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SERIAL DIGITAL COMMUNICATION SYSTEMS WITH SIGNALS ARRANGED IN ORTHOGONAL SETS

by

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A Doctoral Thesis

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ABSTRACT

The investigation is concerned with signal design and detection processes suitable for use in a synchronous serial baseband data-transmission system where the signals are transmitted in orthogonal groups over a channel which is time invariant and known.

A number of different detection processes have been proposed and analysed theoretically for the case where no signal processing is carried out at the transmitter. Adjacent groups of transmitted signal-elements are here separated by gaps of no signal, whose duration is such that the corresponding received groups of signal-elements do not overlap in time. The detection of a group of signals, in the proposed arrangements, is carried out iteratively by a sequence of similar operations which can be performed successively by a simple piece of equipment. The optimum detection process is of limited practical value because of the very large number of sequential operations required, when there are more than a few signal-elements in a group. The more effective of the suboptimum detection processes achieve a tolerance to additive white Gaussian noise approaching that of the optimum detector, but require far fewer sequential operations and can be implemented quite simply. The tolerance to noise of the various detection processes has been assessed by computer simulation for different numbers of signal-elements in a group and with both binary and multi-level signals.

It has been shown that when a linear transformation is applied to the signal at the transmitter, such that there is no intersymbol interference between the individual received signal-elements in the sampled signal at the detector input, the best arrangement uses an

uninterrupted transmitted signal, with no gaps between adjacent groups of signal-elements, and the detector uses only the central group of sample values of each received group of elements for the corresponding detection process. An alternative to this arrangement uses gaps between adjacent transmitted groups of elements, and the linear processing of signals at the transmitter is such that the received signals in a group are orthogonal but usually with considerable intersymbol interference. Each received group of elements is now detected by a set of matched detectors. The tolerance to additive white Gaussian noise of each of these systems is similar to that of the process of optimum linear equalization at the receiver.

Different arrangements of sharing the linear equalization between the transmitter and receiver have been studied. An advantage of up to 3 dB in tolerance to additive white Gaussian noise is gained by the best of these arrangements, over the process of optimum linear equalization at the receiver.

In all the arrangements tested it has been found that a near optimum detection process at the receiver, with no signal processing at the transmitter, always achieves a tolerance to additive white Gaussian noise at least as good as or better than that where the same detection process is used at the receiver but with some linear signal processing at the transmitter.

A study of some techniques using non-linear processing of signals at the transmitter has shown at best only a small improvement in tolerance to additive white Gaussian noise over the arrangement of linear equalization at the receiver.

GLOSSARY OF SYMBOLS AND TERMS

m	number of signal-elements in a group.
$g + 1$	maximum number of successive sample values of the sampled impulse response of the channel, the first and last of which are non-zero.
n	number of sample values corresponding to a group of received signal-elements.
S	m-component row vector whose components carry the element values of the signal-elements of a group.
R	x-component row vector whose components are the sample values of a received group of signal-elements. x is defined in the text.
$y(t)$	impulse response of the channel.
Y	$m \times n$ matrix of rank m whose i^{th} row Y_i is given by $Y_i = \overbrace{0 \; . \; . \; . \; 0}^{i-1} \; \overbrace{y_o \; y_1 \; . \; . \; . \; y_g}^{g+1} \; \overbrace{0 \; . \; . \; . \; 0}^{m-i}$ where $y_o \; y_1 \; . \; . \; . \; y_g$ are the sample values of the sampled impulse response of the channel.
$\lvert x \rvert$	magnitude (absolute value) of x, if x is a scalar.
$\lvert X \rvert$	length (Euclidean norm) of X, if X is a vector.
$\{x_i\}$	the components of X, if X is a vector.
$\{X_i\}$	the rows or columns of X, if X is a matrix.
a_{ij}	the component of matrix A located in i^{th} row and j^{th} column.
A^T	transpose of matrix A.
det A	determinant of matrix A.
A^{-1}	inverse of matrix A.
$D(f)$	Fourier transform of the time function d(t).

$D(z)$ z-transform of a set of sample values given by the components of the vector D.

σ^2 two-sided power spectral density of zero mean additive white Gaussian noise at the input to receiver filter.

W x-component row vector whose components are sample values of a Gaussian random variable with zero mean and variance σ^2. x is defined in the text.

A signal-element is a unit component of a digitally-coded signal.

Two groups of m signal-elements are said to be orthogonal when they are disjoint in time.

Vectors are treatedas matrices having one row or column.

A square matrix A is symmetric if $A = A^T$

A square matrix A is positive definite if all the eigen-values of A are non-zero and positive.

A unit or identity matrix A, is a square matrix where

$$a_{ii} = 1.0$$

and $$a_{ij} = 0 \text{ for } i \neq j.$$

A set of p vectors X_i having n components x_{1i}, x_{2i}, ... x_{ni}, is said to be linearly independent provided that no set of constants $q_1, q_2, \ldots, q_p$ exists (at least one q_i must be non-zero) such that

$$q_1X_1 + q_2X_2 + \ldots + q_pX_p = 0.$$

The rank of a matrix A is the largest square array in A with a non-vanishing determinant.

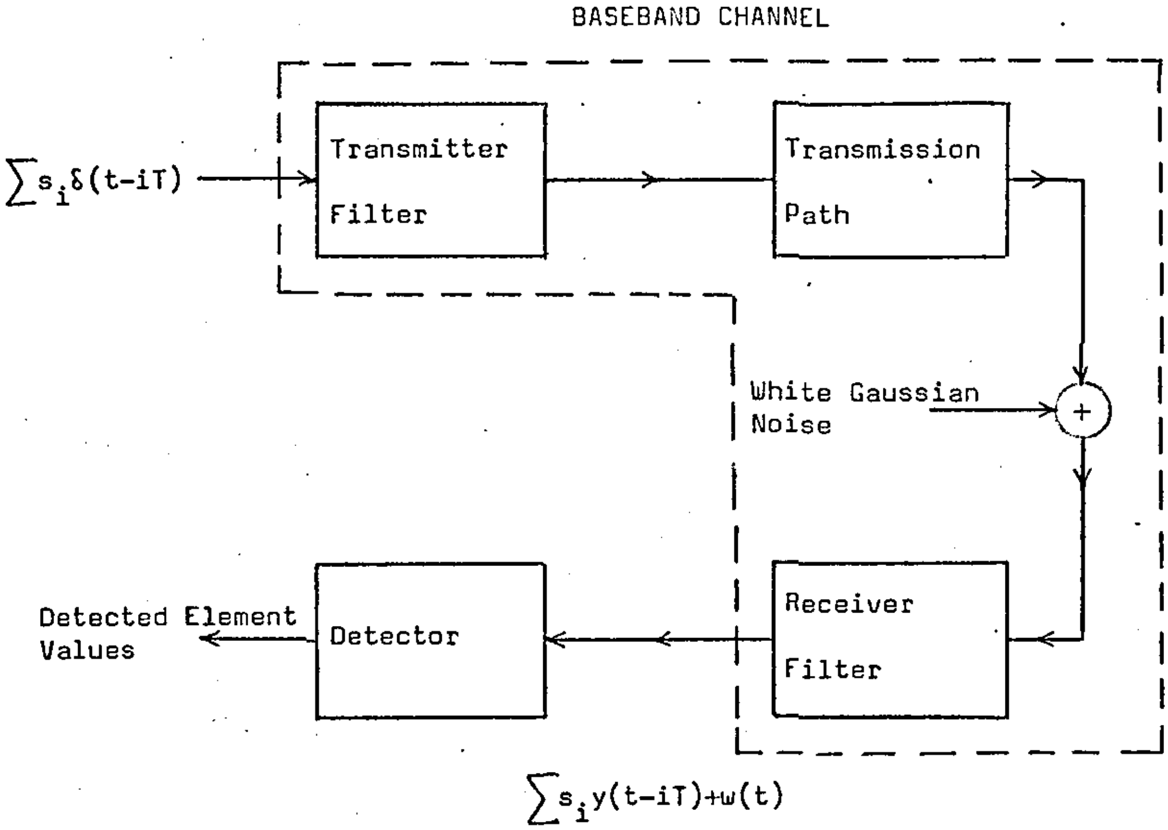
1.0 INTRODUCTION

1.1 Background

In the study of detection processes for distorted digital signals, techniques of both linear and non-linear equalization of the channel have been widely studied.[1-23] The non-linear equalization of the channel[14-21,23] usually gives a better tolerance to additive white Gaussian noise than linear equalization,[1-13,22] normally requiring a lower average signal to noise power ratio for a given error rate. An even better tolerance to noise can be achieved through the use of more sophisticated detection processes,[24-31] which do not equalize the channel. Many of these processes, however, involve considerable equipment complexity.

An interesting technique has recently been proposed which for certain applications can achieve a similar standard of performance as the more sophisticated processes just mentioned, but with relatively simple equipment.[32-35] The arrangement is a synchronous serial data-transmission system in which the transmitted signal-elements are linearly independent and spaced in orthogonal groups.[32-35] A sufficient time interval separates adjacent transmitted groups to eliminate intersymbol interference between the received groups at the detector input. The data-transmission system is shown in Fig. 1.1-1.

Four different arrangements of the data-transmission system have been studied.[33] In the first three of these a process of exact linear equalization is applied to the received sampled signal, and the individual signal-elements are then detected from the signs of the corresponding sample values in the equalized signal. The equalization process may be achieved by a linear network or preferably by an iterative technique which performs the same linear transformation on the received sampled signal as does the linear network. The iterative



<u>Figure 1.1-1</u>

Block diagram of the synchronous serial data-transmission system.

technique involves a sequence of identical or similar operations which are performed successively by a simple piece of equipment. The fourth arrangement employs the optimum detection process which minimizes the probability of error in the simultaneous detection of the signal-elements in a group.

The results of the investigation[33] show that the fourth arrangement sometimes achieves a considerable advantage in tolerance to additive white Gaussian noise over the other three systems, particularly with more severe signal distortions. However, with m binary signal-elements in a group, the first three systems can be implemented quite simply for values of m up to 20 or even 30, whereas the optimum detection process involves 2^m sequential operations. The latter is not a practical arrangement when $m > 10$ and the transmission rate approaches 10,000 bits per second. If 4-or 8-level signal-elements are used the optimum detection process requires 4^m and 8^m sequential operations,respectively. The corresponding increase in the number of sequential operations in the other three systems is relatively insignificant, so that the optimum detection process is now by far the least attractive.

An alternative technique for the transmission of orthogonal groups of signal-elements is to use code-division multiplexing (CDM) in place of the time division multiplexing (TDM) of the transmitted signals assumed in the systems just described. In a CDM system the waveform of two transmitted signal-elements of a group, having the same element values are not simple time shifts of each other as in a TDM system. A CDM system is thus a parallel system whereas a TDM system is a serial system. Various parallel systems using CDM have been studied but it was found that these are much more complex than the serial system just outlined,and do not in fact achieve any useful advantage in tolerance to additive white Gaussian noise.[45,55] The results of this work suggest that serial systems are likely to be more cost-effective than parallel systems.

In all the work just mentioned it is assumed that the channel characteristics may vary slowly with time. The techniques investigated have therefore been strongly influenced by the need to estimate the channel response at the receiver, from the received data signal. Furthermore, since the channel response is estimated at the receiver, any attempt at the total or partial equalization of the channel at the transmitter, requires the transmission of the necessary information from the receiver to the transmitter, thus immediately increasing the equipment complexity. For this reason, in all the systems studied so far, the signal processing involved in the equalization of the channel or the correction of the effects of signal distortion, are concentrated at the receiver, where the channel response is known.

In an application where the channel response is fixed and known, it may be advantageous to perform at least some of the signal processing involved with the correction of signal distortion, at the transmitter. It is, for instance, well known that when the signal processing is achieved by a single transversal equalizer, an advantage can often be gained in tolerance to additive Gaussian noise by suitably sharing the linear equalization between the transmitter and receiver.[1,2]

1.2 Outline of Investigation

The investigation is concerned with the basic principles and methods of operation of the various systems studied. The primary aim has been to obtain a better understanding of these systems and hence to develop the most cost-effective arrangement for the particular application considered here. Since the various systems studied are all arrangements for processing sets of numerical values, these are computer-like systems which are best simulated on a computer rather than tested on a practical model. The latter would simply be a special purpose digital computer with the appropriate analogue/digital interfaces, and would be most costly and time consuming to build and test in the laboratory. The research methods have therefore involved a combination of theoretical analysis and computer simulation. The investigation is not concerned with the detailed practical implementation of the systems studied. It has been assumed throughout the discussion that the impulse response of the baseband channel is known at the receiver and does not vary with time. The transmission rate is assumed to be less than about 10,000 bauds, and the signal-elements may be 2, 4 or 8-level.

In the majority of the cases studied, groups of signal-elements are made orthogonal or disjoint in time by providing a sufficient time gap between adjacent groups so that there may be considerable intersymbol interference between the elements of a received group but no interference between the elements of different groups. In the remaining cases the orthogonality of the different groups of signal-elements is achieved by leaving no gaps between adjacent groups of transmitted signal-elements and using time guard bands between the detection processes for adjacent groups, so that each group of signal-elements is detected from only the central sample values of that group.

The mathematical model of the data-transmission system is discussed in Section 2 which also considers two important transversal equalizers for the equalization of base-band channels.[21,23] The former of the two, is the optimum linear equalizer which minimizes the mean square distortion in the equalized signal,[21] and the latter is a non-linear equalizer which maximizes the ratio of the output signal to additive white Gaussian noise.[21,23] In both these equalization arrangements, no gaps are inserted between adjacent groups of signal-elements at the transmitter and the transmission is a continuous (uninterrupted) serial stream of signals.

In section 3 two important detection processes are analysed. It is assumed for both of these that the appropriate time gap is inserted between adjacent groups of signal-elements at the transmitter. The first of the two detection processes is the optimum process of linear equalization for a received group of signal-elements, which is, of course, not the same as the linear equalization of the channel although it approaches the latter as the group size increases. This arrangement makes the most inefficient use of the available prior knowledge of the received signal, of the various systems studied in Sections 4 and 5, and therefore achieves the lowest tolerance to additive white Gaussian noise. The second of the two detection processes studied in Section 3, is the optimum detection process which at high signal/noise ratios achieves the best tolerance to additive Gaussian noise. It selects the set of element values having the maximum posterior probability of being correct. These two detection processes are of fundamental importance because they set the lower and upper bounds for the tolerances to additive white Gaussian noise achieved by the various arrangements studied in Sections 4 and 5. Section 3 ends with a simple example of groups of two binary signal-elements, to bring out the fundamental principles involved in the two detection processes just mentioned, and the advantages gained by transmitting signals in separate groups rather than in an uninterrupted serial stream.

Section 4 describes the various detection processes which have been developed from the two basic processes described in Section 3. The linear equalization process for a received group of signal-elements is implemented in an iterative process using the Gauss-Seidel method.[32] The other detection processes studied in Section 4 make use of suitable modifications of the Gauss-Seidel method, and in every case the detection of the signal-elements of a group is carried out in a sequence of identical operations which can be performed by a simple piece of equipment. The rate at which signals can be transmitted is, therefore, limited by the time required for the sequential operations in the detection of a group of signal-elements. An interesting technique for detecting multi-level signals is presented in which an initial search is carried out to select from the total number of possible values of each signal-element, the two or three element values which are most likely to be correct. The detection of a group of signal-elements is then completed by an iterative process which operates only on the selected element values, so that it treats the received signal-elements as though these were the corresponding 2 or 3-level elements.

Section 5 presents the results of computer simulation tests which have been used to compare the tolerances to additive white Gaussian noise of the different detection processes described in Sections 3 and 4. The results of the tests are used to elucidate the relationships that exist between the different detection processes. Section 5 ends with an attempt to compare the tolerances to additive white Gaussian noise of the various detection processes using orthogonal groups of signal , studied in Sections 3 and 4, with those described in Section 2 where the signals are transmitted in a continuous (uninterrupted) serieal stream , the information rate being the same in the two cases.

Section 6 describes the class of systems in which some linear processing, that is a linear transformation, is applied to the transmitted signals. Where the linear processing at the transmitter achieves exact equalization of a received group of signal-elements, adjacent groups of transmitted signal-elements are not separated by time gaps and the values of the individual signal-elements of a group are detected directly from the corresponding sample values of the received signal, by comparing these with the appropriate thresholds. Where the linear processing at the transmitter only partially equalizes a received group of elements, separation of adjacent groups of transmitted signal-elements depends upon the linear transformation used at the transmitter. In cases where time gaps are inserted between adjacent groups, a received group of signal-elements at the receiver may here by detected by any of the detection processes described in Sections 3 and 4.

Section 7 presents the results of computer simulation tests which have been used to compare the tolerances to additive white Gaussian noise of the different arrangements described in Section 6. The results of the tests are used to clarify the relationships that exist between the different systems.

In Section 8, Systems employing non-linear processing of signal at the transmitter are described. The non-linear processing here is such that adjacent groups of transmitted signal-elements are separated by time gaps. The tolerance to additive white Gaussian noise of these systems is studied by computer simulation.

2.0 LINEAR AND NON-LINEAR EQUALIZATION OF A BASEBAND CHANNEL

2.1 Model of the Data-Transmission System

The data-transmission system considered here is shown in Fig. 1.1-1. It is a synchronous serial baseband system, where the input signal to the transmitter filter is a stream of regularly spaced impulses, the value or area of each of which carries the value of the corresponding signal-element. Each impulse $s_i\delta(t - iT)$ at the input to the transmitter filter is therefore the corresponding input signal-element and it may be either binary or multi-level.

The transmission path itself could be either a low pass channel with an upper frequency limit no greater than about 10 KHz, or else a typical voice frequency channel with a frequency band no wider than 3000 to 3400 Hz, such as could be obtained over the telephone network. In the latter case the transmission path in Fig. 1.1-1 is assumed to include a linear modulator (at the transmitter) and a linear demodulator(at the receiver) the whole forming a linear baseband channel. An example of such a system is an arrangement using vestigial sideband suppressed carrier amplitude modulation (with a reinserted pilot carrier) at the transmitter, and coherent demodulation of the received signal, the reference carrier being held correctly synchronized (phase locked) to the received signal, with the aid of the received pilot carrier.[44,46,50,54] It may be shown that, with the elimination of the effects of the pilot carrier at the output of the demodulator in the arrangement just described, the modulator and demodulator together with the band-pass channel are equivalent to a single baseband channel.[33,34,46,50,54]

Over practical band-pass channels the characteristics are either not known prior to a transmission, such as over the switched telephone network, or else they may vary considerably (but usually slowly) with time, such as over point-to-point HF radio links. In either case, the impulse response of the channel must be estimated at the receiver from the received signal. Where the channel characteristics do not vary significantly with time, this may be carried out at the start of a transmission,[32-34] and where the channel characteristics vary with time, this must be carried out continuously using the received data signal.[4,5, 32,34] In the latter case a reasonably good estimate of the channel response can be obtained so long as the impulse response of the channel does not vary significantly during the reception of about 100 successive signal-elements. The zero-level elements present during the gaps are not here considered as signal-elements and it is assumed that there are about 16 signal-elements in each group, so that about 6 groups of elements are received before there is a noticeable change in the channel impulse response. At an actual element rate of about 1000 bauds, this implies that there is no significant change in the channel characteristics over a period of 0.1 seconds. This is normally the case over HF radio links where the typical fading rates are 4 to 15 fades per minute.[32,34,47,53,55] Where there is a negligible change in the channel impulse response over a received group of signal-elements and where the receiver has a good estimate of the impulse response, it is immaterial to the detection process whether or not the channel characteristics are varying with time. It is clear therefore that the systems studied here may be used either over a time invariant channel or else over a channel whose characteristics may slowly vary with time. In the latter case, the impulse response of the channel may be estimated in any one of several different ways and these are described in the published literature.[4-11,32,34] This investigation is not concerned with the methods of estimating the channel and it is therefore assumed throughout that the impulse response of the channel is known and furthermore, is time invariant.

The transmitter filter limits the spectrum of the transmitted signal approximately to the available bandwidth of the transmission path. Where the transmission path itself is a band-pass channel, the transmitter filter is assumed to include the low-pass filter equivalent to all filters involved in the linear modulator.

The receiver filter removes the noise components outside a frequency band approximately corresponding to the bandwidth of the received signal, and where the transmission path is a band-pass channel, the receiver filter includes the low-pass filter equivalent to all filters involved in the linear demodulation of the received modulated carrier signal.

The transmission path together with the transmitter and receiver filters, form a linear baseband channel whose impulse response is taken to be $y(t)$. Thus in the absence of noise, the signal at the output of the receiver filter is

$$\sum_i s_i y(t - iT) \qquad (2.1\text{-}1)$$

Over some practical channels such as voice frequency channels using HF radio links, the most important type of noise introduced by the channel is additive noise which can for practical purposes, be taken to be additive white Gaussian noise.[46,47,53] The difference between the two is sufficiently small not to introduce any serious discrepancies in the results, when the noise actually present is taken to be white Gaussian noise. The latter is of course not physically realizable, having infinite bandwidth and therefore infinite power level, for a non-zero power spectral density.

Over telephone circuits, however, the most important source of additive noise is impulsive noise which sometimes resembles short bursts of Gaussian noise. It has been shown that, if one data-transmission system has a better tolerance than another, to additive white Gaussian noise, it will also in general have a better tolerance to the additive noise, over telephone circuits.[44] It follows therefore, that the relative tolerance of two systems to additive white Gaussian noise is a good measure of their relative tolerance to the additive noise over telephone circuits.[44] Furthermore, the effects of additive white Gaussian noise on a digital data-transmission system may readily be analyzed theoretically and studied by computer simulation. For the above reasons, in the model of the data-transmission system, it is assumed that additive white Gaussian noise is introduced at the output of the transmission path. The noise has zero mean and a two sided power spectral density of σ^2, giving the zero mean Gaussian waveform $w(t)$ at the output of the receiver filter. Thus the resultant signal at the output of the receiver filter is

$$r(t) = \sum_i s_i\, y(t - iT) + w(t) \qquad (2.1\text{-}2)$$

The impulse response $h(t)$ of the transmitter and receiver filters in cascade is assumed to be such that $h(0) = 1$ and $h(iT) = 0$ for all non-zero integer values of i. This impulse response is achieved by using the same transfer function $B(f) = H^{\frac{1}{2}}(f)$ for the transmitter and receiver filters, where

$$H(f) = \begin{cases} \frac{1}{2}T(1 + \cos \pi fT) & \text{for } -\frac{1}{T} < f < \frac{1}{T} \\ 0 & \text{elsewhere} \end{cases} \qquad (2.1\text{-}3)$$

The use of the same transfer function for the transmitter and receiver filters is conventional[6,33,34] and enables an easy comparison to be made with other systems. Alternative transfer functions for the filters are, of course, available and some of these make more efficient use of bandwidth.[33]

If C(f) is the transfer function of the transmission path, then the channel transfer function expressed in terms of the transfer functions of the transmission path and filters, is

$$Y(f) = H(f)\, C(f), \qquad (2.1\text{-}4)$$

and the impulse response of the channel y(t) is given by the inverse Fourier transform of Y(f), that is,

$$y(t) = F^{-1}\{Y(f)\} = \int_{-\infty}^{\infty} C(f)\, H(f)\, e^{j2\pi ft} df \qquad (2.1\text{-}5)$$

When no signal distortion is introduced by the transmission path, that is, C(f) = 1,

$$y(t) = \int_{-\infty}^{\infty} H(f)\, e^{j2\pi ft}\, df \qquad (2.1\text{-}6)$$

From Eqn. (2.1-3)

$$y(t) = \tfrac{1}{2} T \int_{-\frac{1}{T}}^{\frac{1}{T}} (1 + \cos \pi f T) e^{j2\pi ft}\, df$$

$$= \tfrac{1}{2} T \int_{-\frac{1}{T}}^{\frac{1}{T}} (1 + \tfrac{1}{2} e^{j\pi fT} + \tfrac{1}{2} e^{-j\pi fT}) e^{j2\pi ft} df$$

$$= \tfrac{1}{2} T \int_{-\frac{1}{T}}^{\frac{1}{T}} \{e^{j\pi f2t} + \tfrac{1}{2} e^{j\pi f(2t+T)} + \tfrac{1}{2} e^{j\pi f(2t-T)}\} df$$

$$= \tfrac{1}{2} T \left[\frac{e^{j\pi f2t}}{j\pi 2t} + \frac{1}{2} \frac{e^{j\pi f(2t+T)}}{j\pi(2t+T)} + \frac{1}{2} \frac{e^{j\pi f(2t-T)}}{j\pi(2t-T)} \right]_{-\frac{1}{T}}^{\frac{1}{T}}$$

$$= \frac{e^{j\pi\frac{2t}{T}} - e^{-j\pi\frac{2t}{T}}}{2j\pi\frac{2t}{T}} + \frac{1}{2} \frac{e^{j\pi(\frac{2t}{T}+1)} - e^{-j\pi(\frac{2t}{T}+1)}}{2j\pi(\frac{2t}{T}+1)}$$

$$+ \frac{1}{2} \frac{e^{j\pi(\frac{2t}{T}-1)} - e^{-j\pi(\frac{2t}{T}-1)}}{2j\pi(\frac{2t}{T}-1)}$$

$$= \frac{\sin \pi\frac{2t}{T}}{\pi\frac{2t}{T}} + \frac{1}{2} \frac{\sin\pi(\frac{2t}{T}+1)}{\pi(\frac{2t}{T}+1)} + \frac{1}{2} \frac{\sin\pi(\frac{2t}{T}-1)}{\pi(\frac{2t}{T}-1)} \qquad (2.1\text{-}7)$$

The shape of y(t) is shown in Fig. 2.1-1. The delay introduced by the filters has for convenience been neglected here. Clearly, when C(f) = 1,

$$y(o) = 1, \; y(\pm \frac{T}{2}) = \tfrac{1}{2} \qquad (2.1\text{-}8)$$

and

$$y(\pm\tfrac{1}{2}\, iT) = 0 \text{ for } i \neq 0 \text{ or } \pm 1 \qquad (2.1\text{-}9)$$

The received signal r(t) at the output of the receiver filter is sampled at time instants t = iT, for all integers i. This assumes that the receiver has the prior knowledge of the time of arrival of each signal-element, that is, the receiver is in element synchronism with the received signal. Techniques for achieving correct element synchronism have been widely studied and can be designed to hold a receiver in correct synchronism even in the presence of quite severe signal distortion.[44,54] The study of these techniques is beyond the scope of this investigation and the problem of maintaining element synchronization will not be considered further.

The i^{th} received signal-element is sampled at time t = iT to give the sample value

$$r(iT) = s_i y(0) + w(iT) \qquad (2.1\text{-}10)$$

or

$$r_i = s_i + w_i \qquad (2.1\text{-}11)$$

where $r_i = r(iT)$ and $w_i = w(iT)$, and it is assumed that C(f) = 1. Thus, when C(f) = 1 and in the absence of noise, the sample value of the i^{th} received element, obtained at time instant t = iT, is s_i.

The energy of the i^{th} transmitted signal-element is

$$\int_{-\infty}^{\infty} s_i^2 \, |B(f)|^2 \, df.$$

$$\int_{-\infty}^{\infty} |B(f)|^2 \, d(f) = \int_{-\infty}^{\infty} |H(f)| \, df$$

$$= \tfrac{1}{2}T \int_{-\frac{1}{T}}^{\frac{1}{T}} (1 + \cos \pi fT)\, df, \text{ from Eqn. (2.1-3)},$$

$$= \tfrac{1}{2}T \left[f + \frac{\sin \pi fT}{\pi T}\right]_{-\frac{1}{T}}^{\frac{1}{T}}$$

$$= 1 \qquad (2.1\text{-}12)$$

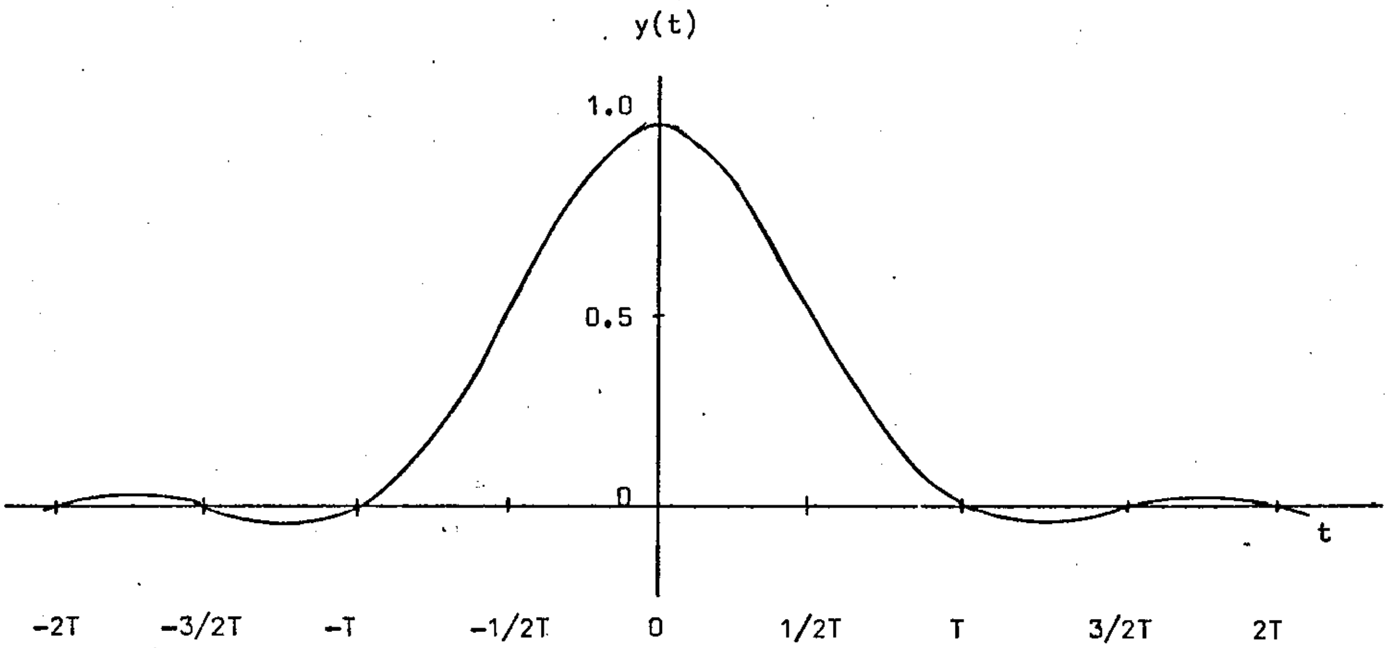


Figure 2.1-1

Impulse response y(t) of the baseband channel when no signal distortion is introduced by the transmission path.

Hence the energy of a single transmitted element is s_i^2.

With additive white Gaussian noise having a two sided power spectral density of σ^2 at the input to the receiver filter, the noise power spectral density at the output of the receiver filter is

$$\sigma^2 \; |B(f)|^2 \;=\; \sigma^2 H(f) \qquad (2.1\text{-}13)$$

so that the mean noise power is

$$\sigma^2 \int_{-\infty}^{\infty} |H(f)| \, df \;=\; \sigma^2 \qquad (2.1\text{-}14)$$

from equation (2.1-12).

Thus w_i in Eqn. (2.1-11) is a sample value of a Gaussian random variable with zero mean and variance σ^2.

From the Wiener-Kinchine Theorem,[21] the autocorrelation function of the noise signal $w(t)$ at the output of the receiver filter is

$$a(\tau) = \int_{-\infty}^{\infty} \sigma^2 \; H(f) \; e^{j2\pi f\tau} \, df$$

$$= \sigma^2 \left[\frac{\sin \pi \frac{2\tau}{T}}{\pi \frac{2\tau}{T}} + \frac{1}{2} \frac{\sin \pi \left(\frac{2\tau}{T} + 1\right)}{\pi\left(\frac{2\tau}{T} + 1\right)} + \frac{1}{2} \frac{\sin \pi \left(\frac{2\tau}{T} - 1\right)}{\pi\left(\frac{2\tau}{T} - 1\right)} \right]$$

from Eqn. (2.1-6). Clearly

$$a(o) = \sigma^2$$

and

$$a(iT) = 0$$

for any non zero integer i. Since the mean value of $w(iT)$ is zero, it follows that the noise component $w(iT)$ is uncorrelated with the noise component $w(hT)$, where the integer $h \neq i$, so that the $\{w_i\}$ are sample values of statistically independent Gaussian random variables with zero mean and variance σ^2.

Suppose now that $C(f) \neq 1$ and signal distortion is introduced in the transmission path. This results in intersymbol interference between the different received signal-elements at the sampling instants $t = iT$. It will be assumed throughout the following discussion that $C(f)$ is such that a received signal-element may introduce intersymbol interference in the sample values of some or all of the g_i

immediately preceding elements and in some or all of the g_2 immediately following elements. Most forms of signal distortion normally experienced are of this general type.[33,34,44,47] Let

$$g = g_1 + g_2.$$

If only the i^{th} element is received, in the absence of noise, then for any integer h,

$$r(hT) = s_i y(hT - iT) \qquad (2.1\text{-}15)$$

or

$$r_h = s_i y_{(h-i)} \qquad (2.1\text{-}16)$$

Where $r_h = r(hT)$ and $y_{(h-i)} = y(hT - iT)$.

$y_{(h-i)}$ is non-zero for some or all values of h in the range $i-g_1$ to $i + g_2$, and is zero for all other values of h. The sample values corresponding to the i^{th} transmitted signal-element are

$$s_i(0 \ldots 0 \; y_{-g_1} \ldots y_0 \ldots y_{g_2} \; 0 \ldots 0).$$

Thus, the sampled impulse response of the baseband channel (i.e. the transmitted filter, transmission path and the receiver filter in cascade) is

$$\sum_{h=-g_1}^{g_2} y_h \, \delta(t - hT) \qquad (2.1\text{-}17)$$

To make this physically realizable, let the first non-zero sample value occur at t = 0, so that the sampled impulse response of the baseband channel becomes

$$\sum_{h=0}^{g} y_h \delta(t - hT) \qquad (2.1\text{-}18)$$

and $y_h = y(hT)$ is now non-zero for some or all values of the integer h in the range 0 to g and is zero for all other values of h. Thus the sampled impulse response of the baseband channel may be simply written as the (g + 1)-component row-vector

$$y_0 \; y_1 \; y_2 \ldots y_g. \qquad (2.1\text{-}19)$$

The sample values of the i^{th} received signal-element now become

$$s_i(0 \ldots 0\ y_0\ y_1 \ldots y_g\ 0 \ldots 0)$$

When a continuous stream of signal-elements is received in the presence of noise, the sample value of the received signal at time $t = hT$ is

$$r(hT) = \sum_{i=h-g}^{h} s_i\, y(hT - iT) + w(hT) \qquad (2.1.20)$$

$$\text{or } r_h = \sum_{i=h-g}^{h} s_i\, y_{h-i} + w_h \qquad (2.1\text{-}21)$$

It is evident that if r_h is used for the detection of any one s_i, there may well be so much inter-symbol interference from the other received elements, that the correct detection of s_i from r_h is not possible, even in the absence of noise.

2.2 Basic Assumptions

In the data-transmission system of Section 2.1 the sampled impulse response of the baseband channel is given by Equation (2.1-18) and its Fourier transform is

$$\sum_{h=0}^{g} y_h\, e^{-j2\pi fhT} \qquad (2.2\text{-}1)$$

where $j = \sqrt{-1}$. The z-transform of the sampled impulse response of the baseband channel is

$$F(z) = \sum_{h=0}^{g} y_h\, z^{-h} \qquad (2.2\text{-}2)$$

where $z = e^{j2\pi fT}$. The coefficients of z^{-h} in Eqn. (2.2-2) are of course the sample values of the sampled impulse response of the baseband channel in Eqn. (2.1-19). The values $\{s_i\}$ of the signal-elements at the input to the transmitter filter are assumed to be either binary or multi-level and furthermore they are statistically independent and equally likely to have any of the possible values (Table 5.2-2).

With the transmission of a continuous (uninterrupted) stream of signal-elements, the i^{th} sample value of the received signal is,

$$r_i = \sum_{j=0}^{g} y_j s_{i-j} + w_i \qquad (2.2\text{-}3)$$

where w_i is sample value of a Gaussian random variable with zero mean and variance σ^2.

The condition for no distortion and delay in transmission is that

$$F(z) = 1 \qquad (2.2\text{-}4)$$

in Eqn. (2.2-2), and this occurs when $y_0 = 1$ and $y_j = 0$ for all j, $j \neq 0$. Thus, in the presence of distortion and delay in transmission, if s_i is detected from r_i, there is, in addition to the noise component w_i, an intersymbol interference component

$$\sum_{j=1}^{g} y_j s_{i-j} \qquad (2.2\text{-}5)$$

added to the wanted signal $y_0 s_i$.

In the detection of s_i from r_i at high signal/noise ratios, the best tolerance to additive white Gaussian noise is achieved through the effective elimination of all intersymbol interference, that is, through the accurate equalization of the baseband channel.[6,21,23] The data-transmission system of Fig. 1.1-1 must now include an equalizer, at the receiver, which equalizes the baseband channel as shown in Fig. 2.2-1. It is assumed throughout the discussions in Sections 2.3, 2.4 and 2.5 that the equalizer in Fig. 2.2-1 is fed with the sample values of the received signal and that it operates entirely on these sample values.

A number of different linear and non-linear equalizers for equalizing the baseband channel are described in the published literature.[1-23] A linear equalizer is usually a feedforward transversal filter,[4-13, 21,22] whereas a non-linear equalizer is usually a combination of a linear feedforward and a non-linear feedback

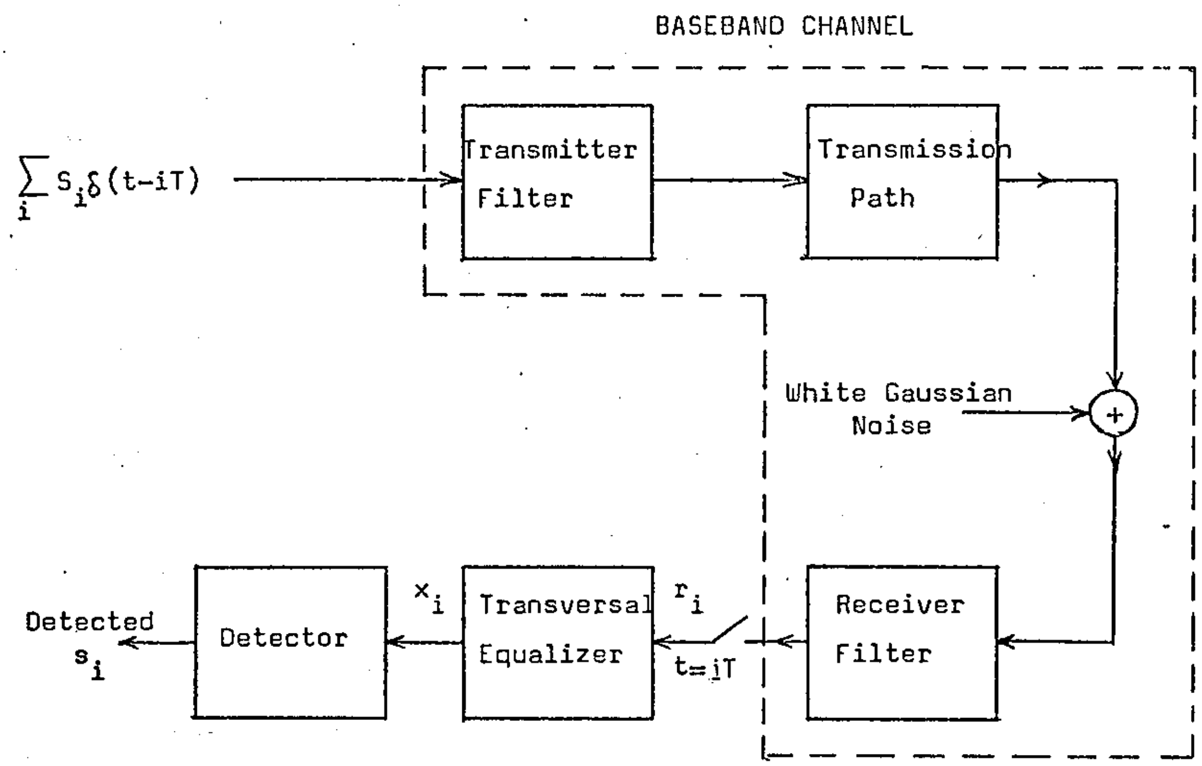


Figure 2.2-1

Synchronous serial basenand data-transmission system with a transversal equalizer at the receiver.

transversal filter.[15-19,21,23] A description of all the techniques of channel equalization using transversal equalizers is not central to the present investigations. Instead, two important equalizers have been considered. The former of the two, described in Section 2.3, is the optimum linear equalizer which minimizes the mean square distortion in the equalized signal[21], and the latter, described in Section 2.4, is the optimum non-linear equalizer which maximizes the output signal/noise ratio[21,23].

2.3 Linear Equalizer

Assume that the equalizer in Fig. 2.2-1 is the linear feed forward transversal filter with p taps as shown in Fig. 2.3-1. Let the i^{th} tap of the filter have a gain d_{i-1}, so that the tap gains of the filter may be represented by the p components of the row-vector.

$$D = d_o\ d_1\ . . . . d_{p-1} \tag{2.3-1}$$

The z-transform of the sampled impulse response of the filter is

$$D(z) = d_o + d_1 z^{-1} + . . . + d_{p-1} z^{-p+1} \tag{2.3-2}$$

The z-transform of the sampled impulse response of the channel and linear equalizer is now

$$\begin{aligned} E(z) &= D(z)F(z) \\ &= e_o + e_1 z^{-1} + . . . + e_{p+g-1} z^{-p-g+1} \end{aligned} \tag{2.3-3}$$

Thus the sampled impulse response of the equalized channel is given by the (p+g)-component row-vector

$$E = e_o\ e_1\ . . . e_{p+g-1} \tag{2.3-4}$$

Let B be the p x (p+g) matrix whose i^{th} row is

$$B_{i-1} = \overbrace{0\ .\ .\ .\ 0}^{i-1}\ \overbrace{y_o\ y_1 \cdots y_g}^{g+1}\ \overbrace{0\ \ldots\ 0}^{p-i} \tag{2.3-5}$$

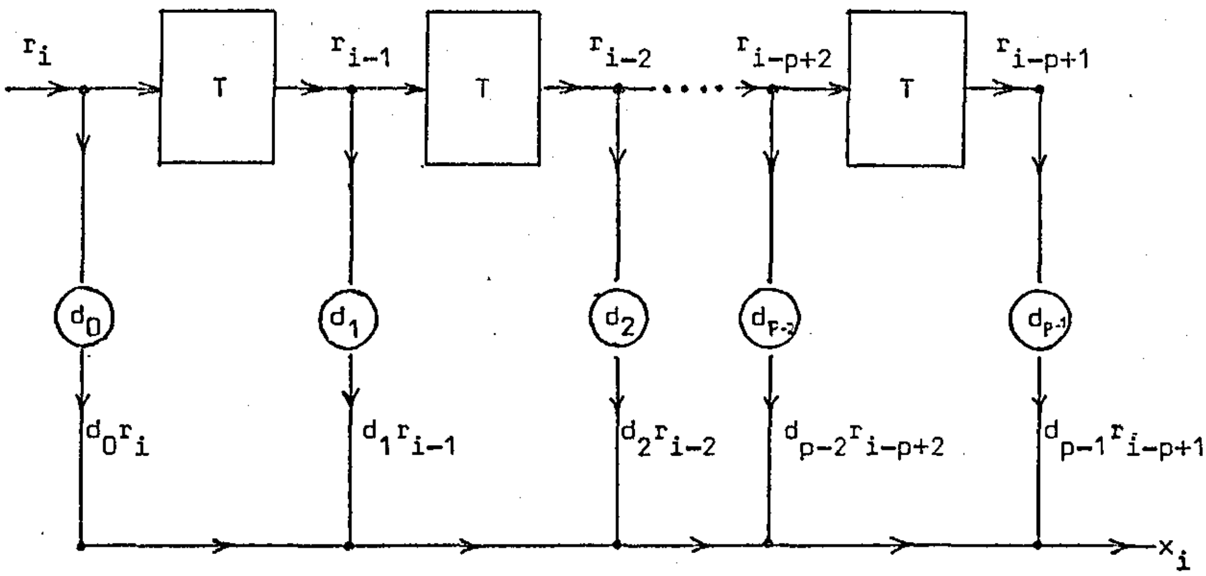


Figure 2.3-1

Linear transversal equalizer for baseband channel.

where, it can be shown, that the p rows of the matrix B are linearly independent.[21] From Eqns. (2.3-3), (2.3-4) and (2.3-5)

$$E = \sum_{i=0}^{p-1} d_i B_i = DB \qquad (2.3\text{-}6)$$

If the channel is exactly equalized, then

$$E(z) = D(z)\ F(z) = z^{-h} \qquad (2.3\text{-}7)$$

where h is a positive integer in the range 0 to p+g-1. The z-transform of the i^{th} received signal-element in the absence of noise, is

$$s_i\ z^{-i}F(z)D(z) = s_i z^{-h-i} \qquad (2.3\text{-}8)$$

For a finite number of filter tap gains the channel is not exactly equalized, and therefore Eqn. (2.3-8) is only approximately satisfied.[21] Let U_h be the (p+g)-component row-vector

$$U_h = \overbrace{0\ .\ .\ .\ 0}^{h}\ 1\ \overbrace{0\ .\ .\ .\ 0}^{p+g-h-1} \qquad (2.3\text{-}9)$$

U_h in Eqn.(2.3-9) is the ideal value of the sampled impulse response of the equalized channel for a total transmission delay of hT seconds, whereas E in Eqn. (2.3-4) is the actual value. Thus the mean square error in the sampled impulse response of the equalized channel is [21]

$$(1-e_h)^2 + \sum_{\substack{i=0 \\ i\neq h}}^{p+g-1} e_i^2 = (U_h - E)(U_h - E)^T \qquad (2.3\text{-}10)$$

$$= |(U_h - E)|^2 \qquad (2.3\text{-}11)$$

where $|(U_h - E)|$ is the length of vector $(U_h - E)$.

Consider now the linear transversal filter which minimizes the mean square error in Eqn.(2.3-11). It is required to find the values of the p tap gains of the linear transversal filter in

Fig. 2.3-1, such that $|(U_h - E)|^2$ and hence $|U_h - E|$ is minimized, the latter being the length of the (p + g)-component row-vector $(U_h - E)$. The vector may of course be represented as a point in a (p + g)-dimensional Euclidean vector space, and the length of the vector is the distance of this point from the origin. It has been shown in reference [21], that the p row-vectors $\{B_i\}$ in Eqn.(2.3-5) are linearly independent, so that the p x (p+g) matrix B has rank p. This means that the vector E = DB is a point in the p-dimensional subspace, of the (p+g)-dimensional vector space, spanned by the $p\{B_i\}$. Since $|U_h - E|$ is the distance between the two (p+g)-component vectors U_h and E, in the (p+g)-dimensional Euclidean vector space containing these vectors, it follows that $|U_h - E|$ is minimum when E is the point in the p-dimensional subspace at the minimum distance from U_h. By the Projection Theorem[21], E is the orthogonal projection of U_h on the p-dimensional subspace. The (p+g)-dimensional Euclidean vector space is shown in Fig. 2.3-2. Thus the p row-vectors given by the rows of B, are orthogonal to the vector $(U_h - E)$, and

$$(U_h - E)\,B^T = 0 \qquad (2.3\text{-}12)$$

or

$$(U_h - DB)B^T = 0 \qquad (2.3\text{-}13)$$

or

$$DBB^T = U_hB^T \qquad (2.3\text{-}14)$$

or

$$D = U_hB^T(BB^T)^{-1} \qquad (2.3\text{-}15)$$

The components of the p-component row-vector D in Eqn.(2.3-15), give the values of the p tap gains of the linear transversal filter which minimizes the mean square error for a given h. The

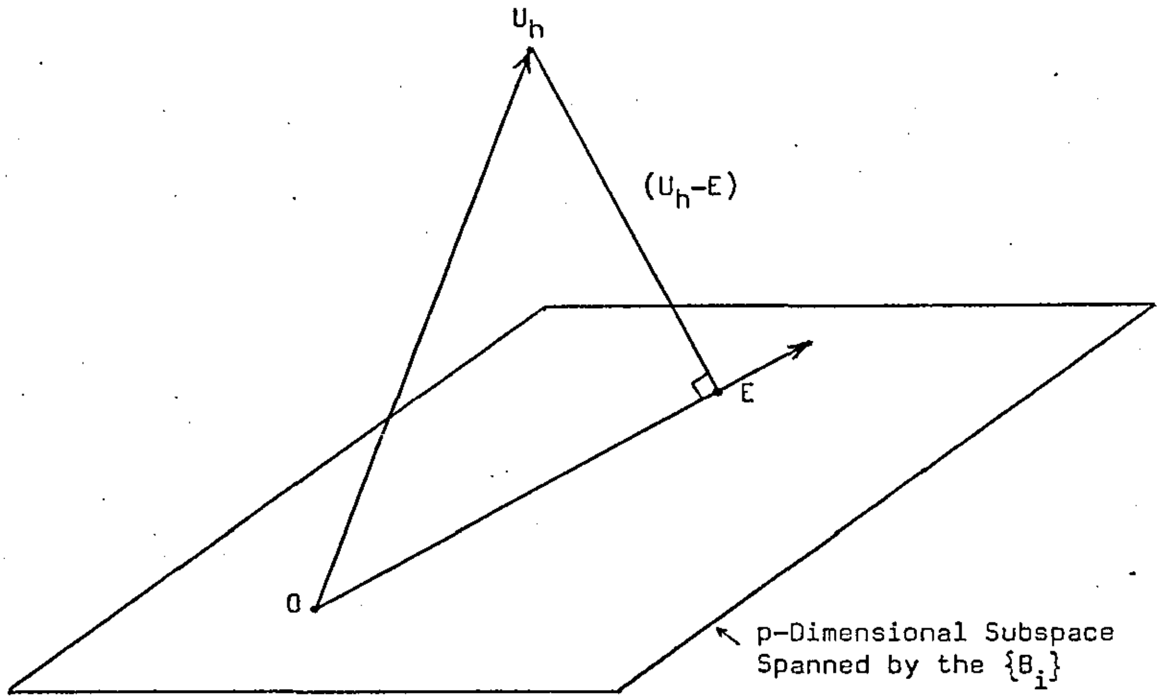


Figure 2.3-2

(p+g)-dimensional Euclidean vector space.

vector D must therefore be determined for each value of h in the range 0 to p+g-1, and the vector D for the required equalizer is that which gives the minimum value of $|U_h - E|$.

From Eqn. (2.3-3) and for sufficiently small value of $|U_h - E|$ the z-transform of the equalized channel becomes

$$E(z) \simeq z^{-h} \qquad (2.3\text{-}16)$$

The mean square distortion in the equalized signal is given by[6,21]

$$\sum_{\substack{i=0 \\ i \neq h}}^{p+g-1} \frac{e_i^2}{e_h^2} \qquad (2.3\text{-}17)$$

It has been shown in reference [21] that if the linear transversal equalizer is designed to minimize the mean square error in the equalized channel subject to the constraint that in Eqn.(2.3-4)

$$e_h = 1 \qquad (2.3\text{-}18)$$

then the filter not only minimizes the mean square error subject to the above constraint, but also minimizes the mean square distortion, and is an optimum linear transversal equalizer in the sense that it achieves the most effective equalization for a given number of tap gains. Furthermore, the sampled impulse response of this optimum linear equalizer is given by the components of the p-component vector

$$G = \frac{1}{e_h} D \qquad (2.3\text{-}19)$$

where D is given by Eqn. (2.3-15). Again G must be determined for each value of h in the range 0 to p+g-1, and the vector G for the required equalizer is that which minimizes $|U_h - E|$, where e_h is now constrained by Eqn. (2.3-18). Thus in the presence of noise and when $|U_h - E|$ is very small, the output of the optimum linear equalizer at time instant (i+h)T is

$$x_{i+h} \approx s_i + v_{i+h} \qquad (2.3\text{-}20)$$

The noise component v_{i+h} is the weighted sum of the corresponding input noise components $\{w_i\}$ and is given by

$$v_{i+h} = \sum_{j=0}^{p-1} w_{i+h-j}\, g_j \qquad (2.3\text{-}21)$$

Since the p$\{w_i\}$ in Eqn. (2.3-21) are sample values of statistically independent Gaussian random variable with zero mean and variance σ^2, it follows that v_{i+h} is a sample value of a Gaussian random variable of zero mean and variance

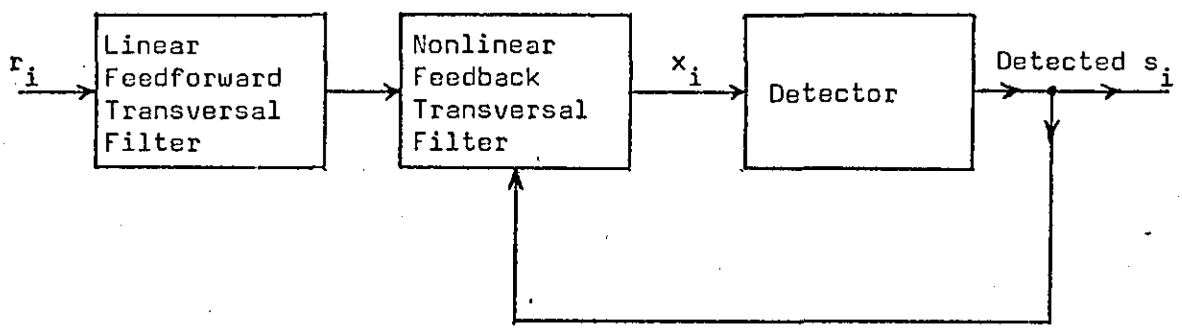
$$\eta^2 = \sum_{i=0}^{p-1} g_i^2 \sigma^2 = \sigma^2 GG^T \qquad (2.3\text{-}22)$$

and s_i is detected by comparing x_{i+h} with the appropriate thresholds.[21]

It is well known[6,21,22] that if one or more roots of the z-transform of the sampled impulse response of the channel have values on the unit circle in the z-plane, a linear transversal feed-forward filter cannot equalize the channel with finite number of tap gains. Thus the optimum linear equalizer having tap gains given by the components of vector G in Eqn. (2.3-19), cannot be used for equalization purposes in cases where the roots of F(z) lie on the unit circle in the z-plane.

2.4 Non-linear Equalizer

Consider now the non-linear equalizer shown in Fig. 2.4-1, which maximizes the output signal/noise ratio. It consists of a linear feedforward transversal filter followed by a non-linear feedback transversal filter. The linear filter performs a process of partial equalization of the baseband channel, the equalization being completed by the non-linear filter which uses decision directed cancellation of intersymbol interference.[16,17,21,23]



<u>Figure 2.4-1</u>

Non-linear equalizer using the linear and non-linear transversal filters.

The non-linear equalizer in Fig. 2.4-1 performs separate processes of linear and non-linear equalization, and the problem here is to determine the particular combination of the linear and non-linear filters that maximizes the signal/noise ratio in the detection of a signal-element, subject to the essentially accurate equalization of the channel.[23]

It was mentioned in Section 2.3 that where the z-transform of the sampled impulse response of the channel has one or more roots on the unit circle in the z-plane, the channel cannot be equalized by a linear equalizer having a finite number of tap gains. Suppose now that

$$F(z) = F_1(z)\, F_2(z) \qquad (2.4\text{-}1)$$

where no roots of $F_1(z)$ and all roots of $F_2(z)$ satisfy $|z| \approx 1$. Clearly $F_1(z)$ has no zeros (roots) on the unit circle in the z-plane, whereas all the roots of $F_2(z)$ are on the unit circle. Both $F_1(z)$ and $F_2(z)$ are assumed to be known at the receiver. Let

$$F_2(z) = f_o + f_1 z^{-1} + \ldots + f_\ell z^{-\ell} \qquad (2.4\text{-}2)$$

where ℓ is a +ve integer. There is no z-transform $G_2(z)$ with only limited number of terms which is such that $F_2(z)G_2(z) \approx z^{-h}$ for some integer h. Thus $F_2(z)$ cannot be equalized by a linear equalizer. However, $F_1(z)$ can be equalized by a linear transversal feedforward filter using the method described in Section 2.3. Let the z-transform of this filter be

$$G_1(z) = c_o + c_1 z^{-1} + \ldots + c_{p-1} z^{-p+1} \qquad (2.4\text{-}3)$$

where the transversal filter has p taps. Clearly

$$F_1(z)\, G_1(z) \simeq z^{-h} \qquad (2.4\text{-}4)$$

and

$$F(z)\, G_1(z) \simeq z^{-h} F_2(z) \qquad (2.4\text{-}5)$$

where h is a positive integer.

The linear transversal filter which precedes the non-linear filter in Fig. 2.4-1, is designed to have a z-transform

$$B(z)\ G_1(z) = D(z) = d_o + d_1 z^{-1} + \ldots + d_{q-1} z^{-q+1} \quad (2.4\text{-}6)$$

so that the linear filter has q taps, the i^{th} of which has a gain d_{i-1} and

$$B(z) = b_o + b_1 z^{-1} + \ldots + b_j z^{-j} \quad (2.4\text{-}7)$$

where

$$q = p + j. \quad (2.4\text{-}8)$$

p is the number of taps required by a linear transversal filter with a z-transform $G_1(z)$ for the satisfactory equalization of $F_1(z)$, and q is the maximum number of taps acceptable for the linear transversal filter which precedes the non-linear filter in Fig. 2.4-1. For reasons which will become clear latter, b_o is constrained to satisfy

$$b_o f_o = 1 \quad (2.4\text{-}9)$$

The remaining coefficients $\{b_i\}$ in Eqn. (2.4-7) may be selected to have any real values. It is now required to find the value of B(z), within the limits of the constraints imposed by Eqns. (2.4-8) and (2.4-9).[23]

The z-transform of the channel and linear filter is

$$F(z)\ B(z)\ G_1(z) \simeq z^{-h} B(z)\ F_2(z) \quad (2.4\text{-}10)$$

where h is a positive integer.

The z-transform of the i^{th} received signal-element at the output of the linear filter is

$$s_i z^{-i} F(z) B(z) G_1(z) \simeq s_i z^{-i-h} B(z) F_2(z) \quad (2.4\text{-}11)$$

Thus the linear filter may be considered to perform a process of partial linear equalization.

The sample value x_{i+h} of the signal at the output of the non-linear filter at time $t = (i + h)T$, contains the first non-zero component of the i^{th} received signal-element, and from Eqn. (2.4-11), has the value

$$x_{i+h} \approx s_i b_o f_o + v_{i+h}$$

$$\text{or} \qquad x_{i+h} \approx s_i + v_{i+h} \tag{2.4-12}$$

from Eqn. (2.4-9) and assuming the correct cancellation of the preceding $(j+\ell)$ signal-elements in the non-linear filter. Eqn.(2.4-9) normalizes the level of the component of the i^{th} signal-element in x_{i+h}. Since

$$v_{i+h} = \sum_{j=0}^{q-1} w_{i+h-j} d_j \tag{2.4-13}$$

where the $q\{w_i\}$ are sample values of statistically independent Gaussian random variable with zero mean and variance σ^2, it follows that v_{i+h} is a sample value of a Gaussian random variable with zero mean and variance

$$\eta^2 = \sum_{i=0}^{q-1} d_i^2 \sigma^2 = \sigma^2 DD^T \tag{2.4-14}$$

where D is a q-component row-vector whose components are

$$d_o \; d_1 \; . \; . \; . \; d_{q-1}, \tag{2.4-15}$$

and d_{i-1} is the coefficient of i^{th} term in D(z) (Eqn. (2.4-6)). The detector now detects the value of the i^{th} signal-element s_i by comparing x_{i+h}, in Eqn. (2.4-12), with the appropriate thresholds, and the non-linear filter then cancels (removes by subtraction) the components of the i^{th} signal-element from the following sample values, thus eliminating the intersymbol interference of the signal-element in the following elements. The process of detection and signal cancellation is then repeated for the next received signal-element, and so on. The combination of the linear and non-linear filters, therefore, eliminates all inter-symbol interference, so long as the signal-elements are correctly detected and cancelled.[23]

In a practical application of the non-linear equalizer, a known sequence of more than $(j+\ell)$ element values is first transmitted and during this sequence, the detector uses its prior knowledge of any particular received sequence of $(j+\ell)$ of these elements, to achieve correct cancellation of these elements without having to detect their actual values. The following received signals can now be detected without intersymbol interference, as previously described, so that the process of detection and signal cancellation can begin.[21]

To maximize the signal/noise ratio at the input to the detector, η^2 in Eqn. (2.4-14) must be minimized. Thus the q tap gains $\{d_i\}$ of the linear transversal filter, in Fig. 2.4-1, must be adjusted, within the constraint imposed by Eqns. (2.4-6) and (2.4-9), to minimize DD^T. Let B be the (j+1)-component vector whose components are

$$b_o \; b_1 \; . \; . \; . \; . \; b_j \qquad (2.4\text{-}16)$$

and let C be the (j+1) x q matrix whose i^{th} row is

$$C_{i-1} = \overbrace{0 \; . \; . \; . \; 0}^{i-1}\overbrace{c_o c_1 \; . \; . \; . \; c_{p-1}}^{p}\overbrace{0 \; . \; . \; . \; 0}^{j-i+1} \qquad (2.4\text{-}17)$$

b_{i-1} is of course the coefficient of i^{th} term in B(z) and c_{i-1} of the i^{th} term in $G_1(z)$. Since C is a convolutional matrix, from Eqns. (2.4-6), (2.4-7) and (2.4-9)[21, 23]

$$D = BC = b_o C_o - LM = \frac{1}{f_o} C_o - LM \qquad (2.4\text{-}18)$$

where, $L = -(b_1, b_2, \; . \; . \; . \; , b_j)$ (2.4-19)

and M is the j x q matrix whose i^{th} row is C_i. M is completely determined by C_o for the given values of j and q. From Eqns. (2.4-14) and (2.4-18)

$$\eta^2 = \sigma^2 (\frac{1}{f_o} C_o - LM)(\frac{1}{f_o} C_o - LM)^T$$

$$= \sigma^2 |\frac{1}{f_o} C_o - LM|^2 \qquad (2.4\text{-}20)$$

where $|\frac{1}{f_o} C_o - LM|$ is the length of the vector $\frac{1}{f_o} C_o - LM$.

The noise variance η^2 is minimum when $|\frac{1}{f_o} C_o - LM|$ is minimum. For given values of F(z), $G_1(z)$ and q, the values of f_o, C_o and M are fixed, leaving L as the only variable in $|\frac{1}{f_o} C_o - LM|$. Thus L must be chosen to minimize this quantity.

$\frac{1}{f_o} C_o$ and LM are q-component vectors and so can be represented as points in a q-dimensional Euclidean vector space. Furthermore,

$$LM = \sum_{i=1}^{j} - b_i C_i \tag{2.4-21}$$

and, it can be shown that the $j\{C_i\}$ are linearly independent[23]. Thus LM is a point in the j-dimensional subspace, of the q-dimensional vector space, spanned by the $j\{C_i\}$, and $|\frac{1}{f_o} C_o - LM|$ is the distance from $\frac{1}{f_o} C_o$ to LM. It follows that $|\frac{1}{f_o} C_o - LM|$ is minimum when LM is the point in the j-dimensional subspace at the minimum distance from $\frac{1}{f_o} C_o$. By the Projection Theorem,[21, 23] LM is the orthogonal projection of $\frac{1}{f_o} C_o$ onto the j-dimensional subspace. Thus each vector C_i, for i = 1, 2, ..., j, is orthogonal to the vector $(\frac{1}{f_o} C_o - LM)$, so that

$$(\frac{1}{f_o} C_o - LM)M^T = 0$$

or

$$LMM^T = \frac{1}{f_o} C_o M^T \tag{2.4-22}$$

or

$$L = \frac{1}{f_o} C_o M^T (MM^T)^{-1} \tag{2.4-23}$$

From Eqns. (2.4-18) and (2.4-23)

$$D = \frac{1}{f_o} C_o - \frac{1}{f_o} C_o M^T (MM^T)^{-1} M$$

$$= \frac{1}{f_o} C_o \{I - M^T (MM^T)^{-1} M\} \tag{2.4-24}$$

where I is a q x q identity matrix, and the required q tap-gains of the linear transversal filter in Fig. 2.4-1 are given by the components of the row-vector D in Eqn. (2.4-24).

When an element is incorrectly detected and therefore incorrectly cancelled the probability of error in the detection of the following elements is greatly increased, so that errors tend to occur in bursts.[21,23,31] However, at high signal/noise ratios (Appendix A2) the average error probability in the presence of these error extension effects is typically only two or three times the value with correct cancellation, and this corresponds to a reduction of a fraction of 1 dB in tolerance to additive white Gaussian noise[21].

2.5 Assessment of the Techniques of Channel Equalization

When one or more roots of the z-transform of the sampled impulse response of the channel lie on the unit circle in the z-plane, a linear transversal equalizer with finite number of tap-gains cannot equalize the channel correctly, however, with the non-linear equalizer exact equalization is achieved in every case. In many practical applications the channel can be equalized (at least approximately) by a linear transversal filter of limited length. Under these conditions, from Equation (2.4-1)

$$F(z) = F_1(z) \qquad (2.5\text{-}1)$$

and

$$F_2(z) = 1 \qquad (2.5\text{-}2)$$

Thus $G_1(z)$ in Eqn. (2.4-3) becomes the z-transform for the linear transversal equalizer for the channel. The p tap-gains of this equalizer are given by the first p components of C_o in Eqn. (2.4-17). Also

$$b_o = f_o = 1 \qquad (2.5\text{-}3)$$

in Eqn. (2.4-9). The application of Eqns. (2.5-1) - (2.5-3) to the analysis of previous section leads to the design and performance of the optimum combination of linear and non-linear equalization, in the case where the channel can be correctly equalized by a linear transversal equalizer.

If such a channel is now equalized by a linear transversal filter with z-transform $G_1(z)$, the z-transform of the i^{th} received signal-element at the output of the linear equalizer is

$$s_i z^{-i} F(z)\, G_1(z) \;\simeq\; s_i z^{-i-h} \tag{2.5-4}$$

from Eqns. (2.4-4) and (2.5-1), so that the i^{th} signal-element is detected from the sample value x_{i+h} at the output of the linear equalizer at time $t = (i+h)T$, by applying the appropriate thresholds. Now

$$x_{i+h} = s_i + v_{i+h} \tag{2.5-5}$$

where v_{i+h} is a sample value of a Gaussian random variable with zero mean and variance

$$\varepsilon^2 = \sum_{i=0}^{p-1} c_i^2 \sigma^2$$

$$= \sigma^2 C_o C_o^T \tag{2.5-6}$$

from Eqn. (2.4-3).

Thus the advantage in tolerance to additive Gaussian noise of the optimum combination of linear and non-linear equalization over the linear equalizer is approximately

$$10 \log_{10} \frac{\varepsilon^2}{\eta^2} = 10 \log_{10} \frac{C_o C_o^T}{DD^T} \tag{2.5-7}$$

expressed in dB, from Eqns. (2.5-6) and (2.4-14). Error extension effects are neglected here.

Where the channel can be equalized linearly, the non-linear equalizer usually gains an advantage in tolerance to additive Gaussian noise over the corresponding linear equalizer. This is so because the non-linear equalizer makes more effective use of the available prior knowledge of the received signal.

Although any channel with a finite impulse response can be equalized by the appropriate equalizer, for certain values of the channel sampled impulse-response, particular sequences of the transmitted element values result in no signal at the output of the receiver filter. No amount of linear or non-linear equalization can give correct operation for the prolonged transmission of such sequences. Consider, for example, the channel impulse response having the z-transform

$$1 + z^{-1} \tag{2.5-8}$$

If the sequence of the element values is such that

$$s_{i+1} = -s_i \text{ for all } i \tag{2.5-9}$$

then the sample values $\{r_i\}$ of the received signal at the output of the receiver filter will be zero, except for the first and the last sample values. The unique detection of such a signal cannot normally be achieved in practice.

An important feature of non-linear equalization by signal cancellation is that only a portion of a received signal-element is used in the detection of that element, the remaining part of the element being removed by subtraction. Clearly, if the whole of the element could be used effectively in its detection, an even better tolerance to additive white Gaussian noise should be obtained. Thus the tolerance to additive Gaussian noise of the non-linear equalizer is often well below the maximum obtainable.[21,34,35]

All the disadvantages of linear and non-linear equalization, mentioned above, can be overcome through the transmission of orthogonal groups of signal-elements.

3.0 THE TWO BASIC DETECTION PROCESSES FOR ORTHOGONAL GROUPS OF SIGNAL-ELEMENTS

3.1 Basic Assumptions

Two groups of signal-elements can be considered to be orthogonal when each of them gives no response in an optimum detection process on the other, that is when they are disjoint in time. The data-transmission system discussed in Section 2.1 is now modified as follows. Following a group of 'm' impulses, at the input to the transmitter filter, the next 'g' impulses are set to zero, so that adjacent groups of m transmitted signal-elements are separated by g zero-level elements. The zero-level elements form gaps (time guard bands) between adjacent groups of transmitted signal-elements and so prevent intersymbol interference between the corresponding groups of elements at the receiver input. Let

$$n = m + g.$$

Since there is no intersymbol interference between different groups of elements at the detector input, for each received group of m elements there are n sample values which are dependent only on the m elements and independent of all other elements. The detector uses these n values in the detection of the m elements.

The detector, of course, requires to know which are the first, second, third, etc. samples in each of the consecutive groups of n samples, and this knowledge must be derived from a suitable training signal sent at the start of each transmission. This is exactly the same requirement as that where the data is sent in characters or words, as for instance in the transmission of alpha-numeric data in binary coded form or else in the transmission of digital data which is coded in an error detecting block code.[32,34,54,55] Once the correct word or block synchronization has been achieved at the start of transmission, it will normally be maintained even in the presence of considerable noise.[34,54] If a slip in synchronization should occur, techniques are available for detecting and correcting it.[54] Techniques for maintaining the correct word synchronization are beyond the scope of this investigation and will not be considered further here.

While one store holds the n sample values for a detection process, another store is receiving the next n sample values, so that nT seconds are available for a detection process. <u>In the detection process, the m elements of a group are detected simultaneously by operating on the corresponding n sample values.</u> Each group of m k-level signal-elements is, in effect, treated as a single element having k^m possible values, that is, as a k^m-level element.

If only the i^{th} signal-element in a group is transmitted, in the absence of noise and with s_i set to unity, the corresponding received n sample values used for the detection of m elements are given by the n-component row-vector

$$Y_i = \overbrace{0 \ . \ . \ . \ 0}^{i-1} \quad \overbrace{y_0 \ y_1 \ . \ . \ . \ y_g}^{g+1} \quad \overbrace{0 \ . \ . \ . \ 0}^{m-i} \qquad (3.1\text{-}1)$$

where y_h must be non-zero for at least one h in the range 0 to g, but it need not of course be non-zero for all h in this range. The row-vector $(y_o, y_1, . . . , y_g)$ is the sampled impulse response of the baseband channel. If there is to be no intersymbol interference between adjacent groups of signal-elements, the non-zero components of the sampled impulse response of the channel must not be spread over more than g+1 consecutive components.

The i^{th} received signal-element is clearly s_iY_i. If all the non-zero components of Y_i are shifted (i-1) places to the left, the vector Y_1 is obtained, so that each Y_i is obtained from every other by a simple time shift of the non-zero components.

The sum of the m received signal-elements in a group, in the absence of noise, is

$$\sum_{i=1}^{m} s_i Y_i = SY, \qquad (3.1\text{-}2)$$

where S is an m-component row-vector whose i^{th} component is s_i and Y is an m x n matrix whose i^{th} row is Y_i given by Equation (3.1-1). Thus

$$Y = \begin{bmatrix} y_o & y_1 & y_2 \cdot\cdot\cdot & y_g & 0 & 0 \cdot\cdot\cdot & 0 \\ 0 & y_o & y_1 \cdot\cdot\cdot & y_{g-1} & y_g & 0 \cdot\cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \cdot\cdot & 0 & y_o & y_1 & \cdot\cdot & y_{g-1} y_g \end{bmatrix} \qquad (3.1\text{-}3)$$

It can be seen that if y_h is the first non-zero component of Y_i, the m x m matrix, formed by the appropriate m adjacent columns of Y such that all the components along its main diagonal are equal to Y_h, is always an upper triangular matrix with non-zero diagonal components, and, therefore, has a non-vanishing determinant of order m. Thus the matrix Y has rank m which means that the row-vectors Y_i, for i=1, . . . , m given by Eqn. (3.1-1), are linearly independent.

Assume for convenience that a received group of m signal-elements is sampled at the time instants T, 2T, ..., nT, so that the sample value of the received signal, at time t = iT, is r_i. Let R be the n-component row-vector whose i^{th} component is r_i, and let W be the n-component row-vector whose i^{th} component is w_i. Then from Eqn. (3.1-2)

$$R = SY + W \qquad (3.1\text{-}4)$$

The vectors R, SY and W may be represented as points in an n-dimensional Euclidean vector space (signal space). Since the $\{w_i\}$, the components of W, are sample values of statistically independent Gaussian random variable with zero mean and variance σ^2, it is shown in Appendix A1, that the value of the orthogonal projection of W onto any given direction in the vector space is a sample value of a Gaussian random variable with zero mean and variance σ^2. It follows that W is equally likely to have any direction in the vector space. Thus for a given vector S, the received vector R in Eqn. (3.1-4), can lie anywhere in the n-dimensional vector space. It is furthermore shown in Appendix A1 that the probability density of the noise vector W is a function only of the length of W and always decreases with the increase in the length of W. Thus the probability of an error in the detection of S from R is minimized by selecting the vector S that minimizes the distance between R and SY. This is illustrated in Fig. 3.1-1 where it is assumed that the detector has the prior knowledge that $S = S_1$ or S_2. Since $R = SY + W$, the noise vectors corresponding to S_1 and S_2 are W_1 and W_2, respectively. Since W_1 is much shorter than W_2, it is clear that $S = S_1$ is more likely to be correct than $S = S_2$.

When the detector knows the matrix Y but in the initial stage of the detection process has no prior knowledge of S_1 or S_2, it knows that SY must lie in the 2-dimensional subspace spanned by Y_1 and Y_2, but it cannot say, before the detection process, that any one vector SY (for <u>any</u> real value of S) is more likely to be correct than any other. The best it can now do is to accept as the initial detected vector S, the vector

$$X = x_1 \; x_2$$

which is selected from the infinite set of all real 2-component row-vectors, such that XY is at the minimum distance from R. This again corresponds to the smallest length of the noise vector W consistent with the available prior knowledge of SY.

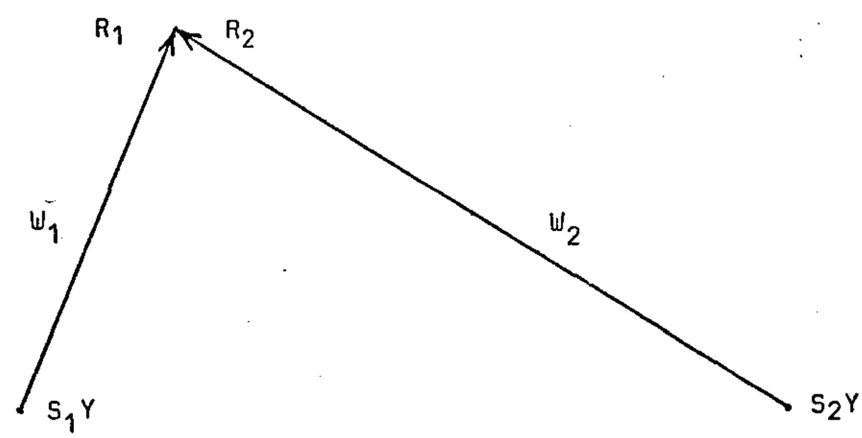


Figure 3.1-1

n-dimensional Euclidean vector space containing the vectors R_1, R_2, W_1, W_2, S_1Y and S_2Y.

In the final stage of the detection process for S, the detector determines S from X. In order to do this the detector must of course have some prior knowledge of the possible values of the vector S. It will be assumed that the detector has exact prior knowledge of the possible values of S in this part of the detection process.

In all detection processes studied here, the detector operates on the received vector R to obtain the detected value of S. In every case it has an exact prior knowledge of the sampled impulse response of the channel in Eqn. (2.1-19).

3.2 The Process of Linear Equalization

This is the optimum linear estimation process for the $m\{s_i\}$, for the case where the detector has prior knowledge of the $m\{Y_i\}$ but has no prior knowledge of the $m\{s_i\}$ or the noise variance.

Consider the n-dimensional Euclidean vector space containing the received vector R. Since the detector knows the $m\{Y_i\}$ it knows the m-dimensional subspace spanned by the $m\{Y_i\}$ and this is of course the m-dimensional subspace containing the vector SY. Since the receiver has no prior knowledge of the $\{S_i\}$, it must assume that any value of S is as likely to be correct as any other. Let the m-component vector

$$Z = z_1 \; z_2 \; \dots \; z_m \tag{3.2-1}$$

be the linear estimate of the vector S. Before the estimation process the detector has no prior knowledge of Z and therefore as far as the detector is concerned ZY is equally likely to lie at any point in the m-dimensional subspace spanned by $\{Y_i\}$. The problem is to determine the best estimate which the receiver can make of S, given the received vector R and using its prior knowledge of Y. This estimate should be such as to maximize P(Z/R), the posterior probability density of Z given R. By Bayes Theorem[50,56]

$$P(Z/R) = \frac{P(Z)}{P(R)} P(R/Z) \qquad (3.2\text{-}2)$$

where P(R/Z) is the conditional probability density of R given Z, and P(Z) and P(R) are, respectively, the probability densities of Z and R. Let

$$ZY = H = h_1 h_2 \ldots h_n \qquad (3.2\text{-}3)$$

Since R is given and, as far as the receiver is concerned, P(Z) is constant for all real values of Z, these being equally likely, the receiver must choose Z to maximize P(R/Z). From Eqns. (3.1-4) and (3.2-3)

$$R = H + W \qquad (3.2\text{-}4)$$

so that

$$r_i = h_i + w_i \qquad (3.2\text{-}5)$$

for i = 1, 2, ..., n, where r_i, h_i and w_i are the i^{th} components of R, H and W, respectively.

Since $\{w_i\}$ are sample values of statistically independent Gaussian random variables with zero mean and variance σ^2, it follows that r_i in Eqn. (3.2-5) is a sample value of a Gaussian random variable with mean value h_i and variance σ^2, and furthermore the $\{r_i\}$ are statistically independent. Hence from Eqn. (3.2-3)

$$\begin{aligned} P(R/Z) &= P(R/H) \\ &= P(r_1, r_2, \ldots, r_n/h_1, h_2, \ldots h_n) \\ &= P(r_1/h_1).P(r_2/h_2) \ldots P(r_n/h_n) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \left[\frac{(r_i - h_i)^2}{2\sigma^2}\right] \\ &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp - \left[\frac{1}{2\sigma^2} \sum_{i=1}^{n} (r_i - h_i)^2\right] \\ &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp - \left[\frac{1}{2\sigma^2} |R - H|^2\right] \end{aligned} \qquad (3.2\text{-}6)$$

where $|R - H|$ is the length of the vector $(R - H)$, and so is the distance between the vectors R and H. Thus in order to maximize P(R/Z) and hence P(Z/R), the receiver must choose Z to minimize $|R - H|$. In other words, the receiver must choose Z to minimize the distance between R and ZY, where Z may have any real value.

Let XY be the orthogonal projection of R on to the m-dimensional subspace spanned by the $\{Y_i\}$. Referring to Fig. 3.2-1, the square of the distance between R and ZY is

$$(R - ZY)\ (R - ZY)^T$$

$$= \ (R - XY + XY - ZY)\ (R - XY + XY - ZY)^T$$

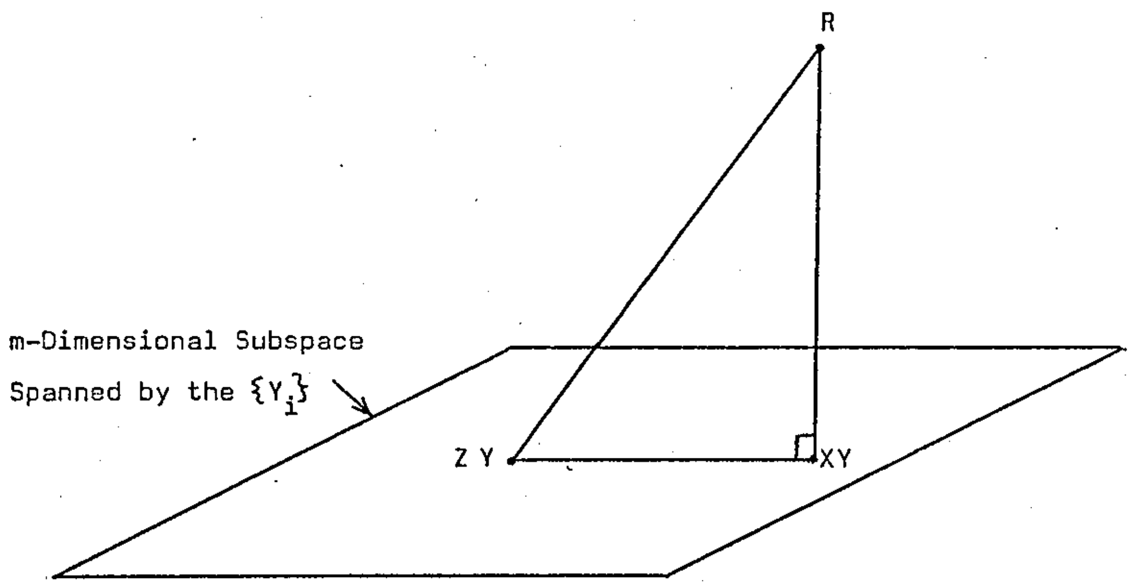
$$= \ (R - XY)(R - XY)^T + (XY - ZY)(XY - ZY)^T \qquad (3.2\text{-}7)$$

since $(R - XY)$ is orthogonal to $(XY - ZY)$.

Now $(R - XY)(R - XY)^T$ is not dependent on Z and $(XY - ZY)(XY - ZY)^T$ is non-negative being the square of the distance between XY and ZY, so that $(R - ZY)(R - ZY)^T$ is minimum when $ZY = XY$ or $Z = X$.

Thus the point in the m-dimensional subspace, spanned by the $\{Y_i\}$, at the minimum distance from R, is the orthogonal projection of R onto this subspace. It follows now that P(R/Z) and therefore P(Z/R) is maximum when $Z = X$.

If the received signal vector R lies in the m-dimensional subspace, then, clearly the most likely value of ZY is R. In general, because of the noise, R will not lie in this subspace, and in this case the best estimate the detector can make of Z is the m-component row-vector X, whose components may have any real values and are such that XY is the orthogonal projection of R onto the m-dimensional subspace spanned by $\{Y_i\}$.



<u>Figure 3.2-1</u>

Orthogonal projection of R onto the m-dimensional subspace containing ZY.

The smallest subspace of the n-dimensional signal space which contains SY for all values of S is the m-dimensional subspace spanned by the $\{Y_i\}$. Since $R = SY + W$, if R is projected onto a subspace which is orthogonal to any of the $\{Y_i\}$, then certain sets of the possible vectors S will give a zero signal component in the projection of R onto this subspace. Such a projection of R, therefore, cannot be used to estimate SY and hence S. It follows that R cannot be projected onto a subspace of dimensionality less than m. Thus the estimate of S that has the greatest a posteriori probability of being correct is obtained by projecting R onto the m-dimensional subspace spanned by the $\{Y_i\}$.

Since $R - XY$ is orthogonal to each of the $\{Y_i\}$

$$(R - XY)Y^T = 0$$

or $$XYY^T = RY^T \qquad (3.2\text{-}8)$$

or $$X = RY^T(YY^T)^{-1} \qquad (3.2\text{-}9)$$

It follows that if the received vector R is fed to the input terminals of the linear network represented by the n x m matrix $Y^T(YY^T)^{-1}$ in Eqn. (3.2-9), then the signals at the m output terminals are the components $\{x_i\}$ of the vector X, where X is the best linear estimate the detector can make of S, under the assumed conditions.

From Eqns. (3.1-4) and (3.2-9)

$$X = RY^T(YY^T)^{-1} = (SY + W)Y^T(YY^T)^{-1}$$

$$= S\,YY^T(YY^T)^{-1} + WY^T(YY^T)^{-1}$$

or $$X = S + U \qquad (3.2\text{-}10)$$

where $$U = WY^T(YY^T)^{-1} \qquad (3.2\text{-}11)$$

The m-component row-vector U is the noise vector at the output of the network $Y^T(YY^T)^{-1}$. The matrix $Y^T(YY^T)^{-1}$ is of course an nxm matrix of rank m. Each component u_i of the noise-vector U is a sample value of a Gaussian random variable with zero mean and a variance which is not normally equal to σ^2 and which may differ from one component to another.

Having obtained X, each transmitted element value s_i can now be detected by comparing the corresponding x_i with the appropriate threshold levels.[57] Since the transmitted signal-elements in a group are statistically independent and equally likely to have any of the possible values, the threshold levels used in the detection of s_i from x_i lie half-way between the adjacent possible values of s_i.[57] Each s_i is detected as its possible transmitted value between the same threshold levels as the corresponding x_i. It may readily be shown that these threshold levels minimize the error probability in the detection of each s_i.[57]

The n x m network $Y^T(YY^T)^{-1}$ performs a process of exact linear equalization on the received vector R to eliminate all inter-symbol interference between the m signals at its m outputs, that is each signal x_i is given by

$$x_i = s_i + u_i \tag{3.2-12}$$

and therefore depends only on the corresponding s_i, together with the noise component u_i, being independent of the remaining element values.

The n x m network $Y^T(YY^T)^{-1}$ could be implemented by the n x m network Y^T feeding the m x m network $(YY^T)^{-1}$ as shown in Fig. 3.2-2. Although more costly to implement in practice[32], this arrangement demonstrates more clearly the nature of the linear transformation performed on R by $Y^T(YY^T)^{-1}$. Now, RY^T is an m-component vector whose i^{th} component is RY_i^T. Let

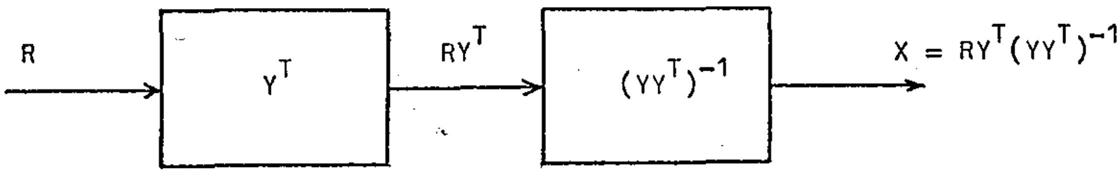


Figure 3.2-2

Expanded form of the linear network $Y^T(YY^T)^{-1}$.

$$Y_i = y_{i1}\ y_{i2}\ \cdot\cdot\cdot\ y_{in} \qquad (3.2\text{-}13)$$

then
$$RY_i^T = \sum_{j=1}^{n} r_j\, y_{ij} \qquad (3.2\text{-}14)$$

RY_i^T is the inner product of the n-component vectors R and Y_i. Furthermore, RY_i^T is the output signal obtained when R is fed to a correlation detector matched to Y_i, since this correlation detector performs the operation described by Eqn. (3.2-14). Evidently the linear network Y^T is a set of m correlation detectors matched to the m vectors $\{Y_i\}$. Each of these performs a process of matched filter detection on the corresponding received signal-element s_iY_i.

The network Y^T is matched to the signal SY and performs a process of matched filter detection on the received vector R. The output signal vector RY^T from the network Y^T uniquely determines the vector XY which, if fed to the input of Y^T, would give this output signal. Thus

$$XYY^T = RY^T \qquad (3.2\text{-}15)$$

Since XY lies in the m-dimensional subspace spanned by the $\{Y_i\}$ and since (3.2-15) is the same as (3.2-8), it follows that XY in (3.2-15) is the orthogonal projection of R onto this subspace.

The network $(YY^T)^{-1}$ transforms the vector RY^T to the vector

$$RY^T(YY^T)^{-1} = XYY^T(YY^T)^{-1} = X \qquad (3.2\text{-}16)$$

so that $(YY^T)^{-1}$ is clearly an inverse network, which reverses the transformation by means of which X has been converted to XYY^T. Clearly, Y^T is equivalent to a matched filter and $(YY^T)^{-1}$ to an inverse filter.

The wanted component in the output signal RY_i^T from the i^{th} correlation detector, in the expanded form of the linear network $Y^T(YY^T)^{-1}$, is $s_iY_iY_i^T$. The correlation detector maximizes the ratio of the energy level of this signal to the average energy level of the noise component WY_i^T in RY_i^T. However, the latter signal also contains m-1 components $\{s_jY_jY_i^T\}$ due to the other components $\{s_j\}$ $j \neq i$, of the m-component row-vector S. The inverse network $(YY^T)^{-1}$ processes the $\{RY_i^T\}$ to eliminate all intersymbol interference, and suitably adjusts the levels of the resultant signals to give the $\{x_i\}$ at its output terminals.

3.3 The Optimum Detection Process

Consider now the optimum detection process for the case where the detector has prior knowledge both of the $\{Y_i\}$ and of the $\{|s_i|\}$. The detector here knows the k^m different possible values of SY, where k is the number of possible levels of the transmitted signal-elements.

It has been shown that where the transmitted signal-elements are statistically independent and equally likely to have any of the possible values, the detector which minimizes the probability of error (that is the probability of one or more element errors) in the detection of m elements of a group, is the detector that determines which of the k^m vectors {SY} is at the minimum distance from the received vector R, in the n-dimension Euclidean vector-space containing R.[34,45,52,57] The detector knows now the exact position of each SY in the vector space. At high signal to noise ratios, this detection process also minimizes the probability of error in the detection of any one of the m elements in a group.

The detection process cannot be implemented by a linear network followed by the appropriate decision thresholds, but is best performed by an iterative process. The receiver generates in turn the vectors {SY} corresponding to the different combinations of the k-level signal-elements in a group. Each vector SY is subtracted from the received vector R. The components of R are stored throughout the detection process for a group of m signal-elements. The components of the difference vector are squared and added, to give the square of the distance between the vector R and the generated vector SY. In the first subtraction process, the distance measure together with the associated vector S are stored. In subsequent subtraction processes no action is taken, unless the distance measure is smaller than that stored. When this occurs, the new distance measure together with the associated vector S replace those stored. Thus at the end of the detection process, the receiver has the vector S which minimizes the distance from SY to R and takes this vector S to give the detected values of the m signal-elements in the received group. Since the separate operations in the detection process are carried out sequentially, these can be performed by a simple piece of equipment.

For any set of signal-elements in a group at high signal to noise ratios (Appendix A2), this detection process achieves a tolerance to additive white Gaussian noise as good as or better than either linear or non-linear equalization, since of course it is the optimum detection process under the assumed conditions. In the particular case where the signal distortion is pure phase distortion it is well known that a linear equalizer achieves the optimum detection of the received signals since it is now also a matched detector, that is, it not only eliminates intersymbol interference but it is also matched to the received signal.[21,22,23] Under these conditions the optimum detector achieves no advantage over a simple linear equalizer. However, pure phase distortion is rarely encountered in practice and most pr-actical channels introduce both phase and attenuation distortions.[44,45] For such channels the optimum detector always achieves an advantage in tolerance to additive white Gaussian noise over the corresponding linear equalizer.[21,34] The latter is either a simple transversal filter or a more complex linear network as described in Sections 2.3 and 3.2, respectively. The reason why the optimum detection process achieves an advantage over the linear equalizer, in tolerance to additive white Gaussian noise, is that it uses more of the available prior knowledge of the received signal. Clearly to achieve the optimum tolerance to additive white Gaussian noise, the detection process must make full use of the available prior knowledge of the received signal.

3.4 Decision Boundaries in Signal Space

For m k-level signal-elements in a group the vector SY has k^m possible values and lies in the subspace spanned by the $m\{Y_i\}$. The linear equalization process can be considered to divide the m-dimensional subspace into k^m decision regions, each corresponding to a different one of the k^m possible values of SY, and the process then determines which of these regions contains the vector XY. The value of S corresponding to this region is the detected value of S.

For the sake of convenience and clarity it will be assumed initially in the following discussion that the transmitted signal-elements are binary antipodal having the possible values $\pm$ 1.

The decision regions in the m-dimensional subspace are defined by m decision boundaries. The i^{th} boundary is the locus of all points traced out by

$$\sum_{j=1}^{m} v_j Y_j \qquad (3.4\text{-}1)$$

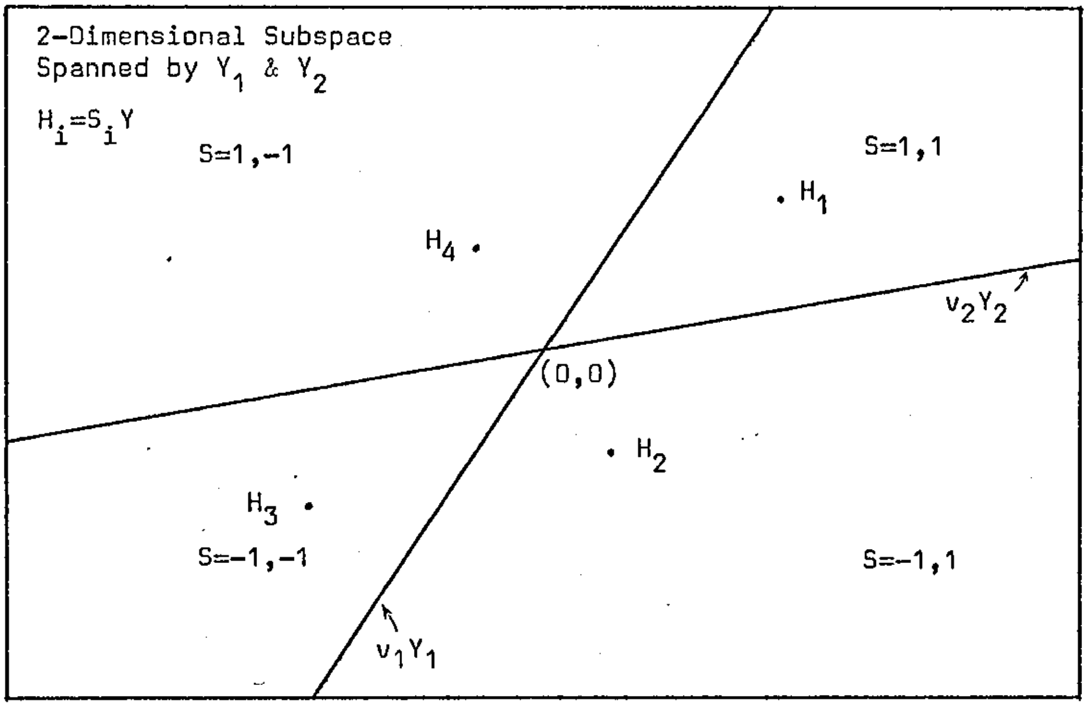
where the $\{v_j\}$ are real scalar quantities such that $v_i = 0$ and the $\{v_j\}$, for $j \neq i$, may vary independently over all real values. The i^{th} decision boundary is therefore traced out by VB_i for all real values of the (m - 1) - component row-vector

$$V = v_1 \ldots v_{i-1} v_{i+1} \ldots v_m,$$

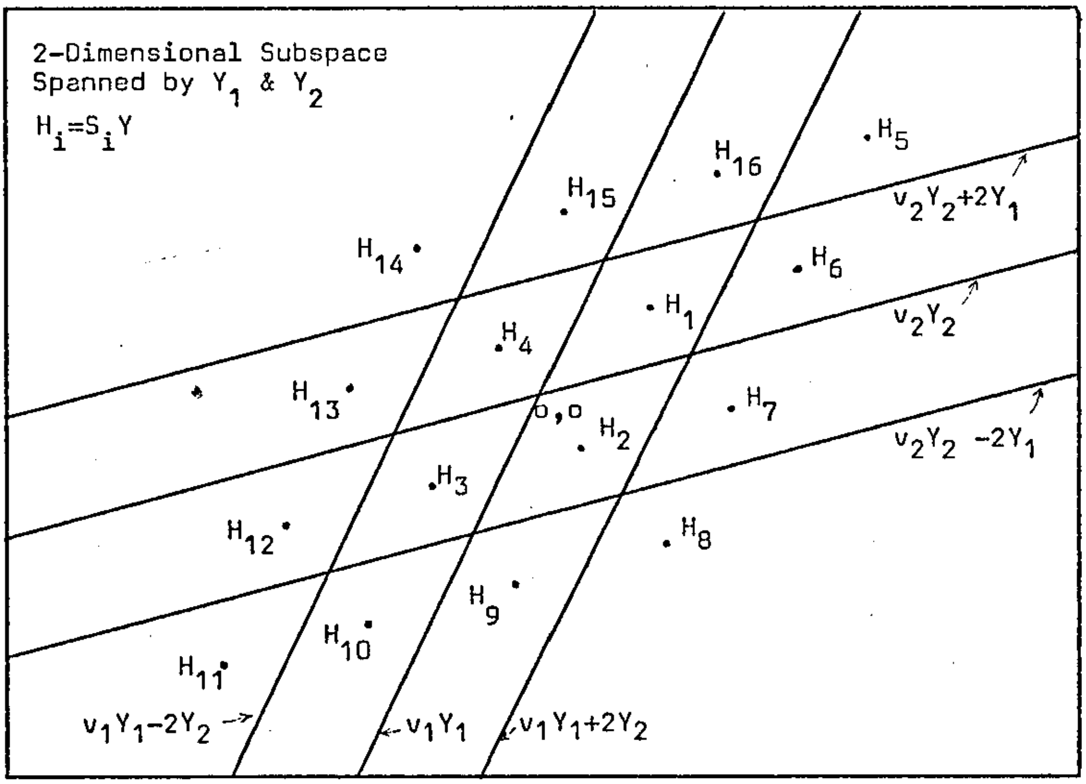
where B_i is the (m - 1) x n matrix obtained by deleting the i^{th} row from Y. The boundary is clearly the (m-1)-dimensional subspace spanned by the (m-1) vectors $\{Y_j\}$ for which $j \neq i$. It divides the m-dimensional subspace spanned by all $\{Y_j\}$ into two regions. If the vector XY lies in one of these, x_i is positive, and if XY lies in the other, x_i is negative. The former will give a detected value of + 1 for s_i and the latter a detected value of -1.

It is difficult to visualize decision boundaries for m greater than two. Fig. 3.4-1(a) shows the decision boundaries and decision regions for a 2-dimensional subspace containing the two binary antipodal signal-elements each having the possible values $\pm$ 1. The lines traced out by v_1Y_1 and v_2Y_2 for all real values of v_1 and v_2, intersect at the origin where $v_1 = v_2 = 0$. It can be seen that if the projection of R onto this space is $XY = x_1Y_1 + x_2Y_2$ and if this lies on the line traced out by v_1Y_1, then $x_1 = v_1$ and $x_2 = 0$. For all $\{XY\}$ to the right of this line, $x_2 > 0$, and for all $\{XY\}$ to the left, $x_2 < 0$. Thus, $v_1Y_1$is the decision boundary which separates the decision regions corresponding to the two possible values of s_2. Similarly, v_2Y_2 is the decision boundary which separates the decision regions corresponding to the two values of s_1.

If now the two signal-elements have say 4 levels instead of 2, such that s_1 and s_2 are each equally likely to have one of the four possible values -3, -1, 1 and 3, and are of course statistically independent, then the decision boundaries become as shown in Fig. 3.4-1(b). There are now six decision boundaries, three separating the four possible values of s_1 and three separating the four possible values of s_2. A decision boundary separating two adjacent values of s_1 is the locus of all points traced out by $v_2Y_2 + \ell Y_1$, where v_2 may have any real value



(a)



(b)

<u>Figure 3.4-1</u>

Decision regions and decision boundaries for the optimum process of linear equalization. (a) Groups of two binary elements. (b) Groups of two 4-level elements.

and $\ell = -2,0,2$ depending upon whether the decision boundary separates the values of s_1 which are -3 and -1, -1 and 1 or 1 and 3, respectively. Similarly a decision boundary separating the two adjacent values of s_2 is the locus of all points traced out by $v_1Y_1 + \ell Y_2$ where v_1 may have any real value and $\ell = -2,0,2$ depending upon whether the decision boundary separates the values of s_2 which are -3 and -1, -1 and 1 or 1 and 3, respectively. For $\ell = 0$ the two decision boundaries v_1Y_1 and v_2Y_2 intersect at the origin where $v_1 = v_2 = 0$. It can be seen from Fig. 3.4-1(b) that the 6 decision boundaries divide the 2-dimensional subspace spanned by Y_1 and Y_2, in $16(4^2)$ different decision regions each corresponding to a different one of the 16 possible values of S. If the orthogonal projection of R on to the 2-dimensional subspace is $XY = x_1Y_1 + x_2Y_2$, then the detected value of the transmitted vector S is that corresponding to the decision region in which the vector XY lies.

Consider now the more general case where there are m k-level signal-elements in a group and they are statistically independent and equally likely to have any of the k possible values which are $k_1, k_2 \ldots k_k$. There are now (k - 1) decision boundaries separating the k possible values of s_1, (k - 1) decision boundaries separating the k possible values of s_2, and so on up to s_m. Thus, in all, there are m(k - 1) decision boundaries and these divide the m-dimensional subspace spanned by the $m\{Y_i\}$ into k^m decision regions each corresponding to a different one of the k^m possible values of S. Following the explanation above, for the binary and 4-level signal-elements, the decision boundary separating any two adjacent of the possible values of s_1, is the locus of the points traced out by

$$v_2Y_2 + v_3Y_3 + \ldots + v_mY_m + \ell_1Y_1$$

$$= VB_1 + \ell_1Y_1 \qquad (3.4\text{-}2)$$

where

$$\ell_1 = \frac{k_j + k_{j+1}}{2} \qquad (3.4\text{-}3)$$

B_1 is an $(m - 1) \times n$ matrix of rank $(m - 1)$, obtained by deleting the first row Y_1 from Y, and V is an $(m - 1)$-component row-vector whose components $\{v_i\}$ may have any real values.

For the optimum detection process the decision boundaries are hyperplanes which perpendicularly bisect the lines joining the different vectors {SY} in the n-dimensional vector space containing the received vector R. The distance of any vector to a decision boundary is half the distance between the two vectors separated by this decision boundary. It can be seen that the vector H = SY which is nearest to R is also the vector H nearest to XY (the orthogonal projection of R onto the 2-dimensional subspace).[34,35] Clearly the decision boundaries, for the optimum detection process, in the 2-dimensional subspace spanned by Y_1 and Y_2 and for possible signal-element values of ± 1, are the $(n - 1)$-dimensional hyperplanes which perpendicularly bisect the lines joining the different pairs of the four vectors $H_i = S_iY$, where S_i are the four possible values of the corresponding vectors at the transmitter. These decision boundaries are shown by the solid lines in Fig. 3.4-2(a). SY and hence S, is detected as the H_i (the value of H) which lies in the same decision region.

If now the two signal-elements have say 4 levels instead of 2, such that s_1 and s_2 are each equally likely to have any of the four values -3, -1, 1 and 3, and are of course statistically independent, then the decision boundaries become as shown in Fig. 3.4-2(b). There are now $16(4^2)$ different vectors H = SY, each corresponding to a different one of the 16 possible values of the transmitted vector S. The different vectors H are shown in Fig. 3.4-2(b). The decision boundaries are, as before, the (n-1)-dimensional hyperplanes that

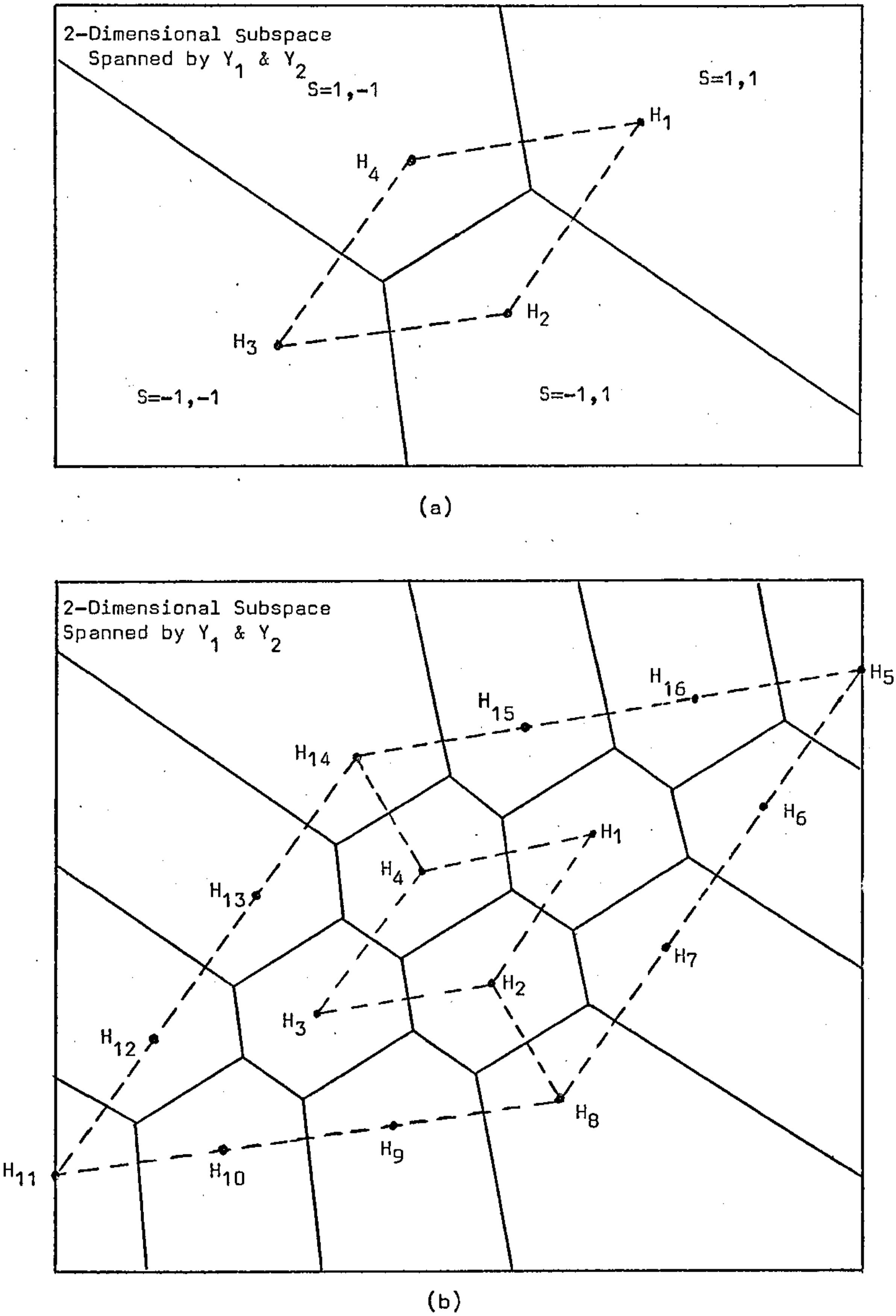


Figure 3.4-2

Decision regions and decision boundaries for the optimum detection process. (a) Groups of two binary elements. (b) Groups of two 4-level elements.

perpendicularly bisect the lines joining the different pairs of the 16 possible vectors H. SY is detected as the value of H which lies in the same decision region as the received vector R.

In the general case where there are m k-level signal-elements in a group, the decision boundary separating any two of the possible vectors {SY}, is now the (n - 1)-dimensional subspace which perpendicularly bisects the line joining the two vectors in the n-dimensional vector space containing the received vector R.[34,35] Consider, for example, two possible vectors $H_b = S_bY$ and $H_c = S_cY$. The decision boundary which separates these two vectors bisects perpendicularly the line $(H_b - H_c)$ joining the two vectors. The perpendicular distance of either vectors H_b or H_c to the decision boundary is

$$d_{bc} = \{\tfrac{1}{2}(H_b - H_c)(H_b - H_c)^T\tfrac{1}{2}\}^{\frac{1}{2}}$$

$$= \{\tfrac{1}{2}(S_b - S_c)YY^T(S_b - S_c)^T\tfrac{1}{2}\}^{\frac{1}{2}} \qquad (3.4\text{-}4)$$

where $\frac{1}{2}(S_b - S_c)$ is an m-component row-vector whose components may have any of the (2k - 1) different values corresponding to the possible values of $s_i - s_j$.

3.5 Probability of Error in a Detection Process

Consider first the process of linear equalization in which the signal-elements are binary coded such that for each i $s_i = \pm 1$. Let the orthogonal projection of SY onto the i^{th} decision boundary be the vector CB_i, where C is an (m - 1)-component row-vector and B_i is the (m - 1) x n matrix obtained from Y by deleting the i^{th} row Y_i. Since the vector $(SY - CB_i)$ is orthogonal to the (m - 1)-dimensional subspace comprising the i^{th} decision boundary, it is also orthogonal to any Y_j where $j \neq i$. Thus,

$$(SY - CB_i)B_i^T = 0 \qquad (3.5\text{-}1)$$

$$\text{or} \qquad CB_iB_i^T = SYB_i^T \qquad (3.5\text{-}2)$$

$$\text{or} \qquad C = SYB_i^T(B_iB_i^T)^{-1} \qquad (3.5\text{-}3)$$

Let d_i be the distance from SY to CB_i. Then

$$d_i^2 = (SY - CB_i)(SY - CB_i)^T$$

$$= (SY - CB_i)Y^TS^T - (SY - CB_i)B_i^TC^T$$

$$= (SY - CB_i)Y^TS^T, \text{ from Eqn. (3.5-1)},$$

$$= (SY - SYB_i^T(B_iB_i^T)^{-1}B_i)Y^TS^T,$$

from Eqn. (3.5-3),

$$\text{or} \quad d_i^2 = SY(I - B_i^T(B_iB_i^T)^{-1}B_i)Y^TS^T \qquad (3.5\text{-}4)$$

where I is an n x n identity matrix.

Let S_i be the (m - 1)-component row-vector obtained by deleting the i^{th} component s_i from S. Then

$$d_i^2 = (s_iY_i + S_iB_i)\{I-B_i^T(B_iB_i^T)^{-1}B_i\}(s_iY_i+S_iB_i)^T$$

$$= \{s_iY_i(I - B_i^T(B_iB_i^T)^{-1}B_i)+S_iB_i - S_iB_i\}(Y_i^Ts_i + B_i^TS_i^T)$$

$$= s_iY_i(I - B_i^T(B_iB_i^T)^{-1}B_i)Y_i^Ts_i$$

$$+ s_iY_i(I - B_i^T(B_iB_i^T)^{-1}B_i)B_i^TS_i^T$$

$$= s_iY_i(I - B_i^T(B_iB_i^T)^{-1}B_i)Y_i^Ts_i + s_iY_i(B_i^TS_i^T - B_i^TS_i^T)$$

$$= s_iY_i(I - B_i^T(B_iB_i^T)^{-1}B_i)Y_i^Ts_i \qquad (3.5\text{-}5)$$

Since $s_i = \pm 1$, from Eqn. (3.5-5)

$$d_i = \{Y_i(I - B_i^T(B_iB_i^T)^{-1}B_i)Y_i^T\}^{\frac{1}{2}} \quad (3.5\text{-}6)$$

Clearly d_i is independent of S and is therefore not a function of the element binary values. It is, however, normally a function of i.

An error occurs in the detection of the i^{th} element when XY and SY lie on opposite sides of the i^{th} decision boundary. Thus the probability of an error in the detection of the i^{th} element is the probability that the orthogonal projection of noise-vector W onto the perpendicular from SY onto the i^{th} decision boundary, is in the direction of the boundary and exceeds the perpendicular distance from SY to the boundary. Since the orthogonal projection of W onto any direction in the m-dimensional subspace spanned by the $\{Y_i\}$, is a Gaussian random vairable with zero mean and variance σ^2, and since d_i is the perpendicular distance from SY to the i^{th} decision boundary, the probability of error in the detection of i^{th} binary element is[34,56]

$$p_i = \int_{d_i}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{w^2}{2\sigma^2}\right) dw$$

$$= \int_{d_i/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) dw \quad (3.5\text{-}7)$$

So that $$p_i = Q\left(\frac{d_i}{\sigma}\right) \quad (3.5\text{-}8)$$

where $$Q(u) = \int_{u}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (3.5\text{-}9)$$

In the general case where d_i is a function of i, different signal-elements in a group of m have different error probabilities. A simple upper bound to the average element error probability is given by the value of p_i for the smallest d_i. Let the minimum value of d_i be d and the corresponding value of p_i be p. Then the average element error probability is less than or equal to

$$p = Q(\bar{d}/\sigma) \tag{3.5-10}$$

Consider now the more general case of multi-level signal-elements where the elements of a group are statistically independent and equally likely to have any of the k different values. In the presence of additive white Gaussian noise it is clear that practically all errors in the detection of the $\{s_i\}$ will involve a transmitted element value being detected as an adjacent value, so that the noise vector W carries R onto the other side of a decision boundary adjacent to the vector SY, where the components of the vector S are the transmitted element values. Suppose that the two adjacent values of s_i are a and a + 2b. s_i is now detected as a when the corresponding $x_i < a + b$ and s_i is detected as a + 2b when $x_i > a + b$, assuming that $x_i > a - b$ and $x_i < a + 3b$. Clearly the decision boundary in the m-dimensional subspace used for the decision as to whether $x_i = a$ or a + 2b, is, from Eqn. (3.4-2), the locus of all points traced out by

$$VB_i + (a + b)Y_i \tag{3.5-11}$$

where V is an (m - 1)-component row-vector whose components may have any real values. The distance from SY to the decision boundary traced out by Eqn. (3.5-11) is clearly the same as the distance of $SY - (a + b)Y_i$ to the hyperplane ((n - 1)-dimensional subspace) VB_i. Let this distance be e. It can be seen from Eqn. (3.5-4) that,

$$e_i^2 = \{SY - (a + b)Y_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\} \times$$

$$\{SY - (a + b)Y_i\}^T$$

Again, let S_i be the vector obtained from S by deleting its i^{th} component s_i, then

$$e_i^2 = \{s_iY_i + S_iB_i - (a+b)Y_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\}$$
$$\{s_iY_i + S_iB_i - (a+b)Y_i\}^T$$
$$= \{(s_i - a - b)Y_i + S_iB_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\}$$
$$\{(s_i - a - b)Y_i + S_iB_i\}^T$$
$$= \{(s_i - a - b)Y_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\}\{(s_i - a - b)Y_i + S_iB_i\}^T$$
$$= \{(s_i - a - b)Y_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\}\{s_i - a - b)Y_i\}^T$$
$$+ \{(s_i - a - b)Y_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\}\{S_iB_i\}^T$$
$$= \{s_i - a - b)Y_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\}\{s_i - a - b)Y_i\}^T$$
$$+ \{(s_i - a - b)Y_i\}\{B_i^TS_i^T - B_i^TS_i^T\}$$
$$= \{(s_i - a - b)Y_i\}\{I - B_i^T(B_iB_i^T)^{-1}B_i\}\{Y_i^T(s_i - a - b)\} \quad (3.5\text{-}12)$$

Since the two values of s_i are assumed to be a and a + 2b,

$$e_i^2 = b^2\{Y_i(I - B_i^T(B_iB_i^T)^{-1}B_i)Y_i^T\} \quad (3.5\text{-}13)$$

and

$$e_i = b\{Y_i(I - B_i^T(B_iB_i^T)^{-1}B_i)Y_i^T\}^{\frac{1}{2}} \quad (3.5\text{-}14)$$

In the particular case where b = 1, Eqn. (3.5-14) reduces to Eqn. (3.5-6). Again, so long as the possible values of each s_i are regularly spaced to give a fixed value of b in Eqn. (3.5-14), as would normally be the case, the distance to the decision boundary is not a function of S. It is, however, normally a function of i.

The probability that the noise vector W carries R into the opposite side of the decision boundary just considered, is

$$p_i = \int_{e_i}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{w^2}{2\sigma^2}\right) dw$$

$$= Q\left(\frac{e_i}{\sigma}\right) \qquad (3.5\text{-}15)$$

When s_i has its most positive or most negative value, there is only one decision boundary at the minimum distance e_i and the error probability here is $Q(e_i/\sigma)$. When s_i has any of its other possible values there are always two decision boundaries at the minimum distance e_i, so that the error probability is now $2Q(e_i/\sigma)$. At high signal/noise ratios the doubling of the error probability corresponds to only a small change in signal/noise ratio (Appendix A2), so that for practical purposes the error probability can be taken as p_i in Eqn. (3.5-15) whatever the value of s_i.

Since e_i in general is a function of i, different signal-elements in a group of m have different error probabilities. A simple upper bound to the average element error probability is given by the value of p_i in Eqn. (3.5-15) for the smallest e_i. Let the minimum value of e_i be e and the corresponding value of p_i be p. Then the average element error probability is less than or equal to

$$p = Q\left(\frac{e}{\sigma}\right) \qquad (3.5\text{-}16)$$

In the case of the optimum detection process the distance to the decision boundary between any two of the k^m possible vectors $H_b = S_bY$ and $H_c = S_cY$ is from Eqn. (3.4-4)

$$d_{bc} = \tfrac{1}{2}\{(S_b - S_c)YY^T(S_b - S_c)^T\}^{\frac{1}{2}} \qquad (3.5\text{-}17)$$

where $\frac{1}{2}(S_b - S_c)$ is an m-component row-vector whose components may have any of the (2k - 1) different values corresponding to the possible values of $s_i - s_j$. $\frac{1}{2}(S_b - S_c)$ has $(2k - 1)^m - 1$ possible values one or more of which will give the minimum distance d to the decision boundaries for all {SY}. In the presence of any significant signal distortion only some of the signal vectors {SY} have decision boundaries at or near the minimum distance d, but some of these vectors may have two or more such decision boundaries. At high signal to noise ratios the reduction in error probability due to the former effect should be approximately offset by theincrease due to the latter.

It must be noted that at high signal to noise ratios with additive white Gaussian noise, even a very small increase in the distance to a decision boundary produces a considerable reduction in the corresponding error probability.[34] Thus the error probability is effectively determined by the nearest decision boundary, the remaining boundaries having in comparison a very small effect on the error probability. Furthermore, if only every second or third possible vectors SY has a decision boundary at a distance equal to the minimum between any value of SY and its associated decision boundaries, or alternatively, if some of the possible vectors SY have two or three decision boundaries at this minimum distance, then in either case the error probability changes by no more than 2 or 3 times, which at high signal to noise ratios represents a change of only a fraction of one dB in the tolerance to the white Gaussian noise, and this can normally be neglected. The distance to the nearest decision boundary for all vectors {SY} is thus a reasonably reliable measure of the tolerance to additive Gaussian noise, so long as the signal to noise ratio remains high.[34]

3.6 A simple Example

A useful comparison between the process of linear equalization and the optimum detection process can be made from a study of a simple case where there are two binary signal-elements in a group.

Suppose that the sampled impulse response of the channel is

$$y_1 \ y_2 \ 0 \ . \ . \ . \ 0 \qquad (3.6\text{-}1)$$

and assume that $m = 2$ and $n = 3$. Also, $s_1 = \pm 1$ and $s_2 = \pm 1$, and these being statistically independent and equally likely to have either binary value.

Now

$$SY = s_1Y_1 + s_2Y_2 \qquad (3.6\text{-}2)$$

So that SY is a point in the 2-dimensional subspace spanned by Y_1 and Y_2. The subspace is shown in Fig. 3.6-1. U_1 and U_2 are the two possible positions of s_1Y_1; V_1 and V_2 are the two possible positions of s_2Y_2. U_1 and V_1 correspond to the positive values of s_1 and s_2, respectively. H_1, H_2, H_3 and H_4 are the four possible positions of SY. The received vector R will not in general lie in the 2-dimensional subspace, but its orthogonal projection onto the subspace is the vector

$$P = XY = x_1Y_1 + x_2Y_2 \qquad (3.6\text{-}3)$$

The linear equalization process detects s_1 and s_2 from the signs of x_1 and x_2.

It can be seen that if P lies any where in the area bounded by AOB in Fig. 3.6-1, $x_1 > 0$ and $x_2 > 0$, so that s_1 and s_2 are both detected as 1 and SY is detected as H_1. Similarly, if P lies in the

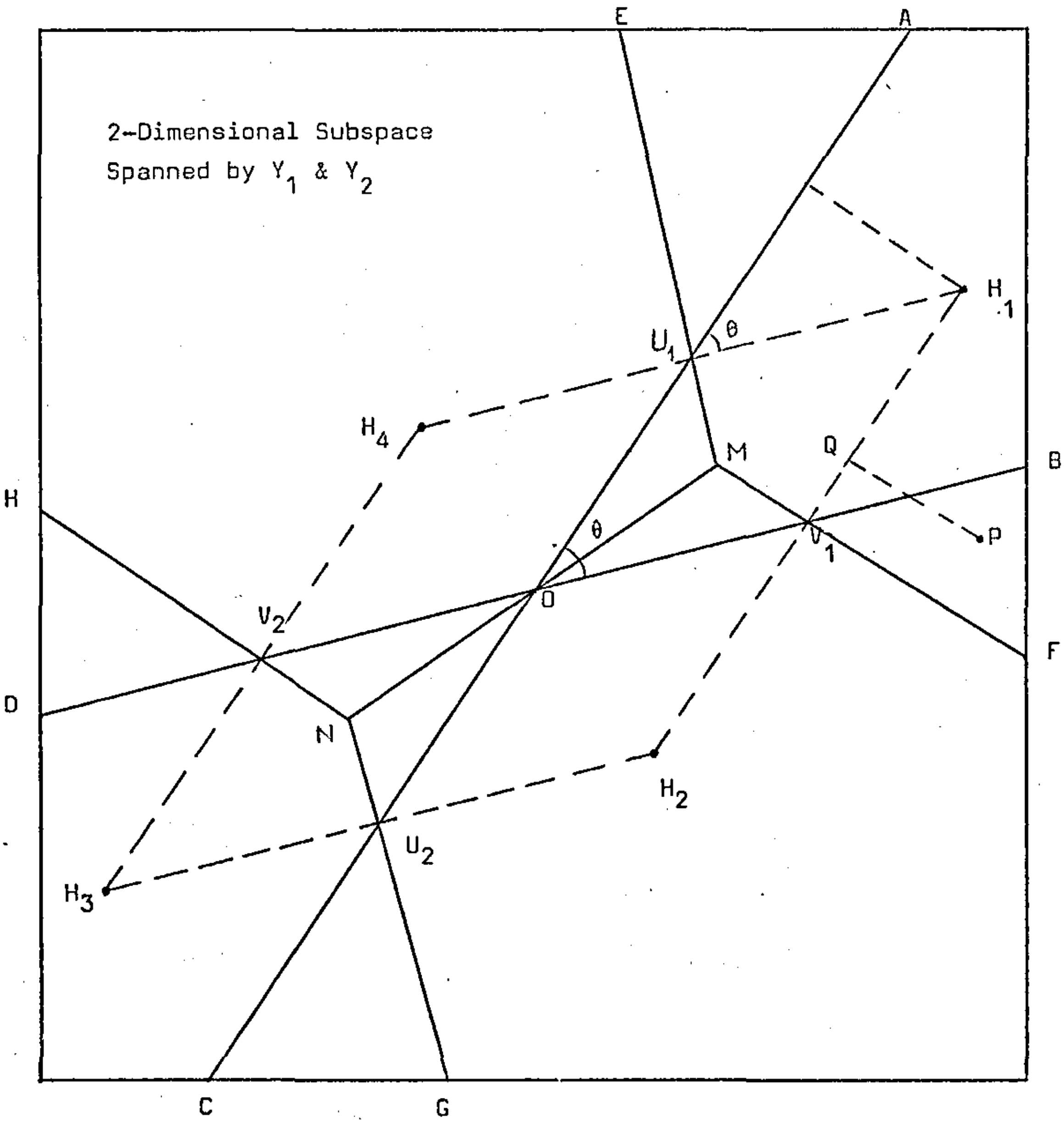


Figure 3.6-1

Decision boundaries for the detection of a group of two binary signal-elements.

area bounded by BOC, COD or DOA, SY is detected as H_2, H_3 or H_4, respectively.

The optimum detection process detects SY as the point H_i at the minimum distance from R and therefore at the minimum distance from P. Thus if P lies in the area bounded by EMF, FMNG, GNH, or HNME, SY is detected as H_1, H_2, H_3 or H_4, respectively. The lines EM, FM, GN, and HN are the perpendicular bisectors of H_4H_1, H_1H_2, H_2H_3 and H_3H_4, respectively.

The vector P is the sum of the received signal-vector H_i = SY and the orthogonal projection of the noise vector W onto the 2-dimensional subspace spanned by Y_1 and Y_2. An error results in the detection of a received signal-element when the projected noise-vector carries P onto the other side of the decision boundary with respect to the received vector H_i. The projected noise vector is equally likely to lie along any direction in the subspace. Furthermore, the probability of its magnitude exceeding a given value depends only on this value and the signal to noise ratio, and decreases rapidly as the value increases. It can be seen in Fig. 3.6-1 that along any direction, from any one of the four $\{H_i\}$, the distance to the nearest decision boundary is in general greater (and never smaller) for the optimum detection process than it is for the process of linear equalization. Thus, for a given signal to noise ratio, the former process has a smaller probability of error than the latter.

When the angle θ in Fig. 3.6-1 is equal to 90^o, that is, when Y_1 and Y_2 are orthogonal, it can be seen that the decision boundaries for the two detection processes are the same, and, therefore, in this case the two processes have the same tolerance to additive Gaussian noise.

3.7 Some Important Conclusions

The prior knowledge of the sampled impulse response of the channel can be represented as a prior knowledge that the received signal vector SY (that is the signal without noise) is confined to the m-dimensional subspace spanned by the m vectors $\{Y_i\}$ in the n-dimensional vector space. The received vector is the n-component vector

$$R = SY + W \qquad (3.7\text{-}1)$$

The m x n matrix Y is known at the receiver and defines the m-dimensional subspace to which SY is confined. In the presence of noise the best linear estimate of S from the received vector R,is obtained by projecting R onto the m-dimensional subspace, followed by a process of matrix inversion. The whole process is achieved by feeding R through the linear network $Y^T(YY^T)^{-1}$ whose m output terminals hold the vector X which is the best linear estimate that can be made of S, under the assumed conditions.

The process of linear equalization discussed in Section 3.2 is equivalent to (but not the same) as the arrangement used with uninterrupted signals. There are however, two advantages here. Firstly, exact equalization is obtained with the n x m network $Y^T(YY^T)^{-1}$, provided that the different received groups of signal-elements are disjoint in time. Since with most practical transmission paths, the impulse response of the channel decays fairly rapidly on each side of the central peak, a good approximation to truly orthogonal groups of elements can be obtained by providing a sufficiently large gap between adjacent groups of transmitted signal-elements. Secondly, for any non-zero sampled impulse response of the channel, SY must be non-zero for all possible values of S. Thus, where there are difficulties in obtaining accurate equalization of the continuous signal or where certain sequences of element values result in excessive attenuation of the received signal, a useful advantage should be gained with the arrangement of orthogonal groups.

The real importance of the arrangement of orthogonal groups, however, is that where the receiver has prior knowledge of the k^m possible values of SY, a considerable advantage in tolerance to additive noise can be obtained by the use of the optimum detection process. Unfortunately, the optimum detection process cannot itself be implemented in practice except when both m and k are small, because of the time required to perform the k^m sequential operations involved in the process. Sub-optimum detection processes must therefore be investigated in the search for a near optimum process, which require far fewer than k^m sequential operations.

4.0 DEVELOPMENT OF THE TWO BASIC DETECTION PROCESSES

4.1 Systems 1 and 2

Consider the process of linear equalization discussed in Section 3.2. The best linear estimate X, of the vector S containing the transmitted signal-element values, is

$$X = R\,Y^T(YY^T)^{-1}$$

This estimate can be obtained by feeding the received vector R to a set of m correlation detectors tuned to the m$\{Y_i\}$, which in turn feed the inverse network $(YY^T)^{-1}$ as shown in figure 4.1-1. Although, the linear n x m network $Y^T(YY^T)^{-1}$ can be implemented as such, much less complex equipment is involved when the transformation $Y^T(YY^T)^{-1}$ is carried out by an iterative process.[32] In such a process the vector X is obtained as a result of a sequence of separate steps, giving successively closer approximation to the required solution.

A large number of different iterative processes are described in the published literature, but the majority of these require considerable equipment complexity and are therefore not suitable for use here.[21,32,34,38,52] There is however one iterative process which is ideally suited to the present application - the point Gauss-Seidel process.[32,38,52] The method of operation of this process will now be described with reference to Fig. 4.1-2.

Let $$D = RY^T \qquad (4.1\text{-}1)$$

and $$A = (YY^T) \qquad (4.1\text{-}2)$$

where D is an m-component row-vector whose components are the m outputs from the correlation detectors $\{Y_i\}$ in Fig. 4.1-1, and A is an m x m real symmetric positive definite matrix.[32,52]

At the start of the detection process, the vector X in Fig. 4.1-2 is set to zero and the received vector R is fed to the input, so that

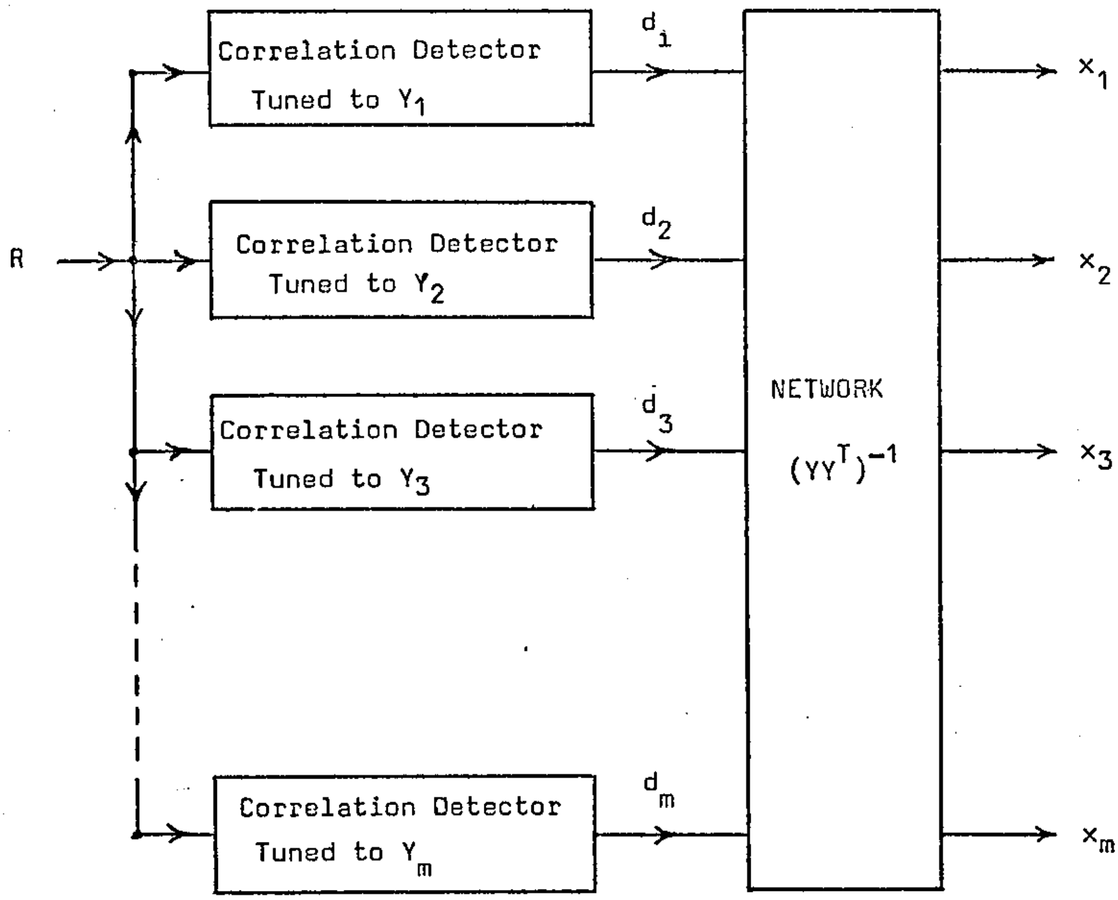
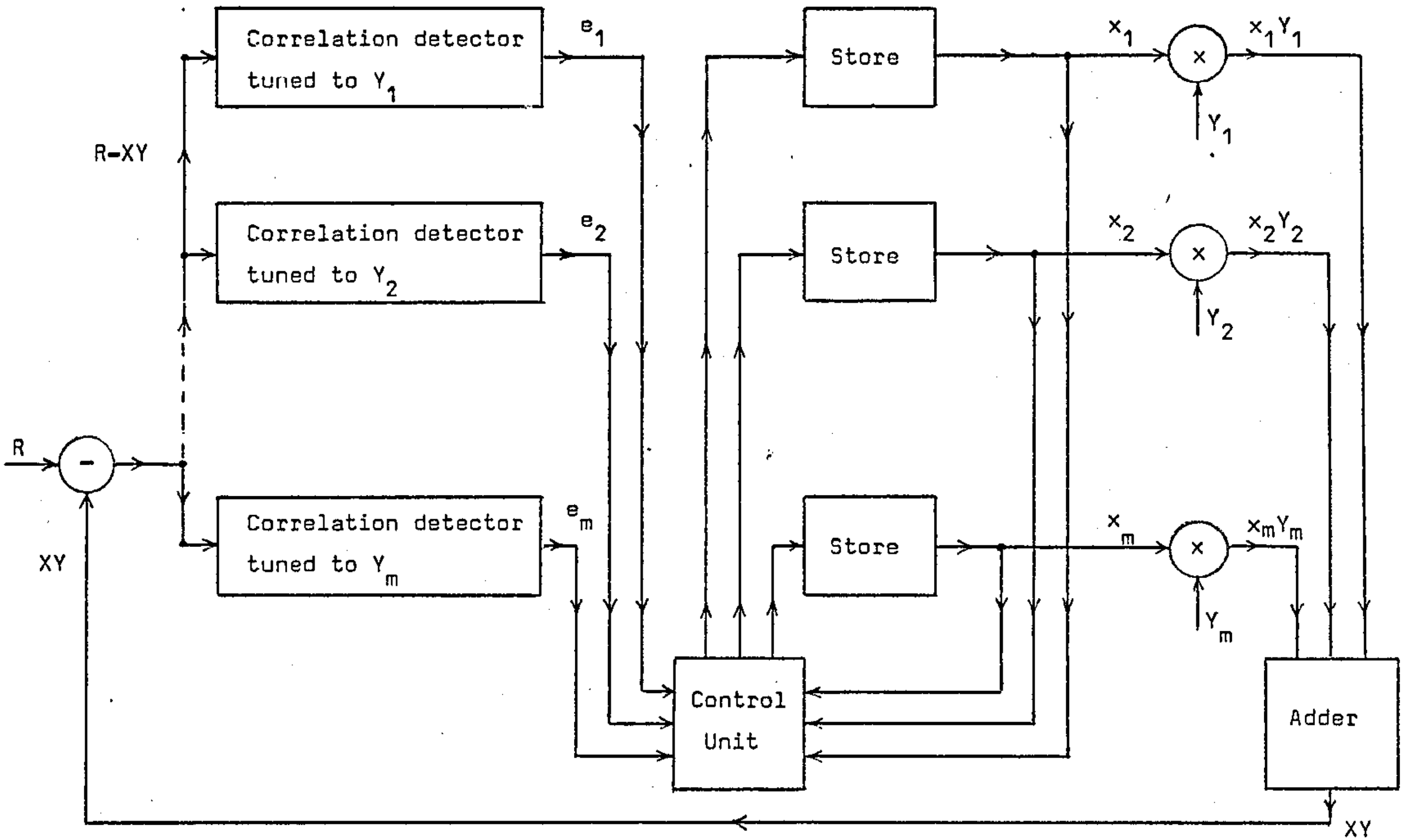


Figure 4.1-1

The process of linear equalization.



<u>Figure 4.1-2</u>

Linear equalization as an iterative process

$$X = 0 \tag{4.1-3}$$

and

$$E = D \tag{4.1-4}$$

where $E = \{e_i\}$ is the output signal vector from the m correlation detectors $\{Y_i^T\}$ during a detection process as shown in Fig. 4.1-2. x_1 is then adjusted so that the output signal e_1 from the first correlation detector is reduced to zero. This in general changes all m output signals $\{e_i\}$ from the correlation detectors. x_2 is now changed so that the output signal from the second correlation detector is reduced to zero, and so on sequentially to x_m, which completes the first cycle of the iterative process. The procedure is then repeated for the second cycle, the $\{x_i\}$ being changed sequentially and in the same order as before, and so on until no further changes in the $\{x_i\}$ are required.

When x_i is adjusted to reduce to zero the output signal from the i^{th} correlation detector, the change in x_i is

$$\Delta x_i = \frac{e_i}{v} \tag{4.1-5}$$

where e_i is the output signal from the i^{th} correlation detector immediately preceding the change and

$$v = Y_i Y_i^T = a_{ii} \text{ for all } i \tag{4.1-6}$$

where a_{ii} is the i^{th} element on the main diagonal of matrix A. Thus, at the end of the process, when all the m $\{x_i\}$ have been adjusted such that

$$E \simeq 0 \tag{4.1-7}$$

X satisfies the relation

$$(R - XY)Y^T = 0$$

or

$$X = RY^T(YY^T)^{-1} \tag{4.1-8}$$

which is the same as that for the process of linear equalization (Eqn. (3.2-9)).

The above iterative process can be modified so that the change in x_i, given by equation (4.1-5), now becomes

$$\Delta x_i = q\frac{e_i}{v} \tag{4.1-9}$$

where q is a constant and

$$0 < q < 2 \tag{4.1-10}$$

The constant q is called the relaxation constant,[38] and Equation (4.1-5) is a special case of (4.1-9).

The above detection process will converge to the required solution vector X, so long as the matrix A is real, symmetric and positive definite and $0 < q < 2$, that is so long as the m $\{Y_i\}$ are linearly independent and $0 < q < 2$.[32,38,52] To obtain the maximum rate of convergence of the iterative process, q should normally have a value equal to or a little greater than 1.[52] A simple implementation of the iterative process is described in reference [32].

Clearly the circuits associated with the correlation detectors in Fig. 4.1-2, perform the same function as the network $(YY^T)^{-1}$ in Fig. 4.1-1, so that the tolerance of the iterative process to the additive Gaussian noise and signal distortion in the channel, should be the same as that of the linear equalization discussed in Section 3.2. The iterative process described above will be referred to as System 1.

The optimum detection process, described in Section 3.3, generates in turn each of the different possible vectors $\{SY\}$ and measures its distance $|R-SY|$ from the received vector R. The detected value of S is that corresponding to the minimum distance. The detection process minimizes the probability of error in the detection of the m elements of a group.[33, 52] It will be referred to as System 2.

4.2 System 3

System 3 is a modification of System 1. The tolerance to additive Gaussian noise of System 1 can be improved by applying the following constraint to the vector X. The values of the $\{x_i\}$ are constrained throughout the iterative detection process so that their values satisfy

$$|x_i| \leq c \quad \text{for each } i \qquad (4.2\text{-}1)$$

where c is the most positive of the possible values of s_i. In the iterative process the constraint overrides and so, if necessary, truncates the change in x_i given by Eqn. (4.1-9).

Consider the example of Section 3.6 and let the vector XY at the end of the iterative process of System 3 be

$$Q = XY = x_1Y_1 + x_2Y_2 \qquad (4.2\text{-}2)$$

Referring to Fig. 3.6-1, P is the orthogonal projection of the received vector R onto the 2-dimensional subspace. When P lies inside the quadrilateral $H_1H_2H_3H_4$, P and Q coincide, whereas when P lies outside $H_1H_2H_3H_4$ then, because of the constraint on the $\{x_i\}$, Q is the point on the quadrilateral at the minimum distance from P, so that Q is the orthogonal projection of P onto the nearest side of $H_1H_2H_3H_4$.[52] x_1 and x_2 now satisfy (4.2-2). If H_1, H_2, H_3 or H_4 is the nearest point on the quadrilateral to P, then this is taken to be the orthogonal projection of P onto the quadrilateral.

If, in Fig. 3.6-1, P lies above EU_1OV_1F, to the right of FV_1OU_2G, below GU_2OV_2H or to the left of HV_2OU_1E, SY is detected as H_1, H_2, H_3 or H_4, respectively. The decision boundaries in this case are a compromise between those for System 1, and those for System 2, so that the tolerance to noise of System 3 should clearly lie somewhere between that of System 1 and that of System 2.

Clark[52] has studied the performance of Systems 1 and 3 in some detail for a more general class of transmitted signals than that assumed here. His computer simulation results suggest that for the type of signals studied, System 3 gains an advantage typically 1 or 2 dB in tolerance to Gaussian noise over that of System 1. Furthermore, the constraint on the $\{x_i\}$, given by Eqn. (4.2-1), not only maintains the convergence of the iterative process but often greatly increases the rate of convergence. The rate of convergence is maximum when q in Equation (4.1-9) is such that $1.25 \lesssim q \lesssim 1.5$.[32,52]

4.3 System 4

In this detection process the detector first generates km vectors $\{Q = XY\}$, each of which is at the minimum distance from R subject to a different set of constraints on X. k is the number of possible element values. For each set of constraints, one of the $m\{x_i\}$ is constrained to have one of the k possible values of s_i (with a different choice of the km possible combinations of the possible values of s_i and i), and the remaining $\{x_i\}$ are estimated using System 3. If P is the orthogonal projection of R onto the m-dimensional space spanned by the $\{Y_i\}$ then the vector Q corresponding to any one set of constraints is the orthogonal projection of P onto the bounded hyperplane specified by these constraints. For each of the vectors $\{X\}$ corresponding to the different vectors $\{Q\}$, the detector determines the vector S at the minimum distance from this vector X, and generates the corresponding vector SY. The detector then determines which of these vectors $\{SY\}$ is at the minimum distance from R, and takes the detected value of S as that giving the minimum distance. The vectors $\{SY\}$ selected in the km iterative processes, are not necessarily all different, since not only can two different vectors $\{Q\}$ give the same selected vector

SY but also the km vectors $\{Q\}$ themselves are not necessarily all different.

In the practical implementation of System 4 a suitably modified arrangement of System 3 selects the vectors $\{SY\}$ in km successive iterative processes, and an arrangement of System 2 then determines the detected vector S from these $\{SY\}$. Consider, for example, the case where the transmitted signal-elements are binary coded, such that $s_i = \pm 1$. The detector uses System 3 to obtain the estimate X of the transmitted signal vector S, with x_1 set at +1. The detected values of the $\{s_i\}$ are then obtained by comparing the $\{x_i\}$ with a threshold level of zero. The detector then generates the vector SY using this detected value of S and subtracts it from the received vector R. The components of the difference vector are squared and added to give the square of the distance between the vector R and the generated vector SY. This distance together with the associated vector S are stored. The whole procedure is then repeated with x_1 set at -1 and again for x_2 set at +1 and then -1 and so on up to x_m. In each iterative process corresponding to a fixed value of one of the $\{x_i\}$, the remaining $\{x_i\}$ are permitted to vary subject to the constraint given by Eqn. (4.2-1). At the end of each iterative process the detector compares the distance just measured with that already stored, and no action is taken unless the former is smaller than the latter. When this occurs, the new distance together with the associated vector S replace those stored. The detector takes the detected value of S as that which remains in the store at the end of the detection process, when all the km iterative processes have been completed.

The method of operation of System 4 can be further clarified by considering the example of Section 3.6. Referring to Fig. 3.6-1, the vectors $\{Q\}$ are here the orthogonal projections of P onto the four lines H_4H_1, H_1H_2, H_2H_3 and H_3H_4, the orthogonal projections onto H_2H_3 and H_3H_4 being H_2 and H_4, respectively. The corresponding vectors $\{SY\}$ are the points $\{H_i\}$ in the same decision regions as the vectors $\{Q\}$, where the decision boundaries are those of System 1. Clearly the selected vectors $\{SY\}$ are H_1, H_2 and H_4. The detected vector H is that member of the selected $\{H_i\}$, which is at the minimum distance from P, and it is therefore H_1.

It can be seen that for the whole detection process of System 4 the decision boundaries in Fig. 3.6-1 are the same as those for System 2, so that for groups of two binary signal-elements, System 4 achieves the same tolerance to additive Gaussian noise as does System 2. Although System 4 is basically more complex than System 2 and in this particular example it requires more sequential operations, the number of sequential operations in System 4 increases very much more slowly with m and k than in System 2, and for larger values of m and k becomes much smaller than in System 2.[35]

The first part of the detection process of System 4 selects from the k^m possible vectors $\{SY\}$ a set of no more than km vectors. The second part of the detection process selects from this set the vector SY at the minimum distance from R. On the other hand, System 2 selects the detected vector SY directly from the whole set of k^m $\{SY\}$.

4.4 Systems 5 and 6

In System 5, the detection process of System 1 is applied to the m received signal-elements of a group, but only the detected value of the first element is accepted. With correct detection, the components (sample values) of the first element s_1Y_1 are known. These are then cancelled (eliminated by subtraction) from the received vector R. The modified vector R now contains sample values due to the remaining (m-1) signal-elements. The detection process of System 1 is now applied to these (m-1) elements to give the detected value of s_2Y_2 which is then cancelled, and so on. In this way each of the m signal-elements of a received group is detected as the first element in its respective group, and, with the correct cancellation of the preceding elements, the error probability in the detection of any element does not drop below the error probability in the detection of the first element s_1Y_1.

The technique applied here is one of non-linear equalization by decision-directed cancellation of intersymbol interference.[15-18,20,21,23,31,34] The incorrect detection of a received signal-element leads of course to the incorrect cancellation of that element. This correspondingly increases its intersymbol interference in the following elements and greatly increases the error probability in their detection. Errors therefore tend to occur in bursts. It must however be borne in mind that since the signal-elements are transmitted in separate groups, a wrong detection and cancellation in any one group of signal-elements does not affect the detection of elements in the following groups. Thus the error bursts [21,31] are contained within the m elements of a group.

System 6 is a modification of System 5, in which System 3 is used in place of System 1 for the detection of each signal-element.

Systems 5 and 6 are applications of a technique studied for a continuous (uninterrupted) stream of signal-elements, where each detected signal-element is cancelled and therefore removed from the received sample values.[21,23,31,34] An important feature of the arrangement of signal detection and cancellation in the case of continuous (uninterrupted) signal is that corresponding to n transmitted signal-elements there are just n sample values of the received signal.

Theoretical analysis has shown that the detection of n elements from the corresponding n sample values reduces the tolerance of the arrangement to additive white Gaussian noise.[21,34] In Systems 5 and 6, since there are n sample values of the received signal corresponding to m elements of a group, where n>m, it follows that the tolerance to additive white Gaussian noise of Systems 5 and 6 is better than that of the corresponding arrangement with continuous (uninterrupted) signal.

4.5 Detection of Multi-Level Signal-Elements

The detection processes described in the previous sections are of interest mainly for the detection of binary signal-elements. They may be applied to multi-level elements but, for useful values of m, they either give an inferior tolerance to noise or else they require an excessive number of sequential operations.

The constraint on the $\{x_i\}$ in System 3 given by Eqn. (4.2-1) becomes less effective with the increase in the number of signal-element levels. This is because, with multi-level signals only a few of the $\{x_i\}$ which correspond to the largest of the possible values of s_i, are affected by this constraint, for the rest of the $\{x_i\}$ the constraint is virtually non-existant. In the limit, therefore, when the number of signal-element levels is very large, the tolerance to Gaussian noise of System 3 will approach that of System 1. Also, for the same reason the tolerance of System 6 to additive Gaussian noise will approach that of System 5 as the number of signal-element levels increases. Again the number of sequential operations required in both Systems 2 and 4 increases rapidly with the number of signal-element levels, and for large values of m, makes them unsuitable for use with multi-level signal-elements.

A promising technique for the detection of multi-level elements is to carry out an initial detection process which selects from the total number of possible values of each signal-element, the two or three element values which are most likely to be correct. The detection of the m signal-elements is then completed by an iterative process which operates only on the selected element values, so that it treats the received signal-elements as though these were the corresponding 2-or 3-level elements. This arrangement often enables a good tolerance to noise to be achieved, without an excessive number of sequential operations. The following detection processes are all based on this technique.

4.6 Systems 7/2, 7/4 and 7/6

In System 7/2 the detection process of System 3 is first applied to give the vector X, as in Eqn. (4.2-2). The value of each x_i is used to determine the nearest two possible values of s_i, to give a set of 2^m likely values of S. The detection process of System 2 is now applied, using only these values of S. In Systems 7/4 and 7/6, the detection process of Systems 4 and 6, respectively, are used for the final detection of S in place of System 2.

4.7 Systems 8/2, 8/4 and 8/6

In System 8/2 the detection process of System 3 is first applied to the received vector R to give an initial (temporary) detected value of the vector S. With each component s_i of S are now associated (where available) the two immediately adjacent of the possible values of s_i. This gives a maximum number of 3^m likely vectors S. The detection process of System 2 is now applied, using only these values of S, to give the final detected value of S. In Systems 8/4 and 8/6, the detection processes of Systems 4 and 6, respectively, are used for the final detection of S, in place of System 2.

The operation of these systems can be explained more simply by considering the example of Section 3.6. The $\{s_i\}$ are now assumed to be 4-level signal-elements instead of binary, with $\pm$ 1 and $\pm$ 3 as the possible four values of s_i. The 2-dimensional subspace spanned by Y_1 and Y_2 is shown in Fig. 4.7-1. Also shown are the decision boundaries and decision regions of Systems 1 and 2, for the 4-level signal-elements considered here. Clearly Fig. 4.7-1 is derived from Figs. 3.4-1(b) and 3.4-2(b). The vectors $\{H_i\}$ where each H_i is one of the 16 possible vectors SY, are shown as points in the 2-dimensional subspace. The values of s_1 and s_2 associated with each H_i are also given. P in Fig. 4.7-1, is the orthogonal projection of R onto the 2-dimensional subspace and is given by

$$P = XY = x_1Y_1 + x_2Y_2 \qquad (4.7\text{-}1)$$

Referring to Fig. 4.7-1 it can be seen that System 1 detects SY as H_1, and hence $s_1 = 1$ and $s_2 = 1$, if P lies anywhere within the area $oa_1a_2a_3o$, while System 2 which is the optimum system, detects SY as H_1 if P lies within the area $b_1b_2b_3b_4b_5b_6b_1$. If P lies in the shaded portions of the area $oa_1a_2a_3o$, System 1 will still detect SY as H_1, but System 2 will now detect SY as either H_2, H_3, H_5 or H_9, depending upon the location of P within the shaded areas. Thus if System 1 detects $s_1 = 1$ and $s_2 = 1$ the possible values of $\{s_i\}$ which can be detected by System 2, are $s_1 = -1$, 1 or 3 and $s_2 = -1$, 1 or 3, and these values include the two immediately adjacent to those detected by System 1. In general, considering all the

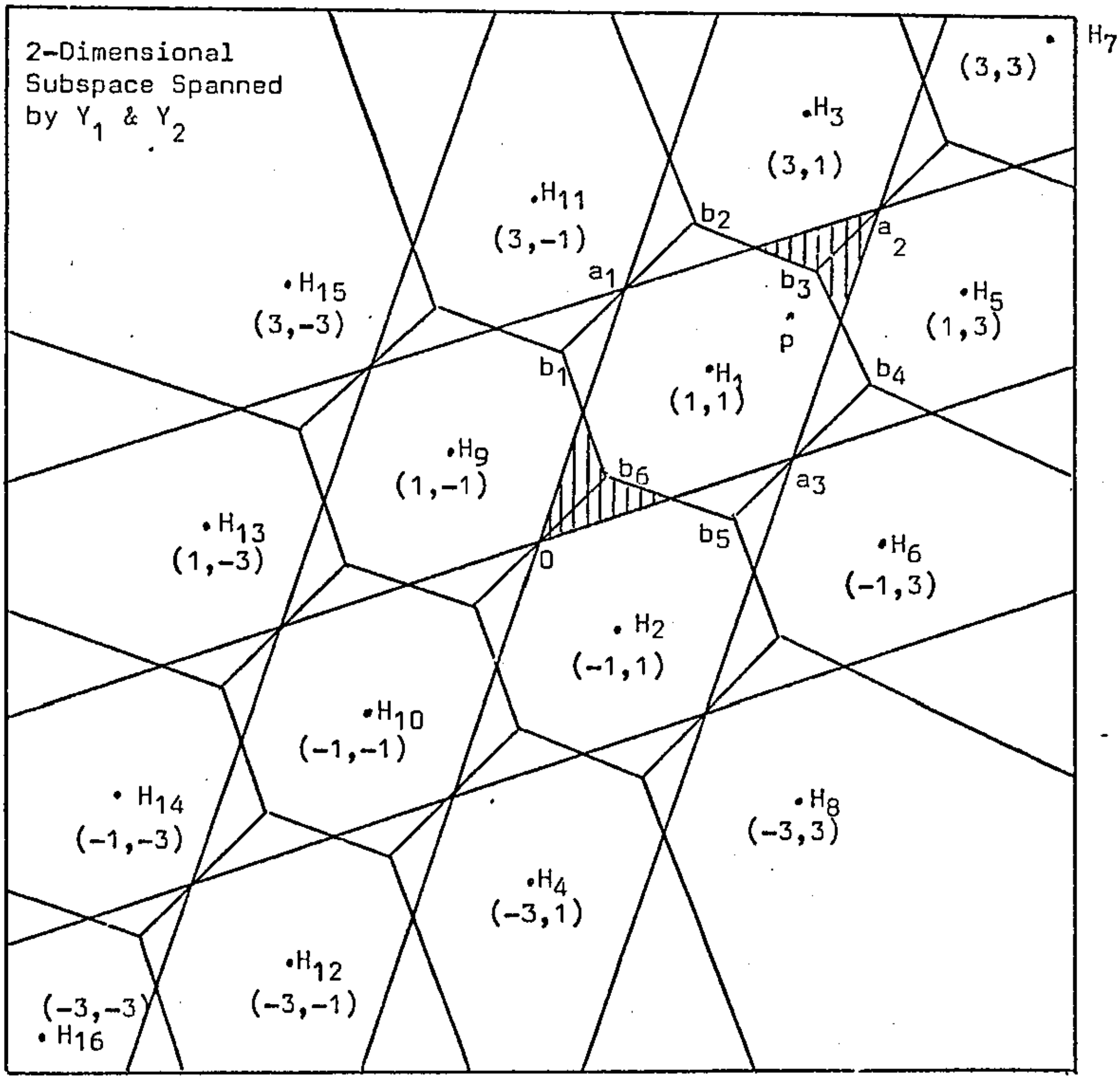


Figure 4.7-1

Decision boundaries of Systems 1 and 2 in 2-dimensional subspace spanned by Y_1 & Y_2. Groups of two 4-level elements.

vectors SY in the 2-dimensional subspace of Fig. 4.7-1, it can be seen that the values of $\{s_i\}$ which can be detected by System 2, given the values detected by System 1, are these values and (where available) the two pairs of values immediately adjacent to those detected by System 1. If now System 2 or 4 is used for the final detection of S, using only the possible values of S selected by using System 1, then, clearly the tolerance to additive Gaussian noise of Systems 8/2 and 8/4 for groups of two 4-level signal-elements, will be similar to that of System 2 and 4, respectively. Evidently this is true also for larger numbers of signal-element levels. Although Systems 8/2, 8/4 and 8/6 use System 3, for the initial detection of $\{s_i\}$ instead of System 1 which has been considered in this example, this does not make much difference because with multi-level signal-elements, the tolerance of System 3 to additive white Gaussian noise is more like that of System 1 as has been mentioned in Section 4.5.

It must be pointed out that before applying System 4 or 6 to the received vector R, for the final detection of S, the constraint on the $\{x_i\}$ given by Eqn. (4.2-1) is modified. If s_i is the initial detected value of the i^{th} element then, the value of the corresponding x_i, during the final detection process of System 4 or 6, is constrained such that

$$s_i' < x_i < s_i'' \text{ for each } i \qquad (4.7\text{-}2)$$

where s_i' and s_i'' are the two values immediately adjacent to s_i (where available), and s_i' is the smaller of the two. Clearly the constraint on the $\{x_i\}$ in Eqn. (4.7-2) is more effective than that given by Eqn. (4.2-1), and therefore System 8/6 should gain some advantage in tolerance to additive Gaussian noise over System 6.

Although Systems 8/2, 8/4 and 8/6 look complicated in theory, their practical implementations are similar to those of Systems 2, 4 and 6, respectively. For each group of received signal-elements, System 3 is first used to detect the initial value of S. With each component s_i are then associated (where available) the two immediately adjacent of the possible values of s_i. The components $\{s_i\}$ of S are now treated as 2- or 3-level signal-elements and the receiver knows these values for each s_i. The vector S is finally detected by using the detection processes of Systems 2, 4 or 6 exactly as described in Sections 4.1, 4.3 and 4.4, respectively. With Systems 4 and 6, however, the constraint given by Eqn. (4.7-2) is now used instead of that given by Eqn. (4.2-1).

Clearly Systems 8/2, 8/4 and 8/6 are modifications of Systems 7/2, 7/4 and 7/6, respectively, which are described in the previous section. It can be seen from Fig. 4.7-1 that at high signal to noise ratios, the two most likely values of s_i in a set of the three possible values, selected after an initial detection of S, are the two values nearest to the corresponding x_i where the $\{x_i\}$ are of course the components of the vector X in Eqn. (4.7-1). Thus at high signal to noise ratios the performance of Systems 7/2, 7/4 and 7/6 for groups of two multi-level signal-elements, can be expected to be similar to that of Systems 8/2, 8/4 and 8/6, respectively.

The implementations of Systems 7/2, 7/4 and 7/6 are similar to those of Systems 8/2, 8/4 and 8/6, respectively. The selected possible values of each s_i are now two, instead of two or three, and are the ones nearest to the corresponding x_i. s_i' and s_i'' in Eqn. (4.7-2) are now the two possible values of s_i. During the initial detection process in any of the systems 7/2, 7/4 and 7/6, $|x_i|$ will never exceed the value max. (s_i), because of the constraint of Eqn. (4.2-1). When x_i has one of the two extreme values of s_i, the two possible values of s_i are taken to be this value and that immediately adjacent to it.

5.0 ASSESSMENT OF THE DETECTION PROCESSES

5.1 Computer Simulation Tests

The relative performances, in the presence of additive white Gaussian noise, of the various detection processes just described have been compared by considering a 2-dimensional subspace. Such an analysis becomes difficult and almost impossible for the higher dimensional subspaces obtained when the number of elements in a group is large. Since, in a practical application, the number of signal-elements in a group is likely to exceed 10 or even 20 with binary or multi-level elements, computer simulation has been used to study the performances of the detection processes in the presence of additive white Gaussian noise.

The tolerances to additive white Gaussian noise of the various detection processes have been compared for the data-transmission system discussed in Section 3.1. The comparison has been made for different values of the sampled impulse response of the baseband channel, for different numbers of elements in a group, and for different numbers of signal-element levels. All the computer simulation programs have been written in FORTRAN IV and run on the ICL 1904 A computer at Loughborough University of Technology.

In every case the energy of an individual transmitted signal-element has been set to unity and the two sided power spectral density σ^2 of the additive white Gaussian noise at the input to the receiver filter is adjusted to obtain a given average element error rate. The value of σ^2 then gives a measure of the tolerance of a system to additive white Gaussian noise. Different error rates have been used for different numbers of signal-element levels in order to allow for the fact that the probability of error in the detection of a k-level signal-element is a function of $\frac{2(k-1)}{k}$ which, although significant at the error rates tested, is not important at the lower error rates usually found in practice. The probability of error in the detection of a multi-level element is considered in more detail in Appendix A8. Thus the results give a better estimate of the relative tolerances to additive white Gaussian noise, at high signal to noise ratios where the error probabilities are no more than 1 in 10^5.

The computer simulation programs are considered in more detail in Appendices A3, A4, A5 and A6.

5.2 Choice of Channel Impulse Response and Transmitted Signals

The sampled impulse response of the baseband channel is specified by a five component row vector L, assuming that an individual signal-element does not cause intersymbol interference in more than four of the neighbouring elements so that $g = 4$. The different vectors tested are shown in Table 5.2-1, each vector is normalized to have unit length. With the exception of the first vector, these are grouped in pairs, each member of a pair in every case causing the same reduction in tolerance to noise as the other. The different values of L have been selected to give a wide range of different signal distortions which include various combinations of amplitude and phase distortions.[33,35,55]

Simulation tests have been performed with 2, 4 and 8-level signal-elements and for both $m = 4$ and $m = 8$. In every case the average transmitted energy per bit is equal to unity. The possible values of signal-elements s_i are shown in Table 5.2-2. The transmitted signal-elements are statistically independent and equally likely to have any of the possible values of s_i. In each simulation test, with

TABLE 5.2-1

Values of the sampled impulse response of the channel

Channel	L
A	(1,0,0,0,0)
B	$2^{-\frac{1}{2}}$(1,1,0,0,0,)
	$2^{-\frac{1}{2}}$(1,-1,0,0,0)
C	$2^{-\frac{1}{2}}$(1,0,1,0,0)
	$2^{-\frac{1}{2}}$(1,0,-1,0,0)
D	$1.5^{-\frac{1}{2}}$(0.5,1,0.5,0,0)
	$1.5^{-\frac{1}{2}}$(-0.5,1,-0.5,0,0)
E	$1.5^{-\frac{1}{2}}$(0.5,1,-0.5,0,0)
	$1.5^{-\frac{1}{2}}$(-0.5,1,0.5,0,0)
F	$1.5^{-\frac{1}{2}}$(1,0.5,0.5,0,0)
	$1.5^{-\frac{1}{2}}$(1,-0.5,0.5,0,0)
G	$1.5^{-\frac{1}{2}}$(1,0.5,-0.5,0,0)
	$1.5^{-\frac{1}{2}}$(1,-0.5,-0.5,0,0)
H	$1.5^{-\frac{1}{2}}$(1,0.667,0.235,0,0)
	$1.5^{-\frac{1}{2}}$(1,-0.667,0.235,0,0)
I	$1.5^{-\frac{1}{2}}$(1,0.667,-0.235,0,0)
	$1.5^{-\frac{1}{2}}$(1,-0.667,-0.235,0,0)
J	$2^{-\frac{1}{2}}$(0.235,0.667,1,0.667,0.235)
	$2^{-\frac{1}{2}}$(0.235,-0.667,1,-0.667,0.235)
K	$2^{-\frac{1}{2}}$(-0.235,0.667,1,0.667,-0.235)
	$2^{-\frac{1}{2}}$(-0.235,-0.667,1,-0.667,-0.235)

TABLE 5.2-2

Possible Values of s_i

No. of Element Levels	Values of s_i
2	± 1
4	$\pm \frac{1}{\sqrt{2.5}}$ and $\pm \frac{3}{\sqrt{2.5}}$
8	$\pm \frac{1}{\sqrt{7}}$, $\pm \frac{3}{\sqrt{7}}$, $\pm \frac{5}{\sqrt{7}}$ and $\pm \frac{7}{\sqrt{7}}$

groups of four and eight binary and groups of four 4-level signal-elements, a total of 4096 signal-elements were transmitted over any given baseband channel. The number of signal-elements transmitted with groups of eight 4-level and groups of four 8-level signal-element was 4000. In cases where the number of signal-elements transmitted was 4096, all possible combinations of the element values in a group were used. However, due to the limitations on the computer time available, it was not possible to do the same with groups of eight 4-level and four 8-level signal-elements, and in these cases the possible values of the signal-elements of a group (Table 5.2-2) were selected such that the elements in a group were statistically independent and equally likely to have any of the possible values.

Systems requiring an unduly large amount of computer time have been tested only with channels A, B, D and J, these channels being the most interesting of those tested. Where a system has been tested over any one of the channels B to K, a computer simulation test has been carried out for each of the two corresponding values of L in Table 5.2-1. Again, because of the limited computer time available, binary and 4-level elements were tested in groups of both four and eight elements, whereas 8-level elements were tested only in groups of four.

5.3 Error Probabilities and Confidence Limits

In the simulation tests, different error probabilities have been used for different numbers of signal-element levels, so that the simulation results give a better estimate of the relative tolerances to additive white Gaussian noise, at high signal to noise ratios, for different numbers of signal-element levels. The values of error probability actually used were 4, 6 and 7 in 10^3 for 2, 4 and 8-level signal-elements, respectively. It has not been possible to test the systems at higher signal to noise ratios (lower error probabilities) because, for a reasonable estimate of the tolerance of a system to noise some 20 or 30 errors must be obtained in a computer simulation test. This implies a very large number of trials when the element error rate is less than 1 in 10^3. The choice of error probabilities and the total number of signal-elements transmitted in each test, is a compromise between the accuracy of the results and the available computer time.

In the case of channel A, the standard deviation σ of the white Gaussian noise at the input of the receiver filter (Fig. 1.1-1), corresponding to a given average element error probability, can of course be derived theoretically. For any baseband channel, the noise variance at the input to the detector is equal to the two sided noise power spectral density σ^2 at the input to the receiver filter. For channel A, when there is no signal distortion, the received signal-elements are orthogonal at the receiver. Furthermore, with binary coded signals, the i^{th} signal-element in a group has only one non-zero component which has the value $\pm$ 1. An error occurs in the detection of this element when the corresponding noise component has a magnitude greater than 1 and the opposite sign to that of the component of the binary element. Hence the probability of error in the detection of the i^{th} element is $Q(1/\sigma)$. For an error probability of 4 in 10^3,

$$Q(1/\sigma) = 4 \times 10^{-3} \qquad (5.3\text{-}1)$$

so that $\sigma = 0.376$, and this is the required value of the standard deviation of the white Gaussian noise at the input to the receiver filter, corresponding to an error probability of 4 in 10^3 over channel A. In the case of 4-level elements transmitted over channel A with an error probability of 6 in 10^3, the standard deviation σ of the noise must satisfy

$$\frac{2(k-1)}{k} Q\left(\frac{d}{\sigma}\right) = 6 \times 10^{-3}$$

as can be seen from Appendix A8, where k is the number of element levels, so that k = 4, and d is the distance to the nearest decision boundary, so that $d = \frac{1}{\sqrt{2.5}}$, from Table 5.2-2. Thus

$$Q\left(\frac{d}{\sigma}\right) = 4 \times 10^{-3}$$

and $$\sigma = 0.238$$

In the case of 8-level elements transmitted over channel A with an error probability of 7 in 10^3, σ must satisfy

$$\frac{2(k-1)}{k} Q\left(\frac{d}{\sigma}\right) = 7 \times 10^{-3}$$

where now k = 8 and $d = 1/\sqrt{7}$. Thus,

$$Q\left(\frac{d}{\sigma}\right) = 4 \times 10^{-3}$$

and $$\sigma = 0.142$$

In each case just considered the transmitted energy per bit is set to unity. It can be seen that the values of σ for 2, 4 and 8-level elements transmitted over channel A are compared for the same value of $Q(\frac{d}{\sigma})$, so that the comparison holds at high signal to noise ratios where a change in error probability by a factor of two corresponds to a negligible change in signal to noise ratio. Evidently under the above conditions, a change from binary to 4-level signals results in a reduction in tolerance to noise of nearly 4 dB, whereas a change from binary to 8-level signals results in a reduction of nearly 8.5 dB.

The value of the standard deviation σ of white Gaussian noise obtained theoretically for channel A has been checked against that obtained by computer simulation, and for every system tested, the two values were found to be in close agreement.

For a given value of the average element error probability p, the number of errors t obtained in a simulation test, is given by

$$t = \ell p \tag{5.3-2}$$

where ℓ is the total number of signal-elements transmitted in a test. In all the systems tested, the actual value of ℓ for groups of four and eight binary elements and groups of four 4-level elements was 4096, while for groups of eight 4-level and four 8-level signal-elements ℓ was equal to 4000.

It has been shown that if the errors are statistically independent, $t > 30$, $p \ll 1$ and if an accuracy of no better than 20% is required for the confidence limits, then it can be assumed that t has a Gaussian probability density with a mean $\mu = t$ and a standard deviation $\eta = \sqrt{t}$. For a given value of $p > 0$, the 95% confidence limits for the value of p are now approximately[52]

$$\pm \frac{2\eta}{\mu} p = \pm \frac{2p}{\sqrt{t}} \tag{5.3-3}$$

where the limits are expressed as deviation from the given value of p.

In any test with orthogonal groups of signals there may be a high degree of dependence between the individual element errors of a group in a detection process. The result of this dependence is to reduce the effective number of independent errors obtained in a test and so to widen the confidence limits. Thus t in Eqn. (5.3-2) does not represent the effective number of independent errors and, therefore, cannot be used to estimate the confidence limits. However, since the signal-elements are transmitted in groups, the element errors in a group being completely independent from those of the other groups, it is reasonable to assume that the effective number of independent errors in a test is equal to or greater than the number of groups of signal-elements in error, and this value provided that it is greater than 30, can be used in Eqn. (5.3-3) to estimate the confidence limits for a given value of p.

From the computer simulation results it was found that the number of groups of signal-elements in error, for any of the detection processes described in Section 4, remained fairly constant for a given element error probability p, a given number of elements in a group m, and a given number of signal-element levels k. The average value of the number of groups in error was, therefore, used to estimate the confidence limits for given values of p, m and k. The 95% confidence limits for different values of m and k are given in Table 5.3-1, and for the given values of m and k these limits are for practical purposes, the same for any of the systems tested. Where the effective number of independent errors j is less than 30 the 95% confidence limits are estimated from the results of reference [49]. For each of the different values of p,m and k, Table 5.3-1 also shows the corresponding 95% confidence limits of σ the standard devation of the white Gaussian noise, and are expressed, in dB, as deviations from the value of σ corresponding to the given value of p.

TABLE 5.3-1

Approximate 95% confidence limits for different values of m and k

Number of signal-elements in a group m	Number of possible signal-element values k	Average element error probability p	Total Number of errors in a simulation test t	Average number of groups of signals in error j	95% confidence limits expressed as deviation from the given value p	95% confidence limits of σ expressed in dB, as deviation from the given value of σ
4	2	4×10^{-3}	16	14	+.0029 -.0022	+0.63 -0.76
8	2	4×10^{-3}	16	12	+.00335 -.00235	+0.72 -0.90
4	4	6×10^{-3}	24	16	+.0039 -.0030	+0.71 -0.90
8	4	6×10^{-3}	24	13	+.0046 -.0033	+0.78 -0.94
4	8	7×10^{-3}	28	18	+.0040 -.0033	+0.77 -0.92

5.4 Results of Computer Simulation Tests

The results of computer simulation tests are shown in Tables 5.4-1 to 5.4-3. The noise power spectral density at the input to the receiver filter, required for a given average element error probability in Tables 5.4-1 to 5.4-3 is quoted in decibels relative to its value when a binary signal is transmitted over channel A with an average element error probability of 4 in 10^3, the noise level here being the same in all cases. As has been said before, the results quoted for channel A have been calculated theoretically and checked by computer simulation.

Tables 5.4-1 to 5.4-3 also show the tolerance to additive white Gaussian noise, for each of the different channels when the optimum linear equalizer, described in Section 2.3, is used at the receiver. These results have been evaluated theoretically, the peak value of the resultant intersymbol interference in the equalized signal, being less than 1% in each case studied. The number of taps used by the linear equalizer for each of the different channels is given in Table 5.4-4. Where * is shown in the tables, the channel cannot be equalized by a linear transversal filter. The reduction in tolerance to noise, caused by any channel, is here unaffected by the values of g and m, and applies also to a continuous (uninterrupted) signal with the same element rate, where each gap is considered to contain g zero-level elements. The tolerance to noise is furthermore not affected by the number of signal-element levels.

The tolerances to noise of System 1 given in Tables 5.4-1 to 5.4-3,have been obtained by two different methods. In the first method the iterative process described in Section 4.1, is used, while in the second method the network $Y^T(YY^T)^{-1}$ in Eqn. (3.2-9) is used. It was found that the two methods give the same results. Furthermore, the tolerances to noise of System 1, for groups of eight binary signal-elements and for each of the different channels, given in Table 5.4-1, agree with those given in reference [55].

Table 5.4-5 gives the theoretical results corresponding to those obtained for System 1 in Tables 5.4-1 to 5.4-3. The results of Table 5.4-5 are obtained by calculating the minimum distances to the decision boundary, according to Eqns. (3.5-6) and (3.5-14).

TABLE 5.4-1

Noise level for an average element error probability of 4 in 10^3, expressed in dB relative to its level with no distortion. Groups of four or eight binary signal-elements

(Results obtained by computer simulation)

Channel	System 1		System 2		System 3		System 4		System 5		System 6		Linear Equalizer
	m=4	m=8	m=4	m=8	m=4	m=8	m=4	m=8	m=4	m=8	m=4	m=8	
A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
B	-3.3	-6.0	-1.0	-1.2	-1.5	-1.6	-1.0	-1.1	-1.5	-1.8	-1.1	-1.3	*
C	-1.1	-3.2	0.0	-0.4	-0.9	-1.6	-0.1	-0.4	-0.8	-1.5	-0.4	-0.6	*
D	-6.6	-13.7	-1.8	-2.4	-3.9	-7.3	-2.1	-2.6	-3.5	-4.9	-2.5	-2.8	*
E	0.0	-0.3	0.0	0.0	0.0	0.0	0.0	0.0	-0.2	0.0	-0.4	0.0	-0.3
F	-1.8	-3.3	0.0	-0.5	-1.2	-1.3	-0.3	-0.9	-1.1	-1.3	-0.6	-1.0	-3.5
G	-0.6	-2.9	0.0	0.0	-0.3	-0.9	0.0	-0.4	-0.2	-0.8	-0.1	-0.4	*
H	-3.0	-3.2	-0.6	-0.7	-1.7	-1.6	-0.6	-0.9	-1.6	-1.6	-1.0	-1.3	-3.5
I	-0.9	-3.7	0.0	0.0	-0.9	-1.6	0.0	-0.4	-0.6	-1.1	0.0	-0.6	-8.2
J	-14.3	-17.6	-3.8	-4.4	-9.9	-13.4	-4.1	-4.9	-8.1	-10.0	-5.8	-7.4	-20.6
K	-3.3	-4.9	-0.7	-1.2	-1.6	-2.5	-0.9	-1.1	-1.6	-2.4	-1.1	-1.6	*

TABLE 5.4-2

Noise level for an average element error probability of 6 in 10^3, expressed in dB relative to its level for an error probability of 4 in 10^3 with binary signal-elements and no distortion. Groups of four or eight 4-level elements.
(Results obtained by computer simulation)

Channel	System													
	1		2		3		4		6		7/2		7/4	
	m=4	m=8	m=4	m=8	m=4	m=8	m=4	m=8	m=4	m=8	m=4	m=8	m=4	m=8
A	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0
B	-7.4	-10.6	-5.3	-	-6.5	-8.9	-5.3	-5.6	-5.8	-7.5	-5.0	-5.7	-5.0	-5.7
C	-5.5	-7.5	-4.3	-	-5.5	-	-4.5	-	-4.6	-	-4.2	-	-4.2	-
D	-12.4	-17.6	-6.8	-	-10.3	-14.5	-6.8	-7.7	-9.2	-12.5	-6.9	-7.8	-6.8	-7.7
E	-4.8	-4.6	-4.5	-	-4.7	-	-4.6	-	-4.6	-	-4.3	-	-4.5	-
F	-6.4	-7.1	-4.9	-	-6.1	-	-5.0	-	-5.8	-	-4.7	-	-4.7	-
G	-5.3	-6.3	-4.2	-	-4.5	-	-4.3	-	-4.6	-	-4.4	-	-4.5	-
H	-7.4	-7.5	-5.2	-	-6.7	-	-5.1	-	-6.4	-	-5.3	-	-5.5	-
I	-5.8	-7.8	-4.6	-	-5.1	-	-4.7	-	-4.9	-	-4.2	-	-4.2	-
J	-19.8	-21.9	-11.2	-11.9	-16.9	-19.5	-11.1	-12.0	-14.6	-16.7	-11.5	-12.2	-11.7	-12.4
K	-7.6	-10.2	-5.0	-	-6.4	-	-5.0	-	-6.8	-	-5.5	-	-5.4	-

Cont'd. ....

TABLE 5.4-2(Cont'd)

Channel	System							
	7/6		8/2	8/4		8/6		Linear Equalizer
	m=4	m=8	m=4	m=4	m=8	m=4	m=8	
A	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0
B	-5.9	-6.4	-5.1	-5.1	-5.7	-5.8	-6.5	*
C	-4.7	-	-4.5	-4.4	-	-4.6	-	*
D	-8.0	-9.1	-6.8	-6.8	-7.8	-7.9	-9.3	*
E	-4.6	-	-4.4	-4.5	-	-4.7	-	-4.2
F	-5.1	-	-4.7	-4.8	-	-5.4	-	-7.5
G	-4.6	-	-4.4	-4.2	-	-4.5	-	*
H	-6.2	-	-5.2	-5.2	-	-5.8	-	-7.5
I	-4.4	-	-4.6	-4.5	-	-4.9	-	-12.1
J	-12.9	-15.5	-11.3	-11.1	-12.1	-12.8	-15.9	-24.6
K	-6.2	-	-5.5	-5.0	-	-5.9	-	*

TABLE 5.4-3

Noise level for an average element error probability of 7 in 10^{3} expressed in dB relative to its level for an error probability of 4 in 10^{3} with binary signal-elements and no distortion. Groups of four 8-level elements.
(Results obtained by Computer Simulation)

Channel	System										
	1	3	4	6	7/2	7/4	7/6	8/2	8/4	8/6	Linear Equalizer
A	-8.5	-8.5	-8.5	-8.5	-8.5	-8.5	-8.5	-8.5	-8.5	-8.5	-8.5
B	-11.8	-11.3	-9.5	-11.2	-9.4	-9.3	-10.8	-9.5	-9.4	-10.9	*
D	-16.9	-15.8	-12.1	-13.3	-12.0	-11.9	-13.1	-12.0	-12.2	-13.0	*
J	-23.8	-22.4	-17.5	-19.7	-17.6	-17.4	-19.0	-17.5	-17.6	-19.2	-29.1

TABLE 5.4-4

Number of taps required for the linear transversal equalizer for different channels.

Channel	No. of taps required for the linear equalizer
A	1
B	*
C	*
D	*
E	30
F	30
G	*
H	30
I	65
J	46
K	*

TABLE 5.4-5

Noise levels in System 1 expressed in dB relative to the noise level for an error probability of 4 in 10^3 with binary signal-elements and no distortion. (Results obtained theoretically)

Channel	binary signals $p = 4 \times 10^{-3}$		4-level signals $p = 6 \times 10^{-3}$		8-level signals $p = 7 \times 10^{-3}$
	m=4	m=8	m=4	m=8	m=4
A	0.0	0.0	-3.98	-3.98	-8.45
B	-3.8	-6.48	-7.78	-10.46	-12.25
C	-1.25	-3.80	-5.23	-7.78	-9.7
D	-8.14	-14.9	-12.12	-18.88	-16.59
E	-0.12	-0.25	-4.1	-4.23	-8.57
F	-2.25	-3.26	-6.23	-7.24	-10.70
G	-1.127	-3.07	-5.11	-7.05	-9.58
H	-3.35	-3.49	-7.32	-7.47	-11.80
I	-2.02	-4.10	-6.00	-8.08	-10.47
J	-14.78	-18.35	-18.78	-22.33	-23.25
K	-3.77	-5.17	-7.75	-9.15	-12.22

5.5 Comparison of Detection Processes

The performances of various detection processes described in Section 4, in the presence of additive white Gaussian noise, were compared by means of a simple model assuming only two elements in a group. Computer simulation results of Tables 5.4-1 to 5.4-3 suggest that the theory developed for the simple case can be extended, much as would be expected, to the higher values of m. The detection processes of Systems 1 to 6 for both binary and multi-level signal-elements, listed in the order of their tolerance to additive Gaussian noise and starting with the best, are 2,4,6,5,3 and 1. The transversal linear equalizer achieves the lowest tolerance to noise. Where there is severe attenuation distortion, System 2 achieves a considerable advantage in tolerance to noise over both System 1 and the linear equalizer. The advantage gained by System 2 over System 1 increases with the value of m and in the case of channel J is as much as 13 dB for eight binary signal-elements in a group. For pure phase distortion, all systems tested achieve the same tolerance to noise as that for no distortion (channel A). The performance of System 4 is very close to that of System 2, and System 6 has a tolerance to noise typically within about 3 dB of that of System 2.

For groups of eight binary signal-elements, transmitted over channel D or J, System 6 gains an appreciable advantage over System 3 in tolerance to noise, but this advantage tends to be somewhat smaller for smaller group sizes or for a greater number of signal levels. System 3 gains an appreciable advantage over System 1, when binary signal-elements are transmitted over channel B, D or J, but the advantage is steadily reduced as the number of signal levels increases as has been explained in Section 4.5. System 6 gains a somewhat smaller advantage over System 5, with binary signal-elements, than does

System 3 over System 1. System 5 has not been tested with multi-level elements, its performance now becoming quite close to that of System 6. The two systems, of course, resemble each other more closely as the number of signal levels increases. System 1 sometimes achieves an advantage of more than 6 dB over the linear equalizer, even though both are processes of linear equalization. The reason for this is that System 1 makes use of the prior knowledge of the g zero-level signal-elements between adjacent groups of m elements, whereas the linear equalizer uses no prior knowledge of the signal-element values.

When m >> g, the operation effectively performed on the received signal by System 1 becomes much the same as that of the linear equalizer for the same signal. Furthermore, the tolerance to noise of a linear transversal equalizer, when correctly designed for the received signal, is not now much affected by whether the signal-elements are transmitted in orthogonal groups or in an uninterrupted stream, so that System 1 has approximately the same tolerance to noise as the linear equalizer for the corresponding uninterrupted serial signal.

Within the limits of the accuracy of computer simulation results (Table 5.3-1), the tolerance to noise of System 1, obtained theoretically, in Table 5.4-5, is in every case, either less than or equal to that given in Tables 5.4-1 to 5.4-3. This is so because the theoretical results are obtained by calculating the minimum distances to the decision boundary (Section 3.5) and therefore provide the lower bounds to the tolerance of System 1 to additive white Gaussian noise, for a given average element error probability.

Correct operation, at the signal-element rate assumed here, cannot be obtained over channels B,C,D,G and K with a simple linear transversal equalizer, whereas correct operation is, in every case, achieved by System 1. This demonstrates the one important advantage of System 1 over a simple linear equalizer, which is that System 1 will operate correctly over any time-invariant or slowly-time-varying linear baseband channel whose impulse response has for practical purposes a finite duration.[32,35]

Table 5.5-1 shows the approximate number of sequential operations required for each of the different detection processes, when orthogonal groups of eight 4-level signal-elements are transmitted over channel J. Apart from System 2, the number of sequential operations in Table 5.5-1 for various detection processes,is arrived at by considering one sequential operation to be equivalent to the estimation or change in estimation of one of the $m\{x_i\}$ in System 1. This operation is of similar complexity and duration to the measurement of $|R - SY|$ for a particular S in System 2. With the exception of System 2, each detection process requires more sequential operations with channel J than with any of the other channels tested.

The detection processes of Systems 1 to 6, in order of the number of sequential operations normally required and starting with the smallest number, are 3,1,6,5,4 and 2. It is assumed here that $m \geqslant 12$ for binary signal-elements, and $m \geqslant 8$ for 4 or 8-level elements. The detection processes involve similar degrees of complexity, Systems 1 and 3 being the least complex and System 4 the most complex.

It was pointed out in Section 4.5 that the detection processes of Systems 1 - 6 are not suitable for use with multi-level signal-elements, since they either give a poor tolerance to noise or else require larger numbers of sequential operations. The results of computer simulation tests, given in Tables 5.4-1 to 5.4-3 confirm this, at least for those cases which have been tested. The tolerance of System 3 to additive Gaussian noise decreases steadily as the number

TABLE 5.5-1

Approximate number of sequential operations required for the detection of a group of eight 4-level elements transmitted over Channel J

System	Number of Operations
1	1,600
2	66,000
3	1,200
4	4,900
5	2,400
6	1,800
7/2	7,800
7/4	3,200
7/6	3,200
8/2	1,500
8/4	2,400
8/6	2,000

of signal-element levels is increased from 2 to 8, and this was expected to happen since the constraint of Eqn. (4.2-1) becomess less and less effective with the increase in the number of possible element values. Also, because of the poorer performance of System 3 with multi-level signals, the tolerance of System 6 to additive white Gaussian noise decreases with the number of levels as can be seen from Tables 5.4-1 to 5.4-3. The tolerance to noise of System 4, on the other hand, remains close to that of System 2. The number of sequential operations required for System 4 are now very much larger than those required for either System 3 or System 6.

The technique for the detection of multi-level signal-elements involving an initial search for the two or three most likely values of an element, described in Section 4.5, seem to work well for higher values of m. As expected, Systems 7/2, 7/4, 8/2 and 8/4 achieve a tolerance to noise similar to that of System 2. Systems 7/6 and 8/6 have a slightly inferior performance, typically within about 3 dB of that of Systems 7/2 and 7/4. There does not appear to be any significant difference between the performance of any one of the Systems 8/2, 8/4 and 8/6 and the corresponding System 7/2, 7/4 or 7/6, although the latter system is, in each case, slightly less complex than the former, and Systems 7/2 and 7/4 require fewer sequential operations in a detection process than do Systems 8/2 and 8/4, respectively. Again, for larger values of m, Systems 8/2 and 8/4 require far fewer sequential operations than Systems 2 and 4, respectively.

Although System 6 does not achieve quite as good a performance as does System 4, either when used on its own for binary signals or else when used as System 7/6 or 8/6 for multi-level signals, it is a simpler system and usually requires fewer sequential operations in a detection process. It appears therefore that of the various systems tested the most cost-effective for binary and multi-level signals are Systems 6 and 7/6, respectively. Where the best available performance is required, without an excessive number of sequential operations in a detection process, the preferred systems for binary and multi-level signals are Systems 4 and 7/4, respectively.

5.6 Comparison of Signals arranged in Separate Groups with the Corresponding Uninterrupted Signals

In Section 5.4, the performance of the linear transversal equalizer in the presence of additive white Gaussian noise, is measured for transmitted signal-elements arranged in separate groups. Since the linear equalizer does not make use of the prior knowledge of the zero-level elements separating adjacent groups of signal-elements, its performance in noise is considerably reduced as compared with that of System 1, particularly for severe signal distortions.

Consider now the case of the synchronous serial base-band data-transmission system of Section 2.1 where the signal-elements are transmitted in a continuous (uninterrupted) serial stream, and let the base-band channel be equalized by a linear transversal equalizer. If now System 1 is to replace the linear equalizer in the arrangement just considered, then the continuous transmission of signal-elements must be modified by inserting the required number of g zero-level elements at the appropriate intervals to give the separate groups of m transmitted elements. If the signal-element rate remains unchanged, the information rate is now m/m+g times that of the original system, which means that more time is needed to transmit a given message. Alternatively, if System 1 is replaced by an arrangement where the signal-elements are transmitted in a continuous (uninterrupted) stream, and where the channel is equalized by a linear equalizer, without now changing the information rate, then clearly the signal-element rate of the latter system is m/m+g times that of System 1. This must normally result in less intersymbol interference in the sample values of the received signal. It follows that for a given channel and information rate, the performance of the linear transversal equalizer in the presence of additive white Gaussian noise, will, in general, be better than that given in Tables 5.4-1 to 5.4-3. Thus, the comparison of the performance of the linear equalizer with that of System 1, carried out in Section 5.4, is not a true comparison. Instead, the tolerance to noise of the linear equalizer should be determined by assuming a continuous stream of transmitted signal-elements with the same information rate as that in the corresponding arrangement of System 1.

In this section an attempt is made to compare the performances of Systems using orthogonal groups of signal-elements (Section 4.0) with those of linear and non-linear equalizers (Section 2.0) where the signal-elements are transmitted in a continuous (uninterrupted) serial stream, the information rate in the two cases being the same. The comparison is here made for binary signal-elements having possible values +1 and -1. For systems using orthogonal groups, it is assumed that there are eight signal-elements in a group, and that every two adjacent transmitted groups are separated by four signal-elements set to zero. This means that $m/m+g = 2/3$ which represents an efficiency of $66\frac{2}{3}$ % for the systems.

The transmitter and receiver filters are assumed to have the resultant transfer function H(f) given by Eqn. (2.1-3) and the resultant impulse response h(t) given by Eqn. (2.1-7) and shown in Fig. 2.1-1. The different channels studied are those with the values of the sampled impulse response given in Table 5.2-1. It is furthermore assumed that the transmission path itself can be represented by the model shown in Fig. 5.6-1. This assumes simple

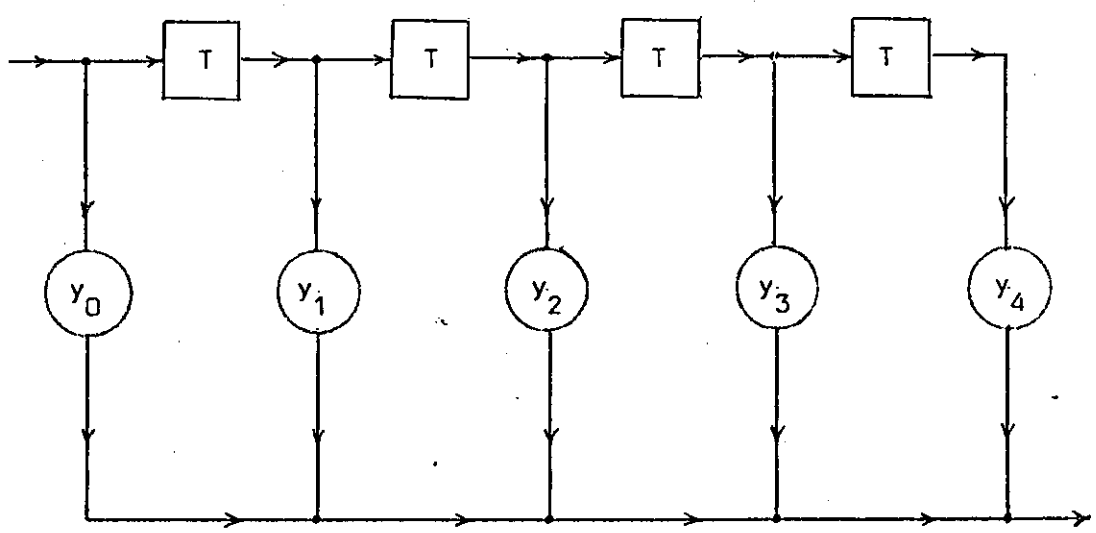


Figure 5.6-1

Model of transmission path.

multipath transmission where all delays are integer multiples of T seconds. Since the sampled impulse response of the transmitter and receiver filters in cascade, in System 1, is simply

$$1 \quad 0 \quad . \; . \; . \quad 0 \qquad (5.6\text{-}1)$$

it can be seen that the sampled impulse response of the baseband channel in Fig. 1.1-1 is now given by

$$y_0 \; y_1 \; y_2 \; y_3 \; y_4 \qquad (5.6\text{-}2)$$

where y_0, y_1, y_2, y_3 and y_4 are the tap gains in Fig. 5.6-1. Although, the model of the transmission path is quite arbitrary, it should enable an interesting comparison to be made between the two systems, on a rather more realistic basis than that used in Section 5.4.

Consider now the data-transmission system of Fig. 1.1-1, described in Sections 2.1 and 3.1. Since adjacent groups of m signal-elements, at the transmitter, are separated by g zero-level elements which carry no information, (m+g)T seconds are required to transmit the information carried by the m signal-elements of a group, where 1/T is the element transmission rate. If orthogonal groups of signal-elements are replaced by a continuous stream of elements, then, for the same information rate, m signal-elements must now be transmitted, with no gaps, over (m+g)T seconds. This means that the element transmission rate in the equivalent continuous transmission, is reduced from 1/T to 1/T' where

$$T' = \left(\frac{m+g}{m}\right)T. \qquad (5.6\text{-}3)$$

Thus the signal-elements in the arrangement using continuous transmission, are now transmitted regularly (with no gaps) at intervals of T' seconds and the received signal, at the output of the receiver filter, is sampled at the time instants $t = iT'$, for all integers i. The transmitter and receiver filters assumed for System 1 have an unnecessarily wide bandwidth at the new sampling rate of 1/T' and introduce some intersymbol interference. Assume therefore that the transmitter and receiver filters in cascade have the impulse response

$$h'(t) = \frac{\sin \pi \frac{2t}{T'}}{\pi \frac{2t}{T'}} + \frac{1}{2} \frac{\sin \pi \left(\frac{2t}{T'} + 1\right)}{\pi \left(\frac{2t}{T'} + 1\right)} + \frac{1}{2} \frac{\sin \pi \left(\frac{2t}{T'} - 1\right)}{\pi \left(\frac{2t}{T'} - 1\right)} \qquad (5.6\text{-}4)$$

This is shown in Fig. 5.6-2, and is clearly the equivalent of h(t) at the new sampling rate.

It is important to note that in comparing System 1 with the other system, the <u>same</u> transmission path, as shown in Fig. 5.6-1, is assumed for the two systems. Thus, since the sampling rates are different in the two systems and since each system uses the transmitter and receiver filters appropriate to its sampling rate, the sampled impulse response of the baseband channel corresponding to any given transmission path, is different for the two systems. In the case of the "orthogonal" system, using separate groups of signal-elements, the sampled impulse response of the baseband channel is the same as the sampled impulse response of the transmission path. Since the sampled impulse response of the baseband channel has here, in every case, been normalized to give a channel vector of unit length, it follows that the sampled impulse response and therefore also the impulse response, assumed for each of the different transmission paths, has also been normalized. In the case of the continuous system, using an uninterrupted stream of transmitted signal-elements at an element rate of 1/T' bauds, the sampled impulse response of the baseband channel does not in general give a channel vector of unit length. Furthermore, since the transmission

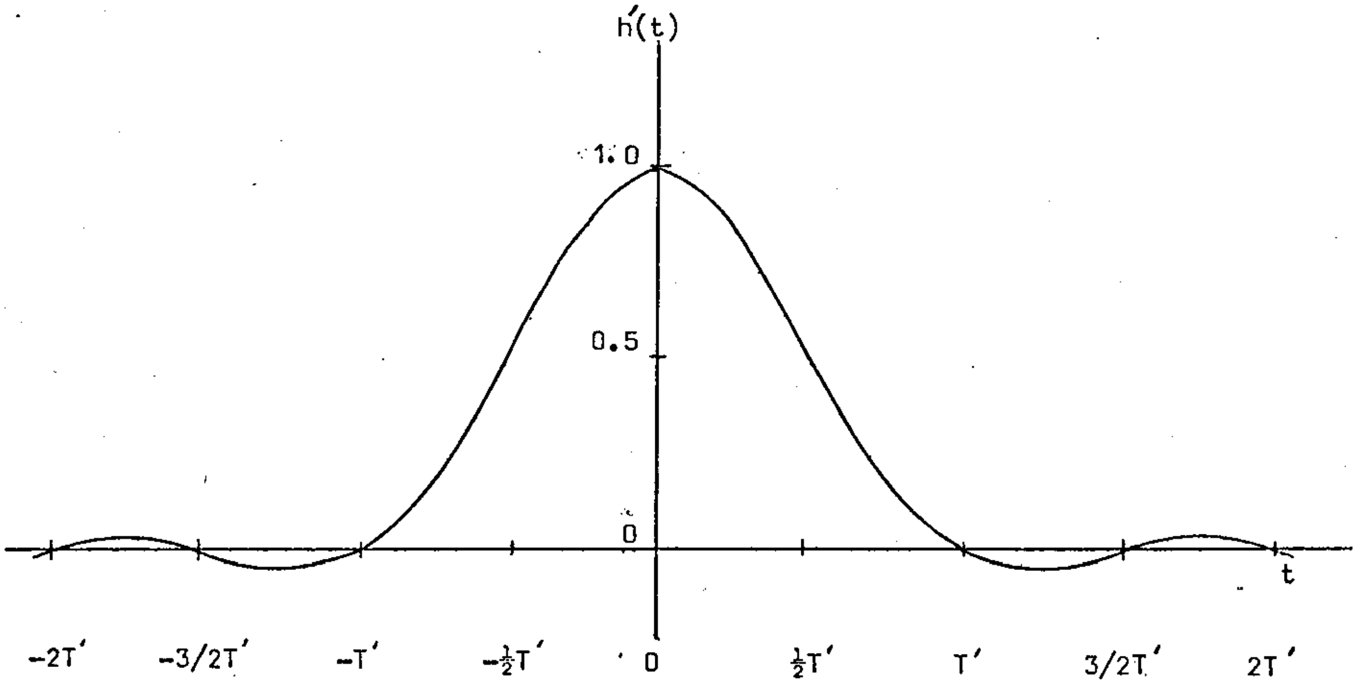


Figure 5.6-2

Impulse response h′(t) of the transmitter and receiver filters in cascade.

path has the impulse response $y_0 + y_1\delta(t-T) + y_2\delta(t-2T) + y_3\delta(t-3T) + y_4\delta(t-4T)$, it is of no value, and indeed incorrect, to consider its sampled impulse response at a sampling rate of 1/T'.

In order to compare the "orthogonal" system, using separate groups of transmitted signal-elements, with the 'continuous' system, using an uninterrupted stream of transmitted signal-elements, not only is the same transmission path used in each comparison but this has an impulse response which is the same as the sampled impulse response of the corresponding baseband channel for the orthogonal system (Table 5.6-1). The impulse response y'(t) of the corresponding baseband channel for the continuous system is determined by convolving the impulse response h'(t) of the transmitter and receiver filters, as shown in Fig. 5.6-2, with the impulse response of the transmission path which is $y_0\delta(t) + y_1\delta(t-T) + y_2\delta(t-2T) + y_3\delta(t-3T) + y_4\delta(t-4T)$. The sampled impulse response of this baseband channel is obtained by sampling y'(t) at intervals of T' seconds, the sampling instants being phased so that one of these coincides with the positive peak of y'(t). The results of some unpublished work on equalizers, carried out at Loughborough University of Technology, suggest that the phase selected here for the sampling instants is the one most likely to maximize the tolerance to additive white Gaussian noise. The continuous system is therefore tested under the conditions most favourable to this system. Table 5.6-2 shows the sampled impulse responses of the different baseband channels, in the case of the continous system. In order to simplify the computations of the sampled impulse responses, it has been assumed that the h'(t) is not in fact as shown in Fig. 5.6-2 but is instead the corresponding raised cosine as shown in Fig. 5.6-3. This approximates quite closely to h'(t) over the time interval - T' to T' but is zero outside this time interval. The approximation does not introduce any significant errors.

TABLE 5.6-1

Sampled impulse responses of different baseband channels in the orthogonal system

Trans-mission Path	Sampled impulse response of baseband channel						
A		(1.0	0	0	0	0	)
B_1	$2.0^{-\frac{1}{2}}$	(1.0	1.0	0	0	0	)
B_2	$2.0^{-\frac{1}{2}}$	(1.0	-1.0	0	0	0	)
C_1	$2.0^{-\frac{1}{2}}$	(1.0	0.0	1.0	0	0	)
C_2	$2.0^{-\frac{1}{2}}$	(1.0	0.0	-1.0	0	0	)
D_1	$1.5^{-\frac{1}{2}}$	(0.5	1.0	0.5	0	0	)
D_2	$1.5^{-\frac{1}{2}}$	(-0.5	1.0	-0.5	0	0	)
E_1	$1.5^{-\frac{1}{2}}$	(0.5	1.0	-0.5	0	0	)
E_2	$1.5^{-\frac{1}{2}}$	(-0.5	1.0	0.5	0	0	)
F_1	$1.5^{-\frac{1}{2}}$	(1.0	0.5	0.5	0	0	)
F_2	$1.5^{-\frac{1}{2}}$	(1.0	-0.5	0.5	0	0	)
G_1	$1.5^{-\frac{1}{2}}$	(1.0	0.5	-0.5	0	0	)
G_2	$1.5^{-\frac{1}{2}}$	(1.0	-0.5	-0.5	0	0	)
H_1	$1.5^{-\frac{1}{2}}$	(1.0	0.667	0.235	0	0	)
H_2	$1.5^{-\frac{1}{2}}$	(1.0	-0.667	0.235	0	0	)
I_1	$1.5^{-\frac{1}{2}}$	(1.0	0.667	-0.235	0	0	)
I_2	$1.5^{-\frac{1}{2}}$	(1.0	-0.667	-0.235	0	0	)
J_1	$2.0^{-\frac{1}{2}}$	(0.235	0.667	1.0	0.667	0.235	)
J_2	$2.0^{-\frac{1}{2}}$	(0.235	-0.667	1.0	-0.667	0.235)	)
K_1	$2.0^{-\frac{1}{2}}$	(-0.235	0.667	1.0	0.667	-0.235	)
K_2	$2.0^{-\frac{1}{2}}$	(-0.235	-0.667	1.0	-0.667	-0.235	)

The Transmission path X has a sampled impulse response, at a sampling rate of 1/T given by the baseband channel X in Table 5.2-1.

TABLE 5.6-2

Sampled impulse responses of different baseband channels in the continuous system

Transmission Path	Sampled impulse response of baseband channel				
A	1.0	0	0	0	0
B_1	0.142	1.075	0.142	0	0
B_2	0.562	-0.562	0	0	0
C_1	0.706	0.538	0.184	0	0
C_2	0.706	-0.538	-0.184	0	0
D_1	0.302	1.03	0.302	0	0
D_2	-0.302	0.62	-0.302	0	0
E_1	0.147	0.92	-0.269	0	0
E_2	-0.269	0.92	0.147	0	0
F_1	0.09	0.94	0.553	0.033	0
F_2	0.73	-0.049	0.130	0	0
G_1	0.09	0.94	-0.212	0.033	0
G_2	0.73	-0.55	-0.130	0	0
H_1	0.135	1.03	0.375	0	0
H_2	0.71	-0.325	0.09	0	0
I_1	0.135	1.03	0.033	0	0
I_2	0.71	-0.545	0.082	0	0
J_1	0.043	0.48	0.97	0.48	0.043
J_2	0.043	-0.205	0.45	-0.205	0.043
K_1	-0.043	0.234	0.97	0.234	-0.043
K_2	-0.043	-0.45	0.45	-0.45	-0.043

The transmission path X has a sampled impulse response, at a sampling rate of 1/T given by the baseband channel X in Table 5.2-1.

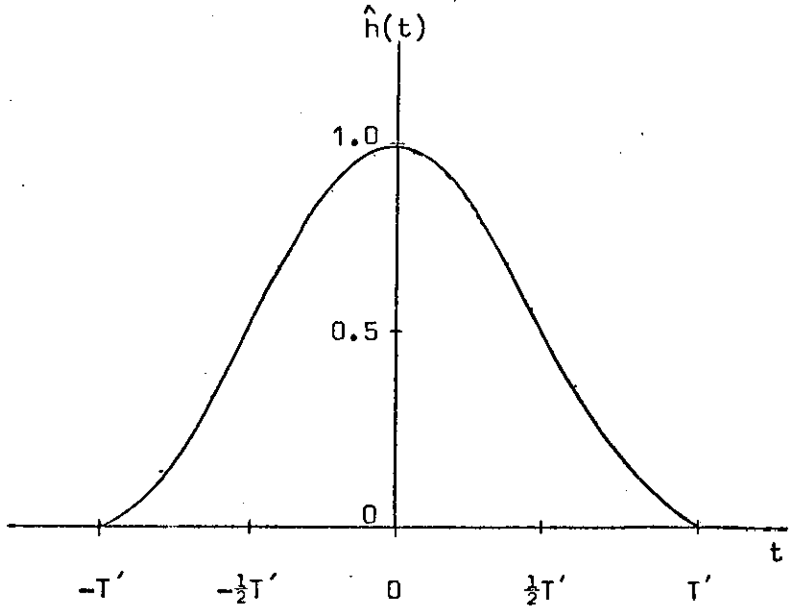


Figure 5.6-3

Approximate impulse response of the transmitter and receiver filters in cascade.

For the transmission paths D_1, D_2, J_1 and J_2, Figs. 5.6-4 to 5.6-7 show the values of the impulse response of the baseband channel in the case of the continuous system.

Table 5.6-3 shows the noise levels of the additive white Gaussian noise required for an average element error probability of 4×10^{-3}, in the case of the continuous system where the baseband channel is equalized by the appropriate linear or non-linear equalizer. The noise level is here quoted in dB relative to its value when binary signal-elements are transmitted over channel A with an error probability of 4×10^{-3}. The number of taps required in each of the equalizers is also given in the table. In the case of the non-linear equalizer, the number of taps refers to the linear filter only. Where * is marked in the table, the channel cannot be equalized by the linear transversal equalizer. The results in Table 5.6-3 are obtained theoretically from Sections 2.3 and 2.4. Table 5.6-4 shows the noise level in the case of Systems 1, 2 and 6 (Section 5.4) for groups of eight binary signal-elements, and an error probability of 4×10^{-3}, and is expressed in dB relative to its corresponding value for the linear and the non-linear equalizers.

Table 5.6-3 shows that the performance of the non-linear equalizer is, in every case, better than that of the linear equalizer. For severe signal distortions, the non-linear equalizer gains a considerable advantage in tolerance to noise, over the linear equalizer. The reason for this is that the non-linear equalizer makes use of the prior knowledge of the signal-element values in cancelling the intersymbol interference of a detected signal-element, from the sample values of the received signal as mentioned in Section 2.5. The baseband channels corresponding to the transmission paths B_2, C_2 and K_2 cannot be equalized by the linear transversal equalizer since the z-transforms of the corresponding sampled impulse responses, have zeros on the unit circle.

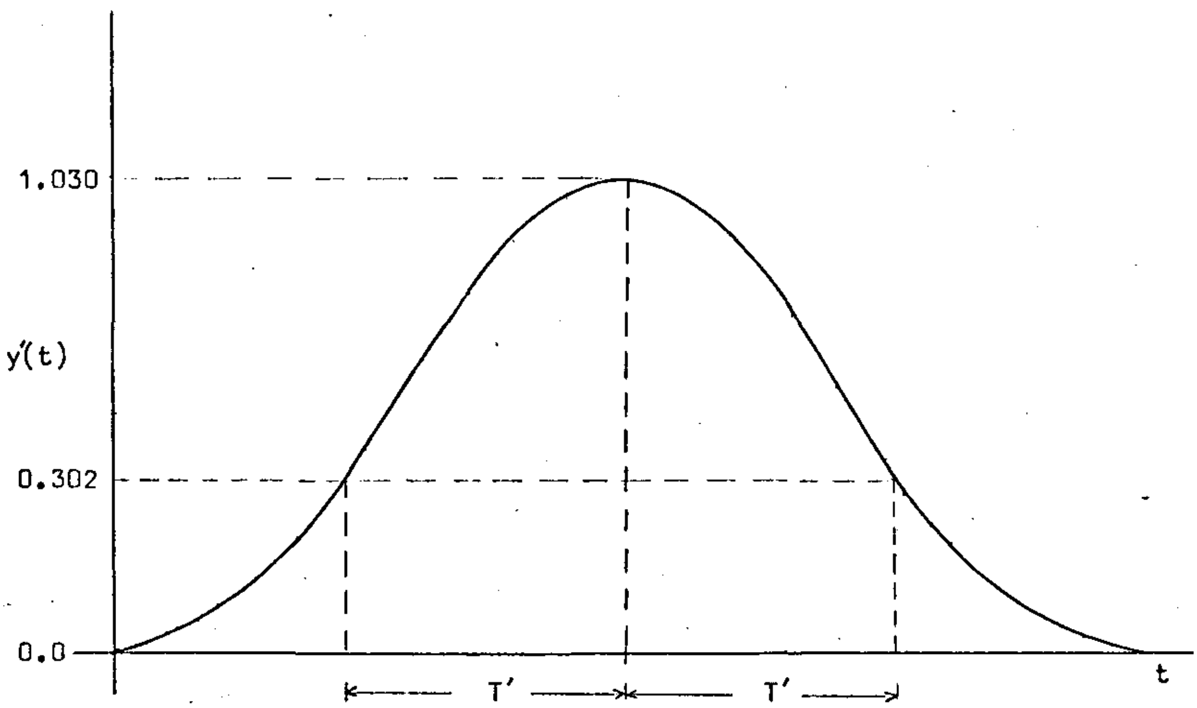


Figure 5.6-4

The impulse response of the baseband channel in the case of the continuous system, where the transmission path has an impulse response

$$1.5^{-\frac{1}{2}}\{0.5\delta(t)+\delta(t-T)+0.5\delta(t-2T)\}$$

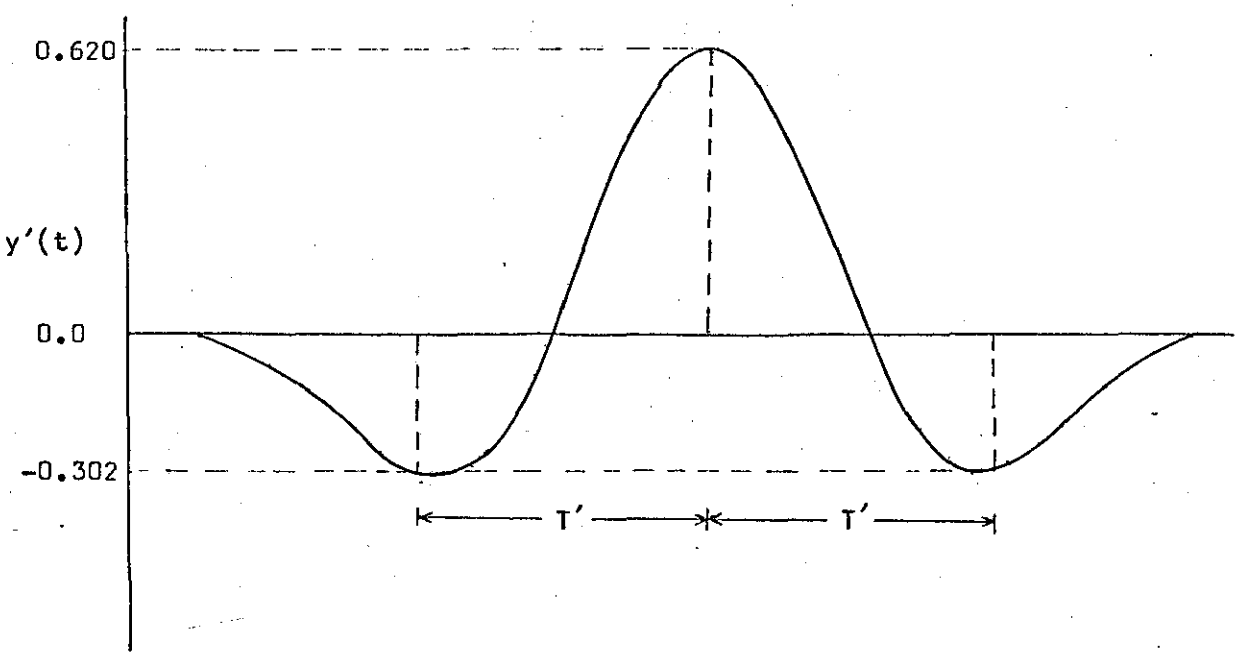


Figure 5.6-5

The impulse response of the baseband channel in the case of the continuous system, where the transmission path has an impulse response

$$1.5^{-\frac{1}{2}}\{-0.5\delta(t)+\delta(t-T)-0.5\delta(t-2T)\}$$

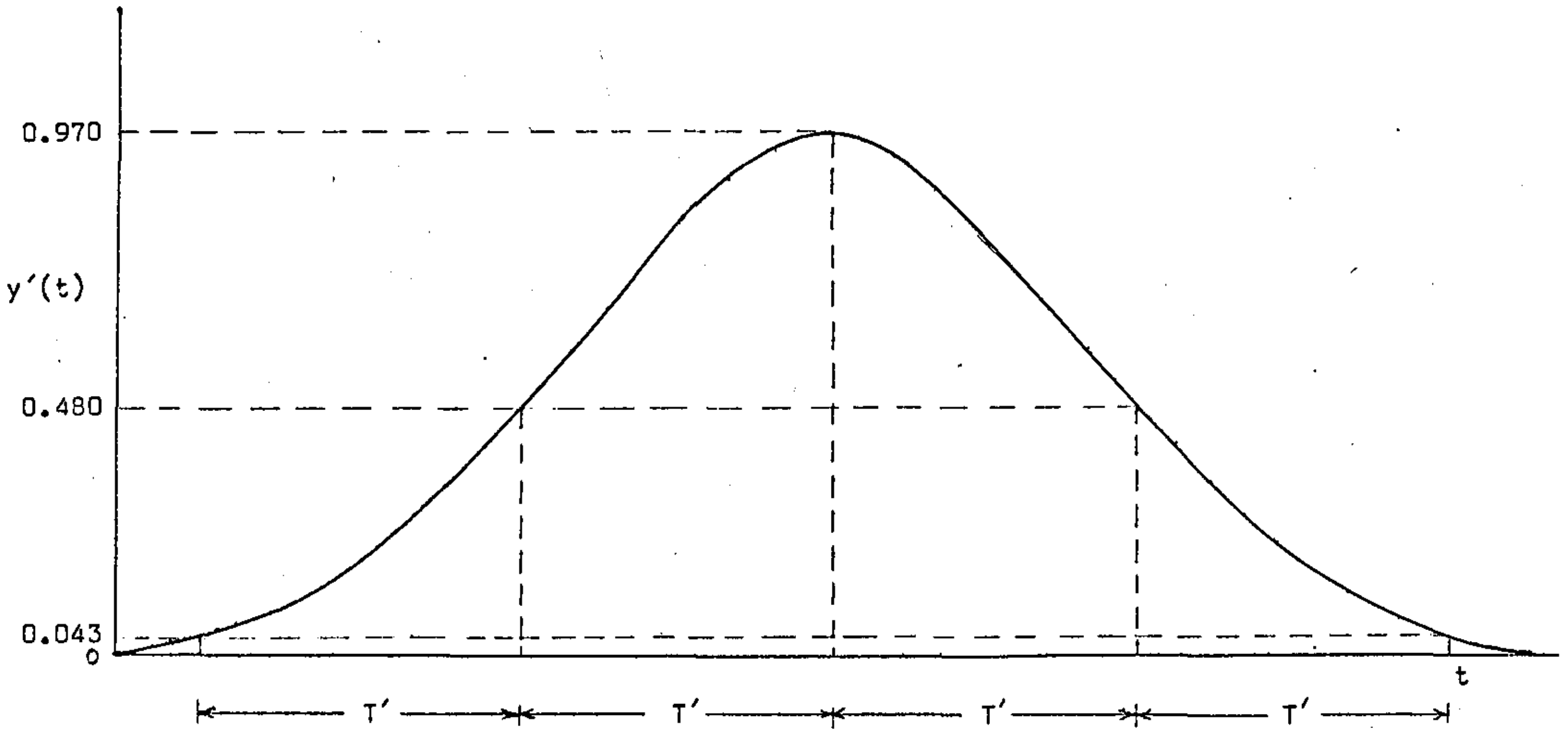


Figure 5.6-6

The impulse response of the baseband channel in the case of the continuous system, where the transmission path has an impulse response

$$2.0^{-\frac{1}{2}}\{0.235\delta(t)+0.667\delta(t-T)+\delta(t-2T)+0.667\delta(t-3T)+0.235\delta(t-4T)\}$$

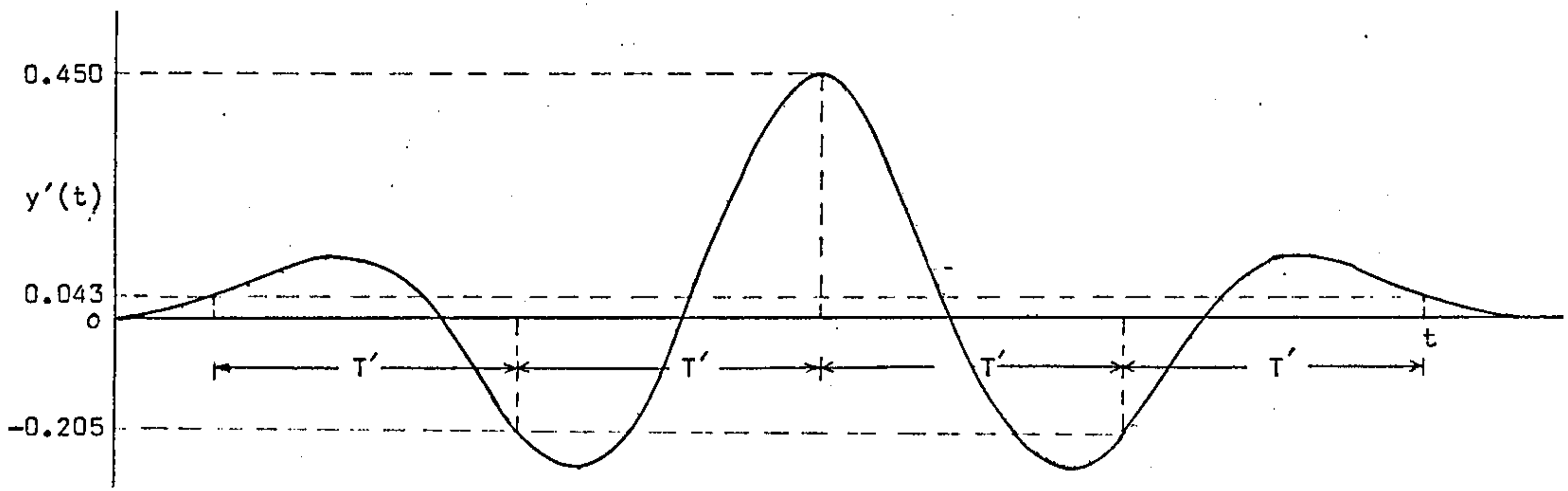


Figure 5.6-7

The impulse response of the baseband channel in the case of the continuous system, where the transmission path has an impulse response

$$2.0^{-\frac{1}{2}}\{0.235\delta(t)-0.667\delta(t-T)+\delta(t-2T)-0.667\delta(t-3T)+0.235\delta(t-4T)\}$$

TABLE 5.6-3

Noise level for an error probability of 4×10^{-3}, expressed in dB relative to its value when a binary signal is transmitted over channel A with the same error probability.

Transmission path	Linear Equalizer		Non-linear Equalizer	
	Number of Tap Gains	Noise Level	Number of Tap Gains	Noise Level
A	1	0.0	1	0.0
B_1	9	+0.21	7	+0.45
B_2	*	*	1	-4.96
C_1	16	-5.3	3	-3.0
C_2	*	*	2	-3.0
D_1	17	-2.47	13	-0.6
D_2	56	-23.54	60	-8.36
E_1	13	-0.4	9	-0.34
E_2	13	-0.4	10	-0.34
F_1	22	-3.5	8	-1.06
F_2	12	-2.9	19	-2.71
G_1	10	-0.43	8	-0.34
G_2	65	-10.76	62	-2.75
H_1	16	-1.44	11	-0.18
H_2	10	-3.84	2	-2.96
I_1	7	+0.1	8	+0.21
I_2	19	-5.82	2	-2.96
J_1	30	-11.76	21	-3.63
J_2	21	-13.04	17	-9.14
K_1	21	-2.45	17	-0.9
K_2	*	*	10	-6.05

TABLE 5.6-4

Noise level in the case of Systems 1, 2 and 6 with groups of eight binary signal-elements and an error probability of 4×10^{-3}, expressed in dB relative to its value in the case of the corresponding binary continuous system with an equalizer at the receiver and an element error probability of 4×10^{-3}.

Transmission path	Noise level relative to that of the linear equalizer			Noise level relative to that of the non-linear equalizer		
	System 1	System 2	System 6	System 1	System 2	System 6
A	0.0	0.0	0.0	0.0	0.0	0.0
B_1	-6.21	-1.41	-1.51	-6.45	-1.65	-1.75
B_2	*	*	*	-1.04	+3.76	+3.66
C_1	+2.1	+4.9	+4.7	-0.2	+2.6	+2.4
C_2	*	*	*	-0.2	+2.6	+2.4
D_1	-11.23	+0.07	-0.33	-13.1	-1.8	-2.2
D_2	+9.84	+21.14	+20.74	-5.34	+5.96	+5.56
E_1	+0.1	+0.4	+0.4	+0.04	+0.34	+0.34
E_2	+0.1	+0.4	+0.4	+0.04	+0.34	+0.34
F_1	+0.2	+3.0	+2.5	-2.24	+0.56	+0.06
F_2	-0.4	+2.4	+1.9	-0.59	+2.21	+1.71
G_1	-2.47	+0.43	+0.03	-2.56	+0.34	-0.06
G_2	+7.86	+10.76	+10.36	-0.15	+2.75	+2.35
H_1	-1.76	+0.74	+0.14	-3.02	-0.52	-1.12
H_2	+0.64	+3.14	+2.54	-0.24	+2.26	+1.66
I_1	-3.8	-0.1	-0.7	-3.91	-0.21	-0.81
I_2	+2.12	+5.82	+5.22	-0.74	+2.96	+2.36
J_1	-5.84	+7.36	+4.36	-13.97	-0.77	-3.77
J_2	-4.56	+8.64	+5.64	-8.46	+4.74	+1.74
K_1	-2.45	+1.25	+0.85	-4.0	-0.3	-0.7
K_2	*	*	*	+1.15	+4.85	+4.45

Table 5.6-5 shows the transmission paths for which the performance of System 1 is either better than, worse than or approximately the same as that of the linear equalizer. The performances of two systems are here considered to be approximately the same if they do not differ from each other by more than one dB. It can be seen from Table 5.6-5 that in the majority of the transmission paths tested, the performance of the linear equalizer is either better than or approximately the same as that of System 1. This shows that for a given information rate, the continuous system with a linear equalizer at the receiver gives, in general, a better tolerance to additive white Gaussian noise than does System 1 of the corresponding arrangement with signals transmitted in separate groups. This is so because the sampled impulse response of the baseband channel in the case of the continuous system generally exhibits a lower level of intersymbol interference than that in the case of the interrupted system, essentially because of the lower element rate over the given transmission path. Except in the case of the transmission path B_1 where the linear equalizer gains an advantage of 1.41 dB and 1.51 dB in tolerance to noise over Systems 2 and 6, respectively, the performance of Systems 2 and 6 in every case, is either better than or approximately equal to that of the linear equalizer.

Table 5.6-6 shows the transmission paths for which the performance of System 2 is either better than, worse than or approximately the same as that of the non-linear equalizer. The performances of two systems are here considered to be the same if they do not differ from each other by more than one dB. For all the transmission paths tested except in the case of the transmission paths B_1 and D_1, the performance of System 2 is either better than or approximately equal to that of the non-linear equalizer. For the transmission paths B_1 and D_1 the non-linear equalizer gains an

TABLE 5.6-5

Table showing the transmission paths for which the performance of System 1 is either better than, worse than or approximately the same as that of the linear equalizer

Better	Worse	Approximately the same
C_1	B_1	E_1
D_2	D_1	E_2
G_2	G_1	F_1
I_2	H_1	F_2
-	I_1	H_2
-	J_1	-
-	J_2	-
-	K_1	-

TABLE 5.6-6

Table showing the transmission paths for which the performance of System 2 is either better than, worse than or approximately the same as that of the non-linear equalizer

Better	Worse	Approximately the same
B_2	B_1	E_1
C_1	D_1	E_2
C_2	-	F_1
D_2	-	G_1
F_2	-	H_1
G_2	-	I_1
H_2	-	J_1
I_2	-	K_1
J_2	-	-
K_2	-	-

advantage of 1.65 dB and 1.8 dB, respetively, in tolerance to noise over System 2. It may be pointed out that the tolerance to noise of the non-linear equalizer is here calculated neglecting the error extension effects (Section 2.4), and its actual value is slightly less (a fraction of one dB) than that given in Table 5.6-3. It can be seen from Tables 5.6-6, 5.6-4 and 5.6-1 that in the cases where System 2 gâins an advantage in tolerance to noise over the non-linear equalizer, one or more of the following conditions are fulfilled -

1. The sampled impulse response of the baseband channel in the case of the continuous system, is such that it cânnot be equalized by a linear transversal equalizer (channels corresponding to the transmission paths B_2, C_2 and K_2).
2. System 1 has a better tolerance to noise than the linear transversal equalizer (channels corresponding to the transmission paths C_1, D_2, G_2 and I_2).
3. The sampled impulse response of the transmission path is such that if y_i is the sample value having the maximum amplitude then one or both of the sample values y_{i-1} and y_{i+1} have signs opposite to that of y_i (transmission paths B_2, D_2, F_2, G_2, I_2, J_2, and K_2).

In the cases where the distortion introduced by the transmission path is nearly pure phase distortion (transmission paths E_1 and E_2), the performances of System 2 and the non-linear equalizer are approximately the same.

Since g/m is probably somewhat larger here than would be used in a practical system, the results of the comparison carried out in this section, investirate that, in practice, a useful advantage in tolerance to noise should be obtained by System 2 over both linear and non-linear equalizers, where the latter are used with uninterrupted signal but the same information rate.

The comparison of the tolerances to additive white Gaussian noise of systems using orthogonal groups with those using the uninterrupted transmission, carried out in this section, is somewhat arbitrary. This is so because the comparison has been made for one particular transfer function of the transmitter and receiver filters. The results do not apply for filter pairs having transfer functions other than that assumed here. Also, the choice of the phase of the sampling instants to obtain the sampled impulse response of the base-band channel for the case of continuous transmission, is arbitrary and is such that, in general, it helps to improve the performance of the linear and non-linear equalizers. If the sampling instants are different than what have been assumed here, then the performance of the two equalizers will, most probably, be lower than that given in Table 5.6-3.

6.0. LINEAR PROCESSING OF THE SIGNAL AT THE TRANSMITTER

6.1 Process of Linear Equalization at the Transmitter

In the arrangement studied in this section, the groups of m transmitted signal-elements are processed at the transmitter so that no signal processing is required at the receiver, other than the comparison of the sampled values of the received group of m signal-elements, with the appropriate thresholds. The processing of a group of m signal-elements is here achieved by placing an m x n linear network F at the transmitter of the serial synchronous baseband data-transmission system discussed in Sections 2.1 and 3.1, which is now modified to the arrangement of Fig. 6.1-1. The m element values $\{s_i\}$ of a group of signal-elements are fed to the m input terminals of the network F, which transforms these values into the corresponding n values given by the components of the n-component row-vector SF, where F is an mxn matrix of rank m defining the linear network. The n output values from the network are sampled in sequence at regular intervals of T seconds, to give the corresponding sequence of n impulses which are fed to the baseband channel. Immediately following the transmission of a group of n impulses, the next set of m elements values are fed to the input of the network F, to give the corresponding set of n output values, which are again sampled in sequence. The process continues in this way and is such that a continuous sequence of regularly spaced impulses is fed to the baseband channel. As before, $n = m + g$.

Let

$$B = SF \qquad (6.1\text{-}1)$$

be the n-component row-vector whose components $\{b_i\}$ are the n values at the n outputs of the m x n network F, when the m values $\{s_i\}$ given by the components of S are fed to the m inputs of F. The n values $\{b_i\}$ at the

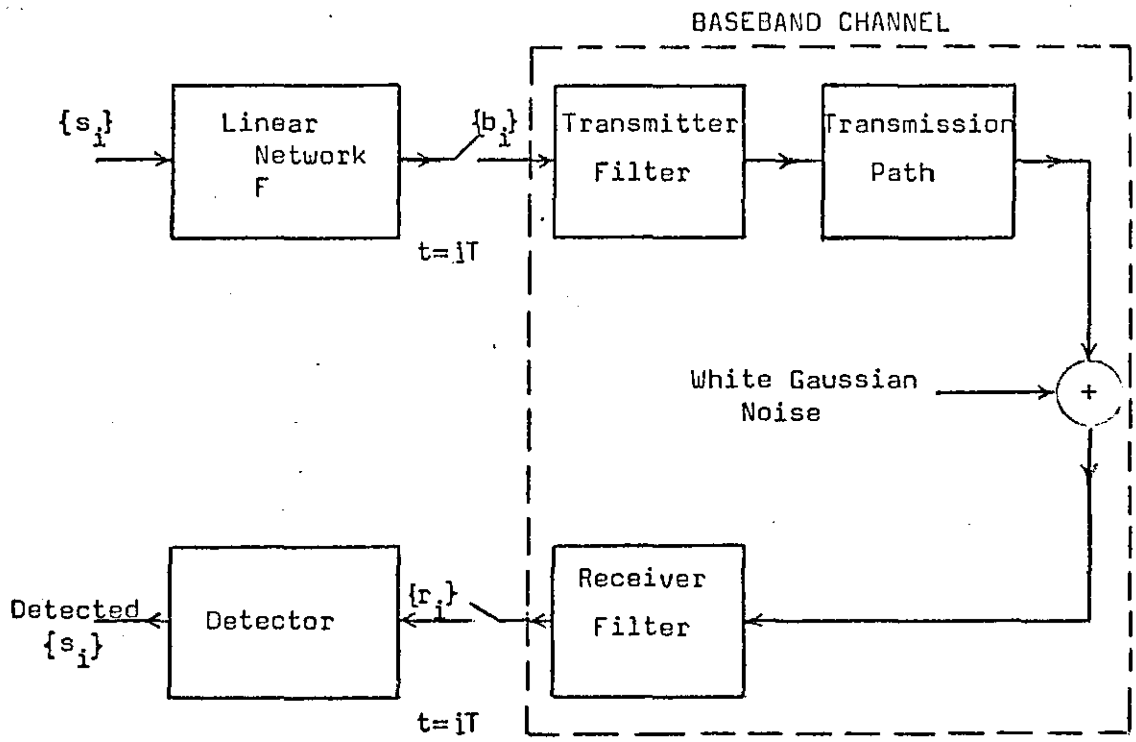


Figure 6.1-1

Serial synchronous baseband data-transmission system with linear signal processing at the transmitter for no signal distortion at the receiver.

outputs of the linear network F are fed, at intervals of T seconds in the form of the corresponding impulses, to the baseband channel which includes the transmitter filter, transmission path and the receiver filter. At the input to the receiver filter, white Gaussian noise with zero mean and a two sided power spectral density of σ^2, is added to the received signal. At the output of the receiver filter, the received signal is sampled at the time instants $t = iT$.

It is assumed that the $g+1$ sample values of the sampled impulse response of the channel,

$$y_0 \; y_1 \; y_2 \; \dots \; y_g \qquad (6.1\text{-}2)$$

are known at the transmitter and the m signal-elements of a group are statistically independent and equally likely to have any of the possible element values. For the sake of convenience, the delay in transmission other than that involved in the time dispersion of the transmitted signal, is ignored.

Consider just a single group of m signal-elements whose values are given by the m components of the row-vector S, at the input to the linear network F in Fig. 6.1-1. The m element values are fed simultaneously to the linear network F over a period of nT seconds, so that over this period the n output terminals of the network F hold the n components of the vector $B = SF$. The output signals from the terminals of F are sampled in order, at regular intervals of T seconds, to give a sequence of impulses $\sum_{i=1}^{n} b_i \delta(t - iT)$ fed to the transmitter filter. It is for convenience assumed here that the first of the n impulses is transmitted at time $t = T$.

The sample values of the received signal, corresponding to a single group of m signal-elements, will normally be a sequence of $(n+g)$ non-zero sample values preceded and followed by zero sample values. The sequence of these $(n+g)$ sample values in the absence of noise is

$$v_i = \sum_{j=1}^{n} b_j \; y_{i-j} \;, \quad i = 1,2,\ldots, n+g \qquad (6.1\text{-}3)$$

where $y_i = o$ for $i < o$ and $i > g$. Let C be the n x (n+g) matrix whose i^{th} row is

$$C_i = \overbrace{o \ldots o}^{i-1} \; \overbrace{y_o \ldots y_g}^{g+1} \; \overbrace{o \ldots o}^{n-i} \qquad (6.1\text{-}4)$$

From Eqn. (6.1-3)

$$V = BC \qquad (6.1\text{-}5)$$

where $V = v_1 \; v_2 \ldots v_{n+g}$ is the (n+g)- component row-vector whose components $\{v_i\}$ are the sample values of the received signal corresponding to a group of m signal-elements, in the absence of noise (Eqn. (6.1-3)), and B is the n-component row-vector in Eqn. (6.1-1). The central m components of the vector V,

$$v_{g+1} \quad v_{g+2} \quad \cdots \quad v_{g+m}$$

are given by BD^T where D is the m x n matrix of rank m, whose i^{th} row is

$$D_i = \overbrace{o \ldots o}^{i-1} \; \overbrace{y_g \; y_{g-1} \ldots y_o}^{g+1} \; \overbrace{o \ldots o}^{m-i} \qquad (6.1\text{-}6)$$

Thus,

$$BD^T = v_{g+1} \quad v_{g+2} \quad \cdots \quad v_{g+m} \qquad (6.1\text{-}7)$$

Assume now that successive groups of signal-elements are transmitted in the arrangement of Fig. 6.1-1, and suppose that one of these groups is that just considered, where the first transmitted impulse of the group occurs at time T seconds. The corresponding n+g received samples which are the n+g components of V and are dependent on the m transmitted elements of the group, are shown in Fig. 6.1-2. Each sample value is here, for simplicity, shown as a positive impulse. It can be seen that the first g components of V are dependent in part on the preceding received group of m signal-elements, and the last g components of V are dependent in part on the following received group of m elements. Thus there is intersymbol interference from adjacent received groups of elements in both the first and last g components of V. However, the central m components of V, which are

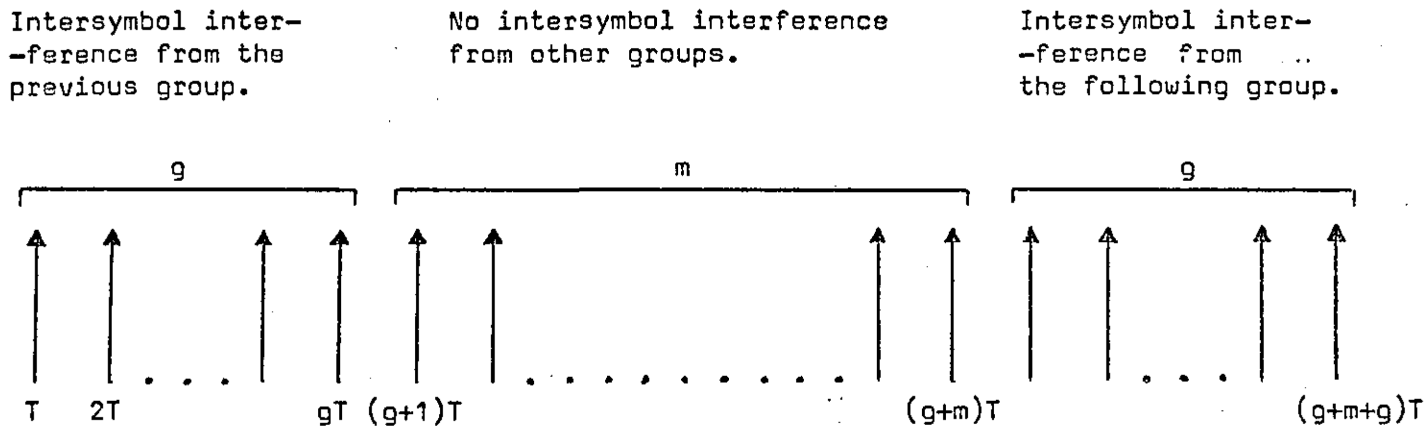


Figure 6.1-2

Sequence of (n+g) sample values of the received signal

due to a single group of m signal-elements.

v_{g+1}, v_{g+2}, ..., v_{g+m}, depend only on the corresponding transmitted group of m elements, and can therefore be used for the detection of these elements with no intersymbol interference from adjacent groups. The first of the central m components of V is v_{g+1} and is the sample value of the received signal at time (g+1) T. The last of the central m components of V is v_{g+m} and is the sample value of the received signal at time (g+m)T. From Eqn. (6.1-7), the m sample values of the received signal which depend only on the corresponding m signal - elements of a group, are, in the absence of noise, the components of the m-component row-vector

$$BD^T \tag{6.1-8}$$

where B = SF is an n-component row-vector and the m $\{s_i\}$ are the values of the m signal-elements of the group at the input to the network F, and are of course the components of the vector S.

Suppose now that the linear network F at the transmitter in Fig. 6.1-1, is such that, corresponding to a group of m signal-elements at the transmitter, the m sample values of the received signal in Eqn. (6.1-8), in the absence of additive noise,are

$$BD^T = S \tag{6.1-9}$$

When noise is present, the m sample values are the m components of the vector

$$R = BD^T + W \tag{6.1-10}$$

where W is an m-component row-vector whose components are sample values of statistically independent Gaussian random variables with zero mean and variance σ^2. Thus the detector in Fig. 6.1-1, can now detect the element values $\{s_i\}$ of the signal-elements by comparing the corresponding $\{r_i\}$ with the appropriate thresholds. Each received group of m signal-elements is detected from the corresponding m sample values, the remaining g sample values being ignored at the detector, and, under these conditions, the received groups of m signal-elements are orthogonal at the detector input.

To maximize the tolerance to noise at the detector input, the n $\{b_i\}$ should be selected such that the total transmitted energy of the n $\{b_i\}$ is minimized. In other words

$$BB^T = |B|^2 \qquad (6.1\text{-}11)$$

must be minimized for the given vector S. Thus the problem is to find an m x n linear network F in Fig. 6.1-1, which minimizes the transmitted element energy and at the same time satisfies the constraint in Eqn. (6.1-9).

From Eqn. (6.1-9)

$$BD_i^T = s_i \quad \text{for } i = 1,2, \ldots, m. \qquad (6.1\text{-}12)$$

Suppose that

$$|D_i| = d \text{ for } i = 1,2, \ldots, m \qquad (6.1\text{-}13)$$

where $|D_i|$ is the length of the vector D_i (i.e. the distance of the point D_i from the origin in an n-dimensional vector space). It can be seen from Eqn. (6.1-6) that $|D_i|$ is independent of i. BD_i^T is the inner product of the vectors B and D_i, so that from Eqn. (6.1-13) it is d times the value of the orthogonal projection of B onto the vector D_i. Thus from Eqn. (6.1-12), B lies on the hyperplane ((n - 1)-dimensional subspace) which contains the point $(s_i/d)D_i$ and which is orthogonal to the vector given by this point, so that the hyperplane is orthogonal to the line joining the origin to $(s_i/d)D_i$. The vectors B and D_i are shown in Fig. 6.1-3, for the case where $d > 1$ and $s_i = 1$. The vector B must, therefore, lie on each of the m hyperplanes given by Eqn. (6.1-8) and as illustrated in Fig. 6.1-3. Thus, the required vector B is the point on these m hyperplanes at the minimum distance from the origin. By the Projection Theorem [21,52] B is the orthogonal projection of the origin onto the (n-m)-dimensional subspace formed by the intersection of the m hyperplanes. Thus B is the inter-section of the m-dimensional subspace spanned by the m $\{D_i\}$ (each of which is

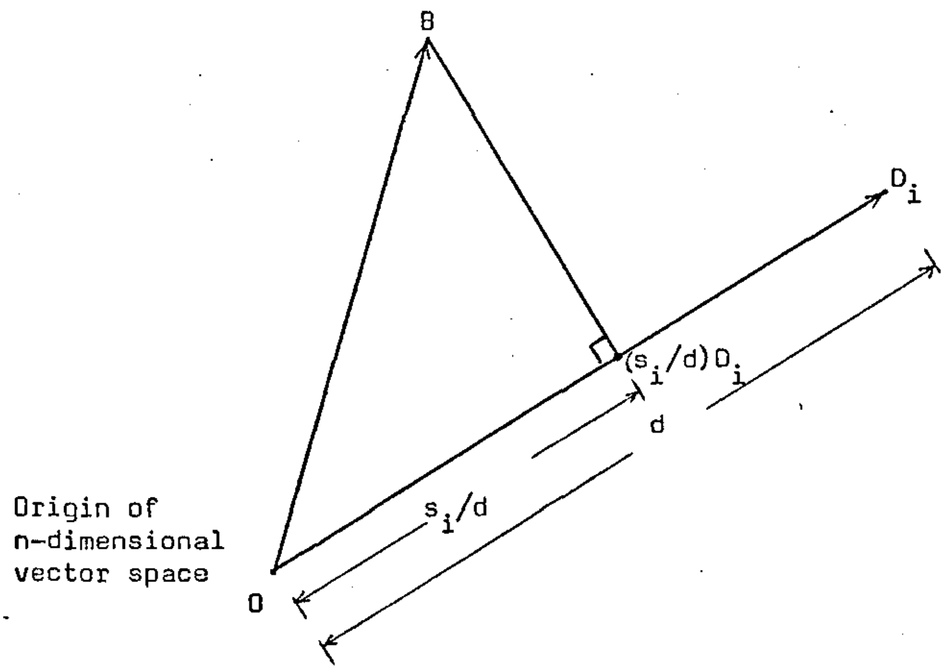


Figure 6.1-3

The vectors B and D_i for d>1 and $s_i = 1$.

orthogonal to the corresponding hyperplane), with the (n-m)-dimensional subspace formed by the intersection of the m hyperplanes. Clearly B can be represented as a linear combination of the m $\{D_i\}$, so that

$$B = \sum_{i=1}^{m} e_i D_i = ED \qquad (6.1\text{-}14)$$

where $$E = e_1 e_2 \ . \ . \ . \ e_m. \qquad (6.1\text{-}15)$$

From Eqn. (6.1-9),

$$S = BD^T = EDD^T \qquad (6.1\text{-}16)$$

Thus $$E = S(DD^T)^{-1} \qquad (6.1\text{-}17)$$

and $$B = S(DD^T)^{-1}D \qquad (6.1\text{-}18)$$

$(DD^T)^{-1}D$ is an m x n matrix of rank m.[37,40] It can be seen that if the impulses given by the components of the vector B are transmitted over the given channel, then the corresponding m sample values at the detector input, in the absence of noise, are given by the m components of the row-vector

$$BD^T = S(DD^T)^{-1}DD^T = S \qquad (6.1\text{-}19)$$

which agrees with Eqn. (6.1-9).

From Eqn. (6.1-1) and (6.1-18), the matrix F representing the linear network at the transmitter in Fig. 6.1-1, is an m x n matrix of rank m, given by

$$F = (DD^T)^{-1}D \qquad (6.1\text{-}20)$$

Thus, under the assumed conditions, the linear network F in Fig. 6.1-1 and given by Eqn. (6.1-20), is such that it maximizes the tolerance to additive white Gaussian noise in the detection of the m signal-elements of a received group from the corresponding m sample values at the input to the detector. The system which employs the optimum linear processing of groups of m-signal elements at the transmitter, just described, is considered in Section 6.2.

6.2 System 1L

In System 1L, the linear processing of groups of m signal-elements, as described in Section 6.1, is used at the transmitter. The operation of the system can be explained with the help of Fig. 6.2-1. Over the appropriate period of nT seconds the values of the m signal-elements of a group, given by the components of the vector S, are fed to the m input terminals of the m x n linear network $(DD^T)^{-1}D$, to give the n values $\{b_i\}$ at the n output terminals of the network. The n $\{b_i\}$ are then sampled in order, at regular intervals of T seconds, and the corresponding impulses are fed to the input of the baseband channel. The signal-elements of a group, at the transmitter, are assumed to be k-level, and they are statistically independent and equally likely to have any of the k possible values. The value of each s_i at the transmitter in Fig. 6.2-1, is divided by a positive scalar quantity ℓ before feeding it to the linear network $(DD^T)^{-1}D$.

The received signal at the output of the baseband channel is sampled at regular intervals of T seconds, and the (g+1)th to the (g+m)th of the (n+g) sample values, dependent on the group of m elements are stored. Thus the detector ignores the first gT seconds of the received waveform corresponding to each group of m elements, and detects the m $\{s_i\}$ from the m stored sample values by comparing these with the appropriate thresholds.

Assume that the transmitted signal-elements are either 2-, 4- or 8- level, and that the possible values of s_i are in each case equally likely and as given in Table 5.2-2, so that the mean square value of s_i is equal to the number of bits per element. Suppose that the m vectors $\{D_i\}$, which are the rows of the matrix D, each have unit length, and let the value of ℓ in Fig. 6.2-1 be such that the average transmitted energy per bit is unity. Since there are m k-level signal-elements in a group,

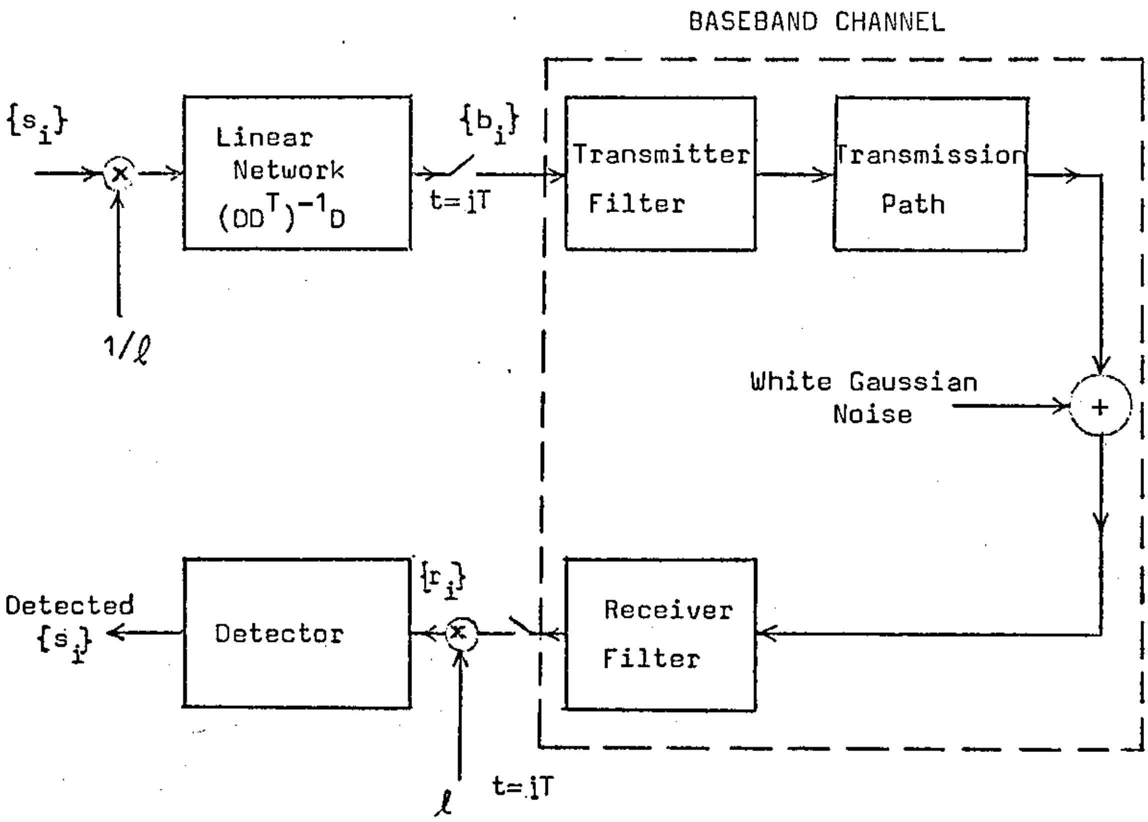


Figure 6.2-1

System 1L.

where k may have the value 2, 4 or 8, the vector S has k^m possible values each corresponding to a different combination of the m k-level signal-elements. It follows, therfore, that the vector $S(DD^T)^{-1}D$ whose components are the values of the corresponding impulses fed to the baseband channel, has k^m different possible values. If e is the total energy of the components of all the k^m possible values of the vector $S(DD^T)^{-1}D$, then clearly

$$\ell = (e / m k^m)^{\frac{1}{2}} \qquad (6.2\text{-}1)$$

The m sample values of the received signal from which the corresponding m $\{s_i\}$ are detected, are the components of the vector

$$R' = \frac{1}{\ell} BD^T + W \qquad (6.2\text{-}2)$$

where $B = S(DD^T)^{-1}D$, and W is an m-component row-vector whose components are sample values of statistically independent Gaussian random variables with zero mean and variance σ^2. In order to make the values of the signal-elements at the detector input, equal to the $\{s_i\}$ at the transmitter, the m sample values which are the components of the vector R', must first be multiplied by ℓ as shown in Fig. 6.2-1. From Eqn. (6.2-2)

$$\begin{aligned} R = \ell R' &= BD^T + \ell W \\ &= S(DD^T)^{-1}DD^T + \ell W \\ &= S + U \qquad (6.2\text{-}3) \end{aligned}$$

where U is an m-component row-vector whose components are sample values of statistically independent Gaussian random variables with zero mean and variance

$$\eta^2 = \ell^2\sigma^2 \qquad (6.2\text{-}4)$$

Thus, the tolerance to noise of System 1L is determined by the value of η^2 given in Eqn. (6.2-4). When there is no signal distortion, that is, when $y_o = 1$ and $y_i = 0$ for all i, $i \neq 0$, $(DD^T)^{-1}$ is an identity matrix. Under these conditions, ℓ has the value unity, so that

$$\eta^2 = \sigma^2 \qquad (6.2\text{-}5)$$

Whereas, in System 1L, the m $\{s_i\}$ are fed to the m x n network $(DD^T)^{-1}D$ at the transmitter to give the n $\{b_i\}$, in System 1, the n $\{r_i\}$ are fed to the n x m network $Y^T(YY^T)^{-1}$ at the receiver to give the m $\{x_i\}$. Furthermore, both D and Y are m x n matrices and the i^{th} row of D is obtained from the i^{th} row of Y simply by reversing the order of the non-zero components y_o, y_1, ..., y_g. It is clear therefore that, just as the Gauss-Siedel process can be used to implement the transformation $Y^T(YY^T)^{-1}$, so, with the appropriate modifications, it can also be used for the transformation $(DD^T)^{-1}D$.[52]

Referring to Fig. 6.2-1, the m x n linear network $(DD^T)^{-1}D$ converts the m element values $\{s_i\}$ such that, under the assumed conditions, the corresponding m sample values at the receiver, have the minimum mean square error due to the presence of additive white Gaussian noise. In this sense they are the best linear estimates of the $\{s_i\}$. Similarly, when the linear network $Y^T(YY^T)^{-1}$ is used at the receiver, with no signal processing at the transmitter, as in System 1, the m sample values at the output of this network have the minimum mean square error due to the presence of additive white Gaussian noise. In this sense they too are the best linear estimates of the $\{s_i\}$. Clearly, Systems 1L and 1 are duals of each other in the sense that each provides the best linear estimate of a received group of m signal-elements, and in System 1 all the signal processing is carried out at the receiver while in System 1L all the signal processing is achieved at the transmitter.

6.3 Linear Equalization Process Shared Between the Transmitter and Receiver

In the arrangement of Fig. 6.3-1, let F_1 be an m x m network at the transmitter of the synchronous serial baseband data-transmission

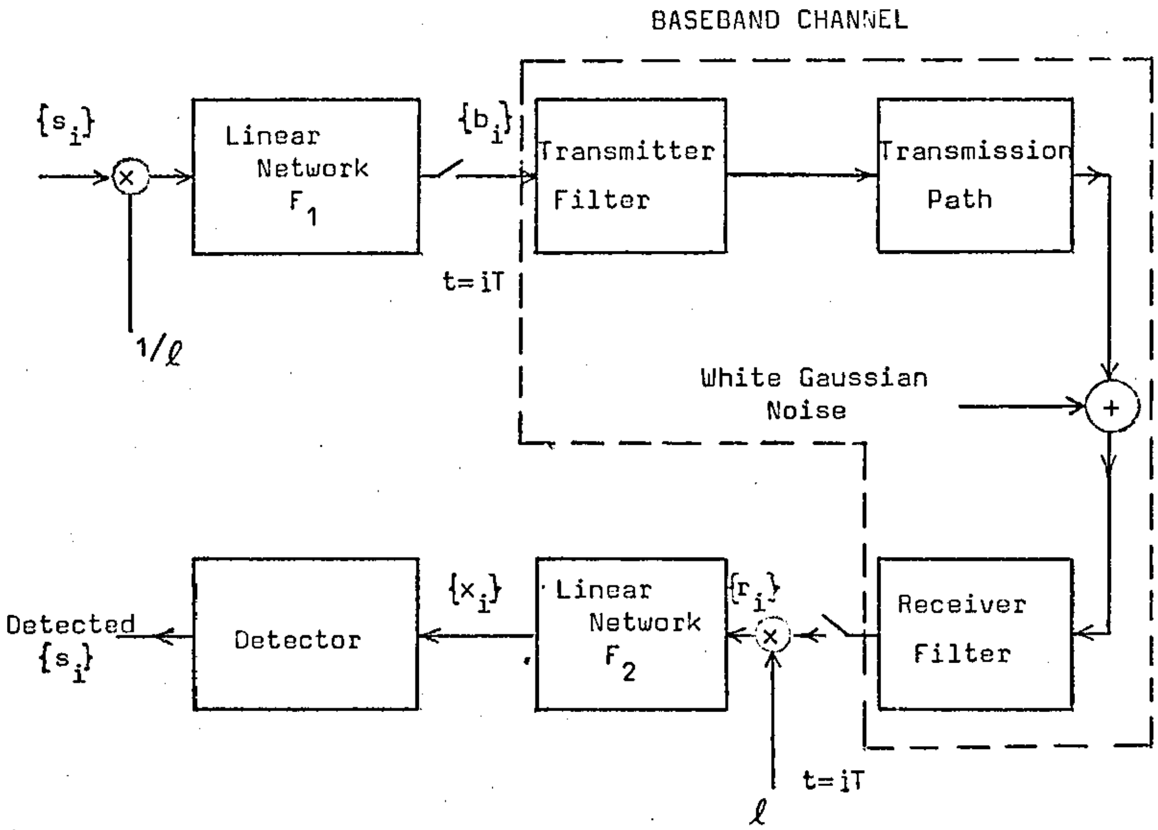


Figure 6.3-1

Process of linear equalization shared between the transmitter and the receiver.

system discussed in Sections 2.1 and 3.1. The m element values $\{s_i\}$ of the m signal-elements of a group are fed to the m input terminals of the linear network F_1 which transforms these values into the m values given by the m components of the row-vector

$$SF_1 = B \tag{6.3-1}$$

where F_1 is an m x m non-singular matrix defining the linear network. The signals at the output of the network F_1 are sampled in order and at regular intervals of T seconds to give a sequence of m impulses which are fed to the input of the baseband channel. The values (areas) $\{b_i\}$ of the m impulses, just mentioned, correspond to the m components of SF_1. Adjacent groups of m impulses at the input to the baseband channel, corresponding to adjacent groups of m signal-elements, are separated by g zero-level impulses. The m signal-elements of a group are assumed to be k-level, where k = 2, 4 or 8 and the possible values of each s_i are as given in Table 5.2-2. The $\{s_i\}$ are statistically independent and equally likely to have any of the k possible values. The value of each s_i is divided by the positive scalar quantity ℓ before feeding it to the network F_1 (Fig.6.3-1).

At the input to the receiver filter, white Gaussian noise with zero mean and a two sided power spectral density σ^2 is added to the received signal. At the output of the receiver filter, the received signal is sampled at regular intervals of T seconds. Since adjacent groups of m impulses at the input to the baseband channel are separated by g zero-level impulses and since n = m + g, it is clear that the n sample values of a received group of m signal-elements will depend only on the corresponding m element values $\{s_i\}$ and not on any other transmitted element. Thus the received groups of signal-elements are orthogonal at the receiver. The receiver stores the n sample values corresponding to a group of m received signal-elements and uses these sample values in the detection of the m elements.

It is assumed here that the (g + 1) sample values of the sampled impulse response of the channel,

$$y_o\ y_1\ \dots\ y_g \qquad (6.3\text{-}2)$$

are known both at the transmitter and at the receiver. For the sake of convenience, the delay in transmission other than that involved in the time dispersion of the transmitted signals is ignored here. The value of ℓ in Fig. 6.3-1 is such that the average transmitted energy per bit is unity. Following the explanation of Section 6.2, if e is the total energy of the components of all the k^m possible values of the vector SF_1, then

$$\ell = (e/m\ k^m)^{\frac{1}{2}} \qquad (6.3\text{-}3)$$

The n sample values corresponding to a received group of m signal-elements are, from Section 3.2, the components of the n-component row-vector

$$R' = \frac{1}{\ell} BY + W \qquad (6.3\text{-}4)$$

where $B = SF_1$ and Y is the m x n matrix of rank m, whose i^{th} row is

$$Y_i = \overbrace{0\ .\ .\ .0}^{i-1}\ \overbrace{y_o\ y_1\ .\ .\ .\ y_g}^{g+1}\ \overbrace{0\ .\ .\ .\ 0}^{m-i} \qquad (6.3\text{-}5)$$

The n components of the row-vector W are sample values of statistically independent Gaussian random variables with zero mean and variance σ^2. As before, assume that $|Y_i| = 1$.

In order to correct for the division of the $\{s_i\}$ by ℓ at the transmitter, the n sample values of a received group of m elements, which are the components of the row-vector R', must now be multiplied by ℓ to give

$$R = \ell R' = BY + \ell W = SF_1 Y + \ell W \qquad (6.3\text{-}6)$$

where ℓW is an n-component row-vector whose components are sample values of statistically independent Gaussian random variables with zero mean and variance

$$\ell^2 \sigma^2 \qquad (6.3\text{-}7)$$

Let F_2 in Fig. 6.3-1, be an n x m linear network such that, when the n sample values of the m received signal-elements of a group are fed to its n input terminals, the m outputs, in the absence of noise, are the m element values $\{s_i\}$ of the received group. Clearly, from Eqn. (6.3-6), in the absence of noise,

$$RF_2 = SF_1YF_2 = S \qquad (6.3\text{-}8)$$

since it is assumed that the two networks F_1 and F_2 together equalize the channel. Thus

$$F_1YF_2 = I \qquad (6.3\text{-}9)$$

where I is an m x m identity matrix. In the presence of noise, the m outputs from the linear network F_2, are the m components of the row-vector

$$X = RF_2 = SF_1YF_2 + \ell WF_2 \qquad (6.3\text{-}10)$$

$$= S + U \qquad (6.3\text{-}11)$$

where

$$U = \ell WF_2 \qquad (6.3\text{-}12)$$

The m-component vector U is the noise-vector at the output of the network F_2. The i^{th} component u_i' of the noise vector U, is a sample value of a Gaussian random variable with zero mean and variance

$$\eta_i^2 = \ell^2\sigma^2(\sum_{j=1}^{n} f_{2ji}^2)$$

$$= \ell^2\sigma^2(F_{2i}F_{2i}^T) \qquad (6.3\text{-}13)$$

from Eqn. (6.3-12), since the n components of W are sample values of statistically independent Gaussian random variables. f_{2ji} is the component in the j^{th} row and i^{th} column of the matrix F_2, and F_{2i} is an n-component vector given by the i^{th} column of the matrix F_2. Each s_i can now be detected by comparing the corresponding x_i with the appropriate thresholds. In general η_i^2 is a function of i, however, at high signal/

noise ratios the tolerance of the arrangement of Fig. 6.3-1, described above, to additive white Gaussian noise is approximately determined by the probability of error in the detection of the s_i for which η_i has the greatest value (Appendix A2).

It is now required to find the n x m linear network F_2 which provides the best linear estimate of the m signal-elements of a received group, given the network F_1 at the transmitter and the received signal-vector R. Let Y' be the m x n matrix given by

$$Y' = F_1 Y \qquad (6.3\text{-}14)$$

Since F_1 is an m x m non-singular matrix and Y has rank m, the m x n matrix Y' is of rank m, and, therefore, the m rows $\{Y_i'\}$ of the matrix Y' are linearly independent. From Eqns. (6.3-6) and (6.3-14)

$$R = SF_1Y + \ell W \qquad (6.3\text{-}15)$$

or

$$R = SY' + \ell W \qquad (6.3\text{-}16)$$

It can be seen that the problem of finding the optimum linear network F_2 is here similar to that in Section 3.2 where the corresponding n sample values of a received group of m signal-elements are the components of the vector (SY + W). Thus, following the procedure of Section 3.2, it is clear that the required network F_2 is given by

$$F_2 = Y'^T(Y'Y'^T)^{-1} \qquad (6.3\text{-}17)$$

$$= (F_1Y)^T\{F_1Y\ (F_1Y)^T\}^{-1}$$

$$= Y^TF_1{}^T\{(F_1(YY^T)F_1{}^T\}^{-1}$$

$$= Y^TF_1{}^T(F_1{}^T)^{-1}(YY^T)^{-1}F_1{}^{-1}$$

$$= Y^T(YY^T)^{-1}F_1{}^{-1} \qquad (6.3\text{-}18)$$

Now

$$F_1YF_2 = F_1YY^T(YY^T)^{-1}F_1{}^{-1} = I \qquad (6.3\text{-}19)$$

which agrees with Eqn. (6.3-9). It can be seen from Eqns. (6.3-16) and (6.3-17), that the n x m network F_2 gives at its m output terminals m linear estimates of the corresponding m $\{s_i\}$, in which the mean square error due to the noise is minimized, given the network F_1 at the transmitter and subject to the other assumed conditions. This follows because the network $Y'^T(Y'Y'^T)^{-1}$ in Fig.6.3-1 is the exact parallel of the network $Y^T(YY^T)^{-1}$ in System 1.

If the m x m matrix F_1 is an identity matrix, that is, if there is no signal processing at the transmitter, then the equalizer network F_2 at the receiver is given by

$$F_2 = Y^T(YY^T)^{-1}I$$

$$= Y^T(YY^T)^{-1} \qquad (6.3\text{-}20)$$

Clearly, under these conditions, the arrangement of Fig. 6.3-1 reduces to that of System 1, and hence System 1 is a special case of the more general class of systems studied here.

Data-transmission systems with different arrangements of the networks F_1 and F_2 in Fig. 6.3-1 will now be described. Since the systems, which are to be described are special cases of the arrangement of Fig. 6.3-1, their basic method of operation and implementation is similar to the arrangement studied in this section.

6.4 Systems 2L, 3L, 4L and 5L

In Systems 2L, 3L, 4L and 5L the arrangement of Fig. 6.3-1, described in Section 6.3, is used. In each system the linear network F_1, at the transmitter, is an m x m network represented by the m x m non-singular matrix F_1, and the n x m linear network at the receiver is represented by the n x m matrix F_2 of rank m. In each case the n sample values corresponding to a group of m received signal-elements are fed to the n input terminals of the network F_2 and the m signal-elements are

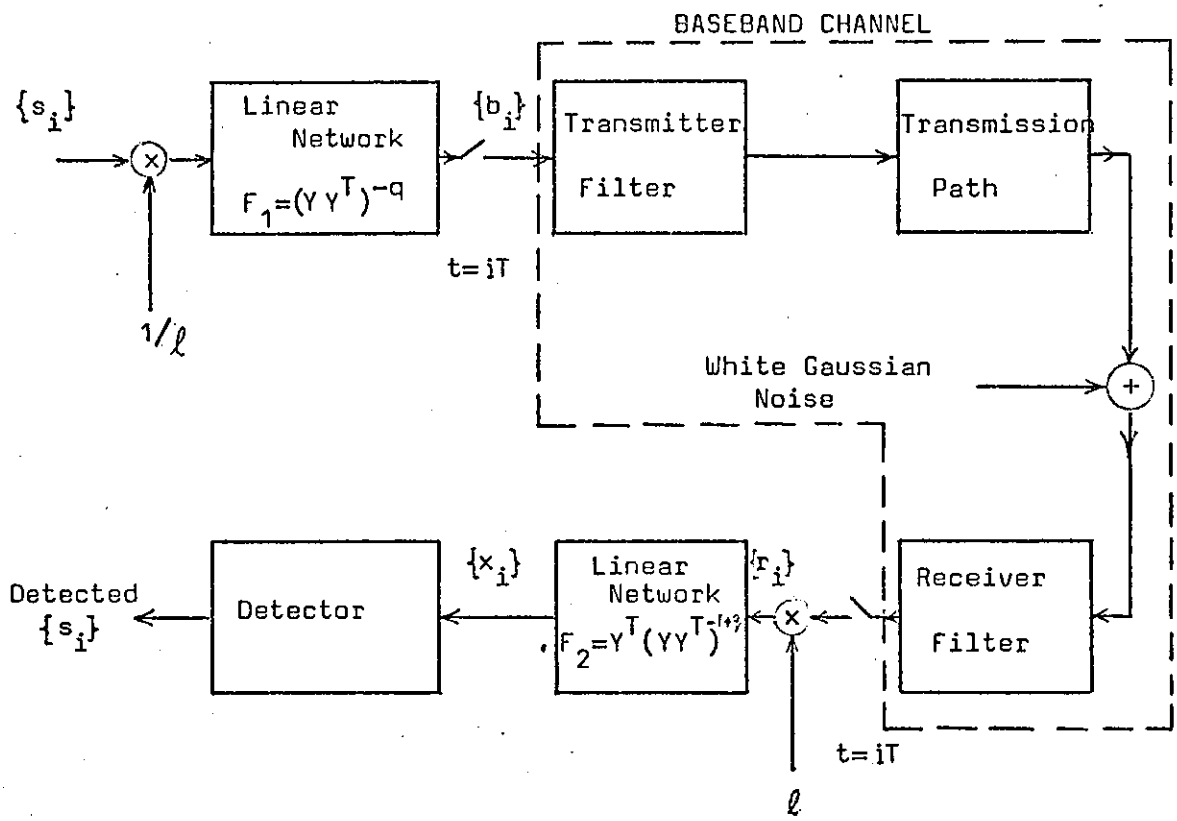


Figure 6.4-1

Systems 2L,3L,4L and 5L.

detected by comparing the m outputs, from the network F_2, with the appropriate thresholds.

In each system

$$F_1 = (YY^T)^{-q} \qquad (6.4\text{-}1)$$

and

$$F_2 = Y^T(YY^T)^{-1+q} \qquad (6.4\text{-}2)$$

The signal-vector at the output of the network F_1 is $\frac{1}{\ell}S(YY^T)^{-q}$, and in the absence of noise, the signal vector at the input to the network F_2 is $S(YY^T)^{-q}Y$, so that the signal vector at the output of F_2 is

$$S(YY^T)^{-q}YY^T(YY^T)^{-1+q} = S \qquad (6.4\text{-}3)$$

since YY^T is non-singular, real and symmetric. Exact equalization of the channel is, therefore, achieved for any real value of q in the range 0 to 1. Since YY^T is a real, symmetric and positive definite matrix, so that $(YY^T)^{-q}$ and $(YY^T)^{-1+q}$ are both real, symmetric and positive definite matrices.[37]

The following arrangements of Fig. 6.4-1 have been studied

System	Value of q
2L	1
3L	3/4
4L	1/2
5L	1/4

In the Systems 2L to 5L, each of the networks F_1 and F_2 would in practice be implemented by the appropriate Gauss-Siedel iterative process.[52]

6.5 Systems 6L, 7L, 8L and 9L

In Systems 6L to 9L, the arrangement of Fig. 6.3-1 described in Section 6.3, is modified to that shown in Fig. 6.5-1. In each System the linear network at the transmitter is an m x n network represented by an m x n matrix of rank m, given by

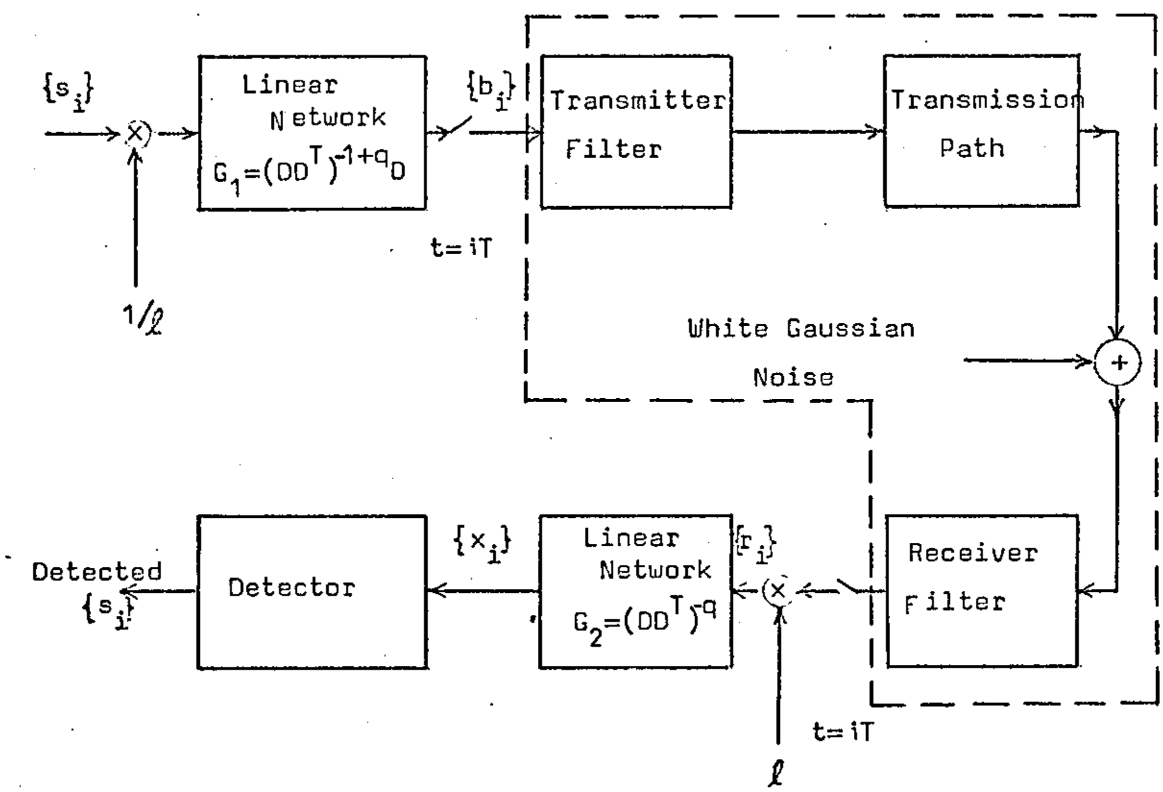


Figure 6.5-1

Systems 6L,7L,8L and 9L.

$$G_1 = (DD^T)^{-1+q}D \qquad (6.5\text{-}1)$$

where D is an m x n matrix of rank m, whose i^{th} row is

$$D_i = \overbrace{0\ \ldots.0}^{i-1}\ \overbrace{y_g y_{g-1}\ \ldots y_o}^{g+1}\ \overbrace{0\ \ldots.0}^{m-i} \qquad (6.5\text{-}2)$$

The linear network at the receiver is an m x m network represented by an m x m non-singular matrix

$$G_2 = (DD^T)^{-q} \qquad (6.5\text{-}3)$$

The arrangement at the transmitter in Fig. 6.5-1, is similar to that of Fig. 6.1-1, described in Section 6.1. There are now n values at the output of the network G_1 and these are sampled in order at regular intervals of T seconds to give the corresponding impulses which are fed to the baseband channel. The values of these impulses are the components of the n-component row-vector SG_1. Adjacent groups of n impulses at the input to the channel, corresponding to adjacent groups of m signal-elements, follow each other with no break in the regular sequence of impulses. It can be seen from Section 6.1, that the (g+1)th to (g+m)th samples of the received signal, corresponding to a received group of m signal-elements, depend only on the corresponding m element values $\{s_i\}$ and are the m components of the row-vector

$$R' = \frac{1}{\ell} SG_1D^T + W \qquad (6.5\text{-}4)$$

As before, ℓ is a positive scalar quantity such that when each s_i at the input to G_1 is divided by ℓ , the average transmitted energy per bit is unity. As in Section 6.3, if e is the total energy of the components of all the k^m possible values of the vector SG_1, then

$$\ell = (e/mk^m)^{\frac{1}{2}} \qquad (6.5\text{-}5)$$

The components of the m-component row-vector W in Eqn. (6.5-4), are sample values of statistically independent Gaussian random variables with zero mean and variance σ^2. The sample values of the received signal are first multiplied by ℓ in order to compensate for the division of the $\{s_i\}$ by ℓ at the transmitter. Thus the m sample values of the received signal which are fed to the m inputs of the m x m network G_2 are the components of the m-component row-vector

$$R = \ell R' = SG_1D^T + \ell W$$

$$= S(DD^T)^{-1+q}DD^T + \ell W \quad (6.5\text{-}6)$$

from Eqn. (6.5-1). In the absence of noise, therefore, the signal vector at the input to the network G_2 is $S(DD^T)^{-1+q}DD^T$, so that the signal vector at the output of G_2 is

$$S(DD^T)^{-1+q}\, DD^T(DD^T)^{-q} = S \quad (6.5\text{-}7)$$

since DD^T is non-singular, real and symmetric. Exact equalization of the channel is, therefore, achieved for any real value of q in the range 0 to 1. Since $(DD^T)^{-1}$ is real, symmetric and positive definite, $(DD^T)^{-1+q}$ and $(DD^T)^{-q}$ are both real, symmetric and positive definite.[37]

The following arrangements of Fig. 6.5-1 have been studied

System	Value of q
6L	1
7L	3/4
8L	1/2
9L	1/4

From Eqns. (6.5-3) and (6.5-6), the signals at the m outputs of the network G_2, in any of the Systems 6L to 9L, are the components of the row-vector

$$X = RG_2 = S(DD^T)^{-1+q}DD^T(DD^T)^{-q} + \ell W(DD^T)^{-q}$$

$$= S + U \quad (6.5\text{-}8)$$

where

$$U = \ell W(DD^T)^{-q} \qquad (6.5\text{-}9)$$

and q has the appropriate value.

The m-component vector U is the noise-vector at the output of the network G_2. The i^{th} component u_i of the noise vector U, is a sample value of a Gaussian random variable with zero mean and variance

$$\eta_i^2 = \ell^2\sigma^2(G_{2i}G_{2i}^T) \qquad (6.5\text{-}10)$$

where G_{2i} is an m-component vector given by the i^{th} column of the matrix $G_2 = (DD^T)^{-q}$. Each s_i can now be detected by comparing the corresponding x_i with the appropriate thresholds. In general, η_i^2 is a function of i, however, at high signal to noise ratios the tolerance of any of the Systems 6L to 9L, to additive white Gaussian noise is approximately determined by the probability of error in the detection of the s_i for which η_i has the greatest value.

If q in Eqn. (6.5-3), is zero, that is, when all the signal processing is carried out at the transmitter, the linear network at the transmitter, from Eqn. (6.5-1), is the m x n network $(DD^T)^{-1}D$. Under these conditions, therefore, the arrangement of Fig. 6.5-1 reduces to that of System 1L described in Section 6.2. It follows that System 1L is a special case of the more general class of systems studied here.

It is shown in Appendix A7 that for any given channel

$$DD^T = YY^T \qquad (6.5\text{-}11)$$

so that

$$G_2 = (YY^T)^{-q} \qquad (6.5\text{-}12)$$

A comparison of Eqns. (6.5-12) and (6.4-1) shows that the m x m network F_1 at the transmitter of any of the Systems 2L to 5L, is the same as the m x m network G_2 at the receiver of the corresponding System 6L to 9L. Furthermore, the channel is exactly equalized in each case. This suggests that Systems 2L, 3L, 4L and 5L are, respectively, duals of Systems 6L, 7L, 8L and 9L, and vice versa.

In the Systems 6L to 9L, each of the networks G_1 and G_2 would in practice be implemented by the appropriate Gauss-Siedel iterative process.

6.6 Orthogonalization of the Sampled Impulse Response of the Baseband Channel

Consider the arrangement of Fig. 6.3-1 described in Section 6.3. It was shown that, given the m x m network F_1 at the transmitter, the linear estimates of the element values $\{s_i\}$ of a group of m signal-elements, are obtained at the receiver, at the m outputs of the n x m network

$$F_2 = Y^T(YY^T)^{-1}F_1^{-1} \qquad (6.6\text{-}1)$$

so that

$$F_1YF_2 = I \qquad (6.6\text{-}2)$$

where I is an m x m identity matrix.

Suppose now that

$$Y' = F_1Y \qquad (6.6\text{-}3)$$

where Y' is an m x n matrix of rank m and has the property that

$$Y'(Y')^T = I \qquad (6.6\text{-}4)$$

and I is an m x m identity matrix. This means that the m rows of Y' have unit length and are orthogonal to each other. Thus the m x m matrix F_1, representing the linear network F_1 in Fig. 6.3-1, transforms the set of m vectors $\{Y_i\}$ to a set of m orthonormal vectors $\{Y_i'\}$.[40] From Eqns. (6.6-3) and (6.6-4)

$$F_1Y(F_1Y)^T = F_1YY^TF_1^T = I \qquad (6.6\text{-}5)$$

or

$$YY^T = F_1^{-1}(F_1^T)^{-1}$$

or $$YY^T = (F_1^T F_1)^{-1}$$

or $$F_1^T F_1 = (YY^T)^{-1} \qquad (6.6\text{-}6)$$

From Eqns. (6.6 -1) and (6.6 -6), the n x m linear network at the receiver, in the arrangement of Fig. 6.3-1 is, now

$$F_2 = Y^T F_1^T F_1 F_1^{-1}$$

$$= Y^T F_1^T$$

$$= (F_1 Y)^T$$

$$= (Y')^T \qquad (6.6\text{-}7)$$

From Eqn. (6.3-6), the n sample values of a received group of m signal-elements at the input to the network F_2 are the components of the n-component row-vector

$$R = SF_1Y + \ell W = SY' + \ell W \qquad (6.6\text{-}8)$$

Since the m rows of Y' are orthonormal, the received signal-elements of a group are orthogonal at the receiver. Under these conditions the optimum detection process for a group of received signal-elements is matched-filter detection. Each signal-element can be thought to be transmitted over a different channel, and the sampled impulse responses of the different channels are, respectively, the m rows of the matrix Y'. Thus, at the receiver all that is required to maximise the signal to noise ratio, in the detection of the m signal-elements, is a set of m correlation detectors or matched filters, matched to the m rows of Y'. The n x m network F_2 in Eqn. (6.6-7), is nothing but a set of m correlation detectors matched to the m rows of Y'. Thus, in the arrangement of Fig. 6.3-1, if F_1 is such that Eqns. (6.6-3) and (6.6-4) are satisfied, then the resulting system is optimum in the sense that no other linear or non-linear detection process, at the receiver, will improve the tolerance of the system to additive white Gaussian noise.

Since the length of each column vector of F_2 is unity, $F_{2i}F_{2i}^T$ in Eqn. (6.3-13) is unity for each i. It follows from Eqn. (6.3-13), that the noise variance at the input to the detector, in the arrangement of Fig. 6.3-1, just considered, is given by

$$\eta^2 = \ell^2\sigma^2 \qquad (6.6\text{-}9)$$

where ℓ is given by Eqn. (6.3-3).

A particular form of the m x m matrix F_1 in Eqn. (6.6-6) is an m x m upper-triangular matrix P such that[36]

$$(YY^T)^{-1} = P^TP \qquad (6.6\text{-}10)$$

Thus

$$Y' = PY \qquad (6.6\text{-}11)$$

is an m x n orthonormal matrix, and from Eqn. (6.6-7)

$$F_2 = (Y')^T$$

$$= Y^TP^T \qquad (6.6\text{-}12)$$

Systems employing the transformation P in Eqn. (6.6-11) are described in Section 6.7.

6.7 Systems 10L and 11L

In System 10L the m x m linear network F_1, at the transmitter in the arrangement of Fig. 6.3-1, is a network represented by the upper triangular matrix P in Eqn. (6.6-11), and the n x m linear network at the receiver, from Eqn. (6.6-12), is

$$F_2 = Y^TP^T \qquad (6.7\text{-}1)$$

The block diagram of System 10L is shown in Fig. 6.7-1. Thus in System 10L, since PY is an m x n orthonormal matrix

$$F_1YF_2 = PYY^TP^T = I \qquad (6.7\text{-}2)$$

which satisfies the condition imposed by Eqn. (6.3-9). The upper triangular matrix P can be evaluated from Eqn. (6.6-10) by the so called "Square Root Method."[36] The practical implementation of the linear transformation P at the transmitter and Y^TP^T at the receiver

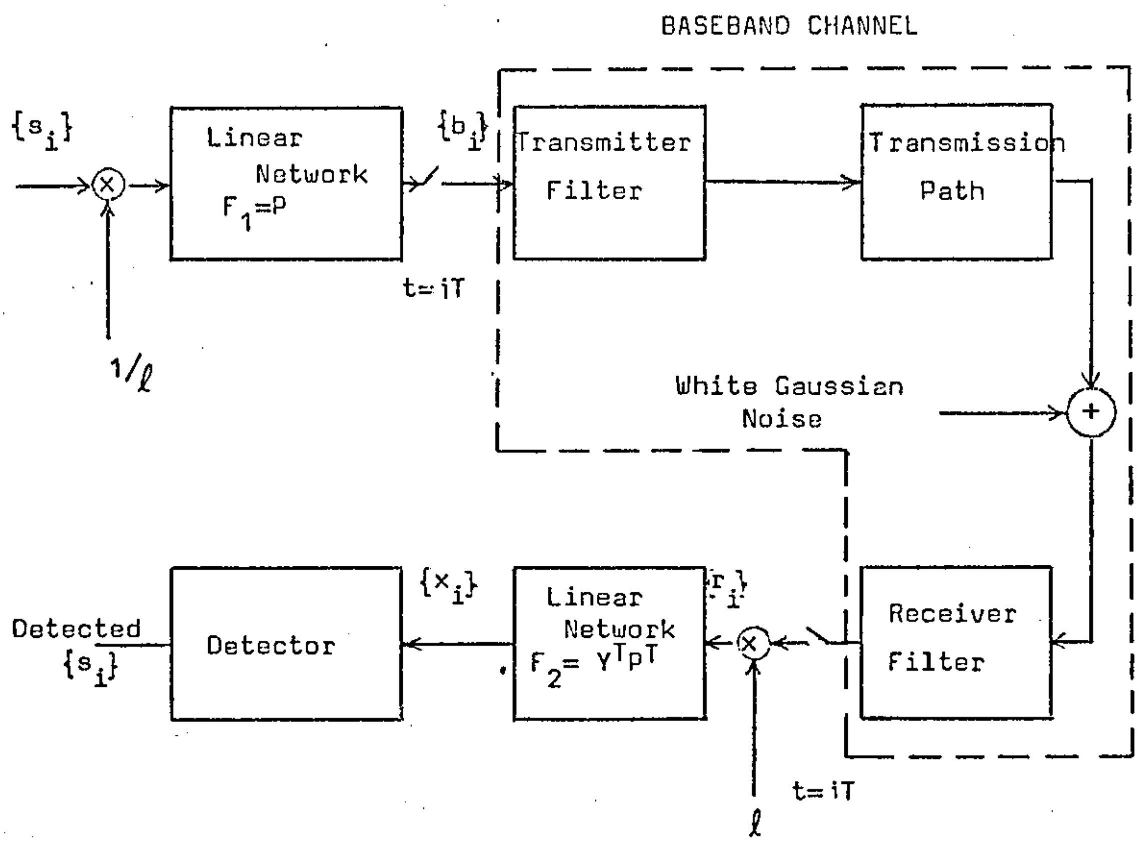


Figure 6.7-1

System 10L.

would normally require the prior knowledge of both P and Y and the components of P and Y^TP^T would simply be stored at the transmitter and receiver, respectively, and used to achieve the appropriate matrix multiplications. P can of course be derived from Y, but a more complex iterative process would now be required than the simple Gauss-Siedel process previously assumed,[52] and this is unlikely to be of any practical value. A description of this process is beyond the scope of the present work and will not be considered here.

It is interesting to note that in System 4L, since $(YY^T)^{-\frac{1}{2}}$ is a symmetric matrix, $F_1 = (YY^T)^{-\frac{1}{2}}$ and $F_2 = Y^T(YY^T)^{-\frac{1}{2}}$, so that

$$F_2 = Y^T F_1^{\ T} \qquad (6.7\text{-}3)$$

and

$$F_1 YY^T F_1^{\ T} = I \qquad (6.7\text{-}4)$$

A comparison of Eqns. (6.7-2) and (6.7-4) shows that the m rows of the m x n matrix $(YY^T)^{-\frac{1}{2}}Y$ are orthonormal. This suggests that Systems 4L and 10L belong to the same class of Systems where the elements of a received group are orthogonal at the receiver.

In System 11L, the arrangement of Fig. 6.5-1 described in Section 6.5 is used. The m x m linear network at the receiver, represented by the m x m non-singular matrix G_2, is such that

$$(G_2^{\ T}D)(G_2^{\ T}D)^T = I \qquad (6.7\text{-}5)$$

where I is an m x m identity matrix, and D is an m x n matrix of rank m, whose i^{th} row is

$$D_i = \overbrace{0\ \ldots.0}^{i-1}\ \overbrace{y_g y_{g-1}\ \ldots y_o}^{g+1}\ \overbrace{0\ \ldots.0}^{m-i} \qquad (6.7\text{-}6)$$

Clearly the m x n matrix $G_2^{\ T}D$ has rank m and is such that its m rows are orthogonal to each other. The m x n linear network at the transmitter of System 11L, which is represented by the m x n matrix G_1 of rank m, is assumed to be such that

$$G_1 = G_2^{\ T}D \qquad (6.7\text{-}7)$$

From Fig. 6.5-1, the n-component output vector from the linear network G_1 is $\frac{1}{\ell}SG_1$, and in the absence of noise, the m-component vector at the input to the network G_2 is SG_1D^T, so that the m component output vector from G_2 is

$$SG_1D^TG_2 = SG_2^TD.D^TG_2$$

$$= SG_2^TD(G_2^TD)^T$$

$$= S \qquad (6.7\text{-}8)$$

from Eqn. (6.7-5). Exact equalization of the channel is therefore achieved in System 11L, so long as the network G_2 is such that Eqn. (6.7-5) is satisfied.

From Eqn. (6.7-5)

$$G_2^TDD^TG_2 = I$$

or

$$DD^T = (G_2^T)^{-1}G_2^{-1}$$

$$= (G_2G_2^T)^{-1}$$

or

$$(DD^T)^{-1} = G_2G_2^T \qquad (6.7\text{-}9)$$

Since for a given sampled impulse response of the channel $DD^T = YY^T$ (Appendix A7), it follows that

$$(DD^T)^{-1} = (YY^T)^{-1} = G_2G_2^T \qquad (6.7\text{-}10)$$

Comparing Eqns. (6.6-10) and (6.7-10), clearly

$$G_2 = P^T \qquad (6.7\text{-}11)$$

Thus the m x m linear network at the receiver of System 11L is represented by an m x m lower triangular matrix P^T, where P^T is such that $P^TP = (YY^T)^{-1}$. The m x n matrix, representing the m x n linear network at the transmitter, is, from Eqn. (6.7-7)

$$G_1 = (P^T)^TD$$

$$= PD \qquad (6.7\text{-}12)$$

The block diagram of System 11L is shown in Fig. 6.7-2. As in the case

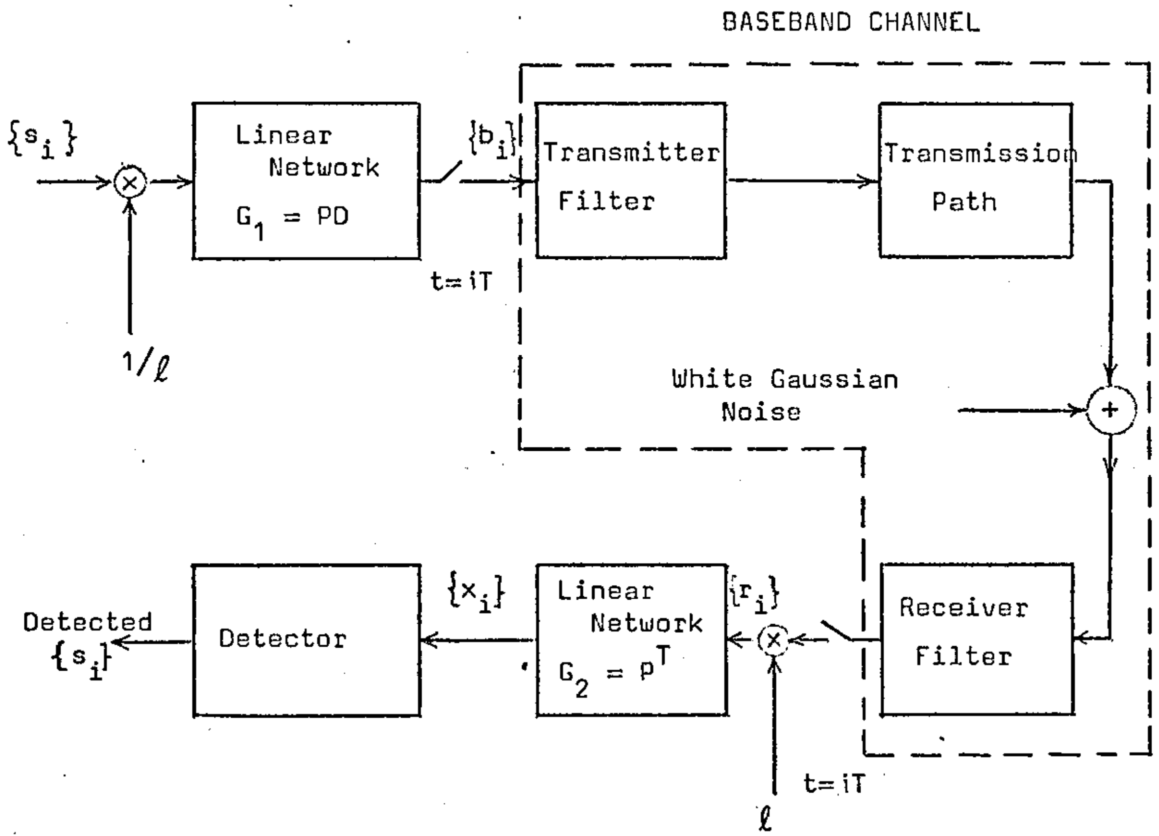


Figure 6.7-2

System 11L.

of System 10L, the practical implementation of the linear transformation PD at the transmitter and P^T at the receiver, would normally require the prior knowledge of both PD and P^T, and the components of PD and P^T would simply be stored at the transmitter and receiver, respectively, and used to achieve the appropriate matrix multiplications.

For a given sampled impulse response of the channel, the m x m linear network at the transmitter of System 10L is the transpose of the linear network at the receiver of System 11L. Furthermore, in both the Systems 10L and 11L the channel is exactly equalized. Systems 10L and 11L are therefore duals of each other.

6.8 Linear Signal Processing at the Transmitter with Non-linear Detection at the Receiver

In Section 6.4, the m x m matrix F_1, representing the linear network at the transmitter of Systems 2L to 5L, is always symmetric and positive definite. The corresponding n x m matrix, representing the linear network at the receiver is given by

$$F_2 = Y^T(YY^T)^{-1}F_1^{-1} \qquad (6.8\text{-}1)$$

It follows, therefore, that the m x m matrix $(YY^T)^{-1}F_1^{-1}$ is also symmetric and positive definite,[37] as is $(YY^T)^{-1}$. It is thus, possible to replace the n x m linear network F_2 at the receiver of Systems 2L to 5L, by the non-linear detection processes of Systems 3, 5 and 6, in order to achieve a better tolerance to noise. Systems employing non-linear detection process at the receiver with linear signal processing at the transmitter, are described in Section 6.9.

6.9 Systems 3LN, 4LN and 5LN

Systems 3LN, 4LN and 5LN are modifications of Systems 3L, 4L and 5L, respectively. In System 3LN, the detection process of System 6 is used at the receiver of System 3L, to detect the m signal-elements of a received group, instead of the linear network $F_2 = Y^T(YY^T)^{-1}$. Similarly, in Systems 4LN and 5LN, the detection process of System 6 is used at the receiver of Systems 4L and 5L, respectively, to detect the m signal-elements of a received group, instead of the respective linear networks.

7.0 ASSESSMENT OF SYSTEMS WITH LINEAR SIGNAL PROCESSING AT THE TRANSMITTER

7.1 Computer Simulation Tests

The tolerances to additive white Gaussian noise of the different systems, employing linear signal processing at the transmitter, have been compared by computer simulation, for different values of the sampled impulse response of the channel. The method of computer simulation is similar to that described in Section 5.1.

In every test, binary signal-elements are assumed such that s_i is equally likely to have the value 1 or -1, the element values in a group being statistically independent. The average transmitted energy per bit is equal to unity, the five component row vector L representing the channel in Table 5.2-1 has unit length, and the two sided power spectral density σ^2 of the additive white Gaussian noise at the input to the receiver filter, is adjusted for an average element error rate of 4 in 10^3. The value of σ^2 then gives a measure of the tolerance of a system to additive white Gaussian noise.

In each computer simulation test, a total of 4096 elements were transmitted over a baseband channel with a fixed value of L. Throughout the test m = 8 and n = m+4. Where a system has been tested over any of the channels B to K, a computer simulation test has been carried out for each of the corresponding values of L in Table 5.2-1. Tests have not been performed with multi-level signals, since exact equalization is, in every case, applied to each group of m signal-elements, and under these conditions a fairly accurate idea of their performances with multi-level signals can be obtained from the results for binary signals.

The tolerances of Systems 1L to 11L to additive white Gaussian noise have also been calculated theoretically using the results derived in Sections 6.2, 6.3 and 6.5 and bearing in mind that for binary signal-elements such that $s_i = \pm 1$, the average probability of error in the detection of s_i from x_i, where x_i is the corresponding output signal from the linear network at the receiver, is approximately $Q(\frac{1}{\eta})$, where η is the largest value of the standard deviation of noise components $\{u_i\}$ at the output of the linear network. These results are given along with those obtained by computer simulation, in Section 7.3

7.2 Error Probabilities and Confidence Limits

In Systems 1L to 11L, for binary coded signals such that $s_i = +1$ or -1, the error probability in the detection of the i^{th} signal-element of a group is

$$p_i = Q(1/\eta_i) \qquad (7.2\text{-}1)$$

where η_i^2 is the corresponding noise variance at the detector input (Section 5.3).

In the case where there is no signal distortion (Channel A) the value of η_i^2 in any of the systems 1L to 11L, is the same for each i and is equal to σ^2 the power spectral density of the additive white Gaussian noise at the input to the receiver filter. Thus, the element error probability, in any of the Systems 1L to 11L with binary signals and no distortion in transmission, is given by

$$p = Q(1/\sigma) \qquad (7.2\text{-}2)$$

Since the value of p is assumed to be 4×10^{-3}

$$Q(1/\sigma) = 4 \times 10^{-3} \quad (7.2\text{-}3)$$

The value of σ, corresponding to the value of $p = 4 \times 10^{-3}$, is 0.376 which is the value of the standard deviation of white Gaussian noise to be added at the input to the receiver filter to obtain an element error probability of 4×10^{-3}. It can be seen that the value of σ obtained here for channel A and binary signal-elements, agrees with the corresponding value of σ obtained in Section 5.3.

From the computer simulation results, it is found that the number of groups of signal-elements in error, in any of the systems tested, is approximately the same as that for System 1, with binary signal-elements (Section 5.3). It follows that the number of independent errors is, in each case, approximately the same as for System 1 with binary signal-elements, so that the 95% confidence limits, for any of the systems tested here, are as given in Table 5.3-1 for m = 8 and k = 2.

7.3 Results of Computer Simulation Tests

The results of the computer simulation tests are shown in Table 7.3-1. The noise power spectral density at the input to the receiver filter, required for an average element error probability of 4×10^{-3}, is quoted in decibels relative to its value when a binary signal is transmitted over channel A with the same error probability, the noise level here being the same in all cases.

Table 7.3-2 gives the theoretical values corresponding to those obtained by computer simulation in Table 7.3-1.

Fig. 7.3-1 shows the variation in the noise level, in the arrangement for Systems 2L - 5L (Fig. 6.3-1) with channel J, as q in Eqns. (6.4-1) and (6.4-2) is varied from 0 to 1. The results have been obtained theoretically.

TABLE 7.3-1

Noise level, for an average element error probability of 4×10^{-3}, expressed in dB relative to its value when a binary signal is transmitted with the same error probability over channel A. Groups of 8 signal-elements.

(Results obtained by Computer Simulation)

Channel	System													
	1L	2L	3L	4L	5L	6L	7L	8L	9L	10L	11L	3LN	4LN	5LN
A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
B	-5.2	-15.0	-10.0	-5.0	-3.5	-16.2	-11.1	-5.7	-3.8	-5.2	-5.4	-9.5	-5.0	-3.1
C	-3.0	-8.6	-5.3	-3.0	-2.4	-9.8	-6.0	-3.2	-2.3	-3.0	-3.2	-5.1	-3.1	-2.2
D	-12.8	-32.1	-22.0	-13.0	-10.0	-33.4	-23.0	-13.4	-9.8	-13.0	-13.6	-20.8	-13.0	-9.1
E	0.0	-0.5	-0.5	0.0	0.0	-0.6	-0.5	0.0	0.0	0.0	0.0	-0.3	0.0	0.0
F	-2.6	-7.0	-4.4	-2.8	-2.1	-8.0	-5.3	-3.0	-2.2	-2.6	-3.0	-4.1	-2.7	-2.2
G	-2.4	-7.9	-4.5	-2.2	-1.7	-8.8	-5.2	-2.6	-2.0	-2.2	-2.8	-3.8	-2.2	-1.7
H	-3.0	-8.2	-5.0	-3.0	-2.6	-8.7	-5.5	-3.2	-2.4	-2.9	-3.2	-5.0	-2.9	-2.5
I	-3.2	-10.0	-6.4	-3.1	-1.9	-11.1	-7.4	-3.3	-2.1	-3.0	-3.4	-5.9	-3.0	-1.6
J	-16.8	-38.0	-26.4	-17.0	-14.0	-39.2	-27.2	-17.5	-14.3	-17.0	-17.6	-26.0	-17.2	-13.2
K	-4.3	-12.1	-7.4	-4.5	-3.2	-13.0	-8.4	-4.7	-3.5	-4.5	-4.6	-8.0	-4.5	-3.0

TABLE 7.3-2

Noise level, for an average element error probability of 4×10^{-3}, expressed in dB relative to its value when a binary signal is transmitted with the same error probability, over Channel A. Groups of 8 signal-elements.

(Results obtained theoretically)

Channel	System										
	1L	2L	3L	4L	5L	6L	7L	8L	9L	10L	11L
A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
B	-5.24	-15.75	-9.95	-5.24	-3.9	-17.84	-11.55	-6.45	-3.9	-5.24	-6.45
C	-3.0	-8.8	-5.5	-3.0	-2.44	-10.16	-6.46	-3.8	-2.44	-3.0	-3.8
D	-12.6	-32.5	-21.9	-12.6	-9.82	-35.4	-24.3	-14.9	-9.82	-12.6	-14.9
E	-0.2	-0.56	-0.4	-0.2	0.0	-0.71	-0.52	-0.26	0.0	-0.2	-0.26
F	-2.68	-7.2	-4.55	-2.68	-2.3	-8.4	-5.56	-3.25	-2.3	-2.68	-3.25
G	-2.32	-8.06	-4.62	-2.32	-1.76	-9.65	-5.88	-3.04	-1.76	-2.32	-3.04
H	-3.06	-8.26	-5.8	-3.06	-2.46	-9.08	-5.79	-3.5	-2.46	-3.06	-3.5
I	-3.15	-10.6	-6.4	-3.15	-2.3	-12.4	-7.72	-4.06	-2.3	-3.15	-4.06
J	-16.6	-38.4	-26.94	-16.6	-14.5	-40.41	-28.1	-18.5	-14.5	-16.6	-18.5
K	-4.42	-12.32	-7.85	-4.42	-3.44	-13.75	-8.88	-5.16	-3.44	-4.42	-5.16

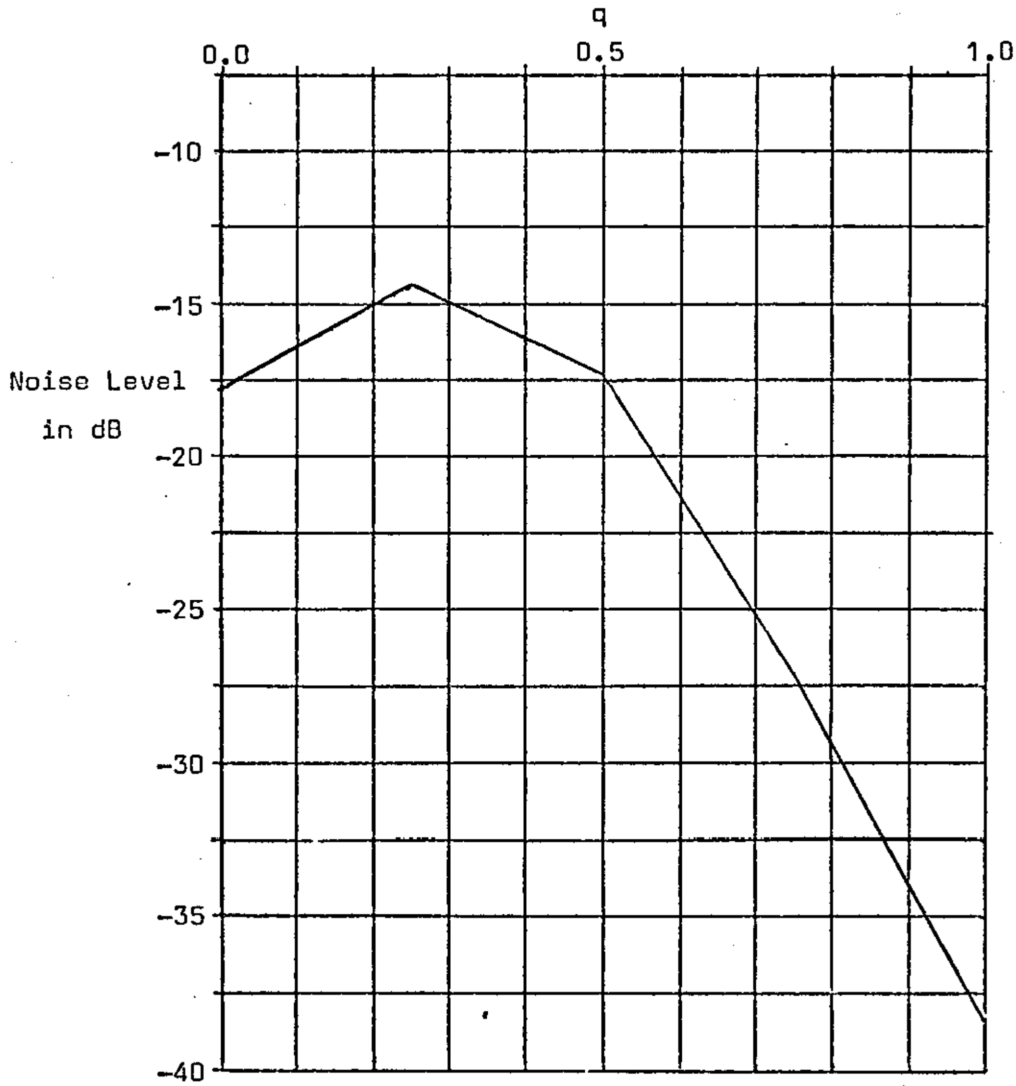


Figure 7.3-1

Variation in noise level for channel 3 and an error probability of 4×10^{-3} as q varies from 0 – 1, expressed in dB relative to the noise level of System 1 with binary signals, no distortion and for an error probability of 4×10^{-3}.

7.4 Comparison of Systems

From the computer simulation results of Table 7.3-1, the best tolerance to additive white Gaussian noise, of the systems studied in Section 6.0, is achieved by Systems 5L, 9L and 5LN, while Systems 2L and 6L give the worst performance. For severe signal distortions Systems 5L and 9L gain an advantage of about 3 dB in tolerance to noise, over System 1L. Within the limits of the accuracy of the computer simulation results (Table 5.3-1) the tolerances of Systems 1L to 11L, to additive white Gaussian noise obtained theoretically in Table 7.3-2, are either less than or equal to their corresponding values obtained by computer simulation. This is so because the theoretical results are obtained by using the maximum value of the noise variance, at the input to the detector, in the detection of the m signal-elements of a group, and therefore, the theoretical results provide the lower bounds to the tolerance of a system to additive white Gaussian noise.

Table 7.4-1 shows the systems studied in Section 6.0, which have approximately the same performances. The performances of two systems are here considered to be the same, approximately, if their respective noise levels do not differ by more than about 1.5 dB. It can be seen from Table 7.4-1 that any of the two systems which have approximately the same tolerance to noise, are also duals of each other, that is, the m x m linear network at the transmitter of one system is the transpose of the m x m network at the receiver of its dual system and in each case the channel is exactly equalized. For example, Systems 5L and 9L are duals of each other, since they both equalize the channel exactly, and the m x m network at the transmitter of System 5L is the transpose of the m x m network at the receiver of 9L. This suggests that Systems which are duals of each other, have approximately the same tolerance to additive white Gaussian noise, at least in the cases studied here.

TABLE 7.4-1

Systems having approximately the same performance, which are also duals of each other

System	Dual system which has approximately the same performance in the presence of additive white Gaussian noise
1	1L
2L	6L
3L	7L
4L	8L
5L	9L
10L	11L

The performances of Systems 4L and 10L are very close to that of System 1L, and can be taken to be the same for practical purposes. There seems to be no significant difference between the performances of Systems 4L and 4LN, even though in the latter case the detection process of System 6 is used at the receiver of System 4L instead of the corresponding linear network. This was expected since in the case of System 4L, the received signal-elements of a group are orthogonal at the receiver, and the optimum detection process is a set of correlation detectors. Therefore no other linear or non-linear technique at the receiver of System 4L can improve its tolerance to additive white Gaussian noise.

Fig. 7.3-1 shows that of the various combinations of the linear networks F_1 and F_2 in the arrangement of Fig. 6.3-1 and for severe signal distortions, the best tolerance to additive white Gaussian noise is achieved by System 5L. This suggests that System 5L is the optimum combination of the linear networks F_1 and F_2 or is at least close to the optimum combination.

Table 7.4-2 shows the approximate number of sequential operations required in the detection of a group of eight binary signal-elements, when transmitted over channel J, at the receiver of Systems 1L to 9L and also System 1. Apart from System 1L which does not require any processing at the receiver, the number of sequential operations required increases with the decrease in the value of q in Eqn. (6.4-1) and with the increase in the value of q in Eqn. (6.5-3), which is to be expected, since under these conditions more and more of the channel equalization is performed at the receiver. Table 7.4-2 also shows the approximate number of sequential operations required in the linear transformation of a group of eight binary signal-elements, in the case

TABLE 7.4-2

Approximate number of sequential operations required for the detection of a group of eight binary signal-elements, transmitted over channel J.

System	Number of sequential operations at the transmitter	Number of sequential operations at the receiver
1L	600	0
2L	500	0
3L	300	80
4L	100	160
5L	50	600
6L	0	700
7L	50	600
8L	100	150
9L	300	100
1	0	850

of channel J, at the transmitter of Systems 1L to 9L and System 1. Apart from System 1 which does not require any processing at the transmitter, the number of sequential operations required increases as the value of q in Eqn. (6.4-1) increases and the value of q in Eqn. (6.5-3) decreases which, again, is to be expected, since under these conditions more and more of the channel equalization is performed at the transmitter. None of the Systems in Table 7.4-2 other than System 1, is suitable for use over time varying channels since in all systems applying a linear transformation to the transmitted signal, a knowledge of the sampled impulse response of the channel is required at the transmitter, which means that this information must be fed from the receiver to the transmitter.

There does not seem to be any useful advantage in tolerance to additive white Gaussian noise gained by Systems 3LN and 5LN over Systems 3L and 5L, respectively, even though in Systems 3LN and 5LN, the non-linear detection process of System 6 replaces the linear network F_2. A partial explanation for this is that after some linear processing of the signal at the transmitter, the individual received signal-elements of a group are no longer simple time shifts of each other (when their element values are the same) but instead each received signal-element will now in general occupy all the available samples of the received group. Under these conditions it may well be that the first (or last) received signal-element of a group no longer has a better tolerance to noise than the signal-elements in the centre of the group. One would however still expect the non-linear constraint used in System 6 to give some advantage in tolerance to noise, although not as much as when all the signal processing is carried out at the receiver. The fact that System 6 gains a considerable advantage in tolerance to additive white Gaussian noise, over Systems 1 and 1L, suggests that to gain the maximum advantage of the detection process of System 6, no linear processing of the signal should be carried out at the transmitter.

The study of systems with linear signal processing at the transmitter, suggests that only a limited advantage in tolerance to additive white Gaussian noise can be gained by the appropriate linear transformations of the signal at the transmitter. Even the best of the systems studied has a tolerance to additive white Gaussian noise well below that of System 2. However, since none of the systems studied so far uses non-linear processing of the signal at the transmitter, it still remains to investigate the tolerance to additive white Gaussian noise likely to be achieved by such systems.

8.0 NONLINEAR PROCESSING OF SIGNAL AT THE TRANSMITTER

8.1 Basic Principles

The synchronous serial baseband data-transmission system, shown in Fig. 1.1-1 and discussed in Section 3.1, is now modified to include nonlinear processing of groups of m signal-elements at the transmitter. The modified arrangement is shown in Fig. 8.1-1. The m signal-elements at the transmitter, whose element values $\{s_i\}$ are the components of the m-component row-vector S, are transformed, non-linearly, into a corresponding set of values given by the components $\{b_i\}$ of the vector B. The $\{s_i\}$ are assumed to be statistically independent and equally likely to have any of the possible values. The m $\{b_i\}$ are sampled in sequence at regular intervals of T seconds to give the corresponding set of impulses with areas given by the $\{b_i\}$ and these are fed to the baseband channel. Adjacent groups of m$\{b_i\}$ are separated by g zero-level impulses. The baseband channel includes a transmitter filter, transmission path and a receiver filter. At the input to the receiver filter, white Gaussian noise with zero mean and a two sided power spectral density of σ^2 is added to the received signal. At the output of the receiver filter, the received signal is sampled at regular intervals of T seconds. Since adjacent groups of m $\{b_i\}$ are separated by gT seconds, it is clear that the n sample values of a received group of m $\{b_i\}$ depend only on the corresponding m signal-elements of a group. The detector, therefore, uses these n sample values in the detection of a received group of m signal-elements.

It is assumed here that the (g + 1) sample values of the sampled impulse response of the baseband channel,

$$y_o y_1 \cdot \cdot \cdot y_g \qquad (8.1\text{-}1)$$

are known at the transmitter and receiver. For the sake of convenience, the delay in transmission, other than that involved in the time dispersion of the transmitted signal, is ignored.

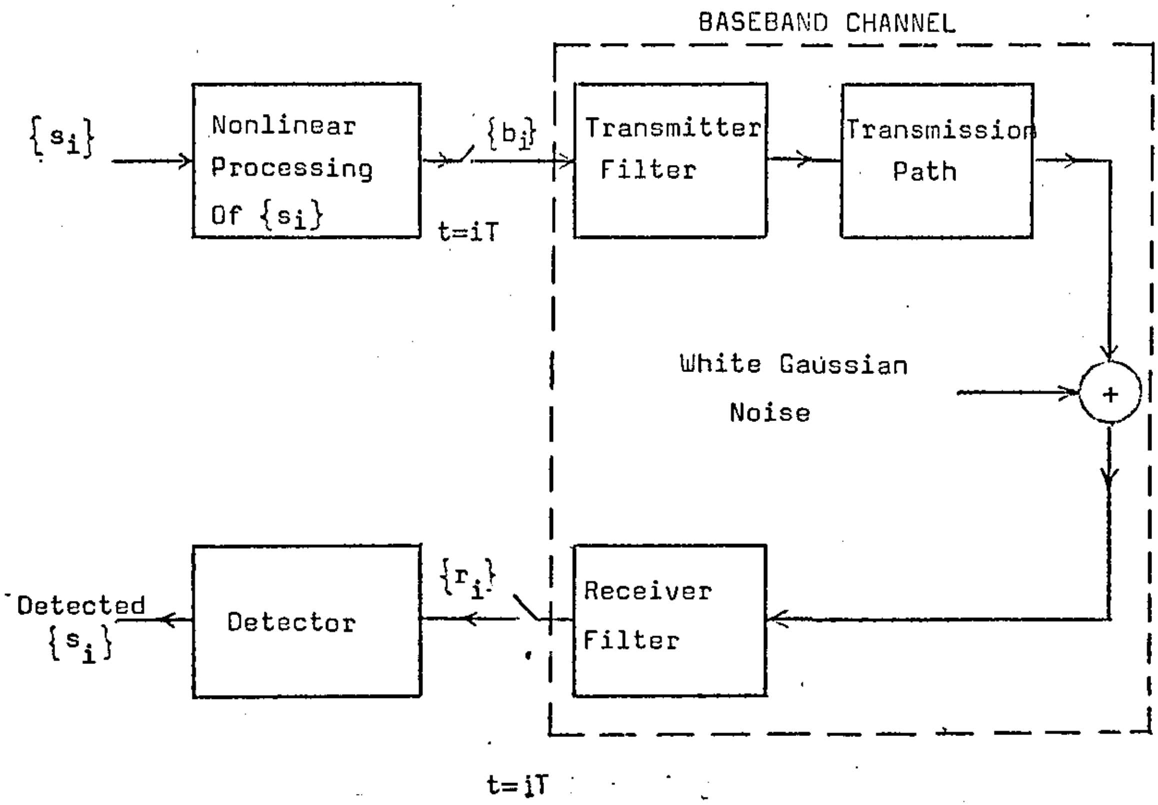


Figure 8.1-1

Non-linear signal processing at the transmitter.

The n sample values of a received group of m signal-elements are, from Section 3.1, the components of the n-component row-vector

$$R = BY + W \qquad (8.1\text{-}2)$$

where Y is the m x n matrix, of rank m, whose i^{th} row is

$$Y_i = \overbrace{0 \ldots 0}^{i-1} \; \overbrace{y_o \ldots . y_g}^{g+1} \; \overbrace{0 \ldots 0}^{m-i} \qquad (8.1\text{-}3)$$

and W is an n-component row-vector whose components are sample values of statistically independent Gaussian random variables with zero mean and variance σ^2. Let the m rows $\{Y_i\}$ of the matrix Y be normalized to have unit length, that is,

$$Y_i Y_i^T = 1.0 \text{ for } i = 1,2 \ldots, m. \qquad (8.1\text{-}4)$$

Let $A = \{a_{ij}\}$ be an m x m matrix, such that

$$A = YY^T \qquad (8.1\text{-}5)$$

The matrix A is non-singular, symmetric and positive definite.[32,52] Also from Eqns. (8.1-4) and (8.1-5),

$$a_{ii} = 1.0 \text{ for } i = 1,2, \ldots, m \qquad (8.1\text{-}6)$$

The detector, in Fig. 8.1-1, has a set of m correlation detectors matched to the m rows of the matrix Y. With the arrangement just described, the m $\{b_i\}$ corresponding to a group of m signal-elements are obtained as follows

1. Set b_m so that

   $a_{11}b_m = s_m$

2. Set b_{m-1} so that

   $a_{11}b_{m-1} + a_{12}b_m = s_{m-1}$

3. Set b_{m-2} so that

   $a_{11}b_{m-2} + a_{12}b_{m-1} + a_{13}b_m = s_{m-2}$ and so on, until finally

   $a_{11}b_1 + a_{12}b_2 + \ldots + a_{1m}b_m = s_1$

If now the $m\{b_i\}$ obtained in the above manner are fed to the baseband channel in the form of the corresponding impulses, then the output signal from the first correlation detector, from Eqns. (8.1-2), (8.1-4) and (8.1-5), is

$$x_1 = RY_1^T$$
$$= BYY_1^T + WY_1^T$$
$$= BA_1 + v_1$$
$$= s_1 + v_1 \qquad (8.1\text{-}7)$$

where A_1 is the first column of the matrix A, and $v_1 = WY_1^T$ is the value of the orthogonal projection of W onto Y_1. s_1 is now detected by comparing x_1 with the appropriate thresholds. Assuming now that b_1 is known at the receiver, b_1Y_1 is cancelled (eliminated by subtraction) from the received signal to form

$$R_1 = R - b_1Y_1 \qquad (8.1\text{-}8)$$

The output signal from the second correlation detector when R_1 is fed at its input, is

$$x_2 = R_1Y_2^T$$
$$= BA_2 - b_1a_{12} + WY_2^T$$
$$= s_2 + v_2 \qquad (8.1\text{-}9)$$

Since A is symmetric, and

$$s_2 = a_{11}b_2 + a_{12}b_3 + \dots + a_{1(m-1)}b_m \qquad (8.1\text{-}10)$$

A_2 is the second column of the matrix A, and $v_2 = WY_2^T$is the value of the orthogonal projection of W onto Y_2. s_2 is then detected by comparing x_2 with the appropriate thresholds. Assuming that b_2 is known at the receiver, b_2Y_2 is cancelled (eliminated by subtraction) from R_1 to form

$$R_2 = R_1 - b_2Y_2 \qquad (8.1\text{-}11)$$

The output signal from the third correlation detector is now

$$x_3 = R_2Y_3^T$$
$$= s_3 + v_3 \qquad (8.1\text{-}12)$$

leading to the detection of s_3, and so on until finally s_m is detected from x_m, where

$$x_m = R_{m-1}Y_m^T$$

$$= s_m + v_m \qquad (8.1\text{-}13)$$

v_m being the value of the orthogonal projection of W onto Y_m.

It is clear that in order to detect s_2, the detector must first cancel b_1 from the received signal. To cancel b_1, it is of course necessary to know its value,but this cannot be determined without first knowing all the $\{s_i\}$. Thus to detect the received signal-elements of a group, using the arrangement just described, the detector must have prior knowledge of the m $\{b_i\}$. Since this is tantamount to a prior knowledge of the m $\{s_i\}$, this is clearly not a practical system. Nevertheless it is interesting to study the performance of this hypothetical system since it clearly provides an upper bound to the performances likely to be obtained from the detection of transmitted signals considered here. Systems based on the scheme just mentioned, are described in Section 8.2.

8.2. Systems 1N, 2N, 3N and 4N

In systems 1N, 2N, 3N and 4N, the m impulses $\{b_i\}$ corresponding to a group of m signal-elements, are obtained at the transmitter in a manner described in Section 8.1. The detection process at the receiver in each of the above systems, is,however, different. The detection process at the receiver of System 1N is similar to that described in Section 8.1. This means that the receiver in System 1N has the prior knowledge of the values of the m $\{b_i\}$ corresponding to a received group of m signal-elements.

In System 2N the receiver has no prior knowledge of the values of the m $\{b_i\}$ corresponding to a received group of m signal-elements, and these values are estimated, at the receiver, from the corresponding n sample values of the received signal. The operation of the detection process at the receiver of System 2N is as follows. From Eqn. (8.1-7), the output signal from the first correlation detector, matched to Y_1, when R is fed at its input, is

$$RY_1^T = s_1 + u_1 \qquad (8.2\text{-}1)$$

where $u_1 = WY_1^T$. s_1, as before, is detected by comparing RY_1^T with the appropriate thresholds. Following the detection of s_1, u_1 and hence u_1Y_1 is known. u_1 is the value of the orthogonal projection of W on the vector Y_1. u_1Y_1 is subtracted from R to give the n-component vector $(R - u_1Y_1)$. The linear estimate of the vector B is now obtained by feeding $(R - u_1Y_1)$ to the linear network $Y^T(YY^T)^{-1}$ (Section 3.2). Thus

$$E_1 = (R - u_1Y_1)Y^T(YY^T)^{-1} \qquad (8.2\text{-}2)$$

where E_1 is an m-component row-vector which is an estimate of B. Let e_{11} be the first component of E_1. $e_{11}Y_1$ is now cancelled (eliminated by subtraction) from $(R - u_1Y_1)$ to give the n-component vector

$$R_1 = R - u_1Y_1 - e_{11}Y_1 = R - (u_1 + e_{11})Y_1 \qquad (8.2\text{-}3)$$

Thus R_1 is obtained from R by cancelling some of the noise and the estimate e_{11} of b_1.

The output from the 2nd correlation detector when R_1 is fed to its input is

$$R_1Y_2^T = s_2 + u_2 \qquad (8.2\text{-}4)$$

where u_2 is the estimate of the projection onto Y_2, of the noise vector in R_1. s_2 is detected by comparing $R_1Y_2^T$ with the appropriate thresholds. The estimate of the noise component u_2Y_2 is now known and is removed from R_1 to give the vector $(R_1 - u_2Y_2)$. The new estimate E_2 of the vector B is now obtained, and is given by

$$E_2 = (R_1 - u_2Y_2)Y^T(YY^T)^{-1} \qquad (8.2\text{-}5)$$

Hence the 2nd component e_{22} of the m-component row-vector E_2, is the estimate of b_2. This is taken to be correct and b_2Y_2 is cancelled from $(R_1 - u_2Y_2)$ to give the n-component vector

$$R_2 = R_1 - u_2Y_2 - e_{22}Y_2 = R_1 - (u_2 + e_{22})Y_2 \qquad (8.2\text{-}6)$$

This leads to the detection of s_3 from $R_2Y_3^T$. The above procedure is repeated for the detection of the following signal-elements of a group, so that s_m is detected by comparing $R_{m-1}Y_m^T$ with the appropriate thresholds, where

$$R_{m-1} = R_{m-2} - (u_{m-1} + e_{(m-1)(m-1)})Y_{m-1} \qquad (8.2\text{-}7)$$

where u_{m-1} is the estimate of the projection onto Y_{m-1}, of the noise vector in R_{m-2}, and $e_{(m-1)(m-1)}$ is the estimate of b_{m-1}. Thus at the end of the detection process

$$R_m = R - (u_1 + e_{11})Y_1 - (u_2 + e_{22})Y_2$$

$$- \ldots. - (u_m + e_{mm})Y_m \qquad (8.2\text{-}8)$$

$$= R - UY - EY \qquad (8.2\text{-}9)$$

where u_i and e_{ii} for $i = 1,2, \ldots,m$ in Eqn. (8.2-8) are, respectively, the components of the m-component row-vectors U and E in Eqn. (8.2-9).

The detection process at the receiver of System 3N is an iterative process. Each cycle of the iterative process is identical to the detection process of System 2N described above. At the beginning of the first cycle of the detection process, in System 3N, the n components of the vector R, in Eqn. (8.2-1), are the n sample values corresponding to a received group of m signal-elements as in System 2N. In the second and subsequent cycles the vector R, at the beginning of the cycle, is taken as

$$R - UY, \qquad (8.2\text{-}10)$$

where, the m-component row vector U (Equation (8.2-9)) was obtained at the end of the previous cycle. The iterative procedure, in the detection process of System 3N, is carried on until there is no further reduction in the number of errors obtained at the end of each iterative cycle.

The detection process at the receiver of System 4N is a modification of the detection process of System 2N. As before, the n-component row-vector R whose components are the sample values of the received signal corresponding to a group of m signal-elements, is fed to the input of the first correlation detector, so that at its output

$$RY_1^T = s_1 + u_1 \tag{8.2-11}$$

where $u_1 = WY_1^T$. s_1 is now detected by comparing RY_1^T with the appropriate thresholds. Again as in System 2N

$$R_1 = R - (u_1 + e_{11})Y_1 \tag{8.2-12}$$

R_1 is now fed to the input of the second correlation detector so that

$$R_1Y_2^T = s_2 + u_2 \tag{8.2-13}$$

from Eqn. (8.2-4). s_2 is detected by comparing $R_1Y_2^T$ with the appropriate thresholds. Again, as in System 2N the new estimate of the vector B is the m-component row-vector

$$E_2 = (R_1 - u_2Y_2)Y^T(YY^T)^{-1} \tag{8.2-14}$$

If e_{11} in Eqn. (8.2-12), is equal to b_1 the first component of the vector B, then clearly the desired cancellation of b_1Y_1 is achieved, and under these conditions the first component e_{21} of the vector E_2 in Eqn. (8.2-14) is zero. However, since e_{11} is the estimate of b_1, it is, in general, not equal to b_1. It follows, therefore, that u_2 is partly $(W - u_1Y_1)Y_2^T$ and partly $\Delta e_{11}Y_1Y_2^T$, where Δe_{11} is the error in estimating b_1 from e_{11}. Furthermore, the first component e_{21} of the vector E_2 is not, in general, equal to zero. $e_{21}Y_1$ and $e_{22}Y_2$ are now removed from $(R_1 - u_2Y_2)$ to give the n-component row-vector

$$\begin{aligned} R_2 &= R_1 - u_2Y_2 - e_{21}Y_1 - e_{22}Y_2 \\ &= R_1 - e_{21}Y_1 - (u_2 + e_{22})Y_2 \end{aligned} \tag{8.2-15}$$

where, of course, e_{21} and e_{22} are respectively, the first and second components of the vector E_2. R_2 is now fed to the input of the third correlation detector, so that

$$R_2Y_3^T = s_3 + u_3 \tag{8.2-16}$$

and s_3 is detected by comparing $R_2Y_3^T$ with the proper thresholds. R_3 is now obtained as

$$R_3 = R_2 - e_{31}Y_1 - e_{32}Y_2 - (u_3 + e_{33})Y_3 \qquad (8.2\text{-}17)$$

where e_{31}, e_{32} and e_{33} are, respectively, the first, second and third components of the m-component row-vector

$$E_3 = (R_2 - u_3Y_3)Y^T(YY^T)^{-1} \qquad (8.2\text{-}18)$$

s_4 is now detected by comparing $R_3Y_4^T$ with the appropriate thresholds. The above procedure is carried on until finally, s_m is detected by comparing $R_{m-1}\ Y_m^T$ with the appropriate thresholds, where

$$R_{m-1} = R_{m-2} - e_{(m-1)1}Y_1 - \ldots - e_{(m-1)(m-2)}Y_{(m-2)}$$

$$- \left[(u_{m-1}) + e_{(m-1)(m-1)}\right] Y_{m-1} \qquad (8.2\text{-}19)$$

and, $e_{(m-1)i}$ for i = 1, 2 .., (m-1), are, respectively, the first (m-1) components of the m-component row-vector

$$E_{m-1} = \left[R_{m-2} - u_{(m-1)}Y_{(m-1)}\right] Y^T(YY^T)^{-1} \qquad (8.2\text{-}20)$$

It may be pointed out that in forming R_2 (Eqn. (8.2-15)) it is not possible to cancel $e_{23}Y_3$, $e_{24}Y_4$, ..., $e_{2m}Y_m$, since this will spoil the relationship

$$R_iY_{(i+1)}^T = s_{i+1} + u_{i+1} ,$$

and this is true in forming of any of the following R_i.

8.3 Results of Computer Simulation Tests

The tolerances of systems just described, to additive white Gaussian noise have been compared by computer simulation for group of eight binary signal-elements. The possible values of the elements are ±1, and they are statistically independent and equally likely to have any of the possible values. The computer simulation tests have been carried out for the different values of the sampled impulse response of the baseband channel , as given in Table 5.6-1, and for groups of eight signal-elements. The method of these tests is similar to that described in Section 5.1.

In every test, the average transmitted element energy per bit is unity. The five component row-vector L representing the sampled impulse response of the channel in Table 5.6-1 has unit length and the two sided power spectral density σ^2 of the additive white Gaussian noise at the input to the receiver filter is adjusted for an average element error probability of 4 in 10^3. The value of σ^2 gives the measure of the tolerance of a system to additive white Gaussian noise.

In each computer simulation test, a total of 4096 elements were transmitted over a baseband channel with fixed value of L.

The results of the computer simulation tests are shown in Table 8.3-1. The noise power spectral density at the input to the receiver filter, required for a given average element error probability of 4×10^{-3}, in Table 8.3-1, is quoted in decibels relative to its value when a binary signal is transmitted over channel A with an average element error probability of 4×10^{-3}. Table 8.3-1 also gives the results for Systems 1 and 2 obtained in Section 5.4.

The results of the computer simulation of System 3N show that, the number of errors detected after the first iterative cycle, is the same as that in System 2N. However, the number of errors at the end of the second and the subsequent iterative cycles, does not converge to a fixed value but varies in a random fashion, and is always greater than its value obtained at the end of the first iterative cycle.

From the computer simulation results, it is found that the number of groups of signals in error, in Systems 1N, 2N and 4N, is approximately the same as that obtained with System 1 and binary signal-elements (Section 5.3). It follows that the number of independent errors is, in each case, approximately the same as for System 1, so that the 95% confidence limits for any of the Systems 1N, 2N and 4N, are as given in Table 5.3-1 for $m = 8$, $k = 2$ and $p = 4 \times 10^{-3}$.

TABLE 8.3-1

Noise level, for an average element error probability of 4×10^{-3}, expressed in dB relative to its level for an error probability of 4×10^{-3} with binary signal-elements and no distortion. Groups of eight binary elements.

Channel	System				
	1N	2N	4N	1	2
A	0.0	0.0	0.0	0.0	0.0
B_1	-1.9	-3.4	-3.5	-6.0	-1.2
B_2	-2.8	-4.3	-4.6	-6.0	-1.2
C_1	-1.6	-2.0	-2.4	-3.2	-0.4
C_2	-2.0	-2.2	-2.2	-3.2	-0.4
D_1	-2.4	-9.2	-9.2	-13.7	-2.4
D_2	-4.0	-10.0	-9.8	-13.7	-2.4
E_1	0.0	0.0	0.0	-0.3	0.0
E_2	0.0	0.0	0.0	-0.3	0.0
F_1	-1.6	-2.4	-2.6	-3.3	-0.5
F_2	-1.8	-2.4	-2.4	-3.3	-0.5
G_1	-0.4	-1.6	-1.6	-2.9	0.0
G_2	-1.2	-2.1	-2.3	-2.9	0.0
H_1	-1.6	-2.6	-2.2	-3.2	-0.7
H_2	-1.2	-2.4	-2.3	-3.2	-0.7
I_1	-1.2	-1.9	-1.9	-3.7	0.0
I_2	-1.5	-2.0	-1.9	-3.7	0.0
J_1	-3.6	-14.3	-14.0	-17.6	-4.4
J_2	-4.1	-14.6	-14.8	-17.6	-4.4
K_1	-1.6	-3.9	-4.0	-4.9	-1.2
K_2	-2.0	-4.1	-4.1	-4.9	-1.2

8.4 Assessment of Systems

From the results of Table 8.3-1, the tolerance to additive white Gaussian noise of System 1N, in every case, is better than that of either System 2N or 4N. For severe signal distortions, System 1N gains an appreciable advantage in tolerance to noise, over Systems 2N and 4N. The advantage in tolerance to noise gained by System 1N is due to the fact that in System 1N, the values of the m $\{b_i\}$ to be cancelled from the received signal, are assumed to be known at the receiver. Hence correct cancellation is achieved in every case and is independent of the detection of the corresponding values $\{s_i\}$ of a received group of signal-elements. The prior knowledge of the m $\{b_i\}$ at the receiver of System 1N, of course, means the prior knowledge of the corresponding values of the signal-elements of a group. Clearly such a situation never arises in actual practice and System 1N is, therefore, a hypothetical system. However, System 1N gives the upper bounds to the tolerances to additive white Gaussian noise which could be achieved by any scheme for detecting the received signal-elements of a group, described in Section 8.1. From Table 8.3-1 which also shows the tolerances to additive white Gaussian noise of Systems 1 and 2, it can be seen that System 1N has approximately, the same tolerance to noise as System 2.

The tolerance to noise of System 2N is very much below that of System 1N, particularly for severe signal distortions. This is because of the fact that, in System 2N, the values of the m $\{b_i\}$ which are to be cancelled, are not assumed to be known at the receiver, and are estimated from the received signal itself. The estimate is obtained linearly, so that the exact cancellation of the $\{b_i\}$ is not achieved here. Thus, as mentioned in Section 8.2, each time the cancellation is performed, an error is introduced in the detection process. The introduction of the errors, at each stage of cancellation in the detection process, therefore, reduces the tolerance of System 2N to additive white Gaussian noise. Clearly to achieve a better performance, the accuracy in the estimates of the m $\{b_i\}$ must be improved.

There does not appear to be any significant difference between the performances of Systems 2N and 4N, and for practical purposes, their tolerances to additive white Gaussian noise are the same. From the results of Table 8.3-1, for severe signal distortions both Systems 2N and 4N gain an advantage of about 3 dB in tolerance to noise over System 1.

In System 3N, the detection process does not converge to a steady value, that is, the number of errors at the end of the second and the subsequent iterative cycles varies randomly and is always greater than that obtained at the end of the first cycle. This happens because the errors introduced during the first iterative cycle, due to the approximate cancellation of the $\{b_i\}$ mentioned above, become significant in the second and the subsequent iterative cycles. The fact that the number of errors at the end of the first cycle, in System 3N, is the same as that obtained in System 2N is obvious, since the first iterative cycle in the detection process of System 3N, is identical to the detection process of System 2N.

Although, for severe signal distortions, Systems 2N and 4N gain an advantage of about 3 dB in tolerance to noise, over System 1, the fact that the sampled impulse response of the channel is required to be known at the transmitter, makes Systems 2N and 4N less attractive in comparison with Systems 4 and 6 (Section 4) where no such knowledge is required at the transmitter and whose performances are better than those of Systems 2N and 4N.

9.0 COMMENTS ON THE RESEARCH PROJECT

9.1 Originality

To the best of the Author's knowledge, the following are the more important of the contributions which are believed to be original. The use of the non-linear technique of detection and cancellation of signals in Systems 5 and 6 (Sections 4.4) and Systems 7/6 and 8/6 for the detection of multi-level signals (Sections 4.6 and 4.7). The theory and development of Systems 2L to 9L (Sections 6.3, 6.4 and 6.5) where the process of linear equalization is shared between the transmitter and the receiver. The signal processing scheme of Section 6.6, in which groups of signal-elements at the transmitter are so arranged that they are orthogonal at the receiver. Systems 3LN, 4LN and 5LN. Systems 2N, 3N and 4N which employ non-linear processing of signal at the transmitter. All computer simulation tests and all computer programs.

9.2 Suggestions for Further Investigations

The main aim of the studies carried out in this project was to develop detection processes for orthogonal groups of signal-elements for use in a synchronous serial baseband data-transmission system, achieving a tolerance to additive white Gaussian noise similar to that of System 2 but not requiring the very large number of sequential operations needed by System 2 when there are many signal-elements in a group. These studies have been carried out from a purely theoretical point of view, and little or no reference is made to the implementation of the detection processes in actual practice. Since many if not most transmission paths are bandpass channels, the work must clearly be developed to include the study of linear modulators and demodulators and in the particular case where two modulated carrier signals are transmitted with the same carrier frequency but in phase quadrature.

Although the detection processes described in Section 4.0, require far fewer sequential operations than those required in the optimum detection process (System 2), the number of such sequential operations can be further reduced by speeding up the convergence rate of the Gauss-Siedel iterative process. This requires the development of more sophisticated non-linear operations on the components of the vector X in Eqn. (4.1-8), during the Gauss-Siedel iterative process.

The detection processes which have been described in this report are developed assuming that the channel is time invariant. It would now be interesting and highly desirable to study their performances with time varying channels.

In Section 5.6, the comparison of orthogonal groups of signals with the equivalent continuous (uninterrupted) transmission, is carried out for arbitrary channels and for a single transfer function of the transmitter and receiver filters. It would be interesting to carry out such a comparison for several different transfer functions of the equipment filters and for more realistic channels.

10.0 CONCLUSIONS

When there is no signal processing at the transmitter of the data-transmission system, adjacent groups of signal-elements at the input to the transmitter filter, are separated by zero-level elements so that there is no intersymbol interference between the elements of different groups, in the received signal. At the receiver the detection of the received signal-elements of a group, is achieved iteratively by a sequence of similar operations. In practical applications, the number of signal-elements in a group is likely to exceed 10 or even 20 with 4- or 8-level elements. Under these conditions System 2 which is the optimum system, requires a very large number of sequential operations and is not a practical arrangement. For binary signals, System 4 achieves a tolerance to noise approaching that of System 2 but requires far fewer sequential operations, and System 6 has a tolerance to noise typically within about 3 dB of that of System 2 and requires even fewer sequential operations than does System 4. For multi-level signals, the tolerances to noise of Systems 8/2 and 8/4 are very close to that of System 2 while the tolerance to noise of System 8/6 is typically within about 3 dB of that of System 2. Systems 8/2, 8/4 and 8/6 require fewer sequential operations than does System 2. The tolerances to noise of Systems 7/2, 7/4 and 7/6 are similar to those of Systems 8/2, 8/4 and 8/6, respectively, but Systems 7/4 and 7/6 require far fewer sequential operations than do Systems 8/4 and 8/6, respectively. The results of computer simulation tests suggest that the preferred systems for groups of 4 and 8 signal-elements are also the preferred systems for the larger group sizes that would normally be used in practice.

In System 1 the process of linear equalization is carried out at the receiver while in System 1L all the linear equalization is achieved at the transmitter. The transmitted signal in System 1L is continuous, with no gaps between adjacent groups of elements, and the signal-elements of a group are detected from the central group of the sample values corresponding to the received group of elements. No signal processing is required at the receiver of System 1L and the system is best suited for situations where a single transmitter feeds many receivers. Both the Systems 1 and 1L have the same tolerance to additive white Gaussian noise.

Of the Systems 2L to 11L, where the process of linear equalization is shared between the transmitter and the receiver, Systems 5L and 9L gain an advantage of about 3 dB in tolerance to noise, over System 1, under conditions of severe signal distortion. This advantage is, however, much smaller than that gained by Systems 4 and 6 over System 1.

Systems 4L and 10L have approximately the same tolerance to noise as System 1. In both the Systems 4L and 10L, the linear processing of signal at the transmitter is such that adjacent groups of transmitted signal-elements are separated by gaps containing zero-level elements and the received signal-elements of a group are orthogonal but with considerable intersymbol interference. The optimum detector in both these systems is a set of correlation detectors.

The performance of System 6 in the presence of additive white Gaussian noise, is better than the performance of any of the Systems 3LN and 5LN. Although in the latter systems the detection process of System 6 is used at the receiver. This suggests that in order to gain the maximum advantage of the detection process of System 6, all the signal processing should be carried out at the receiver.

In both the Systems 2N and 4N, groups of signal-elements are processed non-linearly and adjacent groups of transmitted signal-elements are separated by gaps containing zero-level elements. For severe signal distortions, Systems 2N and 4N gain an advantage of about 3 dB in tolerance to noise over System 1, which is a much smaller advantage than that gained by Systems 4 and 6 over System 1.

The arrangement of orthogonal groups of signals studied here has some useful advantages over a synchronous serial system with continuous (uninterrupted) transmission. Firstly, exact equalization of the channel is, in every case, achieved. Secondly, a complete loss of signal cannot result from an unfortunate combination of signal-element values and channel impulse response. Thirdly, there are no error extension effects from one group of elements to the next, regardless of the detection process used. Finally, detection processes achieving a near optimum tolerance to additive noise can be implemented quite simply. The disadvantage of orthogonal groups of signal-elements is that for a given information rate, the bandwidth required is wider than that required for a continuous (uninterrupted) signal. This reduces the tolerance to noise of the arrangement of orthogonal groups and partly offsets the basic advantages gained by the arrangement. However, when the number of signal-elements in a group is relatively large compared with the number of elements set to zero between adjacent groups, a useful advantage in tolerance to noise should be gained by the better detection processes over a linear or non-linear transversal equalizer, where the latter is used with a continuous (uninterrupted) signal having the same information rate.

From a study of the various systems tested, it appears that the most cost effective systems for binary and multi-level signals are Systems 6 and 7/6, respectively. Where the best available performance is required, without an excessive number of sequential operations in a detection process, the preferred systems for binary and multi-level signals are Systems 4 and 7/4, respectively.

APPENDIX A1

VARIANCE OF THE SAMPLE VALUES OF A GAUSSIAN RANDOM VARIABLE IN AN n-DIMENSIONAL EUCLIDEAN VECTOR SPACE

Consider a unit vector U which may have any direction in the n-dimensional Euclidean vector space containing U. Since U is a unit vector

$$UU^T = 1 \tag{A1-1}$$

Let W be the n-component row-vector, in the vector space, whose components $\{w_i\}$ are sample values of statistically independent Gaussian random variable with zero mean and variance σ^2.

The value of the projection of the noise-vector W onto U is the inner product of the vectors W and U, given by

$$WU^T = UW^T \tag{A1-2}$$

This is a sample value of a Gaussian random variable with zero mean and variance given by the expected value of the square of the inner product of W and U. Now

$$\begin{aligned} E\{(UW^T)^2\} &= E\{UW^TWU^T\} \\ &= \sigma^2 UQU^T \end{aligned} \tag{A1-3}$$

where $E\{\ \cdot\ \}$ represents the expected value, and

$$\sigma^2 Q = E\{W^T W\} \tag{A1-4}$$

is the covariance matrix of the n noise sample values $\{w_i\}$ which are the components of W. From Eqn. (A1-4), the component in the i^{th} row and j^{th} column of the n x n covariance matrix $\sigma^2 Q$ is

$$\sigma^2 q_{ij} = E\{w_i\, w_j\} \tag{A1-5}$$

where w_i and w_j are the i^{th} and j^{th} components, respectively, of the noise vector W.

Since the different noise samples are statistically independent and therefore uncorrelated, and since they have zero mean,

$$E\{w_i\, w_j\} = 0 \quad \text{for } i \neq j \tag{A1-6}$$

and since the noise samples have a variance σ^2

$$E\{w_i\, w_i\} = E\{w_i^{\,2}\} = \sigma^2 \qquad \text{(A1-7)}$$

Thus

$$\sigma^2 Q = \sigma^2 I$$

$$\text{or} \qquad Q = I \qquad \text{(A1-8)}$$

where I is an n x n identity matrix.

From Eqns. (A1-3) and (A1-8) the value of the projection of W onto U and therefore the value of the noise component in the direction of U, is a sample value of a Gaussian random variable with zero mean and variance

$$\begin{aligned} \sigma^2 UQU^T &= \sigma^2 UIU^T \\ &= \sigma^2 UU^T \\ &= \sigma^2 \qquad \text{(A1-9)} \end{aligned}$$

from (A1-1).

It follows that so long as the n components of W are sample values of statistically independent Gaussian random variable, the value of the orthogonal projection of W onto any given direction, in the n-dimensional Euclidean vector space, is a sample value of a Gaussian random variable with zero mean and variance σ^2.

The probability density function of the noise vector W is

$$\begin{aligned} P(W) &= P(w_1, w_2, \ldots, w_n) \\ &= P(w_1)\, P(w_2) \ldots P(w_n) \\ &= \prod_{i=1}^{n} P(w_i) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{w_i^{\,2}}{2\sigma^2} \right) \end{aligned}$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}n}} \quad \exp\left(- \frac{1}{2\sigma^2} \sum_{i=1}^{n} w_i^{\,2} \right)$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}n}} \quad \exp\left(- \frac{1}{2\sigma^2} \; |W|^2 \right) \qquad \text{(A1-10)}$$

Thus $P(w)$ depends only on $|W|$ and increases steadily as $|W|$ decreases.

If the m-component vector S is detected from the received n-component vector

$$R = SY + W$$

where the m x n matrix Y is given by Eqn. (3.1-3), the conditional probability density function of R, given SY, is

$$P(R/SY) = P(W/SY)$$

$$= P(w_1, w_2, \ldots, w_n/SY)$$

$$= P(w_1/SY)\, P(w_2/SY) \ldots P(w_n/SY)$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}n}} \quad \exp\left(- \frac{1}{2\sigma^2} |W|^2 \right)$$

from Eqn. (A1-10), where W is now the noise vector corresponding to the assumed values of R and SY.

Clearly $P(R/SY)$ is maximum when $|W|$ is minimum. But it is well known that when the different possible vectors $\{SY\}$ are equally likely, the detection process that minimizes the probability of error is that which selects the vector SY corresponding to the maximum value of $P(R/SY)$.[45,57] Thus the detection process, that minimizes the error probability, selects the vector SY corresponding to the minimum value of

$$|W| = |R - SY|$$

so that it selects the vector SY nearest to R.

APPENDIX A2

ERROR PROBABILITY AND SIGNAL/NOISE RATIO

When the signal-elements in a group are statistically independent and are equally likely to have the two possible values $\pm$ 1, the probability of error in the detection of the i^{th} signal-element of a group, from Section 3.5, is

$$p_i = Q\left(\frac{d_i}{\sigma}\right) \qquad \text{(A2-1)}$$

where $\hat{\sigma}^2$ is the power spectral density of the additive white Gaussian noise at the input to the receiver filter and d is the distance to the single decision boundary in the detection of the ith element of the group of m. Let p_i be equal to p and d_i be equal to d so that

$$p = Q\left({}^{d}/\sigma\right) \qquad \text{(A2-2)}$$

The variation of the element error probability p with d/σ is obtained from probability distribution tables and is shown in Fig. A2-1.

At high signal/noise ratios, that is, when p has a value around 1×10^{-6}, it can be seen from Fig. A2-1 that for a given change in the error probability the corresponding change in the signal/ratio is relatively small. For, let p have the value 3×10^{-7}, so that the corresponding value of d/σ, from Fig. A2-1, is 5. If now the error probability is doubled, that is, if p now has the value 6×10^{-7}, the corresponding change in tolerance to noise is only 0.26 dB. This shows that at high signal/noise ratios even the doubling of the error probability produces a negligible change in signal/noise ratio. On the other hand a small change in signal/noise ratio produces a relatively large change in the element error probability.

Consider that there are two binary signal-elements in a group, having possible values $\pm$ 1. From Eqn. (A2-1)

$$p_1 = Q\left(\frac{d_1}{\sigma}\right) \text{ and } p_2 = Q\left(\frac{d_2}{\sigma}\right)$$

Assume now that, the signal/noise ratio is high and furthermore

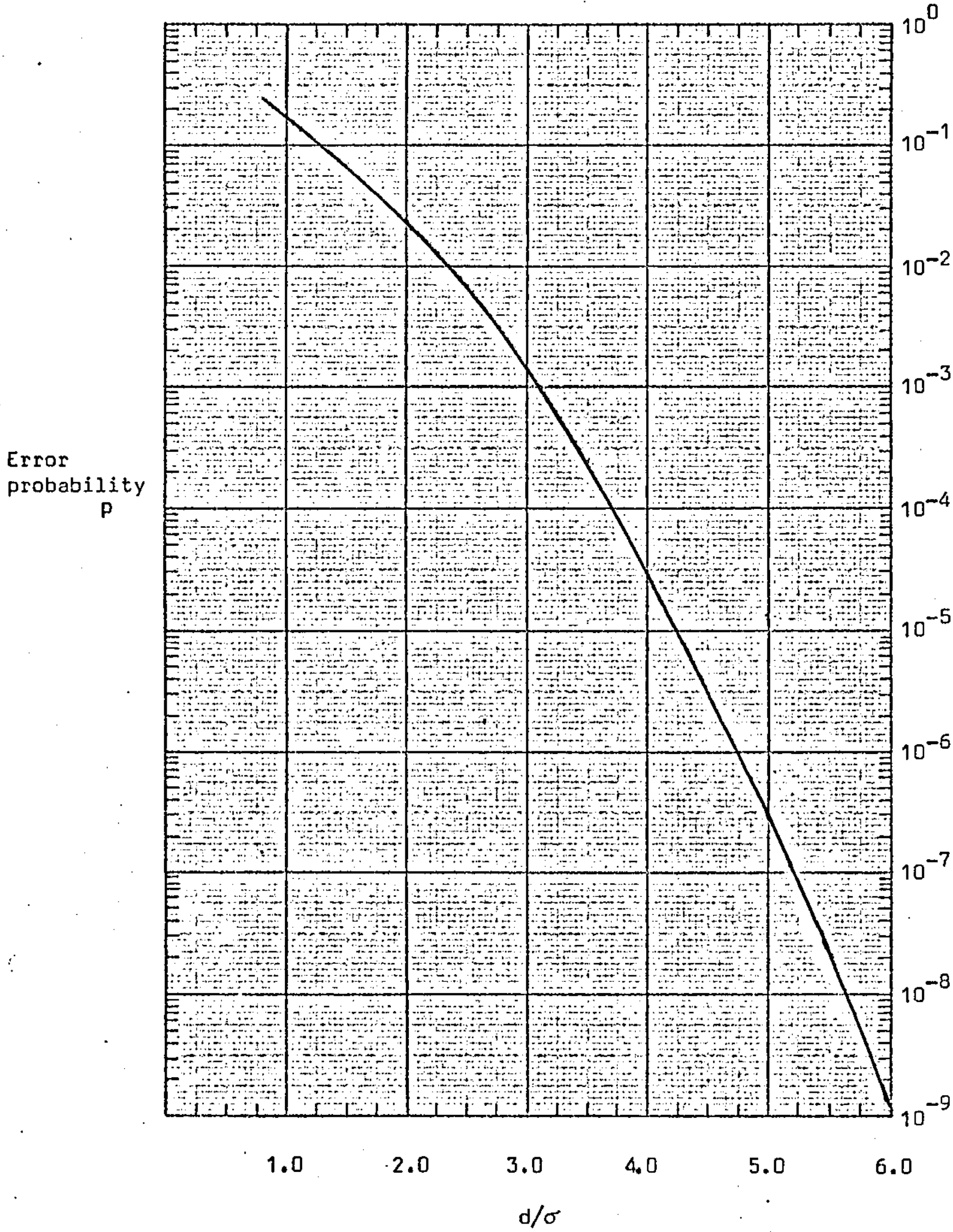


Figure A2-1

Variation of the element error probability p with d/σ.

$d_1/\sigma = 4.0$ (say) and $d_2/\sigma = 5.5$ (say). From Fig. A2-1, p_1 corresponding to $d_1/\sigma = 4.0$, is 3×10^{-5}, and p_2 corresponding to $d_2/\sigma = 5.5$, is 1.7×10^{-8}. It can be seen that $p_1 >> p_2$. It, therefore, follows that the average element error probability in the detection of the two signal-elements of the group is effectively given by p_1 which corresponds to the smaller of the two distances d_1 and d_2 provided, of course, that the signal/noise ratio is high. Again if there are m signal-elements in a group, the average element error probability, at high signal to noise ratios, is approximately given by the p_i in Eqn. (A2-1), which corresponds to the smallest of the d_i.

APPENDIX A3

COMPUTER SIMULATION PROGRAM FOR SYSTEM 2

```
      MASTER FAIZA
C     COMPUTER PROGRAM TO SIMULATE SYSTEM 2 FOR THE
C     DETECTION OF ORTHOGONAL GROUPS OF 4-LEVEL SIGNALS
C     M = NUMBER OF SIGNAL ELEMENTS IN A GROUP
C     N = NUMBER OF SAMPLE VALUES OF A RECEIVED GROUP OF
C            SIGNAL ELEMENTS
C     N1 = MAXIMUM NUMBER OF SAMPLE VALUES OF THE SAMPLED
C            IMPULSE RESPONSE OF THE CHANNEL
C     NBT = TOTAL NUMBER OF ELEMENTS TRANSMITTED
C            = M*4**M*NM
C     ANE = STD. DEVIATION OF THE ADDITIVE GAUSSIAN NOISE
C     THE FUNCTION UTR1(1,1,K1), WHERE K1 IS INITIALLY SET
C     TO ZERO, GENERATES RANDOM NUMBERS HAVING GAUSSIAN
C     DISTRIBUTION WITH ZERO MEAN AND UNIT VARIANCE
C     ANF = NORMALIZING FACTOR  FOR MAKING THE AVERAGE
C              ENERGY PER TRANSMITTED BIT EQUAL TO UNITY
      DIMENSION Y(12,12),Z(256,4),S(256,8),R(12),Z1(8),
     1Z2(8),ER(8)
      WRITE(2,1)
    1 FORMAT(//10X,' COMPUTER SIMULATION OF SYSTEM 2',
     1' WITH 4-LEVEL SIGNAL-ELEMENTS')
      READ(1,2)M,N1,NM
    2 FORMAT(3I0)
      N=M+N1-1
      N2=N1+1
      MM=4**M
      K1=0
      ANF=SQRT(2.5)
C     GENERATE THE MATRIX Z WHOSE ROWS ARE ALL POSSIBLE
C     COMBINATIONS OF SIGNAL ELEMENTS IN A GROUP
      DO 3 I=1,M
    3 Z2(I)=3.0
      DO 7 I=1,MM
      DO 5 J=1,M
      Z2(J)=Z2(J)+2.0
      IF(Z2(J)-3.0)6,6,4
    4 Z2(J)=-3.0
    5 CONTINUE
    6 DO 7 J=1,M
    7 Z(I,J)=Z2(J)/ANF
```

```
C       READ THE CHANNEL IMPULSE RESPONSE

        READ(1,9)(Y(1,I),I=1,N1)
    9   FORMAT(5F0.0)
        WRITE(2,10)
   10   FORMAT(///' IMPULSE RESPONSE OF THE CHANNEL')
        WRITE(2,11)(Y(1,I),I=1,N1)
   11   FORMAT(/5X,5(F12.4))


        DO 12 I=N2,N
   12   Y(1,I)=0.0

C       NORMALIZE CHANNEL IMPULSE RESPONSE

        SUM=0.0
        DO 13 I=1,N1
   13   SUM=SUM+Y(1,I)*Y(1,I)
        BF=SQRT(SUM)
        DO 14 I=1,N1
   14   Y(1,I)=Y(1,I)/BF

C       FORM THE M*N MATRIX Y

        DO 16 I=2,M
        L=I-1
        L1=I+1
        DO 15 K=1,L
   15   Y(I,K)=0.0
        DO 16 K=L1,N
        J=K-I+1
   16   Y(I,K)=Y(1,J)

C       FORM ALL THE POSSIBLE VECTORS S=Z*Y

        DO 17 I=1,MM
        DO 17 J=1,N
        S(I,J)=0.0
        DO 17 K=1,M
   17   S(I,J)=S(I,J)+Z(I,K)*Y(K,J)
        READ(1,18)ASF
   18   FORMAT(F0.0)

C       SET THE ERROR COUNTERS TO ZERO

        TE=0.0
        DO 19 I=1,M
   19   ER(I)=0.0
        DO 100 LL=1,NM
        DO 100 I=1,MM
```

```
C     FORM THE RECEIVED VECTOR R = Z*Y + W

      DO 21 II=1,M
      R(II)=0.0
      DO 20 J=1,M
 20   R(II)=R(II)+Z(I,J)*Y(J,II)
 21   R(II)=R(II)+UTR1(1,1,K1)*ARF

C     COMPUTE THE SQUARE OF THE DISTANCE OF EACH VECTOR
C     S FROM R AND DETECT Z

      DV=1000000.0
      DO 25 J=1,MM
      AN=0.0
      DO 22 K=1,M
 22   AN=AN+(R(K)-S(J,K))*(R(K)-S(J,K))
      IF(DV-AN)25,25,23
 23   DV=AN
      DO 24 K=1,M
 24   Z1(K)=Z(J,K)
 25   CONTINUE


C     COUNT THE ERRORS

      DO 27 J=1,M
      IF(Z1(J)-Z(I,J))26,27,26
 26   TE=TE+1.0
      ER(J)=ER(J)+1.0
 27   CONTINUE
 100  CONTINUE
      NBT=N*MM*NM

C     PRINT THE RESULTS

      WRITE(2,28)M
 28   FORMAT(/' NUMBER OF ELEMENTS PER GROUP = ',I2)
      WRITE(2,29)NBT
 29   FORMAT(/' TOTAL NUMBER OF ELEMENTS TRANSMITTED = ',I5)
      WRITE(2,30)ARF
 30   FORMAT(/' STD. DEVIATION OF NOISE = ',F8.4)
      WRITE(2,31)TE
 31   FORMAT(/' TOTAL NUMBER OF ERRORS = ',F6.1)
      WRITE(2,32)
 32   FORMAT(/' TOTAL ERRORS IN THE INDIVIDUAL ELEMENTS',
     1'OF A GROUP')
      WRITE(2,33)(ER(I),I=1,M)
 33   FORMAT(/5X,8(F8.4))
      STOP
      END
```

APPENDIX A4

COMPUTER SIMULATION PROGRAM FOR SYSTEM 3

```
      MASTER FAIZ

C     COMPUTER PROGRAM TO SIMULATE SYSTEM 3 FOR THE DETECTION
C     OF ORTHOGONAL GROUPS OF BINARY SIGNAL-ELEMENTS.

C     M = NUMBER OF SIGNAL-ELEMENTS IN A GROUP

C     N = NUMBER OF SAMPLE VALUES OF A RECEIVED GROUP OF
C     SIGNAL-ELEMENTS

C     N1 = MAXIMUM NUMBER OF SAMPLE VALUES OF THE SAMPLED
C     IMPULSE RESPONSE OF THE CHANNEL

C     RC = RELAXATION CONSTANT

C     NC = NUMBER OF ITERATIVE CYCLES

C     NBT = TOTAL NUMBER OF ELEMENTS TRANMITTED
C     = N2**M*NM

C     ARE=STD.DEVIATION OF THE ADDITIVE GAUSSIAN NOISE
C     THE FUNCTION UTR1(1,1,K1) , WHERE K1 IS INITIALLY SET
C     TO ZERO, GENERATES RANDOM NUMBERS HAVING GAUSSIAN
C     DISTRIBUTION WITH ZERO MEAN AND UNIT VARIANCE

      DIMENSION Y(15,15),ERI(100,8),ER(100),A(10,10),R(15),
     1Z(10),X(10),D(10),XA(10)
      WRITE(2,1)
    1 FORMAT(/10X,' COMPUTER SIMULATION OF SYSTEM 3 WITH BINARY SIGNALS'
     1)
      M=3
      N1=5
      N=M+N1-1
      N2=N1+1
      NC=50
      RC=1.25
      MM=2**M
      NM=2
      K1=0
```

```
C     READ THE CHANNEL IMPULSE RESPONSE
      READ(1,2)(Y(1,I),I=1,N1)
    2 FORMAT(5F0.0)
      DO 3 I=N2,N
    3 Y(1,I)=0.0
      WRITE(2,4)
    4 FORMAT(///' IMPULSE RESPONSE OF THE CHANNEL')
      WRITE(2,5)(Y(1,I),I=1,N1)
    5 FORMAT(/5X,5(F12.4))
C     NORMALISE CHANNEL IMPULSE RESPONSE
      SUM=0.0
      DO 6 I=1,N1
    6 SUM=SUM+Y(1,I)*Y(1,I)
      BF=SQRT(SUM)
      DO 7 I=1,N1
    7 Y(1,I)=Y(1,I)/BF
C     FORM THE M*M MATRIX Y
      DO 9 I=2,M
      L=I-1
      L1=I+1
      DO 8 K=1,L
    8 Y(I,K)=0.0
      DO 9 K=L1,N
      J=K-I+1
    9 Y(I,K)=Y(1,J)
C     FORM THE MATRIX A=Y*Y(TRANSPOSE)
      DO 10 I=1,M
      DO 10 J=1,M
      A(I,J)=0.0
      DO 10 K=1,N
   10 A(I,J)=A(I,J)+Y(I,K)*Y(J,K)
      READ(1,11)APF
   11 FORMAT(F0.0)
C     SET INITIAL CONDITIONS TO ZERO
      LK=0
      DO 12 I=1,NC
      ER(I)=0.0
      DO 12 J=1,M
   12 ERI(I,J)=0.0
      DO 200 KK=1,NM
      DO 13 I=1,M
   13 Z(I)=1.0
```

```
      DO 200 IL=1,MM
C     GENERATE THE COMPONENTS OF THE SINAL VECTOR Z
      DO 15 I=1,M
      Z(I)=Z(I)+2.0
      IF(Z(I)-1.0)16,16,14
   14 Z(I)=-1.0
   15 CONTINUE
   16 CONTINUE
C     FORM THE RECEIVED VECTOR R=Z*Y
      DO 18 I=1,N
      R(I)=0.0
      DO 17 J=1,M
   17 R(I)=R(I)+Z(J)*Y(J,I)
   18 R(I)=R(I)+UTR1(1,1,K1)*ARF
C     SET INITIAL CONDITIONS FOR THE GAUSS-SEIDEL PROCESS
C     I.E. SET X=0,XA=0 & D=R*Y(TRANSPOSE)
      DO 19 I=1,M
      X(I)=0.0
   19 XA(I)=0.0
      DO 20 I=1,M
      D(I)=0.0
      DO 20 J=1,N
   20 D(I)=D(I)+R(J)*Y(I,J)
      TE=ER(NC)
      DO 150 JK=1,NC
      DO 100 I=1,M
      XX=X(I)
      X(I)=X(I)+(D(I)-XA(I))*RC
C     APPLY THE CONSTRAINT ON X
      IF(ABS(X(I))-1.0)22,22,21
   21 X(I)=SIGN(1.0,X(I))
   22 DX=X(I)-XX
      DO 100 J=1,M
  100 XA(J)=XA(J)+DX*A(I,J)
```

```
C     DETECT Z FROM X AND COUNT THE ERRORS

      DO 24 I=1,M
      RF=X(I)*Z(I)
      IF(RF)23,23,24
   23 ER(JK)=ER(JK)+1.0
      ERI(JK,I)=ERI(JK,I)+1.0
   24 CONTINUE
  150 CONTINUE
      IF(ER(NC).GT.TE)LK=LK+1
  200 CONTINUE

C     PRINT THE RESULTS

      WRITE(2,25)M
   25 FORMAT(/' NUMBER OF ELEMENTS PER GROUP = ',I2)
      NBT=M*MM*NM
      WRITE(2,26)NBT
   26 FORMAT(/' TOTAL NUMBER OF ELEMENTS TANSMITTED = ',I5)
      WRITE(2,27)ARF
   27 FORMAT(/' STD. DEVIATION OF NOISE = ',F8.4)
      WRITE(2,28)
   28 FORMAT(//5X,6H CYCLE,3X,13H TOTAL ERRORS,10X,30H ERRORS IN ELEMENT
     1S OF A GROUP)
      DO 29 I=1,NC
   29 WRITE(2,30)I,ER(I),(ERI(I,J),J=1,M)
   30 FORMAT(/7X,I3,4X,F6.1,3X,8(F8.1))
      WRITE(2,31)LK
   31 FORMAT(/' NUMBER OF INDEPENDENT ERRORS = ',I4)
      STOP
      END
```

APPENDIX A5

COMPUTER SIMULATION PROGRAM FOR SYSTEM 6

```
      MASTER FAIZA

C     COMPUTER PROGRAM TO SIMULATE SYSTEM 6 FOR
C     THE DETECTION OF ORTHOGONAL GROUPS OF BINARY
C     SIGNAL-ELEMENTS.

C     M = NUMBER OF SIGNAL-ELEMENTS IN A GROUP.

C     N = NUMBER OF SAMPLE VALUES OF A RECEIVED GROUP OF
C         SIGNAL-ELEMENTS.

C     N1 = MAXIMUM NUMBER OF SAMPLE VALUES OF THE SAMPLED
C          IMPULSE RESPONSE OF THE CHANNEL.

C     RC = RELAXATION CONSTANT.

C     NC = NUMBER OF ITERATIVE CYCLES REQUIRED IN THE
C          DETECTION PROCESS.

C     NBT = TOTAL NUMBER OF ELEMENTS TRANSMITTED.
C         = M*2**M*NM

C     ARF = STD. DEVIATION OF ADDITIVE WHITE GAUSSIAN NOISE.

C     THE FUNCTION UTR1(1,1,K1), WHERE K1 IS INITIALLY SET
C     TO ZERO, GENERATES RANDOM NUMBERS HAVING GAUSSIAN
C     DISTRIBUTION WITH ZERO MEAN AND UNIT VARIANCE.

      DIMENSION Y(15,15),A(10,10),X(10),Z(10),Z1(10),R(15)
     1,R1(15),ER(10),TIP(10),XA(10),D(10)
      WRITE(2,1)
    1 FORMAT(///10X,' COMPUTER SIMULATION OF SYSTEM 6',
     1' WITH BINARY SIGNAL ELEMENTS')
      READ(1,2)M,N1,NM,NC
    2 FORMAT(4I0)
      N=M+N1-1
      N2=N1+1
      RC=1.25
      MM=2**M
      NBT=M*2**M*NM
      K1=0
```

```
C       READ CHANNEL IMPULSE RESPONSE

        READ(1,3)(Y(1,I),I=1,N1)
  3     FORMAT(5(F0.0))
        WRITE(2,4)
  4     FORMAT(//' CHANNEL IMPULSE RESPONSE')
        WRITE(2,5)(Y(1,I),I=1,N1)
  5     FORMAT(/5X,5(F12.4))
        DO 6 I=N2,N
  6     Y(1,I)=0.0

C       NORMALIZE CHANNEL IMPULSE RESPONSE

        SUM=0.0
        DO 7 I=1,N1
  7     SUM=SUM+Y(1,I)*Y(1,I)
        BF=SQRT(SUM)



        DO 8 I=1,N1
  8     Y(1,I)=Y(1,I)/BF

C       FORM THE M*N MATRIX Y

        DO 10 I=2,M
        L=I-1
        L1=L+1
        DO 9 K=1,L
  9     Y(I,K)=0.0
        DO 10 K=L1,N
        J=K-I+1
 10     Y(I,K)=Y(1,J)

C       FORM THE MATRIX A = Y*Y(TRANSPOSE)

        DO 11 I=1,M
        DO 11 J=1,M
        A(I,J)=0.0
        DO 11 K=1,N
 11     A(I,J)=A(I,J)+Y(I,K)*Y(J,K)
        READ(1,12)ARF
 12     FORMAT(F0.0)

C       SET ERROR COUNTERS TO ZERO

        DO 13 I=1,M
 13     ER(I)=0.0
        TE=0.0
        DO 400 IJK=1,NM
```

```
C     GENERATE THE VECTOR Z

      DO 14 I=1,M
 14   Z(I)=1.0
      DO 400 LJK=1,MM
      DO 16 I=1,M
      Z(I)=Z(I)+2.0
      IF(Z(I)-1.0)17,17,15
 15   Z(I)=-1.0
 16   CONTINUE
 17   CONTINUE

C     FORM THE RECEIVED VECTOR R=Z*Y+W

      DO 19 I=1,M
      R(I)=0.0
      DO 18 J=1,M
 18   R(I)=R(I)+Z(J)*Y(J,I)
 19   R(I)=R(I)+(UTR1(1,1,K1)*ARF)

C     FORM THE VECTOR D=R*Y(TRANSPOSE)

      DO 20 I=1,M
      D(I)=0.0
      DO 20 J=1,M
 20   D(I)=D(I)+R(J)*Y(I,J)

C     START THE PROCESS OF DETECTION AND CANCELLATION

      DO 300 LL=1,M


C     SET INITIAL CONDITIONS TO ZERO

      DO 21 I=LL,M
      X(I)=0.0
 21   XA(I)=0.0

C     DETECT Z USING SYSTEM 3

      DO 100 JK=1,NC
      DO 100 I=LL,M
      XX=X(I)
      X(I)=X(I)+(D(I)-XA(I))*RC
      IF(ABS(X(I))-1.0)23,23,22
 22   X(I)=SIGN(1.0,X(I))
 23   DX=X(I)-XX
      DO 100 J=LL,M
 100  XA(J)=XA(J)+DX*A(I,J)
```

```
C     DETECT Z(I) AND STORE

      IF(X(LL))24,24,25
 24   Z1(LL)=-1.0
      GO TO 26
 25   Z1(LL)=1.0
 26   CONTINUE

C     CANCEL THE DETECTED SIGNAL AND FORM NEW R & D

      LL1=LL+1
      IF(LL.EQ.N)GO TO 29
      DO 27 I=LL1,N
 27   R(I)=R(I)-Z1(LL)*Y(LL,I)
      DO 28 I=LL1,N
      D(I)=0.0
      DO 28 J=LL1,N
 28   D(I)=D(I)+R(J)*Y(I,J)
 300  CONTINUE
 29   CONTINUE

C     COUNT THE NUMBER OF ERRORS

      DO 400 I=1,M
      IF(Z(I)*Z1(I))30,30,400
 30   ER(I)=ER(I)+1.0
      TE=TE+1.0
 400  CONTINUE

C     PRINT THE RESULTS

      WRITE(2,31)M
 31   FORMAT(/' NUMBER OF ELEMENTS IN A GROUP = ',I3)
      WRITE(2,32)NBT
 32   FORMAT(/' TOTAL NUMBER OF ELEMENTS TRANSMITTED = ',I6)
      WRITE(2,33)ARF
 33   FORMAT(/' STD. DEVIATION OF NOISE = ',F10.3)
      WRITE(2,34)
 34   FORMAT(/' TOTAL NUMBER OF ERRORS IN THE INDIVIDUAL',
     11X,' ELEMENTS OF THE GROUPS')
      WRITE(2,35)(ER(I),I=1,M)
 35   FORMAT(/5X,3(F8.1))
      WRITE(2,36)TE
 36   FORMAT(/' TOTAL NUMBER OF ERRORS = ',F6.1)


      STOP
      END
```

APPENDIX A6

COMPUTER SIMULATION PROGRAM FOR SYSTEM 8/4

```
      MASTER SAAD

C     COMPUTER PROGRAM TO SIMULATE SYSTEM 8/4 FOR THE
C     DETECTION OF ORTHOGONAL GROUPS OF  4-LEVEL SIGNALS.

C     M = NUMBER OF SIGNAL-ELEMENTS IN A GROUP

C     N = NUMBER OF SAMPLE VALUES OF A RECEIVED GROUP OF
C           SIGNAL-ELEMENTS

C     N1 = MAXIMUM NUMBER OF SAMPLE VALUES OF THE SAMPLED
C            IMPULSE RESPONSE OF THE CHANNEL

C     RC = RELAXATION CONSTANT

C     NC1 = NUMBER OF ITERATIVE CYCLES FOR THE INITIAL
C             DETECTION PROCESS

C     NC2 = NUMBER OF ITERATIVE CYCLES FOR THE FINAL
C             DETECTION PROCESS

C     NBT = TOTAL NUMBER OF SIGNAL-ELEMENTS TRANSMITTED
C           = M*4**M*NM

C     AMF = STD.DEVIATION OF ADDITIVE GAUSSIAN NOISE

C     THE FUNCTION HRN1(1,1,K1), WHERE K1 IS INITIALLY SET
C     TO ZERO, GENERATES RANDOM NUMBERS HAVING GAUSSIAN
C     DISTRIBUTION WITH ZERO MEAN AND UNIT VARIANCE

C     ANF = NORMALISING FACTOR TO MAKE THE ENERGY PER
C             TRANSMITTED BIT EQUAL TO UNITY

      DIMENSION Y(15,15),A(10,10),R1(2,8),R(15),X(8),X1(8),
     1D(8),R1(12),Z(8),XA(8),ER(8),ANOR(2),Q(8)
      WRITE(2,1)
 1    FORMAT(///10X,' COMPUTER SIMULATION OF SYSTEM 8/4',
     1' WITH 4-LEVEL SIGNAL-ELEMENTS')
      READ(1,2)M,N1,NM,NC1,NC2
 2    FORMAT(5I0)
      N=M+N1-1
      N2=N+1
      RC=1.25
      ANF=SQRT(2.5)
      MM=4**M
      K1=0
      PP=-3.0/ANF
      QQ=-1.0/ANF
      RR=1.0/ANF
      SS1=3.0/ANF
      TT=2.0/ANF
      NBT=M*MM*NM
```

```
C     READ CHANNEL IMPULSE RESPONSE
      READ(1,3)(Y(1,I),I=1,M1)
    3 FORMAT(5(F0.0))
      WRITE(2,4)
    4 FORMAT(//' CHANNEL IMPULSE RESPONSE')
      WRITE(2,5)(Y(1,I),I=1,M1)
    5 FORMAT(/5X,5(F12.4))
      DO 6 I=M2,M
    6 Y(1,I)=0.0
C     NORMALISE CHANNEL IMPULSE RESPONSE
      SUM=0.0
      DO 7 I=1,M1
    7 SUM=SUM+Y(1,I)*Y(1,I)
      SF=SQRT(SUM)
      DO 8 I=1,M1
    8 Y(1,I)=Y(1,I)/SF
C     FORM THE M*M MATRIX Y
      DO 10 I=2,M
      L=I-1
      L1=L+1
      DO 9 K=1,L
    9 Y(I,K)=0.0
      DO 10 K=L1,M
      J=K-I+1
   10 Y(I,K)=Y(1,J)
C     FORM THE MATRIX A = Y*Y(TRANSPOSE)
      DO 11 I=1,M
      DO 11 J=1,M
      A(I,J)=0.0
      DO 11 K=1,M
   11 A(I,J)=A(I,J)+Y(I,K)*Y(J,K)
      READ(1,12)ARF
   12 FORMAT(F0.0)
C     SET ERROR COUNTERS TO ZERO
      DO 13 I=1,M
   13 ER(I)=0.0
      TE=0.0
      DO 300 IJK=1,MM
      DO 14 I=1,M
   14 D(I)=0.0
      DO 300 JJJ=1,MM
```

```
C       GENERATE THE VECTOR Z
        DO 16 I=1,M
        Q(I)=Q(I)+2.0
        IF(Q(I)-3.0)17,17,15
   15   Q(I)=-3.0
   16   CONTINUE
   17   DO 18 I=1,M
   18   Z(I)=Q(I)/ANF
C       FORM THE RECEIVED VECTOR R = ZY + W
        DO 20 I=1,N
        R(I)=0.0
        DO 19 J=1,M
   19   R(I)=R(I)+Z(J)*Y(J,I)
   20   R(I)=R(I)+WTE1(1,1,K1)*ARF


C       FORM THE VECTOR D =R*Y(TRANSPOSE)
        DO 21 I=1,M
        D(I)=0.0
        DO 21 J=1,N
   21   D(I)=D(I)+R(J)*Y(I,J)
C       PERFORM THE INITIAL DETECTION PROCESS USING SYSTEM 3,
C       TO OBTAIN THE 2 MOST LIKELY VALUES OF Z
C        SET INITIAL CONDITIONS
        DO 22 I=1,M
        X(I)=0.0
   22   XA(I)=0.0
        DO 400 IL=1,201
        DO 100 I=1,M
        XY=X(I)
        X(I)=X(I)+(D(I)-XA(I))*RC
        IF(ABS(X(I))-SS1)24,24,23
   23   X(I)=SIGN(SS1,X(I))
   24   DX=X(I)-XY
        DO 100 J=1,M
  100   XA(J)=XA(J)+DX*A(I,J)
        DO 31 I=1,M
        IF(X(I))25,25,28
   25   IF(X(I)+RP)26,26,27
   26   Q1(1,I)=P2
        Q1(2,I)=Q2
        GO TO 31
   27   Q1(1,I)=Q2
        Q1(2,I)=P2
        GO TO 31
```

```
28    IF(Y(I)-R0)29,30,30
29    Q1(1,I)=S0
      Q1(2,I)=R0
      GO TO 31
30    Q1(1,I)=R0
      Q1(2,I)=S1
31    CONTINUE

C     APPLY THE FINAL DETECTION PROCESS USING SYSTEM4 WITH
C     THE TWO MOST LIKELY VALUES OF EACH Z(I) STORED IN
C     Q1(1,I) AND Q1(2,I)

      DO 300 L=2,N
      DO 300 IK=1,2
      X(L-1)=Q1(IK,L-1)
      DO 32 I=L,N
32    X(I)=0.0
      DO 33 I=1,N
      XA(I)=0.0
      DO 33 J=1,N
33    XA(I)=XA(I)+X(J)*A(J,I)
      DO 300 IJ=1,NC2
      DO 300 I=L,N
      XX=X(I)
      X(I)=X(I)+(D(I)-XA(I))*RC

C     APPLY CONSTRAINTS ON X(I)
      IF(X(I)-Q1(1,I))34,35,35


34    X(I)=Q1(1,I)
      GO TO 37
35    IF(X(I)-Q1(2,I))37,37,36
36    X(I)=Q1(2,I)
37    DX=X(I)-XX
      DO 200 J=1,N
200   XA(J)=XA(J)+DX*A(I,J)
      DO 44 I=L,N
      IF(Y(I))38,38,41
38    IF(Y(I)+T2)39,39,40
39    X(I)=S0
      GO TO 44
40    X(I)=R0
      GO TO 44
41    IF(Y(I)-T2)42,42,43
42    X(I)=R1
      GO TO 44
43    X(I)=S1
44    CONTINUE
      DO 45 I=1,N
      D1(I)=0.0
      DO 45 J=1,N
45    D1(I)=D1(I)+X(J)*V(J,I)
```

```
      AN=0.0
      DO 46 I=1,N
46    AN=AN+((R(I)-D1(I))*(R(I)-D1(I)))
      STOR(IK)=AN
300   CONTINUE
      DV=1000000.0
      DO 48 I=1,2
      IF(DV-ANORM(I))48,48,47
47    XB=Y1(I,L-1)
      DV=STOR(I)
48    CONTINUE
      X(L-1)=XB
400   CONTINUE
      DV=1000000.0
      DO 52 K=1,2
      X(M)=Y1(K,M)
      DO 49 I=1,N
      D1(I)=0.0
      DO 49 J=1,M
49    D1(I)=D1(I)+X(J)*Y(J,I)
      AN=0.0
      DO 50 I=1,N
50    AN=AN+((R(I)-D1(I))*(R(I)-D1(I)))
      IF(DV-AN)52,52,51
51    DV=AN
      BL=X(M)
52    CONTINUE
      X(M)=BL

C     COUNT THE NUMBER OF ERRORS

      DO 54 I=1,M
      IF(Z(I)-X(I))53,54,53
53    IE=IE+1.0
      ER(I)=ER(I)+1.0
54    CONTINUE
500   CONTINUE

C     PRINT THE RESULTS


      WRITE(2,55)M
55    FORMAT(/' NUMBER OF ELEMENTS PER GROUP = ',I2)
      WRITE(2,56)NBT
56    FORMAT(/' TOTAL NUMBER OF ELEMENTS TRANSMITTED = ',I5)
      WRITE(2,57)AKF
57    FORMAT(/' STD. DEVIATION OF NOISE = ',F8.4)
      WRITE(2,58)
58    FORMAT(/' TOTAL ERRORS IN THE INDIVIDUAL ELEMENTS',
     */,' OF A GROUP')
      WRITE(2,59)(ER(I),I=1,M)
59    FORMAT(/5X,8(F6.1))
      STOP
      END
```

APPENDIX A7

THE AUTOCORRELATION MATRICES YY^T AND DD^T

Let

$$y_o\ y_1\ y_2\ \dots\ y_g \qquad \text{(A7-1)}$$

be the g+1 sample values of the sampled impulse response of the baseband channel. Let Y and D be the two mxn matrices of rank m, whose i^{th} rows are, respectively, Y_i and D_i, where $n = m + g$ and

$$Y_i = \overbrace{0\ \dots\ 0}^{i-1}\ \overbrace{y_o\ y_1\ \dots\ y_g}^{g+1}\ \overbrace{0\ \dots\ 0}^{m-i} \qquad \text{(A7-2)}$$

and

$$D_i = \overbrace{0\ \dots\ 0}^{i-1}\ \overbrace{y_g\ y_{g-1}\ \dots\ y_o}^{g+1}\ \overbrace{0\ \dots\ 0}^{m-i} \qquad \text{(A7-3)}$$

Since the components of the matrices YY^T and DD^T are, respectively, the autocorrelation coefficients of the sequences $0\ .\ .\ 0\ 0\ y_o\ y_1\ .\ .\ .\ y_g\ 0\ 0\ .\ .\ 0$ and $0\ .\ .\ 0\ 0\ y_g\ y_{g-1}\ .\ .\ .\ y_o\ 0\ 0\ .\ .\ 0$, the matrices YY^T and DD^T are real, symmetric and positive definite.[32,35,52] Furthermore, if a_i, $i = 0,1, \dots, m-1$ are the coefficients of the autocorrelation function of the sampled impulse response of the channel in Eqn. (A7-1), then the matrix YY^T has the form

$$\begin{bmatrix} a_o & a_1 & a_2 & \dots & a_{m-1} \\ a_1 & a_o & a_1 & \dots & a_{m-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m-1} & a_{m-2} & \dots & \dots & a_o \end{bmatrix} \qquad \text{(A7-4)}$$

The matrix DD^T has also the same form as the matrix YY^T. Thus YY^T and DD^T are completely determined by the components of their first rows. It follows that in order to show that the two matrices are equal for a given sampled impulse response of the channel, it is only necessary to show that the components of the first rows of the matrices DD^T and YY^T are equal.

From Eqn. (A 7-2), the m components of the first row of the matrix YY^T are

$$Y_1 Y_i^T \quad \text{for } i = 1, 2, \ldots, m$$

$$= y_{i-1} y_o + y_i y_1 + y_{i+1} y_2 + \ldots + y_g y_{g-i+1}$$

$$i = 1, 2, \ldots m \quad \text{(A7-5)}$$

Also from Eqn. (A7-3), the m components of the first row of matrix DD^T are

$$D_1 D_i^T \quad \text{for } i = 1,2, \ldots, m$$

$$= y_g y_{g-i+1} + y_{g-1} y_{g-i} + \ldots + y_1 y_i + y_o y_{i-1}$$

$$= y_{i-1} y_o + y_i y_1 + \ldots + y_{g-1} y_{g-i} + y_g y_{g-i+1}$$

$$i = 1, 2, \ldots, m \quad \text{(A7-6)}$$

From Eqns. (A7-5) and (A7-6)

$$D_1 D_i^T = Y_1 Y_i^T \quad i = 1,2, \ldots, m \quad \text{(A7-7)}$$

which shows that for a given sampled impulse response of the channel, the m x m metrices YY^T and DD^T are equal.

APPENDIX A8

PROBABILITY OF ERROR IN THE DETECTION OF MULTI-LEVEL SIGNAL-ELEMENTS

Consider the detection of s from x, where $s = a + 2id$, for $i = 0, 1, \ldots, k-1$, a and d being real scalar quantities, and

$$x = s + u. \qquad \text{(A8-1)}$$

u is a sample value of a Gaussian random variable with zero mean and variance σ^2.

It is well known that if s is equally likely to have any of its k possible values, the detection process that minimizes the probability of error in the detection of s from x, accepts the possible value of s at the minimum distance from x.[45,56,57] This is equivalent to comparing x with k-1 thresholds whose values are $a + (2i - 1)d$, for $i = 1, 2, \ldots, k-1$, and accepting the possible value of s between the same thresholds as x. Assume now that this arrangement is used, as shown in Fig. A8-1. Clearly d is the distance from each value of s to the nearest decision boundaries.

If s in Eqn. (A8-1) has one of its two extreme values, that is, a or $a + 2(k-1)d$, then the probability of error in the detection of s from x is

$$\int_{d}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{u^2}{2\sigma^2}\right) du$$

$$= \int_{d/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}u^2\right) du$$

$$= Q\left(\frac{d}{\sigma}\right) \qquad \text{(A8-2)}$$

If s has one of the remaining k-2 values, that is $a + 2id$, for $i = 1, 2, \ldots, k-2$, then the probability of error in the detection of s from x is

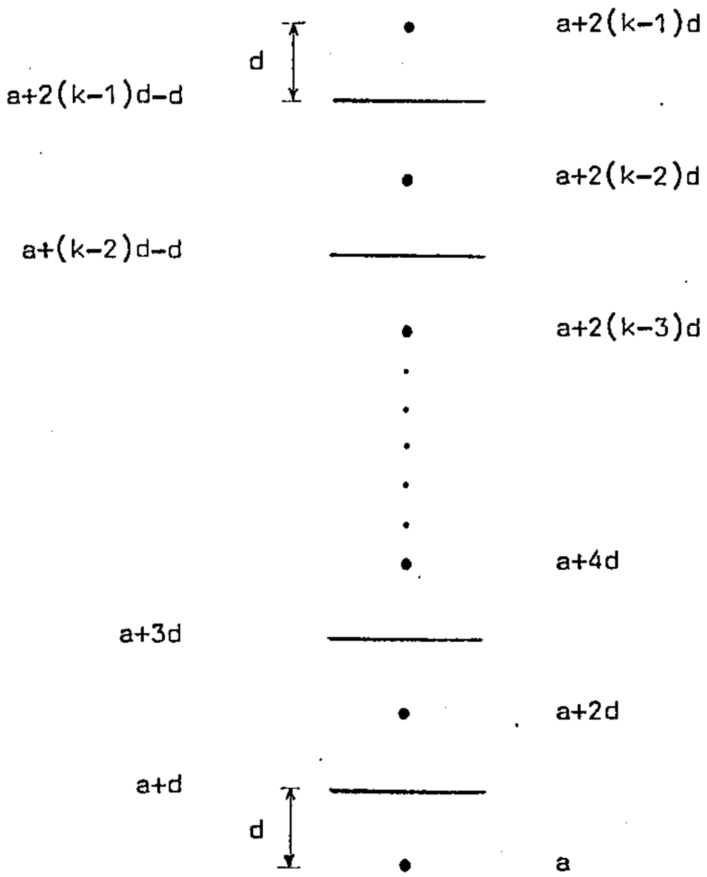


Figure A8-1

Decision boundaries used in the detection of s from x.

• Possible values of s.

——— Decision boundaries.

$$2 \int_{d}^{\infty} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{u^2}{2\sigma^2}\right) du$$

$$= 2Q\left(\frac{d}{\sigma}\right) \qquad \text{(A8-3)}$$

Since s is equally likely to have any of its k possible values, the average probability of error in the detection of s from x is

$$p = \frac{1}{k}\left\{2Q\left(\frac{d}{\sigma}\right) + (k-2)\ 2Q\left(\frac{d}{\sigma}\right)\right\}$$

$$= \frac{2(k-1)}{k}\ Q\left(\frac{d}{\sigma}\right) \qquad \text{(A8-4)}$$

It can be seen that when s has two possible values

$$p = Q\left(\frac{d}{\sigma}\right),$$

when s has four possible values

$$p = 1.5\ Q\left(\frac{d}{\sigma}\right)$$

and when s has eight possible values

$$p = 1.75\ Q\left(\frac{d}{\sigma}\right).$$

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