# Applications of Nevanlinna theory to $q$-difference equations.

## by

## David Barnett

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# Abstract

Recently Nevanlinna theory (the theory of meromorphic functions) has been used as a detector of integrability of difference equations. In this thesis we study meromorphic solutions of so-called $q$-difference equations and extend some key results from Nevanlinna theory to the $q$-difference operator.

The Lemma on the Logarithmic Derivative of a meromorphic function has many applications in the study of meromorphic functions and ordinary differential equations. In this thesis, a $q$-difference analogue of the Logarithmic Derivative Lemma is presented, and then applied to prove a number of results on meromorphic solutions of complex $q$-difference equations. These results include a difference analogue of the Clunie Lemma, as well as other results on the value distribution of solutions.

# Keywords/phrases

# Acknowledgements

# Contents

# Introduction

In this thesis we study meromorphic solutions of $q$-difference equations and extend some key results from Nevanlinna theory to the $q$-difference operator. In particular a $q$-difference analogue of the Logarithmic Derivative Lemma is presented, and then applied to prove a number of results on meromorphic solutions of complex $q$-difference equations.

$q$-difference equations arise naturally in many contexts, such as $q$-Painlevé equations, the Schröder equation and iteration theory. Recent work on $q$-difference equations by Bergweiler, Ishizaki and Yanagihara [7] has shown that any meromorphic solution of a linear $q$-difference equation with rational coefficients has zero-order growth. It should be noted that $q$-difference equations are less likely to admit genuinely meromorphic solutions than difference equations. However as shown at the end of chapter two we are dealing with a non-trivial set.

In chapter one we present the key ideas of Nevanlinna theory so that we can apply them in our study of $q$-difference equations later on, where we find interesting results for the $q$-difference operator. The Nevanlinna theory of meromorphic functions was created by R. Nevanlinna (partially in cooperation with F. Nevanlinna) in 1925. A good introduction to the subject is [23].

Nevanlinna theory is an efficient tool for studying the density of points in the complex plane at which a meromorphic function takes a prescribed value. It also provides a natural way to describe the growth of a meromorphic function. The key tool used in Nevanlinna theory is the characteristic function $T(r, f)$. This measures the average size of a meromorphic function on large circles, taking into account the number of poles inside the circles. Chapter one is a review of known theory.

In chapter two we start to look at how the Nevanlinna theory introduced in chapter one can be applied to studying meromorphic solutions of $q$-difference equations. A autonomous first order non-linear $q$-difference equation is the Schröder equation. In the first part of chapter two we review a standard result which shows there exist meromorphic solutions of the Schröder equation. In the second part we look at the growth of meromorphic solutions of the general linear $q$-difference equation. Linear $q$-difference equations with rational coefficients do not always admit meromorphic solutions, even if the coefficients are constants. Bergweiler, Ishizaki and Yanagihara gave sufficient conditions for the existence of meromorphic solutions of linear $q$-difference equations, and characterized the growth of solutions in terms of the Nevanlinna characteristic $T(r, f)$ [7]. They concluded that all meromorphic solutions $f$ of a linear $q$-difference equation with rational coefficients satisfy

$T(r, f) = O((\log r)^2)$, from which it follows that all solutions are of zero order growth.

Valiron has shown that the non-autonomous Schröder $q$-difference equation

$$f(qz) = R(z, f(z)) = \frac{\sum_{j=0}^{n} a_j(z) f(z)^j}{\sum_{j=0}^{m} b_j(z) f(z)^j}, \tag{1}$$

where the coefficients $a_j(z), b_j(z)$ are meromorphic functions and $q$ is a complex constant, admits a one parameter family of meromorphic solutions, provided that $q \in \mathbb{C}$ is chosen appropriately [44]. It was shown by Gundersen *et al.* [18] that the order of growth of solutions of (1) is equal to $\log_q(\deg_f R)$, where $\log_q$ is the $q$-based logarithm. In section 3 we show how this result implies a $q$-difference analogue of the classical Malmquist's Theorem [32]. That is we show that if (1) possesses a zero-order solution then it is the discrete Riccati equation. These results suggest to us that the development of Nevanlinna theory for zero-order solutions of $q$-difference equations is natural. Chapter two is a review of known theory.

Chapter three contains most of the original research carried out. We develop a $q$-difference analogue of the Lemma on the Logarithmic Derivative. Due to the findings of chapter two our theory is for zero-order meromorphic functions. The Lemma on the Logarithmic Derivative is one of the most important and useful results of Nevanlinna theory, it has applications in the theory of meromorphic functions and in the theory of ordinary differential equations. For example, the Lemma on the Logarithmic Derivative is a key ingredient in the proofs of the Second Main Theorem of Nevanlinna theory [36] and Yosida's generalization [46] of the Malmquist Theorem [32]. In the rest of the chapter we use our result to study zero-order meromorphic solutions of large classes of $q$-difference equations. Applications include a $q$-difference analogue of the Clunie Lemma, (see Lemma 21 or [12]). The original lemma has proved to be an invaluable tool in the study of non-linear differential equations. The $q$-difference analogue gives similar information about the zero-order meromorphic solutions of non-linear $q$-difference equations.

### Historical Background

An important question we sometimes ask about systems of ordinary differential equations is 'How do we know when they can be solved explicitly?' It was observed in the late nineteenth and early twentieth centuries that ordinary differential equations whose general solutions are meromorphic appear to be integrable in that they can be solved explicitly or they are the compatibility conditions of certain types of linear problems. In the 1880s Kovalevskaya [28, 29] observed that all known solutions

of the equations of motion of a spinning top were meromorphic when extended to the complex plane. Searching for further meromorphic solutions she was able to explicitly solve for one further unknown case. No further cases in which these equations can be solved explicitly have been discovered since.

Malmquist looked at equations of the form

$$f' = R(z, f), \tag{2}$$

where $R$ is a rational function of $z$ and $f$. The Malmquist Theorem says that if (2) admits a non-rational meromorphic solution, then $R$ is quadratic in $f$. That is, the equation is the Riccati equation. Equations of this form are special since they can be linearised.

A century ago Painlevé [37, 38], Fuchs [14] and Gambier [15] classified a large class of second order differential equations in terms of a characteristic which is now known as the Painlevé property. An ordinary differential equation is said to possess the Painlevé property if all of its solutions are single-valued about all movable singularities (see, for example, [1]). Painlevé and his colleagues showed that any equation with the Painlevé property of the form

$$f'' = F(z; f, f'),$$

where $F$ is rational in $f$ and $f'$ and locally analytic in $z$, can be transformed to one of fifty canonical equations. Forty four of these were integrable in terms of previously known functions (such as elliptic functions and linear equations). The remaining six are now known as the Painlevé differential equations. During the twentieth century it was confirmed by different authors (and by different methods) that these equations possess the Painlevé property [37, 33, 31, 27, 16].

The Painlevé property is a good detector of integrability. For instance, the six Painlevé differential equations are proven to be integrable by the inverse scattering techniques based on an associated isomonodromy problem, see, for instance, [3]. It is widely believed that all ordinary differential equations possessing the Painlevé property are integrable, although there are examples of equations which are solvable via an evolving monodromy problem but do not have the Painlevé property [10].

Recently there has been much interest in extending the idea's of Painlevé to difference equations. Ablowitz, Halburd and Herbst [2] suggested that the growth (in the sense of Nevanlinna) of meromorphic solutions could be used to identify those equations which are of "Painlevé type". In [20] the existence of one finite order non-rational meromorphic solution was shown to be sufficient to reduce a general class

of difference equations to either one of the known difference Painlevé equations, or to the difference Riccati equation. This indicates that the existence of a finite-order (in the sense of Nevanlinna theory) meromorphic solution of a difference equation is a strong indicator of integrability of the equation.

Ablowitz, Halburd and Herbst [2] considered the question of when a difference equation is integrable (or otherwise). For a large number of difference equations they found that the answer can be found from the singularity structure at infinity. This coupled with the fact that many ordinary difference equations admit meromorphic solutions, suggests that Nevanlinna theory is an important ingredient in the study of integrable discrete or difference systems. This has led to recent work in generalizing the theorems of Nevanlinna theory concerning differential equations to analogous theorems concerning difference equations. When applying Nevanlinna theory to study the growth and value distribution of meromorphic solutions of differential equations, estimates involving logarithmic derivatives have often proved to be useful [16, 30]. Recently, similar tools involving shifts have been developed to study ordinary difference equations [11, 21, 22]. The following theorem by Halburd and Korhonen [21] is among the fundamental results of this type.

**Theorem 1** *Let $f$ be a non-constant finite-order meromorphic function, and $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right)$$

*for any $\delta < 1$, and for all $r$ outside of an exceptional set with finite logarithmic measure.*

Note that Chiang and Feng [11] have also obtained the following theorem which is a similar but weaker estimate.

**Theorem 2** *Let $f(z)$ be a meromorphic function of finite order $\sigma$ and let $\eta$ be a non-zero complex number. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\sigma-1+\epsilon}).$$

Theorem 1 and its corollaries proved to be indispensable when singling out Painlevé type equations from large classes of difference equations [19, 20]. Theorem 1 may also be used to study value distribution of finite-order meromorphic solutions of large classes of difference equations, including difference Riccati and difference Painlevé equations.

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# Chapter 1

# Introduction to Nevanlinna Theory

## Introduction

This chapter is mainly a review of known material, the majority of the ideas come from [30] and [17]. In 1925 R. Nevanlinna (partially in cooperation with F. Nevalinna) created what is now called the Nevanlinna theory of meromorphic functions. A good introduction to the subject is [23].

Nevanlinna theory is an efficient tool for studying the density of points in the complex plane at which a meromorphic function takes a prescribed value. It also provides a natural way to describe the growth of a meromorphic function. In this chapter we present the key ideas of Nevanlinna theory so that we can use them in our later work concerning $q$-difference equations.

## 1.1 The Nevanlinna Characteristic Function and The First Main Theorem

For entire functions $g(z)$ where $z = re^{i\varphi}$, the device we use for measuring the growth is the maximum modulus $M(r, g) = \max_{|z|=r} |g(z)|$. Many of the properties of entire functions are encoded into the maximum modulus. In the study of meromorphic functions a natural analogue of the maximum modulus is needed, this analogue is called the characteristic function, $T$. The characteristic function measures the average size of a meromorphic function on large circles, it takes into account the number of poles inside the circles and is derived from the Poisson-Jensen formula which we state below.

**Theorem 3** *Let $f$ be a meromorphic function such that $f(z) \neq 0, \infty$ and let $a_1, a_2, \ldots$ (resp. $b_1, b_2, \ldots$) denote its zeros (resp. poles), each taken into account*

*according to its multiplicity. Suppose $z = re^{i\varphi}$ and $r < R$. Then*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\varphi - \theta) + r^2} \log|f(Re^{i\theta})|\, d\theta$$

$$+ \sum_{|a_i|<R} \log\left|\frac{R(z-a_i)}{R^2 - \bar{a}_i z}\right| - \sum_{|b_j|<R} \log\left|\frac{R(z-b_j)}{R^2 - \bar{b}_j z}\right|. \qquad (1.1)$$

The Poisson-Jensen formula relates $\log|f(z)|$ to its zeros and poles. For details of the proof see [23].

Taking $z = 0$ in Theorem 3 gives us the Jensen Formula, i.e.

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \sum_{|a_i|<R} \log\frac{R}{|a_i|} + \sum_{|b_j|<R} \log\frac{R}{|b_j|}. \qquad (1.2)$$

In order to make sense of the above formula we make the following definition.

**Definition 4** *For meromorphic functions $f$, the unintegrated counting function denoted $n(r, a, f)$, is the number of times $f(z) = a$ for $|z| \leq r$ (counting multiplicities).*

*Note that $n(r, a) := n(r, a, f)$.*

There are 2 summation terms in the Jensen formula, one for $f$'s zeros and one for $f$'s poles. We want to split the integral term in a similar way so that we have a term that relates to zeros and a term that relates to poles. To do this we define the $\log^+$ function.

**Definition 5** *For $x > 0$ we define*

$$\log^+ x := \max(\log x, 0)$$

The $\log^+$ function satisfies the following.

**Lemma 6** (a) $\log \alpha \leq \log^+ \alpha$;

(b) $\log^+ \alpha \leq \log^+ \beta$ *for* $\alpha \leq \beta$ ;

(c) $\log \alpha = \log^+ \alpha - \log^+ \frac{1}{\alpha}$;

(d) $|\log \alpha| = \log^+ \alpha + \log^+ \frac{1}{\alpha}$;

(e) $\log^+(\prod_{i=1}^n \alpha_i) \leq \sum_{i=1}^n \log^+ \alpha_i$;

(f) $\log^+(\sum_{i=1}^n \alpha_i) \leq \log n + \sum_{i=1}^n \log^+ \alpha_i$;

Using definition 4 and Lemma 6 we can show that the Jensen Formula becomes

$$
\begin{aligned}
\log|f(0)| \;=\;& \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta + \int_0^r \frac{n(t,\infty)}{t}\,dt \\
& - \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f(re^{i\theta})}\right|\,d\theta - \int_0^r \frac{n(t,0)}{t}\,dt.
\end{aligned}
\tag{1.3}
$$

We now look at what happens to equation (1.3) when $f$ can have zeros and poles at the origin. That is we let $f$ be a meromorphic function with the Laurent expansion

$$
f(z) = \sum_{i=m}^{\infty} c_i z^i, \;\; c_m \neq 0, \;\; m \in \mathbb{Z}
$$

at the origin. Then defining $h(z) := f(z)z^{-m}$, we have

$$
h(z) = \sum_{i=m}^{\infty} c_i z^{i-m} = c_m + \cdots
$$

$$
\Rightarrow h(0) = c_m,
$$

and therefore the Jensen formula gives

$$
\begin{aligned}
\log|c_m| \;=\;& \log|h(0)| \\
=\;& \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|\,d\theta + \int_0^r \frac{n(t,\infty,h)}{t}\,dt - \int_0^r \frac{n(t,0,h)}{t}\,dt \\
=\;& \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta + \int_0^r \frac{n(t,\infty)-n(0,\infty)}{t}\,dt + n(0,\infty)\log r \\
& - \left(\int_0^r \frac{n(t,0)-n(0,0)}{t}\,dt + n(0,0)\log r\right).
\end{aligned}
\tag{1.4}
$$

To make equation (1.4) more meaningful we make the following definitions.

**Definition 7** *(Proximity function). For a meromorphic function $f$, we define*

$$
m\left(r,\frac{1}{f-a}\right) := \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{1}{f(re^{i\theta})-a}\right|\,d\theta
$$

*supposing $f \not\equiv a \in \mathbb{C}$ and*

$$
m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta.
$$

The proximity function describes the average "closeness" of $f$ to any poles (or $a$-points) on a circle of radius $r$.

**Definition 8** *(Counting function). For a meromorphic function $f$, we define*

$$
N\left(r,\frac{1}{f-a}\right) := \int_0^r \frac{n(t,a)-n(0,a)}{t}\,dt + n(0,a)\log r
$$

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*supposing $f \not\equiv a \in \mathbb{C}$ and*

$$N(r, f) := \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} \, dt + n(0, \infty) \log r.$$

The counting function, $N(r, f)$, is used to give us a measure of the density of $f$'s poles (or $a$-points) in the disc $|z| \leq r$. We are now able to define the characteristic function, $T$.

**Definition 9** *(Characteristic function). For a meromorphic function $f$, we define*

$$T(r, f) := m(r, f) + N(r, f).$$

$T$ can be understood as an analogue of the logarithm of the maximum modulus of an entire function and therefore it is used as a measure of the growth of a meromorphic function $f$.

The characteristic function satisfies the following properties.

**Proposition 10** *Let $f, f_1, \ldots, f_n$ be meromorphic functions and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\delta - \beta\gamma \neq 0$. Then*

**(a)** $T(r, f_1 \cdots f_n) \leq \sum_{i=1}^n T(r, f_i)$,

**(b)** $T(r, f^n) = nT(r, f), \quad n \in \mathbb{N}$,

**(c)** $T\left(r, \sum_{i=1}^n f_i\right) \leq \sum_{i=1}^n T(r, f_i) + \log n$,

**(d)** $T\left(r, \frac{\alpha f + \beta}{\gamma f + \delta}\right) = T(r, f) + O(1)$,

*assuming $f \not\equiv -\delta/\gamma$.*

The following proposition tells us that when our new tool $T$ is applied to entire functions it behaves similarly to the log of the maximum modulus.

**Proposition 11** *Let $g$ be an entire function and assume that $0 < r < R < \infty$ and that the maximum modulus $M(r, g) = \max_{|z|=r} |g(z)|$ satisfies $M(r, g) \geq 1$. Then*

$$T(r, g) \leq \log M(r, g) \leq \frac{R + r}{R - r} T(R, g).$$

Also the function $T(r, f)$ is an increasing function of $r$ and a convex increasing function of $\log r$. This enables us to define the order of growth of a meromorphic function in a natural way as follows:

$$\rho := \lim_{r \to \infty} \sup \frac{\log T(r, f)}{\log r}.$$

We remark that for entire functions $\rho(f)$ is equal to the classical growth order

$$\sigma(f) := \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

### 1.1.1 Nevanlinna's First Main Theorem

Nevanlinna's First Main Theorem follows from equation (1.4). It states that

$$T(r, f) = T\left(r, \frac{1}{f - a}\right) + O(1) \tag{1.5}$$

for all complex numbers $a$ (for details of the proof see [30]). This implies that if $f$ takes the value $a$ less often than average so that $N(r, \frac{1}{f-a})$ is relatively small, then the proximity function $m(r, \frac{1}{y-a})$ must be relatively large. And vice versa. This reasoning can be illustrated by the exponential function, $e^z$. Since $e^z \neq 0, \infty$ we have that $N(r, e^z) = N(r, \frac{1}{e^z}) = 0$. The First Main Theorem states that $m(r, e^z)$ and $m(r, \frac{1}{e^z})$ must be large and this is certainly true since $m(r, e^z) = m(r, \frac{1}{e^z}) = \frac{r}{\pi}$. This means that on any large circle there must be a large part on which $e^z$ is close to zero and another large part on which $e^z$ is close to infinity. And we can see that this is the case by observing that the exponential is very large in most of the positive half plane and very small in most of the negative half plane.

Using The First Main Theorem we are able to prove the following result

**Theorem 12** *A meromorphic function $f$ is rational if and only if $T(r, f) = O(\log r)$.*

For details of the proof see [30].

## 1.2 Applications to Differential Equations

We are now ready to start looking at how Nevanlinna theory can be developed as a tool to assist in the process of solving differential equations. A useful starting point would be to have an estimate for $T(r, f')$ in terms of $T(r, f)$. Since the Poisson-Jensen formula gives us an expression for $\log |f(z)|$ it turns out that we can estimate the proximity function of the logarithmic derivative $f'/f$ more easily than the proximity function of the derivative $f'$.

Using the Poisson-Jensen formula it can be shown that in a neighbourhood of $z_0$ we have the following expression:

$$\log f(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho' e^{i\theta})| \frac{\rho' e^{i\theta} + z}{\rho' e^{i\theta} - z} \, d\theta$$

$$+ \sum_{i=1}^n \log \frac{\rho'(z - a_i)}{\rho'^2 - \overline{a}_i z} - \sum_{j=1}^m \log \frac{\rho'(z - b_j)}{\rho'^2 - \overline{b}_j z} + ic,$$

where $c$ is a real constant, $0 < r < \rho < R$ and $\rho' = (\rho + r)/2$. Taking derivatives on both sides with respect to $z$ gives

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho' e^{i\theta})| \frac{2\rho' e^{i\theta}}{(\rho' e^{i\theta} - z)^2} \, d\theta$$

$$+ \sum_{i=1}^n \left( \frac{\overline{a}_i}{\rho'^2 - \overline{a}_i z} - \frac{1}{a_i - z} \right) + \sum_{j=1}^m \left( \frac{1}{b_j - z} - \frac{\overline{b}_j}{\rho'^2 - \overline{b}_j z} \right). \quad (1.6)$$

Formula (1.6) is valid in a neighbourhood around $z_0$ and in particular, at the point $z_0$. Since $z_0$ is arbitrary we have that (1.6) is valid everywhere except at the zeros and poles of $f$. But since (1.6) assumes infinity on both sides when there are zeros and poles we have that (1.6) is valid everywhere in the disk $|z| < \rho'$.

Since we want to estimate the proximity function of $f'/f$ we take the modulus of (1.6) and consider each term separately. We get

$$\left| \frac{f'(z)}{f(z)} \right| \leq \frac{4\rho'}{(\rho' - r)^2} \left\{ T(\rho', f) + \log^+ \frac{1}{|f(0)|} \right\} + \frac{n(\rho')}{r} \left\{ \frac{r}{\delta(z)} + \frac{r}{\rho' - r} \right\}.$$

Taking $\log^+$ of both sides, estimating the terms and integrating then gives

$$m \left( r, \frac{f'}{f} \right) \leq \log^+ \rho' + 2 \log^+ \frac{1}{\rho' - r} + \log^+ T(\rho', f) + \log^+ \log^+ \frac{1}{|f(0)|} + \log^+ \frac{1}{r}$$

$$+ 3 \log^+ \rho + 3 \log^+ \frac{1}{\rho - \rho'} + 3 \log^+ T(\rho, f) + 3 \log^+ \log^+ \frac{1}{|f(0)|} + 6 \log 2$$

$$+ \frac{1}{2} + \log^+ \frac{r}{\rho' - r} + 5 \log 2,$$

i.e.

$$m \left( r, \frac{f'}{f} \right) \leq 4 \log^+ T(\rho, f) + 4 \log^+ \log^+ \frac{1}{|f(0)|} + 5 \log^+ \rho$$

$$+ 6 \log^+ \frac{1}{\rho - r} + \log^+ \frac{1}{r} + 14. \quad (1.7)$$

(For more details of the above argument see [17]). Transcendental functions, $f$, have the property that $\log r = o(T(r, f))$ as $r \to \infty$. Therefore we see that the only

important term in the above lemma is the $4\log^+ T(\rho, f)$ term. But it turns out that this term is also insignificant and this is shown using the Borel-Nevanlinna Growth Lemma [9] which is the following.

**Lemma 13** *Let $F(r)$ and $\phi(r)$ be positive, nondecreasing, continuous functions defined for $r_0 \leq r < \infty$, and assume that $F(r) \geq e$ for $r \geq r_0$. Let $\xi(x)$ be a positive, nondecreasing function defined for $e \leq x < \infty$. Let $C > 1$ be a constant, and let $E$ be the closed subset of $[r_0, \infty)$ defined by*

$$E = \left\{ r \in [r_0, \infty) : F\left(r + \frac{\phi(r)}{\xi(F(r))}\right) \geq CF(r) \right\}$$

*Then, for all $R < \infty$,*

$$\int_{E \cap [r_0, R]} \frac{dr}{\phi(r)} \leq \frac{1}{\xi(e)} + \frac{1}{\log C} \int_e^{F(R)} \frac{dx}{x\xi(x)}.$$

**Proof.** To simplify notation set

$$h(r) = \frac{\phi(r)}{\xi(F(r))}.$$

If $E$ is empty then the lemma is proved, hence we can assume $E$ is non empty. First we construct two sequences $\{r_n\}$ and $\{s_n\}$.

The first point in the sequence $\{r_n\}$ is $r_0$ (given above). Suppose $\{r_n\}$ has been defined. Then each $s_n$ is defined in terms of $r_n$ in the following way. If there is no $s > r_n$ such that $F(s) \geq CF(r_n)$ then our sequences are complete. Otherwise, by the continuity of $F$ (see diagram below), there exists an $s > r_n$ such that $F(s) = CF(r_n)$.
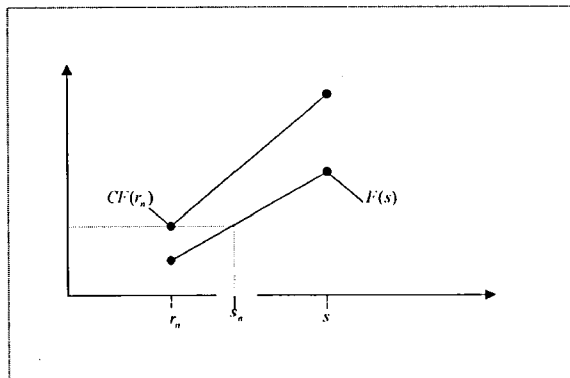


We take

$$s_n := \text{smallest } s \geq r_n \text{ such that } F(s) = CF(r_n).$$

Proceeding inductively we define

$$r_{n+1} := \text{ the smallest } r \in E \text{ with } r \geq s_n.$$

If $r_{n+1}$ does not exist then our sequences are complete.

We now show that either the sequence $\{r_n\}$ is finite or $r_n \to \infty$ as $n \to \infty$. Suppose the sequence $\{r_n\}$ is infinite. Then for all $j$ we have that

$$F(r_{j+1}) \geq F(s_j) = CF(r_j) \quad (\text{ by definition } F \text{ is nondecreasing and } r_{j+1} \geq s_j).$$

Hence,

$$F(r_{n+1}) \geq CF(r_n) \geq C^2 F(r_{n-1}) \geq \cdots \geq C^n F(r_1), \tag{1.8}$$

and this implies $F(r_n) \to \infty$ as $n \to \infty$ and therefore $r_n \to \infty$ as $n \to \infty$, as required.

We now use our sequences to show that $E$ has finite logarithmic measure. Fix $R < \infty$ then take

$$N := \text{ largest } n \text{ such that } r_n < R.$$

If $s_N$ does not exist then let $s_N = R$. By definition we have that the set $E \cap [r_0, R]$ is contained in

$$\bigcup_{n=1}^{N} [r_n, s_n].$$

Hence

$$\int_{E \cap [r_0, R]} \frac{dr}{\phi(r)} \leq \sum_{n=1}^{N} \int_{r_n}^{s_n} \frac{dr}{\phi(r)}$$

$$\leq \sum_{n=1}^{N} \int_{r_n}^{s_n} \frac{dr}{\phi(r_n)} \quad (\phi \text{ is nondecreasing})$$

$$= \sum_{n=1}^{N} \frac{s_n - r_n}{\phi(r_n)}.$$

Now since $r_n \in E$ we have

$$F(r_n + h(r_n)) \geq CF(r_n) = F(s_n),$$

and this implies

$$r_n + h(r_n) \geq s_n,$$

and therefore

$$0 < s_n - r_n \leq h(r_n).$$

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Thus we have

$$
\int_{E \cap [r_0, R]} \frac{dr}{\phi(r)} \leq \sum_{n=1}^{N} \frac{h(r_n)}{\phi(r_n)}
$$

$$
= \sum_{n=1}^{N} \frac{1}{\xi(F(r_n))} \qquad (\text{ by the definition of } h)
$$

$$
\leq \sum_{n=1}^{N} \frac{1}{\xi(C^{n-1}F(r_1))} (\text{ by } (1.8)).
$$

But

$$
\sum_{n=1}^{N} \frac{1}{\xi(C^{n-1}F(r_1))} = \frac{1}{\xi(F(r_1))} + \sum_{n=2}^{N} \frac{1}{\xi(C^{n-1}F(r_1))}
$$

$$
= \frac{1}{\xi(F(r_1))} + \sum_{n=2}^{N} \frac{1}{\xi(\exp(\log F(r_1) + (n-1)\log C))}
$$

$$
\leq \frac{1}{\xi(F(r_1))} + \frac{1}{\log C} \int_{\log F(r_1) + \log C}^{\log F(r_1) + (N-1)\log C} \frac{dv}{\xi(e^v)}.
$$

By (1.8) and the definition of $N$ we have

$$
C^{N-1}F(r_1) \leq F(r_N) \leq F(R).
$$

Hence

$$
\log F(r_1) + (N-1)\log C \leq \log F(R)
$$

and therefore

$$
\sum_{n=1}^{N} \frac{1}{\xi(C^{n-1}F(r_1))} \leq \frac{1}{\xi(F(r_1))} + \frac{1}{\log C} \int_{\log F(r_1)}^{\log F(R)} \frac{dv}{\xi(e^v)}
$$

$$
= \frac{1}{\xi(F(r_1))} + \frac{1}{\log C} \int_{F(r_1)}^{F(R)} \frac{du}{u\xi(u)}.
$$

The lemma follows since $F(r_1) \geq e$ and $\xi(x) \geq \xi(e)$ for $e \leq x < \infty$. $\square$

**Corollary 14** *Let $T : [r_0, \infty) \to [1, \infty)$ be a continuous, nondecreasing function. Then*

$$
T\left(r + \frac{1}{T(r)}\right) < 2T(r) \tag{1.9}
$$

*outside of a possible exceptional set $E_0 \subset [r_0, \infty)$ with linear measure $\leq 2$.*

Note that Corollary 14 is often referred to as the Borel Lemma. Due to the exceptional set in the Borel Lemma we introduce the $S(r, f)$ notation. A quantity which is of the growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure, is denoted by $S(r, f)$. We now have

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**Theorem 15** *Let $f$ be a transcendental meromorphic function. Then*

$$m\left(r, \frac{f'}{f}\right) = S(r, f).$$

*If $f$ is of finite order growth, then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r).$$

**Proof.** By equation (1.7) we have

$$
\begin{aligned}
m\left(r, \frac{f'}{f}\right) \leq\ & 4\log^+ T(\rho, f) + 4\log^+ \log^+ \frac{1}{|f(0)|} + 5\log^+ \rho \\
+\ & 6\log^+ \frac{1}{\rho - r} + \log^+ \frac{1}{r} + 14 \\
=\ & \begin{cases} 4\log^+ T(\rho, f) + S(r, f), & \text{if } f \text{ is transcendental,} \\ 4\log^+ T(\rho, f) + O(\log r), & \text{if } f \text{ is of finite order growth.} \end{cases}
\end{aligned}
$$

If we can show that $4\log^+ T(\rho, f) = S(r, f)$ then we are done. Fix $r$ sufficiently large such that $T(r, f) \geq 1$. Choose $\rho = r + \frac{1}{T(r,f)}$, then by Corollary 14 we have

$$T(\rho, f) = T\left(r + \frac{1}{T(r,f)}, f\right) < 2T(r, f),$$

outside of a possible exceptional set of finite linear measure. This implies

$$\frac{\log^+ T(\rho, f)}{T(r, f)} < \frac{\log^+(2T(r, f))}{T(r, f)} \to 0, \text{ as } r \to \infty,$$

outside of a possible exceptional set of finite linear measure, as required. $\square$

**Corollary 16** *Let $f$ be a transcendental meromorphic function and $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

*and if $f$ is finite order of growth, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

**Corollary 17** *For any transcendental meromorphic function $f$,*

$$T(r, f') \leq 2T(r, f) + S(r, f).$$

**Proof.** The corollary follows by noting that

$$m(r, f') = m\left(r, \frac{f'f}{f}\right) \le m\left(r, \frac{f'}{f}\right) + m(r, f)$$

and

$$N(r, f') \le 2N(r, f). \qquad \square$$

The Lemma on the Logarithmic Derivative is an integral part of the proof of the Second Main Theorem, one of the deepest results of Nevanlinna theory. In addition, logarithmic derivative estimates are crucial for applications to complex differential equations.

Consider equations of the form

$$f' = R(z, f), \tag{1.10}$$

We make the following which comes from [30].

**Definition 18** *Let $R(z, f)$ be rational in $f$ with meromorphic coefficients. A meromorphic solution, $f$, of equation (1.10) is called admissible, if $T(r, \alpha) = S(r, f)$ holds for all coefficients $\alpha(z)$ of $R(z, f)$. Note that if the coefficients $\alpha$ are rational then any meromorphic solution is admissible if and only if it is transcendental.*

By proving the Malmquist Theorem we will find that if (1.10) possesses a non-rational meromorphic solution then it must reduce to

$$f' = \alpha_2(z)f^2 + \alpha_1(z)f + \alpha_0(z), \tag{1.11}$$

where at least one of the coefficients $\alpha_i$ does not vanish. Hence by demanding an admissible solution we are led to the the Riccati equation (1.11). The Riccati equation is special since it can be linearised.

The following result is the Valiron-Mohon'ko Theorem, see [30] Theorem 2.2.5 for the proof, it tells us about the characteristic function of rational functions with meromorphic coefficients in $S(r, f)$.

**Theorem 19** *Let $f$ be a meromorphic function. Then for all irreducible rational functions in $f$,*

$$R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{i=0}^{p} a_i(z)f^i}{\sum_{j=0}^{q} b_j(z)f^j}, \tag{1.12}$$

*with meromorphic coefficients $a_i(z), b_j(z)$ such that*

$$\begin{cases} T(r, a_i) = S(r, f), & i = 0, \ldots, p \\ T(r, b_j) = S(r, f), & j = 0, \ldots, q, \end{cases} \tag{1.13}$$

*the characteristic function of $R(z, f(z))$ satisfies*

$$T(r, R(z, f)) = dT(r, f) + S(r, f), \tag{1.14}$$

*where $d = \max(p, q)$.*

The following theorem is the Malmquist Theorem.

**Theorem 20** *Let $R(z, f)$ be rational in $f$ with meromorphic coefficients. If the differential equation (1.10) possesses an admissible meromorphic solution, then (1.10) reduces to*

$$f' = \alpha_2(z)f^2 + \alpha_1(z)f + \alpha_0(z)$$

*where at least one of the coefficients $\alpha_i(z)$ does not vanish.*

**Proof.** Let

$$R(z, f) = \frac{a_0(z) + a_1(z)f + \cdots + a_p(z)f^p}{b_0(z) + b_1(z)f + \cdots + b_q(z)f^q}$$

and

$$d := \max(p, q).$$

Equation (1.10) and Theorem 19 imply that

$$dT(r, f) + S(r, f) = T(r, R(z, f)) = T(r, f').$$

Hence by Corollary 17 we have

$$dT(r, f) + S(r, f) \leq 2T(r, f),$$

i.e.

$$(2 - d)T(r, f) \geq S(r, f).$$

This implies $d \leq 2$.

With $d \leq 2$ equation (1.10) becomes

$$f' = \frac{a_0(z) + a_1(z)f + a_2(z)f^2}{b_0(z) + b_1(z)f + b_2(z)f^2} := \frac{P}{Q}. \tag{1.15}$$

Let

$$g := \frac{1}{f - \alpha},$$

for some $\alpha \in \mathbb{C}$. Then (1.15) becomes

$$g' = \frac{(a_1(z)\alpha + a_0(z) + a_2(z)\alpha^2)g^4 + (2a_2(z)\alpha + a_1(z))g^3 + a_2(z)g^2}{(b_1(z)\alpha + b_0(z) + b_2(z)\alpha^2)g^2 + (2b_2(z)\alpha + b_1(z))g + b_2(z)} := \frac{\tilde{P}}{\tilde{Q}}.$$

Choosing $\alpha$ such that the $g^4$ coefficient in the numerator is non-zero and using the same argument as above we have that $\deg\left(\frac{\tilde{P}}{\tilde{Q}}\right) \leq 2$. Since we have a degree four polynomial divided by a degree two polynomial that has a maximum degree of two the right hand side must cancel to a degree two polynomial. Since $\frac{P}{Q}$ is irreducible this implies $b_1(z) \equiv b_2(z) \equiv 0$.

Therefore equation (1.15) is actually the Riccati equation. I.e.

$$f' = \alpha_2(z)f^2 + \alpha_1(z)f + \alpha_0(z),$$

where

$$\alpha_j = \frac{a_j(z)}{b_0(z)}, \qquad j = 0, 1, 2. \qquad \square$$

The following theorem is the Clunie Lemma, it has numerous applications to the study of complex differential equations, and beyond.

**Lemma 21** *Let $f$ be a transcendental meromorphic solution of*

$$f^n P(z, f) = Q(z, f),$$

*where $P(z, f)$ and $Q(z, f)$ are polynomials in $f$ and its derivatives with meromorphic coefficients, say $\{a_\lambda : \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in $f$ and its derivatives is $\leq n$, then*

$$m(r, P(z, f)) = S(r, f).$$

**Proof.** We split the proximity function of $P$ into two parts by defining

$$E_1 := \{\varphi \in [0, 2\pi] : |f(re^{i\varphi})| < 1\},$$

$$E_2 := [0, 2\pi] \backslash E_1.$$

Now we have

$$2\pi m(r, P(z, f)) = \int_{E_1} \log^+ |P| \, d\varphi + \int_{E_2} \log^+ |P| \, d\varphi.$$

First we consider $E_1$. Each term of $P$ is of the form

$$a_\lambda(z) f^{l_0} \cdots \left(f^{(\nu)}\right)^{l_\nu}.$$

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Hence $P$ can be expressed in terms of $\lambda = (l_0, \ldots, l_\nu)$. Therefore

$$P(z,f) = \sum_{\lambda \in I} P_\lambda(z) = \sum_{\lambda \in I} a_\lambda(z) f^{l_0}(f')^{l_1} \cdots \left(f^{(\nu)}\right)^{l_\nu}.$$

Therefore

$$
\begin{aligned}
\int_{E_1} \log^+ |P|\, d\varphi &= \int_{E_1} \log^+ \left| \sum_{\lambda \in I} a_\lambda(z) f^{l_0}(f')^{l_1} \cdots \left(f^{(\nu)}\right)^{l_\nu} \right| d\varphi \\
&\leq \sum_{\lambda \in I} \int_{E_1} \log^+ \left| a_\lambda(z) f^{l_0}(f')^{l_1} \cdots \left(f^{(\nu)}\right)^{l_\nu} \right| d\varphi + O(1) \\
&\leq \sum_{\lambda \in I} \int_{E_1} \log^+ \left( |a_\lambda(z)| \left| \frac{f'}{f} \right|^{l_0} \cdots \left| \frac{f^{(\nu)}}{f} \right|^{l_\nu} \right) d\varphi + S(r,f) \\
&= \sum_{\lambda \in I} \int_{E_1} \left( \log^+ |a_\lambda(z)| + l_1 \log^+ \left| \frac{f'}{f} \right| + \cdots + l_\nu \log^+ \left| \frac{f^{(\nu)}}{f} \right| \right) d\varphi \\
&\quad + S(r,f) \\
&= 2\pi \sum_{\lambda \in I} \left[ m(r,a_\lambda) + \sum_{j=1}^{\nu} l_j m\left(r, \frac{f^{(j)}}{f}\right) \right] + S(r,f) \\
&= S(r,f),
\end{aligned}
\tag{1.16}
$$

by our assumption in the lemma and Corollary 16.

Now we consider $E_2$. To do this case we note that

$$P(z,f) = \frac{Q(z,f)}{f^n}.$$

But

$$
\begin{aligned}
Q(z,f) &= \sum_{\lambda \in J} Q_\lambda(z) \\
&= \sum_{\lambda \in J} b_\lambda(z) f^{l_0}(f')^{l_1} \cdots \left(f^{(\nu)}\right)^{l_\nu}.
\end{aligned}
$$

By our assumption $l_1 + \cdots + l_\nu \leq n$ for all $\lambda = (l_0, \ldots, l_\nu) \in J$. Hence we have

$$
\begin{aligned}
\int_{E_2} \log^+ |P| \, d\varphi &= \int_{E_2} \log^+ \left| \frac{1}{f^n} \sum_{\lambda \in J} b_\lambda(z) f^{l_0} (f')^{l_1} \cdots (f^{(\nu)})^{l_\nu} \right| d\varphi \\
&= \int_{E_2} \log^+ \left| \sum_{\lambda \in J} \frac{b_\lambda(z)}{f^{n - l_0 - \cdots - l_\nu}} \frac{f^{l_0}}{f^{l_0}} \frac{(f')^{l_1}}{f^{l_1}} \cdots \frac{(f^{(\nu)})^{l_\nu}}{f^{l_\nu}} \right| d\varphi \\
&\leq \sum_{\lambda \in J} \int_{E_2} \log^+ \left( |b_\lambda(z)| \left| \frac{f'}{f} \right|^{l_1} \cdots \left| \frac{f^{(\nu)}}{f} \right|^{l_\nu} \right) d\varphi + S(r, f) \\
&= \sum_{\lambda \in J} \int_{E_2} \left( \log^+ |b_\lambda(z)| + l_1 \log^+ \left| \frac{f'}{f} \right| + \cdots + l_\nu \log^+ \left| \frac{f^{(\nu)}}{f} \right| \right) d\varphi \\
&\quad + S(r, f) \\
&= 2\pi \sum_{\lambda \in J} m(r, b_\lambda) + \sum_{j=1}^{\nu} l_j m \left( r, \frac{f^{(j)}}{f} \right) + S(r, f) \\
&= S(r, f). \tag{1.17}
\end{aligned}
$$

Combining (1.16) and (1.17) gives us the lemma. $\square$

## 1.3 Nevanlinna's Second Main Theorem

A very important theorem in complex analysis is Picard's Great Theorem [43]. It states that every non constant entire function attains every complex value with at most one exception. Nevanlinna offered a deep generalization of Picard's Great Theorem in the form of his Second Main Theorem. The First Main Theorem tells us that for every complex number $a$ the sum $m(r, a, f) + N(r, a, f)$ is largely independent of $a$. The Second Main Theorem tells us that in general it is the term $N(r, a, f)$ that is dominant in the sum $N + m$ and also that for most values of $a$ the equation $f(z) = a$ has mostly simple roots.

**Theorem 22** *Let $f$ be a non-constant meromorphic function, let $q \geq 2$ and let $a_1, \ldots, a_q \in \mathbb{C}$ be distinct points. Then*

$$
m(r, f) + \sum_{n=1}^{q} m \left( r, \frac{1}{f - a_n} \right) \leq 2T(r, f) + S(r, f).
$$

**Proof.** Throughout this proof we will make use of proposition 10 without reference. Denote

$$
P(f) := \prod_{n=1}^{q} (f - a_n).
$$

Using the First Main Theorem we have

$$\sum_{n=1}^{q} m\left(r, \frac{1}{f-a_n}\right) = \sum_{n=1}^{q} T\left(r, \frac{1}{f-a_n}\right) - \sum_{n=1}^{q} N\left(r, \frac{1}{f-a_n}\right)$$

$$= qT(r,f) - \sum_{n=1}^{q} N\left(r, \frac{1}{f-a_n}\right) + S(r,f).$$

But by the Valiron-Mohon'ko Theorem we have that

$$qT(r,f) = T(r, P(f)) + S(r,f).$$

We also have that

$$\sum_{n=1}^{q} N\left(r, \frac{1}{f-a_n}\right) = N\left(r, \frac{1}{P(f)}\right).$$

Since $\frac{1}{P(f)}$ can be expressed as a partial fraction (for some constants $\alpha_n \in \mathbb{C}$) we have that

$$m\left(r, \frac{f'}{P(f)}\right) = m\left(r, \sum_{n=1}^{q} \alpha_n \frac{f'}{f-a_n}\right)$$

$$\leq \sum_{n=1}^{q} \left[\underbrace{m(r, \alpha_n)}_{=\log^+ \alpha_n} + m\left(r, \frac{f'}{f-a_n}\right)\right] + O(1)$$

$$= S(r,f) \qquad \text{by the logarithmic derivative result, Theorem 15.}$$

Hence we have

$$\sum_{n=1}^{q} m\left(r, \frac{1}{f-a_n}\right) = m\left(r, \frac{1}{P(f)}\right) + S(r,f)$$

$$= m\left(r, \frac{f'}{P(f)}\frac{1}{f'}\right) + S(r,f)$$

$$\leq m\left(r, \frac{1}{f'}\right) + S(r,f).$$

Using The First Main Theorem this implies

$$\sum_{n=1}^{q} m\left(r, \frac{1}{f-a_n}\right) \leq T(r,f') - N\left(r, \frac{1}{f'}\right) + S(r,f)$$

$$= m(r,f') + N(r,f') - N\left(r, \frac{1}{f'}\right) + S(r,f).$$

Now using

$$m(r, f') = m\left(r, f\frac{f'}{f}\right) \leq m(r, f) + m\left(r, \frac{f'}{f}\right) = m(r, f) + S(r, f), \text{ by Theorem 15,}$$

we have

$$\sum_{n=1}^{q} m\left(r, \frac{1}{f - a_n}\right) \leq m(r, f) + N(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f)$$

$$= T(r, f) + N(r, f') - N(r, f) - N\left(r, \frac{1}{f'}\right) + S(r, f).$$

Therefore we have

$$m(r, f) + \sum_{n=1}^{q} m\left(r, \frac{1}{f - a_n}\right) \leq m(r, f) + T(r, f) + N(r, f')$$

$$-N(r, f) - N\left(r, \frac{1}{f'}\right) + S(r, f)$$

$$= 2T(r, f) - \left(N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f')\right)$$

$$+S(r, f).$$

Now if we can show

$$\left(N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f')\right) \geq 0,$$

we are done. To do this we show that

$$N(r, f') = N(r, f) + \bar{N}(r, f),$$

where $\bar{N}(r, f)$ is the counting function for distinct poles of $f$, i.e.

$$\bar{N}(r, f) := \int_0^r \frac{\bar{n}(t, f) - \bar{n}(0, f)}{t} \, dt + \bar{n}(0, f) \log r,$$

where $\bar{n}(r, f)$ is the number of poles (not counting multiplicities) $f$ has in the disc $|z| \leq r$. And we see that this is true by substituting $n(r, f') = n(r, f) + \bar{n}(r, f)$ into the definition for $N(r, f')$. Hence we have

$$2N(r, f) - N(r, f') = 2N(r, f) - N(r, f) - \bar{N}(r, f) = N(r, f) - \bar{N}(r, f) \geq 0,$$

and therefore

$$m(r, f) + \sum_{n=1}^{q} m\left(r, \frac{1}{f - a_n}\right) \leq 2T(r, f) + S(r, f).$$

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as required. $\square$

# Chapter 2

# Existence and Growth of Meromorphic Solutions of some $q$-Difference Equations

## Introduction

This chapter is mainly a review of known material from [42], [7] and [18].

In this chapter we start to look at how the Nevanlinna theory introduced in chapter one can be applied to studying meromorphic solutions of $q$-difference equations. A natural class of equations to consider are linear equations and autonomous first order non-linear equations.

In the first section we look at a natural class of autonomous first order non-linear $q$-difference equations, i.e. the Schröder equation

$$f(qz) = R(f(z)), \tag{2.1}$$

where $q = R'(0)$, $|q| \neq 0, 1$, and $R(f)$ is a rational function in $f$. We show that the Schröder equation (2.1) has a convergent power series solution $f(z) = z + \cdots$ in a neighbourhood of $z = 0$, where $R$ is a rational function in $f$ with constant coefficients and $q$ is not a root of unity.

In the second section we look at the growth of meromorphic solutions of the general linear $q$-difference equation.

$$\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z), \tag{2.2}$$

where $0 < |q| < 1$ is a complex number, and $a_j(z)$, $j = 0, 1, \ldots, n$ and $Q(z)$ are rational functions and $a_0(z) \not\equiv 0$, $a_n(z) \equiv 1$. Linear $q$-difference equations with rational coefficients do not always admit meromorphic solutions, even if the coefficients are constants. Bergweiler, Ishizaki and Yanagihara gave sufficient conditions for the existence of meromorphic solutions of linear $q$-difference equations, and characterized the growth of solutions in terms of the Nevanlinna characteristic $T(r, f)$ [7]. They concluded that all meromorphic solutions $f$ of a linear $q$-difference equation with rational coefficients satisfy $T(r, f) = O((\log r)^2)$, from which it in particular follows that all solutions are of zero order growth. In the next chapter we will develop Nevanlinna theory for the $q$-shift operator acting on zero-order meromorphic functions.

It was shown by Gundersen *et al.* [18] that the order of growth of solutions of the non-autonomous Schröder $q$-difference equation

$$f(qz) = R(z, f(z)),$$

where $R(z, f(z))$ is rational in both arguments, is equal to $\log_q(\deg_f R)$, where $\log_q$ is the $q$-based logarithm. In section 3 we show how their result implies a $q$-difference analogue of the Malmquist Theorem [32].

## 2.1 The Existence of Solutions of Schröder's Equation.

The following is an expanded form of the argument given in [42] .

First take $\alpha$ to be a fixed point of $R$, i.e. $R(\alpha) = \alpha$ (if we assume $R(f(z)) \not\equiv f(z) + k$ for some constant $k$, we know $\alpha$ exists, this is because to find $\alpha$ we must find the roots of a polynomial with degree greater than one). We define $g(z)$ by $g(z) := f(z) - \alpha$, then by (2.1) we get

$$\alpha + g(qz) = R(\alpha + g(z)) = \alpha + \tilde{R}(g(z)),$$

where

$$\tilde{R}(g(z)) = R'(\alpha)g(z) + \cdots .$$

Hence we can transform equation (2.1) to the equation

$$g(qz) = \hat{R}(g(z)),$$

and this equation has solution $g(z) \equiv 0$. Therefore without loss of generality we assume $R(0) = 0$ and look for a series solution of the form

$$f(z) = z + b_2 z^2 + \cdots . \tag{2.3}$$

Equation (2.1) implies that in order for solutions to exist we must have $R'(0) = q$, therefore we make the assumption that $R'(0) = q$ and write $R$ in the form

$$R(z) = qz + a_2 z^2 + a_3 z^3 + \cdots , \qquad q \neq 0. \tag{2.4}$$

### 2.1.1   Formal Series Solution

We seek a solution to (2.1) in terms of a formal power series as in (2.3). Under the assumption that $q$ is not a root of unity, comparison of coefficients will give rise to exactly one solution, which we will call the Schröder series. We now prove that this is the case inductively.

The first coefficient $b_1$ is 1. Let $n \geq 2$, and assume that the coefficients $b_k$ $(1 < k < n)$ have been determined so that both sides of (2.1) agree in terms of order $k < n$.

Subtract $qf(z)$ from both sides of (2.1) to get

$$f(qz) - qf(z) = R(f(z)) - qf(z).$$

Then using (2.3) and (2.4) we get

$$qz + \sum_{l=2}^{\infty} b_l q^l z^l - qz - \sum_{l=2}^{\infty} b_l q z^l = qf(z) + \sum_{l=2}^{\infty} a_l (f(z))^l - qf(z),$$

i.e.

$$\sum_{l=2}^{\infty} (q^l - q) b_l z^l = \sum_{l=2}^{\infty} a_l (f(z))^l. \tag{2.5}$$

On the right hand side the $z^n$ coefficient is a polynomial in $a_l$ $(l = 2, \ldots, n)$ and $b_k$ $(k = 2, \ldots, n-1)$, hence the right hand side is known. On the left hand side the $z^n$ coefficient is $(q^n - q) b_n$. Since $q$ is not a root of unity we have $q^n - q = q(q^{n-1} - 1) \neq 0$ and therefore we can uniquely find each $b_n$ by recursion.

### 2.1.2   Convergence of Formal Series Solution

We will prove the convergence of (2.3) by showing that there is a series of the form

$$F(z) = \sum_{n=1}^{\infty} B_n z^n, \tag{2.6}$$

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where $B_n \geq |b_n|$, $n \in \mathbb{N}$, which converges for $|z| < R_0$ for some $R_0 > 0$. The function $F$ is said to majorize $f$.

Since (2.4) has a non-zero radius of convergence there exists a positive number $a$ such that $|a_{n+1}| < a^n$, $n \in \mathbb{N}$. Substituting $\hat{z} := az$ into (2.4) we get

$$R\left(\frac{\hat{z}}{a}\right) = q\frac{\hat{z}}{a} + a_2\frac{\hat{z}^2}{a^2} + a_3\frac{\hat{z}^3}{a^3} + \cdots.$$

Taking $\hat{R}(\hat{z}) := aR\left(\frac{\hat{z}}{a}\right)$ we get

$$\frac{1}{a}\hat{R}(\hat{z}) = q\frac{\hat{z}}{a} + a_2\frac{\hat{z}^2}{a^2} + a_3\frac{\hat{z}^3}{a^3} + \cdots.$$

Finally taking $\hat{a}_n := \frac{a_n}{a^{n-1}}$ we get

$$\hat{R}(\hat{z}) = q\hat{z} + \hat{a}_2\hat{z}^2 + \hat{a}_3\hat{z}^3 + \cdots.$$

We are therefore able to transform our series (2.4) to a series such that

$$|a_{n+1}| < 1, \qquad \forall n \in \mathbb{N}. \tag{2.7}$$

Therefore without loss of generality we assume (2.7) holds for (2.4). Also since $q$ is not a root of unity and $|q| \neq 1$ we have that there exists $c$ such that

$$|q^{n+1} - q| > c > 0, \qquad (n = 0, 1, 2, \ldots).$$

By (2.5)

$$b_n = \frac{P_n(a_0, \ldots, a_n, b_2, \ldots, b_{n-1})}{q^l - q}$$

for some polynomial $P_n$ for all $n \geq 2$. Hence

$$|b_n| \leq \frac{|P_n(1, \ldots, 1, b_2, \ldots, b_{n-1})|}{c}$$

for all $n \geq 2$. Therefore we define the sequence $B_n$ inductively by taking $B_1 = 1$ and

$$B_n = \frac{|P_n(1, \ldots, 1, B_2, \ldots, B_{n-1})|}{c}.$$

It then follows by induction that $|b_n| \leq B_n$ for all $n \in \mathbb{N}$. By (2.5) we have

$$\sum_{n=2}^{\infty} cB_n z^n = \sum_{n=2}^{\infty}(z + B_2 z^2 + \cdots)^n,$$

i.e.

$$c(F - z) = \sum_{n=2}^{\infty} F^n, \tag{2.8}$$

We have shown that if the solution of equation (2.8) exists then it majorizes $f$.

The series

$$z = F - \frac{1}{c} \sum_{n=2}^{\infty} F^n$$

converges for $|F| < 1$. Differentiating with respect to $F$ we find that

$$\left. \frac{dz}{dF} \right|_{F=0} = 1.$$

Hence by the Inverse Function Theorem the series has an inverse which converges in a neighborhood of $z = 0$. This implies that the series for $f$ converges in a neighborhood of $z = 0$ also. Therefore there exists $\rho > 0$, such that the series for $f$ converges when $|z| < \rho$. Using equation (2.1) we find that the series for $f$ converges when $|z| < |q|\rho$. Arguing inductively the series for $f$ converges when $|z| < |q|^n \rho$ for all $n \in \mathbb{N}$. This implies that we can find $f(z)$ for any $z \in \mathbb{C}$ (since we have assumed $|q| < 1$).

## 2.2 Growth of Meromorphic Solutions of Linear $q$-Difference Equations

The following is an expanded form of the arguments given in [7]. Consider the following linear $q$-difference equation

$$\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z), \tag{2.9}$$

where $0 < |q| < 1$ is a complex number, and $a_j(z)$, $j = 0, \dots, n$ and $Q(z)$ are rational functions and $a_0(z) \not\equiv 0$, $a_n(z) \equiv 1$. We have the following

**Theorem 23** *All meromorphic solutions of (2.9) satisfy $T(r, f) = O((\log r)^2)$.*

**Proof.** Since $Q$ and $a_j$ are rational functions we can choose $R$ so that $Q$ and $a_j$ have no zeros (unless they are identically zero) or poles in the set

$$B_R := \{ z \in \mathbb{C} : |z| \geq R \}.$$

We now show that $n(r, f) = O(\log r)$ by estimating how many poles $f$ has in the set $B_R$.

Let $s = \frac{1}{q}$, $w \in B_R$ and suppose $w, sw, \dots, s^{n-1}w$ are not poles. Then by equation (2.9) $s^n w$ is not a pole. By induction it follows that $s^j w$ is not a pole for any $j \in \mathbb{N}$. (I.e. $w, sw, \dots, s^{n-1}w$ are not poles implies $s^j w$ is not a pole for all $j \in \mathbb{N}$).

Therefore we find that for every pole, $w$, such that $|w| \geq |s|^n R$, there exists a pole in the annulus

$$D_R := \{z \in \mathbb{C} : R \leq |z| \leq |s|^n R\}.$$

Since $f$ is meromorphic, $D_R$ contains a finite number of poles. Call them $w_1, \ldots, w_m$. Hence we have that every pole of $f$ in $B_R$ belongs to the set

$$\{s^j w_l : j \in \mathbb{N}, l \in \{1, \ldots, m\}\}.$$

In order to estimate $n(r, f)$ we fix $r$. Then $n(r, f)$ is less than or equal to $n(R, f)$ (a constant) plus the number of terms of the form $s^j w_l$ ($l \in 1, \ldots, m$) such that $|s^j w_l| < r$. Fixing $l$ we have that $|s^j w_l| < r$ if and only if $j < \frac{\log r - \log |w_l|}{\log |s|}$. Therefore for each $l \in \{1, \ldots, m\}$ we have that

$$j \in \left\{1, \ldots, \left[\frac{\log r - \log |w_l|}{\log |s|}\right]\right\},$$

where here the square brackets are used to denote the integer part. Thus for all $l \in \{1, \ldots, m\}$ there are $O(\log r)$ poles of the form $s^j w_l$. This implies

$$n(r, f) = O(\log r).$$

Therefore

$$
\begin{aligned}
N(r, f) &:= \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r \\
&= \int_0^R \frac{n(t, f) - n(0, f)}{t} \, dt + \int_R^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r \\
&= O((\log r)^2).
\end{aligned}
\tag{2.10}
$$

To estimate the proximity function, $m(r, f)$, we will use the maximum modulus, $M(r, f)$. The maximum modulus is only defined when $f$ does not have poles. As described above for $r \geq R$, $f$'s poles have modulus $|s^j w_l|$, $l = 1, \ldots, m$, $j = 1, \ldots, n$. Hence if we fix $T \in [R, |s|^n R]$ such that $|s^j w_l| \neq T$ for all $j \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, m\}$, we have that $f$ has no pole with modulus $|s|^j T$ for any $j \in \mathbb{N}$. We therefore have the inequality

$$m(|s|^k T, f) \leq \log M(|s|^k T, f) \leq \log M_k := L_k, \tag{2.11}$$

where

$$M_k := \max_{j=0,1,\ldots,k} M(|s|^j T, f) + 1.$$

We now show that $L_k \leq Ck^2$ ($k \in \mathbb{N}$) by first showing that $M_k \leq |s|^{Bk} M_{k-1}$, for some positive constants $B$ and $C$. First let

$$J := \min\{j : M_k = M(|s|^j T, f) + 1\}.$$

We must have that either $J < k$ or $J = k$. If $J < k$ then by definition we have $M_k = M_{k-1}$. If $J = k$ then $M_k = M(|s|^k T, f) + 1$. Therefore we have two cases to consider

**Case 1** $M_k = M_{k-1}$.

In this case we clearly have $M_k \leq |s|^{Bk} M_{k-1}$ straightaway.

**Case 2** $M_k = M(|s|^k T, f) + 1$.

By rearranging equation (2.9) and taking the maximum modulus we have

$$
\begin{aligned}
M(r, f) &= M\left(r, \frac{1}{a_0} \sum_{j=1}^{n} a_j(z) f(q^j z) + Q\right) \\
&\leq \sum_{j=1}^{n} \left[ M\left(r, \frac{a_j}{a_0}\right) M(r, f(q^j z)) \right] + M(r, Q).
\end{aligned}
$$

Since the $a_j$ and $Q$ are rational functions there exists $A$ such that

$$M\left(r, \frac{a_j}{a_0}\right) = O(r^A) \qquad j = 0, \ldots, n$$

and

$$M(r, Q) = O(r^A).$$

Hence

$$M(r, f) \leq r^A \left( \sum_{j=1}^{n} M(|q|^j r, f) + 1 \right). \tag{2.12}$$

It follows that for $k \geq n$ we have

$$
\begin{aligned}
M(|s|^k T, f) &\leq (|s|^k T)^A \left( \sum_{j=1}^{n} M(|s|^{k-j} T, f) + 1 \right) \\
&= (|s|^k T)^A \left( \sum_{j=k-n}^{k-1} M(|s|^j T, f) + 1 \right) \\
&\leq |s|^{kA} T^A \left( \sum_{j=k-n}^{k-1} M_j + 1 \right) \\
&\leq |s|^{kA} T^A (n+1) M_{k-1}.
\end{aligned}
$$

Thus we have

$$M_k \leq |s|^{Bk} M_{k-1}$$

for some $B > A$ and all $k \in \mathbb{N}$. Taking logs we have

$$\log M_k \leq (B \log |s|)k + \log M_{k-1},$$

i.e.

$$L_k \leq Ck + L_{k-1},$$

where $L_k := \log M_k$ and $C := B \log |s|$. We see that $L_1 \leq C + L_0$ and from here prove by induction that

$$
\begin{aligned}
L_k &\leq C \sum_{j=1}^{k} j + L_0 \\
&= C \frac{k(k+1)}{2} + L_0 \\
&\leq Ck^2,
\end{aligned}
$$

for large $k$.

Going back to the inequality (2.11) we have

$$m(|s|^k T, f) \leq \log M(|s|^k T, f) \leq L_k \leq Ck^2.$$

But

$$
\begin{aligned}
k^2 &= k^2 \left( \frac{\log |s|}{\log |s|} \right)^2 \\
&= \left( \frac{\log |s|^k}{\log |s|} \right)^2 \\
&= \left( \frac{\log(|s|^k T) - \log T}{\log |s|} \right)^2,
\end{aligned}
$$

hence

$$m(|s|^k T, f) \leq C \left( \frac{\log(|s|^k T) - \log T}{\log |s|} \right)^2. \tag{2.13}$$

Combining (2.13) and (2.10) we obtain

$$T(r, f) = m(r, f) + N(r, f) = O((\log r)^2)$$

for $r = |s|^k T$, $k \in \mathbb{N}$, $k \to \infty$. But since $T(r, f)$ is an increasing function we are able to prove that the last equation also holds if $r \to \infty$ through any sequence of $r$-values. $\square$

**Theorem 24** *All transcendental meromorphic solutions of equation (2.9) satisfy*

$$(\log r)^2 = O(T(r, f)).$$

**Proof.** As in the proof of Theorem 23 we take $s = \frac{1}{q}$ and choose $R$ so that the $Q$ and $a_j$ have no zeros or poles in the set

$$B_R := \{z \in \mathbb{C} : |z| \geq R\}.$$

We have two cases to consider. Either $f$ has infinitely many poles or $f$ has finitely many poles.

### Case 1: $f$ has infinitely many poles.

The argument at the beginning of the proof of Theorem 23 states that if there are no poles in

$$D_R := \{z \in C : R \leq |z| \leq |s|^n R\},$$

then there are no poles in $B_R$. Therefore if there are infinitely many poles in $B_R$, then there is at least one pole in $D_R$. The same argument works for any $S \geq R$. Thus if for $|z| \geq R$ we partition the complex plane into sets of the form

$$C_i := \{z : R|s|^{(i-1)n} \leq |z| \leq R|s|^{in}\},$$

then each $C_i$ has at least one pole for all $i \in \{1, 2, \ldots\}$.

If we fix $r \geq R$, then we have that there exists $i \in \{1, 2, \ldots\}$ such that

$$R|s|^{(i-1)n} \leq r \leq R|s|^{in}.$$

This implies $n(r, f) \geq i - 1$. We then have

$$
\begin{aligned}
\log r &\leq \log R + in \log |s| \\
&\leq \log R + 2n \log |s| n(r, f),
\end{aligned}
$$

provided our original $r$ was chosen large enough. It immediately follows that

$$\log r = O(n(r, f)).$$

Finally we have

$$
\begin{aligned}
N(r,f) &\geq \int_R^r \frac{n(t,f) - n(0,f)}{t}\,dt \\
&\geq \int_R^r \frac{K\log r - n(0,f)}{t}\,dt,
\end{aligned}
$$

for some $K > 0$. This implies that

$$
(\log r)^2 = O(N(r,f)),
$$

and hence

$$
(\log r)^2 = O(T(r,f)),
$$

as required.

## Case 2 : $f$ has finitely many poles.

Without loss of generality we assume $f$ is entire. If $f$ is not entire then there exists a polynomial, $P(z)$, such that $\tilde{f} := P(z)f(z)$ is entire and solves an equation of the form (2.9). If we can prove the result holds for $\tilde{f}$ then we have that the result holds for $f$, since $T(r,f) = T(r,\tilde{f}) + O(\log r)$.

The idea of the proof here is to prove the result holds for $\log M(r,f)$ and then apply proposition 11.

We write equation (2.9) in the form

$$
a_l(z)f(q^l z) = -\sum_{j=0}^{l-1} a_j(z)f(q^j z) - \sum_{j=l+1}^{n} a_j(z)f(q^j z) + Q(z),
$$

for $0 \leq l \leq n$. For $m \in \mathbb{N}$, $m > l$, we take the modulus and evaluate at $|z| = |s|^m R$ with $z$ taken such that $M(|s|^{m-l}R, f) = |f(q^l z)|$. We then obtain

$$
|a_l(z)|M(|s|^{m-l}R, f) \leq \sum_{j=0}^{l-1} |a_j(z)|M(|s|^{m-j}R, f) + \sum_{j=l+1}^{n} |a_j(z)|M(|s|^{m-j}R, f) + |Q(z)|.
$$

Since the $a_j$ and $Q$ are rational we can assume that

$$
|a_j(z)| \sim c_j |z|^{d_j}
$$

and

$$
Q(z) \sim \lambda |z|^\gamma
$$

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as $|z| \to \infty$, where $c_j, \lambda > 0$ and $d_j, \gamma \in \mathbb{Z}$, $j = 0, 1, \ldots, n$. Also for $i \in \mathbb{N}$ we define

$$T_i = M(|s|^i R, f).$$

By the maximum modulus principle, $(T_i)$ is an increasing sequence and therefore our estimate becomes

$$
\begin{aligned}
& c_l R^{d_l} |s|^{md_l} T_{m-l} \\
\leq \ & (1 + o(1)) \left( \sum_{j=0}^{l-1} c_j R^{d_j} |s|^{md_j} T_m + \sum_{j=l+1}^{n} c_j R^{d_j} |s|^{md_j} T_{m-l-1} \right) + \lambda R^\gamma |s|^{m\gamma}.
\end{aligned}
$$

Considering the above estimate with $m = kl$ where $k \in \mathbb{N}$, $k \geq 2$ yields

$$
\begin{aligned}
& c_l R^{d_l} |s|^{kld_l} T_{(k-1)l} \\
\leq \ & (1 + o(1)) \left( \sum_{j=0}^{l-1} c_j R^{d_j} |s|^{kld_j} T_{kl} + \sum_{j=l+1}^{n} c_j R^{d_j} |s|^{kld_j} T_{(k-1)l-1} \right) + \lambda R^\gamma |s|^{kl\gamma}.
\end{aligned}
$$

Define

$$d := \max\{d_j : j = 0, 1, \ldots, n\}$$

and choose

$$l = \min\{j : d_j = d\}.$$

Then we have $d_j \leq d - 1$ for $0 \leq j \leq l - 1$ and $d_j \leq d$ for $l + 1 \leq j \leq n$. Our estimate then becomes

$$
\begin{aligned}
& c_l R^d |s|^{kld} T_{(k-1)l} \\
\leq \ & (1 + o(1)) \left( \sum_{j=0}^{l-1} c_j R^{d-1} |s|^{kl(d-1)} T_{kl} + \sum_{j=l+1}^{n} c_j R^d |s|^{kld} T_{(k-1)l-1} \right) + \lambda R^\gamma |s|^{kl\gamma}.
\end{aligned}
$$

It follows that

$$T_{(k-1)l} \leq A_1 |s|^{-kl} T_{kl} + A_2 T_{(k-1)l-1} + A_3 |s|^{kl(\gamma-d)} \tag{2.14}$$

with positive constants $A_j$. (Note we chose $l$ in order to get the $|s|^{-kl}$ factor in the $T_{kl}$ term.)

Since $\log M(\alpha r, f)$ is convex in $\log r$ and since $f$ is transcendental,

$$\frac{M(\alpha r, f)}{M(r, f)} \to \infty \quad \text{and} \quad \frac{M(r, f)}{r^\beta} \to \infty$$

as $r \to \infty$ for each $\alpha, \beta \in \mathbb{R}$, $\alpha > 1$. This implies that

$$\frac{T_i}{T_{i-1}} \to \infty \quad \text{and} \quad \frac{T_i}{\mu^i} \to \infty \tag{2.15}$$

for each $\mu > 0$ as $i \to \infty$. Therefore provided we take $k$ large enough it follows that

$$A_2 T_{(k-1)l-1} \leq \frac{1}{3} T_{(k-1)l}$$

and

$$A_3 |s|^{kl(\gamma-d)} = A_3 |s|^{l(\gamma-d)} |s|^{(k-1)l(\gamma-d)} \leq \frac{1}{3} T_{(k-1)l}.$$

Thus for sufficiently large $k$ (2.14) becomes

$$T_{(k-1)l} \leq 3 A_1 |s|^{-kl} T_{kl}.$$

We put $S_k := T_{kl}$, then for $B < l$ we have

$$S_k \geq |s|^{Bk} S_{k-1}$$

for all large $k \in \mathbb{N}$. We now argue in a similar way to the proof of Theorem 23. Taking logs we have

$$\log S_k \geq (B \log |s|) k + \log S_{k-1},$$

i.e.

$$L_k \geq Ck + L_{k-1},$$

where $L_k := \log S_k$ and $C := B \log |s|$. We see that $L_1 \geq C + L_0$ and from here prove by induction that

$$
\begin{aligned}
L_k &\geq C \sum_{j=1}^{k} j + L_0 \\
&= C \frac{k(k+1)}{2} + L_0 \\
&\geq C k^2 \\
&= C \left( \frac{\log |s|^{kl} R - \log R}{l \log |s|} \right)^2
\end{aligned}
$$

for large $k$. Therefore we conclude that

$$\log M(r, f) \geq \left( \frac{C}{(l \log |s|)^2} - o(1) \right) (\log r)^2$$

for $r = |s|^{kl} R$, $k \in \mathbb{N}$, $k \to \infty$. Since the maximum modulus and log functions are increasing we have that the above result holds for any sequence of $r$-values.

Therefore using proposition 11 we arrive at

$$(\log r)^2 = O(T(2r, f))$$

as $r \to \infty$. It then follows that

$$(\log r)^2 = O(T(r, f))$$

as $r \to \infty$, as required. $\square$

Note that if in equation (2.9) we had $|q| > 1$ we can make the change of variables $\zeta = q^n z$. Then we can apply Theorem 23 and Theorem 24 taking our $q$ shift to be $\hat{q} = \frac{1}{q}$.

## 2.3    A $q$-Difference Analogue of the Malmquist Theorem

Ablowitz, Halburd and Herbst [2] considered discrete equations as delay equations in the complex plane which allowed them to analyze the equations with methods from complex analysis. The equations they consider to be of "Painlevé type" possess the property that they have sufficiently many finite-order meromorphic solutions.

Heuristically if we make a change of variables, i.e. if we take $x := q^z$ we are able to transform a difference equation into a $q$-difference equation. This suggests a logarithmic change in the growth of the solution, i.e. it suggests that difference equations that possess finite order growth solutions have $q$-difference analogues that possess zero-order growth solutions. We stress here that this argument is only heuristic and not a proof for the following two reasons. Firstly making the change of variables $x := q^z$ does not necessarily preserve the singularity structure. For example, suppose that $g(x)$ was a meromorphic solution of the difference equation

$$\Omega(x, g(x), g(x + c_1), \ldots, g(x + c_n)) = 0.$$

Making our change of variables gives

$$\Omega\left(\frac{\log z}{\log q}, f(z), f(q_1 z), \ldots, f(q_n z)\right) = 0, \tag{2.16}$$

where $f(z) := g(x)$ and $q_j = q^{c_j}$, $j = 1, \ldots, n$. Equation (2.16) shows that after a change of variables the solution can become branched. The second reason is that our results rely on Nevanlinna theory. In Nevanlinna theory we need our solutions to be valid on disks $|z| < r$. Making the change of variables $x := q^z$ maps disks to

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other regions, so knowledge of how $g$ grows on disks $|z| < r$ does not immediately translate into knowledge of how $f$ grows on disks $|z| < R$ for some $R > 0$. For example suppose $q > 1$, then if $x = q^z$ we have

$$|x| < r$$

if and only if

$$|q^{\operatorname{Re}(z)}| < r$$

if and only if

$$\operatorname{Re}(z) < \frac{\log r}{\log q}.$$

Therefore in this case, $x$ in the disk of radius $r$ corresponds to $z$ in a 'half plane type' region.

In the above section we showed that a general linear $q$-difference equation possesses solutions with zero-order growth. Valiron has shown that the non-autonomous Schröder $q$-difference equation

$$f(qz) = R(z, f(z)) = \frac{\sum_{j=0}^{n} a_j(z) f(z)^j}{\sum_{j=0}^{m} b_j(z) f(z)^j}, \tag{2.17}$$

where the coefficients $a_j(z), b_j(z)$ are meromorphic functions and $q$ is a complex constant, admit a one parameter family of meromorphic solutions, provided that $q \in \mathbb{C}$ is chosen appropriately [44]. We ask the question, what are the necessary conditions for (2.17) to possess meromorphic solutions with zero-order growth.

Let $d := \max\{m, n\}$ where $m$ and $n$ defined as in (2.17). We make the following definition in analogy with the differential case, definition 18.

**Definition 25** *Let $R(z, f)$ be rational in $f$ with meromorphic coefficients. A meromorphic solution, $f$, of equation (2.17) is called admissible, if $T(r, \alpha) = S(r, f)$ holds for all coefficients $\alpha(z)$ of $R(z, f)$.*

In [18] the following lemma and theorem are proved.

**Lemma 26** *Suppose that $|q| \le 1$ and that $f(z)$ is an admissible solution of an equation of the form (2.17). Then $d \le 1$.*

**Theorem 27** *Suppose that $f$ is an admissible solution of an equation of the form (2.17) with $|q| > 1$. Then*

$$\rho(f) = \frac{\log d}{\log |q|}.$$

Combining the above lemma and theorem we have the following

**Theorem 28** *Suppose equation (2.17) admits a zero-order meromorphic solution. Then $d = 1$. I.e.*

$$f(qz) = \frac{a_0(z) + a_1(z)f(z)}{b_0(z) + b_1(z)f(z)}.$$  (2.18)

We substitute

$$f(z) = \frac{a_1\left(\frac{z}{q}\right)}{b_1\left(\frac{z}{q}\right)} \left( \frac{g(z) - g(\frac{z}{q})}{g(z)} \right)$$

into (2.18). This yields

$$-b_1\left(\frac{z}{q}\right) \left( a_1(z) b_1\left(\frac{z}{q}\right) b_0(z) + a_1(z) b_1(z) a_1\left(\frac{z}{q}\right) \right) g(z)$$

$$-b_1\left(\frac{z}{q}\right) \left( -a_1(z) b_1\left(\frac{z}{q}\right) b_0(z) + b_1\left(\frac{z}{q}\right) b_1(z) a_0(z) \right) g(qz)$$

$$+b_1\left(\frac{z}{q}\right) a_1(z) b_1(z) a_1\left(\frac{z}{q}\right) g\left(\frac{z}{q}\right) = 0,$$

which is a linear equation in $g$. Since (2.18) can be linearised, Theorem 27 suggests that the existence of zero-order meromorphic solutions is a good detector of integrable $q$-difference equations. And since (2.18) can be linearised we call it the $q$-discrete Riccati equation.

### Example 1

Consider the following 2-difference equation

$$f(2z) = \frac{1}{z} \frac{f(z)}{f(z) + 1}.$$  (2.19)

Equation (2.19) is an example of a discrete Riccati equation. If we let $g(z) := \frac{1}{f(z)}$, then (2.19) becomes the linear equation

$$g(2z) = z + zg(z).$$  (2.20)

This equation was considered by Wittich [45] who showed it can be solved by

$$g(z) = \sum_{n=1}^{\infty} \frac{z^n}{2^{n(n+1)/2}},$$  (2.21)

which is a transcendental function of zero-order. Hence (2.19) possesses a zero-order meromorphic solution.

### Example 2

Consider the following two $q$-difference equations of the form (2.17) from [18],

- $f(2z) = -1 + 2f(z)^2$, (2.22)
- $f((n+1)z) = e^z f(z)^n$. (2.23)

Equation (2.22) has the solution $f(z) = \cos z$, which is meromorphic. Equation (2.23) has the solution $f(z) = e^z$. Note that both these solutions have order 1, they are not zero-order because the equations are not of the form (2.18).

# Chapter 3

# The $q$-Difference Analogue of the Lemma on the Logarithmic Derivative with Applications to $q$-Difference Equations

## Introduction

This chapter contains most of the original work of the thesis. In it we derive a $q$-difference analogue of the Lemma on the Logarithmic Derivative. The Lemma on the Logarithmic Derivative states that

$$m\left(r, \frac{f'}{f}\right) = o(T(r, f)),$$

outside of a possible small exceptional set. This is one of the most important results in Nevanlinna theory, it has many applications in the theory of meromorphic functions and in the theory of ordinary differential equations. For example it plays a major part in the proofs of the Second Main Theorem of Nevanlinna theory [36] and Yosida's generalization [46] of the Malmquist Theorem [32]. In this chapter we prove the following theorem.

**Theorem 29** *Let $f$ be a non-rational zero-order meromorphic function, $q \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T(r, f)), \tag{3.1}$$

*on a set of logarithmic density 1.*

Theorem 29 is an analogue of the Lemma on the Logarithmic Derivative for $q$-difference equations. It may be used to study zero-order meromorphic solutions of $q$-difference equations in a similar manner as Theorem 1 applies for finite-order

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meromorphic solutions of difference equations. The restriction to zero-order meromorphic functions is analogous to demanding finite order of growth in the ordinary shift case. For instance, all meromorphic solutions of linear and $q$-Riccati difference equations with rational coefficients are of zero order.

Concerning the sharpness of Theorem 29, the exponential function does not satisfy (3.1) for any $q \in \mathbb{C}$, and so the assertion of Theorem 29 cannot be extended to hold for all finite-order meromorphic functions.

In the rest of the chapter we use our result to study zero-order meromorphic solutions of large classes of $q$-difference equations. One result is a $q$-difference analogue of the Clunie Lemma, (see Lemma 21 or [12]). The original lemma has become an invaluable tool in the study of non-linear differential equations. The $q$-difference analogue gives similar information about the zero-order meromorphic solutions of non-linear $q$-difference equations.

# 3.1   Analogue   of   the   Logarithmic   Derivative   Lemma

The natural $q$-difference analogue of the logarithmic derivative is

$$\frac{f(qz) - f(z)}{f(z)} = \frac{f(qz)}{f(z)} - 1.$$

But since the contribution from the $-1$ term is not significant we work with the quantity $\frac{f(qz)}{f(z)}$. As in the proof of the Lemma on the Logarithmic Derivative given by [17] we start with the Poisson-Jensen Formula (1.1). This gives the following

**Lemma 30** *Let $f$ be a meromorphic function such that $f(0) \neq 0, \infty$ and let $q \in \mathbb{C}$ such that $|q| \neq 0$. Then,*

$$
\begin{aligned}
m\left(r, \frac{f(qz)}{f(z)}\right) &\leq \left(n(\rho, f) + n\left(\rho, \frac{1}{f}\right)\right)\left(\frac{|q-1|^\delta(|q|^\delta + 1)}{\delta(1-\delta)|q|^\delta} + \frac{|q-1|r}{\rho - |q|r} + \frac{|q-1|r}{\rho - r}\right) \\
&\quad + \frac{4|q-1|r\rho}{(\rho - r)(\rho - |q|r)}\left(T(\rho, f) + \log^+\left|\frac{1}{f(0)}\right|\right),
\end{aligned}
$$

*where $z = re^{i\phi}, \rho > \max(r, |q|r)$ and $0 < \delta < 1$.*

**Proof:** Using the identity

$$\frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\varphi - \theta) + r^2} = \text{Re}\left\{\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z}\right\}, \qquad z = re^{i\varphi}$$

and the Poisson-Jensen formula with $R = \rho$ we see

$$
\begin{aligned}
\log\left|\frac{f(qz)}{f(z)}\right| &= \int_0^{2\pi} \log\left|f\left(\rho e^{i\theta}\right)\right| \operatorname{Re}\left(\frac{2\rho z e^{i\theta}(q-1)}{(\rho e^{i\theta}-z)(\rho e^{i\theta}-qz)}\right)\frac{d\theta}{2\pi} \\
&\quad + \sum_{|a_n|<\rho} \log\left|\frac{(qz-a_n)(\rho^2-\bar{a}_n z)}{(z-a_n)(\rho^2-\bar{a}_n qz)}\right| \\
&\quad - \sum_{|b_n|<\rho} \log\left|\frac{(qz-b_m)(\rho^2-\bar{b}_m z)}{(z-b_m)(\rho^2-\bar{b}_m qz)}\right| \\
&=: S_1(z) + S_2(z) - S_3(z),
\end{aligned}
$$

where $\{a_n\}$ and $\{b_n\}$ are the zeroes and poles of $f$ respectively.

Integration on the set $E := \{\psi \in [0,2\pi] : \left|\frac{f(qre^{i\psi})}{f(re^{i\psi})}\right| \geq 1\}$ gives us the proximity function,

$$
\begin{aligned}
m\left(r, \frac{f(qz)}{f(z)}\right) &= \int_E \log\left|\frac{f(qz)}{f(z)}\right| \frac{d\psi}{2\pi} \\
&= \int_E \left(S_1(re^{i\psi}) + S_2(re^{i\psi}) - S_3(re^{i\psi})\right)\frac{d\psi}{2\pi} \\
&\leq \int_E \left(|S_1(re^{i\psi})| + |S_2(re^{i\psi})| + |S_3(re^{i\psi})|\right)\frac{d\psi}{2\pi}.
\end{aligned}
$$

We will now proceed to estimate each $\int_0^{2\pi} |S_j(re^{i\psi})|\frac{d\psi}{2\pi}$ separately. Since

$$
\begin{aligned}
|S_1(z)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\operatorname{Re}\left(\frac{2(q-1)z\rho e^{i\theta}}{(\rho e^{i\theta}-qz)(\rho e^{i\theta}-z)}\right)d\theta\right| \\
&\leq \left|\frac{1}{2\pi}\frac{2\rho(q-1)r}{(\rho-|q|r)(\rho-r)}\right|\int_0^{2\pi}\left|\log|f(\rho e^{i\theta})|\right|d\theta \\
&\leq \left|\frac{2\rho(q-1)r}{(\rho-|q|r)(\rho-r)}\right|\left(m(\rho,f)+m\left(\rho,\frac{1}{f}\right)\right),
\end{aligned}
$$

we have

$$
\int_0^{2\pi} |S_1(re^{i\psi})|\frac{d\psi}{2\pi} \leq \frac{4\rho|q-1|r}{(\rho-|q|r)(\rho-r)}\left(T(\rho,f)+\log^+\frac{1}{|f(0)|}\right).
$$

Next we consider the cases $j = 2, 3$ combined together. By denoting $\{c_n\} = \{a_n\} \cup \{b_n\}$, we obtain

$$\int_0^{2\pi} (|S_2(re^{i\psi})| + |S_3(re^{i\psi})|) \frac{d\psi}{2\pi} \leq \sum_{|c_n| < \rho} \int_0^{2\pi} \left| \log \left| \frac{(qre^{i\psi} - c_n)(\rho^2 - \bar{c}_n re^{i\psi})}{(re^{i\psi} - c_n)(\rho^2 - \bar{c}_n qre^{i\psi})} \right| \right| \frac{d\psi}{2\pi}$$

$$\leq \sum_{|c_n| < \rho} \int_0^{2\pi} \log^+ \left| \frac{qre^{i\psi} - c_n}{re^{i\psi} - c_n} \right| \frac{d\psi}{2\pi}$$

$$+ \sum_{|c_n| < \rho} \int_0^{2\pi} \log^+ \left| \frac{re^{i\psi} - c_n}{qre^{i\psi} - c_n} \right| \frac{d\psi}{2\pi}$$

$$+ \sum_{|c_n| < \rho} \int_0^{2\pi} \log^+ \left| \frac{\rho^2 - \bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n qre^{i\psi}} \right| \frac{d\psi}{2\pi}$$

$$+ \sum_{|c_n| < \rho} \int_0^{2\pi} \log^+ \left| \frac{\rho^2 - \bar{c}_n qre^{i\psi}}{\rho^2 - \bar{c}_n re^{i\psi}} \right| \frac{d\psi}{2\pi}$$

$$= \sum_{|c_n| \leq \rho} \int_0^{2\pi} \log^+ \left| 1 + \frac{(q-1)re^{i\psi}}{re^{i\psi} - c_n} \right| \frac{d\psi}{2\pi}$$

$$+ \sum_{|c_n| \leq \rho} \int_0^{2\pi} \log^+ \left| 1 - \frac{(q-1)re^{i\psi}}{qre^{i\psi} - c_n} \right| \frac{d\psi}{2\pi}$$

$$+ \sum_{|c_n| \leq \rho} \int_0^{2\pi} \log^+ \left| 1 + \frac{(q-1)\bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n qre^{i\psi}} \right| \frac{d\psi}{2\pi}$$

$$+ \sum_{|c_n| \leq \rho} \int_0^{2\pi} \log^+ \left| 1 - \frac{(q-1)\bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n re^{i\psi}} \right| \frac{d\psi}{2\pi}.$$

Define

$$g(x) := (1 + x)^\delta - (1 + x^\delta),$$

then $g(0) = 0$ and

$$g'(x) = \delta(1 + x)^{\delta - 1} - \delta x^{\delta - 1} < 0 \text{ since } \delta - 1 < 0.$$

This implies $g(x) \leq 0$, hence

$$\log(1 + x) = \frac{1}{\delta} \log(1 + x)^\delta$$

$$\leq \frac{1}{\delta} \log(1 + x^\delta)$$

$$\leq \frac{x^\delta}{\delta},$$

using the fact that $\log(1 + |x|) \leq |x|$ for all $x$. Therefore we have

$$\int_0^{2\pi} \log^+ \left| 1 + \frac{(q-1)re^{i\psi}}{re^{i\psi} - c_n} \right| \frac{d\psi}{2\pi} \leq \frac{1}{\delta} \int_0^{2\pi} \log^+ \left( 1 + \left| \frac{(q-1)re^{i\psi}}{re^{i\psi} - c_n} \right| \right)^\delta \frac{d\psi}{2\pi}$$

$$\leq \frac{1}{\delta} \int_0^{2\pi} \left| \frac{re^{i\psi}(q-1)}{re^{i\psi} - c_n} \right|^\delta \frac{d\psi}{2\pi}$$

$$= \frac{|q-1|^\delta r^\delta}{2\pi\delta} \int_0^{2\pi} \frac{d\psi}{|re^{i\psi} - c_n|^\delta}.$$

Then using $\left| re^{i\psi} - |c_n| \right| > \frac{2}{\pi} r\psi$ for all $0 \leq \theta \leq \frac{\pi}{2}$ we get

$$\int_0^{2\pi} \log^+ \left| 1 + \frac{(q-1)re^{i\psi}}{re^{i\psi} - c_n} \right| \frac{d\psi}{2\pi} \leq \frac{4|q-1|^\delta r^\delta}{2\pi\delta} \int_0^{\frac{\pi}{2}} \frac{d\psi}{|re^{i\psi} - |c_n||^\delta}$$

$$\leq \frac{4|q-1|^\delta r^\delta}{2\pi\delta} \int_0^{\frac{\pi}{2}} \frac{d\psi}{\left( \frac{2}{\pi} r\psi \right)^\delta}$$

$$= \frac{|q-1|^\delta}{\delta(1-\delta)}, \tag{3.2}$$

and
$$\int_0^{2\pi} \log^+ \left| 1 - \frac{(q-1)re^{i\psi}}{qre^{i\psi} - c_n} \right| \frac{d\psi}{2\pi} \leq \frac{|q-1|^\delta}{|q|^\delta \delta(1-\delta)}. \tag{3.3}$$

Again using the fact that $\log(1 + |x|) \leq |x|$ for all $x$ we have

$$\int_0^{2\pi} \log^+ \left| 1 + \frac{(q-1)\bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n qre^{i\psi}} \right| \frac{d\psi}{2\pi} \leq \int_0^{2\pi} \log^+ \left( 1 + \left| \frac{(q-1)\bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n qre^{i\psi}} \right| \right) \frac{d\psi}{2\pi}$$

$$\leq \int_0^{2\pi} \left| \frac{(q-1)\bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n qre^{i\psi}} \right| \frac{d\psi}{2\pi}$$

$$= |q-1|r \int_0^{2\pi} \left| \frac{c_n}{\rho^2 - \bar{c}_n qre^{i\psi}} \right| \frac{d\psi}{2\pi}.$$

Using the fact that for all $a$ such that $|a| < \rho$,

$$\left| \frac{a}{\rho^2 - \bar{a}re^{i\psi}} \right| \leq \frac{1}{\rho - r}.$$

we obtain
$$\int_0^{2\pi} \log^+ \left| 1 + \frac{(q-1)\bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n qre^{i\psi}} \right| \frac{d\psi}{2\pi} \leq \frac{|q-1|r}{\rho - |q|r}, \tag{3.4}$$

and
$$\int_0^{2\pi} \log^+ \left| 1 - \frac{(q-1)\bar{c}_n re^{i\psi}}{\rho^2 - \bar{c}_n z} \right| \frac{d\psi}{2\pi} \leq \frac{|q-1|r}{\rho - r}. \tag{3.5}$$

Combining (3.2), (3.3), (3.4) and (3.5) gives

$$\int_0^{2\pi} (|S_2| + |S_3|) \frac{d\psi}{2\pi} \leq \left( n(\rho, f) + n\left( \rho, \frac{1}{f} \right) \right) \left( \frac{|q-1|^\delta(|q|^\delta + 1)}{\delta(1-\delta)|q|^\delta} + \frac{|q-1|r}{\rho - |q|r} + \frac{|q-1|r}{\rho - r} \right).$$

The assertion follows by combining the obtained bounds for the $S_i$ terms. $\square$

If $f(z)$ has either a zero or a pole at the origin, then, for a suitable $p \in \mathbb{Z}$, we may write $f(z) = z^p g(z)$ where $g(z)$ is finite and non-zero at the origin. Hence, by taking $K > \max\{1, |q|\}$ and applying Lemma 30 with $\rho = Kr$, we have, for all $r$ sufficiently large,

$$
\begin{aligned}
m\left(r, \frac{f(qz)}{f(z)}\right) \leq\ & D_1\left(n(Kr, f) + n\left(Kr, \frac{1}{f}\right)\right) \\
& + \frac{D_2}{K}T(Kr, f) + O\left(\frac{1}{K}\right),
\end{aligned}
\tag{3.6}
$$

where $D_1$ and $D_2$ are constants independent of $r$ and $K$.

In order to deal with the $T(Kr, f)$ term we use the following result which is a special case of Lemma 4 in [24].

**Lemma 31** *Suppose $T : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function such that*

$$
\varlimsup_{r \to \infty} \frac{\log T(r)}{\log r} = 0.
$$

*Then the set*

$$
E := \{r : T(C_1 r) \geq C_2 T(r)\},
$$

*has logarithmic density 0, where $C_1 > 1$ and $C_2 > 1$.*

**Proof.** We prove this lemma in the following way. First we inductively define a sequence $\{r_n\}$ so that the set $E$ is contained in the intervals $(r_n, C_1 r_n)$. We then show these intervals have logarithmic density 0.

Let
$$
r_1 := \text{the least value of } r \text{ such that } r \geq 1 \text{ and } r \in E.
$$

If $r_1$ does not exist then the lemma is proved. Also we take

$$
r_{n+1} := \text{the least value of } r \text{ such that } r \geq C_1 r_n \text{ and } r \in E.
$$

Therefore we have that
$$
E \subset \bigcup_{n=1}^{\infty} (r_n, C_1 r_n) =: E'.
$$

Now if we can show $E'$ has logarithmic density 0 we are done. But

$$
\begin{aligned}
\overline{\mathrm{logdens}}\, E' &= \varlimsup_{r\to\infty} \frac{1}{\log r} \int_{E'\cap[1,r]} \frac{dr}{r} \\
&= \varlimsup_{r\to\infty} \frac{1}{\log r} \sum_{r_n \le r} \int_{r_n}^{C_1 r_n} \frac{dr}{r} \\
&= \varlimsup_{r\to\infty} \frac{1}{\log r} \sum_{n \le r} \log C_1 \\
&= \varlimsup_{r\to\infty} \frac{1}{\log r} k \log C_1,
\end{aligned} \tag{3.7}
$$

where

$$
k = k(r) := \text{the largest integer, } k, \text{ such that } r_k \le r.
$$

By definition, for all $n$, $r_n \in E$, hence we have

$$
T(r) \ge T(r_k) \ge T(C_1 r_{k-1}) \ge C_2 T(r_{k-1}) \ge \cdots \ge C_2^{k-1} T(r_1) \ge C_2^{k-1} T(1).
$$

This implies

$$
\log \frac{T(r)}{T(1)} \ge (k-1) \log C_2,
$$

and therefore

$$
\begin{aligned}
k \log C_1 &= (k-1) \log C_1 + \log C_1 \\
&\le \frac{\log C_1}{\log C_2} \log \frac{T(r)}{T(1)} + \log C_1 \\
&\le \frac{\log C_1}{\log C_2} \log T(r) + O(1).
\end{aligned} \tag{3.8}
$$

Substituting (3.8) into (3.7) we get

$$
\begin{aligned}
\overline{\mathrm{logdens}}\, E' &\le \frac{\log C_1}{\log C_2} \varlimsup_{r\to\infty} \frac{\log T(r)}{\log r} \\
&= 0,
\end{aligned}
$$

from the assumption in the lemma.

To show that the $n(Kr, f)$ term in equation (3.6) is small we first prove the following lemma.

**Lemma 32** *Suppose $f$ is a meromorphic function with order zero. Then for all $n \in \mathbb{N}$ the set*

$$
E_n := \left\{ r \ge 1 : n(r, f) < \frac{T(r,f)}{2^n} \right\}
$$

*has logarithmic density 1.*

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**Proof:**

$$N(Kr, f) = \int_0^{Kr} \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log Kr$$

$$\geq \int_r^{Kr} \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log Kr$$

$$\geq \int_r^{Kr} \frac{n(t, f)}{t} dt$$

$$\geq n(r, f) \int_r^{Kr} \frac{dt}{t}$$

$$= n(r, f) \log K,$$

By Lemma 31, for any $K > 1$ we have $N(Kr, f) \leq 2N(r, f)$ on a set of logarithmic density 1. Therefore

$$n(r, f) \leq \frac{1}{\log K} N(Kr, f) \leq \frac{2}{\log K} N(r, f) \leq \frac{2}{\log K} T(r, f),$$

on a set of logarithmic density 1. If for a given $n \in \mathbb{N}$ we take $\log K = 2^{n+1}$, i.e. $K = \exp 2^{n+1} > 1$ then for all $n \in \mathbb{N}$ the set

$$E_n := \left\{ r \geq 1 : n(r) < \frac{T(r, f)}{2^n} \right\},$$

has logarithmic density 1, as required. $\square$

The following lemma together with Lemma 32 implies that for all meromorphic functions, $f$, with zero-order, $n(r, f) = o(T(r, f))$, on a set of logarithmic density 1.

**Lemma 33** *Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function, and let $U : \mathbb{R}^+ \to \mathbb{R}^+$. If there exists a decreasing sequence, $\{c_n\}_{n \in \mathbb{N}}$, such that $c_n \to 0$ as $n \to \infty$, and, for all $n \in \mathbb{N}$, the set*

$$F_n := \{ r \geq 1 : U(r) < c_n T(r, f) \}$$

*has logarithmic density 1, then*

$$U(r) = o(T(r, f)),$$

*on a set of logarithmic density 1.*

**Proof:** Since each set $F_n$ has logarithmic density 1 we have

$$\overline{\lim_{r \to \infty}} \frac{1}{\log r} \int_{[1,r] \cap F_n} \frac{dt}{t} = 1,$$

which implies that for all $n$, there exists $r_n$, such that

$$\frac{1}{\log r} \int_{[1,r] \cap F_n} \frac{dt}{t} > 1 - \frac{1}{n}, \qquad \text{for all } r \geq r_n.$$

I.e.

$$\int_{[1,r] \cap F_n} \frac{dt}{t} > \left(1 - \frac{1}{n}\right) \log r,$$

for all $r \geq r_n$. We set $F$ to be the union of the sets $[r_n, r_{n+1}) \cap F_n$ where $n$ runs through all positive integers. Then for all $r \in F$ we have that

$$U(r) < c_{n_r} T(r, f),$$

where $n_r \to \infty$ as $r \to \infty$. Since $c_n \to 0$ as $n \to \infty$, this implies that $U(r) = o(T(r,f))$ on $F$. Therefore if we can show that $F$ has logarithmic density 1 we are done.

Since for all sufficiently large $r$ there is $n$ so that $r_n \leq r \leq r_{n+1}$, we have

$$\int_{[1,r] \cap F} \frac{dt}{t} \geq \int_{[1,r] \cap F_n} \frac{dt}{t} > \left(1 - \frac{1}{n}\right) \log r.$$

Dividing through by $\log r$ and taking the limit as $n \to \infty$ gives us that $F$ has logarithmic density 1, as required. $\square$

The following corollary is an immediate consequence of Lemmas 32 and 33.

**Corollary 34** *For all meromorphic functions, $f$, with zero-order, we have that*

$$n(r, f) = o(T(r, f)),$$

*on a set of logarithmic density 1.* $\square$

Now we are ready to complete the proof of Theorem 29. Applying Corollary 34 to equation (3.6) we get

$$m\left(r, \frac{f(qz)}{f(z)}\right) \leq o(T(Kr, f)) + \frac{D_2}{K} T(Kr, f),$$

on a set of logarithmic density 1.

But by Lemma 31 we have that $T(Kr, f) \leq 2T(r, f)$ on a set of logarithmic density 1. Therefore we have

$$U(r) := m\left(r, \frac{f(qz)}{f(z)}\right) + o(T(r, f)) \leq \frac{2D_2}{K} T(r, f),$$

on a set of logarithmic density 1. By taking $K = 2^n$ and applying Lemma 32 we have that the set

$$F_n := \left\{ r \geq 1 : U(r) < \frac{2D_2}{2^n} T(r, f) \right\}$$

has logarithmic density one. Lemma 33 then immediately implies

$$m \left( r, \frac{f(qz)}{f(z)} \right) = o(T(r, f)).$$

on a set of logarithmic density 1, as required. $\square$

## 3.2  Applications to $q$-Difference Equations

The Lemma on the Logarithmic Derivative is a key component in the proof of the Second Main Theorem, one of the deepest results of Nevanlinna theory. It is also important for applications to complex differential equations. Similarly, Theorem 29 enables an efficient study of complex analytic properties of zero-order meromorphic solutions of $q$-difference equations. In this chapter we are concerned with functions which are polynomials in $f(q_j z)$, where $q_j \in \mathbb{C}$, with coefficients $a_\lambda(z)$ such that

$$T(r, a_\lambda(z)) = o(T(r, f))$$

on a set of logarithmic density 1. Such functions will be called *q-difference polynomials in $f(z)$*.

The following theorem is analogous to the Clunie Lemma [12]. It can be used to study pole distribution of meromorphic zero-order solutions of non-linear $q$-difference equations.

**Theorem 35** *Let $f(z)$ have zero-order growth and be a non-rational meromorphic solution of*

$$f(z)^n P(z, f) = Q(z, f),$$

*where $P(z, f)$ and $Q(z, f)$ are $q$—difference polynomials in $f(z)$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its $q$-shifts is at most $n$, then*

$$m(r, P(z, f)) = o(T(r, f)),$$

*on a set of logarithmic density 1.*

**Proof:** We follow the reasoning behind the original Clunie Lemma, see, for instance, [30], replacing the Lemma on the Logarithmic Derivative with Theorem 29.

In calculating the proximity function of $P$, we split the region of integration into two parts. By defining

$$E_1 := \{\varphi \in [0, 2\pi] : |f(re^{i\varphi})| < 1\}$$

and

$$E_2 := [0, 2\pi]\backslash E_1,$$

we have

$$2\pi m(r, P(z,f)) = \int_{E_1} \log^+ |P|\, d\varphi + \int_{E_2} \log^+ |P|\, d\varphi. \qquad (3.9)$$

First we consider $E_1$. Each term of $P$ is of the form

$$a_\lambda(z) f(z)^{l_0} f(q_1 z)^{l_1} \cdots f(q_\nu z)^{l_\nu}.$$

Hence writing with $\lambda = (l_0, \ldots, l_\nu)$,

$$P(z,f) = \sum_{\lambda \in I} P_\lambda(z,f) = \sum_{\lambda \in I} a_\lambda(z) f(z)^{l_0} f(q_1 z)^{l_1} \cdots f(q_\nu z)^{l_\nu}.$$

For each $\lambda$ we have

$$|P_\lambda(re^{i\varphi})| \le |a_\lambda(re^{i\varphi})| \left|\frac{f(q_1 re^{i\varphi})}{f(re^{i\varphi})}\right|^{l_1} \cdots \left|\frac{f(q_\nu re^{i\varphi})}{f(re^{i\varphi})}\right|^{l_\nu},$$

whenever $\varphi \in E_1$. Therefore for each $\lambda$ we obtain

$$\int_{E_1} \log^+ |P_\lambda(re^{i\varphi})| \frac{d\varphi}{2\pi} \le m(r, a_\lambda) + O\left(\sum_{j=1}^{\nu} m\left(r, \frac{f(q_j z)}{f(z)}\right)\right),$$

and so by Theorem 29 and our assumption in the theorem,

$$\int_{E_1} \log^+ |P(re^{i\varphi}, f)| \frac{d\varphi}{2\pi} = o(T(r,f)) \qquad (3.10)$$

on a set of logarithmic density 1.

Now we consider $E_2$. To do this case we note that

$$P(z,f) = \frac{Q(z,f)}{f^n}.$$

But

$$Q(z,f) = \sum_{\gamma \in J} Q_\gamma(z,f) = \sum_{\gamma \in J} b_\gamma(z) f(q_0 z)^{l_0} \cdots f(q_\mu z)^{l_\mu}.$$

By our assumption $l_1 + \cdots + l_\mu \le n$ for all $\gamma = (l_0, \ldots, l_\mu) \in J$. Hence we have

$$
\begin{aligned}
|P(z,f)| &= \left| \frac{1}{f(z)^n} \sum_{\gamma \in J} b_\gamma(z) f(z)^{l_0} f(q_1 z)^{l_1} \cdots f(q_\mu z)^{l_\mu} \right| \\
&\le \sum_{\gamma \in J} |b_\gamma(z)| \left| \frac{f(q_1 re^{i\varphi})}{f(re^{i\varphi})} \right|^{l_1} \cdots \left| \frac{f(q_\mu re^{i\varphi})}{f(re^{i\varphi})} \right|^{l_\mu}.
\end{aligned}
$$

Therefore by Theorem 29 again,

$$
\int_{E_2} \log^+ |P(re^{i\varphi}, f)| \frac{d\varphi}{2\pi} = o(T(r,f)), \tag{3.11}
$$

on a set of logarithmic density 1. The assertion follows by combining (3.9), (3.10) and (3.11). $\square$

Let $a$ and $f$ be meromorphic zero-order functions such that $T(r,a) = o(T(r,f))$ on a set of logarithmic density 1. Then $a$ is said to be a *slowly moving target* or a *small function* with respect to $f$. In particular, constant functions are always slowly moving compared to any non-constant meromorphic function. The next result can be used as a tool to analyze the value distribution of zero-order meromorphic solutions $f$, with respect to slowly moving targets. It is an analogue of a result due to A. Z. Mohon'ko and V. D. Mohon'ko [34] on differential equations.

**Theorem 36** *Let $f(z)$ have zero order growth and be a non-rational meromorphic solution of*

$$
P(z,f) = 0 \tag{3.12}
$$

*where $P(z,f)$ is a q-difference polynomial in $f(z)$. If $P(z,a) \not\equiv 0$ for a slowly moving target $a$, then*

$$
m\left(r, \frac{1}{f-a}\right) = o(T(r,f))
$$

*on a set of logarithmic density 1.*

**Proof.** By substituting $f = g + a$ into (3.12) we obtain

$$
Q(z,g) + D(z) = 0, \tag{3.13}
$$

where $Q(z,g) = \sum_{\gamma \in J} b_\gamma(z) G_\gamma(z,f)$ is a q-difference polynomial in $g$ such that all of its terms are at least degree one, and $T(r,D) = o(T(r,g))$ on a set of logarithmic density 1. Also $D \not\equiv 0$, since $a$ does not satisfy (3.12).

Using (3.13) we have

$$
\begin{aligned}
m\left(r, \frac{1}{g}\right) &= m\left(r, \frac{D}{Dg}\right) \\
&\leq m\left(r, \frac{D}{g}\right) + m\left(r, \frac{1}{D}\right) \\
&= m\left(r, \frac{Q(z,g)}{g}\right) + m\left(r, \frac{1}{D}\right).
\end{aligned}
\tag{3.14}
$$

Note that the integral to be evaluated vanishes on the part of $|z| = r$ where $|g| > 1$. It is therefore sufficient to consider only the case $|g| \leq 1$. Hence

$$
\begin{aligned}
\left|\frac{Q(z,g)}{g}\right| &= \frac{1}{|g|}\left|\sum_{\gamma \in J} b_\gamma(z) g(z)^{l_0} g(q_1 z)^{l_1} \cdots g(q_\nu z)^{l_\nu}\right| \\
&\leq \sum_{\gamma \in J} |b_\gamma(z)| \left|\frac{g(q_1 z)}{g(z)}\right|^{l_1} \cdots \left|\frac{g(q_\nu z)}{g(z)}\right|^{l_\nu}.
\end{aligned}
\tag{3.15}
$$

Since $Q(z,g)$ is a $q$-difference polynomial we have

$$
m(r, b_\gamma) = o(T(r,g))
$$

on a set of logarithmic density 1 for all $\gamma \in J$. Also by Theorem 29

$$
m\left(r, \frac{g(qz)}{g(z)}\right) = o(T(r,g))
$$

on a set of logarithmic density 1 for all $q \in \mathbb{C}$. Hence by (3.14) and (3.15) and the fact that $\sum_{j=0}^{\nu} l_j \geq 1$ for all $\gamma \in J$ we have

$$
m\left(r, \frac{1}{g}\right) = o(T(r,g))
$$

on a set of logarithmic density 1. Since $g = f - a$ the assertion follows. $\square$

## 3.3   Second Main Theorem

The Lemma on the Logarithmic Derivative plays a key role in the proof of the Second Main Theorem of Nevanlinna theory. Similarly the following theorem is obtained by using an analogue of the standard proof technique behind the Second Main Theorem [35] together with Theorem 29.

Let $f(z)$ be a non-constant meromorphic function of zero-order, let $q \in \mathbb{C}\backslash\{0,1\}$ and let $a \in \mathbb{C}$. By denoting

$$\Delta_q f := f(qz) - f(z),$$

and by applying Theorem 29 with the function $f(z) - a$, we have

$$
\begin{aligned}
m\left(r, \frac{\Delta_q f}{f-a}\right) &= m\left(r, \frac{f(qz)-a}{f(z)-a}\right) + O(1) \\
&= o(T(r, f-a)) + O(1) \quad (3.16)
\end{aligned}
$$

on a set of logarithmic density 1. We denote by $\mathbf{S_q}(f)$ the set of all meromorphic functions $g$ such that $T(r, g) = o(T(r, f))$ for all $r$ on a set of logarithmic density 1. Functions in the set $\mathbf{S_q}(f)$ are called *small compared to* $f$ or *slowly moving* with respect to $f$. Also, if $g \in \mathbf{S_q}(f)$ we denote $T(r, g) = S_q(r, f)$.

**Theorem 37** *Let $q \in C$, and let $f$ be a meromorphic function of zero-order such that $f(qz) \not\equiv f(z)$. Let $p \geq 2$, and let $a_1, \ldots, a_p \in \mathbb{C}$ be distinct points. Then*

$$m(r, f) + \sum_{k=1}^{p} m\left(r, \frac{1}{f-a_k}\right) \leq 2T(r, f) - N_{pair}(r, f) + S_q(r, f)$$

*where*

$$N_{pair}(r, f) := 2N(r, f) - N(r, \Delta_q f) + N\left(r, \frac{1}{\Delta_q f}\right).$$

**Proof.**

Using the First Main Theorem we have

$$
\begin{aligned}
\sum_{k=1}^{p} m\left(r, \frac{1}{f-a_k}\right) &= \sum_{k=1}^{p} T\left(r, \frac{1}{f-a_k}\right) - \sum_{k=1}^{p} N\left(r, \frac{1}{f-a_k}\right) \\
&= pT(r, f) - \sum_{k=1}^{p} N\left(r, \frac{1}{f-a_k}\right) + S_q(r, f). \quad (3.17)
\end{aligned}
$$

But by the Valiron-Mohon'ko Theorem (Theorem 19) we have that

$$pT(r, f) = T(r, P(f)) + S_q(r, f),$$

we also have that

$$\sum_{k=1}^{p} N\left(r, \frac{1}{f-a_k}\right) = N\left(r, \frac{1}{P(f)}\right),$$

where
$$P(f) := \prod_{k=1}^{p} (f - a_k),$$

Now
$$\frac{1}{P(f)} = \sum_{k=1}^{q} \frac{\alpha_k}{f - a_k}.$$

Hence by (3.16), we obtain
$$m\left(r, \frac{\Delta_q f}{P(f)}\right) \le \sum_{k=1}^{p} m\left(r, \frac{\Delta_q f}{f - a_k}\right) + S_q(r, f) = S_q(r, f),$$

and so
$$m\left(r, \frac{1}{P(f)}\right) = m\left(r, \frac{\Delta_q f}{P(f)} \frac{1}{\Delta_q f}\right) \le m\left(r, \frac{1}{\Delta_q f}\right) + S_q(r, f). \qquad (3.18)$$

Hence by the First Main Theorem (3.17) becomes
$$\begin{aligned}
\sum_{k=1}^{p} m\left(r, \frac{1}{f - a_k}\right) &= T(r, P(f)) - N\left(r, \frac{1}{P(f)}\right) + S_q(r, f) \\
&= m\left(r, \frac{1}{P(f)}\right) + S_q(r, f) \\
&\le m\left(r, \frac{1}{\Delta_q f}\right) + S_q(r, f) \\
&\le T(r, \Delta_q f) - N\left(r, \frac{1}{\Delta_q f}\right) + S_q(r, f).
\end{aligned}$$

Therefore we have
$$\begin{aligned}
m(r, f) + \sum_{k=1}^{p} m\left(r, \frac{1}{f - a_k}\right) &\le T(r, f) + N(r, \Delta_q f) + m(r, \Delta_q f) \\
&\quad - N\left(r, \frac{1}{\Delta_q f}\right) - N(r, f) + S_q(r, f).
\end{aligned}$$

But
$$m(r, \Delta_q f) = m\left(r, f \frac{\Delta_q f}{f}\right) \le m(r, f) + m\left(r, \frac{\Delta_q f}{f}\right) = m(r, f) + S_q(r, f)$$

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by equation (3.16). This implies

$$
\begin{aligned}
m(r,f) + \sum_{k=1}^{p} m\left(r, \frac{1}{f - a_k}\right) \;\leq\; & T(r,f) + N(r, \Delta_q f) + m(r,f) \\
& - N\left(r, \frac{1}{\Delta_q f}\right) - N(r,f) + S_q(r,f) \\
=\; & 2T(r,f) + N(r, \Delta_q f) - N\left(r, \frac{1}{\Delta_q f}\right) \\
& - 2N(r,f) + S_q(r,f),
\end{aligned}
$$

as required. $\square$

# Conclusions

In this thesis we have gone a good way in developing Nevanlinna theory for the $q$-difference operator acting on zero-order meromorphic functions. In particular we have found a $q$-difference analogue of the Lemma on the Logarithmic Derivative (see Theorem 29). This result has potentially a large number of applications in the study of $q$-difference equations. Many ideas and methods from the theory of differential equations may now be used together with Theorem 29 to obtain information about zero-order meromorphic solutions of $q$-difference equations. In chapter three we provided examples of this, for example we presented a $q$-difference analogue of the Clunie Lemma (see Theorem 35).

Nevanlinna's Second Main Theorem implies that a non-constant meromorphic function cannot have too many points with high multiplicity. In this thesis we have presented a $q$-difference analogue of Nevanlinna's Second Main Theorem (see Theorem 37).

Our findings are an analogue of the results concerning the difference operator by Halburd and Korhonen in [21, 22]. Historically $q$-difference equations are the most natural class of equations to look at after differential equations and difference equations. The restriction to zero-order meromorphic solutions is natural in light of the results found in chapter two. Here we reviewed results that show that all meromorphic solutions of linear $q$-difference equations have zero-order growth. We also showed that by looking for zero-order solutions in a class of non-linear $q$-difference equations we are led to an analogue of the Malmquist Theorem. I.e. looking for zero-order solutions singles out 'integrable' equations.

In chapter one we review the key ideas of Nevanlinna theory. In particular Nevanlinna's First Main Theorem (see equation (1.5)), the Lemma on the Logarithmic Derivative (see Theorem 15), the Malmquist Theorem (see Theorem 20), the Clunie Lemma (see Lemma 21) and Nevanlinna's Second Main Theorem (see Theorem 22).

# Bibliography

[1] M. J. Ablowitz and P.A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, Cambridge University Press, Cambridge, 1991.

[2] M. J. Ablowitz, R. G. Halburd and B. Herbst, On the extension of the Painlevé property to difference equations, *Nonlinearity* 13 (2000) 889-905.

[3] M. J. Ablowitz and H. Segur, Exact linearization of a Painlevé transcendent, *Phys. Rev. Lett.* 38 (1977) 1103-1106.

[4] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.

[5] D. C. Barnett, R. G. Halburd, R. J. Korhonen and W. Morgan, Nevanlinna theory for the $q$-difference operator and meromorphic solutions of $q$-difference equations, to appear in the Proceedings of the Royal Society of Edinburgh, 2006.

[6] W. Bergweiler and W. Hayman, Zeros of solutions of a functional equation, *Comput. Methods Funct. Theory*, 3:55-78, 2003.

[7] W. Bergweiler, K. Ishizaki and N. Yanagihara, Meromorphic solutions of some functional equations, Methods Appl. Anal. 5 (1998), 248-258. Correction: Methods Appl. Anal. 6(4) (1999).

[8] W. Bergweiler, K. Ishizaki and N. Yanagihara, Growth of meromorphic solutions of some functional equations, I. *Aequationes Math.*, 63:140-151, 2002.

[9] W. Cherry, Z. Ye, Nevanlinna's theory of value distribution, Springer-Verlag, Berlin, 2001.

[10] S. Chakravarty and M. J. Ablowitz, Integrability, monodromy evolving deformations and self-dual Bianchi IX systems, *Phys. Rev. Lett.* 76 (1996) 857-860.

[11] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane, to appear in the Ramanujan Journal, 2005.

[12] J. Clunie, On integral and meromorphic functions, J. London Math. Soc. 37 (1962) 17-27.

[13] A. E. Eremenko and M. L. Sodin, Iterations of rational functions and the distribution of the values of Poincaré functions, *Teor. Funktsii Funktsional. Anal. i Prilozhen.*, 53:18-25, 1990. (Russian); translation in J. Soviet Math. 58 (1992), no. 6, 504-509.

[14] L. Fuchs, Sur quelques équations différentielles linéares du second ordre, *C. R. Acad. Sci. Paris* 141 (1905) 555-558.

[15] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est á points critiques fixes, *Acta Math.* 33 (1910) 1-55.

[16] V. I. Gromak, I. Laine and S. Shimomura, Painlevé differential equations in the complex plane (Walter de Gruyter, Berlin, 2002).

[17] Z. Guan-Hou, Theory of entire and meromorphic functions, deficient and asymptotic values and singular directions, American Mathematical Society, 1993.

[18] G. Gundersen, J. Heittokangas, I. Laine, J. Rieppo and D. Yang, Meromorphic solutions of generalized Schröder equations, Aequationes Math. **63** (2002), no 1-2, 110-135.

[19] R. G. Halburd and R. J. Korhonen, Existence of finite-order meromorphic solutions as a detector of integrabilty in difference equations, Physica D 218 (2006) 191-203.

[20] R. G. Halburd and R. J. Korhonen, Finite order meromorphic solutions and discrete Painlevé equations. *to appear in Proc. London Math. Soc.*, http://arXiv.org, no:nlin.SI/0504026.

[21] R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.*,314:477-487,2006.

[22] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math. 31 (2006) 463-478.

[23] W. K. Hayman, Meromorphic functions, Oxford at the Clarendon Press, Oxford University Press 1964.

[24] W. K. Hayman, On the characteristic of functions meromorphic in the plane and of their integrals, Proc. London Math. Soc., 14a:93-128, 1965.

[25] J. Heittokangas, I. Laine, J. Rieppo, and D. Yang, Meromorphic solutions of some linear functional equations, *Aequationes Math.*, 60:148-166, 2000.

[26] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser Verlag, Basel, 1985.

[27] N. Joshi and M. Kruskal, A direct proof that solutions of the six Painlevé equations have no movable singularities except poles, *Stud. Appl. Math.* 93(1994) 187-207.

[28] S. Kovalevskaya, Sur le problém de la rotation d'un corps solid autour d'un point fixé, Acta Math. **12** 177-232 (1889).

[29] S. Kovalevskaya, Sur une propriété d'un systém d'équations différ entielles qui definit la rotation d'un corps solide autor d'un point fixé, Acta Math. **14** 81-93, (1889).

[30] I. Laine, Nevanlinna theory and complex differential equations, Walter de Gruyter Berlin, New York 1993.

[31] B. Malgrange, Sur les déformations isomonodromiques. i. singularités réguliéres, *Progr. Math.* 37 (1983) 401-426.

[32] J. Malmquist, Sur les fonctions á un nombre fini des branches définies par les équations différentielles du premier ordre, *Acta Math.* 36 (1913) 297-343.

[33] T. Miwa, Painlevé property of monodromy preserving deformation equations and the analyticity of $\tau$ functions, *Publ. Res. Inst. Math. Sci.* 17 (1981) 703-721.

[34] A. Z. Mohon'ko and V. D. Mohon'ko, Estimates of the Nevanlinna characteristics of certain classes of meromorphic functions, and their applications to differential equations, *Sibirsk. Mat. Zh.* 15 (1974) 1305-1322, (Russian).

[35] R. Nevanlinna, Eindeutige analytische funktionen, Volume 46 of Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 1974.

[36] R. Nevanlinna, Zur Theorie der meromorphen Funktionen, *Acta Math.* 46 (1925) 1-99.

[37] P. Painlevé, Mémoire sur les équations différentielles dont l'intégrale générale est uniform, *Bull.Soc. Math. France* 28 (1900) 201-261.

[38] P. Painlevé, Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, *Acta Math.* 25 (1902) 1-85.

[39] J. P. Ramis, About the growth of entire functions solutions of linear algebraic *q*-difference equations, *Ann. Fac. Sci. Toulouse Math.*, 1:53-94,1992.

[40] J. Ritt, Transcendental transcendency of certain functions of Poincaré, Math. Ann. 95 (1925/26), 671-682.

[41] L. Rubel, Some research problems about algebraic differential equations, Trans. Amer. Math. Soc. 280 (1983), 43-52.

[42] C. L. Siegel, J. K. Moser,Lectures on celestial mechanics, Springer-Verlag Heidelberg New York 1971.

[43] N. Steinmetz, Eine Verallgemeinerung des zweiten Nevanlinnaschen Hauptsatzes. J. Reine Angew. Math., 368:134-141, 1986.

[44] G. Valiron, Fonctions analytiques, Press. Univ. de France, Paris, 1952.

[45] H. Wittich, Sur les fonctions entiéres d'ordre nul et d'ordre fini et en particulier les fonctions a correspondance réguliére, Ann. Fac. Sci. (Toulouse) 5 (1913), 117-257.

[46] K. Yosida, A generalization of Malmquist's Theorem, *J. Math.* 9 (1933) 253-256.