

Collaborative task design with student partners in a STEM foundation mathematics course: visual support for the multiplication of matrices

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This paper concerns part of a collaborative project involving ex-students from a university Foundation Studies Programme working with teacher-researchers and analytic assistants in designing computer-based tasks for Foundation level tutorial sessions. This paper focuses on the design and use of a GeoGebra file to assist with students becoming proficient with matrix multiplication. The visual support of highlighting particular rows and columns of two matrices which are to be multiplied together proved helpful for students to develop success with carrying out matrix multiplication but also had a negative effect as well. A particular issue arose concerning the lack of variation in the size of matrices within the first set of questions. This led to difficulties with later questions involving matrices of different sizes.

Keywords: technology; matrices; collaborative; imagery.

Introduction and background

GeoGebra (<https://www.geogebra.org/>) is an open-source piece of software which has versatility to be used for many topics within mathematics. Examples of its use can be found in modelling (Hidiroğlu & Bukova Güzel, 2013) and problem situations which relate to functions (Ofra & Tabach, 2013). In our case, the GeoGebra program is used differently. Firstly, we use it in the context of matrix multiplication. Secondly, we created a series of questions, with sliders and buttons which did not require the learner to know or learn any of the functionality of GeoGebra itself (more details about the file created are given below).

Relatively little research has been carried out on the learning of arithmetic with matrices. Chang (2011) offered lesson notes for working with students on a linear algebra course and used questioning alongside the transformation of the image of a face to support a physical meaning for matrix multiplication. She offered two ways to define matrix multiplication. Larson (2010) explored how students think about the multiplication of a matrix with a vector including seeing it as either a matrix acting on a vector or a vector acting on a matrix. Hannah, Stewart and Thomas (2014) analysed students' views as their university teacher experimented with different orders of presentation on topics, including matrices and solving matrix equations of the form $A\mathbf{x}=\mathbf{b}$ (where A was a matrix and \mathbf{x} and \mathbf{b} were vectors). Students favoured the use of pictures and examples over being given definitions. Imagery is an important aspect of what technology can bring to a topic and this includes the possibility of dynamic images rather than just static versions. Taylor, Pountney and Malabar (2007) looked at topics at undergraduate level, including matrix multiplication, and compared animated images

with static versions of these. With matrix multiplication, numbers from the first matrix were multiplied by certain numbers in the second matrix. Their animation involved these numbers coming together through copies of numbers from the first matrix moving to be alongside the appropriate numbers in the second matrix. Findings suggested this aided students to see which numbers from the first matrix should be multiplied by which numbers from the second. In addition, students considered that the animated version speeded up their understanding of what was involved compared to the static versions.

This paper seeks to add to the limited literature of using visual imagery to assist students' learning of matrix multiplication. It is also part of a wider project (2016-18 The Catalyst Project) to (a) incorporate computer-based tasks into a university Foundation Studies course and (b) involve ex-students from Foundation Studies in the design and implementation of computer tasks for that course. As such this was one part of a highly collaborative project. It involved recruiting ex-students from the previous year's course, who responded to an initial announcement and who were selected following an interview. These we call Student Partners. We also involved current PhD students as Analytic Assistants, who helped with the data collection and analysis of that data. Lastly, there were three teacher-researchers who were members of staff, one of whom taught the Foundation course. Teachers can sometimes fail to see tasks from the students' perspective (Choy, 2016) and Johnson, Coles and Clarke (2017) suggest that both teachers and students should be part of the task design process. This was a crucial aspect of our research project. The overall aim was to investigate how the co-development of computer-based tasks with Student Partners could enhance the students' learning on the course. In particular, this paper focuses on matrix multiplication as this was one aspect of the course the Student Partners identified as being problematic. The particular research question related to this paper was to find out to what extent interactive visual supports enhance students' ability to multiply matrices.

Methodology

During the academic year 2016-2017, three tutorial sessions on matrices took place in a computer laboratory, each lasting 50 minutes. Every student on the Foundation Studies course attended one of these tutorials. Across the three tutorials there were 18 computers set-up with screen-capture software. This captured the screen and mouse movements along with conversations between the students. Two students did not complete any of the matrix multiplication tasks; hence we collected data for this paper from 16 screens. In addition we had, for the purpose of analysis, (i) audio data and photographs from the task design meetings carried out with the Student Partners (SPs), analytic assistants and teacher-researchers, (ii) reflections from SPs, (iii) feedback questionnaires from 13 Foundation Students commenting on the computer tasks and one interview with a Foundation student who attended the matrices tutorial.

Initially, a real-time analysis of each screen-capture was carried out where the video was played in real time and a factual summary of what happened was noted along with timings. Rich conversations, related to the task or reflecting upon the visual imagery, were identified for later transcription. Following transcription, the videos were viewed in greater detail at specific points which were either (a) key moments when mistakes were made; (b) significant movements of the sliders controlling the visual support; (c) where there was change from correct to incorrect entries or vice versa; or (d) rich conversations. A grounded theory (Strauss & Corbin, 1990) approach was taken where codes were developed related to (a) – (d) above. A factual list for each computer screen was created showing whether successive answers entered were right or wrong

and whether the visual support was positioned correctly each time. A focus was then taken on the developing success, or otherwise, of the students across their tutorial time on the matrix multiplication questions and, in particular, the role of the visual support offered from the design of the GeoGebra file.

The design of the matrix tasks

Four design meetings took place between the teacher-researchers, SPs and the analytical assistants to design tasks based upon the use of Autograph and GeoGebra. The purpose of the design meetings was to identify topic areas which the SPs felt would benefit from additional support. Once topics were identified the SPs were involved with contributing to the design of computer tasks, offering visual support for the Foundation Students on the course.

Prior to the first design meeting the SPs were asked to revisit the two topics of complex numbers and matrices and identify what they recalled having difficulty with. One topic identified was matrix multiplication as a topic where it was easy to make mistakes. One SP (SPs will be labelled, SP1, SP2, etc.), SP1, reflected prior to the first meeting that remembering the rules of matrix multiplication was an area of difficulty. At the design meeting SP2 commented that it was confusing as to whether you go “down and along or along and along”. After some discussion about it was decided that GeoGebra might be an appropriate tool to produce a useful file for matrix multiplication. The ideas for two GeoGebra files were developed and collectively discussed in the first design meeting and draft versions of the files brought to the second design meeting. One concerned the multiplication of matrices and is the focus of this paper.

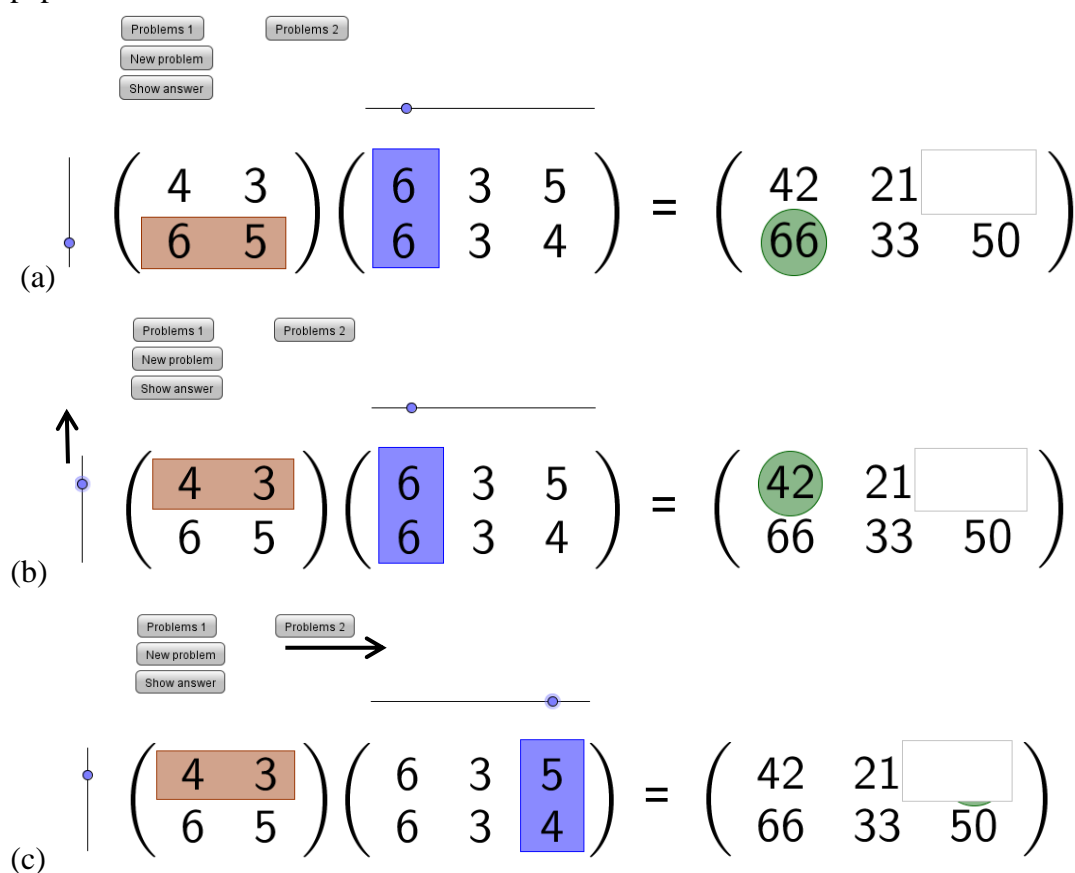


Figure 1 (a-c): Movement of sliders to correct position behind the empty box in the answer matrix.

The GeoGebra file showed two matrices, with randomly generated positive numbers, which were to be multiplied together. All but one value of the answer matrix was shown. The task was to enter in the remaining number. There were shaded rectangles offering visual support and a circle in the answer matrix indicated where the resultant number appeared for the given coloured rectangles. There were sliders alongside the matrices to move the coloured rectangles, with the circle moving automatically as a consequence (see Figure 1(a-c)).

At the first design meeting SP3 was positive about the initial idea and suggested the highlighting of related elements. SP3 felt you could “see in front of your eyes which one is multiplied with which”. SP2 felt it would help visualise the process which they had originally identified as something which could be confusing. In particular they mentioned it helped “forge the link between the... cause and effect”. SP3, however, pointed out that this computer task encourage robotic thinking and it was acknowledged that this task would not be about developing understanding but about assisting with the process.

At the second design meeting a re-worked version of the file was shown where randomly generated positive integers from one to seven appeared in the matrices which were to be multiplied. This version incorporated two new aspects. The first was a button which, when pressed, showed the full answer matrix. This allowed the learner to compare the number they entered with the correct number in the answer matrix. The second aspect was a ‘Problems 2’ (P2) set of questions in addition to the original P1 (see Figure 1) set of questions. Here larger randomly chosen matrices, involving negative as well as zero and positive integers, were involved without the visual support of sliders or coloured rectangles. This showed the partially completed answer matrix with two missing numbers, which were to be entered. As with P1 the complete answer matrix could be revealed as well, and an infinite series of problems of a similar nature produced.

Analysis

Prior to the tutorial sessions, the students on the Foundation course had received a lecture and course notes which included how to multiply matrices. However, seven of the 16 screen-captures revealed students needing help from one of their fellow students in order to even make a start with multiplying matrices. Table 1 below gives an overview of the eventual success of the students on the 16 screen-captures, irrespective of initial difficulties.

Consistent success with P1 & P2	11	Inconsistent success	1
Success with P1 but not with P2	1	Consistently incorrect	1
Success with P1. Did not attempt P2	1	Too little data to comment	1

Table 1: Eventual success of the students, irrespective of initial difficulties

When errors were made most students took time to look at the correct answer matrix and their own, incorrect, entries. With the P1 set of questions, with the exception of one student, every time a student entered an incorrect number this was followed by getting the next task correct. The one exception was a student who did not use the sliders to move the coloured rectangles into their correct place on three consecutive occasions; thus not using the visual support. It was only when the student used the sliders, so that

the coloured rectangles were in the appropriate position, that they obtained correct answers. The remainder of the questions on P1 were then answered correctly.

Power of the visual

The GeoGebra file was designed to offer a strong visual cue to help students recall the process of multiplying matrices. However, on some occasions the students did not use the sliders to move the highlighted rectangles to the correct row and column for the particular missing number in the answer matrix. Instead, they incorrectly carried out the calculation based on the incorrect row and column already highlighted. Thus the visual lure of the highlighted rectangles was stronger than the logic of considering if the row and column were appropriate.

Generally, however, the imagery appeared to offer useful support for many of the students. There were examples of students spending time when the highlighted boxes were positioned so that the circle in the answer matrix was over a number which could be seen (for example see Figure 1(a)). In this way they could see an answer and check how it could be arrived at from the highlighted row and column. Mouse movements were seen ‘pointing’ to the relevant numbers which had to be multiplied. Students then moved the sliders so that the highlighted boxes were in the correct position and then entered a correct answer. At times when mistakes were made, it triggered a discussion with fellow students who were working on a nearby computer, particularly when students struggled with the P2 problems, which no longer provided the visual support of the coloured rectangles. Students were heard helping others by saying “Imagine how it was before [with the highlighted boxes]”, “I kind of imagine... the boxes on the screen”. Many students started off moving the highlighted rectangles to the correct position before answering, thus making use of the visual support. After a while they no longer bothered moving the rectangles as they could answer correctly without that support.

Transition from P1 to P2

The transition from P1 questions (which involved a 2 by 2 matrix multiplied by a 2 by 3 matrix with visual support) to P2 questions (which involved a 3 by 3 matrix multiplied by a 3 by 2 matrix including negative numbers and no visual support) was difficult for some students. Students made mistakes which appeared to relate to difficulties with calculations with negative numbers. However, an interesting finding came from the way in which some students transferred the imagery from the P1 questions to the larger matrices in P2. Many students had difficulty with the first P2 question even after they had been successful with the P1 questions. Some students gave a verbal outburst on seeing the first P2 question on the screen, such as “Oh my lord”. Five students switched back to look again at P1 before returning to P2. Three of these appeared then to transfer literally what they did with the P1 questions; they started off with the larger matrices in P2 by multiplying just the first two pairs of numbers as they had done with P1, rather than the three pairs needed for the larger matrices.

Feedback from Foundation Students

Following the tutorial sessions, we gathered feedback about the matrices tasks from the Student Partners, written feedback from 11 of the Foundation Students who engaged with the tasks in the tutorial and an interview with one Foundation Student. The comments from the Foundation Students were all very positive. One said “I have to

admit I missed the lectures on that [matrices] and hadn't done it before I came to the tutorial. And I can now do it. So I think that just proves that it was helpful... it does give you a visual understanding of how it all works." Several others also mentioned the visual support. For example one commented: "The boxes indicating which columns and rows were good to get me started. After a while I stopped using these." The screen-capture analysis showed that there was a clear issue about the transition from P1 to P2 questions. Although one student commented that they liked the fact that the visual hints were no longer visible for P2 questions, another student suggested that having "an option to add or remove the sliders would be good because then students of varying abilities can use the program". We intend to meet again with the Student Partners to reflect upon this very useful recommendation and all the tasks and to consider changes for their use in future years.

Discussion

The GeoGebra file allowed students the freedom to move sliders in order to see how different rows and columns were associated with particular positions within the answer matrix. There was evidence that some students made good use of these sliders and positioned them so that they could work on how a visible number in the answer matrix might have been obtained. At other times the act of moving the sliders, so that the circle in the answer matrix was in the position of the missing number, meant that the students' attention was taken to the relevant row and column to be multiplied. There was evidence that students were successful when moving the sliders to the correct position when previously they had entered incorrect answers without having moved the sliders.

The GeoGebra file offered instant feedback by showing the correct answer matrix underneath the matrix which had a missing number for the students to enter. Thus, students could immediately see whether they were correct or not. Most students took time, after entering an incorrect number, to see where they had gone wrong. This led to success with the next question. This finding aligns with Lozano (2017) who talked about future actions being shaped due to immediate feedback from the computer. Yildiz and Baltaci (2016) also commented upon students correcting errors due to feedback from GeoGebra and reported that making mistakes created discussion between the students. This was also a feature from our study as on some occasions students consulted another student when they were making mistakes.

Success with P1 questions (which had visual support) did not always translate into success with the P2 questions where the matrices were larger and there was no visual support. Even though students were getting success with P1 questions, the question arises as to what sense of generality was carried forward when meeting the first P2 question. As teachers we might feel that the P1 questions offered examples of the generality of multiplying any two matrices together. However, students do not necessarily know this generality and have to make sense of what they are seeing (Caglayan, 2014), and try to construct rules from the particular examples they meet (Mason, 1996). The fact that quite a few students did not find the transition straightforward, with some multiplying only two pairs of numbers together despite the matrices being larger, indicated there may not be sufficient variation within the P1 questions for students to develop appropriate generality.

Conclusions and ways forward

There is evidence that the visual support offered helped 11 of the 16 groups of students to gain consistent success with multiplying matrices with both the P1 and P2 problems. Only one student was consistently incorrect with their answers. The feedback from the Foundation Students indicates that they found the visual support helpful. In particular, the act of moving the sliders to the correct position resulted in correct answers being entered. However, there were clear difficulties when moving from questions where visual support is provided to questions where this is no longer the case and the matrices are a different size. It appears that there is not enough variation in the size of matrices with the P1 questions. All P1 questions involve multiplying a 2 by 2 matrix with a 2 by 3 matrix. The size of the matrix does not change. This was mainly done due to technical difficulties with producing a file where the size of the matrix can vary. This raises the issue of technical considerations which are a practical reality when designing computer tasks, and the educational variation which is desired for the task to be effective. We feel that the result of this study has shown that although the current file did enhance the students' ability to multiply matrices, more variation within the P1 questions will make the tasks more effective. The GeoGebra file offers a new kind of visual support for students learning to multiply matrices; our research adds to the limited literature in this area.

Seven of the 16 screen captures showed students having no idea of how to even start to multiply matrices. The fact that the Foundation Students had a lecture already and notes that clearly showed how to multiply matrices means that these workshop tasks were invaluable. As such the Student Partners were accurate in identifying matrix multiplication as an important area where additional support could be useful. As teacher-researchers we have found the presence of the student voice in the whole process invaluable - from task initiation, design and reflection on the process. This will feed into the re-designing of the tasks for future use whilst we continue to use the Student Partners in re-working the computer tasks.

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