr.\alpha}(t_2)$, of equation

$$\dot{s} = \omega_2(t,s) \tag{C_2}$$

is such that

$$s_M(t) \leqq V_m^{Fr.\beta}(t), \text{ all } t \in (t_2, t_o+T) \tag{3.3.10}$$

and

$$s_M(t_o+T) < V_m^{Fr.\beta}(t_o+T), \text{ if } J = \bar{I} \tag{3.3.11}$$

Proof of Theorem 3.3.3:

(a) Bearing in mind that

$$K \cap \left(\bar{\beta} - I(\alpha)\right) \subset \bar{\beta} \tag{3.3.12}$$

we see that conditions (i), (iii), (v) of Theorem 3.3.3 correspond to conditions (ii) - (iii) - (iv) of Theorem 3.3.1, with $t_1 = t_o$, $\gamma = \alpha$. Hence, system (EC) is finally-stable with respect to the sets $(\alpha, \beta, U, t_o \in J)$, i.e., the conditions $x_o \in \alpha$, $u(\cdot) \in U$, imply that

$$x(t) = x\left(t;\, x_o, t_o, u(\cdot)\right) \in \beta, \text{ all } t \in J, \tag{3.3.13}$$

(b) Using (3.3.13), one can see that

$$\bar{K}/\gamma \subseteq \bar{\beta}/\gamma = \bar{\beta} - \gamma \tag{3.3.13}$$

and hence

$$V_m^{\bar{K}/\gamma}(t_o+T) \geqq V_m^{\bar{\beta}-\gamma}(t_o+T) \tag{3.3.14}$$

Thus the functions $V[x,t]$ and $\omega_1(t,r)$ satisfy the assumptions of Theorem 3.1.1 - II, which means that the system (EC) is finally-stable with respect to the sets (α, γ, U, J). <u>This completes the proof of the theorem</u>.

We note that the above theorem is still valid in the case where $\alpha \subseteq \gamma \subset \beta$, $\bar{\alpha} \subset \beta$, β open. Thus, a similar result can be stated concerning the expansive-stability of system (EC) with respect to the sets $(\alpha, \beta, \gamma, J)$.

Remark 3.3.1:

Since Theorem 3.1.1 is used in the proof of the above theorem, we have to assume that the set γ is open, if $J = I$. The assumption that β is open is due to the discussion given in the beginning of the section.

(3) We note that we have avoided until now the discussion of the case where $\alpha = \beta$. This is because the theorem established in (1) cannot be applied to this case. Thus, the following theorems are justified.

Theorem 3.3.4:

System (EC) is finally-stable with respect to the sets (α, β, U, $t_1 \varepsilon J$), if there exist two functions $V[x,t] \varepsilon L[K,J]$, $\omega(t,r) \varepsilon \Omega$, such that

(i) $V_M^{\alpha}(t_o)$ and $V_m^{K/\beta}(t)$, all $t \varepsilon J$, $t \geqq t_1$, are finite.

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, all $t \varepsilon I$, $x \varepsilon K$, $u(\cdot) \varepsilon U$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha}(t_o)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t) < V_m^{K/\beta}(t), \text{ all } t \varepsilon J, t \geqq t_1 \tag{3.3.15}$$

Theorem 3.3.5: *(β open)*

System (EC) is stable with respect to the sets (α, β, U, J), $\alpha \subseteq \beta$, if there exist two functions $V[x,t] \varepsilon L[\bar{\beta},J]$, and $\omega(t,r) \varepsilon \Omega^*$, such that

(i) $V_M^{\alpha}(t_o)$ and $V_m^{Fr.\beta}(t)$, $t \varepsilon J$, $t > t_o$, are finite.

(ii) $V^*[x,t] < \omega(t,V[x,t])$, all $t \varepsilon I$, $x \varepsilon \bar{\beta}$, $u(\cdot) \varepsilon U$.

(iii) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha}(t_o)$, of equation (C) is such that

$$r_M(t) \leqq V_m^{Fr.\beta}(t), \text{ all } t \in (t_o, t_o+T) \tag{3.3.16}$$

and

$$r_M(t_o+T) < V_m^{Fr.\beta}(t_o+T), \text{ if } J = \bar{I} \tag{3.3.17}$$

(4) Finally, we note that in both theorems 3.3.1 and 3.3.4, β cannot be considered as a point $x_1 \in R^n$. This is because we require, in Theorem 3.3.1, the existence of a set γ, $\bar{\gamma} \subset \beta$, besides the assumption that β is open; and because, in Theorem 3.3.4, one can show that

$$V_m^K(t) < V_m^{K/\beta}(t), \text{ all } t \in J, \ t \geqq t_1.$$

This means that one cannot apply the theory established in this section to conclude any results concerning controllability as defined in 1.4.II. One can, however, apply this theory by modifying the definition of (α, β, $t_1 \in J$)-controllability in an appropriate manner, i.e., given $x_o \in R^n$, $x_1 \in R^n$, then system (EC) is ($\{x_o\}$, $\{x_1\}$, $t_1 \in J$)-controllable, if there exists an open neighbourhood β of x, such that the system is ($\{x_o\}$, β, $t_1 \in J$)-controllable (in the sense of Definition 1.4.5-II).

On the other hand, one can apply the theory established in Section 2.3-II, bearing in mind that system (EC) is finally-stable with respect to the sets (α, β, U, $t_1 \in J$), if and only if it is semi-finally stable with respect to the sets (α, β, U, $t_2 \in J$), for all $t_2 \in J$, $t_2 \geqq t_1$.

(5) Finally, one notes that a system (EC) is not finally-stable with respect to the sets (α, β, U, t_1 ε J), if and only if there exists a t_2 ε J, $t_2 \geqq t_1$, such that the system is not semi-finally stable with respect to the sets (α, β, U, t_2 ε J). Hence, a theory concerning this aspect of final-stability can be deduced by an appropriate application of the theory established in Section 2.4-II.

3.4 Strong final-stability

One can establish a theory similar to the one presented in 2.5-II. We shall, however, consider a different approach. Moreover, we shall limit the discussion to the case of strong final-stability with respect to the sets (α, β, U, J).

1. *Theorem 3.4.1:*

System (EC) is strongly finally-stable with respect to the sets (α, β, U, J), if

(a) it is finally-stable with respect to the sets (α, β, U, J),

and

(b) there exist two families of functions $V[x,t;\ x_o] \ \varepsilon \ L[K(x_o), J]$, $\omega(t,r;\ x_o) \ \varepsilon \ \Omega$, all $x_o \ \varepsilon \ \alpha^c$, where

$$K(x_o) = K(\{x_o\},\ U,\ J). \quad [\text{Ref.}(3.1.1)] \tag{3.4.1}$$

such that

(i) for each fixed $x_o \ \varepsilon \ \alpha^c$, $V_m^{K(x_o)\cap\beta}(t;\ x_o)$ is finite for all t ε J, $t \geqq t(x_o)$, for some $t(x_o)$ ε J; where

$$V_m^{K(x_o)\cap\beta}(t;\ x_o) = \sup_{x\varepsilon K(x_o)\cap\beta} V[x,t;\ x_o] \tag{3.4.2}$$

(ii) for each fixed $x_o \ \varepsilon\ \alpha^c$,

$$V^*[x,t;\ x_o] \leqq \omega(t,V[x,t;\ x_o]),\quad \text{all } t\ \varepsilon\ J,$$

$x\ \varepsilon\ K(x_o)$, $u(\cdot)\ \varepsilon\ U$; where

$$V^*[x,t;\ x_o] = \limsup_{h\to 0^+} \frac{V[x + hg,\ t + h;\ x_o] - V[x,t;\ x_o]}{h} \tag{3.4.3}$$

(iii) for each fixed $x_o\ \varepsilon\ \alpha^c$, the maximal solution $r_M(t;\ x_o)$ of

$$\dot{r} = \omega(t,r;\ x_o),\quad r(t_o;\ x_o) = V[x_o,t_o;\ x_o]$$

is such that

$$r_M(t;\ x_o) < V_m^{K(x_o)\cap\beta}(t;\ x_o),\quad \text{all } t\ \varepsilon\ J,\ t \geqq t(x_o). \tag{3.4.4}$$

2. On the other hand, the following theorem does not consider any special functions. In fact, it is a new formulation of the definition of strong final-stability [Ref. Definition 1.2.12-II].

Theorem 3.4.2:

System (EC) is strongly finally-stable with respect to the sets $(\alpha,\ \beta,\ U,\ J)$, if

(i) it is finally-stable with respect to the sets $(\alpha,\ \beta,\ U,\ J)$, and

(ii) to each set $\alpha^* \subseteq \alpha^c$, there corresponds a set $\beta^* = \beta^*(\alpha^*) \subseteq \beta^c$, such that the system is finally-stable with respect to the sets $(\alpha^*,\ \beta^*,\ U,\ J)$.

§4. APPLICATIONS

The aim of this section is to show the application of some of the theory established in the previous sections. For reasons of space, we limit our discussion to the cases of (non)semi-final stability with respect to the sets $(\alpha, \beta, U, t_1 \varepsilon J)$.

4.1 Stability results:

Consider the differential system

$$\dot{x} = F(x,t)x + G(x,t)u \tag{FG}$$

where $F(x,t)$ is an $n \times n$ matrix, $x \varepsilon R^n$, $G(x,t)$ an $m \times n$ matrix and u an m-vector. All the functions involved are assumed to be smooth enough to ensure that the system satisfies the required properties.

Let α and β be any two connected sets in R^n. We shall seek to establish the sufficient conditions for system (FG) to be semi-finally stable with respect to the sets $(\alpha, \beta, U, t_1 \varepsilon J)$, for some control subset U which will be determined later.

Let $V[x,t]$ be defined by

$$V[x,t] = x^T S(t)x \tag{4.1.1}$$

where $S(t)$ is an $n \times n$ time-varying matrix with the following properties:

(i) $S(t) \varepsilon C^1(J^*)$, $J^* = [t_o, t_1)$, i.e., the elements of $S(t)$ are continuously differentiable over J^*.

(ii) $\sup_{x\varepsilon\alpha} x^T S(t_o)x$ and $\inf_{x\varepsilon\beta^c} x^T S(t_1)x$ are finite.

A matrix S(t) satisfying the above properties will be said to be of class $A(t_1, \alpha, \beta)$.

Along trajectories of (FG),

$$\dot{V} = x^T(\dot{S} + F^T S + SF)x + u^T G^T Sx + x^T SGu \tag{4.1.2}$$

Suppose, furthermore, that there exists a function $\mu(t,r) \in \Omega$ such that the maximal solution $r_M(t)$ of

$$\dot{r} = \mu(t,r) \quad r(t_o) = \sup_{x\epsilon\alpha} . x^T S(t_o)x \tag{4.1.3}$$

satisfies the inequality

$$r_M(t_1) < \inf_{x\epsilon\beta^c} x^T S(t_1)x \tag{4.1.4}$$

and let $U = U(F,G,S,\mu)$ be the set of all (real) functions $u(\cdot)$ satisfying

$$u^T G^T Sx + x^T SGu \leqq \mu(t, x^T Sx) - x^T(\dot{S} + F^T S + SF)x \tag{4.1.5}$$

for all $t \in J^*$, $x \in R^n$. Then,

$$\dot{V} \leqq \mu(t,V[x,t]) \tag{4.1.6}$$

and one can see that the conditions of Theorem 2.3.2-II are satisfied, provided U is not empty. If we put

$$U(F,G) = \{U(F,G,S,\mu) : S \in A(t_1, \alpha, \beta),\ \mu \in M(S \in A)\} \qquad (4.1.6)$$

where $M(S \in A)$ is the set of all functions $\mu(t,r)$ with the above properties; then we conclude that

Result 4.1.1:

System (FG) is semi-finally stable with respect to the sets (α, β, U, $t_1 \in J$) if $U \subseteq U(F,G,S,\mu)$, for some $U(F,G,S,\mu) \in U(F,G)$.

Result 4.1.2:

System (FG) is (α, β, $t_1 \in J$) semi-controllable if $U(F,G) \neq \emptyset$. In particular, system (FG) is (α, β, $t_1 \in J$) semi-controllable in the two following cases:

Case 1: If there exists a matrix $S(t) \in A(t_1, \alpha, \beta)$ such that:

(i) $(G^T Sx + x^T SG)_i \neq 0$, for some $1 \leq i \leq m$

(ii) $M(S \in A) \neq \emptyset$.

For, in this case, (4.1.5) becomes

$$\sum_{i=1}^{m} (G^T Sx + x^T SG)_i u_i \leq \mu(t, x^T Sx) - x^T(\dot{S} + F^T S + SF)x \qquad (4.1.7)$$

for some $\mu \in M(S \in A)$, by (ii). By (i), the inequality (4.1.7) admits at least one solution $u(\cdot)$. Hence $U(F,G,S,\mu)$ is not empty.

Case 2: If there exists a matrix $S(t) \in A(t_1, \alpha, \beta)$, such that

(iii) $x^T(SG + G^T S)x > 0$, all $x \in R^n$, $t \in J^*$, $x \neq 0$.

(iv) $M(S \in A) \neq \emptyset$.

It will be sufficient, in this case, to take $U = \{u(\cdot) | u = cx, x \in R^n\}$, where c is a scalar satisfying

$$c \leq \frac{\mu(t, x^T Sx) - x^T(\dot{S} + F^T S + SF)x}{x^T(SG + G^T S)x} \qquad (4.1.8)$$

for some $\mu \in M(S \in A)$.

Result 4.1.3:

System

$$\dot{x} = F(x,t)x \tag{F}$$

is semi-finally stable with respect to the sets (α, β; $t_1 \in J$), if for some $S \in A(t_1, \alpha, \beta)$ and some $\mu \in M(S \in A)$, we have

$$\mu(t, x^T Sx) \geq x^T(\dot{S} + F^T S + SF)x \tag{4.1.9}$$

for all $t \in J^*$, $x \in R^n$.

For, in this case, we can set $u(\cdot) \equiv 0$ in system (FG), since $U = \{u(\cdot) \equiv 0\} \subseteq U(F,G,S,\mu)$.

We note that, according to remark 2.3.3-II, we cannot consider the case where β is a point $x_1 \in R^n$. It is however possible to modify the definition of the (α, β, $t_1 \in J$) semi-controllability in an appropriate manner: for example, such modification is to say that system (EC) is ($\{x_o\}$, $\{x_1\}$, $t_1 \in J$) semi-controllable if there exists a neighbourhood β of x_1 such that the system (EC) is ($\{x_o\}$, β, $t_1 \in J$) semi-controllable in the sense of Definition 1.4.2-II. Moreover, a theory can be established using Theorem 2.3.3-II where the above difficulty is removed.

Examples:

Example 4.1.1:

Consider system

$$\dot{x} = F(x,t)x \tag{F}$$

with

(i) $x^T(F^T + F)x \leqq \lambda(t)x^Tx$, all $x \in R^n$, $t \in [t_o, t_1)$,

(ii) $\int_{t_o}^{t_1} \lambda(t)dt < 2 \log_e \frac{b}{a}$, $a > 0$, $b > 0$

Then, system (F) is semi-finally stable with respect to the sets (α, β, $t_1 \in J$), where

$$\alpha: \ ||x|| < a, \qquad \beta: \ ||x|| < b \tag{4.1.10}$$

If, moreover, (ii) is valid for all $t \in J$, $t \geqq t_1$, then system (F) is finally-stable with respect to the sets (α, β, $t_1 \in J$). Note, however, that the relation (ii) need not hold for all $t \in J$, $t > t_1$. If, furthermore, $a \leqq b$, then (ii) may hold for all $t \in J$, $t > t_o$; in this case, the system (F) is stable with resepct to the sets (α, β, J).

Proof of Example 4.1.1:

Let, in result 4.1.3, $S(t) = I_n$ (the n-dimensional identity matrix), and $\mu(t,r) = \lambda(t)r$. Obviously, $I_n \in A(t_1, \alpha, \beta)$ and $\mu \in M(I_n \in A)$, by (ii).

Moreover,

$$x^T(\dot{S} + F^TS + SF)x = x^T(F^T + F)x \leqq \lambda(t)x^Tx = \mu(t, x^TSx)$$

Thus, inequality (4.1.9) is satisfied.

Remark 4.1.1:

If $J = \bar{I}$, and $t_1 \not\equiv t_o+T$; then, the conclusion is that the system is finally-stable with respect to the sets (α, β, J). But, in this case, relation (ii) implies the existence of a $t^* \in (t_o, t_o+T)$ such that

$$\int_{t_o}^{t} \lambda(s)ds < 2 \log_e \frac{b}{a}, \quad \text{all } t \in [t^*, t_o+T]$$

which means that the system (F) is uniformly finally-stable with respect to the sets (α, β, J).

Example 4.1.2:

If there exists a matrix B(t) such that

(i) $x^T(B^T + B)x \leq \lambda(t)x^T x - x^T(F^T + F)x$, all $x \in R^n$, $t \in [t_o, t_1)$

(ii) $\int_{t_o}^{t_1} \lambda(t)dt < 2 \log_e \frac{b}{a}$, $b > 0$, $a > 0$

then system (FG) is semi-finally stable with respect to the sets (α, β, U, t_1 ε J) where α and β are as in Example 4.1.1, and

$$U = \{u(\cdot) | u(x,t) = G^{-1}(x,t)B(t)x, \ t \in [t_o, t_1), \ x \in R^n\}$$

Example 4.1.3:

Consider the system (VPC)

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = -x_1 + \sigma(x_1^2 - 1)x_2 \qquad \text{(VPC)}$$

and let

$$\alpha: ||x|| < a, \quad \beta: ||x|| < b, \ a > 0, \ b > 0$$

then, system (VPC) is (α, β, t_1 ε J) semi-controllable provided

$$\sigma > \frac{1}{t_1 - t_o} \log_e \frac{a}{b} \qquad (4.1.11)$$

Moreover, if a < b, and

$$\frac{1}{t_1 - t_o} \log_e \frac{a}{b} < \sigma \leqq 0 \tag{4.1.12}$$

then, system (VP)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \sigma(x_1^2 - 1)x_2 \tag{VP}$$

is semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \varepsilon J)$.

Proof:

Let $S(t) = I_n$, then $I_n \varepsilon A(t_1, \alpha, \beta)$. Let $\mu(t,r) = -2\sigma r$, then $\mu \varepsilon M(I_n \varepsilon A)$; for the (maximal) solution $r_M(t)$ of

$$\dot{r} = -2\sigma r, \quad r(t_o) = \sup_{x\varepsilon\alpha} x^T S(t_o)x = a^2$$

is given by

$$r_M(t) = a^2 e^{-2\sigma(t-t_o)}$$

and therefore, satisfies the condition

$$r_M(t_1) = a^2 e^{-2\sigma(t_1-t_o)} < b^2 = \inf_{x\varepsilon\beta^c} x^T S(t_1)x \quad \text{by (4.1.11)}$$

Now,

$$F = \begin{bmatrix} 0 & 1 \\ -1 & \sigma(x_1^2 - 1) \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{so that}$$

$$U(F,G,-2\sigma r, I_n) = \{u(\cdot) \mid 2x_1u_1 \leqq -2\sigma(x_1^2 + x_2^2) - 2\sigma(x_1^2 - 1)x_2^2, \quad t \varepsilon [t_o, t_1), \ x \varepsilon R^n\}.$$

For example, we can take

$$\begin{cases} u_1 \leqq -\sigma x_1(1 + x_2^2), & x_1 > 0 \\ u_1 \geqq -\sigma x_1(1 + x_2^2), & x_1 < 0 \\ u_1 = 0, & x_1 = 0. \end{cases}$$

We note that if $\sigma \leqq 0$, then we can take $u_1 \equiv 0$, all $x \in R^n$, which means that (VPC) reduces to system (VP). <u>This completes the proof</u>.

4.2 <u>Instability theory</u>:

We shall use Theorem 2.4.1-II. For this purpose let

$$V = -x^T R(t) x \tag{4.2.1}$$

where R(t) is an n × n matrix such that

(i) $R(t) \in C^1(J^*)$,

(ii) $\sup_{x \in \beta} x^T R(t_1) x$ is finite.

Let $B(t_1, \beta)$ be the set of such matrices.

Along trajectories of (FG), we get

$$\dot{V} = -x^T(\dot{R} + F^T R + RF)x - u^T G^T R x - x^T R G u \tag{4.2.2}$$

If, now, there exists a function $\mu(t,r)$ with the property that the maximal solution $r_M(t)$ of

$$\dot{r} = -\mu(t,r) \qquad r(t_o) = -x_o^T S(t_o) x_o \tag{4.2.3}$$

(for some $x_o \in \alpha$) is such that

$$r_M(t_1) < -\sup_{x \in \beta} x^T S(t_1) x \tag{4.2.4}$$

then

$$\dot{V} \leqq -\mu(t, V[x,t]) \tag{4.2.5}$$

for all $t \in J^*$, $x \in R^n$, and all $u(\cdot)$ satisfying

$$u^T G^T R x + x^T R G u \geqq \mu(t, -x^T R x) - x^T(\dot{R} + F^T R + RF)x \tag{4.2.6}$$

for all $t \in J^*$, $x \in R^n$. Let $m(\alpha, R \in B)$ denote the set of all functions $\mu(t,r)$ with the above properties, and $U^c(F,G,R,\mu)$ be the set of all functions $u(\cdot)$ satisfying the inequality (4.2.6). If U^c is not empty, then we conclude the following results:

Result 4.2.1:

System (FG) is not semi-finally stable with respect to the sets (α, β, U, $t_1 \in J$) if $U \cap U^c \neq \emptyset$, for some $U^c = U^c(F,G,R,\mu)$.

Furthermore, let

$$U^c(F,G) = \{U^c(F,G,R,\mu) : R \in B(t_1,\beta), \text{ and } \mu \in m(\alpha, R \in B)\} \quad (4.2.7)$$

then

Result 4.2.2:

System (FG) is not (α, β, $t_1 \in J$) semi-controllable if

$$S^m \subseteq U^c(F,G)$$

Examples:

Example 4.2.1:

Let

$$\dot{x} = F(x,t)x \quad \text{(F)}$$

be such that

(i) $x^T(F^T + F)x \geq \mu(t)x^Tx$, all $t \in [t_o,t_1)$, $x \in R^n$.

(ii) $\int_{t_o}^{t_1} \mu(t)dt > 2 \log_e \frac{b}{||x_o||}$, for some x_o, such that $||x_o|| < a$

then, system (F) is not semi-finally stable with respect to the sets $(\alpha,\ \beta,\ t_1\ \varepsilon\ J)$,

where

$$\alpha:\ ||x|| < a\ , \qquad \beta:\ ||x|| < b \tag{4.2.8}$$

Proof

Apply result 4.2.1 to system

$$\dot{x} = F(x,t)x + G(x,t)u \tag{FG}$$

Let $R(t) \equiv I_n$, then $I_n\ \varepsilon\ B(t_1,\beta)$, for $\sup_{x\varepsilon\beta} x^T R(t_1)x = b^2$.

Let

$$\mu(t,r) \equiv -\mu(t)r$$

then $\mu\ \varepsilon\ m(\alpha, I_n \varepsilon B)$, for the (maximal) solution $r_M(t)$ of

$$\dot{r} = +\mu(t)r \qquad r(t_o) = -x_o^T x_o = -||x_o||^2$$

[where $x_o\ \varepsilon\ \alpha$ is given by (ii)] is given by

$$r_M(t) = -||x_o||^2\, e^{\int_{t_o}^{t} \mu(s)ds}$$

and therefore satisfies

$$r_M(t_1) = -||x_o||^2\, e^{\int_{t_o}^{t_1} \mu(s)ds} < -||x_o||^2\, e^{2\log_e \frac{b}{||x_o||}} = -b^2$$

$$= -\sup_{x\varepsilon\beta} x^T R(t)x$$

Hence,

$$U^c\left(F, G, I_n, -\mu(t)r\right) = \{u(\cdot)\,|\,u^T G^T x + x^T G u \geq \mu(t)x^T x - x^T(F^T + F)x\}$$

which implies, by (i), that $\{u(\cdot) \equiv 0\}$ is contained in $U^c\left(F,G,I_n,-\mu(t)r\right)$. We conclude that system (FG) is not semi-finally stable with respect to the sets $(\alpha, \beta, \{0\}, t_1 \in J)$. This is equivalent to saying that system (F) is not semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in J)$.

Comparing this result with the result of Example 4.1.1, we see that condition (ii) of Example 4.2.1 implies that

$$\int_{t_o}^{t_1} \mu(t)dt > 2 \log_e \frac{b}{||x_o||} > 2 \log \frac{b}{a}$$

which means that any $\lambda(t)$ satisfying condition (i) of Example 4.1.1 cannot satisfy condition (ii) of the same example for the same $t_1 \in J$.

Example 4.2.2:

Consider the system

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = -x_1 + q(x_1^2 - 1)x_2 \qquad \text{(VPC)}$$

and let

$$\alpha : ||x|| < a, \quad \beta : ||x|| < b$$

then, system (VPC) is not semi-finally stable with respect to the sets $\left(\alpha, \beta, \{u(\cdot)\}, t_1 \in J\right)$, for all $u(\cdot) \in U$, where

$$U : \begin{array}{ll} u_1 \geqq -\sigma\, x_1(1 + x_2^2) & x_1 > 0 \\ u_1 \leqq -\sigma\, x_1(1 + x_2^2) & x_1 < 0 \\ u_1 \equiv 0 & x_1 = 0 \end{array} \qquad (4.2.9)$$

provided

$$\sigma < \frac{1}{t_1 - t_o} \log_e \frac{||x_o||}{b}, \text{ for some } x_o \text{ such that}$$

$$||x_o|| < a \qquad (4.2.10)$$

Proof:

Let $R(t) \equiv I_n$, then $R(t) \in B(t_1,\beta)$, for $\sup_{x\in\beta} x^T R(t_o)x = b^2$. Let

$\mu(t,r) = 2\sigma r$, then $\mu \in m(\alpha, I_n \in B)$, for the (maximal) solution

$r_M(t)$, $r_M(t_o) = -||x_o||^2$, x_o given by (4.2.10), of

$$\dot{r} = -\mu(t,r) = -2\sigma r \tag{4.2.11}$$

is given by

$$r_M(t) = -||x_o||^2 e^{-2\sigma(t-t_o)} \tag{4.2.12}$$

and hence satisfies

$$r_M(t_1) < -||x_o|| e^{-2\log_e \frac{||x_o||}{b}} = -b^2 = -\sup_{x\in\beta} x^T R(t_1)x.$$

Now,

$$F = \begin{bmatrix} 0 & 1 \\ -1 & \sigma(x_1^2 - 1) \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

so that

$$U^c(F,G,I_n,2\sigma r) = \{u(\cdot) | x_1 u_1 \geqq -\sigma x_1^2(1 + x_2^2)\}$$

Thus we see clearly that $U \subseteq U^c(F,G,I_n, 2\sigma r)$.

This completes the proof.

Example 4.2.3:

If $||x_o|| > b$, and

$$0 \leqq \sigma < \frac{1}{t_1 - t_o} \log_e \frac{||x_o||}{b}$$

then system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \sigma(x_1^2 - 1)x_2 \tag{VP}$$

is not semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in J)$.

This example is an immediate consequence of Example 4.2.2.

Remark 4.2.1:

In the above examples, use has been made of diagonal forms only, i.e., $R(t) \equiv I_n$. This is not however necessary as the following (trivial) example will show

Example 4.2.4:

Consider system

$$\dot{x}_1 = \lambda x_1, \quad \dot{x}_2 = \mu x_2 \tag{4.2.13}$$

where λ and μ are any real constants. Let

$$\alpha : (x_{10}, x_{20}), \quad x_{10}x_{20} > 0 \tag{4.2.14}$$

and

$$\beta : (x_{11}, x_{21}), \quad x_{11}x_{21} \leqq 0 \tag{4.2.15}$$

then system (4.2.13) is not semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in J)$.

In fact, the result holds for all $t_1 \in R^+$, $t_1 > t_o$, as we shall see from the proof. This conclusion is in line with the fact that any trajectory of (4.2.13) starting in any quadrant of the (x_1,x_2)-plane will remain in the same quadrant for all subsequent values of $t \in R^+$.

Proof:

Let $V = -x_1x_2$, then $V_m^{\beta}(t_1) = -x_{11}x_{21} \geqq 0$ and along trajectories of

(4.2.13),

$$\dot{V} = -(\lambda + \mu)\ x_1x_2 = (\lambda + \mu)V \tag{4.2.16}$$

Thus, taking $\omega(t,r) = (\lambda + \mu)r$, we can see that the (maximal) solution $r(t)$, $r(t_o) = V[x_o, t_o] = -x_{10}x_{20}$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is given by

$$r(t) = -x_{10}x_{20}e^{(\lambda+\mu)t} \tag{4.2.17}$$

and hence satisfies

$$r(t_1) < V_m^\beta(t_1) \tag{4.2.18}$$

Thus, the conditions of Theorem 2.4.1-II are satisfied.

Note that we can apply result 4.2.1 by setting

$$R = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

CHAPTER III

Discrete Systems

§1. INTRODUCTION AND DEFINITIONS

1.1 Introduction

We consider systems of the form

$$x(j + 1) = f[x(j),j] \qquad \text{(DS)}$$

defined over the interval $J = [j_o, j_o + 1, \ldots, j_o + j_N]$, where j_o, j_N are a non-negative number and a positive integer respectively, and where $x(j) \in R^n$, $j \in J$, is a real vector which represents the state of the system at times j_o, j_o+1, ..., j_o+j_N. It will be assumed that the n-vector function $f[x(j),j]$ possesses all the necessary properties, so that there is no difficulty with the questions of existence, uniqueness and continuity of solutions with respect to initial conditions.

We intend to extend to discrete systems of the form (DS) the theory of final-stability established in the preceding chapters. We note that the concept of final-stability will include the concept of finite-time stability introduced in [30].

Let $V[x(j), j]$ denote a mapping

$$V : R^n \times J \to R^1 \qquad (1.1.1)$$

Accordingly, we define the following functions

$$V_m^\alpha(j) = \operatorname*{Inf}_{x\varepsilon\alpha} V[x(j),j], \quad V_M^\alpha(j) = \operatorname*{sup.}_{x\varepsilon\alpha} V[x(j),j], \; j \in J \qquad (1.1.2)$$

Define the total difference of $V[x(j),j]$ along the trajectory $x(j)$ of (DS) as

$$\Delta V[x(j),j] = V[x(j + 1), j + 1] - V[x(j),j] \qquad (1.1.3)$$

and

$$\Delta x(j) = x(j + 1) - x(j) \tag{1.1.4}$$

1.2 Definitions of semi-final stability

Definition 1.2.1:

System (DS) is semi-finally stable with respect to the sets (α, β, J), if, for any trajectory $x(j)$ the condition

$$x(j_o) \in \alpha \tag{1.2.1}$$

implies the existence of $j_p \in J$, such that

$$x(j_p) \in \beta \tag{1.2.2}$$

where j_p may depend on the particular trajectory $x(j)$.

Definition 1.2.2:

System (DS) is not semi-finally stable with respect to the sets (α, β, J), if there exists a trajectory $x(j)$ with initial condition $x(j_o) \in \alpha$ and satisfying

$$x(j) \in \beta^c, \quad \text{for all } j \in J \tag{1.2.3}$$

Definition 1.2.3:

System (DS) is semi-finally stable with respect to the sets $(\alpha, \beta, j_k \in J)$, if, for any trajectory $x(j)$, the condition $x(j_o) \in \alpha$ implies that $x(j_k) \in \beta$.

Definition 1.2.4:

System (DS) is not semi-finally stable with respect to the sets $(\alpha, \beta, j_k \varepsilon J)$, if there exists a trajectory $x(j)$ satisfying the condition

$$x(j_o) \ \varepsilon \ \alpha, \text{ and } x(j_k) \ \varepsilon \ \beta^c \tag{1.2.4}$$

Definition 1.2.5:

System (DS) is uniformly semi-finally stable with respect to the sets (α, β, J), if there exists a $j_k \ \varepsilon \ J$, such that the system is semi-finally stable with respect to the sets $(\alpha, \beta, j_k \ \varepsilon \ J)$.

Definition 1.2.6:

System (DS) is strongly semi-finally stable with respect to the sets (α, β, J), if

(i) it is semi-finally stable with respect to the sets (α, β, J), and

(ii) for any trajectory $x(j)$, the condition $x(j_o) \ \varepsilon \ \alpha^c/\beta$, implies that

$$x(j) \ \varepsilon \ \beta^c, \quad \text{all } j \ \varepsilon \ J \tag{1.2.5}$$

Definition 1.2.7:

System (DS) is strongly semi-finally stable with respect to the sets $(\alpha, \beta, j_k \ \varepsilon \ J)$, if

(i) the system is semi-finally stable with respect to the sets (α, β, J), and

(ii) for any trajectory $x(j)$, the condition $x(j_o) \ \varepsilon \ \alpha^c$ implies that $x(j_k) \ \varepsilon \ \beta^c$.

1.3 Definitions of final-stability:

Definition 1.3.1:

System (DS) is finally-stable with respect to the sets (α, β, J), if, for any trajectory $x(j)$, the condition

$$x(j_o) \in \alpha$$

implies the existence of a $j_p \in J$, such that

$$x(j) \in \beta, \quad \text{all } j \in J, \ j \geqq j_p \qquad (1.3.1)$$

where j_p may depend on the particular trajectory.

Definition 1.3.2:

System (DS) is not finally-stable with respect to the sets (α, β, J), if there exists a trajectory $x(j)$, $x(j_o) \in \alpha$, such that

$$x(j) \in \beta^c, \text{ all } j \in J, \ j \geqq j_k, \quad \text{for some } j_k \in J \qquad (1.3.2)$$

Definition 1.3.3:

System (DS) is finally-stable with respect to the sets $(\alpha, \beta, j_k \in J)$, for any trajectory $x(j)$, $x(j_o) \in \alpha$, we have

$$x(j) \in \beta, \quad \text{all } j \in J, \ j \geqq j_k \qquad (1.3.5)$$

Definition 1.3.4:

System (DS) is not finally-stable with respect to the sets $(\alpha, \beta, j_k \in J)$, if there exists a trajectory $x(j)$, $x(j_o) \in \alpha$, such that

$$x(j_p) \in \beta^c \tag{1.3.4}$$

for some $j_p \in J$, $j_p \geqq j_k$.

The system is said to be uniformly finally-stable if there exists a $t_k \in J$, such that the system is finally-stable with respect to the sets $(\alpha, \beta, j_k \in J)$.

Definition 1.3.5:

System (DS) is strongly finally-stable with respect to the sets (α, β, J), if

(i) it is finally-stable with respect to the sets (α, β, J), and

(ii) for any trajectory $x(j)$, the condition $x(j_o) \in \alpha^c$ implies the existence of a $j^* \in J$ such that

$$x(j) \in \beta^c, \quad \text{for all } j \in J,\ j \geqq j^* \tag{1.3.5}$$

where j^* may depend on the particular trajectory.

Definition 1.3.6:

System (DS) is strongly finally-stable with respect to the sets $(\alpha, \beta, j_k \in J)$, if

(i) it is finally-stable with respect to the sets (α, β, J), and

(ii) for any trajectory $x(j)$, the condition $x(j_o) \in \alpha^c$ implies that $x(j) \in \alpha^c$, all $j \in J$, $j \geqq j_k$.

§2. GENERAL THEOREMS ON THE DIFFERENT TYPES OF SEMI-FINAL STABILITY

2.1 Semi-final stability:

Theorem 2.1.1:

System (DS) is semi-finally stable with respect to the sets (α, β, J), $\alpha \not\subseteq \beta$, if there exist two real-valued functions $V[x(j),j]$, $\phi(j)$, defined for all $x(j) \in \beta^c$, and $j \in J$, such that

(i) $V_M^{\alpha/\beta}(j_o)$ and $V_m^{\beta^c}(j_o+j_N)$ are finite.

(ii) $\Delta V[x(j),j] < \phi(j)$, $j \in J$, along any trajectory $x(j)$, $x(j_o) \in \alpha/\beta$, as long as $x(j) \in \beta^c$.

(iii) $\sum_{k=j_o}^{k=j_o+j_N-1} \phi(k) \leqq V_m^{\beta^c}(j_o+j_N) - V_M^{\alpha/\beta}(j_o)$

Proof of Theorem 2.1.1:

(a) Any trajectory $x(j)$ emanating from $\alpha \cap \beta$, if $\alpha \cap \beta \neq \emptyset$, has a point in β.

(b) Suppose, contrary to the expected conclusion, that there exists a trajectory $x(j)$, $x(j_o) \in \alpha/\beta$, satisfying

$$x(j) \in \beta^c, \quad \text{all } j \in J \tag{2.1.1}$$

But then,

$$V[x(j_o+j_N), j_o+j_N] = V[x(j_o),j_o] + \sum_{k=j_o}^{k=j_o+j_N-1} \Delta V[x(k),k]$$

Using (i), (ii), and (iii) we get

$$V[x(j_o+j_N), j_o+j_N] < V_m^{\beta^c}(j_o+j_N) \tag{2.1.2}$$

But $x(j_o+j_N) \in \beta^c$, by (2.1.1). Hence, the last inequality constitutes a contradiction in view of the definition of $V_m^{\beta^c}(j)$. Thus, the original assumption is false and there exists a $j_p \in J$, such that $x(j_p) \in \beta$.

From (a) and (b), we conclude that system (DS) is semi-finally stable with respect to the sets (α, β, J). This completes the proof of Theorem 2.1.1.

2.2 Non semi-final stability:

Since the question of non semi-final stable systems (DS) with respect to the sets (α, β, J), $\alpha \subseteq \beta$, does not arise, we limit our discussion to the case $\alpha \not\subseteq \beta$.

We shall use the following additional notation:

$$d(x,y) = ||x - y|| \tag{2.2.1}$$

where $x,y \in R^n$, and $||.||$ is the euclidean norm.

$$d(x,\beta) = \inf_{y \in \beta} (x,y) \tag{2.2.2}$$

where β is any connected set in R^n, unless otherwise stated. It is clear that $d(x,\beta) = 0$, for all $x \in \bar{\beta}$. We define the set $\gamma(\beta,e)$, $e > 0$, as follows

$$\gamma(\beta,e) = \{x | d(x,\beta) < e\} \tag{2.2.3}$$

that is, the set of all points $x \in R^n$ whose distance from the set β is less than the positive number e. It is clear that

$$\gamma(\beta,e) \supset \bar{\beta} \tag{2.2.4}$$

Theorem 2.2.1:

System (DS) is not semi-finally stable with respect to the sets $(\alpha, \beta, J), \alpha \not\subseteq \beta$, if there exist a positive constant e, and two functions $V[x(j),j]$, $\phi(j)$, defined for all $x(j) \in \beta^c$, and $j \in J$, such that

(a) $\alpha/\gamma(\beta,e) \neq \emptyset$.

(b) there exists a point $x_o \in \alpha/\gamma(\beta,e)$ such that

(i) $||\Delta x(j)|| < e$, for all $j \in J$, along the trajectory $x(j)$, $x(j_o) = x_o$, as long as $x(j) \in \beta^c$.

(ii) $\Delta V[x(j),j] > \phi(j)$, $j \in J$, along the trajectory $x(j)$, $x(j_o) = x_o$, as long as $x(j) \in \beta^c$.

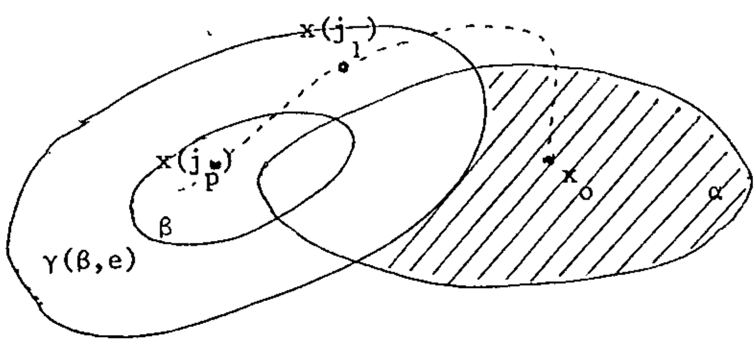
(iii) $V_M^{\gamma(\beta,e)/\beta}(j)$, $j \in J$, $j > j_o$, is finite.

(iv) $$\sum_{k=j_o}^{k=j_1-1} \phi(k) \geqq V_M^{\gamma(\beta,e)/\beta}(j_1) - V[x_o,j_o],$$

for all $j_1 \in J$, $j_1 > j_o$, $j_1 \leqq j_o+j_N-1$.

Proof of Theorem 2.2.1:

Suppose, contrary to the expected conclusion, that system (DS) is semi-finally stable with respect to the sets (α, β, J). Then, for the trajectory $x(j)$, $x(j_o) = x_o$,



$\alpha/\gamma(\beta,e)$ = shaded area.

Figure 2.2.1

there exists a $j_p \in J$, $j_o < j_p \leq j_o + j_N$, such that

$$x(j_p) \in \beta \tag{2.2.5}$$

Let j_p be a first such point, and let $j_1 = j_p - 1$, then a moment of reflexion will yield

$$j_o < j_1 < j_p, \quad x(j_1) \in \gamma(\beta,e)/\beta, \text{ and}$$

$$x(j) \in \beta^c, \text{ all } j \in [j_o, \ldots, j_1] \tag{2.2.6}$$

But, then

$$V[x(j_1),j_1| = V[x(j_o),j_o] + \sum_{k=j_o}^{j_1-1} \Delta V[x(t),t] > V_M^{\gamma(\beta,e)/\beta}(j_1) \tag{2.2.7}$$

by (ii), (iii) and (iv). Inequality (2.2.7) constitutes a contradiction to (2.2.6) in view of the definition of $V_M^{\gamma(\beta,e)/\beta}(j)$. Thus, the original assumption is false and system (DS) is not semi-finally stable with respect to the sets (α, β, J). This completes the proof of Theorem 2.2.1.

2.3 Semi-final stability with respect to the sets (α, β, $j_1 \in J$).

1. The case $\alpha \subseteq \beta$:

Theorem 2.3.1:

System (DS) is semi-finally stable with respect to the sets (α, β, $j_1 \in J$), $\alpha \subseteq \beta$, $j_1 > j_o$, if there exist a positive constant e, no matter how large, and two functions $V[x(j),j]$, $\phi(j)$, defined for all $j \in [j_o, \ldots, j_1]$, and all $x(j) \in \gamma(\beta,e)$, such that

(i) $||\Delta x(j)|| < e$, all $j \in [j_o, \ldots, j_1]$, along any trajectory. $x(j)$, $x(j_o) \in \alpha$, as long as $x(j) \in \gamma(\beta,e)$.

(ii) $V_M^\alpha(j_o)$ and $V_m^{\gamma(\beta,e)/\beta}(j)$, $j \in (j_o,\ldots, j_1]$, are finite.

(iii) $\Delta V[x(j),j] < \phi(j)$, $j \in [j_o, \ldots, j_1]$, along any trajectory $x(j)$, $x(j_o) \in \alpha$, as long as $x(j) \in \gamma(\beta,e)$.

(iv) $\sum_{k=j_o}^{k=j_2-1} \phi(k) \leqq V_m^{\gamma(\beta,e)/\beta}(j_2) - V_M^\alpha(j_o)$, all $j_2 \in (j_o, \ldots, j_1]$

Remark 2.3.1:

If the number e is difficult to determine, then obviously Theorem 2.3.1 is not applicable. The following theorem overcomes this difficulty to the detriment of the other conditions.

Theorem 2.3.2:

System (DS) is semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, $\alpha \subseteq \beta$, $j_1 > j_o+1$, if there exist four functions $V_1[x(j),j]$, $\phi_1(j)$, defined for all $x(j) \in \beta$, $j \in [j_o,\ldots, j_1)$, and $V_2[x(j),j]$, $\phi_2(j)$, defined for all $x(j) \in \beta^c$, and $j \in (j_o,\ldots j_1]$, such that

(i) $V_{1M}^\alpha(j_o)$, $V_{1m}^\beta(j)$, $j \in (j_o, \ldots, j_1)$, $V_{2M}^{\beta^c}(t)$, $j \in (j_o, \ldots, j_1)$ and $V_{2m}^{\beta^c}(j_1)$ are finite.

(ii) $\Delta V_1[x(j),j] < \phi_1(j)$, $j \in [j_o, \ldots, j_1)$, along any trajectory $x(j)$, $x(j_o) \in \alpha$, as long as $x(j) \in \beta$.

(iii) $\Delta V_2[x(j),j] < \phi_2(j)$, along any trajectory $x(j)$, $x(j_o) \in \alpha$, as long as $x(j) \in \beta^c$, $j \in (j_o, \ldots, j_1]$.

(iv) $\sum_{k=j_o}^{k=j_2-1} \phi_1(k) \leqq V_{1m}^\beta(j_2) - V_{1M}^\alpha(j_o)$, $j_2 \in (j_o, \ldots, j_1)$

(v) $$\sum_{k=j_3}^{k=j_1-1} \phi_2(k) \leqq V_{2m}^{\beta^c}(j_1) - V_{2M}^{\beta^c}(j_3),\ j_3 \in (j_o, \ldots, j_1)$$

Proof of Theorem 2.3.2:

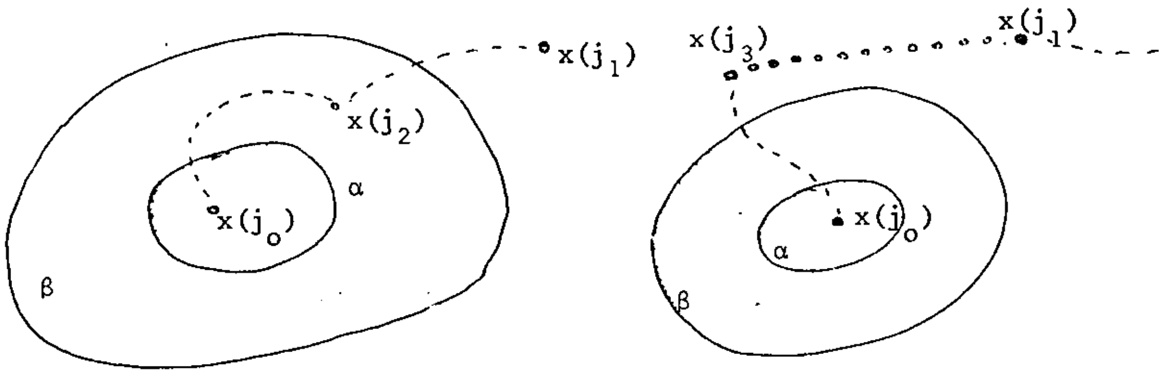


Figure 2.3.1(a): $x(j) \in \beta$, all $j \in [j_o, j_2]$

Figure 2.3.1(b): $x(j) \in \beta^c$, all $j \in [j_3, j_1]$

Suppose contrary to the expected conclusion, that there exists a trajectory $x(j)$, $x(j_o) \in \alpha$, and such that

$$x(j_1) \in \beta^c \tag{2.3.1}$$

Since $j_1 > j_o+1$, then two possible situations may occur:

(a) there exists a $j_2 \in (j_o, \ldots, j_1)$, such that

$$x(j) \in \beta, \text{ all } j \in [j_o, \ldots, j_2] \tag{2.3.2}$$

Or (and)

(b) there exists a $j_3 \in (j_o, \ldots, j_1)$, such that

$$x(j) \in \beta^c, \text{ all } j \in [j_3, \ldots, j_1]$$

(Fig. 2.3.1 illustrates the two possibilities in R^2)

We note that, as long as (2.3.1) holds, the two situations may occur at the same time.

In the case (a), we have

$$V_1[x(j_2),j_2] = V_1[x(j_o),j_o] + \sum_{k=j_o}^{k=j_2-1} \Delta V_1[x(k),k]$$

Using (i), (ii) and (iv), we get

$$V_1[x(j_2),\ j_2] < V_{1m}^{\beta}(j_2) \tag{2.3.3}$$

But $x(j_2) \in \beta$, by (2.3.2). Hence, the last inequality (2.3.3) constitutes a contradiction. Thus, if the original assumption (2.3.1) is still considered valid, the possibility (a) cannot occur. This means that the situation (b) must occur. But then

$$V_2[x(j_1),j_1] < V_{2m}^{\beta^c}(j_1) \tag{2.3.4}$$

by (i), (iii) and (v). But this last inequality constitutes a contradiction to (2.3.1). Hence, the original assumption (2.3.1) is false. <u>This completes the proof of Theorem 2.3.2</u>, bearing in mind the fact that $x(j)$, $x(j_o) \in \alpha$, is chosen arbitrarily.

Remark 2.3.2:

Conditions (i), (ii) and (iv) of the above theorem imply that any trajectory $x(j)$, $x(j_o) \in \alpha$, cannot satisfy

$$x(j_o+1) \in \beta \tag{2.3.5}$$

which narrows its range of application.

2. The general case:

Theorem 2.3.3:

System (DS) is semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \varepsilon J)$, $j_1 > j_o$, if there exist four functions $V_1[x(j),j]$, $\phi_1(j)$, defined for all $x(j) \varepsilon \beta^c$, $j \varepsilon [j_o, \ldots, j_1]$, and $V_2[x(j),j]$, $\phi_2(j)$ defined for all $x(j) \varepsilon \gamma(\beta,e)$, $j \varepsilon [j_o, \ldots, j_1]$, where e is a positive number given by (i), such that

(i) $||\Delta x(j)|| < e$, for all trajectories $x(j)$, $x(j_o) \varepsilon \alpha$, and all $j \varepsilon [j_o, \ldots, j_1]$.

(ii) $V^{\alpha}_{1M}(j_o)$, $V^{\beta^c}_{1m}(j_1)$, $V^{\beta}_{2M}(j)$, $j \varepsilon [j_o, \ldots, j_1)$ and $V^{\gamma(\beta,e)/\beta}_{2m}(j)$, $j \varepsilon (j_o, j_1]$ are finite.

(iii) $\Delta V_1[x(j),j] < \phi_1(j)$, $j \varepsilon [j_o, \ldots, j_1]$ along any trajectory $x(j)$, $x(j_o) \varepsilon \alpha$, as long as $x(j) \varepsilon \beta^c$.

(iv) $\Delta V_2[x(j),j] < \phi_2(j)$, $j \varepsilon [j_o, \ldots, j_1]$ along any trajectory $x(j)$, $x(j_o) \varepsilon \alpha$, as long as $x(j) \varepsilon \gamma(\beta,e)$.

(v) $\sum_{k=j_o}^{k=j_1-1} \phi_1(k) \leqq V^{\beta^c}_{1m}(j_1) - V^{\alpha}_{1M}(j_o)$

(vi) $\sum_{k=j_2}^{k=j_3-1} \phi_2(k) \leqq V^{\gamma(\beta,e)/\beta}_{2m}(j_3) - V^{\beta}_{2M}(j_2)$ for all $j_2 \varepsilon [j_o, \ldots, j_1)$ and all $j_3 \varepsilon (j_2, \ldots, j_1]$.

Proof of Theorem 2.3.3:

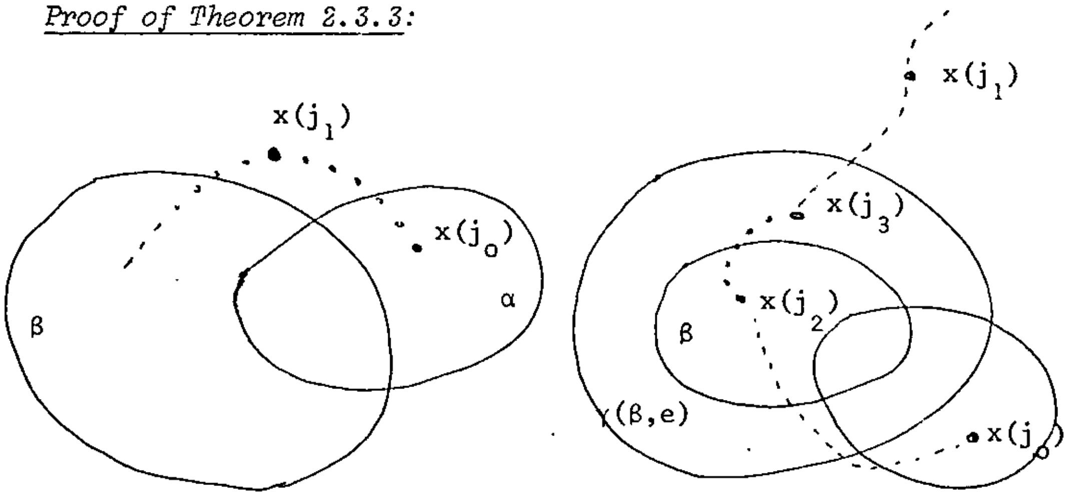


Fig. 2.3.2

Suppose, contrary to the expected conclusion, that there exists a trajectory $x(j)$, $x(j_o) \in \alpha$, such that

$$x(j_1) \in \beta^c \tag{2.3.6}$$

then, two possibilities may arise, either

(a) the trajectory is such that

$$x(j) \in \beta^c, \text{ all } j \in [j_o, \ldots, j_1] \tag{2.3.7}$$

{Fig. 2.3.2(a)}

or

(b) there exists a $j_2 \in [j_o, \ldots, j_1)$ such that

$$x(j_2) \in \beta \tag{2.3.8}$$

{Fig. 2.3.2(b)}

Suppose, for a moment, that the second situation occurs, then by (i) and (2.3.6), there exists a $j_3 \in (j_2, \ldots, j_1]$ such that

$$x(j_3) \in \gamma(\beta,e)/\beta, \; x(j) \in \gamma(\beta,e), \text{ all } j \in [j_2, \ldots, j_3] \tag{2.3.9}$$

But then,

$$V_2[x(j_3),j_3] = V_2[x(j_2),j_2] + \sum_{k=j_2}^{k=j_3-1} \Delta V[x(k),k] .$$

Using (ii), (iv) and (vi), we get

$$V_2[x(j_3),j_3] < V_{2m}^{\gamma(\beta,e)/\beta}(j_3) \tag{2.3.10}$$

But $x(j_3) \varepsilon \gamma(\beta,e)/\beta$, by (2.3.9). Hence, the last inequality (2.3.10) constitutes a contradiction in view of the definition of $V_{2m}^{\gamma(\beta,e)/\beta}(j)$. Thus, the situation (b) is not possible. The only other possibility is (a), but in this case

$$V[x(j_1),j_1] < V_{1m}^{\beta^c}(j_1) \tag{2.3.11}$$

by (ii), (iii) and (v). Inequality (2.3.11) gives the required contradiction. The theorem follows.

Remark 2.3.3:

Other theorems may be established avoiding most of the probable complications of the above theorems. This can be done in a manner similar to that adopted for differential systems with the appropriate modifications.

2.4 Non semi-final stability with respect to the sets (α, β, j_1 ε J):

The proofs of the theorems of this section, and the following section, will be omitted for their similarity to the ones given in the preceding sections.

1. <u>The case $\alpha \subseteq \beta$:</u>

Obviously, any system (DS) is semi-finally stable with respect to the sets $(\alpha, \beta, j_o \in J)$, as long as $\alpha \subseteq \beta$. So, we assume that $j_1 > j_o$.

Theorem 2.4.1:

System (DS) is not semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, $\alpha \subseteq \beta$, $j_1 > j_o$, if there exist a positive number e, two functions $V[x(j),j]$, $\phi(j)$ defined for all $x(j) \in \gamma[\alpha,(j_1 - j_o)e]$, $j \in [j_o, \ldots, j_1]$, and a point $x_o \in \alpha$, such that

(i) $||\Delta x(j)|| < e$, $j \in [j_o, \ldots, j_1]$, along the trajectory $x(j)$, $x(j_o) = x_o$.

(ii) $\gamma[\alpha,(j_1-j_o)e] / \beta \neq \emptyset$

(iii) $V_M^{\gamma[\alpha,(j_1-j_o)e]\cap\beta}(j_1)$ is finite.

(iv) $\Delta V[x(j),j] > \phi(j)$, $j \in [j_o, \ldots, j_1]$, along the trajectory $x(j)$, $x(j_o) = x_o$.

(v) $\sum_{k=j_o}^{k=j_1-1} \phi(k) \geqq V_M^{\gamma[\alpha,(j_1-j_o)e]\cap\beta}(j_1) - V[x_o,j_o]$.

(vi) there exists no positive number e^*, such that $||\Delta x(j)|| < e^*$, for all $j \in [j_o, \ldots, j_1]$, and

$$\gamma[\alpha,(j_1-j_o)e^*]/\beta = \emptyset \tag{2.4.1}$$

Note that (vi) implies condition (ii). From (i), the trajectory $x(j)$ emanating from $x_o \in \alpha$ at $j = j_o$, remains in $\gamma[\alpha,(j_1-j_o)e]$ for all $j \in [j_o, \ldots, j_1]$. This can be explained as follows:

$$x(j) - x(j_o) = \sum_{k=j_o}^{k=j-1} \Delta x(j)$$

and for any $j \in (j_o, j_1]$, we get

$$||x(j) - x(j_o)|| \leqq \sum_{k=j_o}^{k=j-1} ||\Delta x(j)|| < e.(j-j_o) \leqq (j_1-j_o)e$$

Thus, if (vi) does not hold, i.e., there exists a e^*, such that $||\Delta x(j)|| < e^*$, and (2.4.1) holds, then we get

$$x(j) \in \beta, \quad \text{all } j \in [j_o, \ldots, j_1]$$

which means that the system is, in fact, finally-stable with respect to the sets $(\alpha, \beta, j_o \in J)$. Hence, condition (vi) is essential.

The existence of positive number e, no matter how large, may be a restriction. That is, it is possible that there exists a system (DS) not semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, without satisfying condition (i). The following version of the theorem avoids this restriction. We shall use the additional notation

$$\delta(x_o, J^*) = \{x | x = x(j),\ x(j_o) = x_o, \text{ for all } j \in J^*\},\ J^* \subseteq J \qquad (2.4.2)$$

i.e., the set of all points x contained in the trajectory $x(j)$, $x(j_o) = x_o$.

Theorem 2.4.2:

System (DS) is not semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, $\alpha \subseteq \beta$, $j_1 > j_o$, if there exist two functions $V[x(j),j]$, $\phi(j)$, defined for all $x(j) \in \delta(x_o, J^*)$, $j \in J^*$, $J^* = [j_o, \ldots, j_1]$, such that

(i) $V_M^{\delta(x_o,j^*)\cap\beta}(j_1)$ is finite.

(ii) $\Delta V[x(j),j] > \phi(j)$, $j \in J^*$, along the trajectory $x(j)$, $x(j_o) = x_o$.

(iii) $\sum_{k=j_o}^{k=j_1-1} \phi(k) \geqq V_M^{\delta(x_o,J^*)\cap\beta}(j_1) - V[x_o,j_o]$.

(iv) there exists no positive number $e^* > 0$ such that

$||\Delta x(j)|| < e^*$, all $j \in [j_o, \ldots, j_1]$, and

$$\gamma[\alpha,(j_1 - j_o)e^*]/\beta = \emptyset \qquad (2.4.1)$$

Let us note, immediately, that conditions (ii), (vi), and condition (iv) of Theorem 2.4.1 and Theorem 2.4.2 respectively, do not present any difficulty in the application of the above theorems. It will be sufficient to show that the other conditions hold, so that the trajectory $x(j)$, $x(j_o) = x_o \in \alpha$, will be such that $x(j_1) \in \beta^c$. This will imply that the conditions under consideration hold. The advantage of mentioning these conditions can be explained as follows: If one can show the existence of such a number e^*, then we know immediately that the system is semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, or, at least, the trajectory $x(j)$, $x(j_o) \in \alpha$, is such that $x(j_1) \in \beta$.

2. <u>The case $\alpha \not\subset \beta$</u>

Taking into account the discussion which follows Theorem 2.4.2, one can show that it is valid in the general case. So we have the following:

Theorem 2.4.3:

System (DS) is not semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, $\alpha \not\trianglelefteq \beta$, if there exist a point $x_o \in \alpha$, and two functions $V[x(j),j]$, $\phi(j)$ defined for all $x(j) \in \delta(x_o,J^*)$, $j \in J^*$, $J^* = [j_o, \ldots, j_1]$, such that

(i) $V_M^{\delta(x_o,J^*)\cap\beta}(j_1)$ is finite.

(ii) $\Delta V[x(j),j] > \phi(j)$, $j \in [j_o, \ldots, j_1]$, along the trajectory $x(j)$, $x(j_o) = x_o$.

(iii) $\sum_{k=j_o}^{k=j_1-1} \phi(k) \geqq V_M^{\delta(x_o,J^*)\cap\beta}(j_1) - V[x_o,j_o]$.

Remark 2.4.1:

The theory established in this section suggests the idea of replacing whenever possible, the whole space R^n with the sets $\delta(x_o,J)$ or $\delta(\alpha,J)$ where

$$\delta(\alpha,J) = \{\delta(x_o,J): \text{ all } x_o \in \alpha\} \qquad (2.4.3)$$

2.5 Strong semi-final stability:

Theorem 2.5.1:

System (DS) is strongly semi-finally stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, $j_1 > j_o$, if

(a) it is semi-finally stable with respect to the sets (α, β, J), and

(b) there exist two functions $V[x(j),j]$, $\phi(j)$, defined for all

$x(j) \in R^n$, $j \in [j_o, \ldots, j_1]$, such that

(i) $V_M^{\alpha^c}(j_o)$ and $V_m^{\beta}(j_1)$ are finite.

(ii) $\Delta V[x(j), j] < \phi(j)$, $j \in [j_o, \ldots, j_1]$, along any trajectory $x(j)$, $x(j_o) \in \alpha^c$.

(iii) $\sum_{k=j_o}^{k=j_1-1} \phi(k) \leqq V_m^{\beta}(j_1) - V_M^{\alpha^c}(j_o)$.

If the conditions of the above theorem hold for all $x_o \in \alpha^c/\beta$ instead of α^c, and all $j_1 \in J$, $j_1 > j_o$, then system (DS) is strongly semi-finally stable with respect to the sets (α, β, J).

§3. GENERAL THEOREMS ON THE DIFFERENT TYPES OF FINAL-STABILITY

3.1 Final-stability:

1. *Theorem 3.1.1:*

System (DS) is finally-stable with respect to the sets (α, β, J), if there exist a positive constant e, two functions $V[x(j),j]$, and $\phi(j)$ defined for all $x(j) \in \gamma(\alpha, j_N e)$, $j \in J$, such that

(i) $||\Delta\, x(j)|| < e$, $j \in J$, along any trajectory $x(j)$, $x(j_o) \in \alpha$.

(ii) $V_M^{\alpha}(j_o)$ and $V_m^{\gamma(\alpha,j_N e)/\beta}(j_o+j_N)$ are finite.

(iii) $\Delta V[x(j),j] < \phi(j)$, $j \in J$, along any trajectory $x(j)$, $x(j_o) \in \alpha$.

(iv) $\sum_{k=j_o}^{k=j_o+j_N-1} \phi(k) \leq V_m^{\gamma(\alpha,j_N e)/\beta}(j_o+j_N) - V_M^{\alpha}(j_o)$.

We note that condition (ii) implies that the set $\gamma(\alpha, j_N e)/\beta$ is not empty. If it is empty, then $\gamma(\alpha, j_N e) \subseteq \beta$, and hence system (DS) is necessarily finally-stable with respect to the sets (α, β, J). In fact, since $\gamma(\alpha, j_N e)$ contains α in its interior, then $\alpha \subseteq \beta$; in this case, the system is stable (in the sense given in [30]) with respect to the sets (α, β, J).

Proof of Theorem 3.1.1:

Let $x(j)$ be any trajectory of system (DS) with $x(j_o) \in \alpha$, and assume, contrary to the expected conclusion, that

$$x(j) \in \beta^c, \text{ for all } j \in [j^*, \ldots, j_o+j_N] \tag{3.1.1}$$

for some $j^* \varepsilon J$. Since $x(j) \varepsilon \gamma(\alpha, j_N e)$, by (i), (3.1.1) gives

$$x(j) \varepsilon \gamma(\alpha, j_N e)/\beta, \text{ all } j \varepsilon [j^*, \ldots, j_o+j_N] \qquad (3.1.2)$$

But then

$$V[x(j_o+j_N), j_o+j_N] = V[x(j_o),j_o] + \sum_{k=j_o}^{k=j_o+j_N-1} \Delta V[x(k),k]$$

Using (iii) and (iv), we get

$$V[x(j_o+j_N), j_o+j_N] < V_m^{\gamma(\alpha,j_N e)/\beta}(j_o+j_N) \qquad (3.1.3)$$

which is the required contradiction. <u>The theorem follows</u>.

2. *Theorem 3.1.2:*

System (DS) is not finally-stable with respect to the sets (α, β, J), if there exist a point $x_o \varepsilon \alpha$, two functions $V[x(j),j]$, and $\phi(j)$ defined for all $x(j) \varepsilon \delta(x_o,J)$, $j \varepsilon J$, such that

(i) $V_M^{\delta(x_o,J)\cap\beta}(j_o+j_N)$ is finite.

(ii) $\Delta V[x(j),j] > \phi(j)$, $j \varepsilon J$, along the trajectory $x(j)$, $x(j_o) = x_o$.

(iii) $\sum_{k=j_o}^{k=j_o+j_N-1} \phi(k) \geqq V_M^{\delta(x_o,J)\cap\beta}(j_o+j_N) - V[x_o,j_o]$

Condition (i) implies that $\delta(x_o,J)\cap\beta \neq \emptyset$. In fact, if $\beta \cap \delta(x_o,J) = \emptyset$, then system (DS) is certainly not finally-stable with respect to the sets (α, β, J).

3.2 Final-stability with respect to the sets $(\alpha, \beta, j_1 \varepsilon J)$:

A few additional notations are needed in this case: let β be any connected set in R^n, then

$$\gamma^-(Fr.\beta,e) = \{x | x \varepsilon \beta,\ d(x,Fr.\beta) < e\} \tag{3.2.1}$$

$$\gamma^+(Fr.\beta,e) = \{x | x \varepsilon \beta^c,\ d(x,Fr.\beta) < e\} \tag{3.2.2}$$

where e is a positive number.

1. *Theorem 3.2.1:*

System (DS) is finally-stable with respect to the sets $(\alpha, \beta, j_1 \varepsilon J)$, $j_1 < j_o + j_N$, if there exist a positive constant e, a set $\gamma \subset \beta$, and two functions $V[x(j),j]$, $\phi(j)$ defined for all $x(j) \varepsilon \beta/\gamma$ and all $j \varepsilon [j_1, \ldots, j_o+j_N]$, such that

(i) $||\Delta x(j)|| < e$, $j \varepsilon [j_1, \ldots, j_o+j_N]$, along any trajectory $x(j)$, $x(j_o) \varepsilon \alpha$.

(ii) $d(Fr.\beta, Fr.\gamma) \geqq 2e$.

(iii) $V_M^{\gamma^+(Fr.\gamma,e)}(j)$ and $V_m^{\gamma^-(Fr.\beta,e)}(j)$, $j \varepsilon (j_1, \ldots, j_o+j_N)$, are finite.

(iv) $\Delta V[x(j),j] < \phi(j)$, $j \varepsilon [j_1, \ldots, j_o+j_N]$, along any trajectory $x(j)$, $x(j_o) \varepsilon \alpha$, as long as $x(j) \varepsilon \beta/\gamma$.

(v) $\sum_{k=j_2}^{k=j_3-1} \phi(k) \leqq V_m^{\gamma^-(Fr.\beta,e)}(j_3) - V_M^{\gamma^+(Fr.\gamma,e)}(j_2)$,

all $j_2 \varepsilon (j_1, \ldots, j_o+j_N)$, $j_3 \varepsilon (j_2, \ldots, j_o+j_N)$.

(vi) the system (DS) is semi-finally stable with respect to the sets $(\alpha, \gamma, j_1 \varepsilon J)$.

The case $j_1 = j_o+j_N$ is included in the case of final-stability with respect to the sets (α, β, J). Condition (v) implies that $(j_1+1, \ldots, j_o+j_N)$ is not empty. On the other hand, if $j_1 = j_o+j_N-2$, then conditions (i), (ii) and (vi) imply that $x(j_o+j_N) \in \beta$.

Proof of Theorem 3.2.1:

Let $x(j)$, $x(j_o) \in \alpha$, be any trajectory of (DS), then

$$x(j_1) \in \gamma, \text{ by (vi)}, \tag{3.2.3}$$

Suppose, contrary to the expected conclusion, that there exists a first $j_q \in (j_1, \ldots, j_o+j_N]$, such that

$$x(j_q) \in \beta^c, \tag{3.2.4}$$

then, by (i), there exists a $j_3 = j_q - 1$, $j_3 \in (j_1, \ldots, j_o+j_N)$, such that

$$x(j_3) \in \gamma^-(\text{Fr}.\beta, e) \tag{3.2.5}$$

Also, there exists a <u>last</u> $j_p \in [j_1, \ldots, j_q)$, such that

$$x(j_p) \in \gamma \tag{3.2.6}$$

then, by (i), there exist a $j_2 = j_p + 1$, $j_2 \in (j_1, \ldots, j_q)$, such that

$$x(j_2) \in \gamma^+(\text{Fr}.\gamma, e) \tag{3.2.7}$$

By taking (i) and (ii) into account, it follows that

$$j_o \leqq j_1 \leqq j_p < j_2 < j_3 < j_q \leqq j_o+j_N \tag{3.2.8}$$

and that

$$x(j) \in \beta/\gamma, \text{ all } j \in [j_2, \cdots j_3] \tag{3.2.9}$$

But then,

$$V[x(j_3),j_3] = V[x(j_2),j_2] + \sum_{k=j_2}^{k=j_3-1} \Delta V[x(k),k]$$

which gives, using (iv) and (v),

$$V[x(j_3),j_3] < V_m^{\gamma^-(Fr.\beta,e)}(j_3) \tag{3.2.10}$$

which is in obvious contradiction to (3.2.5).

Thus, the original assumption (3.2.4) is false, and there is no j_q such that $x(j_q) \in \beta^c$.

Since the above argument is independent of the exact value of $x(j_o) \in \alpha$ and the particular trajectory chosen, it holds for all trajectories emanating from α at $j = j_o$, and the theorem is proved.

Remark 3.2.1:

Assuming that the system (DS) is finally-stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, and suppose that there exists a first $j_k \in (j_1, \ldots, j_o+j_N)$, such that

$$x(j_k) \in \gamma^-(Fr.\beta,e)$$

then there exists j_4 such that $x(j_4) \in \gamma^+(Fr.\gamma,e)$.
So let $j_5 \in (j_1, \ldots, j_k)$ be such that

$$x(j_5) \in \gamma^+(Fr.\gamma,e), \quad x(j) \in \beta/\gamma, \quad j \in [j_5, \ldots, j_k].$$

But then, the conditions of Theorem 3.2.1 give the contradiction

$$V[x(j_k),j_k] < V_m^{\gamma^-(Fr.\beta,e)}(j_k).$$

So, there is no $j_k \in (j_1, \ldots, j_o+j_N)$ such that $x(j_k) \in \gamma^-(Fr.\beta,e)$. We can conclude then that the system (DS) is finally-stable with respect to the sets $\{\alpha, \beta/\gamma^-(Fr.\beta,e), j_1 \in [j_o, \ldots, j_o+j_N-1]\}$. To avoid this restriction, we can replace (iii) and (v) by

(iii)* $V_M^{\gamma^+(Fr.\gamma,e)}(j),\ j \in (j_1, \ldots, j_o+j_N)$,

$V_m^{\gamma^+(Fr.\beta,e)}(j),\ j \in (j_1, \ldots, j_o+j_N]$, are finite.

(iv)* $\sum_{k=j_2}^{j_3-1} \phi(k) \leqq V_m^{\gamma^+(Fr.\beta,e)}(j_3) - V_M^{\gamma^+(Fr.\gamma,e)}(j_2)$

$j_2 \in (j_1, \ldots, j_o+j_N)$

$j_3 \in (j_2, \ldots, j_o+j_N]$

provided $V[x(j),j]$ is defined over $\gamma^+(Fr.\beta,e)/\gamma$. Moreover, according to this remark, condition (ii) may be relaxed to (ii)* $d(Fr.\beta, Fr.\gamma) \geqq e$, provided (iii)* and (iv)*. These modifications make it possible to apply the theorem in the case where diam.$\beta < 2e$, where

$$\text{diam.}\beta = \sup_{\substack{x\in\beta \\ y\in\beta}} d(x,y).$$

for, in this case, (ii) of Theorem 3.2.1 cannot be satisfied for any set $\gamma \subseteq \beta$.

2. Let $\bar{\alpha} \subset I(\beta)$, and let

$$Q = d(Fr.\beta, Fr.\alpha) \tag{3.2.11}$$

be positive. In this case, we have the following theorem, which follows from Theorem 3.2.1, by putting $\gamma = \alpha$, $e = Q/2$, and $j_1 = j_o$ {Condition (vi) is automatically satisfied}. We note, finally, that this theorem is a direct generalisation of the stability theorem established in [30].

Theorem 3.2.2:

System (DS) is stable with respect to the sets (α, β, J), $\bar{\alpha} \subset I(\beta)$, i.e., finally-stable with respect to the sets $(\alpha, \beta, j_o \in J)$; if there exist two functions $V[x(j),j]$, $\phi(j)$ defined for all $x(j) \in \beta/\alpha$, and $j \in J$, such that

(i) $||\Delta x(j)|| < Q/2$, $j \in J$, along any trajectory $x(j)$, $x(j_o) \in \alpha$.

(ii) $V_M^{\gamma^+(Fr.\alpha,e)}(j)$ and $V_m^{\gamma^-(Fr.\beta,e)}(j)$, $j \in (j_o, \ldots, j_o+j_N)$, $e = Q/2$. are finite.

(iii) $\Delta V[x(j),j] < \phi(j)$, $j \in J$, along any trajectory $x(j)$, $x(j_o) \in \alpha$, as long as $x(j) \in \beta/\alpha$.

(iv) $\sum_{k=j_2}^{k=j_1-1} \phi(k) \leq V_m^{\gamma^-(Fr.\beta,e)}(j_1) - V_M^{\gamma^+(Fr.\alpha,e)}(j_2)$, $e = Q/2$ all $j_2 \in (j_o, \ldots, j_o+j_N)$, and $j_1 \in (j_2, \ldots, j_o+j_N)$.

(One can deduce a different result by means of Remark 3.2.1.)

3. We note that Theorem 3.2.1 does not permit us to conclude anything about the final-stability with respect to the sets $(\alpha, \beta, j_o \in J)$, $\alpha \subseteq \beta$, i.e., stability [30] with respect to the sets (α, β, J). Thus, the following theorem is important. We note that there is no corresponding theorem in [30].

Theorem 3.2.3:

System (DS) is stable [30] with respect to the sets (α, β, J), $\alpha \subseteq \beta$, i.e., finally-stable with respect to the sets $(\alpha, \beta, j_o \in J)$; if there exist a positive number e, two functions $V[x(j),j]$, $\phi(j)$ defined for all $x(j) \in \gamma(\beta,e)$, $j \in J$, such that

(i) $||\Delta x(j)|| < e$, $j \in J$, along any trajectory $x(j)$, $x(j_o) \in \alpha$.

(ii) $V_M^{\alpha}(j_o)$ and $V_m^{\gamma^+(Fr.\beta,e)}(j)$, $j \in (j_o, \ldots, j_o+j_N]$, are finite.

(iii) $\Delta V[x(j),j] < \phi(j)$, $j \in J$, along any trajectory $x(j)$, $x(j_o) \in \alpha$, as long as $x(j) \in \gamma(\beta,e)$.

(iv) $\sum_{k=j_o}^{k=j_1-1} \phi(K) \leqq V_m^{\gamma^+(Fr.\beta,e)}(j_1) - V_M^{\alpha}(j_o)$, all $j_1 \in (j_o, \ldots, j_o+j_N]$.

4. *Theorem 3.2.4:*

System (DS) is not finally-stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, $j_1 \in (j_o, \ldots, j_o+j_N)$, if there exist a point $x_o \in \alpha$, a point $t_2 \in [j_1, \ldots, j_o+j_N)$, two functions $V[x(j),j]$, and $\phi(j)$ defined for all $x(j) \in \delta(x_o,J_2)$, $j \in J_2 = [j_o, \ldots, j_2]$, such that

(i) $V_M^{\delta(x_o,J_2)\cap\beta}(j_2)$ is finite.

(ii) $\Delta V[x(j),j] > \phi(j)$, $j \in [j_o, \ldots, j_2]$, along the trajectory $x(j)$, $x(j_o) = x_o$.

(iii) $\sum_{k=j_o}^{k=j_2-1} \phi(k) \geqq V_M^{\delta(x_o,J_2)\cap\beta}(j_2) - V[x_o,j_o]$

This theorem can be proved in the usual manner.

3.3 Strong final-stability

A theory concerning this aspect of final-stability can be established in a similar way. But, we limit our discussion to the statement of two theorems which may constitute a new formulation of the definitions of strong final-stability:

Theorem 3.3.1:

System (DS) is strongly finally-stable with respect to the sets (α, β, J), if

(i) it is finally-stable with respect to the sets (α, β, J), and

(ii) to every set $\alpha^* \subset \alpha^c$, there corresponds a set $\beta^* = \beta^*(\alpha^*) \subseteq \beta^c$, such that the system is finally-stable with respect to the sets (α^*, β^*, J).

Theorem 3.3.2:

System (DS) is strongly finally-stable with respect to the sets $(\alpha, \beta, j_1 \in J)$, if

(i) it is finally-stable with respect to the sets (α, β, J), and

(ii) to each $\alpha^* \subset \alpha^c$, there corresponds a set $\beta^* \subseteq \beta^c$, such that the system (DS) is finally-stable with respect to the sets $(\alpha^*, \beta^*, j_1 \in J)$.

§4. CONCLUSION

One notes the similarity between the theory established in the preceding sections and that developed in Chapter I concerning differential systems without the influence of perturbing forces. The extension of this theory to discrete systems under the influence of perturbing forces is possible and follows the same lines as the theory established in Chapter II.

However, some essential modifications were obviously necessitated by the nature of the discrete system (DS). For example,

(i) the total derivative of V had to be replaced by its total difference, and

(ii) the use of the continuity of the trajectories with respect to the independent variable t had to be abandoned for the simple reason that this property is no longer true.

Finally, an obvious reason for the importance of the theory of final-stability for discrete systems is that, in many practical applications, one has to make use of the difference system related to the differential system under consideration.

CHAPTER IV

Dynamical Systems

§1. INTRODUCTION AND DEFINITIONS

1.1 Introduction

Let E be a metric space with the metric distance $\rho(p,q)$, $p \in E$, $q \in E$. Then,

Definition 1.1.1: [3,4,50]

A dynamical system (or continuous flow) on E, is the triplet (E,R,f), where $f: E \times R \to E$ is a mapping from the product space $E \times R$ into E satisfying the following axioms:

(i) $f(p,0) = p$, for all $p \in E$

(ii) $f\big(f(p,t_1),t_2\big) = f(p,t_1+t_2)$, for every $p \in E$, and $t_1,t_2 \in R$.

(iii) f is continuous.

The above axioms are usually referred to as the identity, hom¢omorphism, and continuity axioms, respectively.

In the sequel, we shall generally drop the symbol f. Thus the image $f(p,t)$ of a point $(p,t) \in E \times R$ will be written simply as pt. The identity and hom¢omorphism axioms then read

(i)* $po = p$, all $p \in E$

(ii)** $pt_1(t_2) = p(t_1+t_2)$, all $p \in E$, and all $t_1,t_2 \in R$.

In the line with the above notation, if $\alpha \subseteq E$, and $J \subseteq R$, then we set

$$\alpha J = \{q \mid q = pt: \ p \in \alpha, \ t \in J\} \qquad (1.1.1)$$

If α is a singleton $\{p\}$ then the segment trajectory over $J \subseteq R$ is defined as follows:

$$pJ = \{q | q = pt, \ t \in J\} \tag{1.1.2}$$

and the trajectory through p is then denoted by pR. We shall, moreover, use the notation $\alpha(J)$, $J = [t_o, t_o+T]$, to denote the set

$$\alpha(J) = \{q | q = pt, \ p \in E, \ t \in J, \ \text{s.t. } pt_o \in \alpha\} \tag{1.1.3}$$

It is to be noted that $\alpha(J) = \alpha J$, if $t_o = 0$.

We shall limit our discussion to the closed interval

$$J = [t_o, t_o+T], \ t_o \in R, \ T > 0 \tag{1.1.4}$$

We also define the set

$$J(t_1,t_2) = [t_1,t_2], \ t_1 \in R, \ t_2 \in R \tag{1.1.5}$$

(if $t_2 = +\infty$, then $J(t_1,t_2) = [t_1,t_2)$.)

Though it is possible to extend the theory established in Chapter I to dynamical systems by replacing, in the corresponding theorems of Chapter II, $V^*[x,t]$ by

$$D^+V[pt,t] = \limsup_{h \to 0^+} \frac{V[p(t+h),t+h] - V[pt,t]}{h} \tag{1.1.6}$$

We shall, however, establish a different approach to the theory considering only closed connected sets α, β, etc. in E and the closed interval J.

Also, it would be possible to extend this theory to dynamical systems defined by the mapping

$$\pi: \; E \times S \times R \to E$$

where S is the control space.

The theory established, in this chapter, will be limited to the basic types of final-stability, i.e., we shall not consider the cases of strong (semi-)final-stability which can be done in a similar way.

1.2 Definitions of final-stability:

Definition 1.2.1:

System (E,R,f) is semi-finally stable with respect to the sets (α, β, J), if, for any trajectory pR the condition

$$\rho(pt_o,\alpha) = 0 \qquad (1.2.1)$$

implies the existence of a $t_1 \in J$, such that

$$\rho(pt_1,\beta) = 0 \qquad (1.2.2)$$

where t_1 may depend on the trajectory pR.

$\left(\rho(q,\gamma) = \min_{x\epsilon\gamma}.(q,x), \gamma \subset E\right)$

Definition 1.2.2:

System (E,R,f) is not semi-finally stable with respect to the sets (α, β, J), if there exists a trajectory p^*R, $p^* \in E$, such that

$$\rho(p^*t_o,\alpha) = 0 \qquad (1.2.3)$$

and

$$\rho(p^{*}t,\beta) > 0 \quad \text{all } t \in J \tag{1.2.4}$$

Definition 1.2.3:

System (E,R,f) is semi-finally stable with respect to the sets (α, β, $t_1 \in J$), if for any trajectory pR, $p \in E$, the condition

$$\rho(pt_o,\alpha) = 0 \tag{1.2.1}$$

implies that

$$\rho(pt_1,\beta) = 0 \tag{1.2.2}$$

Definition 1.2.4:

System (E,R,f) is not semi-finally stable with respect to the sets (α, β, $t_1 \in J$), if there exists a trajectory $p^{*}R$, $p^{*} \in E$, such that

$$\rho(p^{*}t_o,\alpha) = 0, \quad \rho(p^{*}t_1,\beta) > 0 \tag{1.2.5}$$

Definition 1.2.5:

System (E,R,f) is uniformly semi-finally stable with respect to the sets (α, β, J), if there exists a $t_1 \in J$, such that the system is semi-finally stable with respect to the sets (α, β, $t_1 \in J$).

Definition 1.2.6:

System (E,R,f) is finally-stable with respect to the sets (α, β, J), if for any trajectory, pR, the condition

$$\rho(pt_o,\alpha) = 0 \tag{1.2.1}$$

implies the existence of a $t_1 \in J$, such that

$$\rho(pt,\beta) = 0, \quad \text{all } t \in J(t_1, t_o+T) \qquad (1.2.6)$$

where $J(t_1, t_o+T) = [t_1, t_o+T]$, and t_1 may depend on the particular trajectory.

Definition 1.2.7:

System (E,R,f) is not finally-stable with respect to the sets (α, β, J) if there exists a trajectory p^*R, $p^* \in E$, and a point $t^* \in J$, such that

$$\rho(pt_o, \alpha) = 0, \quad \rho(p^*t,\beta) > 0, \quad \text{all } t \in J(t^*,t_o+T) \qquad (1.2.7)$$

Definition 1.2.8:

System (E,R,f) is finally-stable with respect to the sets (α, β, $t_1 \in J$) if, for any trajectory pR, the condition

$$\rho(pt_o,\alpha) = 0 \qquad (1.2.1)$$

implies that

$$\rho(pt,\beta) = 0, \quad \text{all } t \in J(t_1, t_o+T) \qquad (1.2.6)$$

Definition 1.2.9:

System (E,R,f) is not finally-stable with respect to the sets (α, β, $t_1 \in J$), if there exists a trajectory p^*R, such that

$$\rho(p^*t_o,\alpha) = 0, \quad \rho(p^*t_2,\beta) > 0 \qquad (1.2.8)$$

for some $t_2 \in J(t_1, t_o+T)$.

Definition 1.2.10:

System (E,R,f) is uniformly finally-stable with respect to the sets (α, β, J), if there exists a t_1 ε J, such that the system is finally-stable with respect to the sets (α, β, t_1 ε J).

1.3 Lemmas

In order to establish the intended theory, we need to state some essential definitions and lemmas. Let $V[s,t]$ denote a mapping

$$V: R^+ \times J \rightarrow R^1 \tag{1.3.1}$$

Definition 1.3.1:

$$V_m^\alpha(t;\beta) = \inf_{q\varepsilon\alpha}.V[\rho(q,\beta),t], \quad V_M^\alpha(t;\beta) = \sup_{q\varepsilon\alpha}.V[\rho(q,\beta),t] \tag{1.3.2}$$

Definition 1.3.2:

$V[s,t] \;\varepsilon\; C(B,D)$, $B \subseteq R^+$, $D \subseteq J$, indicates that the function $V[s,t]$ is continuous over $B \times D$.

Definition 1.3.3:

For any continuous function c(t), we define

$$D^+V[c(t),t] = \limsup_{h\to o^+}. \frac{V[c(t+h),t+h] - V[c(t),t]}{h} \tag{1.3.3}$$

$$D^-V[c(t),t] = \liminf_{h\to o^+}. \frac{V[c(t+h),t+h] - V[c(t),t]}{h} \tag{1.3.4}$$

Definition 1.3.4:

A function $\omega: J \times R^1 \to R^1$ is said to be of class $\Omega(D,R)$, $D \subseteq J$, if it is smooth enough to ensure the existence of the maximal and the minimal solutions of

$$\dot{r} = \omega(t,r) \tag{C}$$

over D.

Lemma 1.3.1: [4,13,18.2]

Let $\omega(t,r) \in \Omega(D,R)$, $D \subseteq J$, and let $\mu(t)$ be a continuous function such that $(t,\mu(t)) \in D \times R$, for $t \in D$, and

$$D^+\mu(t) = \lim\sup_{h \to 0^+} \frac{\mu(t+h) - \mu(t)}{h} \leq \omega(t,\mu(t)) \tag{1.3.5}$$

for all $t \in D$. Then, if $\mu(t^*) \leq r^*$, $t^* \in D$, we have $\mu(t) \leq r_M(t)$, all $t \in D$, $t \geq t^*$, where $r_M(t)$ is the maximal solution of (C), with initial condition $r_M(t^*) = r^*$.

Lemma 1.3.2: [4,13,18.2]

Let $\omega(t,r) \in \Omega(D,R)$, $D \subseteq J$, and let $\mu(t)$ be continuous and such that $(t,\mu(t)) \in D \times R$, for $t \in D$, and

$$D^-\mu(t) = \lim\inf_{h \to 0^+} \frac{\mu(t+h) - \mu(t)}{h} \geq \omega(t,\mu(t)) \tag{1.3.6}$$

for all $t \in D$, then if $\mu(t^*) \geq r^*$, $t^* \in D$, we have $\mu(t) \geq r_m(t)$, all $t \in D$, $t \geq t^*$, where $r_m(t)$ is the minimal solution of (C), with initial condition $r_m(t^*) = r^*$.

Remark 1.3.1:

If D is an interval $[t_1,t_2]$, $t_1 \in J$, $t_2 \in J$, then we may assume that the conditions (1.3.5) and (1.3.6) hold for all $t \in [t_1,t_2)$.

§2. SEMI-FINAL STABILITY THEORY

2.1 Semi-final stability:

(1) The case $\alpha \subseteq \beta$ is trivial; so we assume that $\alpha \not\subseteq \beta$. Let

$$k_1 = \sup_{q\varepsilon\alpha^*(J)} \rho(q,\beta), \quad \alpha^* = \alpha/\beta \tag{2.1.1}$$

Theorem 2.1.1:

System (E,R,f) is semi-finally stable with respect to the sets (α, β, J), $\alpha \not\subseteq \beta$, if there exist two functions $V[s,t] \varepsilon C(J(0,k_1),J)$, $\omega(t,r) \varepsilon \Omega(J,R)$, such that

(i) $V_M^{\alpha/\beta}(t_o;\beta)$ is finite.

(ii) $V[s,t_o]$ is positive definite for all $s \varepsilon [0, \sup_{q\varepsilon\alpha/\beta}(q,\beta)]$, and $V[s,t]$ is such that

$$V[s,t] = 0 \Rightarrow s = 0, \quad \text{all } t \varepsilon J$$

(iii) $D^+V[\rho(pt,\beta),t] \leq \omega(t,V[\rho(pt,\beta),t])$, all $t \varepsilon J$, $\rho \varepsilon E$, s.t. $pt_o \varepsilon \alpha/\beta$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha/\beta}(t_o;\beta)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t_1) \leq 0, \quad \text{some } t_1 \varepsilon J,\ t_1 > t_o \tag{2.1.2}$$

Proof of Theorem 2.1.1:

Let pR, $p \in E$, be any trajectory such that

$$\rho(pt_o,\alpha) = 0 \tag{2.1.3}$$

If $pt_o \in \alpha \cap \beta$, the conclusion of the theorem is readily proved. Assume then that $pt_o \in \alpha/\beta$, then

$$0 < \rho(pt_o,\beta) \leqq \sup_{q\epsilon\alpha/\beta} \rho(q,\beta) \tag{2.1.4}$$

and

$$0 < V\left[\rho(pt_o,\beta),t_o\right] \leqq V_M^{\alpha/\beta}(t_o;\beta) \tag{2.1.5}$$

by (ii), (2.1.4), and (i). Using (iii), (2.1.5) and Lemma 1.3.1 - IV, with $\mu(t) = V\left[\rho(pt,\beta),t\right]$, and bearing in mind that

$$0 \leqq \rho(pt,\beta) \leqq k_1, t \in J \tag{2.1.6}$$

we get

$$V\left[\rho(pt,\beta),t\right] \leqq r_M(t), \quad \text{all } t \in J \tag{2.1.7}$$

where $r_M(t)$ is given by (iv). By (2.1.2),

$$V\left[\rho(pt_1,\beta),t_1\right] \leqq 0 \tag{2.1.8}$$

By (2.1.5) and (2.1.8), we conclude there exists a $t^* \in (t_o,t_1]$ such that

$$V\left[\rho(pt^*,\beta),t^*\right] = 0 \tag{2.1.9}$$

which gives, by (ii), $\rho(pt^*,\beta) = 0$, i.e., $pt^* \in \beta$.

Since the above argument is independent of the choice of p, $pt_o \in \alpha/\beta$, it holds for all trajectories pR, $pt_o \in \alpha/\beta$. This completes the proof of Theorem 2.1.1.

Remark 2.1.1:

We note, from the proof of Theorem 2.1.1, that any trajectory pR, $pt_o \in \alpha$ will enter β before or at $t = t_1$. This is a stronger result than required, and may be avoided by setting $t_1 = t_o + T$. On the other hand, condition (iv) can be modified in the following manner.

(iv)* to each $r_o \in B$, there corresponds a $t(r_o)$ such that the maximal solution $r_M(t;r_o)$, $r_M(t_o;r_o) = r_o$, of equation (C) is such that

$$r_M\left(t(r_o);r_o\right) \leq 0.$$

where B is the set such that $V\left[\rho(pt_o,\beta),t_o\right] \in B$, all $p \in E$, such that $pt_o \in \alpha/\beta$.

In this case (i) may be relaxed to the following statement:

$V\left[\rho(pt_o,\beta),t_o\right]$ is finite for each fixed $p \in E$, such that $pt_o \in \alpha/\beta$.

(2) Any system (E,R,f) is semi-finally stable with respect to the sets (α, β, J), $\alpha \subseteq \beta$. So the following theorem is concerned with the case $\alpha \not\subseteq \beta$.

Theorem 2.1.2:

System (E,R,f) is not finally-stable with the sets (α, β, J), $\alpha \not\subseteq \beta$, if there exists a point $p^* \in E$, $p^* t_o \in \alpha/\beta$, two functions $V[s,t] \in C\left(J(0,k_1),J\right)$, $\omega(t,r) \in \Omega(J,R)$, such that

(i) $V\left[\rho(p^* t_o,\beta), t_o\right]$ is non-negative and finite.

(ii) $V[s,t] \neq 0, \Rightarrow s \neq 0$, all $t \in J$

(iii) $D^{-}V[\rho(p^{*}t,\beta),t] \geqq \omega(t,V[\rho(p^{*}t,\beta),t])$, all $t \in J$.

(iv) the minimal solution $r_m(t), r_m(t_o) = V[\rho(p^{*}t_o,\beta),t_o]$, of equation (C) is such that

$$r_m(t) > 0, \quad \text{all } t \in J,\ t > t_o \qquad (2.1.10)$$

Proof of Theorem 2.1.2:

Consider the trajectory $p^{*}R$. Applying (i), (iii), and Lemma 1.3.2 - IV, we get

$$V[\rho(p^{*},t,\beta),t] \geqq r_m(t),\ t \in J \qquad (2.1.11)$$

By (2.1.10), we get

$$V[\rho(p^{*}t,\beta),t] > 0, \quad \text{all } t \in J,\ t > t_o \qquad (2.1.12)$$

which gives, using (ii),

$$\rho(p^{*}t,\beta) > 0, \quad \text{all } t \in J,\ t > t_o \qquad (2.1.13)$$

This completes the proof of the theorem.

Remark 2.1.2:

(1) k_1 may be replaced by

$$K^{*} = \sup_{t \in J} . \rho(p^{*}t,\beta). \qquad (2.1.4)$$

(2) Condition (i) of Theorem 2.1.2 is implied by condition (iv); for, by (2.1.10), we conclude that

$$r_m(t_o) = V[\rho(p^{*}t_o,\beta),t_o] \geqq 0.$$

2.2 Semi-final stability with respect to the sets $(\alpha, \beta, t_1 \varepsilon J)$:

(1) Let $J_1 = J(t_o, t_1)$ and

$$k_2 = \sup_{q \varepsilon \alpha(J_1)} \rho(q,\beta) \tag{2.2.1}$$

Theorem 2.2.1:

System (E,R,f) is semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \varepsilon J)$, if there exist two functions $V[s,t] \varepsilon C(J(0,k_2),J_1)$, $\omega(t,r) \varepsilon \Omega(J_1,R)$ such that

(i) $V_M^\alpha(t_o;\beta)$ is finite.

(ii) $V[s,t_1]$ is positive definite for all $s \varepsilon J(0,k_2)$.

(iii) $D^+V[\rho(pt,\beta),t] \leqq \omega(t,V[\rho(pt,\beta),t])$, all $t \varepsilon J(t_o,t_1)$, $p \varepsilon E$, such that $pt_o \varepsilon \alpha$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^\alpha(t_o;\beta)$, of equation (C) is such that $r_M(t_1) = 0$.

Proof of Theorem 2.2.1:

Let pR, $p \varepsilon E$, be any trajectory of (E,R,f), with $pt_o \varepsilon \alpha$, then

$$V[\rho(pt_o,\beta),t_o] \leqq V_M^\alpha(t_o;\beta) \tag{2.2.2}$$

Application of (2.2.2), (iii) and Lemma 1.3.1-IV gives

$$V[\rho(pt,\beta),t] \leqq r_M(t), \text{ all } t \varepsilon J(t_o,t_1) \tag{2.2.3}$$

then by (iv),

$$V[\rho(pt_1,\beta),t_1] \leqq 0$$

But $V[\rho(pt_1,\beta),t_1] \geqq 0$, by (ii), then

$$V[\rho(pt_1,\beta),t_1] = 0$$

which gives, using (ii) again, $\rho(pt_1,\beta) = 0$. The theorem follows

(2) The following theorem avoids the restriction that $V[s,t_1]$ must be positive-definite.

Theorem 2.2.2:

System (E,R,f) is semi-finally stable with respect to the sets (α, β, $t_1 \varepsilon J$), if there exist three functions $V[s,t] \varepsilon C(J(0,k_2),J_1)$, $\omega_1(t,r) \varepsilon \Omega(J_1,R)$, and $\omega_2(t,r) \varepsilon \Omega(J_1,R)$, such that

(i) $V[s,t_1] = 0 \Rightarrow s = 0$.

(ii) $V^{\alpha}_{M}(t_o;\beta)$ and $V^{\alpha}_{m}(t_o;\beta)$ are finite.

(iii) $D^{+}V[\rho(pt,\beta),t] \leqq \omega_1(t,V[\rho(pt,\beta),t])$,

$D^{-}V[\rho(pt,\beta),t] \geqq \omega_2(t,V[\rho(pt,\beta),t])$,

all $t \varepsilon J$, $p \varepsilon E$ such that $pt_o \varepsilon \alpha$.

(iv) the minimal solution $r_m(t)$, $r_m(t_o) = V^{\alpha}_{m}(t_o;\beta)$, of equation

$$\dot{r} = \omega_2(t,r) \qquad (C_2)$$

is such that $r_m(t_1) = 0$.

(v) the maximal solution $r_M(t)$, $r_M(t_o) = V^{\alpha}_{M}(t_o;\beta)$, of equation

$$\dot{r} = \omega_1(t,r) \qquad (C_1)$$

is such that $r_M(t_1) = 0$.

Proof of Theorem 2.2.2:

It is easy to show that

$$r_m(t) \leq V[\rho(pt,\beta)t] \leq r_M(t) \tag{2.2.4}$$

all $t \in J_1$, $p \in E$ with $pt_o \in \alpha$. Condition (iv), (v) and (2.2.4) give

$$V[\rho(pt_1,\beta)t_1] = 0 \tag{2.2.5}$$

Which gives, by (i), $\rho(pt_1,\beta) = 0$. The theorem follows.

(3) We give below a result concerning the non semi-final stability with respect to the sets $(\alpha, \beta, t_1 \in J)$. It can be proved in the same way as above.

Theorem 2.2.3: $(t_1 > t_o)$

System (E,R,f) is not semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in J)$, if there exist a point $p^* \in E$, with $p^*t_o \in \alpha$, two functions $V[s,t] \in C(J(0,k_2)$, $\omega(t,r) \in \Omega(J_1,R)$, such that

(i) $V[s,t_1] \neq 0 \Rightarrow s \neq 0$

(ii) $D^- V[\rho(p^*t,\beta),t] \geq \omega(t,V[\rho(p^*t,\beta),t])$, all $t \in J_1$.

(iii) $V[\rho(p^*t_o,\beta), t_o]$ is finite, and the minimal solution $r_m(t)$ of

$$\dot{r} = \omega(t,r) \quad r(t_o) = V[\rho(p^*t_o,\beta)t_o]$$

is such that $r_m(t_1) > 0$.

§3. FINAL-STABILITY THEORY

3.1 Final-Stability:

(i) *Theorem 3.1.1:*

System (E,R,f) is finally stable with respect to the sets (α, β, J), if there exist two functions $V[s,t] \in C(J(0,K_3),J)$, $\omega(t,r) \in \Omega(J,R)$, where

$$K_3 = \sup_{q\in\alpha(J)} \rho(q,\beta). \tag{3.1.1}$$

such that

(i) $V[s,t]$ is positive-definite with respect to s, for all $s \in J(0,K_3)$, and $t \in J$.

(ii) $V[\rho(pt_o,\beta),\ t_o]$ is finite, for all $p \in E$, $pt_o \in \alpha$.

(iii) $D^+V[\rho(pt,\beta),t] \leq \omega(t,V[\rho(pt,\beta),t])$, all $t \in J$, $p \in E$, with $pt_o \in \alpha$.

(iv) for each $r_o \in B$, there corresponds a $t(r_o) \in J$ such that the maximal solution $r_M(t;r_o)$, $r_M(t_o;r_o) = r_o$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_M(t;r_o) = 0, \quad \text{all } t \in J(t(r_o),t_o+T) \tag{3.1.2}$$

$B \subseteq R$ is the set such that

$$V[\rho(pt_o,\beta),t_o] \in B, \text{ all } p \in E \text{ s.t. } pt_o \in \alpha \tag{3.1.3}$$

Proof of Theorem 3.1.1:

Let pR, $p \in E$, $pt_o \in \alpha$, be any trajectory of (E,R,f), and let $t_p = t\left(V\left[\rho(pt_o,\beta),t_o\right]\right)$ as given by (iv). Using (ii), (iii) and Lemma 1.3.1 - IV, we get

$$r_M\left(t;\ V\left[\rho(pt_o,\beta),t_o\right]\right) \geqq V\left[\rho(pt,\beta),t\right] \tag{3.1.4}$$

for all $t \in J$. By (3.1.2),

$$V\left[\rho(pt,\beta),t\right] \leqq 0 \tag{3.1.5}$$

for all $t \in J(t_p,t_o+T)$. By (i) and (3.1.5), we conclude that

$$\rho(pt,\beta) = 0, \quad \text{all } t \in J(t_p,t_o+T) \tag{3.1.6}$$

The theorem follows.

(2) The following theorem avoids the restriction that $V[s,t]$ is positive-definite.

Theorem 3.1.2:

System (E,R,f) is finally-stable with respect to the sets (α, β, J), if there exist three functions, $V[s,t] \in C\left(J(0,K_3),J\right)$, $\omega_1 \in \Omega(J,R)$, and $\omega_2 \in \Omega(J,R)$, such that

(i) $V\left[\rho(pt_o,\beta),t_o\right]$ is finite, for all $p \in E$, such that $pt_o \in \alpha$, and

$$V[s,t] = 0 \Rightarrow s = 0, \text{ all } t \in J. \tag{3.1.7}$$

(ii) $D^+V\left[\rho(pt,\beta),t\right] \leqq \omega_1\left(t,V\left[\rho(pt,\beta),t\right]\right)$, $D^-V\left[\rho(pt,\beta),t\right] \geqq \omega_2\left(t,V\left[\rho(pt,\beta),t\right]\right)$, all $t \in J$, $p \in E$, such that $pt_o \in \alpha$.

(iii) to each $r_o \in B$ there corresponds a $t(r_o) \in J$, such that:

(a) the maximal solution $r_M(t;r_o)$, $r_M(t_o;r_o) = r_o$, of equation

$$\dot{r} = \omega_1(t,r) \tag{C_1}$$

is such that

$$r_M(t;r_o) = 0, \text{ all } t \in J\big(t(r_o),t_o+T\big) \tag{3.1.8}$$

(b) the minimal solution $r_m(t;r_o)$, $r_m(t_o;r_o) = r_o$, of equation

$$\dot{r} = \omega_2(t,r) \tag{C_2}$$

is such that

$$r_m(t;r_o) = 0, \text{ all } t \in J\big(t(r_o),t_o+T\big) \tag{3.1.9}$$

where B is given by (3.1.3).

Remark 3.1.1:

Condition (3.1.7) can be assumed to hold only for all $t \in J(t^*,t_o+T)$, for some $t^* \in J$.

(3) *Theorem 3.1.3:*

System (E,R,f) is not finally-stable with respect to the sets (α, β, J), if there exist a point $p^* \in E$, $p^*t_o \in \alpha$, two functions $V[s,t] \in C\big(J(0,K^*),J\big)$, $\omega(t,r) \in \Omega(J,R)$, such that

(i) $V[s,t] \neq 0 \Rightarrow s \neq 0$, for all $t \in J(t^*,t_o+T)$, for some $t^* \in J$.

(ii) $D^-V[\rho(p^*t,\beta),t] \geq \omega\big(t,V[\rho(p^*t,\beta),t]\big)$, all $t \in J$.

(iii) the minimal solution $r_m(t)$, $r_m(t_o) = V[\rho(p^* t_o,\beta),t_o]$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that

$$r_m(t) > 0 \tag{3.1.10}$$

all $t \in J(t^{**}, t_o+T)$, for some $t^{**} \in J$, where K^* is given by (2.1.14).

3.2 Final-stability with respect to the sets (α, β, $t_1 \in J$):

Theorem 3.2.1:

System (E,R,f) is finally-stable with respect to the sets (α, β, $t_1 \in J$), if there exist two functions $V[s,t] \in C(J(0,K_3),J)$, $\omega(t,r) \in \Omega(J,R)$, such that

(i) $V[s,t]$ is positive-definite with respect to s, for all $t \in J(t_1,t_o+T)$, $s \in J(0,K_3)$.

(ii) $V_M^\alpha(t_o;\beta)$ is finite.

(iii) $D^+V[\rho(pt,\beta),t] \leq \omega(t,V[\rho(pt,\beta),t])$, all $t \in J$, $p \in E$, $pt_o \in \alpha$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^\alpha(t_o;\beta)$, of equation

$$\dot{r} = \omega(t,r) \tag{C}$$

is such that $r_M(t) = 0$, all $t \in J(t_1, t_o+T)$; where K_3 is given by (3.1.1).

The following theorem avoids the question of positive-definite functions.

Theorem 3.2.2:

System (E,R,f) is finally-stable with respect to the sets $(\alpha,\ \beta,\ t_1 \in J)$, if there exist three functions $V[s,t] \in C(J(0,K_3),J)$, $\omega_1(t,r) \in \Omega(J,R)$, and $\omega_2(t,r) \in \Omega(J,R)$, such that

(i) $V_M^{\alpha}(t_o;\beta)$ and $V_m^{\alpha}(t_o;\beta)$ are finite.

(ii) $V[s,t] = 0 \Rightarrow s = 0$, all $t \in J(t_1,\ t_o+T)$.

(iii) $D^+V[\rho(pt,\beta),t] \geqq \omega_2(t,V[\rho(pt,\beta),t])$, $D^-V[\rho(pt,\beta),t] \geqq \omega_2(t,V[\rho(pt,\beta),t])$, all $t \in J$, $p \in E$, $pt_o \in \alpha$.

(iv) the maximal solution $r_M(t)$, $r_M(t_o) = V_M^{\alpha}(t_o;\beta)$, of equation

$$\dot{r} = \omega_1(t,r) \qquad (C_1)$$

is such that $r_M(t) = 0$, all $t \in J(t_1,\ t_o+T)$.

(v) the minimal $r_m(t)$, $r_m(t_o) = V_m^{\alpha}(t_o;\beta)$, of equation

$$\dot{r} = \omega_2(t,r) \qquad (C_2)$$

is such that $r_m(t) = 0$, all $t \in J(t_1,\ t_o+T)$.

Theorem 3.2.3:

System (E,R,f) is not finally-stable with respect to the sets $(\alpha,\ \beta,\ t_1 \in J)$ if there exist a point $p^* \in E$, $p^*t_o \in \alpha$, two functions $V[s,t] \in C(J(0,K^*),J)$, $\omega(t,r) \in \Omega(J,R)$, where K^* is given by (2.1.14) such that

(i) $V[s,t_2] \neq 0 \Rightarrow s \neq 0$, for some $t_2 \in J(t_1,\ t_o+T)$.

(ii) $V[\rho(p^*t_o,\beta),t_o]$ is finite.

(iii) $D^{-}V[\rho(p^{*}t,\beta),t] \geq \omega(t,V[\rho(p^{*}t,\beta),t])$, all $t \in J$.

(iv) the minimal solution $r_m(t)$, $r_m(t_o) = V[\rho(p^{*}t_o,\beta),t_o]$, of equation

$$\dot{r} = \omega(t,r) \qquad \text{(C)}$$

is such that $r_m(t_2) > 0$, for some t_2 satisfying (i).

§4. CONCLUSION

(1) As mentioned in the introduction, the theory established in this chapter is limited to the basic definitions of final-stability. One can establish a theory related to the different types of strong (semi-)final stability of system (E,R,f).

(2) Obviously, an important class of ordinary and partial differential systems can be studied by means of the above theory. We note, however, that many ordinary (and partial) differential systems do not define dynamical systems. Hence, the theory of final-stability established in the first two chapters is relevant.

(3) One notices that we have not used the full strength of the definition of dynamical systems. So, it would be possible to extend the theory to more general systems provided, of course, the adequate modifications are made.

(4) The extension of the above to dynamical systems with control is probably possible and may be done in a way similar to that used in Chapter II.

(5) The extension of the theory to discrete dynamical systems is also possible. One has however to consider the essential modifications dictated by the nature of the system.

CHAPTER V

Conclusions, References, and Notation

CONCLUSIONS

In the thesis we have tried to establish as many results as possible. Our purpose has been to give the interested reader as large a field of research as possible; so he may investigate the possible openings and the probable applications of the different results. As a contribution from our part to possible future studies, we discuss below some ideas which we believe may be useful:

1. A possible field of research would be the amelioration of the above results: minimising the conditions whenever possible, or using other techniques to prove the existing results.

   The idea is not void of sense, since it was possible to establish different theorems yielding similar conclusions.

2. Generalising the concept of final-stability furthermore by restricting the controls in a different manner, we give below an example of such a possibility:

Definition 1:

System (EC) is practically finally-stable with respect to the sets $(\alpha, \beta, U, t_1 \in J)$, $U \subseteq S^m$, if for any $x_o \in \alpha$, there exists a control $u_o(\cdot) \in U$, such that a corresponding trajectory (at least) $x(t) = x(t;x_o,t_o.u_o(\cdot))$ satisfies the relation

$$x(t;\ x_o,t_o,u_o(\cdot)) \in \beta,\ t \in J,\ t \geqq t_1. \tag{1}$$

The difference between this type of final-stability and the usual concept of final-stability is that the latter implies the former, i.e., a system (EC) which is finally-stable with respect to the sets $(\alpha,\ \beta,\ U,\ t_1 \in J)$ is certainly practically finally-stable with respect to the sets $(\alpha,\ \beta,\ U,\ t_1 \in J)$, but the converse need not be true. Of course, the methods used above are no longer applicable. It is however possible to modify them to apply here. For example:

Theorem 1:

System (EC) is practically finally-stable if there exist two families of functions $V[x,t;x_o] \in L[R^n,J]$, $\omega(t,r;x_o) \in \Omega$, all $x_o \in \alpha$, such that

(i) $V[x_o,t_o;x_o]$ and $V_m^{\beta^c}(t;x_o)$, $t \in J$, $t \geqq t_1$, are finite, for each $x_o \in \alpha$, where

$$V_m^{\beta^c}(t;x_o) = \inf_{x \in \beta^c} V[x,t;x_o] \tag{2}$$

(ii) to each $x_o \in \alpha$, there corresponds a $u_o(\cdot) \in U$, such that

$$V^*[x,t;x_o] \leqq \omega(t,V[x,t;x_o];x_o),$$

all $t \in J$, $x \in R^n$, $u(\cdot) = u_o(\cdot)$, where

$$V^*[x,t;x_o] = \lim.\sup_{h \to 0^+} \frac{V[x + hg,\ t + h;\ x_o] - V[x,t;x_o]}{h} \tag{3}$$

(iii) for each $x_o \in \alpha$, the maximal solution $r_M(t;x_o)$ of

$$\dot{r} = \omega(t,r;x_o) \qquad r(t_o;x_o) = V[x_o,t_o;x_o] \tag{4}$$

is such that

$$r_M(t;x_o) < V_m^{\beta^c}(t;x_o), \text{ all } t \in J,\ t \geqq t_1.$$

3. An important field of research, in theory at least, is the problem of establishing converse theorems. The following is an attempt to generalise some known results [16,17,35].

L. Weiss [35] introduced the concept of uniform finite-time stability as follows:

Definition 2:

System

$$\dot{x} = f(x,t) \tag{5}$$

is uniformly stable with respect to the sets (α, β, J), $\alpha \subset \beta$, if any trajectory $x(t) = x(t;x^*,t^*)$, $t^* \varepsilon J$, $x^* \varepsilon \alpha$, is such that

$$x(t;t^*,x^*) \varepsilon \beta \tag{6}$$

all $t \varepsilon J$, $t \geq t^*$; where

$$\alpha : ||x|| < a, \quad \beta: ||x|| < b, \quad a < b. \tag{7}$$

This definition is obviously different from the definition of uniform final-stability given in the previous chapters. There is, however, no possibility of confusion, since stability with respect to the sets (α, β, J) is, in fact, uniform final-stability of a particular type; that is, stability with respect to the sets (α, β, J) is equivalent to final-stability with respect to the sets $(\alpha, \beta, t_o \varepsilon J)$. This leads us to propose the following generalisation of Definition 2. It is, however, more convenient to use the term "monotonic" instead of "uniform". Thus, no possible confusion will arise.

Definition 3:

System (5) is monotonically finally-stable with respect to the sets $(\alpha, \beta, t_1 \in J)$, if

(i) it is semi-finally stable with respect to the sets $(\alpha, \beta, t_1 \in J)$, and

(ii) any trajectory $x(t;x^*,t^*)$, $t^* \in J$, $t^* \geqq t_1$, $x^* \in \beta$, satisfies

$$x(t;x^*,t^*) \in \beta$$

all $t \in J$, $t \geqq t^*$.

But in order to be able to generalise the result given by Weiss [35], we need to restrict the above definition in the following manner.

Definition 4:

System (5) is strictly monotonically finally-stable with respect to the sets $(\alpha, \beta, t_1 \in J)$, if

(i) it is semi-finally stable with respect to the sets $(\alpha, \gamma, t_1 \in J)$, for some set γ, $\bar{\gamma} \subset I(\beta)$, and

(ii) any trajectory $x(t;x^*,t^*)$, $t^* \in J$, $t^* \geqq t_1$, $x^* \in \gamma$, satisfies

$$x(t;x^*,t^*) \in \beta$$

all $t \in J$, $t \geqq t^*$.

We are now in a position to propose the following theorem:

Theorem 2: (J = I)

Let f(x,t) satisfy a local Lipschitz condition with respect to x, for all $t \in J$, $t \geqq t_1$, in addition to the other usual requirements. Then system (5) is strictly monotonically finally-stable with respect to the sets (α, β, $t_1 \in J$), if and only if

(a) it is semi-finally stable with respect to the sets (α, γ, $t_1 \in J$), for some set γ, $\bar{\gamma} \subset I(\beta)$, and

(b) there exist two functions $V[x,t] \in L[\bar{\beta} - I(\gamma), J(t_1)]$, $\omega(t,r) \in \Omega$, where $J(t_1) : t \in J, t \geqq t_1$, such that

(i) $V_M^{Fr.\gamma}(t)$ and $V_m^{Fr.\beta}(t)$ are finite, all $t \in J(t_1)$,

(ii) $V^*[x,t] \leqq \omega(t,V[x,t])$, $t \in [t_1,t_o+T)$, $x \in \bar{\beta} - I(\gamma)$.

(iii) for each $t_2 \in (t_1,t_o+T)$, the maximal solution $r_M(t)$ of

$$\dot{r} = \omega(t,r) \qquad r(t_2) = V_M^{Fr.\gamma}(t_2)$$

is such that

$$r_M(t) < V_m^{Fr.\beta}(t), \quad t \in J(t_1),\ t > t_2.$$

That the conditions are sufficient is obvious from the theory established in this thesis. On the other hand, it is possible to show that the conditions are necessary in the case where γ and β are given by

$$\gamma : ||x|| \leqq c, \quad \beta : ||x|| < b, \quad c < b$$

bearing in mind that strict monotonic final-stability will imply that the system is uniformly stable (in the sense of Weiss) with respect to the sets $(\gamma, \beta, J(t_1))$. We note that the case $J = \bar{I}$ is considered separately by Weiss [35].

On the other hand, some results established in [16,17] lead us to propose the following definition and theorem.

Definition 5:

System (5) is strictly finally-stable with respect to the sets $(\alpha, \beta, t_1 \varepsilon J)$, if

(i) it is semi-finally stable with respect to the sets $(\alpha, \gamma, t_1 \varepsilon J)$, for some γ, $\bar{\gamma} \subset I(\beta)$, and

(ii) it is stable with respect to the sets $\{\gamma, \beta, J(t_1)\}$, $\{$where $J(t_1) : t \varepsilon J, t \geqq t_1\}$, i.e., for any trajectory $x(t)$, the condition $x(t_1) \varepsilon \gamma$ implies that $x(t) \varepsilon \beta$, all $t \varepsilon J(t_1)$.

It is to be noted that condition (ii) is stronger than:

(ii)* it is finally-stable with respect to the sets $(\alpha, \beta, t_1 \varepsilon J)$.

Theorem 3: [16] $(J = I)$

System (5) is strictly-finally stable with respect to the sets $(\alpha, \beta, t_1 \varepsilon I)$, if and only if

(a) it is semi-finally stable with respect to the sets $(\alpha, \gamma, t_1 \varepsilon I)$, for some set γ, $\bar{\gamma} \subset I(\beta)$, and

(b) there exist two functions $V[x,t] \varepsilon L[\bar{\beta}, J(t_1)]$, $\omega(t,r) \varepsilon \Omega$, such that

(i) $V^*[x,t] \leqq \omega(t, V[x,t])$, all $t \varepsilon J(t_1)$, $x \varepsilon \bar{\beta}$.

(ii) for each set $\delta \subset \gamma$, the maximal solution $r_M(t)$ of

$$\dot{r} = \omega(t,r) \qquad r(t_1) = V_M^{Fr.\delta}(t_1)$$

is such that

$$r_M(t) < V_m^{Fr.\beta}(t), \text{ all } t \in J(t_1), \; t > t_1.$$

[The function $f(x,t)$ is assumed to be locally Lipschitzian in x.]

Finally, it would be worthwhile to consider the problem of establishing converse theorems for most of the types of final-stability.

4. Following Weiss and Infante [37], we say that one of the desirable goals in the development of any theory of stability is to be able to determine the stability properties of a complicated system by knowing the stability properties of lower order subsystems which, when coupled together in an appropriate fashion, form the original system.

In general, this is rather difficult to achieve, but certain results along this line are immediately available in the case of finite-time stability [30,37]. As a further contribution to such a field of possible investigations, we propose below some definitions and a result.

The system which we consider here is of the following form

$$\dot{x} = \eta(x,t) \tag{8}$$

with decomposition

$$\dot{w}^{(i)} = \eta^{(i)}\left(w^{(1)}, \ldots, w^{(k)}, t\right) \tag{9}$$

$$(i = 1, \ldots, k)$$

where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = w = \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(k)} \end{bmatrix}, \quad x \in R^n, \; w^{(i)} \in R^{\eta_i} \quad \sum_{i=1}^{k} \eta_i = n \tag{10}$$

The usual assumptions which ensure existence, continuous dependence on initial conditions are assumed to hold over the usual interval J. As before, small greek letters will denote connected sets. Moreover, let

$$D_{\alpha_1 x \ldots x\alpha_k} = \{x \mid \in D_{\alpha_1 x \ldots x\alpha_k}, \text{ if and only if } w^{(i)} \in \alpha_i, \quad i = 1,\ldots,k.\} \tag{11}$$

$$D^c_{\alpha_1 x \ldots x\alpha_k} = \{x \mid w^{(i)} \in \beta_i^c, \text{ for some } 1 \leq i \leq k\} \tag{12}$$

Definition 6:

System (8) with decomposition (9) is semi-finally stable with respect to the sets $(D_{\alpha_1 x \ldots x\alpha_k}, D_{\beta_1 x \ldots x\beta_k}, J)$, if, for any trajectory $x(t)$, the condition

$$x(t_o) \in D_{\alpha_1 x \ldots x\alpha_k}$$

implies that

$$x(t_1) \in D_{\beta_1 x \ldots x\beta_k}$$

where $t_1 \in J$ may depend on the particular trajectory.

A weaker form of semi-final stability can be defined as follows:

Definition 7:

System (8) with decomposition (9) is said to be partially semi-finally stable with respect to the sets $(D_{\alpha_1 x \ldots x\alpha_k}, D_{\beta_1 x \ldots x\beta_k}, J)$, if, for any trajectory $x(t) = [w^{(i)}(t), i = 1, \ldots, k]$, the condition

$$x(t_o) \in D_{\alpha_1 x \ldots x\alpha_k}$$

implies the existence of $t_1^{(i)} \in J$, $i = 1, \ldots, k$, such that

$$w^{(i)}\left(t^{(i)}\right) \in \beta_i, \; i = 1, \ldots, k$$

where $t_1^{(i)}$, $i = 1, \ldots, k$, may depend on the particular trajectory.

We note immediately that semi-final stability, in Definition 6, implies partial semi-final stability, but the converse in not necessarily true.

Definition 8:

System (8) with decomposition (9) is not semi-finally stable with respect to the sets $(D_{\alpha_1 x \ldots x\alpha_k}, D_{\beta_1 x \ldots x\beta_k}, J)$, if there exists a trajectory $x(t)$, $x(t_o) \in D_{\alpha_1 x \ldots x\alpha_k}$, and such that

$$x(t) \in D^c_{\beta_1 x \ldots x\beta_k}, \quad \text{all } t \in J.$$

It is to be noted that this definition does not exclude the possibility of the system being partially semi-finally stable with respect to the same sets; for the above trajectory $x(t) = [w^{(i)}(t), i = 1, \ldots, k]$ may satisfy the relations

$$W^{(i)}(t_i) \in \beta_i, \ i = 1, \ldots, k$$

provided $t_i \neq t_j$ for some $i \neq j$.

Such distinctions do not arise in the following definitions.

Definition 9:

System (8) with decomposition (9) is semi-final stable with respect to the sets $(D_{\alpha_1 x \ldots x \alpha_k}, D_{\beta_1 x \ldots x \beta_k}, t_1 \in J)$, if, for any trajectory $x(t)$, the condition

$$x(t_o) \in D_{\alpha_1 x \ldots x \alpha_k}$$

implies that

$$x(t_1) \in D_{\beta_1 x \ldots x \beta_k}.$$

Definition 9 suggests a new definition of semi-final stability which might be useful and is less restrictive.

Definition 10:

System (8) with decomposition (9) is semi-finally stable with respect to the sets $(D_{\alpha_1 x \ldots x \alpha_K}, D_{\beta_1 x \ldots x \beta_K}, t_1 \in J, \ldots, t_k \in J)$, if, for any trajectory $x(t) = [W^{(i)}(t), i = 1, \ldots, k]$, the condition

$$x(t_o) \in D_{\alpha_1 x \ldots x \alpha_k}$$

implies that

$$W^{(i)}(t_i) \in \beta_i, \text{ all } i = 1, \ldots, k.$$

One can define the other types of final-stability in a similar way. But we terminate these definitions by the following one which is given by Weiss and Infante [37]. Let $\alpha_i \subseteq \beta_i$, for all $i = 1, \ldots, k$.

Definition 11:

System (8) with decomposition (9) is stable with respect to the sets $(D_{\alpha_1 x \ldots x\alpha_k}, D_{\beta_1 x \ldots x\beta_k}, J)$, if, for any trajectory $x(t)$, the condition

$$x(t_o) \in D_{\alpha_1 x \ldots x\alpha_k}$$

implies that

$$x(t) \in D_{\beta_1 x \ldots x\beta_k}, \text{ all } t \in J.$$

Weiss and Infante [37] proved some results in the case $k = 2$, but it is probable that the extension to some complicated systems is possible. Motivated by a result concerning stability of discrete systems [30], we propose the following theorem

Theorem 4:

We consider system (8) with the following decomposition

$$\dot{w}^{(1)} = g^{(1)}\left(w^{(1)}, w^{(k)}, t\right) \tag{13.1}$$

$$\dot{w}^{(2)} = g^{(2)}\left(w^{(2)}, w^{(1)}, t\right) \tag{13.2}$$

....

....

$$\dot{w}^{(k)} = g^{(k)}\left(w^{(k)}, w^{(k-1)}, t\right) \tag{13.k}$$

Let (i_1) subsystem (13.1) be stable with respect to the sets

$(\alpha_1, \beta_1, U_1, J), \alpha_1 \subseteq \beta_1, U_1 \equiv \bar{\beta}_k;$

(i_2) subsystem (13.2) be stable with respect to the sets

$(\alpha_2, \beta_2, U_2, J), \alpha_2 \subseteq \beta_2, U_2 \equiv \bar{\beta}_1,$ ....

....

(i_K) subsystem (13.k) be stable with respect to the sets

$(\alpha_k, \beta_k, U_k, J), \alpha_k \subseteq \beta_k, U_k \equiv \bar{\beta}_{k-1}.$

Then, system (8) is stable with respect to the sets $(D_{\alpha_1 x \ldots x\alpha_k}, D_{\beta_1 x \ldots x\beta_k}, J)$.

Proof (We limit the case to open sets β_i, i = 1, ..., k, and J = I.)

Let $x(t) = w^{(i)}(t)$, i = 1, ..., k. be an arbitrary trajectory such that

$$x(t_o) \in D_{\alpha_1 x \ldots x\alpha_k} \tag{14}$$

Suppose that there exist a $t_1 \in (t_o, t_o+T)$, the first such time at which $w^{(k)}(t_1) \in Fr.\beta_k$, then

$$w^{(k)}(t) \in \beta_k,\ t \in [t_o, t_1) \tag{15}$$

But, by (i_1), it follows that

$$w^{(1)}(t) \in \beta_1,\ t \in [t_o, t_1] \tag{16}$$

Furthermore, by (i_2), it follows that

$$w^{(2)}(t) \in \beta_2,\ t \in [t_o, t_1] \tag{17}$$

Repeating this (k - 1) times, observe by the (i_{k-1}) hypothesis of the theorem that subsystem $\{13.(k-1)\}$ is stable with respect to the sets $(\alpha_{k-1}, \beta_{k-1}, U_{k-1}, J)$, $U_{k-1} \equiv \bar{\beta}_{k-2}$.

Now, if t_2 is the first value of $t \in J$ for which

$$w^{(k-1)}(t_2) \in \beta^c_{k-1} \tag{18}$$

then, it follows that $t_2 > t_1$. Now, consider subsystem (13.k); by (i_k), it is stable with respect to the sets $(\alpha_k, \beta_k, U_k, J)$, $U_k \equiv \bar{\beta}_{k-1}$, then

$$w^{(k)}(t_1) \in \beta_k \tag{19}$$

which is in obvious contradiction to (15). Thus, $w^{(k)}(t) \in \beta_k$, all $t \in J$.

Next, by repeating the above argument for each of the k-subsystems, one can show that

$$w^{(i)}(t) \in \beta_i \text{ all } t \in J.$$

This completes the proof of Theorem 4.

5. A further field of research would be to establish connections with the theory of boundedness of the solutions of systems such as system (5). Also, one can investigate the possible relationships between the theory of final-stability and the theory of Liapunov's stability.

We propose below some theorems concerning some types of boundedness and stability. For the required definitions and the definitions of the other types of boundedness and stability we refer the reader to the following references: [2,9,10,19,26,27,32,38,39-49,50]. The theory was suggested to us by a transformation

$$t = \frac{s}{T - s} + t_o \tag{20}$$

given in [34] where the idea was to apply Liapunov's direct method to the theory of finite-time stability. It is, however, possible to use the methods of final-stability to establish some results on boundedness and Liapunov's stability by means of transformations similar to (20). The following are illustrative examples: Let, for any $T > 0$, $J(T) = [0,T)$. Assume furthermore that the uniqueness property is satisfied, then

Theorem 5:

System (5) has a bounded solution if there exist $t_o \in R^+$, $x_o \in R^n$, $T > 0$, and two functions $V[x,s] \in L[\bar{\beta},J(T)]$, $\omega(t,r) \in \Omega^*$, where β is a bounded set containing x_o in its interior, such that

(i) $V[x_o,0]$ and $V_m^{Fr.\beta}(s)$, $s \in (0,T)$, are finite.

(ii) $$\limsup_{h \to 0^+} \frac{1}{h}\left\{V\left[x + \frac{hT}{(T-s)^2} f\left(x, \frac{s}{T-s} + t_o\right), s + h\right] - V[x,s]\right\}$$

$$< \omega(s,V[x,s]), \text{ all } s \in J(T),\ x \in \bar{\beta}.$$

(iii) the maximal solution $r_M(s)$ of

$$\frac{dr}{ds} = \omega(s,r) \qquad r(0) = V[x_o,0] \tag{21}$$

is such that

$$r_M(s) \leqq V_m^{Fr.\beta}(s), \qquad s \in (0,T) \tag{22}$$

Proof:

Consider the transformation (20), then system (5) becomes

$$y' = f^*(y,s) \tag{23}$$

where $y' = \frac{dy}{ds}$, $f^*(y,s) = \frac{T}{(T - s)^2} f(y, \frac{s}{T - s} + t_o)$, and

$y(s) \equiv x(t(s))$, where $x(t)$ is a trajectory of (5), with $x(t_o) = x_o$, and $y(s)$ is a trajectory of (23), with $y(0) = x_o$. It is easy to see that the functions $V[y,s]$ and $\omega(s,r)$ satisfy the following conditions:

(i)* $V_M^\alpha(0)$ and $V_m^{Fr.\beta}(s)$, $s \in (0,T)$, are finite. $(\alpha = \{x_o\})$.

(ii)* $V^*[y,s] < \omega(s,V[y,s])$, all $y \in \bar{\beta}$, $s \in J(T)$

(iii)* the maximal solution $r_M(s)$, $r_M(0) = V_M^\alpha(0)$, of equation

$$\frac{dr}{ds} = \omega(s,r)$$

satisfies (22).

Thus, system (23) is stable with respect to the sets $(\alpha, \beta, J(T))$. The theorem follows. (Ref. Theorem 3.3.5 - II)

The possibility of showing the existence of at least one bounded solution of a given system (5) is important as far as two-dimensional systems are concerned. An important application is the existence of a periodic solution of system (5); for, following Massera [28,29], the existence of a periodic solution is implied by the existence of all solutions in the future, and the existence of a bounded solution, provided n = 2.

The following theorem would be useful, in theory at least, if one wanted to show the existence of an unbounded solution of system (5).

Theorem 6:

System (5) has an unbounded solution if there exist a strictly increasing sequence $\{b_j\}$, $b_j > 0$, $b_j \to \infty$, as $j \to \infty$, a point $x_o \in \beta_1$, $\beta_j : ||x|| < b_j$, and two functions $V[x,t] \in L[R^n, J(t_o,\infty)]$, $\omega(t,r) \in \Omega^*$, for some $t_o \in R^+$, $J(t_o,\infty) = [t_o,\infty)$, such that

(i) $V[x_o,t_o]$ is finite.

(ii) $V^*[x,t] < \omega(t,V[x,t])$, all $t \in J(0,\infty)$, $x \in R^n$.

(iii) there exists a strictly increasing sequence $\{t_j\}$, $t_j > 0$, $t_j \to \infty$, as $j \to \infty$, such that

(a) $V_m^{\beta_j}(t_o+t_j)$ is finite,

(b) the maximal solution $r_M(t)$ of

$$\dot{r} = \omega(t,r) \qquad r(t_o) = V[x_o,t_o]$$

is such that

$$r_M(t_o+t_j) \leq V_m^{\beta_j}(t_o+t_j), \text{ all } j = 0, \ldots, \infty.$$

It is possible to prove the above theorem by means of Theorem 2.4.1 - II, by setting $\alpha \equiv \{x_o\}$, $J_j = [0,T_j)$, $T_j = t_o + t_j + \varepsilon$, $\varepsilon > 0$. So, using the above mentioned theorem one can show that system (5) is not semi-finally stable with respect to the sets $(\alpha, \beta_j, t_o + t_j \in J_j)$, for all $j = 0, \ldots, \infty$. The theorem follows from the uniqueness of the solution through (x_o,t_o).

We note however that the uniqueness requirement is not necessary, bearing in mind that Theorem 2.4.1 - II shows, in fact, that any trajectory emanating from (x_o,t_o) is such that $x(t_1) \in \beta^c$.

Finally, we state a theorem concerning the Liapunov stability of the equilibrium $x = 0$ of system (5), i.e., we require that $f(0,t) = 0$, all $t \varepsilon R^+$. Let $R(a)$ be the set

$$R(a) : ||x|| < a \tag{24}$$

Theorem 7:

The equilibrium of (5) is stable if there exist a neighbourhood $R(h)$ of the origin, two functions $V[x,s] \varepsilon L[\overline{R(h)}, R^+]$, and $\omega(s,r) \varepsilon \Omega^*$, such that to each $0 < \varepsilon < h$, and each $t_o \varepsilon R^+$, there correspond two positive numbers $\delta = \delta(t_o,\varepsilon) < \varepsilon$, $T = T(t_o,\varepsilon)$, with the following properties:

(i) $V_M^{R(\delta)}(0)$ and $V_m^{Fr.R(\varepsilon)}(s)$, $s \varepsilon (0,T)$, are finite.

(ii) $$\limsup_{h \to 0^+} \frac{1}{h}\left\{V\left[x + \frac{hT}{(T-s)^2} f(x, \frac{s}{T-s} + t_o), s + h\right] - V[x,s]\right\} < \omega(s,V[x,s])$$

for all $s \varepsilon [0,T]$, $x \varepsilon \overline{R(\varepsilon)}$, where $T = T(t_o,\varepsilon)$.

(iii) the maximal solution $r_M(s; t_o,\varepsilon)$ of

$$\frac{dr}{ds} = \omega(s,r), \quad r(0; t_o,\varepsilon) = V_M^{R(\delta)}(0)$$

is such that

$$r_M(s; t_o,\varepsilon) \leqq V_m^{Fr.R(\varepsilon)}(s), \quad s \varepsilon (0,T)$$

where $T = T(t_o,\varepsilon)$, and $\delta = \delta(t_o,\varepsilon)$.

To prove this theorem, one may use, for each $t_o \varepsilon R^+$ and $0 < \varepsilon < h$, the transformation $t = \frac{s}{T-s} + t_o$, with $T = T(t_o,\varepsilon)$; and then show by Theorem 3.3.5 - II that the system obtained is stable with respect to the sets $(R(\delta), R(\varepsilon), [0,T))$.

6. Another approach to the theory of final stability is the use of an m-vector function $V = (V_1, \ldots, V_m)$ instead of a scalar function. The idea was suggested to us by the work of Lakshmikantham and Leela [18.2] concerning Liapunov's stability. Two obvious advantages of this approach are the following:

   (a) the elegant presentation of the theory whenever several Liapunov-like functions are used.

   (b) it would be more probable to find m Liapunov-like functions V_i, $i = 1, \ldots, m$, satisfying each K_i, $i = 1, \ldots, m$, hypotheses than to find a single scalar function W satisfying all the $\sum_{i=1}^{m} K_i$ hypotheses.

7. Finally, we note that the above discussion and suggestions can be extended to both discrete and dynamical systems.

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NOTATION

Chapter I

R^n	:	the n-dimensional real euclidean spaces.
R	:	the set of all real numbers.
R^1	:	one-dimensional real euclidean spaces.
R^+	:	the set of all non-negative real numbers.
$\|\|x\|\|$	=	$\left[\sum_{i=1}^{n} x_i^2\right]^{\frac{1}{2}}$
I	=	$[t_o, t_o+T)$, $t_o \varepsilon R^+$, $T \varepsilon R^+$, $T > 0$.
$\alpha, \beta, \ldots,$ etc.		are connected sets in R^n.
$\alpha \cap \beta$	:	intersection of α and β, i.e., $x \varepsilon \alpha \cap \beta \Rightarrow x \varepsilon \alpha$, and $x \varepsilon \beta$.
$\alpha \cup \beta$	:	union of α and β, i.e., $x \varepsilon \alpha \cup \beta \Rightarrow x \varepsilon \alpha$, or $x \varepsilon \beta$.
$x \varepsilon \alpha$	:	x is an element of α.
$x \notin \alpha$	:	x is not an element of α.
α^c	:	complement of α in R^n.
$I(\alpha)$	:	interior of α.
$\bar{\alpha}$	:	closure of α.
$Fr.\alpha$	:	boundary of α.
α/β	=	$\alpha - \alpha \cap \beta$
$I^c(\alpha)$	:	complement of the interior of $I(\alpha)$ in R^n.
$\alpha \subset \beta$	:	$x \varepsilon \alpha \Rightarrow x \varepsilon \beta$, but there exist some $x \varepsilon \beta$, such that $x \notin \alpha$.
$\alpha \subseteq \beta$	:	$x \varepsilon \alpha \Rightarrow x \varepsilon \beta$
$\alpha \not\subset \beta$	:	either $\alpha = \beta$, or $\alpha/\beta \neq \emptyset$, where
$\emptyset$	:	the empty set in R^n.

$\alpha \not\subseteq \beta$: neither $\alpha \subset \beta$ nor $\alpha = \beta$.

$V[x,t]$: $R^n \times I \to R^1$

$V[x,t] \in C^1[\alpha \times I^*]$: $V[x,t]$ and its partial derivatives $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial x_1}, \ldots$

$\ldots, \frac{\partial V}{\partial x_n}$ are continuous over $\alpha \times I^*$.

$V_M^\alpha(t) = \sup_{x \in \alpha} . V[x,t]$

$V_m^\alpha(t) = \inf_{x \in \alpha} . V[x,t]$

$\cdot = \frac{d}{dt}$

$\phi(t) \in R[I^*]$: $\phi(t)$ is Riemann-integrable over I^*.

Chapter II:

(EC) : $\dot{x} = g(x,t,u)$, (EG) : $\dot{x} = f(x,t) + G(x,t)u$.

J : either $I = [t_o, t_o+T)$, or $\bar{I} = [t_o, t_o+T]$

S^m : $\{u(x,t) | u : R^n \times I \to R^m\}$, the set of admissible controls $u(\cdot)$.

U : a given subset of S^m.

$V[x,t] \in L(\gamma, I^*)$: $V[x,t]$ is continuous over $\gamma \times I^*$ and satisfies a local Lipschitz condition in x, for each $t \in I^*$.

$$V^*[x,t] = \limsup_{h \to o^+} \frac{V[x + hg(x,t,u), t + h] - V[x,t]}{h}$$

$\omega(t,r)$: $J \times R^1 \to R^1$

$\omega \in \Omega$: if it is smooth enough so that the maximal solutions of

(C) : $\dot{r} = \omega(t,r)$ exist over J.

$\omega \in \Omega^*$: if in addition $\omega(t,r)$ is monotonic increasing in r, for each fixed $t \in J$.

$W(\alpha, \beta, J, U) = \{x | x = x(t; x_o, t_o, u(\cdot)), x_o \in \alpha/\beta, u(\cdot) \in U, t \in J\}$.

$W_o(x_o, u_o, J) \supseteq \{x | x = x(t; x_o, t_o, u_o(\cdot)), t \in J\}$

$Z(\alpha, J^*, U) = \{x | x = x(t; x_o, t_o, u(\cdot)),$ all $x_o \in \alpha$, $t \in J^*$, $u(\cdot) \in U\}$, $J^* = [t_o, t_1)$.

$W_o^* \quad = W_o(x_o, u^*, J^*)$.

$N(\alpha, \beta, U, J) = \{x | x = x(t; x_o, t_o, u(\cdot)),$ all $x_o \in \alpha^c/\beta, t \in J,$ $u(\cdot) \in U\}$.

$K(\alpha, U, J) = \{x | x = x(t; x_o, t_o, u(\cdot)), x_o \in \alpha, t \in J, u(\cdot) \in U\}$.

$K(x_o) \quad = K(\alpha, U, J), \quad \alpha \equiv \{x_o\}$.

$V[x,t; x_o] \in L[\gamma, J]$: for each x_o, $V[x,t; x_o]$ is continuous over $\gamma \times J$, and satisfies a local Lipschitz condition in x, for each fixed $t \in J$.

$$V^*[x,t; x_o] = \lim_{h\to o^+} \sup. \frac{V[x + hg, t + h; x_o] - V[x,t; x_o]}{h}$$

$$V_M^\alpha(t; x_o) = \sup_{x\epsilon\alpha}. V[x,t; x_o]$$

$$V_m^\alpha(t; x_o) = \inf_{x\epsilon\alpha}. V[x,t; x_o]$$

$A(t_1, \alpha, \beta)$: the set of all continuous differentiable matrices $S(t)$, over $J^* = [t_o,t_1)$, such that $\sup_{x\epsilon\alpha}. x^TS(t_o)x$, $\inf_{x\epsilon\beta^c}. x^TS(t_1)x$ are finite.

$M(S \in A)$: the set of all functions $\mu(t,r) \in \Omega$ such that (for the given S) the maximal solution $r_M(t)$ of

$$\dot{r} = \mu(t,r), \quad r(t_o) = \sup_{x\epsilon\alpha}. x^TS(t_o)x \quad ,$$

is such that

$$r_M(t) < \inf_{x\epsilon\beta^c}. x^TS(t_1)x.$$

(FG) : $\dot{x} = F(x,t)x + G(x,t)u$.

(F) : $\dot{x} = F(x,t)x$.

$U(F,G) = \{U(F, G, S, \mu) | s \in A(t_1, \alpha, \beta)$, and $\mu \in M(S \in A)\}$, where

$U(F, G, S, \mu) = \{u(\cdot) | u^T G^T Sx + x^T SGx \leqq \mu(t, x^T Sx) - x^T(\dot{S} + F^T S + SF)x,$

$x \in R^n,\ t \in [t_o, t_1)\}.$

I_n : the identity matrix in R^n.

(VPC) : $\dot{x}_1 = x_2 + u_1,\ \ \dot{x}_2 = -x_1 + \sigma(x_1{}^2 - 1)x_2$

(VP) : $\dot{x}_1 = x_2,\ \ \dot{x}_2 = -x_1 + \sigma(x_1{}^2 - 1)x_2.$

$B(t_1, \beta)$: the set of all continuously differentiable matrices R such that $\sup_{x\varepsilon\beta}.\ x^T R(t_1)x$ is finite.

$m(\alpha, R \varepsilon B)$: the set of all $\mu(t,r)$ such that the maximal solution $r_M(t)$ of $\dot{r} = -\mu(t,r)$, $r(t_o) = -x_o{}^T R(t_o)x_o$, is such that

$$r_M(t_1) < -\sup_{x\varepsilon\beta}.\ x^T R(t_1)x$$

for some $x_o \varepsilon \alpha$.

$U^c(F, G, R, \mu) = \{u(\cdot) | u^T G^T Rx + x^T RGu \geqq \mu(t, -x^T Rx) - x^T(\dot{R} + F^T R +$

$RF)x$, all $x \varepsilon R^n,\ t \varepsilon [t_o, t_1)\}.$

$U^c(F, G) = \{U^c(F, G, R, \mu) | R \varepsilon B(t_1, \beta),\ \mu \varepsilon m(\alpha, R \varepsilon B)\}.$

Chapter III

$J = [j_o, j_o+1, j_o+2, \ldots\ldots, j_o+j_N].$

(DS) : $x(j + 1) = f[x(j), j].$

$\Delta V[x(j),j] = V[x(j + 1), j + 1] - V[x(j), j]$

$\Delta x(j) = x(j + 1) - x(j).$

$d(x,y) = ||x - y||,\ x \varepsilon R^n,\ y \varepsilon R^n$

$d(x,\beta) = \inf_{y\varepsilon\beta}.\ (x,y),\ d(x, \beta) = 0$ if $x \varepsilon \bar{\beta}$

$\text{diam. } \beta \quad = \quad \sup_{\substack{x\varepsilon\beta \\ y\varepsilon\beta}} d(x,y)$

$\gamma(\beta, e) \quad = \quad \{x | d(x,\beta) < e\}, \ \gamma(\beta,e) \supset \bar{\beta}.$

$\delta(x_o, J^*) \quad = \quad \{x | x = x(j), \ x(j_o) = x_o, j \ \varepsilon \ J^*\}, \ J^* \subseteq J.$

$\delta(\alpha, J^*) \quad = \{\delta(x_o, J^*) : x_o \ \varepsilon \ \alpha\}.$

$\gamma^-(Fr.\beta, e) = \{x | x \ \varepsilon \ \beta, \ d(x, Fr.\beta) < e\}.$

$\gamma^+(Fr.\beta, e) = \{x | x \ \varepsilon \ \beta^c, \ d(x, Fr.\beta) < e\}.$

<u>Chapter IV</u>

E : a metric space

$\rho(p,q)$: $p \ \varepsilon \ E$, $q \ \varepsilon \ E$, metric distance.

$\alpha, \beta, \ldots,$ etc. : closed connected sets in R^n

$J \quad = \quad [t_o, \ t_o+T]$

$J(t_1,t_2) \quad = \quad [t_1,t_2]$, $t_1, t_2 \ \varepsilon \ R$. If $t_1 = -\infty$, then $J(t_1,t_2) = (t_1,t_2]$, and if $t_2 = +\infty$, then $j(t_1,t_2) = [t_1,t_2)$.

$\rho(p,\beta)$: metric distance from $p \ \varepsilon \ E$ to $\beta \ \varepsilon \ E$.

$\alpha J \quad = \quad \{q | q = pt : p \ \varepsilon \ \alpha, \ t \ \varepsilon \ J\}.$

$pJ \quad = \quad \{q | q = pt : t \ \varepsilon \ J\}.$

$\alpha(J) \quad = \quad \{q | q = pt : p \ \varepsilon \ E, \ t \ \varepsilon \ J, \ \text{s.t. } pt_o \ \varepsilon \ \alpha\}.$

$V[s,t]$: $R^+ \times J \to R^1$.

$V_m^\alpha(t;\beta) \quad = \quad \inf_{q\varepsilon\alpha} V[\rho(q,\beta),t]$

$V_M^\alpha(t;\beta) \quad = \quad \sup_{q\varepsilon\alpha} V[\rho(q,\beta),t]$

$V[s,t] \ \varepsilon \ C(B,D)$: function $V[s,t]$ continuous over $B \times D$.

ω : $J \times R^1 \to R^1$.

$\omega \in \Omega(D,R)$: if it is smooth enough to ensure the existence of maximal and minimal solutions of the differential equation (C) over D.

$$D^{+}V[c(t),t] = \limsup_{h\to o^{+}} \frac{V[c(t+h),\ t+h] - V[c(t),t]}{h}$$

$$D^{-}V[c(t),t] = \liminf_{h\to o^{+}} \frac{V[c(t+h),\ t+h] - V[c(t),t]}{h}$$

$$k_1 = \sup_{q\epsilon\alpha^*(J)} \rho(q,\beta),\ \alpha^* = \alpha/\beta.$$

$$k^* = \sup_{t\epsilon J} \rho(p^* t,\beta).$$

$$k_2 = \sup_{q\epsilon\alpha(J_1)} \rho(q,\beta).$$

$$k_3 = \sup_{q\epsilon\alpha(J)} \rho(q,\beta)$$

Chapter V

$$D_{\alpha_1 x \ldots x\alpha_k} = \{x | x \in D_{\alpha_1 x \ldots x\alpha_k},\ \text{if and only if}\ w^{(i)} \in \alpha_i,\ i = 1, \ldots, k\},\ \text{where}$$

$$x = (x_1, \ldots, x_n) = [w^{(i)}, \ldots, w^{(k)}],\ w^{(i)} \in R^{n_i},\ \sum_{i=1}^{k} n_i = n.$$

$$D^{c}_{\alpha_1 x \ldots x\alpha_k} = \{x | w^{(i)} \in \beta_i^c,\ \text{for some}\ 1 \leqq i \leqq k\}.$$

$$J(T) = [0,\ T).$$

$R(a)$: $||x|| < a$

$$J_j = [0,\ t_o + t_j + \epsilon),\ \epsilon > 0.$$