EXPLAINING THE BAYES' THEOREM GRAPHICALLY

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Abstract

The Bayes' theorem on conditional probabilities is normally presented to students in introductory courses/modules on Statistics and Probability. This because most STEM students will make use of conditional probabilities in their professional lives with or without noticing. However, maybe because of the unfamiliar notation or because of the variety of ways in which this theorem can be formulated, most students have trouble understanding it. Moreover, when it comes to practical applications and problem exercises, most students (who have generally memorised its manifold ways of rearranging the conditional probabilities formula along with a few applications) struggle even more to come up with correct solutions.

By means of a completely graphical approach, this paper presents an alternative way of explaining the Bayes' theorem to STEM students. By means of diagrams and schematics the students can see the conditional probabilities represented as areas in a square. Simple geometric operations with these areas (additions and multiplications mostly) allow them, not just to understand this theorem far quicker, but to apply it confidently in almost any possible problem configuration. Overall, this paper offers an alternative or complementary way of explaining this important theorem more clearly to students that take probability courses by conveying it graphically instead of with the traditional mathematical formulae.

Through a representative case study, this paper deals provides first-hand evidence about how confusing to understand the Bayes' theorem might be at first even in simple problems, and how the understanding of this theorem is dramatically improved when presenting it graphically.

Keywords: student, visual learning, statistics, probabilities, Bayes' theorem.

1 INTRODUCTION

In daily decision making, we usually update our estimates depending on the events that we see happening or not. For example, we might decide to buy a car when we find out about a government campaign that partially subsidises the price of cars, or we might have higher chances of not taking an exam when we see that some of our capable colleagues are failing to pass it themselves.

This behaviour emanates from the fact that when we observe that some events occur (or not), others are more (or less) likely to happen too. In other words, there are events (as we call them in statistics) that might be correlated to the occurrence of other events. This is modelled in probability theory by the formula of Bayes, also named the Bayes' theorem, Bayes' rule and Bayes' law.

The Bayes' theorem describes the probability of an event, based on prior knowledge of conditions that might be related to the event. For example, if the probabilities of losing hair are higher for the older male population, we can infer a more accurate probability of someone being bald when we know that he is a man who is 60 years old. This, compared to when we don't know how old our subject is and/or when we don't even know if it is a man or a woman.

The Bayes' theorem is also highly relevant in Bayesian inference, a particular branch of statistical inference, that deals with different probability interpretations depending on the occurrence of a subset of prior events.

Bayes' theorem was proposed by Reverend Thomas Bayes who lived between 1701 and 1761. However, his work was not properly discovered and applied until the early 1800s when Pierre-Simon Laplace published his "Théorie analytique des probabilités" in 1812. The importance of this theorem is such that some mathematicians concede that the "Bayes' theorem is to theory of probability what the Pythagorean theorem is to geometry" [1].

In conclusion, Bayes' theorem is highly relevant and frequently used by all of us, no matter we are (or not) aware of it. However, making the students understand the Bayes' theorem is a totally different issue. After having tried to explain this theorem for many years and see how our students still struggle to grasp his significance and mastering its endless applications, we decided to explain it in a visually instead of mathematically. This paper deals with the formulation of the Bayes' theorem graphically, which has proved to significantly increase students' understanding and retention.

2 FORMULATION OF THE BAYES' THEOREM

Bayes' theorem deals with revising decision making when there are sequential events whereby new additional information is obtained for a subsequent event. Then, that new information is used to revise the probability of the initial event(s).

In a more formal way, this is like saying that additional data can be used to update prior (initial) probabilities to new posterior (final) probabilities. Therefore, a prior probability is an initial probability value obtained before any additional information is known. A posterior probability is a probability value that has been updated using later information. Hence, in these terms, the Bayes' theorem is stated like this:

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B)} \tag{1}$$

In (1) P(A) and P(B) are the prior probabilities of events A and B, respectively, happening without regard to one another. P(A|B) and P(B|A) are the posterior probabilities. Particularly, P(A|B) represents the probability of A happening once it is know that B is happening (or has already happened). P(B|A) represents the probability of B happening once it is know that A is happening, again, or has already happened.

As can be seen, the Bayes' theorem is not particularly mathematically complex. However, it is an easy subject of the playground of mathematics [2]. For example, as (1) can also be expressed as P(B|A)= P(A|B)*P(B)/P(A), P(A)= P(A|B)*P(B)/ P(B|A) or P(B)= P(B|A)*P(A)/ P(A|B), then the following is also true: P(A|B)=P(A and B)/P(B) and P(B|A)=P(A and B)/P(A). Also, P(A and B)= P(A|B)*P(B)= P(B|A)*P(A). And this is just the beginning, there are endless combinations that can be established and here is when the students start to get lost.

2.1 An example of application

Common applications of the Bayes' theorem involve the calculation of posterior probabilities in medicine. Arguably, Bayes' theorem is easy to understand when it is applied in context where we consider event A the probability of being affected by a disease or illness, and event B the probability of a test detecting that disease.

A well-known example that many instructors use nowadays is taken from Deborah J. Bennett's book "Randomness" [3], which is a famous quiz that was presented to medical doctors:

"*A test of a disease presents a rate of 5% false positives. The disease strikes 1/1,000 of the population. People are tested at random, regardless of whether they are suspected of having the disease. A patient's test is positive. What is the probability of the patient being stricken with the disease?*"

Despite it is not mentioned, for solving this problem it is necessary to assume that the rate of false negatives is zero.

Well, when this quiz was presented, most doctors said the answer was 95%. This happened because most of them believed that the test had a 95% accuracy rate. However, the real answer is a little below than 2%. This happens all the time. No medical test is perfect (positive always whenever there is a disease in the patient and negative always when there is not) therefore, there are high chances that, whenever a test is positive, still the chances of actually having the disease are quite low. Let us work out how we come to that conclusion by applying the Bayes' theorem.

Let A be the event that the patient actually has the disease and B be the event that the test gives positive. It is known that P(A)=1/1000=0.001 and currently P(B) is unknown. We also know that the rate of false positives is 5%, that is, P(B|not A)=0.05, therefore P(not B|not A)=0.95. We should also know that, since the rate of false negatives is 0, this means that P(not B|A)=0, therefore P(B|A)=1.

We know from Bayes that P(A|B)= P(B|A)*P(A)/P(B), but since we don't know P(B), then P(A|B) (which is what we are looking for: the probability of having the disease when the test has resulted positive) cannot be calculated yet.

We also know that P(B)=P(A and B)+P(not A and B), which is after Bayes, P(B|A)*P(A)+P(B|not A)*P(not A)=1*0.001+0.05*(1-0.001)=0.05095.

Now we can complete the calculations with the original Bayes' theorem P(A|B)= P(B|A)*P(A)/P(B)= 1*0.001/0.05095=0.019627≈2%.

Well, if you got lost at some point while handling so many mathematical expressions, you are not the only one. Most students have the same problem and I can say that this is one of the simplest applications of the Bayes' theorem in real-life.

What is worse, after trying to re-explain this theorem in many ways and providing numerous exercises, the students still struggle to make sense of the mathematical apparatus of this theorem. That is why, the authors considered at some point to resort to explain it with a totally different strategy.

3 VISUAL REPRESENTATION OF THE BAYES' THEOREM

The Bayes theorem is explained in both pre-university and introductory university courses. It is also clear that the mathematical formulation does not seem to be particularly daunting either. However, apply it is a different animal. There are so many ways of reformulating the problem that it ends up being very confusing. At least that is the problem the authors had throughout many years of teaching decision sciences courses. And that is why we came up with a graphical approach.

It has been known for a long time that most of us are predominantly visual learners, that is, we understand much better the information when seen by means of pictures or graphs, compared to when we are just presented with textual information [4,5]. So, apparently, presenting the Bayes' theorem graphically seemed a good idea. However, to the best of our knowledge, we could not find a visual representation of this theorem (with this exception [6] which we consider rather too advanced for first-year students). Therefore, we had to came up with one ourselves.

3.1 Probability of independent events

Before explaining the Bayes' theorem, it is common to explain what independent events are. When two events are said to be independent of each other, what this means is that the probability that one event occurring in no way affects the probability of the other event occurring [7]. For example, someone cast a die and flips a coin. Both event outcomes will be totally independent because the result of each experiment will not be influenced with each other.

Therefore, when we explain what two independent events A and B are we present the students with Figure 1.

Figure 1 represent in both axes the probabilities of happening (positive) or not (negative) two different events. A might be considered the primary event and B the secondary. The probabilities of each event not happening are just one minus the probabilities of the event happening. Hence, all the edges in this square measure one unit (like the total probability).

We know that event B is independent from event A because the probability of B happening once A has happened, that is P(B|A), is exactly the same as the probability of B happening if A had not happened, that is, P(B|not A). That makes that the grey rectangles (① and ②) in Figure 1 have the exact same height. It is worth noting that the probabilities of A happening (P(A)) and not happening A (P(not A)) do not have to be the same, nor the probabilities of B happening or not happening (P(B|A) vs P(not B|A) and P(B|not A) vs P(not B|notA)) either.

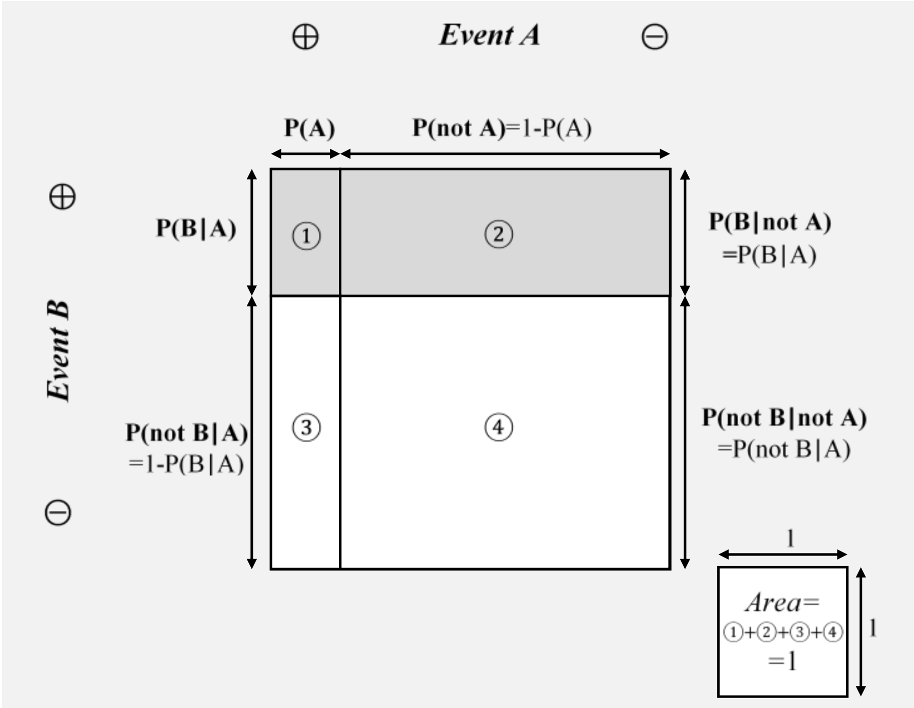

Figure 1. Distribution of probabilities of two independent events with two possible outcomes each.

3.2 Probability of dependent events. Law of conditional probabilities

Once the students understand what independent events are, they move on to events which are dependent. Here is when the Bayes' theorem becomes useful, as knowing that a main event A has taken place or not, allows us to improve the estimation of the probability of a secondary event B happening or not. This is exactly what Figure 2 represents and the major outcome of this paper. Again, the square in Figure has edges whose length is one, therefore the area of this square also equals one.

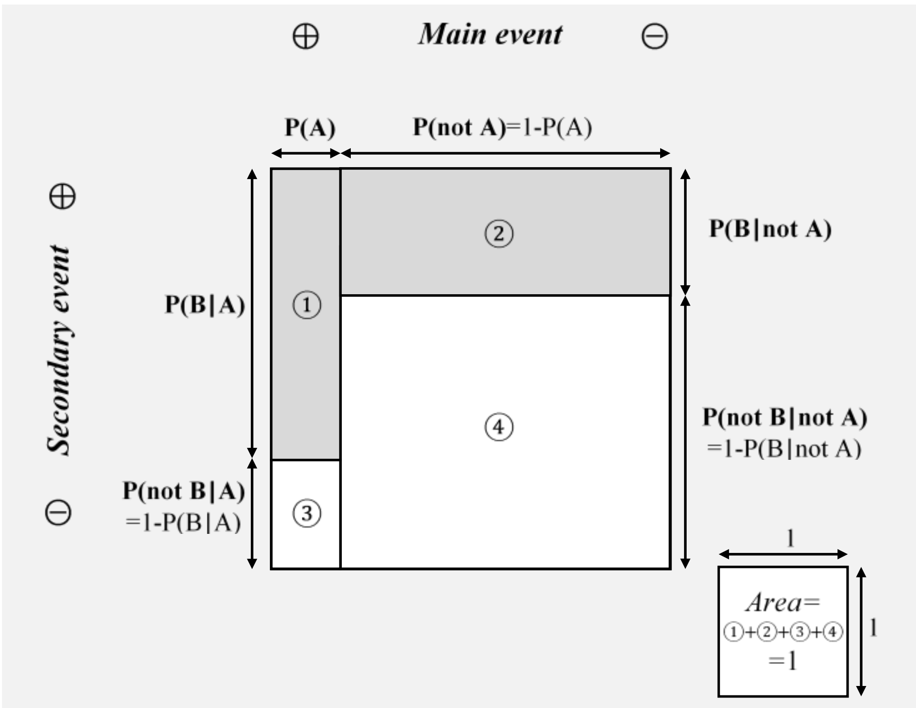

Figure 2. Distribution of probabilities of two correlated events with two possible outcomes (see how the conditional probabilities of B change depending on A happening or not).

Normally, most examples of the Bayesian inference involve trying to figure out the probability of P(B|A), that is, the probability that B happens once we have found out that A has happened. It is also frequent having to assess the probability of B happening once has not happened, that is P(B|not A). However, neither one can be seen in Figure 2. How do we calculate it?

To answer this question, let us revisit the problem which medical doctors had to answer. Figure 3 particularises Figure 2 for the quiz mentioned above about a test having being positive on an individual when testing if that person was suffering from a disease.

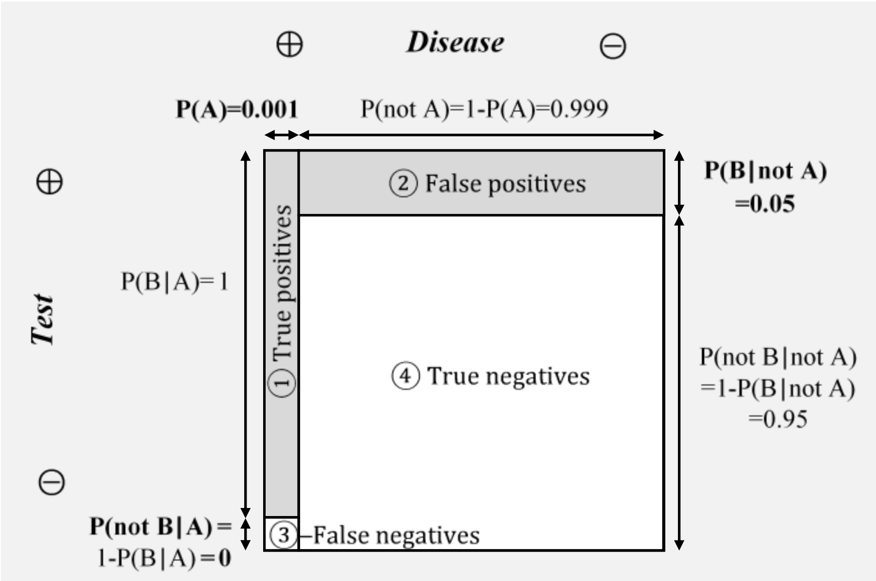

Figure 3. Problem solution expressed graphically.

As can be seen, all the probabilities information around the square in bold come from the data directly provided by the problem wording. The other probabilities are very easily inferred just by knowing that they are the complementary probabilities of the probabilities in bold.

OK, so our subject had a positive in his/her test. That means he/she is in one of the two squares shaded in grey (he actually has the disease, square ① or P(A U B), or he does not have it despite the test was positive, square ② or P(B U not A). There is no other possibility as the test was indeed positive. Well, so what is the probability of him/her having the disease then? This is the same as asking, what are the chances of being within square ① when we can be within square or ① or ②. And the answer is as easy as ①/(①+②).

In fact, we are trying to calculate P(A|B), that is, the way Bayes' theorem generally arranged to answer as in equation (1). However, here this is much easier to understand. Therefore, Bayes' theorem can be expressed just expressed as:

$$\mathbf{P(A|B)} = \frac{\mathbf{P(B|A) \cdot P(A)}}{\mathbf{P(B)}} = \frac{\mathrm{P(B\ U\ A)}}{\mathrm{P(B\ U\ A) + P(B\ U\ not\ A)}} = \frac{①}{① + ②}$$

Isn't this approach much straightforward for answering the problem? What is more, the students can now much better assess in which square/s they are depending on what the problem wording says. There is no need to start rearranging the Bayes' formula in endless ways, nor to resort to other supplementary formulae. Basically, everything you need to answer any two-event Bayesian problem is within the schematic of Figure 2 and with it, once you know the values of some of the probabilities around the edges, you just need to calculate a few rectangle areas.

However, we admit that statistics problems can have many ways of presenting the same information and many ways of playing with the same variables. That is why, we can also show the students all the possible variations Bayes' problems like these can have. This is represented in Figure 4, which actually represents almost all the possible problem combinations that the students might have to work out from the variables stated in Figure 2. Actually, once Figure 4 is understood, there are no more secrets with conditional probabilities.

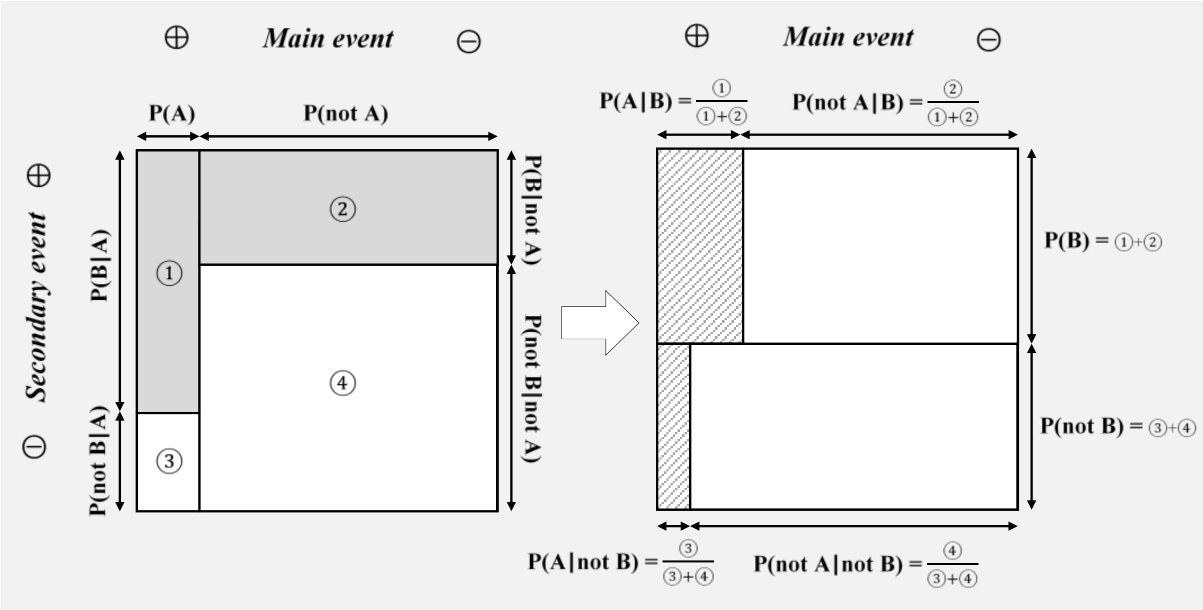

Figure 4. Geometric transformations to express some of the more common probability outcomes of the Bayes' theorem.

3.3 Bayes' theorem with three events

However, previous figures had represented the Bayes' theorem in two dimensions, that is, for two dependent events A and B. But the beauty of this approach is that it also allows the student to visualise the conditional probabilities in cases with three dependent events.

Despite admittedly this requires many more lines and drawing effort that for the 2-event case, Figure 5 represents how to conceptualise conditional probability cases with three events.

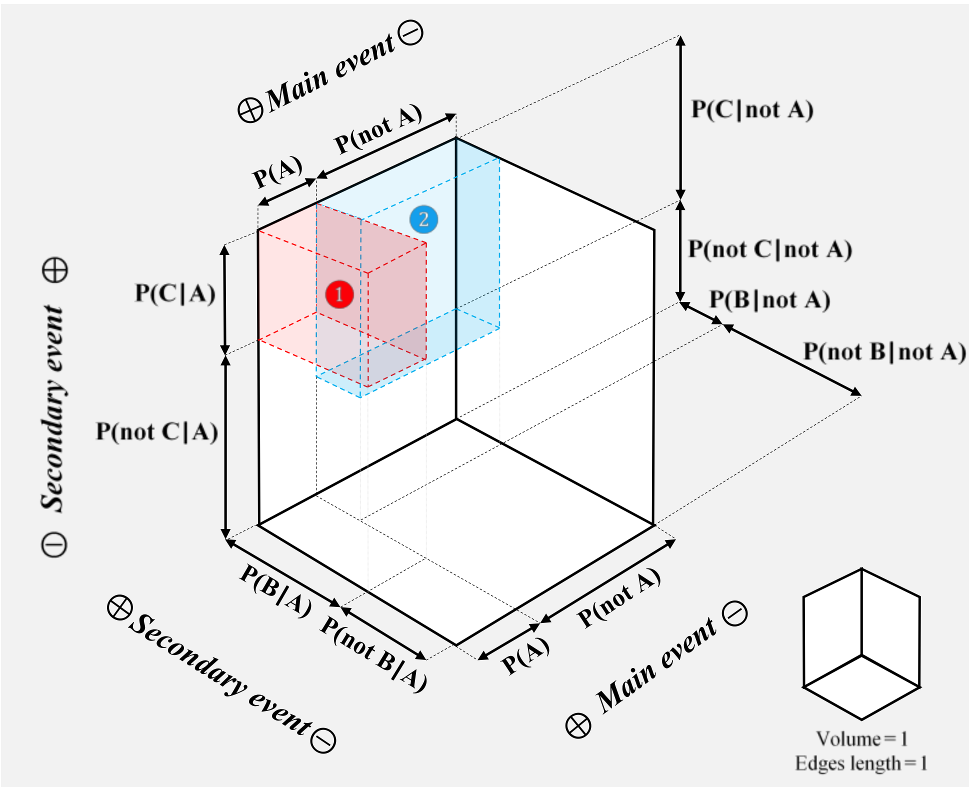

Figure 5. Graphical representation of the Bayes' theorem with three variables.

For instance, Figure 5 might be representing the case of a patient whose results in two different tests (events B and C) have been positive. Now we might want to know the probabilities of him/her having the disease. Again, this is just a matter of calculating the probabilities of being within the red prism (1) once we know that we are either in the red prism (1) or the blue prism (2). In other words, the solution of this problem is again the Bayes theorem for three events, that is:

$$\mathbf{P}(\mathbf{A}|\mathbf{B},\mathbf{C}) = \frac{1}{1 + 2}$$

However, if you had to develop the calculations manually without the assistance of Figure 5, this problem might have been far more confusing.

4 CONCLUSIONS

Bayes' theorem or the Law of Conditional probabilities describes how the probabilities of some events can be calculated when prior information from other non-independent events is known. However, despite its unarguable importance in sciences and engineering, most students struggle to make sense of this theorem. This is particularly more acute when they try to apply it effectively in the wide variety of real-life and exam problems afterwards.

In the absence of other adequate resources for explaining this theorem more effectively, the authors of this paper considered developing a graphical approach. This approach allows any student to assess the posterior probabilities of independent events in whichever form just by calculating the areas of simple rectangles. Hence, the graphical representation proposed promotes understanding and retention, but it also allows working with more complex probability problems like those involving more than two events.

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REFERENCES

[1] H. Jeffreys "Scientific Inference" (3rd edition) Cambridge University Press, UK. p. 31. ISBN 978-0-521-18078-8, 1973.

[2] M. Vanhoucke, "Taking sound business decisions. From rich data to better solutions." (1st edition) OR-AS, Merelbeke, Belgium, 2015.

[3] D.J. Bennett, "Randomness" (1st edition) Harvard University Press, USA, 1999.

[4] R.M. Felder & L.K. Silverman, "Learning and Teaching Styles in Engineering Education," *Engineering Education*, vol. 78, no. 7, pp. 674–681, 1988.

[5] R.M. Felder & R. Brent, "Teaching and learning STEM. A practical guide." (1st ed.) San Francisco, Jossey-Bass, 2016.

[6] J. López-Puga, M. Krzywinski & N. Altman, "Points of Significance: Bayes' theorem," *Nature Methods*, vol.12, pp. 277–278, 2015.

[7] Wyzant, "Further Concepts in Probability," *Last accessed 15 September 2017*. Retrieved from: https://www.wyzant.com/resources/lessons/math/statistics_and_probability/probability/further_concepts_in_probability